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Zeros of Real Random Polynomials Spanned by Bergman Polynomials

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Abstract

Let $f_n(z) = \sum_{j=0}^{n} \eta_j p_j(z)$, where $\{\eta_j\}$ are real-valued independent and identically distributed standard normal random variables, and $\{p_j(z)\}$ are Bergman polynomials on the unit disk of the form $p_j(z) = \sqrt{(j+1)/\pi} z^j$, $j \in \{0,1,\ldots,n\}$. From well-known formulas for the expected number of real zeros and purely complex zeros of random polynomials, we prove that the expected number of zeros of $f_n(z)$ in the unit disk is asymptotic to $2n/3$, and of these zeros, asymptotically $\sqrt{2}\log n/\pi$ of them are on $[-1,1]$.

Introduction

The study of the zeros of polynomials of the form

$$k_n(z) = \eta_n z^n + \eta_{n-1} z^{n-1} + \cdots + \eta_1 z + \eta_0,$$

where $\{\eta_k\}_{k=0}^{n}$ are random variables has a rich history (c.f. the books by Bharucha-Reid and Sambandham [1] and Farahmand [3] for a great reference and many early results). Let $\mathbb{E}$ denote the expectation, $N_n(\Omega)$, $\Omega \subset \mathbb{C}$, denote the number of zeros of k_n in Ω. When $\{\eta_k\}_{k=0}^{n}$ are independent and identically distributed (i.i.d.) standard (i.e. with mean zero and variance one) normal random variables, in 1943, Kac [4] gave a formula for the expected number of real zeros $k_n(z)$, and proved that

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{2+o(1)}{\pi} \log n, \quad \text{as} \quad n \to \infty.$$

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In the above, we are using the "little o" Landau notation. Which in the above result means that, after distributing $o(1)\log n = o(\log n)$, there is a function $g(n)$ such that $g(n)/\log n \to 0$ as $n \to \infty$. We remark that in Kac's work, he showed that the expected number of zeros in $[-1,1]$ is asymptotic to $\log n/\pi$ as $n \to \infty$.

In what follows, we fix $\{\eta_k\}_{k=0}^n$ to be real-valued i.i.d. standard normal random variables. Nearly half a decade after Kac's result, Shepp and Vanderbei [9] gave a formula for the purely complex zeros of $k_n(z)$. From their work it follows that

$$\mathbb{E}[N_n(\mathbb{D})] = (1+o(1))\frac{n}{2},$$

where $\mathbb{D}$ denotes the unit disk.

We are interested in what happens when we change the monominal spanning basis, i.e. $\{z^k\}_{k=0}^n$, to a different basis. And if we do so, will these mentioned asymptotics change? Before going into our change of spanning basis, we first make light on a property of the monomials that we would like to somewhat preserve. Namely, $\{z^k\}_{k=0}^n$ are orthogonal on the unit circle with respect to normalized arc length measure. That is, denoting $\mathbb{T}$ as the unit circle, and $z = e^{i\theta}$ since $z \in \mathbb{T}$, and the "overlined bar" as complex-conjugation, it follows that

$$\int_{\mathbb{T}} z^n \overline{z^m}\,\frac{dz}{2\pi} = \int_{-\pi}^{\pi} e^{in\theta} e^{-im\theta}\,\frac{d\theta}{2\pi} = \begin{cases} 1, & \text{when } n = m, \\ 0, & \text{when } n \neq m. \end{cases}$$

Retaining an orthogonality property, we will be taking a spanning basis from a class of functions that are orthogonal on the unit disk with respect to area measure. Specifically, we consider random polynomials of the form

$$f_n(z) = \sum_{j=0}^{n} \eta_j p_j(z) \tag{1}$$

where $\{\eta_j\}$ are real-valued i.i.d. standard normal, and $p_j(z) = \sqrt{(j+1)/\pi}\,z^j$, $j = 0,1,\ldots,n$. Note that these spanning p_j's are indeed orthogonal on the unit disk. To see that this is so, observe that

$$\begin{aligned}
&\int_{\mathbb{D}} \sqrt{\frac{n+1}{\pi}} z^n \overline{\sqrt{\frac{m+1}{\pi}} z^m}\, dA(z) \\
&\qquad = \int_0^1 \int_0^{2\pi} \sqrt{\frac{n+1}{\pi}} (re^{i\theta})^n \overline{\sqrt{\frac{m+1}{\pi}} (re^{i\theta})^m}\, r\, d\theta\, dr \\
&\qquad = \frac{\sqrt{(n+1)(m+1)}}{\pi} \int_0^1 \int_0^{2\pi} r^{n+m+1} e^{i\theta(n-m)}\, d\theta\, dr \\
&\qquad = \begin{cases} 1, & \text{when } n = m, \\ 0, & \text{when } n \neq m. \end{cases}
\end{aligned}$$

We remark that the basis functions $\{p_j(z)\}$ are known as Bergman polynomials and are used as basis functions for representing analytic functions in the unit disk [11].

Below are two plots of zeros of the different random polynomials.

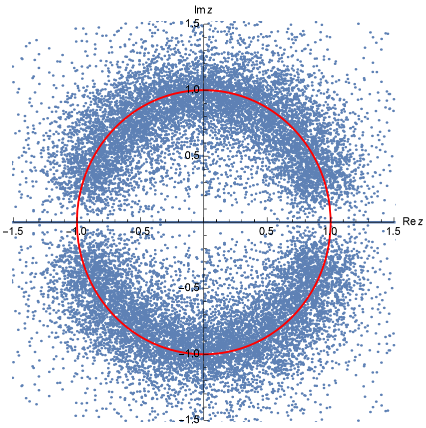

Figure 1:In red is the unit circle and in blue are the zeros in $[-1.5, 15] \times [-1.5, 1.5]$ for 2500 different random polynomials of the form $P_{10}(z) = \sum_{k=0}^{10} \eta_k z^k$.

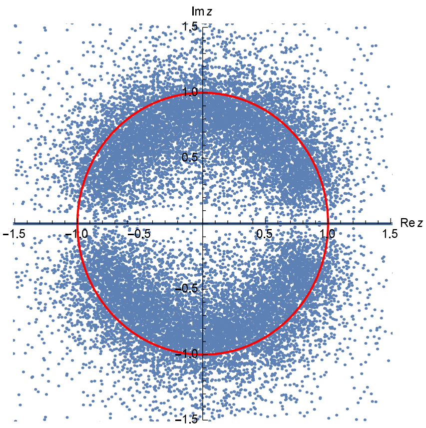

Figure 2: In red is the unit circle and in blue are the zeros in $[-1.5, 15] \times [-1.5, 1.5]$ for 2500 different random polynomials of the form $P_{10}(z) = \sum_{k=0}^{10} \eta_k \sqrt{(k+1)/\pi} z^k$.

From these images, it appears that by switching the basis to the Bergman polynomials, there are more zeros in the unit disk than in the monomial case. Our work will indeed show that this is so. We remark that in both cases, it is known the zeros accumulate near the unit circle (see Pritsker and Yeager [7]). Our approach will be to use known formulas for the expected number of real zeros and the expected number of complex zeros of $f_n(z)$, and then to find asymptotics as $n \to \infty$ for these formulas.

Random polynomials spanned by Bergman polynomials on the unit disk have not been as widely studied as the cases when the spanning functions are orthogonal on the real line or on the unit circle. We also note that recently it was shown that if the random variables $\{\eta_k\}$ are i.i.d. complex-valued standard normal, $f_n(z)$ does in fact have more zeros in the unit disk (and actually has the same asymptotic as our results, c.f. [5]). We are motivated by these situations for our research.

Main Results

The formula for the expected number of real zeros given by Edelman and Kostlan [2] (with alternative proofs given by Vanderbei [10] , Pritsker Lubinsky Xie [6], and Yeager [12]) provides an explicit formula for $\mathbb{E}[N_n(\Omega)]$ when $\Omega \subset \mathbb{R}$. Denoting $\rho_n(x)$ as the density function (see the Proofs Section for the explicit form), that is

$$\mathbb{E}[N_n(\Omega)] = \int_\Omega \rho_n(x)\, dx,$$

our first result allows us to simplify $\rho_n(x)$ associated to the expected number of real zeros of $f_n(z)$.

Lemma 1. *The density function for the real zeros for* (1) *simplifies as*

$$\rho_n(x) = \frac{1}{\pi}\sqrt{\frac{2}{(1-x^2)^2} - \frac{(n+1)(n+2)x^{2n}(x^{2n+4}-(n+2)x^2+n+1)}{(1+x^{2n+2}((n+1)x^2-(n+2)))^2}}. \tag{2}$$

From the simplified shape of ρ_n, we are able to find the following asymptotic with an explicit upper bound constant.

Theorem 1. *The expected number of real zeros of* (1) *in the line segment* $[-1,1]$ *satisfies*

$$\mathbb{E}[N_n([-1,1])] = \left(\frac{\sqrt{2}}{\pi} + o(1)\right)\log n, \quad \textit{as} \quad n \to \infty, \tag{3}$$

where in the upper bound of the above we have

$$C = \frac{1}{\pi}\left(\sqrt{2}\log 2 + 2\sqrt{\frac{1}{2} - \frac{e^2+1}{(e^2-3)^2}}\right) = 0.473729\ldots.$$

For our next result, observe that

$$\mathbb{E}[N_n(\mathbb{D})] = \mathbb{E}[N_n(\mathbb{D}\setminus[-1,1])] + \mathbb{E}[N_n([-1,1])].$$

Taking into account (3) and that $\log n/n \to 0$ as $n \to \infty$, we see that

$$\frac{\mathbb{E}[N_n(\mathbb{D})]}{n} = \frac{\mathbb{E}[N_n(\mathbb{D}\setminus[-1,1])]}{n} + o(1), \quad \text{as} \quad n \to \infty. \tag{4}$$

Thus to find an asymptotic for the expected number of zeros in the unit disk, it suffices to find one for the purely complex zeros in the unit disk. Using a formula provided by Vanderbei [10] for these purely complex zeros (see Proofs section for this explicit formula), we are obtain the following:

Theorem 2. *For random polynomial given in* (1)*, it follows that*

$$\mathbb{E}[N_n(\mathbb{D})] = \left(\frac{2}{3} + o(1)\right) n, \quad as \quad n \to \infty.$$

The Proofs

The mentioned formula for the expected number of real zeros provided by Edelman and Kostlan et. al. is the following

$$\mathbb{E}[N_n(\Omega)] = \int_\Omega \rho_n(x)\, dx$$

where

$$\rho_n(x) = \frac{1}{\pi}\sqrt{\frac{K_n(x,x)K_n^{(1,1)}(x,x) - K_n^{(1,0)}(x,x)^2}{K_n(x,x)^2}} \tag{5}$$

with

$$K_n(z,w) = \sum_{j=0}^{n} p_j(z)\overline{p_j(w)} \tag{6}$$

$$K_n^{(1,0)}(z,w) = \sum_{j=0}^{n} p_j'(z)\overline{p_j(w)} \tag{7}$$

$$K_n^{(1,1)}(z,w) = \sum_{j=0}^{n} p_j'(z)\overline{p_j'(w)}. \tag{8}$$

We now show how to simplify $\rho_n(x)$.

