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# The Additive Property of the Sum-of-Divisors Function 

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#### Abstract

In the Fall 2017 issue of The Pentagon [3], Anand Prakash posed some general questions concerning the "Sum of Divisors of Integers." We will be presenting some results addressing these questions. More specifically, if $\sigma(n)$ is the sum of divisors function: 1. Can we characterize all pairs $(m, n)$ such that $$
2(m+n)=\sigma(m)+\sigma(n)
$$ and 2. Can it be shown that there are an infinite number of solutions of the form $(m, k p)$ where $k$ is a perfect number and $p$ is a prime number with $\operatorname{gcd}(k, p)=1$ ? In his note, Mr. Prakash showed that for certain values of $m$ and $k$ there are an infinite number of pairs $(m, k p)$ satisfying the equation in (1). Though we were unable to characterize all pairs $(m, n)$, we show that there are infinitely many values of $m$ and $k$ that satisfy the equation in (1).


## The Questions

The sum-of-divisor function is a number-theoretic function and is defined by $\sigma(n)=\sum_{d \mid n} d$ where $n$ and $d$ are positive integers. A perfect number, $n$, is defined as one such that $\sigma(n)=2 n$. A couple of examples are $\sigma(6)=1+2+3+6=2 * 6=12$ and $\sigma(28)=1+2+4+7+14+28=$ $2 * 28=56$. Here we are seeking to determine for what positive integers $m$ and $n$ is it the case that

$$
\begin{equation*}
2(m+n)=\sigma(m)+\sigma(n) . \tag{1}
\end{equation*}
$$

Obviously, if $m$ and $n$ are both perfect numbers then $\sigma(m)+\sigma(n)=$ $2 m+2 n=2(m+n)$. However, there are other integer pairs $(m, n)$ that satisfy this property. For example, $(13,30),(56,22)$, and $(118,84)$. This raises the questions which we address in this paper. Namely:
Can we characterize all pairs of positive integers $(m, n)$ such that $\sigma(m)+$ $\sigma(n)=2(m+n)$ ?
Can we show that there are an infinite number of solutions to that characterization?

## Properties Used

Some elementary properties of the sum-of-divisor function are established and stated in this section.

It is easy to see that the sum-of-divisor function for any prime number $p$, is

$$
\sigma(p)=p+1
$$

Theorem 1 For any integer of the form $p^{n}$ where $p$ is a prime number and $n$ is a positive integer, the sum-of-divisors function can be expressed as

$$
\sigma\left(p^{n}\right)=1+p+p^{2}+p^{3}+\cdots+p^{n}=\frac{p^{n+1}-1}{p-1} .
$$

Proof. We will use proof by induction. Since $\sigma\left(p^{1}\right)=1+p=\frac{p^{2}-1}{p-1}$, the result holds for $n=1$. Now assume that the desired result is true for a fixed integer $k$, that is, assume

$$
\sigma\left(p^{k}\right)=1+p+p^{2}+p^{3}+\cdots+p^{k}=\frac{p^{k+1}-1}{p-1} .
$$

We need to prove that

$$
\sigma\left(p^{k+1}\right)=1+p+p^{2}+p^{3}+\cdots+p^{k+1}=\frac{p^{k+2}-1}{p-1} .
$$

By substitution we have

$$
\begin{aligned}
\sigma\left(p^{k+1}\right) & =1+p+p^{2}+p^{3}+\cdots+p^{k}+p^{k+1} \\
& =\frac{p^{k+1}-1}{p-1}+p^{k+1} \\
& =\frac{p^{k+1}-1+p^{k+1}(p-1)}{p-1} \\
& =\frac{p^{k+2}-1}{p-1} .
\end{aligned}
$$

Therefore, the claim holds for all natural numbers $n$.
Corollary 1 For the prime number $p=2$ and any positive integer $n$ we have

$$
\sigma\left(2^{n}\right)=\frac{2^{n+1}-1}{2-1}=2^{n+1}-1
$$

Theorem 2 The function $\sigma(n)$ is a multiplicative number-theoretic function, that is, for the non-negative integers $m$ and $n$ where $\operatorname{gcd}(m, n)=1$,

$$
\sigma(m \cdot n)=\sigma(m) \cdot \sigma(n)
$$

Proof. See [1]

## Key Results

We now present a couple of theorems confirming that there are, in fact, methods of generating an infinite number of pairs of integers ( $m, n$ ) satisfying $\sigma(m)+\sigma(n)=2(m+n)$.

Theorem 3 Let $k$ be a perfect number and let $p$ be a prime number such that $\operatorname{gcd}(k, p)=1$. Then, if $q=2 k-1+2^{r+1}$ is an odd prime number for some non-negative integer $r$ then $m=2^{r} q$ and $n=k p$ satisfies $\sigma(m)+$ $\sigma(n)=2(m+n)$.

Proof. The theorem can be proven by using the earlier stated properties of the divisor function as well as substitution, factorization, and expansion. From Theorem 2 and Corollary 1,

$$
\sigma(m)+\sigma(n)=\left(2^{r+1}-1\right)(q+1)+2 k(p+1)
$$

SO

$$
\begin{aligned}
\sigma(m)+\sigma(n)= & 2^{r+1}-q-1+2 k+2^{r+1} q+2 k p \\
= & 2^{r+1}-\left(2 k-1+2^{r+1}\right)-1+2 k \\
& \quad+2^{r+1}\left(2 k-1+2^{r+1}\right)+2 k p \\
& \quad\left(\text { since } q=2 k-1+2^{r+1}\right) \\
= & 2^{r+1}\left(2 k-1+2^{r+1}\right)+2 k p \\
= & 2^{r+1} q+2 k p \quad\left(\text { since } q=2 k-1+2^{r+1}\right) \\
& \quad 2\left(2^{r} q+k p\right) \\
= & 2(m+n) .
\end{aligned}
$$

Examples of pairs $(m, n)$ satisfying Theorem 1 can be found in Table 1 (see Appendix).

Theorem 4 Let $n=2^{k} p$ and $m=2^{r} q$ where $k$ and $r$ are both positive integers and $p$ and $q$ are odd prime numbers. Then, for every $r$ and $k$ such that $p+q=2\left(2^{k}+2^{r}-1\right), \sigma(m)+\sigma(n)=2(m+n)$ is satisfied.

Proof. We have

$$
\begin{aligned}
& \sigma(m)+\sigma(n)=\left(2^{k+1}-1\right)(p+1)+\left(2^{r+1}-1\right)(q+1) \\
&=2^{k+1} p+2^{k+1}-p-1+2^{r+1} q+2^{r+1}-q-1 \\
&=2 m+2 n+2^{k+1}+2^{r+1}-2-(p+q) \\
&=2 m+2 n+2^{k+1}+2^{r+1}-2-\left(2^{k+1}+2^{r+1}-2\right) \\
& \quad \quad \quad\left(\text { since } p+q=2\left(2^{k}+2^{r}-1\right)\right) \\
&=2 m+2 n \\
&=2(m+n)
\end{aligned}
$$

Goldbach's conjecture posits that every even integer greater than 4 can be expressed as the sum of two odd prime ([2], p. 33). Goldbach's conjecture has been confirmed for integers up to $4 \times 10^{18}$. Therefore, Goldbach's conjecture provides the existence of such a $p$ and $q$ for Theorem 4. Some examples utilizing Theorem 4 are $(m, n)=\left(2^{3} \cdot 7,2^{1} \cdot 11\right)$, $(m, n)=\left(2^{3} \cdot 11,2^{1} \cdot 7\right)$, or $(m, n)=\left(2^{4} \cdot 23,2^{3} \cdot 23\right)$. More examples can be found in Table 2.

In both of the above theorems at least one of $m$ or $n$ are even integers. We now turn our investigation into pairs $(m, n)$ where both are odd positive integers. First we give a few necessary results.

Lemma 1 The function $f(x)=\frac{c x}{x-1}$ is decreasing for all $c>0$ and $x \neq 1$.

Proof. The derivative of $f(x)$ is $f^{\prime}(x)=\frac{-c}{(x-1)^{2}}$ which is negative for a $c>0$.

Lemma 2 Let $p$ and $q$ be odd primes with $p<q$, and $a$ and $b$ positive integers. Then:
$\sigma(p)<2 p$;
If $m=p^{a}$, then $\sigma(m)<2 m$;
If $m=p^{a} q^{b}$, then $\sigma(m)<2 m$.
Proof. (1) We have $\sigma(p)=p+1<p+p=2 p$.
(2) If $m=p^{a}$ then, by Theorem 1 ,

$$
\sigma(m)=\sigma\left(p^{a}\right)=\frac{p^{a+1}-1}{p-1}<\frac{p^{a} p}{p-1}=\frac{m p}{p-1} .
$$

By lemma 1 , since $p \geq 3, \frac{m p}{p-1} \leqslant \frac{3 m}{2}$. Therefore

$$
\sigma(m) \leqslant \frac{3 m}{2}<2 m .
$$

(3) If $m=p^{a} q^{b}$, then by Theorem 1 and Theorem 2

$$
\begin{aligned}
\sigma(m) & =\sigma\left(p^{a} q^{b}\right) \\
& =\frac{p^{a+1}-1}{p-1} \cdot \frac{q^{b+1}-1}{q-1} \\
& <\frac{p^{a} p q^{b} q}{(p-1)(q-1)} \\
& =\frac{m p q}{(p-1)(q-1)} .
\end{aligned}
$$

Since $p \geq 3$ and $q>p$ we have $q \geq 5$. From lemma 1 above it follows that $\frac{m p}{p-1}<\frac{3 m}{2}$ and $\frac{q}{q-1}<\frac{5}{4}$. Therefore

$$
\sigma(m)<\frac{3 m}{2} \cdot \frac{5}{4}=\frac{15 m}{8}<2 m .
$$

Corollary 2 If $p, q, r$ and $s$ are distinct odd primes and $a, b, c$ and $d$ are positive integers and $(m, n)=\left(p^{a} q^{b}, r^{c} s^{d}\right)$ then $\sigma(m)+\sigma(n)<2(m+n)$.

