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Groups, geometry and a common lemma for Bézout's identity and Euclid's lemma

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Abstract

The aim of this note is to give proofs of Bézout's identity and Euclid's lemma, based on a common algebraic lemma which characterizes the additive subgroups of $\mathbb{Z}$. Although proofs using similar arguments are probably known, it seems difficult (and maybe impossible) to find a reference where both propositions are obtained from this common lemma. In fact, most of the time, Euclid's lemma is obtained from Bézout's identity (see the reference below). Using the same algebraic lemma, we will also prove a geometric variant of Euclid's lemma which, unfortunately, is rarely mentioned in arithmetic courses.

The Results

In the proofs of this note, we will apply the following common lemma:

Lemma. *The only additive subgroups of $\mathbb{Z}$ (non-empty subsets which are invariant under additions and the multiplication by -1) are the sets $\{XK_0, X \in \mathbb{Z}\}$, where $K_0 \in \mathbb{Z}$.*

Proof. The converse implication is trivial. The direct implication is trivial when the subgroup is $\{0\}$. When the subgroup G is not $\{0\}$, we will show that $G = \{XK_0, X \in \mathbb{Z}\}$, where K_0 is its least positive element (as a matter of fact, the set of positive elements of G is a non-empty subset of $\mathbb{Z}$, bounded from below). The converse inclusion is trivial. In order to prove the direct inclusion, let us denote by X the quotient and by R the remainder of the Euclidean division by K_0 of every element N of G. We can easily see that $R = N - XK_0 \in G$. On the other hand, the division algorithm gives: $0 \leq R < K_0$. Thus, $R = 0$ and so $N = XK_0$. Hence, all subgroups have the form $\{XK_0, X \in \mathbb{Z}\}$. ■

Using this lemma, we can prove Bézout's identity and Euclid's lemma:

Theorem (Bézout's identity). *Let A, B be two relatively prime integers. Then there exist two integers U, V such that $AU + BV = 1$.*

Proof. Let us consider the following subset of $\mathbb{Z}$:

$$F = \{AU + BV, (U,V) \in \mathbb{Z}^2\}\cdot$$

For all (U_1,V_1), $(U_2,V_2) \in \mathbb{Z}^2$, we have:

$$(AU_1 + BV_1) + (AU_2 + BV_2) = A(U_1 + U_2) + B(V_1 + V_2) \in F,$$
$$-(AU_1 + BV_1) = A(-U_1) + B(-V_1) \in F\cdot$$

noindent So F is a subgroup of $\mathbb{Z}$ and, by the initial lemma, there are integers U_0, V_0 such that:

$$F = \{X(AU_0 + BV_0), X \in \mathbb{Z}\}\cdot$$

Since A,$B \in F$, the integer $AU_0 + BV_0$ divides A and B, which are relatively prime: it has to be equal to -1 or 1. Therefore: $A(\pm U_0) + B(\pm V_0) = 1$. ■

Theorem (Euclid's lemma). *Let A, B be two relatively prime integers. Let C be an integer such that A divides BC. Then A divides C.*

Proof. Let us consider the following subset of $\mathbb{Z}$:

$$G = \{P \in \mathbb{Z}, A \text{ divides } BP\}\cdot$$

For all P_1, $P_2 \in G$, since A divides BP_1 and BP_2, it also divides $BP_1 + BP_2 = B(P_1 + P_2)$ and $-BP_1 = B(-P_1)$. Therefore: $P_1 + P_2$, $-P_1 \in G$. So G is a subgroup of $\mathbb{Z}$ and, by the initial lemma, there is an integer P_0 such that:

$$G = \{XP_0, X \in \mathbb{Z}\}\cdot$$

As A, $C \in G$, we can find an integer Y such that $A = YP_0$ and an integer X such that $C = XP_0$. Since $P_0 \in G$, the integer YP_0 divides BP_0. If $P_0 = 0$, then A trivially divides $C = X.0 = 0$. If $P_0 \neq 0$, then Y divides B, as well as A. As A and B are relatively prime, we have: $Y = \pm 1$. Thus: $A = YP_0 = \pm P_0$ and $C = XP_0 = \pm XA$. ■

Applying the initial lemma, we can also prove a geometric variant of Euclid's lemma:

Theorem (Euclid's geometric lemma). *Let A, B be two relatively prime integers and let C, D be two integers such that (A,B) and (C,D) are collinear (which is equivalent to saying that $AD = BC$, or that the vector (C,D) is on the same vector line as (A,B)). Then there exists an integer X such that $(C,D) = X(A,B)$.*

We can easily obtain Euclid's lemma from Euclid's geometric lemma (taking $D = BC/A$), and obtain Euclid's geometric lemma from Euclid's lemma (taking

$X = C/A$). Here, we will prove it directly, taking advantage of the group structure of a set of abscissae of the set:

$$\begin{aligned} H' &= \{(U,V) \in \mathbb{Z}^2, (A,B) \text{ and } (U,V) \text{ are collinear} \\ &= \{(U,V) \in \mathbb{Z}^2, AV = BU\}, \end{aligned}$$

which generalizes the set $\mathbb{Z}$, corresponding to the case where $(A,B) = (1,0)$.

Proof. Using the inner product, let us define an integer abscissa φ in H': $\varphi(U,V) = (A,B).(U,V) = AU + BV$. The function φ is one-to-one:

$$\begin{aligned} &\begin{cases} (U_1,V_1),(U_2,V_2) \in H' \\ \varphi(U_1,V_1) = \varphi(U_2,V_2) \end{cases} \\ &\Rightarrow \begin{cases} AV_1 - BU_1 = AV_2 - BU_2 (= 0) \\ \quad AU_1 + BV_1 = AU_2 + BV_2 \end{cases} \\ &\Rightarrow \begin{cases} B(U_2 - U_1) - A(V_2 - V_1) = 0 \\ A(U_2 - U_1) + B(V_2 - V_1) = 0 \end{cases} \\ &\Rightarrow \begin{cases} (A^2 + B^2)(U_2 - U_1) = A(A(U_2 - U_1) + B(V_2 - V_1)) \\ \qquad + B(B(U_2 - U_1) - A(V_2 - V_1)) = 0 \end{cases} \\ &\Rightarrow (U_1,V_1) = (U_2,V_2) \end{aligned}$$

as $(A,B) \neq (0,0)$, since A and B are relatively prime. Let $H = \varphi(H')$. If vectors (U_1,V_1), (U_2,V_2) are collinear with (A,B), then so are (U_1+U_2, V_1+V_2), $(-U1, -V1)$, and we have:

$$\begin{aligned} (AU_1 + BV_1) + (AU_2 + BV_2) &= A(U_1 + U_2) + B(V_1 + V_2) \in H, \\ -(AU_1 + BV_1) &= A(-U_1) + B(-V_1) \in H\cdot \end{aligned}$$

So H is a subgroup of $\mathbb{Z}$ and, by the initial lemma, there is a vector $(A_0,B_0) \in H'$ such that:

$$H = \{X\varphi(A_0,B_0), X \in \mathbb{Z}\} = \{\varphi(X(A_0,B_0)), X \in \mathbb{Z}\}\cdot$$

This implies that, for all $(U,V) \in H'$, there exists $X \in \mathbb{Z}$ such that $\varphi(U,V) = \varphi(X(A_0,B_0))$. As φ is one-to-one, we have: $(U,V) = X(A_0,B_0)$. In other words: $H' = \{X(A_0,B_0), X \in \mathbb{Z}\}$. Using this, we can find an integer Y such that $(A,B) = Y(A_0,B_0)$ and an integer X such that $(C,D) = X(A_0,B_0)$. As A and B are relatively prime, we have: $Y = \pm 1$. Thus: $(A,B) = \pm(A_0,B_0)$, $(C,D) = \pm X(A,B)$. ∎

In the previous proof, we obtained the identity:
$H' = \{X(A_0,B_0), X \in \mathbb{Z}\}$ applying the initial lemma to $H = \varphi(H')$. But we can also prove it by extending the arguments of the proof of the initial lemma, which

is generalized by this identity in dimension 2. We just have to define (A_0,B_0) (instead of K_0) as the point of H' where φ achieves its least positive value. Then, we need to extend the Euclidean division of $\mathbb{Z}$ to H', associating to every $(U,V) \in H'$ a quotient $X \in \mathbb{Z}$ and a remainder $(R,S) \in H'$ such that: $(U,V) = X(A_0,B_0) + (R,S)$ and $0 \leq \varphi(R,S) < \varphi(A_0,B_0)$.

The previous proof still holds if we define the abscissa φ as the inner product by any vector which is not orthogonal to (A,B).

- In the particular case where this vector is $(1,0)$ instead of (A,B), we obtain a “projection” of the previous proof onto the horizontal axis, which is equivalent to the proof we gave for Euclid’s non-geometric lemma.

- In the particular case where the vector which defines the abscissa φ is the vector provided by Bézout’s identity (which was not used in the previous proof), we do not need the initial lemma any more when showing that the set H' has the form $\{X(A_0,B_0), X \in \mathbb{Z}\}$, and we can take $(A_0,B_0) = (A,B)$ without having to prove that $Y = \pm 1$. We just have to see that, for all $(U,V) \in H'$:

$$\varphi(U,V) = \varphi(U,V).1 = \varphi(U,V)\varphi(A,B) = \varphi((\varphi(U,V))(A,B))\cdot$$

  Thus, since φ is one-to-one: $(U,V) = (\varphi(U,V))(A,B) = X(A,B)$. The proof for this case is equivalent to the classical proof of Euclid’s lemma as a corollary of Bézout’s identity.

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Another proof of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

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Abstract

In this paper we present a new short and simple solution to the Basel problem using a complex line integral of $\frac{\log(1+z)}{z}$.