Proof of Lemma 1. Observe that the kernels are

$$K_n(z,w) = \sum_{k=0}^{n}\left(\frac{k+1}{\pi}\right)(z\overline{w})^k = \frac{1+(z\overline{w})^{n+1}((n+1)z\overline{w}-(n+2))}{\pi(1-z\overline{w})^2}, \tag{9}$$

$$K_n^{(1,0)}(z,w) = \sum_{k=0}^{n}\left(\frac{k+1}{\pi}\right)kz^{k-1}\overline{w}^k = \frac{2\overline{w}K_n(z,w)}{1-z\overline{w}} - \frac{(n+1)(n+2)z^n\overline{w}^{n+1}}{\pi(1-z\overline{w})}, \tag{10}$$

$$K_n^{(1,1)}(z,w) = \sum_{k=0}^{n}\left(\frac{k+1}{\pi}\right)k^2z^k\overline{w}^k = \frac{2(1+2z\overline{w})K_n(z,w)}{(1-z\overline{w})^2} - \frac{(n+1)(n+2)z^n\overline{w}^n(1+2z\overline{w})}{\pi(1-z\overline{w})^2} - \frac{n(n+1)(n+2)z^n\overline{w}^n}{\pi(1-z\overline{w})}. \tag{11}$$

After much algebraic simplification one sees that

$$\begin{aligned}
K_n(x,x)K_n^{(1,1)}(x,x) - |K_n^{(1,0)}(x,x)|^2 = \\
&\frac{2K_n(x,x)^2}{(1-x^2)^2} - \frac{n(n+1)(n+2)x^{2n}K_n(x,x)}{\pi(1-x^2)} \\
&\quad - \frac{(n+1)(n+2)x^{2n}(1-2x^2)K_n(x,x)}{\pi(1-x^2)^2} \\
&\quad - \frac{(n+1)^2(n+2)^2x^{4n+2}}{\pi^2(1-x^2)^2}.
\end{aligned}$$

Using (5) then further simplifying gives

$$\begin{aligned}
\pi\rho_n(x) &= \sqrt{\frac{K_n(x,x)K_n^{(1,1)}(x,x) - |K_n^{(0,1)}(x,x)|^2}{K_n(x,x)^2}} \\
&= \sqrt{\frac{2}{(1-x^2)^2} - \frac{(n+1)(n+2)x^{2n}(x^{2n+4}-(n+2)x^2+n+1)}{(1+x^{2n+2}((n+1)x^2-(n+2)))^2}}. \qquad (12)
\end{aligned}$$

■

Proof of Theorem 1. From (12), it is clear that $\rho_n(-x) = \rho_n(x)$. Thus

$$\mathbb{E}[N_n([-1,1])] = \int_{-1}^{1} \rho_n(x)\,dx = 2\int_0^1 \rho_n(x)\,dx.$$

Upper Estimate: For our upper estimate, we write

$$\mathbb{E}[N_n([-1,1])] = 2\int_0^{1-1/n} \rho_n(x)\,dx + 2\int_{1-1/n}^{1} \rho_n(x)\,dx := I_1 + I_2.$$

To estimate I_1 from above, we would like to conclude that for $x \in [0, 1-1/n]$, we have

$$\begin{aligned}
\rho_n(x) &= \frac{1}{\pi}\sqrt{\frac{2}{(1-x^2)^2} - \frac{(n+1)(n+2)x^{2n}(x^{2n+4}-(n+2)x^2+n+1)}{(1+x^{2n+2}((n+1)x^2-(n+2)))^2}} \qquad (13) \\
&\le \frac{\sqrt{2}}{\pi} \cdot \frac{1}{1-x^2}. \qquad (14)
\end{aligned}$$

That is, we need to argue that

$$0 \le \frac{(n+1)(n+2)x^{2n}(x^{2n+4}-(n+2)x^2+n+1)}{(1+x^{2n+2}((n+1)x^2-(n+2)))^2}. \qquad (15)$$

Since the denominator is clearly non-negative, that leaves us to focus on the numerator. As

$$(n+1)(n+2)x^{2n} \ge 0,$$

we actually need to show that

$$x^{2n+4} - (n+2)x^2 + n + 1 \geq 0.$$

We first make the observation that since $0 \leq x \leq 1 - 1/n$, it follows that $x^2 \leq 1 - 2/n + 1/n^2$. Thus

$$x^{2n+4} - (n+2)x^2 + n + 1 \geq -(n+2)\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) + n + 1 = \frac{3n-2}{n^2} + 1 > 0,$$

as desired given that n is a natural number.

Thus (15) holds true, so that we have justified our bound (13). Therefore

$$I_1 \leq \frac{2}{\pi}\int_0^{1-1/n} \frac{\sqrt{2}}{1-x^2}\,dx = \frac{\sqrt{2}}{\pi}\left(\log n + \log(2 - 1/n)\right) \leq \frac{\sqrt{2}}{\pi}\left(\log n + \log 2\right). \tag{16}$$

Turning now to I_2, changing the variable $x = 1 - y/n$ we have

$$I_2 = 2\int_{1-1/n}^1 \rho_n(x)\,dx = 2\int_0^1 \frac{\rho_n(1-y/n)}{n}\,dy.$$

As we seek an asymptotic for I_2, we will be estimating this integral in the limiting sense. Moreover, we would like to pass the limit as n tends to infinity over the integral. To achieve this goal, as $[0,1]$ is a closed interval, we need to know that the limit would hold uniformly for $y \in [0,1]$ as $n \to \infty$.

To this end, since it is known that $(1 - y/n)^n$ converges uniformly to e^{-y} for $y \in [0,1]$ as $n \to \infty$, using (12) and properties of uniform convergence (e.g. differences, products, quotients, etc.), and finally doing some algebra, we achieve uniformly for $y \in [0,1]$ that

$$\lim_{n\to\infty} \frac{\rho_n(1-y/n)}{n} = \frac{1}{\pi}\sqrt{\frac{1}{2y^2} + \frac{e^{2y}(1-2y)-1}{(e^{2y}-2y-1)^2}}.$$

Hence we indeed have

$$\begin{aligned}\lim_{n\to\infty} I_2 &= 2\int_0^1 \lim_{n\to\infty} \frac{\rho_n(1-y/n)}{n}\,dy\\ &= \frac{2}{\pi}\int_0^1 \sqrt{\frac{1}{2y^2} + \frac{e^{2y}(1-2y)-1}{(e^{2y}-2y-1)^2}}dy\\ &= \int_0^1 t(y)\,dy.\end{aligned}$$

where

$$t(y) := \frac{2}{\pi}\sqrt{\frac{1}{2y^2} + \frac{e^{2y}(1-2y)-1}{(e^{2y}-2y-1)^2}}.$$

While $t(y)$ does appear to have a singular point at $y=0$, this point is removable as

$$\lim_{y\to 0} t(y) = \frac{\sqrt{2}}{3\pi}.$$

Thus $t(y)$ is a continuous function. Since $t(y)$ is a continuous function on $[0,1]$, by the extreme value theorem $t(y)$ attains its maximum over this interval. As the graph below shows that $t(y)$ is increasing on $[0,1]$ (we note that this can also be seen by $t'(y)>0$ over $y\in[0,1]$), it must take its largest value at $y=1$.

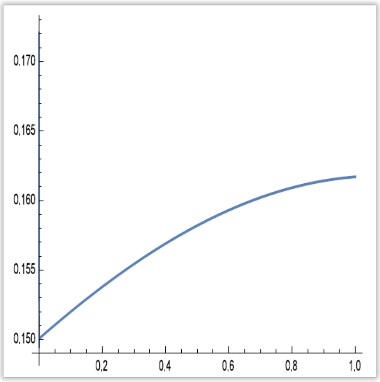

Figure 3: In blue is the limiting value $t(y)$ of the integrand for I_2 over its domain of integration $[0,1]$.

Hence we achieve

$$\lim_{n\to\infty} I_2 \le \frac{2}{\pi}\int_0^1 \max_{0\le y\le 1}\left(\sqrt{\frac{1}{2y^2}+\frac{e^{2y}(1-2y)-1}{(e^{2y}-2y-1)^2}}\right)dy \le \frac{2}{\pi}\sqrt{\frac{1}{2}-\frac{e^2+1}{(e^2-3)^2}}. \tag{17}$$

Combining (16) and (17) we see that

$$\mathbb{E}[N_n([-1,1])] \le \frac{\sqrt{2}}{\pi}\log n + C, \quad \text{as} \quad n\to\infty, \tag{18}$$

where

$$C = \frac{1}{\pi}\left(\sqrt{2}\log 2 + 2\sqrt{\frac{1}{2}-\frac{e^2+1}{(e^2-3)^2}}\right) = 0.473729\ldots.$$

Lower Estimate: To begin our lower bound, we first write

$$\begin{aligned}\rho_n(x) &= \frac{1}{\pi}\sqrt{\frac{2}{(1-x^2)^2}-\frac{(n+1)(n+2)x^{2n}(x^{2n+4}-(n+2)x^2+n+1)}{(1+x^{2n+2}((n+1)x^2-(n+2)))^2}}\\ &= \frac{\sqrt{2-h_n(x)}}{\pi(1-x^2)},\end{aligned}$$

where

$$h_n(x) = \frac{(1-x^2)^2(n+1)(n+2)x^{2n}(x^{2n+4}-(n+2)x^2+n+1)}{(1+x^{2n+2}((n+1)x^2-(n+2)))^2}.$$

We now let $0 < \delta, \varepsilon, \gamma < 1$ be arbitrary. Our first estimate is

$$\begin{aligned}\mathbb{E}[N_n([-1,1])] &= 2\int_0^1 \rho_n(x)\,dx \\ &= \frac{2}{\pi}\int_0^1 \frac{\sqrt{2-h_n(x)}}{1-x^2}\,dx \\ &> \frac{2}{\pi}\int_0^{1-n^{\delta-1}} \frac{\sqrt{2-h_n(x)}}{1-x^2}\,dx. \end{aligned} \tag{19}$$

We now begin to estimate $h_n(x)$. Since $0 \le x \le 1-n^{\delta-1} < 1$, for the numerator of $h_n(x)$ it follows that

$$\begin{aligned}(1-x^2)^2(n+1)(n+2)x^{2n}&(x^{2n+4}-(n+2)x^2+n+1) \\ &\le 2(n+1)(n+2)x^{2n}(x^{2n+4}+n+1) \\ &\le 2(n+1)(n+2)x^{2n}(1+n+1) \\ &\le 2(n+1)(n+2)^2(1-n^{\delta-1})^{2n}.\end{aligned}$$

As the product of $(1-n^{\delta-1})^n$ and any power of n goes to zero as n tends to infinity, the above can be made smaller than ε for large enough n. Thus

$$(1-x^2)^2(n+1)(n+2)x^{2n}(x^{2n+4}-(n+2)x^2+n+1) \le \varepsilon. \tag{20}$$

In a similar fashion, turning to the denominator $h_n(x)$, observe that for large enough n we have

$$1+x^{2n+2}((n+1)x^2-(n+2)) \ge 1-(n+2)(1-n^{\delta-1})^{2n+2} > 1-\gamma,$$

so that

$$\frac{1}{(1+x^{2n+2}((n+1)x^2-(n+2)) \ge 1-(n+2)(1-n^{\delta-1})^{2n+2})^2} \le \frac{1}{(1-\gamma)^2}. \tag{21}$$

Combining (20) and (21) we see that

$$h_n(x) \le \frac{\varepsilon}{(1-\gamma)^2} := \zeta,$$

where $\zeta > 0$ is arbitrary small for large n.