Proof. By part (c) of Lemma 2, $\sigma(m)<2 m$ and, similarly, $\sigma(n)<2 n$ so $\sigma(m)+\sigma(n)<2(m+n)$.

Corollary 3 If the pair $(m, n)$ is such that $\sigma(m)+\sigma(n)=2(m+n)$ and both $m$ and $n$ are odd positive integers, then at least one of $m$ or $n$ must have 3 or more different prime factors.

Proof. Part (1) of Lemma 2 tells us we cannot have both $m$ and $n$ as odd primes, part (2) and part (3) of the lemma give us that $m$ and $n$ cannot be products of two odd primes or products of powers of two odd prime numbers.

Some examples of such pairs are $(945,31),(2205,37),(4095,547)$, (2835, 139). More examples can be found in Table 3.

We have found a fair number of such pairs. In each of the pairs we found one is an odd prime and the other is an abundant odd number. Abundant numbers $n$, are defined to be those which have $\sigma(n)>2 n$. A list of such odd abundant numbers can be found athttp://oeis.org/ wiki/Odd_abundant_numbers.

## Further research

So far we have shown that there are an infinite number of pairs $(m, n)$ satisfying $\sigma(m)+\sigma(n)=2(m+n)$. However, there is still some work to be done in order to classify all such pairs. Further research regarding abundant numbers and the possible compositions of the integer pairs could also lead to more results.

## Appendix

## Table 1

This table was generated by an appropriate pair of integers $m$ and $n$ and then check that $q$ is a prime as required.

| $\boldsymbol{k}$ | $\boldsymbol{r}$ | $\boldsymbol{q}=\mathbf{2 k - 1}$ <br> $\mathbf{2}^{r+1}$ | $\boldsymbol{m}=\mathbf{2}^{r} \boldsymbol{q}$ | $\boldsymbol{n}=\boldsymbol{k p}$ | $(\boldsymbol{m}, \boldsymbol{n})$ | $\boldsymbol{p}=$ <br> $\boldsymbol{p r i m e}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 6 | 0 | 13 | $2^{0 *} 13$ | $2^{*} 3^{*} p$ | $(13,6 \mathrm{p})$ | $\neq 2,3$ |
| 6 | 2 | 19 | $2^{2 *} 19$ | $2^{*} 3^{*} p$ | $(76,6 \mathrm{p})$ | $\neq 2,3$ |
| 6 | 4 | 43 | $2^{4 *} 43$ | $2^{*} 3^{*} p$ | $(688,6 \mathrm{p})$ | $\neq 2,3$ |
| 6 | 6 | 139 | $2^{6 *} 139$ | $2^{*} 3^{*} p$ | $(8896,6 \mathrm{p})$ | $\neq 2,3$ |
| 6 | 8 | 523 | $2^{8 *} 523$ | $2^{*} 3^{*} p$ | $(133888,6 \mathrm{p})$ | $\neq 2,3$ |
| 28 | 1 | 59 | $2^{1 *} 59$ | $2^{2 *} 7^{*} p$ | $(118,28 \mathrm{p})$ | $\neq 2,7$ |
| 28 | 7 | 311 | $2^{7 *} 311$ | $2^{2 *} 7^{*} p$ | $(39808,28 p)$ | $\neq 2,7$ |
| 496 | 11 | 5087 | $2^{11 *} 5087$ | $2^{4 *} 31^{*} p$ | $(10418176,496 p)$ | $\neq 2,31$ |
| 8128 | 5 | 16319 | $2^{5 *} 16319$ | $2^{6 *} 127^{*} p$ | $(522208,8128 p)$ | $\neq 2,127$ |
| 33550336 | 3 | 67100687 | $2^{3 *} 67100687$ | $2^{12 *}\left(2^{13}-1\right)^{*} p$ | $\left(2^{3} q, 33550336 p\right)$ | $\neq 2$, |
| 8191 |  |  |  |  |  |  |

## Table 2

This table was generated by an appropriate pair of integers $m$ and $n$ and then check that $p+q=2\left(2^{k}+2^{r}-1\right)$ as required.

| $m$ | $n$ |
| :---: | :---: |
| $2^{3 *} 7$ | $2^{*} 11$ |
| $23^{*} 11$ | $2^{*} 7$ |
| $2^{4 *} 23$ | $2^{3 *} 23$ |
| $2^{3 *} 17$ | $2^{3 *} 13$ |
| $2^{2 *} 19$ | $2^{4 *} 19$ |
| $2^{5 *} 11$ | $2^{6 *} 179$ |
| $2^{6 *} 11$ | $2^{5 * 179}$ |

## Table 3

This table was generated by an odd prime $n$ and an abundant odd number $m$.

| $m$ | $n$ |
| :---: | :---: |
| $3^{3 *} 5^{*} 7$ | 31 |
| $3^{2 *} 5^{*} 7^{2}$ | 37 |
| $3^{2 *} 5^{*} 7^{*} 13$ | 547 |
| $3^{4 * 5 * 7}$ | 139 |
| $3^{2} * 5 * 7 * 17$ | 523 |
| $3^{2} * 5 * 7 * 13$ | 547 |

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[2] Pettofrezzo, A. J. \& Byrkit, D. R. (1970), Elements of Number Theory, Englewood Cliffs, NJ, Prentice-Hall. Inc.
[3] Prakash, A. (2017), Sums of Divisors of Integers, The Pentagon,77 (1), 20-22.

# Counting Odd Numbers in Truncations of Pascal's Triangle 

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#### Abstract

A "truncation" of Pascal's triangle is a triangular array of numbers that satisfies the usual Pascal recurrence but with a boundary condition that declares some terminal set of numbers along each row of the array to be zero. Presented here is a family of natural truncations of Pascal's triangle that generalize a kind of Catalan triangle. The numbers in each array are realized as differences of binomial coefficients; as counts of certain lattice paths and tableaux; and as entries of representing matrices for certain linear transformations of polynomial spaces. Lucas's theorem is applied to determine precisely those truncations for which the number of odd entries on each row is a power of two.


## 1. Introduction

Observe that the following conditions (i), (ii), and (iii) uniquely determine an integer-valued function $A$ on $\mathbb{Z} \times \mathbb{Z}$ :
(i) $A(0,0)=1$,
(ii) $A(n, k)=0$ if $n<0, k<0$, or $k>\lfloor n / 2\rfloor$, and
(iii) $A(n, k)=A(n-1, k-1)+A(n-1, k)$ for all other integer pairs $(n, k)$ when $n>0$.

The output numbers of interest are those within the triangular array $(A(n, k))$ indexed by integer pairs $(n, k)$ for which $0 \leq k \leq n$. When we display this array, we get a kind of "truncation" of Pascal's triangle. Here are the
first ten rows:

|  |  |  |  |  |  |  |  | 1 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  | 1 |  | 2 | 1 |
|  |  |  |  |  | 1 |  | 3 |  | 2 |
|  |  |  |  | 1 |  | 4 |  | 5 |  |
|  |  |  | 1 |  | 5 |  | 9 |  | 5 |
|  |  | 1 |  | 6 |  | 14 |  | 14 |  |
|  | 1 |  | 7 |  | 20 |  | 28 |  | 14 |
| 1 |  | 8 |  | 27 |  | 48 |  | 42 |  |



Figure 1
The numbers of this array are very well-known in enumerative combinatorics. For example, the sequence of numbers in the rightmost nonzero "column" of the array is the famous sequence of Catalan numbers. The nonzero entries of this array are called "ballot numbers," as they count the number of ways one candidate can defeat another candidate in a twoperson election, under certain constraints. For further explication of this and other well-known phenomena related to this Catalan array, see for example [1] and references therein.

In this paper, we generalize this Catalan array by simply and naturally varying the "boundary condition" (ii) above. We will have one such array for each positive integer $t$, where $t$ identifies the first row of the array that no longer fully agrees with Pascal's triangle, i.e. the first "truncated" row. So, for example, the $t=1$ array is the Catalan array depicted above. The nonzero numbers in these more general triangular arrays are shown to be differences of binomial coefficients as well as counts of certain lattice paths. The fourth author, in consultation with the first author, studied these arrays in an undergraduate student honors thesis [13] concerning differential operators on function spaces. A version of the motivating problem of that thesis is presented below, and in Theorem 4 it is shown how the numbers in our truncated Pascal arrays are coefficients for certain polynomials which arise in the study of differential operators. However, in [10], Reuveni independently presented the so-called "Catalan's trapezoids," which are the same as our truncated Pascal's triangles but indexed and formatted somewhat differently. In [11], these trapezoids are applied in a probabilistic analysis of certain lattice-gas flow models.

We close this introduction with some descriptive comments about the content of the paper. We think these Pascal-like arrays are inherently pretty and provide for an excellent enumerative example or exercise: We have a recurrence, an explicit formula, combinatorial interpretations, and a polynomial algebra context for these numbers, as summarized in Theorem 4. This would seem to place us well within a salutary enumerative environment as envisioned by Stanley in Chapter 1 of his classic text [14]. Our main result - Theorem 7 - is a (modest) enumerative application of these
arrays that generalizes the well-known problem of counting odd numbers in the rows of Pascal's triangle; this new theorem, which appears in [3], was obtained by the third author in consultation with the first and second authors. For Pascal's triangle, solutions to this odd-counting problem and other related problems are entertainingly recounted in [4]. Our work in this paper leaves open the possibility of generalizing other such problems from Pascal's triangle to the truncations of Pascal's triangle presented here.