1. Introduction

The Basel problem is a famous problem in number theory, first posed by Pietro Mengoli in 1644, and asks for the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots .$$

The problem remained open for 90 years, until Leonhard Euler found the exact sum to be $\frac{\pi^2}{6}$, in 1734. He would eventually propose three separate solutions to the problem during his lifetime.

Mathematicians have found different solutions to the Basel problem drawing from diverse domains of mathematics as complicated analysis, calculus, probability, and the theory of Hilbert space [1-13]. A good survey of such proofs that mathematicians have discovered can be found in [14-15]. In this paper we present a simple solution to the Basel problem using the Cauchy Integral Theorem applied to the complex line integral of $\frac{\log(1+z)}{z}$.

2. The Proof

We first take note of the Taylor series expansion of $\log(1+x)$ which is as follows:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .$$

The series converges for $|x| < 1$. Now if we divide the expression by $x \neq 0$ we get

$$\frac{\log(1+x)}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots . \tag{1}$$

Integrating both sides of (1) from -1 to 1, we get the following equality:

$$\begin{aligned}\int_{-1}^{1} \frac{\log(1+x)}{x} dx &= \int_{-1}^{1} \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \cdots\right) dx \\ &= 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.\end{aligned} \tag{2}$$

The integrand on the L.H.S. of (2) has a removable singularity at 0. This singularity can be removed by putting $\left.\frac{\log(1+x)}{x}\right|_{x=0}$ as $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$.

Here, term-wise integration is allowed because the series on the R.H.S. of (2) converges uniformly in $[-1,1]$ and each individual term in the series is integrable in $[-1,1]$. The uniform convergence can be proved by showing that the sequence of functions $f_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^k}{k+1}$ converges uniformly to a limiting function $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k+1}$ on the set $[-1,1]$. Indeed, given any $\varepsilon > 0$ let $N = \frac{1}{\varepsilon} - 1$ and note that for any $n > N$ and x in $[-1,1]$, since $f(x)$ is the sum of an alternating series we have

$$|f(x) - f_n(x)| \le \left|\frac{x^{n+1}}{n+1}\right| \le \frac{1}{n+1} < \frac{1}{N+1} = \varepsilon.$$

The series on the R.H.S. of (2) can be written as

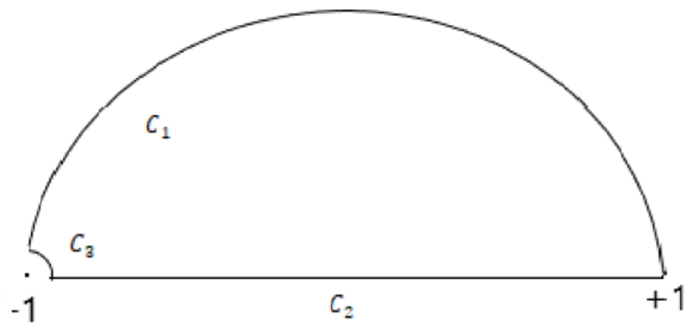
$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\ &= \left(1 - \frac{1}{4}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.\end{aligned}$$

Therefore we can get the sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ by knowing the definite integral $\int_{-1}^{1} \frac{\log(1+x)}{x} dx$. Let us evaluate this integral by using complex variables. For our convenience, we change the variable of integration to z. Therefore our only task remaining is to compute the integral

$$\int_{-1}^{+1} \frac{\log(1+z)}{z} dz. \tag{3}$$

Notice that the integrand in (3) has a removable singularity at 0 as we have discussed earlier.

Now let us consider the semi-circle $\gamma(x) = e^{i\theta}, 0 \le \theta \le \pi$. Instead of integrating along the straight line from -1 to 1 we can integrate it along the arc of the semi-circle if we knew the contour integral over the full semi- circle. Notice that function $\frac{\log(1+z)}{z}$ has a non-removable singularity at -1, so we have to bend our contour around the pole -1 so that there is no pole inside the contour. The bend can be modeled as the arc of the circle of radius ε having centre at -1 having one end point at $-1+\varepsilon$ the other end point of the arc is the intersection of the bigger and the smaller circle.



The closed contour is defined as

$$C := C_1 + C_2 + C_3$$

Here, C_1 is a circular arc at centre 0 spanning from 0 to $\pi - 2\sin^{-1}\left(\frac{\varepsilon}{2}\right)$, C_2 is a circular arc at centre -1 having radius ε, and C_3 is the straight line from$-1+\varepsilon$ to 1. With these notations we can write

$$\int_C \frac{\log(1+z)}{z}dz = \int_{C_1} \frac{\log(1+z)}{z}dz + \int_{C_2} \frac{\log(1+z)}{z}dz + \int_{C_3} \frac{\log(1+z)}{z}dz. \quad (4)$$

Here, the direction of the integration in the arc is anticlockwise. Now we shall use the Cauchy Integral Theorem. Before using that let us first state the theorem.

Theorem (Cauchy Integral Theorem). *Let $U \subseteq \mathbb{C}$ be a simply connected open set, and let $f : U \to \mathbb{C}$ be a holomorphic function. Let $\gamma : [a,b] \to U$ be a smooth closed curve. If γ is homotopic to a constat curve, then*

$$\int_\gamma f(z)dz = 0.$$

Let us put $f(z) = \frac{log(1+z)}{z}$. Using the Cauchy Integral Theorem we have

$$\int_C \frac{\log(1+z)}{z}dz = 0. \quad (5)$$

We can then write, using equations (4) and (5), that

$$\begin{aligned}\int_{-1}^{1} \frac{\log(1+z)}{z}dz &= \lim_{\varepsilon\to 0}\int_{C_2} \frac{\log(1+z)}{z}dz \\ &= -\lim_{\varepsilon\to 0}\int_{C_1} \frac{\log(1+z)}{z}dz - \lim_{\varepsilon\to 0}\int_{C_3} \frac{\log(1+z)}{z}dz.\end{aligned} \quad (6)$$

C_3 lies on the circle or radius ε centered at -1 and this circle can be written as $-1+\varepsilon e^{it}$. Therefore we can write

$$\begin{aligned}\int_{C_3} \frac{\log(1+z)}{z}dz &= i\varepsilon \int_0^{\frac{\pi-\varepsilon}{2}} \frac{\log(\varepsilon e^{it})}{-1+\varepsilon e^{it}} e^{it} dt \\ &= i\varepsilon \int_0^{\frac{\pi-\varepsilon}{2}} \frac{\log(\varepsilon)+it}{-1+\varepsilon e^{it}} e^{it} dt \\ &= i\varepsilon\log(\varepsilon) \int_0^{\frac{\pi-\varepsilon}{2}} \frac{e^{it}}{-1+\varepsilon e^{it}}dt - \varepsilon \int_0^{\frac{\pi-\varepsilon}{2}} \frac{te^{it}}{-1+\varepsilon e^{it}}dt\end{aligned}$$

Both the integrals are bounded above because we know, by the ML inequality,

$$\left|\int_0^{\frac{\pi-\varepsilon}{2}} \frac{e^{it}}{-1+\varepsilon e^{it}}dt\right| \leq \left|\frac{e^{it}}{-1+\varepsilon e^{it}}\right| |dt| = \frac{\pi-\varepsilon}{2(1-e)}$$

and

$$\left|\int_0^{\frac{\pi-\varepsilon}{2}} \frac{te^{it}}{-1+\varepsilon e^{it}}dt\right| \leq \left|\frac{te^{it}}{-1+\varepsilon e^{it}}\right| |dt| = \frac{(\pi-\varepsilon)^2}{4(1-e)}.$$

Therefore the first integral is multiplied by $\varepsilon(\log\varepsilon)$ and the second one is multiplied by ε. Both of them tend to zero as ε tends to zero and hence $\int_{C_3} \frac{\log(1+z)}{z}dz$ also tends to zero as ε tends to zero. Here, the direction of the integration in the semi-circular arc C_1 is anti-clockwise. Now if we put $z = e^{it}$, we can rewrite the analytic function $g(z) = \frac{\log(1+e^{i\theta})}{e^{i\theta}}$ as

$$\frac{\log(1+e^{i\theta})}{e^{i\theta}} = \frac{\log\left(2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right)}{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)} = \frac{\log e^{i\left(\frac{\theta}{2}\right)} + \log\left(2\cos\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}.$$

Now the contour integral around C_1 shall be

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{C_1} \frac{\log(1+z)}{z}dz &= \lim_{\varepsilon\to 0}\int_0^{\pi-2\sin^{-1}\left(\frac{\varepsilon}{2}\right)} \frac{\log(1+e^{i\theta})}{e^{i\theta}}d\theta \\
&= \int_0^{\pi} \frac{\log e^{i\left(\frac{\theta}{2}\right)} + \log\left(2\cos\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)} \left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right) id\theta \\
&= \int_0^{\pi} \left(\log e^{i\left(\frac{\theta}{2}\right)} + \log\left(2\cos\frac{\theta}{2}\right)\right) id\theta \\
&= \int_0^{\pi} \left(-\frac{\theta}{2} + i\log\left(2\cos\frac{\theta}{2}\right)\right) d\theta
\end{aligned}$$

since $\lim_{\varepsilon\to 0}\left(\pi-\sin^{-1}\left(\frac{\varepsilon}{2}\right)\right) = \pi$. Now we get

$$\begin{aligned}
\int_{C_1} \frac{\log(1+z)}{z}dz &= \int_0^{\pi} \left(-\frac{\theta}{2} + i\log\left(2\cos\frac{\theta}{2}\right)\right) d\theta \\
&= -\frac{\pi^2}{4} + \int_0^{\pi} i\log\left(2\cos\frac{\theta}{2}\right) d\theta \\
&= -\frac{\pi^2}{4} + I,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
I &= \int_0^{\pi} i\log\left(2\cos\frac{\theta}{2}\right) d\theta \\
&= i\int_0^{\pi} \left(\log(2) + \log\cos\frac{\theta}{2}\right) d\theta \\
&= i(\pi\log 2 + J).
\end{aligned}$$