Therefore (19) is bounded below by

$$\int_0^{1-n^{\delta-1}} \frac{\sqrt{2-\gamma}}{\pi}\cdot\frac{1}{1-x^2}dx > \frac{(\sqrt{2-\gamma})(1-\delta)}{2\pi}\log n.$$

Combining the above with (18) gives the desired result that

$$\mathbb{E}[N_n([-1,1])] = \left(\frac{\sqrt{2}}{\pi} + o(1)\right)\log n, \quad \text{as} \quad n\to\infty.$$

■

Proof of Theorem 2. From (4), to compute an asymptotic for $\mathbb{E}[N_n(\mathbb{D})]/n$ it suffices to find one for $\mathbb{E}[N_n(\mathbb{D}\setminus[-1,1]))/n$. To this end, using our notation of the kernels given in (9) and (10), applying Proposition 2.1 of [10] it follows that

$$\frac{\mathbb{E}[N_n(\mathbb{D}\setminus[-1,1])]}{n} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F_n(z)}{n}\, dz, \tag{22}$$

where

$$F_n(z) = \frac{K_n^{(1,0)}(z,z)\sqrt{(K_n(z,z)^2 - |K_n(z,\bar{z})|^2} + K_n(z,z)K_n^{(1,0)}(z,z) - \overline{K_n(z,\bar{z})}K_n^{(1,0)}(z,\bar{z})}{K_n(z,z)\sqrt{(K_n(z,z)^2 - |K_n(z,\bar{z})|^2} + (K_n(z,z))^2 - |K_n(z,\bar{z})|^2}.$$

We will be passing the limit as $n\to\infty$ over the contour integral (22). As noted by Vanderbei (cf. Lemma 2.3 in [10]), $F_n(z)$ has a (bounded) jump discontinuity when $K_n(z,z) = K_n(z,\bar{z})$, which occurs when $z = \pm 1$ (i.e. when $\theta = \pm\pi$). This problem will be handled as follows. Let $\varepsilon > 0$. It is clear that

$$\lim_{\varepsilon\downarrow 0}\int_{-\pi+\varepsilon}^{\pi-\varepsilon} \frac{F_n(z)}{n}\, dz = \int_{-\pi}^{\pi}\frac{F_n(z)}{n}\, dz = \int_{\mathbb{T}}\frac{F_n(z)}{n}\, dz.$$

If we can show that $F_n(z)/n$ converges uniformly to some $F(z)$ on $[-\pi+\varepsilon, \pi-\varepsilon]$ (we note which implies that viewing the sequence $\{\int F_n/n\}$ converges uniformly to $\{\int F\}$ on $[-\pi+\varepsilon,\pi-\varepsilon]$), then appealing to Moore-Osgood Theorem (Rudin Theorem 7.11 [8]), we will have

$$\begin{aligned}
\lim_{n\to\infty}\lim_{\varepsilon\downarrow 0}\int_{-\pi+\varepsilon}^{\pi-\varepsilon}\frac{F_n(z)}{n}\, dz &= \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty}\int_{-\pi+\varepsilon}^{\pi-\varepsilon}\frac{F_n(z)}{n}\, dz \\
&= \lim_{\varepsilon\downarrow 0}\int_{-\pi+\varepsilon}^{\pi-\varepsilon}\lim_{n\to\infty}\frac{F_n(z)}{n}\, dz \\
&= \lim_{\varepsilon\downarrow 0}\int_{-\pi+\varepsilon}^{\pi-\varepsilon} F(z)\, dz \\
&= \int_{\mathbb{T}} F(z)\, dz.
\end{aligned} \tag{23}$$

We now work to show that $F_n(z)/n$ is uniformly convergent on $\mathbb{T}\setminus\{\pm 1\}$. First, we factor $F_n(z)$ the following way

$$F_n(z) = \frac{K_n^{(1,0)}(z,z)\left(\sqrt{1-a_n(z)} + 1 - b_n(z)c_n(z)\right)}{K_n(z,z)\left(\sqrt{1-a_n(z)} + 1 - a_n(z)\right)},$$

where

$$a_n(z) = \frac{|K_n(z,\bar{z})|^2}{(K_n(z,z))^2}, \qquad b_n(z) = \frac{\overline{K_n(z,\bar{z})}}{K_n(z,z)}, \qquad c_n(z) = \frac{K_n^{(1,0)}(z,\bar{z})}{K_n^{(1,0)}(z,z)}.$$

We note that when $z \in \mathbb{T} \setminus \{\pm 1\}$, we have $z = e^{i\theta}$ and $|z| = 1$, so that

$$K_n(z,z) = \sum_{k=0}^{n} \frac{k+1}{\pi} |z|^{2k} = \sum_{k=0}^{n} \frac{k+1}{\pi} = \frac{(n+1)(n+2)}{2\pi}, \tag{24}$$

and

$$\begin{aligned} K_n(z,\bar{z}) &= \sum_{k=0}^{n} \frac{k+1}{\pi} z^{2k} \\ &= \sum_{k=0}^{n} \frac{k+1}{\pi} e^{2ki\theta} = \frac{1 + e^{2i\theta(n+1)}(e^{2i\theta}(n+1) - (n+2))}{\pi(1-e^{2i\theta})^2}. \end{aligned} \tag{25}$$

As

$$\lim_{n\to\infty} \frac{e^{2i\theta(n+1)}}{n} = 0, \quad \text{and} \quad \lim_{n\to\infty} \frac{e^{2i\theta}(n+1) - (n+2)}{n} = e^{2i\theta} - 1, \tag{26}$$

both hold uniformly for $\theta \in \mathbb{T} \setminus \{\pm 1\}$, using (24) and (25), it follows that

$$\lim_{n\to\infty} \frac{K_n(z,\bar{z})}{K_n(z,z)} = \lim_{n\to\infty} \frac{2(1 + e^{2i\theta(n+1)}(e^{2i\theta}(n+1) - (n+2)))}{(n+1)(n+2)(1-e^{2i\theta})} = 0$$

uniformly for $\theta \in \mathbb{T} \setminus \{\pm 1\}$. Similarly, we have

$$\lim_{n\to\infty} \frac{\overline{K_n(z,\bar{z})}}{K_n(z,z)} = \lim_{n\to\infty} \frac{2(1 + e^{-2i\theta(n+1)}(e^{-2i\theta}(n+1) - (n+2)))}{(n+1)(n+2)(1-e^{-2i\theta})} = 0.$$

Thus $a_n(z) \to 0$ and $b_n(z) \to 0$ both uniformly for $\theta \in \mathbb{T} \setminus \{\pm 1\}$ as $n \to \infty$.

Turning now to $c_n(z)$, $z \in \mathbb{T} \setminus \{\pm 1\}$, we have

$$\begin{aligned} K_n^{(1,0)}(z,z) &= \sum_{k=0}^{n} \frac{k(k+1)}{\pi} z^{k-1} \overline{z^k} \\ &= z^{-1} \sum_{k=0}^{n} \frac{k(k+1)}{\pi} |z|^{2k} \\ &= e^{-i\theta} \sum_{k=0}^{n} \frac{k(k+1)}{\pi} \\ &= \frac{\bar{z}}{\pi} \cdot \frac{n(n+1)(n+2)}{3}. \end{aligned} \tag{27}$$

As

$$K_n^{(1,0)}(z,\bar{z}) = \sum_{k=0}^{n} \left(\frac{k+1}{\pi}\right) k z^{k-1} z^k = \frac{2zK_n(z,\bar{z})}{1-z^2} - \frac{(n+1)(n+2)z^{2n+1}}{\pi(1-z^2)},$$

replacing $z = e^{i\theta}$, and dividing the above by (27), and appealing to the limits (26), we see that $c_n(z) \to 0$ uniformly for $z \in \mathbb{T} \setminus \{\pm 1\}$ as $n \to \infty$.

Therefore we achieve that

$$\lim_{n\to\infty} \frac{F_n(z)}{K_n^{(1,0)}(z,z)/K_n(z,z)} = 1$$

uniformly for $z \in \mathbb{T} \setminus \{\pm 1\}$. Since

$$\frac{K_n^{(1,0)}(z,z)}{K_n(z,z)} = \frac{2n}{3}\bar{z},$$

our above limit says

$$\lim_{n\to\infty} \frac{F_n(z)}{n} = \frac{2}{3}\bar{z}.$$

Appealing now to (22) and (23), we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mathbb{E}[N_n(\mathbb{D} \setminus [-1,1])]}{n} &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{2}{3}\bar{z}\, dz \\
&= \frac{2}{3\pi} \int_{\mathbb{D}} \frac{d}{d\bar{z}}(\bar{z})\, dA(z) \\
&= \frac{2}{3\pi} \int_{\mathbb{D}} 1\, dA(z) \\
&= \frac{2}{3},
\end{aligned}$$

where we have applied the Complex Version of Green's Theorem in the second equality above. Thus the expected number of purely complex zeros of (1) is asymptotic to $2n/3$. Therefore, by observation (4), we have

$$\mathbb{E}[N_n(\mathbb{D})] = \left(\frac{2}{3} + o(1)\right) n, \quad \text{as} \quad n \to \infty.$$

■

References

[1] A. T. Bharucha-Reid and M. Sambandham, Random polynomials, Academic Press, Orlando, 1986.

[2] A. Edelman and E. Kostlan, How many zeros of a random polynomial are real?, Bull. Amer. Math. Soc. 32 (1995) 1-37.

[3] K. Farahmand, Topics in random polynomials, Pitman Res. Notes Math. 393 (1998).

[4] M. Kac, On the average number of real roots of a random algebraic equation, Bull. Amer. Math. Soc. 49 (1943) 314–320.

[5] M. Landi, K. Johnson, G. Moseley, and A. Yeager, "Zeros of complex random polynomials spanned by Bergman polynomials," Involve: J. Math. (2021), Vol. 14, no. 2, 271–281.

[6] D. Lubinsky, I. Pritsker, and X. Xie, Expected number of real zeros for random linear combinations of orthogonal polynomials, Proc. Amer. Math. Soc. 144 (2016) 1631–1642.

[7] I. Pritsker and A. Yeager, Zeros of polynomials with random coefficients, J. Approx. Theory 189 (2015), 88–100.

[8] W. Rudin, Principles of Mathematical Analysis, McGraw Hill, New York, Third Ed., 2013.

[9] L. Shepp and R. Vanderbei, The complex zeros of random polynomials, Trans. Amer. Math. Soc. 347 (1995), 4365–4384.