## 2. A family of truncations of Pascal's triangle

For the rest of the paper, $n$ and $k$ are integer variables and $t$ denotes a fixed positive integer, which we think of informally as designating the first truncated row of the associated Pascal-like integer array. Consider a function $\mathfrak{a}_{t}: \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ satisfying
(i) $a_{t}(0,0)=1$,
(ii) $\mathrm{a}_{t}(n, k)=0$ if $n<0, k<0$, or $k>\min \left\{\left\lfloor\frac{n-1+t}{2}\right\rfloor, n\right\}$, and
(iii) $\mathrm{a}_{t}(n, k)=\mathrm{a}_{t}(n-1, k-1)+\mathrm{a}_{t}(n-1, k)$
for all other integer pairs $(n, k)$. Figure 2 displays the first ten rows of the array $\left(\mathrm{a}_{4}(n, k)\right)$ when viewed as a truncation of Pascal's triangle. We call an array $\left(\mathrm{a}_{t}(n, k)\right)_{0<k<n<\infty}$ a truncated Pascal's triangle. Of course, the Catalan array $(A(n, \bar{k}))$ of $\S 1$ is just the $t=1$ version of $\left(\mathrm{a}_{t}(n, k)\right)$. In the next section we offer various interpretations of and contexts for the numbers in these truncated Pascal's triangles.


Figure 2: The first ten rows of the truncated Pascal's triangle $\left(a_{4}(n, k)\right)$.

## 3. Algebraic-combinatorial aspects of truncated Pascal's triangles

In this section, we aim to give several different descriptions of the numbers appearing in our given truncated Pascal's triangle ( $a_{t}(n, k)$ ), which will culminate in Theorem 4. These new descriptions will be denoted by $\mathrm{b}_{t}(n, k), \mathrm{c}_{t}(n, k), \mathrm{c}_{t}^{\prime}(n, k)$, and $\mathrm{d}_{t}(n, k)$. To start, declare that

$$
\begin{equation*}
\mathrm{b}_{t}(n, k):=\binom{n}{k}-\binom{n}{k-t}, \tag{2}
\end{equation*}
$$

with the usual understanding that the binomial coefficient $\binom{p}{q}$ is zero unless
$0 \leq q \leq p$. For example,

$$
\mathrm{b}_{4}(7,5)=\binom{7}{5}-\binom{7}{1}=\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{5!}-\frac{7}{1!}=21-7=14 .
$$

Note that the latter quantity agrees with $\mathrm{a}_{4}(7,5)$, which is the $(n, k)=$ $(7,5)$ entry of the example array $\left(a_{4}(n, k)\right)$ depicted in $\S 2$ above.

Next, we count lattice paths. An NE-path from $\left(x_{1}, y_{1}\right)$ to $\left(x_{2}, y_{2}\right)$ in the plane will be a continuous path starting at $\left(x_{1}, y_{1}\right)$, ending at $\left(x_{2}, y_{2}\right)$, and consisting of a finite number of unit steps in the north and east directions only. Say an NE-path from $(0,0)$ to $(k, n-k)$ is an $(n, k)$-NE-path, and call such a path $t$-admissible if it does not intersect the line $y=x-t$; in such a case we say the path stays weakly above $y=x-t+1$. For example, when $t=4$ and $(n, k)=(7,5)$, then $(k, n-k)=(5,2)$. As we can see in the pictures below, the number of 4 -admissible ( 7,5 )-NE-paths is 14 , i.e. there are fourteen NE-paths from $(0,0)$ to $(5,2)$ that stay weakly above $y=x-3$. For now, one can ignore the numbers assigned to the horizontal steps of each lattice path, although the pattern in which they are assigned should be evident.











So, this count agrees with $a_{4}(7,5)$ and $b_{4}(7,5)$. In general, we set

$$
\begin{equation*}
\mathrm{c}_{t}(n, k):=\mid\{t \text {-admissible }(n, k) \text {-NE-paths }\} \mid \tag{3}
\end{equation*}
$$

For integers $x_{1}, y_{1}, x_{2}$, and $y_{2}$, the number of all NE-paths from $\left(x_{1}, y_{1}\right)$ to $\left(x_{2}, y_{2}\right)$ is easily seen to be $\binom{y_{2}-y_{1}+x_{2}-x_{1}}{x_{2}-x_{1}}$ : Of the $y_{2}-y_{1}+$ $x_{2}-x_{1} \mathrm{~N}$ or E steps required to move from $\left(x_{1}, y_{1}\right)$ to $\left(x_{2}, y_{2}\right)$, exactly $x_{2}-x_{1}$ must be E's. Consider for the moment those NE-paths from $(0,0)$ to $(k, n-k)$ which cross the line $y=x-t+1$ and therefore cross or touch the line $y=x-t$. At the first point of intersection with $y=x-t$, reflect the initial part of each such path across that line to obtain an NE-path from $(t,-t)$ to $(k, n-k)$. For example, here is how we reflect a $(7,5)$-NE-path that is not 4-admissible:


This reflection procedure, often called André's reflection principle, can be reversed and therefore shows that the set of NE-paths from $(0,0)$ to $(k, n-k)$ which cross the line $y=x-t-1$ is in one-to-one correspondence with the set of all NE-paths from $(t,-t)$ to $(k, n-k)$. That is, the number of NE-paths from $(0,0)$ to $(k, n-k)$ that cross the line $y=x-t-1$ is
$\binom{n-k+t+k-t}{k-t}$. Therefore,
$\mathrm{c}_{t}(n, k)=\binom{n}{k}-\binom{n-k+t+k-t}{k-t}=\binom{n}{k}-\binom{n}{k-t}=\mathrm{b}_{t}(n, k)$.
This shows:
Lemma 5 For all integers $n$ and $k$, we have $\mathrm{b}_{t}(n, k)=\mathrm{c}_{t}(n, k)$.
For more about the well-known observation recorded above as Lemma 1, see Chapter 1 of [6]. For a recent and readable survey of lattice path enumeration with references to many closely related results, see [5].

In algebraic combinatorics, objects called "tableaux" are often used as row and column indices for collections of matrices that represent algebraic structures such as groups or Lie algebras. ${ }^{1}$ Such tableaux generally take the form of an array of boxes of some specified shape filled with integer entries subject to certain rules. Next, we offer a re-interpretation of $t$-admissible NE-paths as columnar tableaux; some possible algebraiccombinatorial contexts for such tableaux are briefly mentioned in the closing section of the paper.

An $(n, k)$-columnar tableau $T=\left(T_{1}, T_{2}, \ldots, T_{k}\right)$ is a strictly increasing $k$-tuple of integers with $\left\{T_{1}, T_{2}, \ldots, T_{k}\right\} \subseteq\{1,2, \ldots, n\}$. We typically visualize such a tableau as a vertical column of $k$ boxes filled from top to bottom with the entries $T_{1}, T_{2}, \ldots, T_{k}$ :

$$
T=\begin{array}{|c|}
\hline T_{1} \\
\hline T_{2} \\
\hline \vdots \\
\hline T_{k} \\
\hline
\end{array}
$$

An $(n, k)$-columnar tableau $T=\left(T_{1}, \ldots, T_{k}\right)$ is $t$-admissible if, whenever $t-1<k$, then $T_{t-1+j} \geq t-1+2 j$ for any $j \in\{1,2, \ldots, k-t+1\}$. For example, when $t=4$ and $(n, k)=(7,5)$, then $k-t+1=2$, and our $j$ 's are therefore from the set $\{1,2\}$. When $j=1, T_{t-1+j}=T_{4} \geq 5$, and when $j=2, T_{t-1+j}=T_{5} \geq 7$. Thus, the $(7,5)$-columnar tableau $(1,2,3,4,6)$ is not 4 -admissible. Indeed, one can easily check that the 4 -admissible $(7,5)$-columnar tableaux are precisely these:

[^0]a total of 14 columnar tableaux. This count agrees with $a_{4}(7,5), b_{4}(7,5)$, and $c_{4}(7,5)$. At this point, a correspondence with the fourteen 4 -admissible (7,5)-NE-paths presented above should be clear. Now let
\[

$$
\begin{equation*}
\mathrm{c}_{t}^{\prime}(n, k):=\mid\{t \text {-admissible }(n, k) \text {-columnar tableaux }\} \mid . \tag{4}
\end{equation*}
$$

\]

The proof of our next lemma is obtained by producing an explicit bijection between the $t$-admissible $(n, k)$-columnar tableaux and the $t$-admissible $(n, k)$-NE-paths. This bijection merely formalizes what we have observed in our example correspondence between the 4 -admissible ( 7,5 )-NE-paths and the 4 -admissible $(7,5)$-columnar tableaux. The proof can be found in §5.

Lemma 6 For all integers $n$ and $k$, we have $\mathrm{c}_{t}(n, k)=\mathrm{c}_{t}^{\prime}(n, k)$.
Finally, we consider another set of numbers $\mathrm{d}_{t}(n, k)$ which arise as coefficients of certain polynomials or, from another viewpoint, as entries of representing matrices for certain linear transformations on polynomial vector spaces. It appears this can be viewed within the context of Rota's finite operator calculus (see [7]), but we use more direct and elementary reasoning here. To set things up, let $\left\{x_{j}\right\}_{j \geq 0}$ be the basis for the polynomial vector space $\mathbb{R}[x]$ (polynomials in the indeterminate $x$ and with real coefficients) given by $x_{j}:=x^{j} / j$ !. Let $S$ be the subspace of $\mathbb{R}[x]$ spanned by $\left\{x_{j}\right\}_{j \geq 1}$. The linear transformation $D: S \longrightarrow \mathbb{R}[x]$ will be the differential operator $D(y):=y^{\prime}+y^{\prime \prime}$. So for any $j \geq 1$, we have $D\left(x_{j}\right)=x_{j-1}+x_{j-2}$ if we regard $x_{-1}:=0$. In fact, $D$ is a vector space isomorphism. For any positive integer $N$, set $D^{-N}:=\left(D^{-1}\right)^{N}$.