Now we know that

$$\begin{aligned} J &= \int_0^{\pi} \left(\log\cos\frac{\theta}{2}\right) d\theta \\ &= 2\int_0^{\pi} \left(\log\cos\frac{\theta}{2}\right) d\left(\frac{\theta}{2}\right) \\ &= 2\int_0^{\frac{\pi}{2}} (\log\cos(v))\, d\,(v) \\ &= 2K, \end{aligned} \tag{8}$$

subsituting $v = \cos\frac{\theta}{2}$. Now observe the following equalities:

$$\int_0^{\frac{\pi}{2}} \log\cos v dv = \int_0^{\frac{\pi}{2}} \log\sin v dv$$

and

$$\int_0^{\pi} \log\sin v dv = 2\int_0^{\frac{\pi}{2}} \log\sin v dv.$$

Therefore

$$\begin{aligned} 2K &= \int_0^{\frac{\pi}{2}} \log\cos v dv \\ &= \int_0^{\frac{\pi}{2}} \log\sin v dv \\ &= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2v}{2}\right) dv \\ &= \int_0^{\frac{\pi}{2}} \log\sin 2v dv - \int_0^{\frac{\pi}{2}} \log 2 dv \\ &= \int_0^{\frac{\pi}{2}} \log\sin 2v dv - \frac{\pi\log 2}{2} \\ &= \frac{1}{2}\int_0^{\pi} \log\sin v dv - \frac{\pi\log 2}{2}. \end{aligned} \tag{9}$$

But we know that $\int_0^{\pi} \log\sin v dv = 2\int_0^{\frac{\pi}{2}} \log\sin v dv$ so we get $2K = \frac{1}{2}(2K) - \frac{\pi\log 2}{2}$ and hence $K = -\frac{\pi\log 2}{2}$. Therefore from (9) we have $J = \pi\log 2$. We can thus write

$$\begin{aligned} I &= \int_0^{\pi} i\log\left(2\cos\frac{\theta}{2}\right) d\theta \\ &= i\int_0^{\pi} \left(\log(2) + \log\cos\frac{\theta}{2}\right) d\theta \\ &= i\,(\pi\log 2 + J) \\ &= i\,(\pi\log 2 - \pi\log 2) = 0. \end{aligned} \tag{10}$$

Therefore, from equations (6), (7), and (10), we get

$$\int_{-1}^{1} \frac{\log(1+x)}{x} dx = -\int_{C_1} \frac{\log(1+z)}{z} dz = -\left(-\frac{\pi^2}{4} + I\right) = \frac{\pi^2}{4}.$$

From equation (2) we know that

$$\sum_0^\infty \frac{1}{(2n+1)^2} = \frac{\pi^2}{8},$$

but we also know that

$$\frac{3}{4}\sum_0^\infty \frac{1}{n^2} = \sum_0^\infty \frac{1}{(2n+1)^2}$$

and so we get

$$\sum_0^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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Editor's note: The following note presents some results on special cases of the Collatz Conjecture, a famous open problem in mathematics. While the results may be of interest in and of themselves, we include this note to also introduce the conjecture to those not familiar with its statement and provide a few references to recent work on the problem.

Step Count Considerations in the $3n+1$ *Problem*

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Abstract

Given integers of a specific form, we compare the number of steps in their Collatz processes.

Introduction

The Collatz conjecture is a famous statement about natural numbers which has been an intriguing mathematical problem since it was posed by Lothar Collatz in 1937. The statement of the conjecture is deceptively simple: consider the following process, beginning with a natural number n:

1. if n is even, divide by 2;
2. if n is odd multiply n by 3 and then add 1;
3. repeat.

Call this the *Collatz process*.

e.g. Take $n = 15$:

1. $3(15)+1=46$	10. $40/2=20$
2. $46/2=23$	11. $20/2=10$
3. $3(23)+1=70$	12. $10/2=5$
4. $70/2=35$	13. $3(5)+1=16$
5. $3(35)+1=106$	14. $16/2=8$
6. $106/2=53$	15. $8/2=4$
7. $3(53)+1 \;=160$	16. $4/2=2$
8. $160/2=80$	17. $2/2=1$
9. $80/2=40$	

Notice that once the Collatz process outputs a 1, the cycle 1, 4, 2, 1 will repeat *ad infinitum*.

Collatz Conjecture: *For any natural number n, the Collatz process will eventually result in a 1.*

Results generated by humans and computers have demonstrated that the conjecture is true for values up to 2^{68}. Distinguished mathematicians such as Paul Erdos and Jeffrey Lagarias have stated that this is an exceptionally difficult problem [5, 6]. John Conway has proved that the inability to find a generalization of the Collatz problem is the main obstacle to proving or disproving the conjecture by arithmetic means [4]. However, Terence Tao and Gerhard Opfer have recently proved that the conjecture is "almost true" for "almost all" natural numbers [7].

We define the *step-count* to be the number of repetitions of the process are required to produce a 1.

e.g. For $n = 15$, the step-count is 17.

This paper will consider various results regarding the step-count.

Four Special Cases

Theorem 1. *– If $i \in \mathbb{N}$ and the step-count for $2^{2i}3^{3i} - 1$ is N, then the step-count for $2^{3i}3^{2i} - 1$ is $N + 2i$.*

Proof. Let $a_0 = 2^{2i}3^{3i} - 1$, $b_0 = 2^{3i}3^{2i} - 1$ and b_k be the k^{th} step in the Collatz process starting with b_0. Then

$$
\begin{aligned}
b_0 =& 2^{3i}3^{2i} - 1 \\
b_1 =& 2^{3i}3^{2i+1} - 2 \\
b_2 =& 2^{3i-1}3^{2i+1} - 1 \\
b_3 =& 2^{3i-1}3^{2i+2} - 2 \\
b_4 =& 2^{3i-2}3^{2i+2} - 1.
\end{aligned}
$$

By induction (the reader should check this), for any $k, 0 \leq k < 3i$,

$$
\begin{aligned}
b_{2k} =& 2^{3i-k}3^{2i+k} - 1 \\
b_{2k+1} =& 2^{3i-k}3^{2i+k+1} - 2.
\end{aligned}
$$

Then $b_{2i} = 2^{3i-i}3^{2i+i} - 1 = 2^{2i}3^{2i} - 1 = a_0$. Therefore the step-count for b_0 is $N + 2i$. ■

e.g.

i	$2^{2i}3^{3i} - 1$	step-count	$2^{3i}3^{2i} - 1$	step count	$N + 2i$
1	107	100	71	102	100+2(1)
2	11663	50	5183	54	50+2(2)
5	14693286787	414	1934917631	424	414+2(5)

Theorem 2. *If $i \in \mathbb{N}$ and the step-count for $2^{2i+2}3^{2i+3} - 1$ is N, then the step-count for $3^{2i+2}2^{2i+3} - 1$ is $N + 2$.*

Proof. Let $a_0 = 2^{2i+2}3^{2i+3} - 1, b_0 = 3^{2i+2}2^{2i+3} - 1$ and b_k be the k^{th} step in the Collatz process starting with b_0. Then

$$
\begin{aligned}
b_0 =& 2^{2i+3}3^{2i+2} - 1 \\
b_1 =& 2^{2i+3}3^{2i+3} - 2 \\
b_2 =& 2^{2i+2}3^{2i+3} - 1 = a_0.
\end{aligned}
$$

Therefore the step-count for b_0 is $N + 2$. ■

e.g.

i	$2^{2i+2}3^{2i+3}-1$	step-count	$2^{2i+3}3^{2i+2}-1$	step count	$N+2$
1	3887	51	2591	53	51+2
2	139967	144	9311	146	144+2

Theorem 3. *If $i \in \mathbb{N}$ and the step-count for $2^{5+2i}3^i+1$ is N, then the step-count for $2^{6+2i}3^i+1$ will be $N+1$.*

Proof. Let $a_0 = 2^{5+2i}3^i+1, b_0 = 2^{6+2i}3^i+1$ and a_k and b_k be the k^{th} step in the Collatz process starting with a_0 and b_0, respectively. Here

$$a_0 = 2^{5+2i}3^i+1$$

$$\begin{aligned} a_1 &= 2^{5+2i}3^{i+1}+4 \\ a_2 &= 2^{5+2i-1}3^{i+1}+2 \\ a_3 &= 2^{5+2i-2}3^{i+1}+1 \\ a_4 &= 2^{5+2i-2}3^{i+2}+4 \\ a_5 &= 2^{5+2i-3}3^{i+2}+2 \\ a_6 &= 2^{5+2i-4}3^{i+2}+1. \end{aligned}$$

By induction (again, the reader should check this), for any $k \in \mathbb{N}$,

$$\begin{aligned} a_{3k} &= 2^{2i+5-2k}3^{i+k}+1 \\ a_{3k+1} &= 2^{2i+5-2k}3^{i+k+1}+4 \\ a_{3k+2} &= 2^{2i+4-2k}3^{i+k+1}+2. \end{aligned}$$

For $k_0 = i+2$ we will have $2i+4-2k_0 = 0$ and i and k_0 are either both even or are both odd. Thus $a_{3k_0+2} = 3^{i+k_0+1}+2$ and, since $i+k_0+2$ is even,

$$\begin{aligned} a_{3k_0+3} &= 3^{i+k_0+2}+7 \\ &\equiv (-1)^{i+k_0+2}+3 \pmod 4 \\ &\equiv 0 \pmod 4. \end{aligned}$$

Therefore, $a_{3k_0+5} = \frac{3^{i+k_0+2}+7}{4}$.