[10] R. Vanderbei, The complex zeros of random sums, arXiv: 1508.05162v1 Aug. 21, 2015.

[11] H. Stahl and V. Totik, General Orthogonal Polynomials, Cambridge Univ. Press, New York, 1992.

[12] A. Yeager, Real Zeros of Random Sums with I.I.D. Coefficients, to appear in Colloq. Math. (2019).

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A Simple Solution for an Interesting Triangle Construction

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Abstract

We establish a simple construction of a triangle given an internal angle α, the length t_A of the angle bisector of α, and the difference $b-c$ of the lengths b and c of the sides forming α, using a property of harmonic range.

Introduction

In 2016, Paris Pamfilos used the detection of a parabola to give a solution of the following construction problem [3]. Let ABC be a triangle. Denote the lengths of the sides as: $a=|BC|$, $b=|CA|$, and $c=|AB|$ (see Figure 2). Let $\angle BAC=\alpha$, and $t_A=|AD|$, where D is the intersection of side BC and the angle bisector of α. The problem is to construct the triangle ABC given α, $b-c$, t_A, with $b>c$.

In 2018, Martina Stepanova gave a shorter solution for this construction [4].

In this paper, we present another simple and new solution for this construction problem using a property of the harmonic range.

Properties of Harmonic Range

First of all, we would like to reiterate the formal definition of harmonic range as follows

Definition. *Collinear points A, B, C, D form a* ***harmonic range*** *of points* (A,B,C,D) *if and only if a direction is established on the line and* $\frac{CA}{CB}=-\frac{DA}{DB}$, *where it is understand that CA represents the directed distance from* **C** *to A, and similarly for the other distances.*

In this section, we give a characterization of harmonic range [1, 2] as follows:

Proposition 3. *Let (A, P, E, C) be a set of points on a line. Let M and N be midpoints of* $\overline{AP}$ *and* $\overline{EC}$ *respectively. Then, (A, P, E, C) is a harmonic range if and only if*

$$4MN^2=AP^2+EC^2.$$

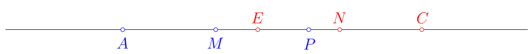

Figure 1: Proof of Proposition 1

Proof. (See Figure 1). Let the coordinates of points A, P, E, and C on a given axis be a, p, e, and c, respectively. Then (A, P, E, C) is a harmonic range if and only if

$$\begin{aligned}
&\frac{EA}{EP} = -\frac{CA}{CP} \\
&\qquad \iff \frac{a-e}{p-e} = -\frac{a-c}{p-c} \\
&\qquad \iff (a-c)\cdot(p-e) = -(a-e)\cdot(p-c) \\
&\qquad \iff 2(ap+ec) = (a+p)(e+c) \\
&\qquad \iff -4(ap+ec) = -2(a+p)(e+c) \\
&\qquad \iff (a^2-2ap+p^2)+(e^2-2ec+c^2) = \\
&\qquad\quad (a^2+2ap+p^2)+(e^2+2ec+c^2)-2(a+p)(e+c) \\
&\qquad \iff (a-p)^2+(e-c)^2 = (a+p)^2+(e+c)^2-2(a+p)(e+c) \\
&\qquad \iff (a-p)^2+(e-c)^2 = 4\left(\frac{a+p}{2}-\frac{e+c}{2}\right)^2 \\
&\qquad \iff AP^2+EC^2 = 4MN^2.
\end{aligned}$$

From this, we obtain (A, P, E, C) is the harmonic range if and only if

$$AP^2+EC^2 = 4MN^2.$$

■

Suppose for a given triangle ABC that $\overline{AD}$ is the angle bisector of $\angle BAC$, with D in $\overline{BC}$. If we let E be the reflection of B through the line-segment $\overline{AD}$, then $|CE| = b-c$.

Proposition 4. *Let P be the point on $\overline{AC}$ of triangle DEC so that $\overline{DP}$ is the angle bisector of $\angle EDC$. Then $\overline{AD} \perp \overline{DP}$, and (A,P,E,C) forms a harmonic range.*

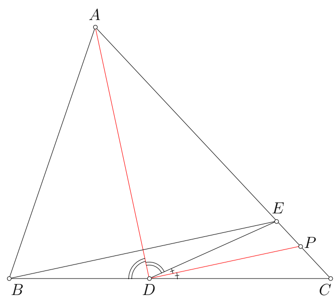

Figure 2: Proof of Proposition 4

Proof. (See Figure 2). Notice that E is on the line-segment $\overline{AC}$ since $\overline{AD}$ is an internal angle bisector of $\angle ABC$ and $b > c$. From $\triangle ABD \cong \triangle AED$, we see that

$\overline{DA}$ is the angle bisector of $\angle BDE$. Therefore two angle bisectors $\overline{DP}$ and $\overline{DA}$ are perpendicular because because $\angle BDE$ and $\angle CDE$ are complementary adjacent angles.

Using properties of angle bisector in triangle with note that $\overline{DP}$ and $\overline{DA}$ are internal and external bisector at vertex D of triangle DEC, we have

$$\frac{PE}{PC} = -\frac{|DE|}{|DC|} = -\frac{AE}{AC}.$$

Thus, (A,P,E,C) forms a harmonic range. ∎

A New Triangle Construction

Suppose a point A is given, along with an angle α at A and length "$b-c$" ($b>c$) representing the (positions) difference between the length of the two sides of the angle at A, and the length t_A from A to the point D such that $\overline{AD}$ is the angle bisector of α. We need to determine the locations of points B and C so that $\triangle ABC$ can be constructed.

Let E be the reflection point of B through the line-segment $\overline{AD}$. Notice that E is on the line-segment $\overline{AC}$ since $\overline{AD}$ is an internal angle bisector of $\angle BAC$ and $b>c$ ($\triangle ABD \cong \triangle AED$). Suppose the perpendicular line to $\overline{AD}$ through D meets $\overline{AC}$ at P. Using Proposition 4, (A,P,E,C) is a harmonic range. Let M, N be the midpoints of $\overline{AP}$ and $\overline{EC}$ respectively. By Proposition 3,

$$AP^2 + EC^2 = 4MN^2.$$

Now we can deduce the positions of B and C.

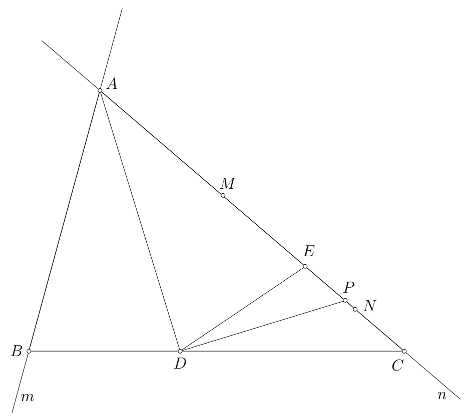

Figure 3: New construction of a triangle under the given conditions

So, we use the following steps of constructions (see Figure 3):

1. Draw a line-segment $\overline{AD}$ of length t_A.
2. Draw lines m and n through A such that they form angles $\frac{\alpha}{2}$ with $\overline{AD}$.
3. Draw a perpendicular line to $\overline{AD}$ at D, and let this line meet n at P.

4. Draw midpoint M of $\overline{AP}$.
5. Draw point N on ray $\overrightarrow{AP}$ such that $|MN| = \frac{1}{2}\sqrt{AP^2 + (b-c)^2}$, and M is between A and N.
(Note that N is correctly constructed by the observation in Proposition 3)
6. Draw circle $(N, \frac{b-c}{2})$ which meets line n at E and C such that E is inside the segment $\overline{AP}$.
7. Let B be the reflection of E in line-segment $\overline{AD}$.

Since we have located points B and C, we can complete the construction of $\triangle ABC$.

References

[1] E. W. Weisstein, Harmonic Range from *MathWorld–A Wolfram Web Resource*, `http://mathworld.wolfram.com/HarmonicRange.html`.

[2] C. V. Durell, Harmonic Ranges and Pencils, Ch. 6 in Modern Geometry: The Straight Line and Circle. London: Macmillan, pp. 65–67, 1928.

[3] P. Pamfilos, The Triangle Construction $\{\alpha,\ b-c,\ t_A\}$, *Forum Geom.*, **16** (2016) pp. 115–117.

[4] M. Stepanova, New Constructions of Triangle from α, $b-c$, t_A, *Forum Geom.*, **18** (2018) pp. 125–132.

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The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before December 31, 2023. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2023 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 911 - 919

Problem 911. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*
Solve for real numbers:
$$\begin{cases} 2\sin x + 1 = 2\sin y + 2\sin z \\ (\sin x + \sin y - \sin z)^2 + (\sin x - \sin y + \sin z)^2 + 1 = \sin x + \sin y + \sin z. \end{cases}$$

Problem 912. *Proposed by Mihaly Bencze, Brasov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Solve in real numbers the following equation:

$$x^2 - 5x - 2\sqrt{x-2} + 7 + \log_2 \frac{x^2-5x+8}{\sqrt{x-2}} + \log_3 \frac{x^2-5x+8}{2\sqrt{x-2}} = 0.$$

Problem 913. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Find $\lim\limits_{n\to\infty} \left(\left(\frac{\pi^2}{6} - \sum\limits_{k=1}^{n} \frac{1}{k^2} \right) * e^{x_n} \right)$ where $x_n = \sum\limits_{k=1}^{n} \frac{1}{k}$.

Problem 914. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Compute $\lim\limits_{n\to\infty} n\sqrt[n]{(2n-1)!!F_n}\sin\frac{1}{n^2}$ where F_n is the n^{th} Fibonacci number.

Problem 915. *Proposed by Toyesh Prakash Sharma (student), Agra College, Agra, India.*
Evaluate the following integral:

$$\int \ln(1+x)\cdot(e^x+\frac{1}{e^x})dx+\int\frac{1}{x+1}\cdot(e^x-\frac{1}{e^x})dx.$$

Problem 916. *Proposed by Raluca Maria Caraion and Forică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania.*
Find: $\Omega=\lim\limits_{p\to\infty}\frac{1}{p^a}\cdot\sum\limits_{m=1}^{p}\sum\limits_{n=1}^{m}\sum\limits_{k=1}^{n}\frac{k^2}{2k^2-2nk+n^2}.$

Problem 917. *Proposed by Marian Ursărescu, "Roman Voda" College, Roman, Neamt, Romania, and Florică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania.*
Let $(a_n)_{n\geqslant 1},(b_n)_{n\geqslant 1}$ be two sequences of real numbers defined by

$$a_n=\int\limits_1^n\left[\frac{n^2}{x}\right]dx;\ b_1>1,\ b_{n+1}=1+\log(b_n)$$

where [*] denotes the greatest integer function. Find $L=\lim\limits_{n\to\infty}\frac{a_n\cdot\log\sqrt[n]{b_n}}{\log n^n}.$

Problem 918. *Proposed by Seán Stewart, King Abdullah University of Science and Technology, Saudi Arabia.*
If $k>0$, evaluate $\int\limits_0^1\frac{\log(1+x^k+x^{2k})}{x}dx.$

Problem 919. *Proposed by the editor*
Find the error in the following proof: We want to find $\lim\limits_{n\to\infty}\frac{4^n}{3^n}$. This is an $\frac{\infty}{\infty}$ form so we can apply L'Hopital's Rule. Let $L=\lim\limits_{n\to\infty}\frac{4^n}{3^n}$. Then

$$\begin{aligned} L=\lim_{n\to\infty}\frac{4^n}{3^n}&=\lim_{n\to\infty}\frac{4^n\cdot\ln 4}{3^n\cdot\ln 3} \qquad \text{by L'Hopital's Rule}\\ &=\lim_{n\to\infty}\frac{4^n}{3^n}\lim_{n\to\infty}\frac{\ln 4}{\ln 3}=L\cdot\frac{\ln 4}{\ln 3}.\end{aligned}$$

Subtracting L from both sides gives, $0=L\cdot(\frac{\ln 4}{\ln 3}-1)$ but $\frac{\ln 4}{\ln 3}-1$ is not 0. Therefore $L=0$.