It is easy to see by induction on $N$ that for all positive integers $N$, $D^{-N}\left(x_{t-1}\right)$ is in $\operatorname{span}_{\mathbb{R}}\left\{x_{j}\right\}_{j=1}^{t-1+N}$. (For the basis step of the induction argument, prove that $D^{-1}\left(x_{t-1}\right)=\sum_{i=0}^{t-1}(-1)^{i} x_{t-i}$ by instead checking that $\left.x_{t-1}=\sum_{i=0}^{t-1}(-1)^{i} D\left(x_{t-i}\right).\right)$ This means that for any positive integer
$N$ we can write

$$
\begin{equation*}
D^{-N}\left(x_{t-1}\right)=\sum_{i=0}^{t-1+N-1}(-1)^{i} \mathrm{~d}_{t}(i+N-1, i) x_{t-1+N-i} \tag{5}
\end{equation*}
$$

for some real numbers $\mathrm{d}_{t}(i+N-1, i)$. Declare $\mathrm{d}_{t}(n, k)$ to be zero for any integer pair $(n, k)$ such that for all $N>0$ and $0 \leq i \leq t-1+N-1$, $(n, k) \neq(i+N-1, i)$, i.e. $\mathrm{d}_{t}(n, k)=0$ if $(n, k)$ does not index any term appearing in any of the sums shown in equation (4) above when $N \geq 1$. Thus, $d_{t}$ is a real-valued function defined on all of $\mathbb{Z} \times \mathbb{Z}$.

As an example, consider $t=4$ and $(n, k)=(7,5)$, so that $(i+N-1, i)=(n, k)$ exactly when $i=5$ and $N=3$. Then $\mathrm{d}_{4}(7,5)$ will be the coefficient of $x_{t-1+N-i}=x_{4-1+3-5}=x_{1}$ in the expansion of $D^{-N}\left(x_{t-1}\right)=D^{-3}\left(x_{4-1}\right)=D^{-3}\left(x_{3}\right)$, multiplied by $(-1)^{i}=(-1)^{5}$. Starting with $D^{-1}\left(x_{3}\right)=x_{4}-x_{3}+x_{2}-x_{1}$, it is easy to confirm that
$D^{-3}\left(x_{3}\right)=D^{-1}\left(D^{-1}\left(D^{-1}\left(x_{3}\right)\right)\right)=x_{6}-3 x_{5}+6 x_{4}-10 x_{3}+14 x_{2}-14 x_{1}$, whence $\mathrm{d}_{4}(7,5)=(-1)^{5} \cdot(-14)=14$. That is, $\mathrm{d}_{4}(7,5)$ agrees with $\mathrm{a}_{4}(7,5), \mathrm{b}_{4}(7,5), \mathrm{c}_{4}(7,5)$, and $\mathrm{c}_{4}^{\prime}(7,5)$.

The content of the next lemma, which is proved in $\S 5$, is that the $\mathrm{d}_{t}(n, k)$ 's satisfy the defining relations for the truncated Pascal's triangle $\left(\mathrm{a}_{t}(n, k)\right)$.

Lemma 7 We have $\mathrm{d}_{t}(0,0)=1$. For all integers $n$ and $k$, we have
(i) $\mathrm{d}_{t}(n, k)=0$ if $n<0, k<0$, or $k>\min \left\{\left\lfloor\frac{n-1+t}{2}\right\rfloor, n\right\}$, and otherwise
(ii) $\mathrm{d}_{t}(n, k)=\mathrm{d}_{t}(n-1, k-1)+\mathrm{d}_{t}(n-1, k)$ as long as $n>0$.

The preceding lemmas furnish the key steps in the proof of the main result of this section, whose value is not so much the novelty of the results (which are routine to enumeration experts) but rather the pleasantness and illustrative utility of the results taken together as an assemblage.

Theorem 8 For all integers $n$ and $k$, we have

$$
\mathrm{a}_{t}(n, k)=\mathrm{b}_{t}(n, k)=\mathrm{c}_{t}(n, k)=\mathrm{c}_{t}^{\prime}(n, k)=\mathrm{d}_{t}(n, k) .
$$

Proof. The $\mathrm{b}_{t}(n, k)$ 's are easily seen to satisfy the defining conditions (i), (ii), and (iii) which uniquely determine the $\mathrm{a}_{t}(n, k)$ 's, so $\mathrm{b}_{t}(n, k)=$ $\mathrm{a}_{t}(n, k)$. The $\mathrm{d}_{t}(n, k)$ 's satisfy these same conditions by Lemma 3 , hence $\mathrm{d}_{t}(n, k)=\mathrm{a}_{t}(n, k)$. And by Lemmas 1 and 2 we have $\mathrm{b}_{t}(n, k)=$ $c_{t}(n, k)=c_{t}^{\prime}(n, k)$.

## 4. Counting odd numbers in truncations of Pascal's triangle

It is a well-known phenomenon that the number of odds on any given row of Pascal's triangle is a power of two. A classical proof of this fact utilizes Lucas's Theorem and is recapitulated in Corollary 6 below, cf. $\$ 8.4$ of [2]. Our goal is to generalize this result to truncations of Pascal's triangle. Before we proceed, we fix some notation. For a prime $p$ and a nonnegative integer $m$, let $l_{p}(m)$ be 0 when $m=0$ and $\left\lfloor\log _{p}(m)\right\rfloor$ otherwise. For $i \in\left\{0,1, \ldots, l_{p}(m)\right\}$, let $m_{i}^{(p)}$ denote the $i^{\text {th }}$ digit of the base $p$ representation of $m$, and let $\mathcal{D}_{p}(m)$ be the subset of $\left\{0,1, \ldots, l_{p}(m)\right\}$ for which $i \in \mathcal{D}_{p}(m)$ if and only if $m_{i}^{(p)} \neq 0$. Further, let $d_{p}(n):=\left|\mathcal{D}_{p}(m)\right|$. Our interest is mainly in the case that $p=2$, but we state Lucas's Theorem in its full generality in order to encourage the reader (and the authors) to keep in mind the possibility of extending some of the ideas of this section to other primes.

Theorem 9 (Lucas's Theorem) Let p be any prime, and fix any nonnegative integers $n$ and $k$. Let $l:=\max \left(l_{p}(n), l_{p}(k)\right)$. Then,

$$
\binom{n}{k} \equiv \prod_{i=0}^{l}\binom{n_{i}^{(p)}}{k_{i}^{(p)}}(\bmod p) .
$$

From here on, we suppress the " $p$ " superscripts and subscripts from the notation introduced above and fix $p=2$ as our prime. So, for example, " $d(n)$ " means $d_{2}(n), " l(n)$ " means $l_{2}(n)$, " $\mathcal{D}(n)$ " means $\mathcal{D}_{2}(n)$, etc.

Corollary 10 Let $n$ be a nonnegative integer. Then the number of odds on the $n^{\text {th }}$ row of Pascal's triangle is $2^{d(n)}$.

Proof. Suppose $0 \leq k \leq n$. Then by Lucas' Theorem, $\binom{n}{k}$ is odd if and only if $n_{i}=1$ whenever $k_{i}=1$. So, $\binom{n}{k}$ is odd if and only if $\mathcal{D}(k) \subseteq \mathcal{D}(n)$. Of course, there are $2^{|\mathcal{D}(n)|}=2^{d(n)}$ choices for such subsets.

Now we turn our attention to truncations of Pascal's triangle. Corollary 6 and the reasoning exhibited in its proof will be used in several of the lemmas that follow. These lemmas support the proof of the following theorem, which is the main result of this paper.

Theorem 11 The number of odds on each row of the Pascal triangle truncation $\left(\mathrm{a}_{t}(n, k)\right)$ is a power of two if and only if $t$ is a power of two. In this case, when $n$ is a nonnegative integer, the number of odds on row $n$ of the array is precisely $\frac{1}{2} \cdot 2^{d(n+t)}$.

The proof of Theorem 7 is at the end of this section and will be easily deduced from the lemmas we establish next.

Lemma 12 Suppose $t$ is not a power of two. Then the number of odds on row $t$ of the truncated Pascal's triangle $\left(\mathrm{a}_{t}(n, k)\right)$ is an odd number greater than one and therefore not a power of two.

Proof. Let $n=t$. If $0 \leq k \leq n-1$, then

$$
\mathbf{a}_{t}(n, k)=\binom{n}{k}-\binom{n}{k-t}=\binom{n}{k}-\binom{n}{k-n}=\binom{n}{k} .
$$

And if $k=n$, then $\mathrm{a}_{t}(n, k)=\binom{n}{n}-\binom{n}{0}=0$. So the $n^{\text {th }}$ row of the given truncated Pascal array is the same as the $n^{\text {th }}$ row of Pascal's triangle with the sole exception of the $n^{\text {th }}$ entry, which is a 1 in Pascal's triangle and a 0 in the truncated Pascal array. So, the number of odds on row $n$ of the truncated Pascal array is, by Corollary $6,2^{d(n)}-1$. This odd number is a power of two if and only if $d(n)=0$ if and only if $n$ is a power of two. But since $t=n$ is not a power of two, we conclude that the number of odds on row $n$ is an odd number greater than one.

The simple observations of Lemmas 9 and 10 are needed for our proof of Lemma 4.8.

Lemma 13 Let $m$ be a nonnegative integer. The binomial coefficient $\binom{2(m+1)}{m+1}$ is even. The binomial coefficient $\binom{2 m+1}{m}$ is odd if and only if there is a positive integer $q$ such that $2 m+1=2^{q}-1$.