Now

$$\begin{aligned}
b_0 =& 2^{6+2i}3^i + 1\\
b_1 =& 2^{6+2i}3^{i+1} + 4\\
b_2 =& 2^{6+2i-1}3^{i+1} + 2\\
b_3 =& 2^{6+2i-2}3^{i+1} + 1\\
b_4 =& 2^{6+2i-2}3^{i+2} + 4\\
b_5 =& 2^{6+2i-3}3^{i+2} + 2\\
b_6 =& 2^{6+2i-4}3^{i+2} + 1.
\end{aligned}$$

By induction (once again, the reader should check this), for any $k \in \mathbb{N}$,

$$\begin{aligned}
b_{3k} =& 2^{2i+6-2k}3^{i+k} + 1\\
b_{3k+1} =& 2^{2i+6-2k}3^{i+k+1} + 4\\
b_{3k+2} =& 2^{2i+6-2k-1}3^{i+k+1} + 2\\
b_{3k+3} =& 2^{2i+6-2k-2}3^{i+k} + 1 = 2^{2i+4-2k}3^{i+k} + 1.
\end{aligned}$$

Then $b_{3k_0+3} = 3^{i+k_0+1} + 1$ and we note that, since $i + k_0 + 1$ is odd:

$$b_{3k_0+3} = 3^{i+k_0+1} + 1 \equiv (-1)^{i+k_0+1} + 1 \equiv 0 (\bmod 4)$$

but

$$b_{3k_0+3} = 3^{i+k_0+1} + 1 \equiv (-5)^{i+k_0+1} + 1 \not\equiv 0 (\bmod 8).$$

Therefore $b_{3k_0+5} = \frac{3^{i+k_0+1}+1}{4}$, which is odd and hence

$$b_{3k_0+6} = 3\left(\frac{3^{i+k_0+1}+1}{4}\right) + 1 = \frac{3^{i+k_0+2}+7}{4} = a_{3k_0+5}.$$

It follows then that the step-count for $2^{6+2i}3^i + 1$ is one more thatn that for $2^{5+2i}3^i + 1$. ∎

e.g.

i	$2^{5+2i}3^i + 1$	step-count	$2^{6+2i}3^i + 1$	step count	$N + 1$
0	33	26	65	27	26+1
1	385	50	769	51	50+1)
2	4609	46	9217	47	46+1

Theorem 4. *If $i \in \mathbb{N}$ and the step-count for $2^{2i+2}3^{2i+4} - 1$ is N, then the step-count for $2^{2i+4}3^{2i+2} - 1$ will be $N + 4$.*

Proof. Let $a_0 = 2^{2i+2}3^{2i+4} - 1, b_0 = 2^{2i+4}3^{2i+2} - 1$ and a_k and b_k be the k^{th} step in the Collatz process starting with a_0 and b_0, respectively. Then

$$\begin{aligned}
b_0 &= 2^{2i+4}3^{2i+2} - 1 \\
b_1 &= 2^{2i+4}3^{2i+3} - 2 \\
b_2 &= 2^{2i+3}3^{2i+3} - 1 \\
b_3 &= 2^{2i+3}3^{2i+4} - 2 \\
b_4 &= 2^{2i+2}3^{2i+4} - 1 = a_0.
\end{aligned}$$

Thus it takes b_0 four more steps to reach 1 that it will for a_0 . ■

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The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before December 31, 2021. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2021 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 870 - 880

Problem 870. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*
Find all positive integers x, y such that $x^3 - y^3 - xy = 113$.

Problem 871. *Proposed by Seán Stewart, Bomaderry, NSW, Australia.*
Evaluate $\int_0^{\pi/2} \csc x \log^3\left(\frac{1+\cos x+\sin x}{1+\cos x-\sin x}\right) dx$.

Problem 872. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*
Let $x, y, z \in (0, 1)$ with $xy + yz + zx = 1$. Prove that

$$4(x+y+z) \leq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 9xyz.$$

Problem 873. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Prove that $a^2\tan^k x + b^2\sin^k x > 2abx^k$ for all $x \in (0, \frac{\pi}{2})$ and positive integer k.

Problem 874. *Proposed by Abhijit Bhattacharjee, ex Msc student in BHU, India.*
Prove that the equation $a^2 + (a+n)^2 = b^2$ with $a, b, n \in \mathbb{N}$ has infinitely many

solutions for each n.

Problem 875. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*
Calculate the following integral:

$$\int_0^\infty \frac{\arctan x}{\sqrt{2x^4+x^2+2}}dx.$$

Problem 876. *Proposed by Ankush Kumar Parcha (student), Indira Gandhi National Open University, New Delhi, India and Toyesh Prakash Sharma (student), St. C.F Andrews School, Agra, India.*
If $x = \sum_{n=1}^{\infty}\left(x^{2n}+\frac{1}{x^{2n}}\right)$ and $y = \sum_{n=0}^{\infty}\frac{1+x^{2n+1}}{x^n}$, compute the value of x^y.

Problem 877. *Proposed by Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain.*
Let x be a real number. For any positive integer n, find closed forms for the following sums:

$$S_n^e = \sum_{k\,even}\binom{n+1}{k}x^k, \quad S_n^o = \sum_{k\,odd}\binom{n+1}{k}x^k$$
$$F_n^e = \sum_{k\,even}\binom{n+1}{k}F_k, \quad F_n^o = \sum_{k\,odd}\binom{n+1}{k}F_k$$
$$L_n^e = \sum_{k\,even}\binom{n+1}{k}L_k, \quad L_n^o = \sum_{k\,odd}\binom{n+1}{k}L_k$$

where F_n and L_n are respectively the n^{th} Fibonacci and Lucas numbers defined both by the recurrence relation $u_{n+2} = u_{n+1} + u_n$ with initial values $F_0 = 0, F_1 = 1, L_0 = 2$ and $L_1 = 1$.

Problem 878. *Proposed by Mihaly Bencze, Brasov, Romania.*
Let F_n and L_n be respectively the n^{th} Fibonacci and Lucas numbers as defined above. Prove the following two inequalities:

1. $\prod_{k=1}^{n}\left(\frac{2F_n}{F_k}-1\right) \geq \left(\frac{2nF_n}{F_{n+2}-1}-1\right)^n$

2. $\prod_{k=1}^{n}\left(\frac{2L_n}{L_k}-1\right) \geq \left(\frac{2nL_n}{L_{n+2}-1}-1\right)^n.$

Problem 879. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*
Let $a > b > 0$. Evaluate $\int_0^\pi \frac{\sin^n x}{(a+b\cos x)^{n+1}}dx$ for $n = 0, 1, 2, \ldots$

Problem 880. *Proposed by Dorin Marghidanu, Colegiul National 'A.I. Cuza', Corabia, Romania.*
Let $n \geq a_k, b_k > 0$ with n and p integers ≥ 2. Prove that

$$\frac{1}{\sqrt[p]{a_1 a_2 \ldots a_n}} + \frac{1}{\sqrt[p]{b_1 b_2 \ldots b_n}} \geq \frac{2\sqrt[p]{2^n}}{\sqrt[p]{(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)}}$$

SOLUTIONS TO PROBLEMS 849 - 858

Problem 849. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*
Prove that in an acute ΔABC the following relationship holds:

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} + \frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} > 6\sqrt{2}.$$

Solution *by Titu Zvonaru, Comăneşti, Romania.*

By items 2.45 and 2.49 from O. Bottema, *Geometric Inequalities*, Groningen, 1969, it is known that

$$\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \geq 2\sqrt{3}$$

and

$$\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C} \geq 6.$$

Since $6 + 2\sqrt{3} > 6\sqrt{2} \Leftrightarrow 36 + 12 + 24\sqrt{3} > 72 \Leftrightarrow \sqrt{3} > 1$ which is clearly true.
Note: The inequality is true for any triangle (not just for acute triangles).

Also solved by Brian Bradie, Christopher Newport University, Newport News, *VA; Pratik Donga, India; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Toyesh Prakash Sharma (student), St. C.F Andrews School, Agra, India; SQ Problem Solving Group, Yogyakarta, Indonesia; Albert Stadler, Herrliberg, Switzerland; Daniel Văcaru, Economic College "Maria Teiuleanu", Piteşti, Romania; and the proposer.*

Problem 850. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*
If $x,y,z>0$ and $x+y+z=2\pi$, prove

$$\frac{\cos^4 x}{y+z}+\frac{\cos^4 y}{z+x}+\frac{\cos^4(x+y)}{x+y}\geqslant\frac{9}{64\pi}.$$

Solution *by Angel Plaza, University of Las Palmas de Gran Canaria, Spain.*

Since $\cos^4(x+y)=\cos^4(2\pi-x-y)=\cos^4 z$, the proposed inequality may be written as

$$\frac{\cos^4 x}{2\pi-x}+\frac{\cos^4 y}{2\pi-y}+\frac{\cos^4(z)}{2\pi-z}\geq\frac{9}{64\pi}.$$

By the Cauchy-Schwarz inequality in Engel form, the left-hand side of the inequality, say LHS, is

$$LHS\geq\frac{\left(\cos^2 x+\cos^2 y+\cos^2 z\right)}{2\pi-x+2\pi-y+2\pi-z}=\frac{\left(\cos^2 x+\cos^2 y+\cos^2 z\right)}{4\pi}$$

Therefore, it is enough to prove that $\left(\cos^2 x+\cos^2 y+\cos^2 z\right)\geq\frac{3}{4}$, where $x,y,z>0$ and $x+y+z=2\pi$. This follows easily by Lagrange multipliers. Notice that the given inequality becomes equality for $x=y=z=\frac{2\pi}{3}$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Daniel Văcaru, Economic College "Maria Teiuleanu", Piteşti, Romania; Titu Zvonaru, Comăneşti, Romania; and the proposer.