SOLUTIONS TO PROBLEMS 891 - 900

Problem 891. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*
Let $1 \leq m, n \leq 2022$ be integers such that $(n^2 - mn - m^2)^2 = 1$. Determine the maximum value of $m^2 + n^2$.

Solution *by John Zerger, Catawba College, Salisbury, NC.*

The answer is $1597^2 + 987^2$.
We first find all solutions (n,m). If $m = n$ then $(1,1)$ is the only solution. Assume without loss of generality that $n > m$. Note first that if (n,m) is a solution then so is $(m+n,n)$ since

$$(m+n)^2 - (m+n)n - n^2 = -(n^2 - mn - m^2) = \pm 1.$$

This gives us solutions $(2,1),(3,2),(5,3),(8,5),(13,8),(21,13),(34,21),$
$(55,34),(89,55),(144,89),(233,144),(377,233),(610,337),(987,610),$
$(1597,987),(2584,1597),\cdots$
Now we show these are the only solutions. Let (n,m) be a solution with $n > m$. If $m > 1$ and $m \leq n–m$ then $2m \leq n$ which gives $2m^2 \leq nm \leq n(n-m)$. Subtracting m^2 gives $1 < m^2 \leq n^2 - mn - m^2$ which is a contradiction to (n,m) being a solution. Therefore, for solutions with $m > 1$ we have $m > n–m$. Hence we consider solutions with $1 \leq n - m < m < n$. Note that if (n,m) is a solution then so is $(m, n-m)$ since

$$m^2 - m(n-m) - (n-m)^2 = -(n^2 - mn - m^2) = \pm 1.$$

Therefore, for any solution (n,m) with $m > 1$ we have a smaller solution $(m, n-m)$. Since we are only interested in positive integer solutions, the process will terminate with solution $(2,1)$. Thus, the solution must be in the list obtained above (pairs of consecutive Fibonacci numbers).
The pair maximizing $m^2 + n^2$ is $(1597, 987)$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; and the proposer.

Problem 892. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*
Find $\Omega = \int_{15\cos(2\arctan 3)}^{20\sin(2\arctan 3)} \frac{\sin^3 x + \sin^5 x}{1 + \cos^2 x + \cos^4 x} dx$.

Solution *by Marian Ursărescu "Roman-Voda" National College, Roman, Romania.*

We have $\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$ and $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$ which gives

$$\begin{aligned} 20\sin(2\arctan 3) &= 20\cdot 2\sin(\arctan 3)\cdot\cos(\arctan 3) = 40\cdot\frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}} \\ &= 12 \end{aligned}$$

and

$$\begin{aligned} 15\cos(2\arctan 3) &= 15(\cos^2(\arctan 3) - \sin^2(\arctan 3)) = 15(\frac{1}{10}-\frac{9}{10}) \\ &= -12. \end{aligned}$$

So the integral is $\Omega = \int_{-12}^{12}\frac{sin^3x+sin^5x}{1+cos^2x+cos^4x}\,dx$. But the function is odd and $\Omega = 0$.

Also solved by Radu Diaconu, "Ioan Slavici" High School, Sibiu and Daniel Văcaru, "Maria Teiuleanu" National Economic College, Piteşti, Romania; Kee-Wai Lau, Hong Kong, China; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; and the proposer.

Problem 893. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*
Solve for real numbers x,y,z the equations:

$$\begin{aligned} \sin x+\sin y &= \sqrt{2+2\sin z} \\ \sin y+\sin z &= \sqrt{2+2\sin x} \\ \sin z+\sin x &= \sqrt{2+2\sin y}. \end{aligned}$$

Solution *by Daniel Vacaru, "Maria Teiuleanu" National Economic College, Piteşti, Romania.*

We have

$$\begin{aligned} &(\sin x+\sin y)-(\sin y+\sin z) = \sqrt{2+2\sin z}-\sqrt{2+2\sin z} \\ &\Leftrightarrow \sin x-\sin z = \frac{(\sqrt{2+2\sin z}-\sqrt{2+2\sin x})}{(\sqrt{2+2\sin z}+\sqrt{2+2\sin x})}(\sqrt{2+2\sin z}+\sqrt{2+2\sin x}) \\ &\Leftrightarrow \sin x-\sin z = \frac{(\sqrt{2+2\sin z})^2-(\sqrt{2+2\sin x})^2}{(\sqrt{2+2\sin z}+\sqrt{2+2\sin x})} \\ &\Leftrightarrow \sin x-\sin z = \frac{\sin z-\sin x}{(\sqrt{2+2\sin z}+\sqrt{2+2\sin x})}. \end{aligned}$$

It follows that $\sin x = \sin y = \sin z$. We obtain the generic equation $2\sin x = \sqrt{2+2\sin x}$ with $\sin x \geq 0$. It follows that

$$4\sin^2 x = 2+2\sin x \Leftrightarrow 2\sin^2 x-\sin x-1 = 0.$$

We get $\sin x = 1$ or $\sin x = -\frac{1}{2}$.

Also solved by Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; and the proposer.

Problem 894. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*
(1) Prove that for any natural number n, the polynomial

$$f_n(x) = [A(x)]^n - [B(x)]^n - [C(x)]^n + [D(x)]^n$$

is divisible by $g(x) = 2x^2 + x - 1$, where:

$$\begin{aligned} A(x) =& 2x^3 + 10x^2 - 11x + 4, \\ B(x) =& x^2–2x + 2, \\ C(x) =& -x^3 + 14x^2 - 3x + 5, \\ D(x) =& 7x3 + 8x2 + 4. \end{aligned}$$

(2) Prove that for any natural number n, the quantity

$$\alpha_n = (-292)^n - 82^n - 1437^n + (-3068)^n$$

is divisible by 119.

(3) Prove that for any natural number n, the quantity

$$\beta_n = 65^n - 82^n - 9^n + 502^n$$

is divisible by 119.

Solution *by Brian Bradie, Christopher Newport University, Newport News, VA.*

(1) Note

$$A(x) - B(x) = 2x^3 + 9x^2 - 9x + 2 = (2x-1)(x^2 + 5x - 2)$$

and

$$D(x) - C(x) = 8x^3 - 6x^2 + 3x - 1 = (2x-1)(4x^2 - x + 1).$$

Therefore $A(x) - B(x) - C(x) + D(x)$ is divisible by $2x - 1$. For any natural number $n > 1$, $[A(x)]^n - [B(x)]^n$ is divisible by $A(x) - B(x)$ and $-[C(x)]^n + [D(x)]^n$ is divisible by $D(x) - C(x)$ so

$$f_n(x) = [A(x)]^n - [B(x)]^n - [C(x)]^n + [D(x)]^n$$

is divisible by $2x - 1$ for any natural number n. Next

$$A(x) - C(x) = 3x^3 - 4x^2 - 8x - 1 = (x+1)(3x^2 - 7x - 1)$$

and

$$D(x) - B(x) = 7x^3 + 7x^2 + 2x + 2 = (x+1)(7x^2+2).$$

Therefore $A(x) - B(x) - C(x) + D(x)$ is divisible by $x+1$. For any natural number $n > 1$, $[A(x)]^n - [C(x)]^n$ is divisible by $A(x) - C(x)$ and $-[B(x)]^n + [D(x)]^n$ is divisible by $D(x) - B(x)$ so

$$f_n(x) = [Ax]^n - [B(x)]^n - [C(x)]^n + [D(x)]^n$$

is divisible by $g(x) = (2x-1)(x+1)$ for any natural number n.

(2) Substitute $x = -8$ into the result from part (1). Because $A(-8) = -292$, $B(-8) = 82$, $C(-8) = 1437$, $D(-8) = -3068$, and $g(-8) = 119$, it follows that $\alpha_n = (-292)^n - 82^n - 1437^n + (-3068)^n$ is divisible by 119 for any n.

(3) Because $65 = -292 + 3(119)$, $9 = 1437 - 12(119)$ and $502 = -3068 + 30(119)$, it follows that $65^n \equiv (-292)^n (\bmod 119)$, $9^n \equiv 1437^n (\bmod 119)$, and $502^n \equiv (-3068)^n (\bmod 119)$. Therefore, $\beta_n = 65^n - 82^n - 9^n + 502^n$ is divisible by 119 for any n.

Also solved by Albert Stadler, Herrliberg, Switzerland; Hyunbin Yoo, South Korea; and the proposer.

Problem 895. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*
Calculate the following integral:

$$\int_0^\infty \frac{\sqrt{x}\ln x}{x^4+x^2+1}dx.$$

Solution *by the proposer.*

Let us denote $I = \int\limits_0^\infty \frac{\sqrt{x}\ln x}{x^4+x^2+1}dx$, $A = \int\limits_0^1 \frac{\sqrt{x}\ln x}{x^4+x^2+1}dx$, and $B = \int\limits_1^\infty \frac{\sqrt{x}\ln x}{x^4+x^2+1}dx$. We consider the integral A. We make the variable change $x^2 = y; x = \sqrt{y}$. We have successively:

$$\begin{aligned}
A &= \frac{1}{4}\int_0^1 \frac{(1-y)y^{-1/4}\ln y}{1-y^3}dy \\
&= \frac{1}{4}\left(\int_0^1 \frac{y^{-1/4}\ln y}{1-y^3}dy - \int_0^1 \frac{y^{3/4}\ln y}{1-y^3}dy\right) \\
&= \frac{1}{4}\left(\int_0^1 \sum_{n=0}^{\infty} y^{3n-1/4}\ln y\,dy - \int_0^1 \sum_{n=0}^{\infty} y^{3n+3/4}\ln y\,dy\right) \\
&= \frac{1}{4}\sum_{n=0}^{\infty}\left(\int_0^1 y^{3n-1/4}\ln y\,dy - \int_0^1 y^{3n+3/4}\ln y\,dy\right).
\end{aligned}$$