Proof. Since $\binom{2(m+1)}{m+1}=\binom{2 m+1}{m+1}+\binom{2 m+1}{m}$, which is even since $\left(\begin{array}{c}\binom{m+1}{m+1}=, ~=~\end{array}\right.$ $\binom{2 m+1}{m}$. Now assume $2 m+1=2^{q}-1$ for a positive integer $q$. If $q=1$, then $\binom{2 m+1}{m}=\binom{1}{0}=1$, which is odd. Next assume $q>1$. Now, $l(2 m+1)=q-1, l(m)=q-2, \mathcal{D}(2 m+1)=\{0,1, \ldots, q-1\}$, and $\mathcal{D}(m)=\{0,1, \ldots, q-2\}$. So by Lucas's Theorem,

$$
\binom{2 m+1}{m} \equiv\binom{1}{0}\binom{1}{1}\binom{1}{1} \cdots\binom{1}{1}(\bmod 2),
$$

hence $\binom{2 m+1}{m}$ is odd. Finally, assume $\binom{2 m+1}{m}$ is odd. If $m=0$, then $2 m+1=1=2^{1}-1$. So now assume $m>0$. Set $r-2:=l(m)$ (hence $r>0$ ) and write

$$
m=m_{r-2} 2^{r-2}+m_{r-3} 2^{r-3}+\cdots+m_{1} 2^{1}+m_{0} 2^{0},
$$

where of course $m_{r-2}=1$. So,

$$
2 m+1=m_{r-2} 2^{r-1}+m_{r-3} 2^{r-2}+\cdots+m_{0} 2^{1}+1 \cdot 2^{0} .
$$

Since $\binom{2 m+1}{m}$ is odd, Lucas's Theorem requires that $\binom{m_{r-3}}{m_{r-2}}=\binom{m_{r-4}}{m_{r-3}}=$ $\cdots=\binom{m_{0}}{m_{1}}=1$. Based on these equalities, we observe that $m_{r-2}=1$ forces $m_{r-3}=1$, which in turn forces $m_{r-4}=1$, etc. We conclude that

$$
m_{r-2}=m_{r-3}=m_{r-4}=\cdots=m_{1}=m_{0}=1
$$

Then $m=2^{r-1}-1$, so $2 m+1=2^{r}-1$.
Lemma 14 Let $n$ be a nonnegative integer. All entries on the $n^{\text {th }}$ row of Pascal's triangle are odd if and only if there is a nonnegative integer $q$ such that $n=2^{q}-1$.

Proof. Suppose all entries on the $n^{\text {th }}$ row are odd. If $n=0$, then $n=$ $2^{0}-1$. If $n>0$, then by Lemma 9 , there is a positive integer $q$ with $n=2^{q}-1$. Conversely, suppose $n=2^{q}-1$ for a nonnegative integer $q$. If $n=0=2^{0}-1$, then all entries on this row are odd, since the only entry on this row is $\binom{0}{0}=1$. Now say $q$ is positive, so

$$
n=n_{q-1} 2^{q-1}+n_{q-2} 2^{q-2}+\cdots+n_{1} 2^{1}+n_{0} 2^{0}
$$

with $n_{q-1}=n_{q-2}=\cdots=n_{1}=n_{0}=1$. Let $0 \leq k \leq n$, and write

$$
k=k_{q-1} 2^{q-1}+k_{q-2} 2^{q-2}+\cdots+k_{1} 2^{1}+k_{0} 2^{0} .
$$

Then by Lucas's Theorem,

$$
\binom{n}{k} \equiv\binom{1}{k_{q-1}}\binom{1}{k_{q-2}} \cdots\binom{1}{k_{0}}(\bmod 2) \equiv 1(\bmod 2) .
$$

So all entries on the $n^{\text {th }}$ row are odd.
The following binomial coefficient identity is a version of Vandermonde's Identity, cf. Identity 132 of [2].

Lemma 15 (Vandermonde's Identity) Let m, $l$, and $j$ be nonnegative integers. Then

$$
\sum_{i=0}^{l}\binom{l}{i}\binom{m-l}{j-i}=\binom{m}{j} .
$$

Lemma 16 Suppose $t=2^{q}$ for some nonnegative integer $q$. Then the quantities $\mathrm{a}_{t}(n, k)$ and $\binom{n+t}{k}$ have the same parity.

Proof. By Lemma 4.7, $\binom{n+t}{k}=\sum_{i=0}^{t}\binom{t}{i}\binom{n+t-t}{k-i}$. Then

$$
\binom{n+t}{k} \equiv \sum_{i=0}^{t}\binom{t}{i}\binom{n}{k-i}(\bmod 2)
$$

Since $t=2^{q}$ for a nonnegative integer $q$, then by Lemma 10 , all entries on row $t-1$ are odd. Then all entries on row $t$ except the first and last are even. So,

$$
\sum_{i=0}^{t}\binom{t}{i}\binom{n}{k-i} \equiv\binom{n}{k}+\binom{n}{k-t}(\bmod 2) .
$$

And,

$$
\binom{n}{k}+\binom{n}{k-t} \equiv\binom{n}{k}-\binom{n}{k-t}(\bmod 2) .
$$

Since $\mathbf{a}_{t}(n, k)=\binom{n}{k}-\binom{n}{k-t}$ by Theorem 4, we conclude that

$$
\binom{n+t}{k} \equiv \mathrm{a}_{t}(n, k)(\bmod 2) .
$$

We are now ready to prove Theorem 7.

## Proof of Theorem 7:

Lemma 8 shows that if the number of odds on each row of the Pascal triangle truncation $\left(\mathrm{a}_{t}(n, k)\right)$ is a power of two, then $t$ must be a power of two. Conversely, let us now suppose that $t=2^{q}$ for some nonnegative integer $q$. We aim to demonstrate the following claim: When $n$ is a nonnegative integer, the number of odds on row $n$ of the array is precisely $\frac{1}{2} \cdot 2^{d(n+t)}$.

We begin by assuming $n$ is odd. Write $n=2 m+1$ for a nonnegative integer $m$. The last nonzero entry on row $n$ occurs at position $k=\min \left\{\left\lfloor\frac{t-1+n}{2}\right\rfloor, n\right\}=\min \left\{\left\lfloor\frac{2^{q}+2 m}{2}\right\rfloor, 2 m+1\right\}$. If $q=0$, then $\left\lfloor\frac{2^{q}+2 m}{2}\right\rfloor=m$, so $k=m$. Of course, we now have $t=2^{0}=1$. By Lemma 4.8, the parity of entry $\mathrm{a}_{1}(n, j)$ of the $n^{\text {th }}$ row of our array (where $0 \leq j \leq k=m$ ) is the same as the parity of the binomial coefficient $\binom{n+1}{j}=\binom{2 m+2}{j}$. Since $\binom{2 m+2}{m+1}$ is even by Lemma 9, then the number of odds on the $(n+1)^{\text {st }}$ row of Pascal's triangle is twice the number of odds amongst the entries entry $\binom{n+1}{j}$ for $0 \leq j \leq k=m$. Now, the number of odds on the $(n+1)^{\text {st }}$ row of Pascal's triangle is $2^{d(n+1)}$ by Corollary 6 . Therefore, the number of odds on the $n^{\text {th }}$ row of our truncated Pascal array is $\frac{1}{2} \cdot 2^{d(n+1)}$, confirming our desired claim when $q=0$.

Continuing with the assumption that $n$ is odd, now assume that $q>0$. Then $\left\lfloor\frac{2^{q}+2 m}{2}\right\rfloor=2^{q-1}+m$. If $2^{q-1}+m \geq 2 m+1$ (and hence $2^{q}>$ $2 m+1$ ), then $k=2 m+1$, so the entries of the $n^{\text {th }}$ row of our array coincide with the entries of the $n^{\text {th }}$ row of Pascal's triangle. This shared
number of odds is therefore $2^{d(n)}$, by Corollary 6 . But

$$
d(n+t)=d\left(n+2^{q}\right)=d(n)+1
$$

since $2^{q}>2 m+1=n$. Then the number of odds on the $n^{\text {th }}$ row of our array is $2^{d(n)}=\frac{1}{2} \cdot 2^{d(n)+1}=\frac{1}{2} \cdot 2^{d(n+t)}$, again confirming our claim. So now consider the case that $2^{q-1}+m<2 m+1$. The $n^{\text {th }}$ row entry $\mathrm{a}_{t}(n, j)$ of our truncated Pascal array (where $0 \leq j \leq k$ ) has the same parity as the entry $\binom{n+t}{j}$ of Pascal's triangle, by Lemma 4.8. Now, $k=2^{q-1}+m$ while $n+t=2^{q}+2 m+2=2\left(2^{q-1}+m+1\right)$. Since $\binom{n+t}{2^{q-1}+m+1}=\binom{2\left(2^{q-1}+m+1\right)}{2^{q-1}+m+1}$ is even by Lemma 9 , then the number of odds on the $(n+t)^{\mathrm{th}}$ row of Pascal's triangle is twice the number of odds amongst the entries entry $\binom{n+t}{j}$ for $0 \leq j \leq k=2^{q-1}+m$. So, the number of odds on the $n^{\text {th }}$ row of our truncated Pascal array is $\frac{1}{2} \cdot 2^{d(n+t)}$, completing the confirmation of our claim when $n$ is odd.