Problem 851. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*
Let $a<b$ and $f:\ [a,b]\to(0,\infty)$ be continuous. Prove

$$3(b-a)\int_a^b f^2(x)dx+(b-a)^2\geqslant 2(b-a)\int_a^b f(x)dx+2\left(\int_a^b f(x)dx\right)^2.$$

Solution *by Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC.*

Let $\omega = \int_a^b f(x)dx$. The Cauchy-Schwarz inequality gives

$$\omega \leq \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}} \left(\int_a^b 1dx\right)^{\frac{1}{2}} = \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}} (b-a)^{\frac{1}{2}}$$

and squaring produces

$$\omega^2 \leq (b-a)\int_a^b f^2(x)dx.$$

Therefore,

$$\begin{aligned}
3(b-a)\int_a^b f^2(x)dx + (b-a)^2 - 2(b-a)\int_a^b f(x)dx - 2\left(\int_a^b f(x)dx\right)^2 \\
&= 3(b-a)\int_a^b f^2(x)dx + (b-a)^2 - 2(b-a)\omega - 2\omega^2 \\
&\geq \omega^2 - 2(b-a)\omega + (b-a)^2 \\
&= (\omega - (b-a))^2 \geq 0.
\end{aligned}$$

Hence the inequality is proved.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, , Herrliberg, Switzerland; and the proposer.

Problem 852. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*
If $a,b,c > 0$ and $a+b+c = 3$, prove

$$\sum_{cyc} \frac{a^3c(b+1)+b^3c(a+1)}{a^2b(b+1)+b^2a(a+1)} \geqslant 3.$$

Solution *by Daniel Văcaru, Economic College "Maria Teiuleanu", Piteşti, Romania.*

We have

$$\begin{aligned}
\frac{a^3c(b+1)+b^3c(a+1)}{a^2b(b+1)+b^2a(a+1)} &= \frac{a^3bc+ab^3c+a^3c+b^3c}{a^2b+2a^2b^2+ab^2} \\
&\geq \frac{a^3bc+ab^3c+a^2bc+ab^2c}{ab(a+b+2ab)} \\
&= \frac{abc(a^2+b^2+a+b)}{ab(a+b+2ab)} \\
&= c\frac{a^2+b^2+a+b}{a+b+2ab} \geq c
\end{aligned}$$

because $a^3+b^3 \geq a^2b+ab^2$ and $a^2+b^2 \geq 2ab$. Then by summation, we have

$$\sum_{cyc} \frac{a^3c(b+1)+b^3c(a+1)}{a^2b(b+1)+b^2a(a+1)} \geq \sum_{cyc} c = 3.$$

Also solved by Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; and the proposer.

Problem 853. *Proposed by Marcel Chirita, Bucharest, Romania.*
Let $x \in \mathbb{Z}$. If $x^5+5x^3+15x^2 > 21x$, prove $x^5+5x^3+15x^2-21x \geqslant 30$.

Solution *by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.*

We write

$$\begin{aligned}
x^5+5x^3+15x^2-21x &= (x^5-x)+5(x^3-x)+15(x^2-x)\\
&= (x-1)x(x^2+1)+5(x-1)x(x+1)+15x(x-1)\\
&= (x-2)(x-1)x(x+1)(x+2)+5(x-1)x(x+1)+5(x-1)x(x+1)\\
&\quad +15x(x-1)\\
&= (x-2)(x-1)x(x+1)(x+2)+10(x-1)x(x+1)+15x(x-1)
\end{aligned}$$

We now claim this is a multiple of 30 using the well-known fact that the product of any k consecutive integers is a multiple of k. The second term is a multiple of 3 and 10 and so is a multiple of 30. The last term is a multiple of 2 and 15. Finally, the first term, as a product of 5 consecutive integers, is a multiple of 2, 3, and 5. Thus $x^5+5x^3+15x^2-21x$ is a sum of positive multiples of 30 and so is greater than or equal to 30. (The given inequality implies that $x \geq 2$, guaranteeing that all terms are nonnegative.)

Also solved by Brian Beasley, Presbyterian College, Clinton, SC; Brian Bradie, Christopher Newport University, Newport News, VA; Pratik Donga, India; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; and the proposer.

Problem 854. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
If $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 2019$ and $3f(x) = f(x+y)+2f(x-y)+y$ for any $x, y \in \mathbb{R}$, then compute $\int_e^{\pi} f(x)dx$.

Solution *by the Spring 2020 North Carolina Wesleyan College Senior Seminar Class members Stephany Bryant, Victoria Burkhart, Cody Fogelman, Jovan Pope, Jadéjah Robinson, Julia Trimmer, Abigail Wooten, Rocky Mount, NC.*

Let $x \in \mathbb{R}$. If $y = x$, then by substitution we have

$$\begin{aligned} 3f(x) &= f(x+y) + 2f(x-y) + y \\ &= f(x+x) + 2f(x-x) + x \\ &= f(2x) + 2f(0) + x. \end{aligned}$$

If $y = -x$, then by substitution we have

$$\begin{aligned} 3f(x) &= f(x+y) + 2f(x-y) + y \\ &= f(x-x) + 2f(x+x) - x \\ &= f(0) + 2f(2x) - x. \end{aligned}$$

Setting the 2 equations equal yields

$$f(2x) + 2f(0) + x = f(0) + 2f(2x) - x.$$

Solving for $f(2x)$ gives

$$f(2x) = 2x + f(0).$$

We are given $f(0) = 2019$ and we can replace $2x$ by x to get
$f(x) = x + 2019$. Integrating,

$$\int_e^\pi (x+2019)dx = \left(\frac{x^2}{2} + 2019x\right)\Bigg|_e^\pi = \frac{\pi^2 - e^2}{2} + 2019(\pi - e).$$

Also solved by Abhijit Bhattacharjee, Banaras Hindu University, Varanasi, India; Brian Bradie, Christopher Newport University, Newport News, VA; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, Bomaderry, NSW, Australia; and the proposers.

Problem 855. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Let $\{a_n\}, \{b_n\}, \{c_n\}$ be positive real sequences such that
$\lim\limits_{n\to\infty} \frac{a_{n+1}}{na_n} = a > 0$, $\lim\limits_{n\to\infty} \frac{b_{n+1}}{nb_n} = b > 0$, and $\lim\limits_{n\to\infty} \frac{c_{n+1}}{nc_n} = c > 0$.
Compute $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{a_n^3}} \sum\limits_{k=1}^{n} (b_k c_k)^{1/k}$.

Solution *by Ioannis Sfikas, National and Kapodistrian University of Athens, Greece.*

Let $L = \lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{a_n^3}} \sum\limits_{k=1}^{n} (b_k c_k)^{1/k}$. Using the Cesaro-Stolz theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^3}{\sqrt[n]{a_n^3}} &= \lim_{n\to\infty} \left(\frac{n}{\sqrt[n]{a_n}}\right)^3 = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{a_n}}^3 \\
&= \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{a_{n+1}} . \frac{a_n}{n^n}\right]^3 \\
&= \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^n \frac{(n+1)}{n} . \frac{n a_n}{a_{n+1}}\right]^3 = \frac{e^3}{a^3}.
\end{aligned}$$

Also

$$\begin{aligned}
\lim_{n\to\infty} \frac{b_1c_1 + \sqrt{b_2c_2} + \ldots + \sqrt[n]{b_nc_n}}{n^3} &= \lim_{n\to\infty} \frac{\sqrt[n+1]{b_{n+1}c_{n+1}}}{(n+1)^3 - n^3} \\
&= \lim_{n\to\infty} \frac{\sqrt[n+1]{b_{n+1}c_{n+1}}}{3n^2 + 3n + 1} \\
&= \lim_{n\to\infty} \frac{n^2}{3n^2 + 3n + 1} . \frac{\sqrt[n]{b_nc_n}}{n^2} \\
&= \frac{1}{3} \lim_{n\to\infty} \sqrt[n]{\frac{b_nc_n}{n^{2n}}} = \frac{1}{3} \lim_{n\to\infty} \frac{b_{n+1}c_{n+1}}{(n+1)^{2n+2}} \\
&= \frac{1}{3} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{2n} . \frac{n^2}{(n+1)^2} . \frac{b_{n+1}}{nb_n} . \frac{c_{n+1}}{nc_n} = \frac{bc}{3e^2}.
\end{aligned}$$

Finally,

$$L = \frac{e^3}{a^3} \cdot \frac{bc}{3e^2} = \frac{bce}{3a^3}.$$

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Toyesh Prakash Sharma (student), St. C.F Andrews School, Agra, India; Albert Stadler, Herrliberg, Switzerland; and the proposers.

Problem 856. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*
Each 1×1 square of a 7×211 rectangle is painted either black or white. Prove that it is possible to choose four rows and four columns of the rectangle so that the sixteen 1×1 squares in which they intersect are painted with the same color.

Solution *by Albert Stadler, Herrliberg, Switzerland.*

We say that a column is black if it has more black than white squares, and vice versa we call it white if it has more white than black squares. Since 7 is odd every column is either black or white. Therefore, there are at least $106 = \frac{212}{2}$ columns that are black or at least 106 columns that are white. Suppose there are at least 106 columns that are black. There are $\binom{7}{4} = 35$ ways of painting 4 squares out of 7 in black. Since every black column has by definition at least 4 black squares, we may pick 4 black squares and assign to it in a unique way a number between 1 and 35 and that number denotes the coloring of that column. Two black columns with the same coloring number have 4 rows with black squares. In total we have 106 black columns so there are at least 4 columns that have the same coloring number, since 106 = 3*35+1 and the coloring number uniquely defines the four rows we need to select.

Also solved by Angel Plaza, University of Las Palmas de Gran Canaria, Spain; and the proposer.