We will now use the following relationship: $\int_0^1 x^a \ln x\,dx = -\frac{1}{(a+1)^2}$ where $a \geq 0$. We obtain:

$$A = \frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{1}{\left(3n+\frac{7}{4}\right)^2} - \frac{1}{\left(3n+\frac{3}{4}\right)^2}\right] = \frac{1}{4}\sum_{n=0}^{\infty}\left[\frac{\frac{1}{9}}{\left(n+\frac{7}{12}\right)^2} - \frac{\frac{1}{9}}{\left(n+\frac{3}{12}\right)^2}\right].$$

We now use the following relationship: $\Psi_1(x) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$ where $\Psi_1(x)$ is the trigamma function. We obtain the value of the integral A :

$$A = \frac{1}{36}\left[\Psi_1\left(\frac{7}{12}\right) - \Psi_1\left(\frac{3}{12}\right)\right].$$

We consider the integral B. We make the variable change $x = \frac{1}{y}$; $y = \frac{1}{x}$. We obtain: $B = -\int_0^1 \frac{y\sqrt{y}\ln y}{y^4+y^2+1}dy$. By proceeding similarly to the integral A, we obtain:

$$B = \frac{1}{36}\left[\Psi_1\left(\frac{5}{12}\right) - \Psi_1\left(\frac{9}{12}\right)\right].$$

Result:

$$I = A + B = \frac{1}{36}\left[\Psi_1\left(\frac{5}{12}\right) + \Psi_1\left(\frac{7}{12}\right) - \Psi_1\left(\frac{1}{4}\right) - \Psi_1\left(\frac{3}{4}\right)\right].$$

We use the reflection formula: $\Psi_1(x) + \Psi_1(1-x) = \frac{\pi^2}{\sin^2(\pi x)}$ which gives $\Psi_1(\frac{5}{12}) + \Psi_1(\frac{7}{12}) = 4\pi^2(2-\sqrt{3})$. The following special values are known: $\Psi_1(\frac{1}{4}) = \pi^2 + 8G$; $\Psi_1(\frac{3}{4}) = \pi^2 - 8G$ where G is Catalan's constant. Finally:

$$I = \frac{1}{18}\pi^2(3 - 2\sqrt{3}).$$

Also solved by Henry Ricardo, Westchester Area Math Circle, New York; and Albert Stadler, Herrliberg, Switzerland.

Problem 896. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

If $a,b,c,d,x_i,y_i > 0$, for all $i = 1,\ldots,m$ and $x_{m+i} = x_i$ with $m \geq 2$, prove that

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{(ax_i+bx_{i+1})^2}{cx_{i+2}+dy_k}\right) \geq \frac{(a+b)^2n^2X_m^2}{cnX_m+dmY_n}$$

where $X_m = \sum_{i=1}^{m} x_i$ and $Y_n = \sum_{k=1}^{n} y_k$.

Solution *by Albert Stadler, Herrliberg, Switzerland.*

By the Cauchy-Schwarz inequality,

$$\sum_{i=1}^{m}\frac{(ax_i+bx_{i+1})^2}{cx_{i+2}+dy_k}\sum_{i=1}^{m}(cx_{i+2}+dy_k) \geq (\sum_{i=1}^{m}(ax_i+bx_{i+1}))^2,$$

hence

$$\sum_{i=1}^{m}\frac{(ax_i+bx_{i+1})^2}{cx_{i+2}+dy_k} \geq \frac{(a+b)^2X_m^2}{cX_m+dmy_k}.$$

The function $y \to \frac{u}{v+wy}$ with $u,v,w > 0$ is convex in $\{y | y \geq 0\}$. So, by Jensen's inequality,

$$\frac{1}{n}\sum_{k=1}^{n}\frac{(a+b)^2X_m^2}{cX_m+dmy_k} \geq \frac{(a+b)^2X_m^2}{cX_m+dm\frac{1}{n}\sum\limits_{k=1}^{n}y_k} = \frac{(a+b)^2nX_m^2}{cnX_m+dmY_n}.$$

Hence,

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{(ax_i+bx_{i+1})^2}{cx_{i+2}+dy_k}\right) \geq \sum_{k=1}^{n}\frac{(a+b)^2X_m^2}{cX_m+dmy_k} \geq \frac{(a+b)^2n^2X_m^2}{cnX_m+dmY_n}.$$

Also solved by Henry Ricardo, Westchester Area Math Circle, New York; Daniel Vacaru, "Maria Teiuleanu" National Economic College, Piteşti, Romania; and the proposer.

Problem 897. *Proposed by Toyesh Prakash Sharma (student), Agra College, Agra, India.*

With F_n and L_n being the Fibonacci and Lucas numbers, show that when $n > 0$

$$\frac{{F_n}^{F_n}+{F_{n+1}}^{F_{n+1}}+{L_n}^{L_n}+{L_{n+1}}^{L_{n+1}}}{4} \geq \left(\frac{F_{n+3}}{2}\right)^{\frac{F_{n+3}}{2}}.$$

Solution *by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.*

The inequality follows by Jensen's Inequality. Consider the function $f(x) = x^x$ defined for positive real numbers. Since $f(x)$ is convex, by Jensen's Inequality

$$\begin{aligned}\frac{{F_n}^{F_n} + {F_{n+1}}^{F_{n+1}} + {L_n}^{L_n} + {L_{n+1}}^{L_{n+1}}}{4} &\geq \left(\frac{F_{n+2}+L_{n+2}}{4}\right)^{\frac{F_{n+2}+L_{n+2}}{4}} \\ &= \left(\frac{F_{n+3}}{2}\right)^{\frac{F_{n+3}}{2}}\end{aligned}$$

because $F_n + F_{n+1} = F_{n+2}$, $L_n + L_{n+1} = L_{n+2}$, $F_{n+2} + L_{n+2} = 2F_{n+3}$.

Also solved by Albert Stadler, Herrliberg, Switzerland; and the proposer.

Problem 898. *Proposed by Florică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania.*

Let (b_n) be defined by $b_n = \frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}} - \frac{n^2}{\sqrt[n]{n!}}$ (Bătineţu's sequence). Find

$$L = \lim_{n\to\infty}\left(1+\frac{b_n}{n}\right)^{\frac{1}{n^{n-2}}\sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k}}.$$

Solution *by Henry Ricardo, Westchester Area Math Circle, New York.*

Page 22 of the Spring 2014 issue of *The Pentagon* contains a proof that $b_n \to e$ as $n \to \infty$. Setting $f(x) = \sum_{k=0}^{n}\binom{n}{k}x^k = (1+x)^n$, it follows that $\sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k} = \frac{1}{n}\int_0^n f(x)dx = \frac{(1+n)^{n+1}-1}{n(n+1)}$. Then

$$x_n = \frac{1}{n^{n-1}}\sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k} = \frac{(1+n)^{n+1}-1}{n^n(n+1)} = \left(1+\frac{1}{n}\right)^n - \frac{1}{n^n(n+1)} \to e$$

as $n \to \infty$. Finally, using the known result that $z_n \to z$ implies $(1+\frac{z_n}{n})^n \to e^z$, we have

$$\begin{aligned}L = \lim_{n\to\infty}\left(1+\frac{b_n}{n}\right)^{\frac{1}{n^{n-2}}\sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k}} &= \lim_{n\to\infty}\left(1+\frac{b_n}{n}\right)^{nx_n} \\ &= \lim_{n\to\infty}\left[\left(1+\frac{b_n}{n}\right)^n\right]^{x_n} \\ &= [e^e]^e = e^{e^2}.\end{aligned}$$

Also solved by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Albert Stadler, Herrliberg, Switzerland; Daniel Vacaru, "Maria Teiuleanu" National Economic College, Piteşti, Romania; and the proposer.

Problem 899. *Proposed by Mihaly Bencze, Brasov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Solve in the set of real numbers the following equation:

$$8x^3+17x+\frac{4}{x}+\log_2^2\left(x+\frac{4}{x}\right)=x^4+20x^2+4+2^{-x^2+4x-2}.$$

Solution *by the proposers.*

Rewritten the equation to solve is:

$$\log_2^2\left(x+\frac{4}{x}\right)+x+\frac{4}{x}=(-x^2+4x-2)^2+2^{-x^2+4x-2}.$$

Using the injectivity of the function $f(t)=2^t+t^2$, $t>0$, we obtain

$$\log_2\left(x+\frac{4}{x}\right)=-x^2+4x-2.$$

Since we have $x+\frac{4}{x}\geq 4$ and $-x^2+4x-2\leq 2 \Leftrightarrow (x-2)^2\geq 0$, we have only one solution, $x=2$.

Also solved by Albert Stadler, Herrliberg, Switzerland.

Problem 900. *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Prove that in any triangle ABC, the following inequality holds:

$$\frac{2r}{R}+\sum\frac{a^2}{bc}\geq 4.$$

Solution *by Kee-Wai Lau, Hong Kong, China.*

Let s be the semiperimeter of the triangle. It is well known that $ABC=4sRr$ and $\sum a^3=2s(s^2-6Rr-3r^2)$. It is also known that $R\geq 2r$ (entry 5.1 on page 48 of [1]) and $s^2\geq 16Rr-5r^2$ (entry 5.8 on page 50 of [1]). Hence

$$\frac{2r}{R}+\sum\frac{a^2}{bc}=\frac{s^2-6Rr+r^2}{2Rr}=\frac{(s^2-16Rr+5r^2)+2r(R-2r)}{2Rr}+4\geqslant 4.$$

Reference: [1] O.Bottema, R.R, Janić, D.S. Mitrinović, P.M. Vasić, *Geometric Inequalities*, Groningen, 1969.

Also solved by Albert Stadler, Herrliberg, Switzerland; Daniel Vacaru, "Maria Teiuleanu" National Economic College, Piteşti, Romania; and the proposer.

Kappa Mu Epsilon News

Edited by Mark Hughes, Historian
Updated information as of March 2023

News of chapter activities and other noteworthy KME events should be sent to

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KAPPA MU EPSILON
Installation Report
California Theta, William Jessup University
Rocklin, California

The installation of the California Theta Chapter of Kappa Mu Epsilon was held in the Commons on William Jessup University campus on Monday, October 17, 2022, at three o'clock in the afternoon.

KME National President Don Tosh was the installing officer. Faculty member Michelle Clark, already a member of KME, was the main driving force in establishing this chapter of KME and will serve as the chapter's Faculty Sponsor. She served as the conductor during the ceremony. Faculty member Bradley Wagner was initiated during the ceremony and will serve as the Corresponding Secretary. During the ceremony, new members were initiated and officers were installed. Don Tosh declared the organization to be the California Theta Chapter of Kappa Mu Epsilon and presented the chapter's charter and crest to the Corresponding Secretary.

The student charter members of California Theta are: Michael Beishline, Hannah-Jeanne Bethards, Stephen Chau, Mikayla Erickson, Lizzie Salvato, Samuel Smith, and Rachel Weaver. Each initiate was invited to sign the California Theta Chapter Roll and was presented with a membership certificate and a KME pin.

The officers of California Theta installed during the ceremony are: Michelle Clark (Faculty Sponsor), Bradley Wagner (Corresponding Secretary), Lizzie Salvato (President), Stephen Chau (Vice-President), Mikayla Erickson (Secretary), and Samuel Smith (Treasurer). Each officer was charged with the responsibilities of his/her new office, and each chose to accept those responsibilities.