Next, assume $n$ is even, and write $n=2 m$ for some nonnegative integer $m$. As before, the last nonzero entry on row $n$ occurs at position
$k=\min \left\{\left\lfloor\frac{t-1+n}{2}\right\rfloor, n\right\}=\min \left\{\left\lfloor\frac{2^{q}-1+2 m}{2}\right\rfloor, 2 m\right\}$. Say $q=0$, so $t=1$. Then $k=\min \left\{\left\lfloor\frac{2 m}{2}\right\rfloor, 2 m\right\}=m$. By Lemma 4.8, the parity of entry $\mathrm{a}_{1}(n, j)$ of the $n^{\text {th }}$ row of our array (where $0 \leq j \leq k=m$ ) is the same as the parity of the binomial coefficient $\binom{n+1}{j}=\binom{2 m+1}{j}$. Since the entries $\binom{n+1}{0},\binom{n+1}{1}, \ldots,\binom{n+1}{m}$ comprise exactly half of the entries of said row of Pascal's triangle, then there are $\frac{1}{2} \cdot 2^{d(n+1)}$ odds amongst these entries. So there are $\frac{1}{2} \cdot 2^{d(n+1)}$ odd entries on the $n^{\text {th }}$ row of our truncated Pascal array, completing the confirmation of our claim when $q=0$.

Keeping the hypothesis that $n$ is even, now assume that $q>0$. Then $\left\lfloor\frac{2^{q}-1+2 m}{2}\right\rfloor=2^{q-1}+m-1$. If $2^{q-1}+m-1 \geq 2 m$ (and hence $2^{q}>2 m$ ), then $k=2 m$, so the entries of the $n^{\text {th }}$ row of our array coincide with the entries of the $n^{\text {th }}$ row of Pascal's triangle. This shared number of odds is therefore $2^{d(n)}$, by Corollary 6 . But $d(n+t)=d\left(n+2^{q}\right)=d(n)+1$ since $2^{q}>2 m=n$. Then the number of odds on the $n^{\text {th }}$ row of our array is $2^{d(n)}=\frac{1}{2} \cdot 2^{d(n)+1}=\frac{1}{2} \cdot 2^{d(n+t)}$, again confirming our claim. So now consider the case that $2^{q-1}+m-1<2 m$. The $n^{\text {th }}$ row entry $\mathrm{a}_{t}(n, j)$ of our truncated Pascal array (where $0 \leq j \leq k$ ) has the same parity as the entry $\binom{n+t}{j}$ of Pascal's triangle, by Lemma 4.8. Now, $k=2^{q-1}+m-1$ while $n+t=2^{q}+2 m=2\left(2^{q-1}+m\right)$. The central coefficient $\binom{2\left(2^{q-1}+m\right)}{2^{q-1}+m}$ of the $(n+t)^{\text {th }}$ row of Pascal's triangle is even by Lemma 9 , so this entry does not contribute to the tally of odd numbers on this row. Therefore the number of odds in row $n$ of our array is exactly one-half the number of odds on the $(n+t)^{\text {th }}$ row of Pascal's triangle, which is $\frac{1}{2} \cdot 2^{d(n+t)}$. This completes the confirmation of our claim when $n$ is odd.

## 5. Some proofs deferred from §3

## Proof of Lemma 2.

Let $\mathcal{P}_{t}(n, k)$ be the set of $t$-admissible $(n, k)$-NE-paths, and let $\mathcal{T}_{t}(n, k)$ be the set of $t$-admissible $(n, k)$-columnar tableaux. In this proof, we identify an $(n, k)$-NE-path $\mathbf{s}$ with the sequence $\mathbf{s}=\left(\left(x_{i}(\mathbf{s}), y_{i}(\mathbf{s})\right)\right)_{i=1}^{k}$ consisting of the $k$ successive endpoints of the horizontal, or easterly, steps of the path. For example, for the first 4 -admissible (7,5)-NE-path depicted above, the sequence of horizontal endpoints is

$$
((1,0),(2,0),(3,0),(4,1),(5,2)) .
$$

Given a $t$-admissible $(n, k)$-NE-path $\mathbf{s}=\left(\left(x_{i}(\mathbf{s}), y_{i}(\mathbf{s})\right)\right)_{i=1}^{k}$, set
$\phi(\mathbf{s}):=\left(x_{i}(\mathbf{s})+y_{i}(\mathbf{s})\right)_{i=1}^{k}$. Let $T=\left(T_{1}, \ldots, T_{k}\right)=\phi(\mathbf{s})$. Since $x_{i}(\mathbf{s})=i$, then $i \leq x_{i}(\mathbf{s})+y_{i}(\mathbf{s})=T_{i}$. In particular, $1 \leq T_{1}$. Also, $y_{i}(\mathbf{s}) \leq y_{i+1}(\mathbf{s})$, then

$$
x_{i}(\mathbf{s})+y_{i}(\mathbf{s})=T_{i}<T_{i+1}=x_{i+1}(\mathbf{s})+y_{i+1}(\mathbf{s})
$$

when $i \in\{1,2, \ldots, k-1\}$. Since $x_{k}(\mathbf{s})=k$ and $y_{k}(\mathbf{s}) \leq n-k$, then $T_{k} \leq n$. So $T$ is an $(n, k)$-columnar tableau. Now suppose $t-1<k$, and let $j \in\{1,2, \ldots, k-t+1\}$. Since s is $t$-admissible, then we have
$j=(t-1+j)-t+1=x_{t-1+j}(\mathbf{s})-t+1 \leq y_{t-1+j}(\mathbf{s})$.
So,

$$
\begin{aligned}
t-1+2 j=(t-1+j)+j & =x_{t-1+j}(\mathbf{s})+j \\
& \leq x_{t-1+j}(\mathbf{s})+y_{t-1+j}(\mathbf{s}) \\
& =T_{t-1+j} .
\end{aligned}
$$

Thus $T$ is a $t$-admissible $(n, k)$-columnar tableau. We can therefore regard $\phi: \mathcal{P}_{t}(n, k) \longrightarrow \mathcal{T}_{t}(n, k)$ as a well-defined function.

Now suppose that $T=\left(T_{1}, \ldots, T_{k}\right)$ is a $t$-admissible $(n, k)$-columnar tableau. Declare that $\psi(T):=\left(\left(i, T_{i}-i\right)\right)_{i=1}^{k}$. Set

$$
\mathbf{s}=\left(\left(x_{i}, y_{i}\right)\right)_{i=1}^{k}:=\psi(T)
$$

To prove that $\mathbf{s}$ is an $(n, k)$-NE-path, it suffices to check that

$$
0 \leq y_{1} \leq y_{2} \leq \cdots \leq y_{k} \leq n-k .
$$

Since $1 \leq T_{1}$, then $0 \leq T_{1}-1=y_{1}$. When $i \in\{1,2, \ldots, k-1\}$, then
$T_{i}<T_{i+1}$ so

$$
y_{i}=T_{i}-i \leq T_{i+1}-(i+1)=y_{i+1} .
$$

Also, $y_{k}=T_{k}-k \leq n-k$ since $T_{k} \leq n$. Next, we check that $\mathbf{s}$ is $t$-admissible by showing that each $y_{i} \geq x_{i}-t+1$, assuming $t-1<k$. Suppose for the moment that $i>t-1$, so that $i=t-1+j$ with $j \in\{1,2, \ldots, k-t+1\}$. Then

$$
\begin{aligned}
y_{i}=T_{i}-i & =T_{t-1+j}-(t-1+j) \\
& \geq t-1+2 j-t+1-j \\
& =j=i-t+1 \\
& =x_{i}-t+1 .
\end{aligned}
$$

On the other hand, if $i \leq t-1$, then $x_{i}-t+1=i-t+1 \leq 0 \leq y_{i}$. This reasoning shows that s is $t$-admissible.

We can therefore regard $\psi: \mathcal{T}_{t}(n, k) \longrightarrow \mathcal{P}_{t}(n, k)$ as a well-defined function. Clearly $\phi$ and $\psi$ are inverses, so $\mathcal{P}_{t}(n, k)$ and $\mathcal{T}_{t}(n, k)$ are equinumerous, which is what we needed to show.

## Proof of Lemma 3.

For (i), consider the $i=0$ term in the expression for $D^{-1}\left(x_{t-1}\right)$ given in the paragraphs preceding the lemma statement. Then

$$
1=\mathrm{d}_{t}(i+N-1, i)=\mathrm{d}_{t}(0,0) .
$$

For (ii), we observe that an integer pair $(n, k)$ is indeed a pair $(i+N-1, i)$ corresponding to a term in the sum shown in equation (4) above if and only if $k=i$ and $n=k+N-1$ for some $N>0$ and $0 \leq i \leq t-1+N-1$. Now simply check inequalities to see that $k \geq 0, n \geq 0, k \leq n$, and $k \leq\left\lfloor\frac{n-1+t}{2}\right\rfloor$.

For (iii), we apply $D$ to each side of equation (4). First,

$$
\begin{align*}
D\left(D^{-N}\left(x_{t-1}\right)\right) & =D^{-(N-1)}\left(x_{t-1}\right) \\
& =\sum_{i=0}^{t-1+N-2}(-1)^{i} \mathrm{~d}_{t}(i+N-2, i) x_{t-1+N-1-i} . \tag{6}
\end{align*}
$$

On the other hand,

$$
\begin{aligned}
& \left(D^{-N}\left(x_{t-1}\right)\right) \\
& =D\left(\sum_{i=0}^{t-1+N-1}(-1)^{i} \mathrm{~d}_{t}(i+N-1, i) x_{u+N-i}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{i=0}^{t-1+N-1}(-1)^{i} \mathrm{~d}_{t}(i+N-1, i)\left(x_{t-1+N-1-i}+x_{t-1+N-2-i}\right) \\
& =\sum_{i=0}^{t-1+N-1}(-1)^{i}\left[\mathrm{~d}_{t}(i+N-1, i)-\mathrm{d}_{t}(i+N-2, i-1)\right] x_{t-1+N-1-i},
\end{aligned}
$$

where the latter is obtained by expanding and reindexing. Then by equating coefficients in the latter expression with coefficients for the expression obtained in equation (5), we see that
$\mathrm{d}_{t}(t-1+2 N-2, t-1+N-1)-\mathrm{d}_{t}(t-1+2 N-3, t-1+2 N-2)=0$
and that for all $0 \leq i \leq t-1+N-2$ we have

$$
\mathrm{d}_{t}(i+N-1, i)-\mathrm{d}_{t}(i+N-2, i-1)=\mathrm{d}_{t}(i+N-2, i) .
$$

The latter formula actually becomes the former when $i=t-1+N-1$, as $\mathrm{d}_{t}(t-1+2 N-3, t-1+N-1)$ evaluates to zero. Therefore,

$$
\begin{equation*}
\mathrm{d}_{t}(i+N-1, i)=\mathrm{d}_{t}(i+N-2, i-1)+\mathrm{d}_{t}(i+N-2, i) \tag{7}
\end{equation*}
$$

for all $N>0$ and $0 \leq i \leq t-1+N-1$. Now if $(n, k)$ is an integer pair with $n \geq 0$ and $0 \leq k \leq \min \left\{\left\lfloor\frac{n-1+t}{2}\right\rfloor, n\right\}$, then set $i=k$ and $N=n+1-k$. As in the previous paragraph we can see that $(i+N-1, i)$ corresponds to a term from equation (4). Then from equation (6), we get $\mathrm{d}_{t}(n, k)=\mathrm{d}_{t}(n-1, k-1)+\mathrm{d}_{t}(n-1, k)$, as desired.