Problem 857. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*
Let a, b, c, d be four positive real numbers. Find the maximum value of

$$\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}}{\sqrt[4]{a+b+c+d}}.$$

Solution *by Brian Bradie, Christopher Newport University, Newport News, VA.*

By the power mean inequality,

$$\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}}{4} \le \left(\frac{a+b+c+d}{4}\right)^{1/4},$$

with equality holding if and only if $a = b = c = d$. Rearranging this inequality yields

$$\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}}{\sqrt[4]{a+b+c+d}} \le \frac{4}{4^{1/4}} = 2\sqrt{2}.$$

Thus, the maximum value of $\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}}{\sqrt[4]{a+b+c+d}}$ is $2\sqrt{2}$, which is attained when $a = b = c = d$.

Also solved by Pratik Donga, India; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Angel Plaza, University of Las Palmas de Gran Canaria, Spain; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; and the proposer.

Problem 858. *Proposed by Pedro H.O. Pantoja, University of Campina Grande, Brazil.*
Let $M = \frac{8}{7}\sin^2\left(\frac{\pi}{7}\right) + \frac{\sqrt{7}}{7}\cot\left(\frac{\pi}{7}\right)$. Is M irrational?

Solution *by Brian Beasley, Presbyterian College, Clinton, SC.*

We show that M is rational, since $M = 1$. Let $M = \sin\left(\frac{\pi}{7}\right)$. Then

$M = \frac{8}{7}x^2 + \frac{\sqrt{7}}{7} \cdot \frac{\sqrt{1-x^2}}{x}$, so $M = 1$ if and only if

$$56x^3 = 49x - 7\sqrt{7(1-x^2)},$$

or equivalently

$$7x - 8x^3 = \sqrt{7(1-x^2)}.$$

Since $0 < x < \sqrt{\frac{7}{8}}$ implies that $7x - 8x^3 > 0$, the latter condition holds if and only if

$$64x^6 - 112x^4 + 49x^2 = 7(1-x^2)$$

or

$$64x^6 - 112x^4 + 56x^2 - 7 = 0.$$

Using the Chebyshev polynomial

$$T_7(x) = x\left(64x^6 - 112x^4 + 56x^2 - 7\right),$$

we have

$$\cos(7\theta) = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

for all real numbers θ. Since $x = \cos\left(\frac{5\pi}{14}\right)$, we let $\theta = \frac{5\pi}{14}$ to obtain

$$0 = x\left(64x^6 - 112x^4 + 56x^2 - 7\right).$$

But $x \neq 0$, so we conclude that

$$64x^6 - 112x^4 + 56x^2 - 7 = 0$$

and hence $M = 1$.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Pratik Donga, India; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Seán Stewart, Bomaderry, NSW, Australia; and the proposer.

Kappa Mu Epsilon News

Edited by Cynthia Huffman, Historian
Updated information as of January 2021

News of chapter activities and other noteworthy KME events should be sent to

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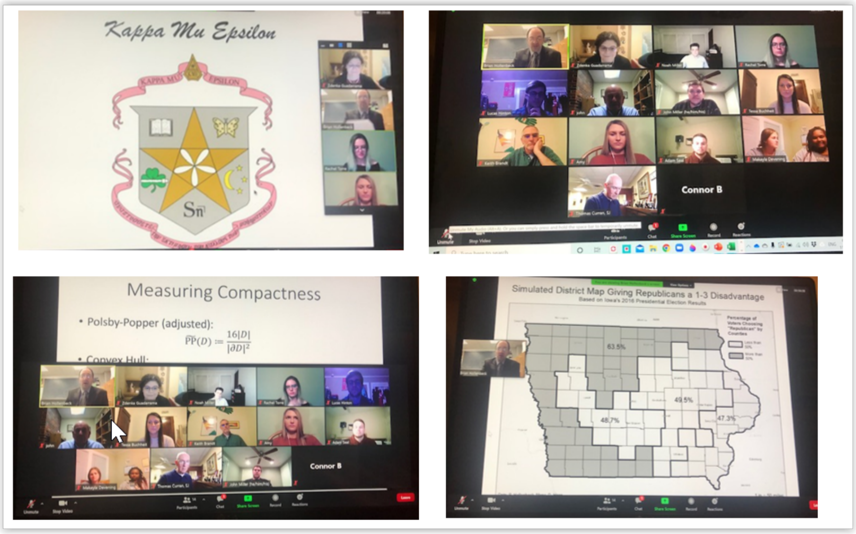
KAPPA MU EPSILON
Installation Report
Missouri Omicron, Rockhurst University
Kansas City, Missouri

The Missouri Omicron Chapter of Kappa Mu Epsilon was installed at 6 p.m. on November 13, 2020 at a ceremony held via Zoom. The meeting was conducted by Professor of Mathematics and Chair, Dr. Zdeňka Guadarrama. KME national president, Dr. Brian Hollenbeck, served as the installing officer.

Nine students and two faculty members were initiated as the charter members of the Missouri Omicron Chapter. The two faculty are Professor of Mathematics, Dr. Zdeňka Guadarrama, and Assistant Professor of Mathematics, Dr. John Miller. The nine students are Adam Seal, Amy O'Neal, Camryn Keaton, Connor Berry, Lucas Hinton, Makayla Devening, Noah Miller, Rachel Torre, and Tessa Buchheit. The first officers of the chapter were installed: Noah Miller, President; Camryn Keaton, Vice President; Rachel Torre, Secretary; Makayla Devening, Treasurer; John Miller, Corresponding Secretary; and Zdeňka Guadarrama, faculty sponsor.

Father Thomas Curran, President of Rockhurst University, also attended the ceremony and voiced his support. Following the installation ceremony, Dr. Hollenbeck presented a talk titled, *Mathematics vs. Gerrymandering*.

Some images from the meeting:



Missouri Omicron

Chapter News

AL Gamma – University of Montevallo
Corresponding Secretary – Scott Varagona; 8 New Members; 707 Total Members
New Initiates – KayCee Creel, Blaze Cruce, Shyan Flack, Benjamin Glass, Paul Hannon, Jovanna Kloss, Allison Laatsch, and Madison Lawrence.

AL Zeta – Birmingham-Southern College
Chapter President – Annie Dial; 8 New Members; 619 Total Members
Other Fall 2020 Officers: Mac DeLay, Vice President; Claude Hall, Secretary; Caroline Meffert, Treasurer; and Dr. Allie Ray, Corresponding Secretary and Faculty Sponsor
We held a virtual induction ceremony in October to welcome 8 new members to KME. We also helped celebrate #BlackInMathWeek by posting pictures and biographies of Black mathematicians throughout our building on campus.

AL Theta – Jacksonville State University
Chapter President – Hannah Davis; 289 Total Members
Other Fall 2020 Officers: Hannah Boozer, Vice President; Marcus Shell, Secretary; Bronte Ray, Treasurer; and Dr. David Dempsey, Corresponding Secretary and Faculty Sponsor
The Alabama Theta chapter had to resort to online meetings during Fall 2020. We met in September to elect new officers, since campus closed before our spring initiation ceremony. We used the Board Game Arena online platform to do some group gaming during September and October virtual meetings.

CA Gamma – California Polytechnic State University - San Luis Obispo
Corresponding Secretary – Dr. Robert W. Easton; 28 New Members; 1204 Total Members
New InitiatesNew Initiates – Nick Beaver, Laura Bialozynski, Maxwell Brewer, Emma Callant, Grace Carlson, Pheng Ang Chiv, Julia Coss, Jackie Driscoll, Madeline Farhani, David Harler, Justin Huynh, Sidra Knox, Hayden Lewis, Danielle Long, Emily Manning, Zoe Nepsa, Mason O'Leary, Naomi Park, Riley Prendergast, Nan Relan, Kate Robinson, Rachael Ryan, Alexis Saucerman, Caroline Semmens, Cody Shanahan. Kendall Sparks. River Stambaugh, and Harry Yan.

CT Beta – Eastern Connecticut State University
Corresponding Secretary and Faculty Sponsor – Dr. Mehdi Khorami; 501 Total Members

HI Alpha – Hawaii Pacific University
Corresponding Secretary – Dr. Tara Davis; 10 New Members; 111 Total Members
New Initiates – Laken Barber, Elena M. Emmert, Elizabeth Fischer, Cooper Gowan, Adriana C. Hernandez, Tan Hua, Sarah Nunez, Chloe Todd, Uiyeol Yoon, and Andy Yu.

IA Alpha – University of Northern Iowa
Chapter President – Matthew Adams; 3 New Members; 1095 Total Members
Other Fall 2020 Officers: Ashley DeWispelaere, Secretary; Stephanie Peiffer, Treasurer; and Mark D. Ecker, Corresponding Secretary and Faculty Sponsor
Due to the COVID-19 pandemic, no KME meetings were held during the Fall 2020 semester at the University of Northern Iowa. However, four new members were initiated into KME this semester.
New Initiates – Nick Fairley, Ryan Oswald, and Joseph Van Zante.

IL Zeta – Dominican University
Corresponding Secretary – Mihaela Blanariu; 438 Total Members
We had no activities this fall. We are planning to induct new members in Spring 2021.

IN Beta – Butler University
Chapter President – Haley Niemann; 12 New Members; 432 Total Members
Other Fall 2020 Officers: Abby Craig, Secretary/Treasurer; Rena Duerksen, Corresponding Secretary; and Dr. William Johnston, Professor Lacey Echol, Faculty Sponsors
New Initiates – Jayme Brickley, Jacob Charboneau, Megan Costello, Walker Demel, Mason Lovett, Chloe Makdad, Eric Nofziger, Benjamin Rempfer, Jenna Repkin, Hannah Robison, Vishnu Vaid, and DeJuan Winters.