Dr. John Jackson, William Jessup University President, concluded the ceremony with some closing remarks.

A time of congratulations and fellowship with refreshments was enjoyed by all following the installation ceremony.

California Theta
Back row: Bradley Wagner, Samuel Smith, Stephen Chau, Michael Beishline, Hannah-Jeanne Bethards, Don Tosh
Front row: Michelle Clark, Mikayla Erickson, Lizzie Salvato, Rachel Weaver

Chapter News

AL Gamma – University of Montevallo
Corresponding Secretary and Faculty Sponsor – Dr. George Lytle; 717 Total Members
The Alabama Gamma chapter will hold its annual initiation ceremony in the spring semester. We're still in the process of acquiring officers.

AL Theta – Jacksonville State University (Spring 2022)
Chapter President – Hannah Davis; 324 Total Members; 21 New Members
Other Spring 2022 Officers: Bronte Ray, Vice President; Dakota Heathcock, Secretary; Evan Parton, Treasurer; Dr. David Dempsey, Corresponding Secretary; and Dr. Jason Cleveland, Faculty Sponsor.
The Alabama Theta chapter met at least monthly in person during Spring 2022 and replaced officers who graduated in December. At long last, we held an in-person initiation ceremony on April 8, 2022, for 21 new members. We had a great

road trip with 2 faculty and 3 students to the KME Southeastern Regional Convention on April 22–23.

AL Theta – Jacksonville State University
Chapter President – Dakota Heathcock; 324 Total Members
Other Fall 2022 Officers: Nicholas Covalsen, Vice President; Adam Parton, Secretary; Suneet Sharma, Treasurer; Dr. David Dempsey, Corresponding Secretary; and Dr. Jason Cleveland, Faculty Sponsor.
The Alabama Theta chapter met at least monthly in person during Fall 2022 and elected new officers. One highlight was an outing for bowling and mini-golf (studying the varied trajectories of both large and small spheres!). We look forward to a spring initiation ceremony and travel to the 2023 KME National Convention.

CA Theta – William Jessup University
Chapter President – Elizabeth Salvato; 9 Total Members; 8 New Members
Other Fall 2022 Officers: Stephen Chau, Vice President; Mikayla Erickson, Secretary; Samuel Smith, Treasurer; Bradley Wagner, Corresponding Secretary; and Michelle Clark, Faculty Sponsor.
This past semester our chapter, California Theta, began its existence. In a ceremony on Oct 17, 2022, KME President Dr. Don Tosh came to Jessup to install the California Theta chapter, with 7 charter student members - Michael Beishline, Hannah-Jeanne Bethards, Stephen Chau, Mikayla Erickson, Lizzie Salvato, Samuel Smith, and Rachel Weaver. The chapter was sponsored by Faculty Member Michelle Clark, who herself was a Kappa Mu Epsilon student member as an undergraduate. Dr. Tosh also installed faculty member Bradley Wagner as the Corresponding Secretary.

On Nov 17, 2022, we held a game night open to the University community, as our first event. The event was well-attended, with over 15 attendees participating in math-themed games, ice-breakers and snacks. The game night was a great opportunity for us to promote the interest of mathematics among undergraduate students and to connect with fellow math enthusiasts.

We would like to express our gratitude to Dr. Don Tosh, faculty members Michelle Clark and Bradley Wagner, and all the members who supported us during the semester. We are excited for all the opportunities and challenges that the next semester brings and looking forward to a productive year.

CT Beta – Eastern Connecticut State University
Corresponding Secretary and Faculty Sponsor – Dr. Mehdi Khorami; 547 Total Members

CT Gamma – Central Connecticut State University
Corresponding Secretary – Gurbakhshash Singh; 78 Total Members
Other Fall 2022 Officer: Nelson Castaneda, Faculty Sponsor.

GA Zeta – Georgia Gwinnett College
Chapter President – Hope Doherty; 66 Total Members
Other Fall 2022 Officers: Gabriel Amat, Vice President; William Watts, Secretary; Matt Elenteny, Treasurer; Dr. Jamye Curry Savage, Corresponding Secretary and Faculty Sponsor; and Dr. Livy Uko, Faculty Sponsor.
The members of the GA Zeta Chapter have been working on new ideas/events to incorporate social events, such as math-related games/activities and study sessions with chapter meetings.

GA Theta – College of Coastal Georgia
Chapter President – Marianela Landi; 18 Total Members; 6 New Members
Other Fall 2022 Officers: Monique Deschenes, Vice President; Darius Hammond, Secretary; Kaelyn Tyler, Treasurer; Aaron Yeager, Corresponding Secretary and Faculty Sponsor.
In the Fall 2022 semester we initiated six new members into the Georgia Theta Chapter of KME. Three of the members in the chapter, Garrett Moseley, Darius Hammond, and Madison Ellis, graduated from CCGA in the Fall. Garrett received a Bachelors in Mathematics and is starting graduate school in the spring. Darius and Madison were the first CCGA students to graduate with a Bachelors in Data Science. Darius is starting graduate school in the spring and Madison is starting a job in industry. We held three meetings over the fall semester.

IA Alpha – University of Northern Iowa
Chapter President – Jacob Metzen; 1117 Total Members; 5 New Members
Other Fall 2022 Officers: Grace Croat, Vice President; Emily Moore, Secretary; Lydia Butters, Treasurer; and Dr. Mark D. Ecker, Corresponding Secretary and Faculty Sponsor.
Eleven student members of KME and three faculty met face-to-face on Wednesday, December 7, 2022 in Wright Hall for the John Cross Fall KME Banquet. Student member Jacob Metzen presented his senior seminar project entitled “2022 Fantasy Football Quarterback Prices” and five new student members were initiated at the banquet.

IA Gamma – Morningside University
Chapter President – Taylor Pierce; 442 Total Members
Other Fall 2022 Officers: Taylor Pierce, Vice President; Isaiah Hinners, Secretary and Treasurer; and Dr. Eric Canning, Corresponding Secretary and Faculty Sponsor.
There were no new initiates in Fall 2022, as we are having an initiation ceremony this Spring. Our KME math club met 7 different evenings during the Fall semester. At these meetings, 3 times there was a guest speaker, we watched the movie *October Sky*, had a craft night, and a couple of game and pizza nights.

IL Zeta – Dominican University
Corresponding Secretary and Faculty Sponsor – Mihaela Blanariu; 458 Total Members
We have not initiated any new members in Fall 2022.

IN Beta – Butler University
Chapter President – Kelly Ryan; 444 Total Members
Other Fall 2022 Officers: Nicole Dickson, Vice President; Aaron Marshall, Secretary; Rasitha Jayasekare, Corresponding Secretary and Faculty Sponsor.
Our main fall activity included election of new officers to the chapter. We are hoping to add more activities in the upcoming year.

KS Beta – Emporia State University
Chapter President – Joey Feuerborn; 1539 Total Members; 7 New Members
Other Fall 2022 Officers: Jeanna Hill, Vice President; Austin Crabtree, Secretary; Kaley Dembowski, Treasurer; Tom Mahoney, Corresponding Secretary; and Brian Hollenbeck, Faculty Sponsor.
The Kansas Beta chapter enjoyed a very social semester. After the February meeting, we had a bowling night. In March, we initiated seven new members at a Pizza Ranch. In April, students hosted an “art night” of acrylic painting. And in May, we gathered for a Color Run.

KS Delta – Washburn University
Chapter President – Ajar Basnet; 834 Total Members
Other Fall 2022 Officers: Sanskar Neupane , Vice President; Graci Postma, Secretary; Sarah Johnson, Treasurer; and Sarah Cook, Corresponding Secretary and Faculty Sponsor.

MD Delta – Frostburg State University
Chapter President – Adam Sullivan; 543 Total Members
Other Fall 2022 Officers: Kaitlyn Henderson-Adams, Vice President; Faith James Sergent, Secretary; Mark Hughes, Corresponding Secretary and Faculty Sponsor; and Frank Barnet, Faculty Sponsor.
We had three meetings during the semester where we enjoyed puzzles, math videos, and pizza. We also represented the Mathematics Department at our university’s annual Majors Fair.

MI Beta – Central Michigan University
Chapter President – Kelsey Knoblock; 1761 Total Members
Other Fall 2022 Officers: Maleia Thompson, Vice President; Jenna Wazny, Secretary; Jeremy Proksch, Treasurer; and Dmitry Zakharov, Corresponding Secretary and Faculty Sponsor.
We had a Bingo and a Jeopardy night. These games have members solving problems from many different math classes to win prizes. We also did a math scav-

enger hunt, which had us working in teams to solve harder problems quickly. One of the favorites for last semester was Dr. Gilsdorf's talk on cultural mathematics.

MO Beta – University of Central Missouri
Chapter President – Haleigh Clark; 1553 Total Members; 4 New Members
Other Fall 2022 Officers: Page Van Barclum, Vice President; Luke Elliott, Secretary and Treasurer; Steven Shattuck, Corresponding Secretary and Faculty Sponsor; and Blaise Heider, Faculty Sponsor.
Missouri Beta had monthly meetings for Fall 2022. At our first meeting, students with REUs and internships shared their experiences and discussed how students could apply for these opportunities. At our second meeting, students played math games. At our third meeting, we had our initiation. At our fourth meeting, we had a pizza party, discussed the national convention and played math games.

MO Theta – Evangel University
Chapter President – Peter Russell; 298 Total Members
Other Fall 2022 Officers: Jack Lin, Vice President; and Dianne Twigger, Corresponding Secretary and Faculty Sponsor.
Missouri Theta held regular, in person meetings for the fall 2022 term. Two KME students, Peter Russell and Hayden Pyle, were invited to present their research at a joint KME meeting with Missouri Alpha at Missouri State University in November. Student engagement appears to be back to pre-Covid levels. Faculty and students are planning on attending the national conference this spring.

MO Iota – Missouri Southern State University (Spring and Fall 2022)
Chapter President – Ashley Stokes; 439 Total Members; 5 New Members
Other Spring 2022 and Fall 2022 Officer: Dr. Amila Appuhamy, Treasurer, Corresponding Secretary and Faculty Sponsor.
We held our initiation ceremony on April 12, 2022, followed by a dinner. A total of five new members were initiated and the total number of participants in the event was twenty.

MO Kappa – Drury University
Chapter President – Julian Fisher; 330 Total Members
Other Fall 2022 Officers: Levi Graham, Vice President; Kelsi Gelle, Secretary and Corresponding Secretary; Brooke Weider, Treasurer; and Colin T. Baker, Faculty Sponsor.
We have begun a weekly meeting to discuss any and all questions that arise from our daily lives and how to apply math(s) to them.

MS Alpha – Mississippi University for Women
Chapter President – Autumn Bigham; 839 Total Members; 2 New Members
Other Fall 2022 Officers: Annie Sparks, Vice President; Dr. Joshua Hanes, Secretary and Treasurer; Dr. Joshua Hanes, Corresponding Secretary and Faculty

Sponsor.
In the Fall semester we initiated two new members, Autumn Bigham and Annie Sparks.