## Some thoughts on extending this work

Given that our proofs employ elementary techniques, perhaps these proofs can be modified to obtain more general results (which is, indeed, a crucial function of rigorous proof in mathematics). One possible direction is to consider patterns in truncated Pascal arrays modulo other primes or prime powers, cf. [4]. Also, when $t=1$, the truncated Pascal array is just the Catalan triangle. In this case, as mentioned in $\S 3$, the nonzero numbers in any given row are known to be dimensions of certain fundamental representations of the associated symplectic Lie group. With $t=1$, the $t$-admissible ( $n, k$ )-columnar tableaux we presented in $\S 3$ coincide (after a simple change in the alphabet of tableaux entries) with columnar symplectic tableaux of [9]. It might be interesting to consider what similar algebraic contexts might be found for $t$-admissible ( $n, k$ )-columnar tableaux when $t>1$.

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## The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before November 1, 2020. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2020 issue of The Pentagon. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

## NEW PROBLEMS 849-858

Problem 849. Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, Mehedinti, Romania.

Prove that in an acute $\triangle A B C$ the following relationship holds:

$$
\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}+\frac{1}{\cos A}+\frac{1}{\cos B}+\frac{1}{\cos C}>6 \sqrt{2}
$$

Problem 850. Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, Mehedinti, Romania.

If $x, y, z>0$ and $x+y+z=2 \pi$, prove

$$
\frac{\cos ^{4} x}{y+z}+\frac{\cos ^{4} y}{z+x}+\frac{\cos ^{4}(x+y)}{x+y} \geqslant \frac{9}{64 \pi}
$$

Problem 851. Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, Mehedinti, Romania.

Let $a<b$ and $f:[a, b] \rightarrow(0, \infty)$ be continuous. Prove

$$
3(b-a) \int_{a}^{b} f^{2}(x) d x+(b-a)^{2} \geqslant 2(b-a) \int_{a}^{b} f(x) d x+2\left(\int_{a}^{b} f(x) d x\right)^{2} .
$$

Problem 852. Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, Mehedinti, Romania.

If $a, b, c>0$ and $a+b+c=3$, prove

$$
\sum_{c y c} \frac{a^{3} c(b+1)+b^{3} c(a+1)}{a^{2} b(b+1)+b^{2} a(a+1)} \geqslant 3 .
$$

Problem 853. Proposed by Marcel Chirita, Bucharest, Romania.
Let $x \in \mathbb{Z}$. If $x^{5}+5 x^{3}+15 x^{2}>21 x$, prove $x^{5}+5 x^{3}+15 x^{2}-21 x \geqslant 30$.
Problem 854. Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.

If $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(0)=2019$ and $3 f(x)=f(x+y)+2 f(x-y)+y$ for any $x, y \in \mathbb{R}$, then compute $\int_{e}^{\pi} f(x) d x$.

Problem 855. Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.

Let $\left\{a_{n}\right\},\left\{b_{n}\right\},\left\{c_{n}\right\}$ be positive real sequences such that
$\lim _{n \rightarrow \infty} \frac{a_{n+1}}{n a_{n}}=a>0, \lim _{n \rightarrow \infty} \frac{b_{n+1}}{n b_{n}}=b>0$, and $\lim _{n \rightarrow \infty} \frac{c_{n+1}}{n c_{n}}=c>0$.
Compute $\lim _{n \rightarrow \infty} \frac{1}{\sqrt[n]{a_{n}^{3}}} \sum_{k=1}^{n}\left(b_{k} c_{k}\right)^{1 / k}$.

Problem 856. Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.

Each $1 \times 1$ square of a $7 \times 211$ rectangle is painted either black or white. Prove that it is possible to choose four rows and four columns of the rectangle so that the sixteen $1 \times 1$ squares in which they intersect are painted with the same color.

Problem 857. Proposed by José Luis Díaz-Barrero, School of Civil
Engineering, Barcelona Tech - UPC, Barcelona, Spain.
Let $a, b, c, d$ be four positive real numbers. Find the maximum value of

$$
\frac{\sqrt[4]{a}+\sqrt[4]{b}+\sqrt[4]{c}+\sqrt[4]{d}}{\sqrt[4]{a+b+c+d}}
$$

Problem 858. Proposed by Pedro H.O. Pantoja, University of Campina Grande, Brazil.

Let $M=\frac{8}{7} \sin ^{2}\left(\frac{\pi}{7}\right)+\frac{\sqrt{7}}{7} \cot \left(\frac{\pi}{7}\right)$. Is $M$ irrational?

## SOLUTIONS TO PROBLEMS 829-839

Problem 829. Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, Mehedinti, Romania.

Let $\Omega_{n}=\binom{n}{7}+2\binom{n-1}{7}+3\binom{n-2}{7}+\cdots(n-6)\binom{7}{7}$ for all $n \geqslant 7$. Find $\Omega=\lim _{n \rightarrow \infty} \sqrt[n]{\Omega_{n}}$.

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Let $n \geq 7$ and

$$
\begin{aligned}
\Omega_{n} & =\binom{n}{7}+2\binom{n-1}{7}+3\binom{n-2}{7}+\cdots(n-6)\binom{7}{7} \\
& =\sum_{j=7}^{n} \sum_{i=7}^{j}\binom{i}{7} .
\end{aligned}
$$

By the Hockey Stick Identity,

$$
\sum_{i=7}^{j}\binom{i}{7}=\binom{j+1}{8} \text { and } \sum_{j=7}^{n}\binom{j+1}{8}=\sum_{j=8}^{n+1}\binom{j}{8}=\binom{n+2}{9} .
$$

Thus

$$
\begin{aligned}
\Omega_{n} & =\frac{(n+2)(n+1) n(n-1) \ldots(n-6)}{9!} \\
& =\frac{1}{9!} n^{9}\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n}\right) \ldots\left(1-\frac{6}{n}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega & =\lim _{n \rightarrow \infty} \sqrt[n]{\Omega_{n}} \\
& =\lim _{n \rightarrow \infty}(\sqrt[n]{n})^{9} \sqrt[n]{\frac{1}{9!}\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n}\right) \ldots\left(1-\frac{6}{n}\right)} \\
& =1 .
\end{aligned}
$$

Also solved by Abhijit Bhattacharjee (student), Banaras Hindu University, India; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; and the proposer.

Problem 830. Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, Mehedinti, Romania.

If $x \in\left(0, \frac{\pi}{2}\right)$, prove that $2(\sin x)^{1-\sin x} \cdot(1-\sin x)^{\sin x} \leqslant 1$.
Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.
Noting that $0<\sin x<1$ for $x \in(0, \pi / 2)$, we apply the weighted AGM inequality twice to see that

$$
\begin{aligned}
2(\sin x)^{1-\sin x} \cdot(1-\sin x)^{\sin x} & \leqslant 2[(1-\sin x) \sin x+\sin x(1-\sin x)] \\
& =4 \sin x(1-\sin x) \\
& \leqslant 4\left(\frac{\sin x+(1-\sin x)}{2}\right)^{2}=1,
\end{aligned}
$$

with equality when $x=\pi / 6$.
Also solved by Brian Bradie, Christopher Newport, Newport News, VA; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Jalil Hajimir, Canada; Mokhtar Khassani, Mostaganem, Algerie; Missouri State University Problem Solving Group, Springfield, MO; Rovsen Pirguliyev, Sumgait, Azerbaijan; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Ioannis Sfikas, National and Kapodistrian University of

Athens, Greece; Remus Florin Stanca, Romania; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comănesti, Romania; Daniel Văcaru, "Maria Teiuleanu" National Economic College, Pitesti, Romania; and the proposer.

Problem 831. Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, Mehedinti, Romania.

If $\triangle A B C \sim \Delta A^{\prime} B^{\prime} C^{\prime}$, prove that

$$
\sum \frac{\left(a^{\prime}+b^{\prime}\right)\left(a^{\prime}+c^{\prime}\right)}{b^{\prime} c^{\prime}}+3 \geqslant \frac{15(b+c)\left(c^{\prime}+a^{\prime}\right)\left(a^{\prime}+b^{\prime}\right)}{8 a b^{\prime} c^{\prime}} .
$$

Solution by Daniel Văcaru, "Maria Teiuleanu" National Economic College, Pitesti, Romania.