KS Alpha – Pittsburg State University
Faculty Sponsor – Dr. Scott Thuong; 2144 Total Members

Other Fall 2020 Officer: Tim Flood, Corresponding Secretary
The KS Alpha chapter had to cancel the North Central KME Regional Convention originally scheduled for April 17-18, 2020, due to the pandemic. We hope to host a regional convention in 2022.

KS Beta – Emporia State University
Corresponding Secretary – Tom Mahoney; 10 New Members; 1526 Total Members
Other Fall 2020 Officer: Brian Hollenbeck, Faculty Sponsor
New Initiates – Ryan August, Zihui Chen, Ian Coleman-Hull, Dr. Fred Coon, Austin Crabtree, Shelby Hettenbach, Jeanna Hill, Gabe Moore, Amanda Pritchard, and Mackinzee Olson.

KS Delta – Washburn University
Chapter President – Abigail Beliel; 810 Total Members
Other Fall 2020 Officers: Paul Ennekin, Vice President; Clare Bindley, Secretary; Madison Henle, Treasurer; and Sarah Cook, Corresponding Secretary and Faculty Sponsor
The Kansas Delta chapter of Kappa Mu Epsilon met with our Club Mathematica for two Zoom meetings during fall 2020. Dr. Ron Wasserstein, Executive Director of the American Statistical Association, presented “Moving to a World Beyond $p < 0.05$”. Faculty member Dr. Sarah Cook presented “The Mathematics Behind the Electoral College”. Also in the fall, KME members participated in the online “A Murder Mystery Event” sponsored by the Campus Activities Board.

MD Alpha – Notre Dame of Maryland University
Chapter President – Hannah Campbell; 397 Total Members
Other Fall 2020 Officers: Ericka Kaschak, Vice President; Vanessa Dunn, Secretary; Breonna Morten, Treasurer; and Charles Buehrle, Corresponding Secretary and Faculty Sponsor

MD Beta – Assumption College
Chapter President – Nicole Averinos; 3 New Members; 441 Total Members
Other Fall 2020 Officers: Moira DiGiacomantonio, Vice President; Greta Ouimette, Secretary; Luke Shuck, Treasurer; Spencer Hamblen, Corresponding Secretary and Faculty Sponsor

MD Delta – Frostburg State University
Chapter President – Bailey Brewer; 537 Total Members
Other Fall 2020 Officers: Madison Green, Vice President; Ashley Armbruster, Secretary; Jay Collins, Treasurer; Mark Hughes, Corresponding Secretary and Faculty Sponsor; and Frank Barnet, Faculty Sponsor
This fall we were unable to carry out any in person activities due to the current pandemic. Fortunately, some members of Maryland Delta chapter took the ini-

tiative to organize some online activities. In particular, a new group of officers was chosen with their positions to take effect this semester. Serving as chapter president is Bailey Brewer with Madison Green as vice president, Ashley Armbruster as secretary, and Jay Collins as treasurer. Later in the semester, members were able to meet virtually for some online games. Thanks to Bailey Brewer and Ashley Armbruster for organizing this.

MI Beta – Central Michigan University
Chapter President – Austin Konkel; 1754 Total Members
Other Fall 2020 Officers: Emily Naegelin, Vice President; Kelsey Knoblock, Secretary; Robert Mason, Treasurer; Camilla Madacki, Public Relations; and Dr. Ben Salisbury, Corresponding Secretary and Faculty Sponsor
During the 2020–2021 academic year, KME met once every two weeks on average. The following is a list of the events which took place during the Fall 2020 semester. Due to coronavirus, all KME events were held virtually.

Fall 2020

1. KME participated in the virtual CMU MainStage event at the start of Fall 2020.

2. The first meeting of the semester was held virtually on August 25th. We all introduced ourselves, discussed our goals for the semester, and did a few icebreakers so new and old members could become acquainted. Then we played a math Scriblio (online Pictionary) that the E-Board put together.

3. On September 8th, KME Vice President Emily Naegelin gave a talk about polyhedra and led the group in making models of polyhedra.

4. On September 22nd, Lauren Hutter, a senior majoring in mathematics, presented her results from her summer research and discussed her experience with research in mathematics.

5. On October 6th, the meeting was devoted to the interests of the E-Board. Each officer gave a brief presentation on a mathematical idea that intrigues them.

6. On October 27th, KME Treasurer Robert Mason gave a presentation on unsolved math mysteries.

7. With the semester winding down, KME held a game night on November 10. The game was Math Bingo.

MI Delta – Hillsdale College
Chapter President – Nicholas West; 363 Total Members
Other Fall 2020 Officers: Benjamin Hufford, Vice President; Aaron Jacobson,

Secretary; Abigail Price, Treasurer; and Dr. Kevin Gerstle, Corresponding Secretary and Faculty Sponsor

MI Epsilon – Ketterling University
Corresponding Secretary – Matt Causley; 34 New Members; 1021 Total Members
New Initiates – Zaria Alexander, Andrea Bade, Hayvin Bolton, Brendan Bozyk, Brandon Brinks, Denver Campbell, Drake Coy, Julia Daniels, Jack Dembovsky, Julia Dezio, Divya Drolia, Alexander Eby, Bradley Gardner, Ephraim Gibson, Anna Jackson, Colin Jones, Abdul Kheil, Hannah Kuperus, Antoney Mckinder, Rebecca Miller, Peter Parsons, Alec Phelps, Paul Resch, Alessandro Roma, Blake Rowe, Nicole Shaw, Thomas Shook, Candace Ulett, Jillian Ulinski, Ethan Whitney, Cameron Wise, Chelsea Wright, Madison Wright, and Zachary Ziegenfelder.

MO Beta – University of Central Missouri
Chapter President – Alec McClendon; 8 New Members; 1537 Total Members
Other Fall 2020 Officers: Haleigh Clark, Vice President; Riley Meyer, Secretary; Briana Ward, Historian; Rhonda McKee, Corresponding Secretary; and Steve Shattuck, Faculty Sponsor
The Missouri Beta chapter adapted to the COVID-19 pandemic by holding meetings in a large room where we could distance ourselves and also by inviting members to join by Zoom, if they preferred. Programs included a virtual visit from an alumnae who spoke about her career and a virtual breakout room. Our chapter is looking forward to hosting the National Convention in April. Participants will be able to join either on campus or virtually. Should be a lot of fun!
New Initiates – Mia Bellanca, Laura Haney, Michael Nimmer, Garret Pemberton, Eli Phillips, Lauryn Smith, Emiline Stewart, and Paige VanBlarcum.

MO Theta – Evangel University
Chapter President – Peter Russell; 9 Current Members
Other Fall 2020 Officers: Hannah Tower, Vice President; and Don Tosh, Corresponding Secretary and Faculty Sponsor
Meetings were held three times during the semester with reduced attendance. We are hoping for more participation and involvement in the spring.

MO Iota – Missouri State Southern University
Corresponding Secretary – Amila Appuhamy; 1 New Member; 433 Total Members
New Initiate – Mallorie Keltz.

MO Mu – Harris-Stowe State University
Corresponding Secretary – Ann Podleski; 7 New Members; 108 Total Members
New Initiates – Jordan Barrett Elder, Elise Celeste Burton, Austin Dixon, Dianne Lam, Derek McFarland Jr., Claire Nicole Tomaw, and Alex Yentumi.

NE Gamma – Chadron State College
Chapter President – Dylan Koretko; 7 New Members; 534 Total Members
Other Fall 2020 Officers: Kyeisha Garza, Vice President; Manou Mbombo, Secretary; and Louis Christopher, Treasurer; and Gregory Moses, Corresponding Secretary and Faculty Sponsor

NE Delta – Nebraska Wesleyan University
Corresponding Secretary – Dr. Melissa Erdmann; 8 New Members; 293 Total Members
We initiated 8 new KME members in the fall. Usually our initiation is in the spring and the picnic is family-style with all sorts of dishes brought by faculty members. This fall we ordered boxed dinners from the cafeteria. Although unusual, it was very enjoyable. Later in the fall we had a well-attended bingo event.

OH Gamma – Baldwin-Wallace University
Corresponding Secretary – David Calvis
The OH Gamma chapter had planned on having an initiation ceremony during the Spring 2020 semester. However, due to the pandemic, it was cancelled.

OK Delta – Oral Roberts University
Corresponding Secretary – Enrique Francisco Valderrama-Araya; 4 New Members; 207 Total Members
New Initiates – Morgan N. Crane, Caleb J. Mazzei, Lydia G. Solomon, and Esther M. Spear.

PA Epsilon – Kutztown University
Corresponding Secretary – Dr. Lyn McQuaid; 9 New Members; 683 Total Members
New Initiates – Caelan Brooks, Kaitlin Eltz, Emily Groshek, Liannah Kim, Collin Levis, Alexander Miller, Alyssa Miller, Lindsey Moyer, and Courtney O'Connell.

PA Eta – Grove City College
Corresponding Secretary – Dale L. McIntyre; 12 New Members; 864 Total Members
New Initiates – Kathryn Balcom, Jonathan Crawford, Jenna Donor, Cody Gustafson, Peter Lowrance, Dana Reigle, Caleb Scutt, Grace Shook, Aric Smith, Logan Stahl, Joshua Tatum, and Anna Truman.

PA Nu - Bloomsburg University
Corresponding Secretary – Dr. Nicholas Scoville; 1 New Member; 256 Total Members
New Initiate – Thomas Mark Mease.

PA Rho – Thiel College
Chapter President – Macy Siefert; 136 Total Members
Other Fall 2020 Officers: Emily Groves, Vice President; Cassandra Bown, Secretary; Kara Baumgardner, Treasurer; Dr. Russ Richins, Corresponding Secretary; and Dr. Jie Wu, Faculty SponsorC
Because of COVID-19 restrictions on campus, we were only able to have one meeting held outdoors at the beginning of the semester. None of our usual activities were accomplished, unfortunately. We're hoping that conditions will improve in the spring and we will be able to do more of our usual activities.