NC Zeta – Catawba College
Chapter President – Olek Malul; 92 Total Members; 4 New Members
Other Fall 2022 Officers: Jaxon Wheeler, Vice President; Leon Heiermann, Treasurer; and Dr. Katherine Baker, Corresponding Secretary and Faculty Sponsor.

NE Beta – University of Nebraska Kearney
Corresponding Secretary and Faculty Sponsor – Dr. Katherine Kime; 937 Total Members
We are planning to send out notices for an initiation this spring. KME student member Brooke Carlson is here, finishing her majors in Math and Physics, and is applying to graduate schools in mechanical engineering. She is a player on the women's basketball team and has had an extra year due to the pandemic. Brooke has been a great student and athlete. Dr. Kaye Sorensen, who was an undergraduate at UNK (1980s or earlier) and is a member of KME from that time, has been a senior lecturer in our department and is retiring this spring. In the last few years she has pioneered the position of College of Arts and Sciences Math Specialist, which focusses on students who are retaking General Studies courses. She will be missed by the department.

OK Alpha – Northeastern State University
Chapter President – Cade Clickenbeard; 1875 Total Members; 3 New Members
Other Fall 2022 Officers: Parker Childers, Vice President; Mark Buckles, Secretary, Treasurer, Corresponding Secretary, and Faculty Sponsor.
We had an initiation ceremony November 16, 2022 at both our Tahlequah campus and Broken Arrow campus.

PA Theta – Susquehanna University
Corresponding Secretary and Faculty Sponsor – Alathea Jensen; 615 Total Members

PA Iota – Shippensburg University
Chapter President – Luis Melara; 765 Total Members; 4 New Members
Other Fall 2022 Officers: Grant Innerst, Vice President; Dr. Paul Taylor, Corresponding Secretary, and Dr. Ji Young Choi, Faculty Sponsor.
Our activity roster of members was wiped out by student apathy during the ovid19 pandemic. We're trying to rebuild now.

PA Mu – Saint Francis University
Chapter President – Michael Gallagher; 510 Total Members
Other Fall 2022 Officers: Morgan Kiesewelter, Vice President; Regina Edgington,

Secretary; Jared Ohler, Treasurer; Brendon LaBuz, Corresponding Secretary and Faculty Sponsor.

On Wednesday October 26 the Pennsylvania My chapter welcomed KME National Historian Dr. Mark Hughes for a presentation. Dr. Hughes presented on some historical applications of Cavalieri's Principle. In the seventeenth century, Bonaventura Cavalieri developed a way to compute areas and volumes that is reminiscent of modern integral techniques. What makes it so impressive is that he developed these techniques decades before Newton and Leibniz published their treatment of calculus. Dr. Hughes explained how Cavalieri's principle can be used to compute the area under the cycloid and the volume of Gabriel's horn, two geometric figures that are interesting historically and also of interest in modern mathematics.

PA Pi – Slippery Rock University

Chapter President – Spencer Kahley; 145 Total Members

Other Fall 2022 Officers: Boris Brimkov, Corresponding Secretary; and Amanda Goodrick, Faculty Sponsor.

We did not have any activities in Fall 2022 but we will be sending out invitations in the next couple months to start planning the next initiation ceremony.

PA Rho – Thiel College

Chapter President – Hunter Gray; 145 Total Members

Other Fall 2022 Officers: Devin Bossard, Vice President; Cassy Brown, Secretary; Brandon Forrest, Treasurer; Dr. Russell Richins, Corresponding Secretary; and Dr. Jie Wu, Faculty Sponsor.

This fall we had several chapter meetings and our usual Challenge 24 competition as a fundraiser for the local food bank. Over 40 students participated.

PA Sigma – Lycoming College

Chapter President – Jordan Kinley; 156 Total Members

Other Fall 2022 Officers: Haley Seebold, Vice President; Laurel Stoppard, Secretary; Derek Lewis, Treasurer; Dr. Andrew Brandon, Corresponding Secretary and Faculty Sponsor.

RI Beta – Bryant University

Corresponding Secretary – Prof. John Quinn; 200 Total Members

Other Fall 2022 Officer: Prof. Gao Niu, Faculty Sponsor.

We have our KME initiation ceremony every spring semester and we are planning to do the same for spring 2023. We are also planning to have two students attend the KME 44th Biennial Convention to present a paper on building a recommendation system for songs for Spotify using K-Nearest Neighbors clustering method.

TX Lambda – Trinity University

Corresponding Secretary – Dr. Hoa Nguyen; 315 Total Members; 8 New Members

VA Beta – Radford University (Spring 2022)
Chapter President – Lauren Craddock; 594 Total Members; 6 New Members
Other Spring 2022 Officers: Abby Ramos Cortez, Vice President; Charlene Galvez, Secretary; Adam Downs, Treasurer; Eric P. Choate, Corresponding Secretary and Faculty Sponsor.

WV Alpha – Bethany College
Chapter President – Cullen J. Wise; 198 Total Members
Other Fall 2022 Officers: Lauren E. Starr, Vice President; Patrick M. Gleason, Secretary; Ian A. Nelson Treasurer; and Dr. Adam C. Fletcher, Corresponding Secretary and Faculty Sponsor.
West Virginia Alpha, like many other chapters across the country, continues to adjust to push for a return to in-person conferences and professional gatherings. West Virginia Alpha chapter and our local Mathematics and Computer Science Club continue to attend meetings virtually and host small chess and gaming tournaments on campus. They are busily planning the in-person return of the annual Math/Science Day Competition for local high school students. The chapter eagerly awaits the next in-person KME meetings in the spring.

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 Apr 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
AL Alpha	Athens State University, Athens	5 Mar 1935
NM Alpha	University of New Mexico, Albuquerque	28 Mar 1935
IL Beta	Eastern Illinois University, Charleston	11 Apr 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 Apr 1937
OH Alpha	Bowling Green State University, Bowling Green	24 Apr 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	University of Central Missouri, Warrensburg	10 Jun 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 Jun 1941
MI Beta	Central Michigan University, Mount Pleasant	25 Apr 1942
NJ Beta	Montclair State University, Upper Montclair	21 Apr 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 Mar 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 Jun 1947
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 Apr 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 Apr 1965
AL Epsilon	Huntingdon College, Montgomery	15 Apr 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
TN Gamma	Union University, Jackson	24 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	McDaniel College, Westminster	30 May 1965
IL Zeta	Dominican University, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 Mar 1971
KY Alpha	Eastern Kentucky University, Richmond	27 Mar 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 Apr 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973

NY Kappa	Pace University, New York	24 Apr 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State University, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sep 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis University, Loretto	14 Sep 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury University, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 Mar 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 Apr 1986
TX Iota	McMurry University, Abilene	25 Apr 1987
PA Nu	Ursinus College, Collegeville	28 Apr 1987
VA Gamma	Liberty University, Lynchburg	30 Apr 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 Apr 1990
CO Delta	Mesa State College, Grand Junction	27 Apr 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 Apr 1991
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 Mar 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 Apr 1997
MI Delta	Hillsdale College, Hillsdale	30 Apr 1997
MI Epsilon	Kettering University, Flint	28 Mar 1998
MO Mu	Harris-Stowe College, St. Louis	25 Apr 1998
GA Beta	Georgia College and State University, Milledgeville	25 Apr 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
PA Pi	Slippery Rock University, Slippery Rock	19 Apr 1999
TX Lambda	Trinity University, San Antonio	22 Nov 1999
GA Gamma	Piedmont College, Demorest	7 Apr 2000
LA Delta	University of Louisiana, Monroe	11 Feb 2001
GA Delta	Berry College, Mount Berry	21 Apr 2001
TX Mu	Schreiner University, Kerrville	28 Apr 2001
CA Epsilon	California Baptist University, Riverside	21 Apr 2003
PA Rho	Thiel College, Greenville	13 Feb 2004
VA Delta	Marymount University, Arlington	26 Mar 2004
NY Omicron	St. Joseph's College, Patchogue	1 May 2004
IL Iota	Lewis University, Romeoville	26 Feb 2005
WV Beta	Wheeling Jesuit University, Wheeling	11 Mar 2005
SC Epsilon	Francis Marion University, Florence	18 Mar 2005
PA Sigma	Lycoming College, Williamsport	1 Apr 2005
MO Nu	Columbia College, Columbia	29 Apr 2005
MD Epsilon	Stevenson University, Stevenson	3 Dec 2005
NJ Delta	Centenary College, Hackettstown	1 Dec 2006
NY Pi	Mount Saint Mary College, Newburgh	20 Mar 2007
OK Epsilon	Oklahoma Christian University, Oklahoma City	20 Apr 2007
HA Alpha	Hawaii Pacific University, Waipahu	22 Oct 2007
NC Epsilon	North Carolina Wesleyan College, Rocky Mount	24 Mar 2008
NY Rho	Molloy College, Rockville Center	21 Apr 2009
NC Zeta	Catawba College, Salisbury	17 Sep 2009
RI Alpha	Roger Williams University, Bristol	13 Nov 2009
NJ Epsilon	New Jersey City University, Jersey City	22 Feb 2010
NC Eta	Johnson C. Smith University, Charlotte	18 Mar 2010
AL Theta	Jacksonville State University, Jacksonville	29 Mar 2010
GA Epsilon	Wesleyan College, Macon	30 Mar 2010
FL Gamma	Southeastern University, Lakeland	31 Mar 2010
MA Beta	Stonehill College, Easton	8 Apr 2011
AR Beta	Henderson State University, Arkadelphia	10 Oct 2011
PA Tau	DeSales University, Center Valley	29 Apr 2012
TN Zeta	Lee University, Cleveland	5 Nov 2012
RI Beta	Bryant University, Smithfield	3 Apr 2013
SD Beta	Black Hills State University, Spearfish	20 Sept 2013
FL Delta	Embry-Riddle Aeronautical University, Daytona Beach	22 Apr 2014
IA Epsilon	Central College, Pella	30 Apr 2014
CA Eta	Fresno Pacific University, Fresno	24 Mar 2015
OH Theta	Capital University, Bexley	24 Apr 2015
GA Zeta	Georgia Gwinnett College, Lawrenceville	28 Apr 2015
MO Xi	William Woods University, Fulton	17 Feb 2016
IL Kappa	Aurora University, Aurora	3 May 2016
GA Eta	Atlanta Metropolitan University, Atlanta	1 Jan 2017
CT Gamma	Central Connecticut University, New Britan	24 Mar 2017
KS Eta	Sterling College, Sterling	30 Nov 2017
NY Sigma	College of Mount Saint Vincent, The Bronx	4 Apr 2018
PA Upsilon	Seton Hill University, Greensburg	5 May 2018

KY Gamma	Bellarmine University, Louisville	23 Apr 2019
MO Omicron	Rockhurst University, Kansas City	13 Nov 2020
AK Gamma	Harding University, Searcy	27 Apr 2021
GA Theta	College of Coastal Georgia, Brunswick	22 Oct 2021
CA Theta	William Jessup University, Rocklin	17 Oct 2022