PA Sigma – Lycoming College
Corresponding Secretary – Dr. Christopher Reed; 9 New Members; 142 Total Members
New Initiates – Kyle Bennett, Itashna Geerwar, Ricardo Gonzales, Tianquan Jiang, Allen Martin, Catherine McCarty, Kayla Nowak, Adarsh Ramnauth, and Nina Amanda Sousa.

RI Beta – Bryant University
Chapter President – Christopher Ethier; 13 New Members; 171 Total Members
Other Fall 2020 Officers: Constance Tang, Vice President; Alexandra Sherman, Secretary; Liam Mahler, Treasurer; Professor John Quinn, Corresponding Secretary; and Professor Alan Olinsky, Faculty Sponsor
Due to the pandemic, we were unable to hold any KME activities during the fall 2020 semester. However, we will review nominations for new KME members in the spring 2021 semester. We also are looking to see if any of our students could present papers virtually at the national convention in the spring.
New Initiates – Kayla Bauerlein, Alexander Eby, Bridget Healy, Kaitlyn Fales, Jamie Fischer, Vanessa Leone, Riley Lynch, Michael Matkowski, Taylor McKinley, Nicole Page, Haley Pollock, Kevin Stefanick, and Jacklyn Sullivan.

TN Gamma – Union University
Corresponding Secretary – Dr. Bryan Dawson
Other Fall 2020 Officer: Dr. Matt Lunsford, Faculty Sponsor
We usually elect officers at the spring initiation banquet, but we didn't have one this year. We plan to have an initiation in the spring, one way or another.

TN Zeta – Lee University
Chapter President – Audrey Royer; 5 New Members; 72 Total Members
Other Fall 2020 Officers: Lauren Lester, Vice President; Joshua Schlabach, Secretary; Lauren Lester Treasurer; John Mayer; Dr. Caroline Maher-Boulis, Corresponding Secretary and Faculty Sponsor

TX Kappa – University of Mary Hardin-Baylor
Corresponding Secretary – Dr. Peter H. Chen; 7 New Members; 281 Total Members

New Initiates – Zachary Crawford, Samuel Ivy, Zane Magee, Cody Ponce, Nicholas Taylor, Sriyan Wickramasuriya, and Yao Zhang.

VA Delta – Marymount University
Chapter President 2019-20 – Jasmine Roy; 4 New Members; 55 Total Members
Other 2019-20 Officers: Anna Moon, Vice President; Jhoselyn Cordova, Secretary; Collene Corbet, Treasurer; Jacquelyn Rische, Faculty Sponsor and Corresponding Secretary
News for academic year 2019-2020: In September 2019, we attended the talk "Untold Stories of Black Mathematicians with Scott Williams" at the MAA Carriage House in Washington, DC. In November, we took a field trip to the National Cryptologic Museum in Annapolis Junction, MD. We had a movie night on February 7, where we watched *The Imitation Game*. We held our initiation virtually on April 25.
New Initiates – Colleen Corbet, Jhoselyn Cordova, Anna Moon, and Joseph Sanz.

WI Alpha – Mount Mary University
Chapter President – Melissa Golo; 304 Total Members
Other Fall 2020 Officers: Hannah Ashbach, Vice President and Treasurer; Melissa Golo, Secretary; Sherrie Serro, Corresponding Secretary and Faculty Sponsor
Mount Mary University's WI Alpha Chapter of KME sponsored the virtual talk: *Is a Burrito a Sandwich? Exploring the World with a Mathematical Mindset*, by Dr. Rebeccah MacKinnon of UW-Parkside on Friday, November 13. The talk was well-attended by faculty, staff and students.

WV Alpha – Bethany College
Chapter President – Joseph A Makowski; 184 Total Members
Other Fall 2020 Officers: Ethan J Young, Vice President; Lena A Grogan, Secretary/Treasurer; Dr. Adam C. Fletcher, Faculty Sponsor and Corresponding Secretary
West Virginia Alpha chapter, like many other chapters across the country, has had an interesting semester. The College remained predominately on-campus throughout the fall semester, with strict COVID protocols in place throughout. The local and national restrictions canceled or postponed a number of the chapter's usual activities, like the Putnam and the Virginia Tech Regional mathematics competitions and a number of professional meetings. West Virginia Alpha chapter and our local Mathematics and Computer Science Club did, however, have a handful of members take part in the American Mathematical Society's virtual Section Meetings and hosted a virtual chess tournament on campus. The chapter eagerly awaits the Biennial meeting in Missouri in April!

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 Apr 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
AL Alpha	Athens State University, Athens	5 Mar 1935
NM Alpha	University of New Mexico, Albuquerque	28 Mar 1935
IL Beta	Eastern Illinois University, Charleston	11 Apr 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 Apr 1937
OH Alpha	Bowling Green State University, Bowling Green	24 Apr 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	University of Central Missouri, Warrensburg	10 Jun 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 Jun 1941
MI Beta	Central Michigan University, Mount Pleasant	25 Apr 1942
NJ Beta	Montclair State University, Upper Montclair	21 Apr 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 Mar 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 Jun 1947
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 Apr 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 Apr 1965
AL Epsilon	Huntingdon College, Montgomery	15 Apr 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
TN Gamma	Union University, Jackson	24 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	McDaniel College, Westminster	30 May 1965
IL Zeta	Dominican University, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 Mar 1971
KY Alpha	Eastern Kentucky University, Richmond	27 Mar 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 Apr 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973

NY Kappa	Pace University, New York	24 Apr 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State University, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sep 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis University, Loretto	14 Sep 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury University, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 Mar 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 Apr 1986
TX Iota	McMurry University, Abilene	25 Apr 1987
PA Nu	Ursinus College, Collegeville	28 Apr 1987
VA Gamma	Liberty University, Lynchburg	30 Apr 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 Apr 1990
CO Delta	Mesa State College, Grand Junction	27 Apr 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 Apr 1991
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 Mar 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 Apr 1997
MI Delta	Hillsdale College, Hillsdale	30 Apr 1997
MI Epsilon	Kettering University, Flint	28 Mar 1998
MO Mu	Harris-Stowe College, St. Louis	25 Apr 1998
GA Beta	Georgia College and State University, Milledgeville	25 Apr 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
PA Pi	Slippery Rock University, Slippery Rock	19 Apr 1999
TX Lambda	Trinity University, San Antonio	22 Nov 1999
GA Gamma	Piedmont College, Demorest	7 Apr 2000
LA Delta	University of Louisiana, Monroe	11 Feb 2001
GA Delta	Berry College, Mount Berry	21 Apr 2001
TX Mu	Schreiner University, Kerrville	28 Apr 2001
CA Epsilon	California Baptist University, Riverside	21 Apr 2003
PA Rho	Thiel College, Greenville	13 Feb 2004
VA Delta	Marymount University, Arlington	26 Mar 2004
NY Omicron	St. Joseph's College, Patchogue	1 May 2004
IL Iota	Lewis University, Romeoville	26 Feb 2005
WV Beta	Wheeling Jesuit University, Wheeling	11 Mar 2005
SC Epsilon	Francis Marion University, Florence	18 Mar 2005
PA Sigma	Lycoming College, Williamsport	1 Apr 2005
MO Nu	Columbia College, Columbia	29 Apr 2005
MD Epsilon	Stevenson University, Stevenson	3 Dec 2005
NJ Delta	Centenary College, Hackettstown	1 Dec 2006
NY Pi	Mount Saint Mary College, Newburgh	20 Mar 2007
OK Epsilon	Oklahoma Christian University, Oklahoma City	20 Apr 2007
HA Alpha	Hawaii Pacific University, Waipahu	22 Oct 2007
NC Epsilon	North Carolina Wesleyan College, Rocky Mount	24 Mar 2008
NY Rho	Molloy College, Rockville Center	21 Apr 2009
NC Zeta	Catawba College, Salisbury	17 Sep 2009
RI Alpha	Roger Williams University, Bristol	13 Nov 2009
NJ Epsilon	New Jersey City University, Jersey City	22 Feb 2010
NC Eta	Johnson C. Smith University, Charlotte	18 Mar 2010
AL Theta	Jacksonville State University, Jacksonville	29 Mar 2010
GA Epsilon	Wesleyan College, Macon	30 Mar 2010
FL Gamma	Southeastern University, Lakeland	31 Mar 2010
MA Beta	Stonehill College, Easton	8 Apr 2011
AR Beta	Henderson State University, Arkadelphia	10 Oct 2011
PA Tau	DeSales University, Center Valley	29 Apr 2012
TN Zeta	Lee University, Cleveland	5 Nov 2012
RI Beta	Bryant University, Smithfield	3 Apr 2013
SD Beta	Black Hills State University, Spearfish	20 Sept 2013
FL Delta	Embry-Riddle Aeronautical University, Daytona Beach	22 Apr 2014
IA Epsilon	Central College, Pella	30 Apr 2014
CA Eta	Fresno Pacific University, Fresno	24 Mar 2015
OH Theta	Capital University, Bexley	24 Apr 2015
GA Zeta	Georgia Gwinnett College, Lawrenceville	28 Apr 2015
MO Xi	William Woods University, Fulton	17 Feb 2016
IL Kappa	Aurora University, Aurora	3 May 2016
GA Eta	Atlanta Metropolitan University, Atlanta	1 Jan 2017
CT Gamma	Central Connecticut University, New Britan	24 Mar 2017
KS Eta	Sterling College, Sterling	30 Nov 2017
NY Sigma	College of Mount Saint Vincent, The Bronx	4 Apr 2018
PA Upsilon	Seton Hill University, Greensburg	5 May 2018

KY Gamma	Bellarmine University, Louisville	23 Apr 2019
MO Omicron	Rockhurst University, Kansas City	13 Nov 2020