With $\triangle A B C \sim \Delta A^{\prime} B^{\prime} C^{\prime}$, we have $a=k a^{\prime}, b=k b^{\prime}, c=k c^{\prime}$ and obtain

$$
\frac{\left(a^{\prime}+b^{\prime}\right)\left(a^{\prime}+c^{\prime}\right)}{b^{\prime} c^{\prime}}=\frac{(a+b)(a+c)}{b c}
$$

and

$$
\frac{15(b+c)\left(c^{\prime}+a^{\prime}\right)\left(a^{\prime}+b^{\prime}\right)}{8 a b^{\prime} c^{\prime}}=\frac{15(b+c)(c+a)(a+b)}{8 a b c} .
$$

That is

$$
\begin{aligned}
& \sum_{\text {by } \frac{(a+b)(a+c)}{b c}+3 \geqslant \frac{15(b+c)(c+a)(a+b)}{(b+c)(c+a)(a+b)}, \text { we obtain }}^{8 a b c} .
\end{aligned}
$$

$$
\sum \frac{a}{b+c}+\frac{3 a b c}{(a+b)(b+c)(c+a)} \geqslant \frac{15}{8}
$$

and we write this as

$$
\sum \frac{a}{b+c}-\frac{3}{2} \geqslant \frac{3}{8}-\frac{3 a b c}{(a+b)(b+c)(c+a)} .
$$

By a calculation, we obtain the LHS is equal to $\frac{\sum\left[(a+b)(a-b)^{2}\right]}{2(a+b)(b+c)(c+a)}$ and the RHS is equal to $\frac{\sum a(b-c)^{2}}{8(a+b)(b+c)(c+a)}$. But

$$
4(a+b) \geqslant c \Rightarrow 4(a+b)(a-b)^{2} \geqslant c(a-b)^{2}
$$

which implies the required inequality.
Also solved by Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; and the proposer.

Problem 832. Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.

Prove that in any triangle ABC the following holds:

$$
\frac{a}{a+b+c} \geqslant \frac{2 \sqrt{3}}{9} \sin A
$$

Solution by Scott Brown, Auburn University, Montgomery, AL.
According to Bottema, Geometric Inequalities, 1968, we have $\sin A=\frac{a}{2 R}$ so the inequality can be written as $\frac{a}{a+b+c} \geqslant \frac{2 \sqrt{3}}{9} \cdot \frac{a}{2 R}$. But this can be further simplified as $\frac{9 R}{2} \geqslant \sqrt{3} \cdot \frac{F}{r}$ where $\frac{F}{r}=\frac{a+b+c}{2}$. Now write the inequality as $27 \operatorname{Rr}^{2} \geqslant 6 \sqrt{3} F$ which can be found on page 63 of Bottema.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comǎnesti, Romania; Daniel Vǎcaru, "Maria Teiuleanu" National Economic College, Pitesti, Romania; and the proposer.

Problem 833. Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.

Show that the equation $x^{6}-5 x^{5}-6 x^{4}+2 x^{3}+9 x^{2}-17 x+1=0$ has no negative roots.

Solution by Sarah Seales, Prescott, AZ.
Let $p(x)=x^{6}-5 x^{5}-6 x^{4}+2 x^{3}+9 x^{2}-17 x+1$. We will show that when $x$ is negative, $p(x)>0$. Let $x=-a$ for some positive real $a$. Then $p(-a)=a^{6}+5 a^{5}-6 a^{4}-2 a^{3}+9 a^{2}+17 a+1$. By the AM-GM inequality, $a^{6}+9 a^{2} \geqslant 2 \sqrt{9 a^{8}}=6 a^{4}$ and $5 a^{5}+17 a \geqslant 2 \sqrt{85 a^{6}}=2 \sqrt{85} a^{3}$. So $p(-a)>0$. Thus $p(x)$ has no negative roots.

Also solved by Brian Beasley, Presbyterian College, Clinton, SC; Brian Bradie, Christopher Newport University, Newport News, VA; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Ravi Prakash, New Delhi, India; Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Henry Ricardo, Westchester Area Math Circle, Purchase, NY; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Michael Sterghiou, Greece; Marian Ursǎrescu, Romania; Daniel Vǎcaru, "Maria Teiuleanu" National Economic College, Pitesti, Romania; and the proposer.

Problem 834. Proposed by D.M. Bǎtinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, Neculai Stanciu, "George Emil

## Palade" School, Buzău, Romania.

Let $\left(F_{n}\right)$ be the Fibonacci sequence, i.e.

$$
F_{0}=0, F_{1}=1, F_{n+2}=F_{n+1}+F_{n}, \text { for all } n \geqslant 0 .
$$

If $e_{n}=\left(1+\frac{1}{n}\right)^{n}$, prove that $\left(\sum_{k=1}^{n} e_{k} F_{2 k-1}\right)\left(\sum_{k=1}^{n} \frac{F_{2 k-1}}{e_{k}}\right) \leqslant \frac{(e+2)^{2}}{8 e} F_{2 n}^{2}$.

## Solution by Marian Ursărescu, Romania.

We use the Kantorovich inequality: $x_{k} \in[m, M]$

$$
\begin{aligned}
x_{k} \in[m, M], & 0<m<M, t_{k}>0 \\
& \Rightarrow \sum_{k=1}^{n} t_{k} x_{k} \cdot \sum_{k=1}^{n} \frac{t_{k}}{x_{k}} \leqslant \frac{(m+M)^{2}}{4 m M}\left(\sum_{k=1}^{n} t_{k}\right)^{2} .
\end{aligned}
$$

Let $e_{k}=\left(1+\frac{1}{k}\right)^{k}$. Then
$2<e_{k}<e$

$$
\Rightarrow\left(\sum_{k=1}^{n} e_{k} F_{2 k-1}\right)\left(\sum_{k=1}^{n} \frac{F_{2 k-1}}{e_{k}}\right) \leqslant \frac{(e+2)^{2}}{8 e}\left(\sum_{k=1}^{n} F_{2 k-1}\right)^{2} .
$$

But $F_{2 k-1}=F_{2 k}-F_{2 k-2}$ and so $\sum_{k=1}^{n} F_{2 k-1}=\sum_{k=1}^{n} F_{2 k}-\sum_{k=1}^{n} F_{2 k-2}=F_{2 n}-$ $F_{0}=F_{2 n}$, and the inequality is proved.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Henry Ricardo, Westchester Area Math Circle, Purchase, NY; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Daniel VVcaru, "Maria Teiuleanu" National Economic College, Pitesti, Romania; and the proposers.

Problem 835. Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.

Let $\left(L_{n}\right)$ be the Lucas sequence, i.e.

$$
L_{0}=2, L_{1}=1, L_{n+2}=L_{n+1}+L_{n}, \text { for all } n \geqslant 0 .
$$

Prove that $n^{n-2}(n-1) \sum_{k=1}^{n} L_{k}^{n}+n^{n-1} \prod_{k=1}^{n} L_{k}>\left(L_{n+2}-3\right)^{n}$ for all $n \geqslant 2$.

Solution by Ioannis Sfikas, National and Kapodistrian University of Athens,

Greece.
We may write the inequality as

$$
\begin{equation*}
(n-1) \sum_{k=1}^{n} L_{k}^{n}+n \prod_{k=1}^{n} L_{k}>\frac{\left(L_{n+2}-3\right)^{n}}{n^{n-2}} . \tag{1}
\end{equation*}
$$

By Janos Suranyi's inequality, if $x_{k}>0$ then

$$
(n-1) \sum_{k=1}^{n} x_{k}^{n}+n \prod_{k=1}^{n} x_{k} \geqslant\left(\sum_{k=1}^{n} x_{k}\right)\left(\sum_{k=1}^{n} x_{k}^{n-1}\right) .
$$

In the case of (1), we have

$$
(n-1) \sum_{k=1}^{n} L_{k}^{n}+n \prod_{k=1}^{n} L_{k} \geqslant\left(\sum_{k=1}^{n} L_{k}\right)\left(\sum_{k=1}^{n} L_{k}^{n-1}\right) .
$$

We need to show that

$$
\left(\sum_{k=1}^{n} L_{k}\right)\left(\sum_{k=1}^{n} L_{k}^{n-1}\right)>\frac{\left(L_{n+2}-3\right)^{n}}{n^{n-2}} .
$$

By Holder's inequality:

$$
\sum_{k=1}^{n} L_{k}^{n-1} \geqslant \frac{\left(\sum_{k=1}^{n} L_{k}\right)^{n-1}}{n^{n-2}}
$$

But

$$
\sum_{k=1}^{n} L_{k}=\sum_{k=1}^{n} L_{k+2}-\sum_{k=1}^{n} L_{k+1}=L_{n+2}-L_{2}=L_{n+2}-3 .
$$

Putting these together gives the desired inequality.
Also solved by Marian Ursărescu, Romania; Daniel Văcaru, "Maria Teiuleanu" National Economic College, Pitesti, Romania; and the proposers.

Problem 836. Proposed by Abhijit Bhattacharjee (student), Banaras Hindu University, India.

Prove that the equation $1+\frac{x}{1!}+\frac{x^{2}}{2!}+\cdots+\frac{x^{n}}{n!}=0$ has exactly one real root if $n$ is odd and no real root if $n$ is even.

Solution by Missouri State University Problem Solving Group, Springfield, MO.

Let $T_{n}(x)=1+\frac{x}{1!}+\frac{x^{2}}{2!}+\ldots+\frac{x^{n}}{n!}$. Note that 0 is not a root of $T_{n}(x)$ for any $n$. Suppose that $n$ is an even integer. Note that $T_{n}{ }^{\prime}(x)=T_{n-1}(x)$ and $T_{n}(x)=T_{n}{ }^{\prime}(x)+\frac{x^{n}}{n!}$. Since $T_{n}{ }^{\prime}(x)$ has odd degree, we know it has at least one real root. Let $r$ be any of the real roots of $T_{n}{ }^{\prime}(x)$. Then $T_{n}(r)=T_{n}{ }^{\prime}(r)+\frac{r^{n}}{n!}$. Since $n$ is even and $r \neq 0$, then $T_{n}(r)>0$. Because $T_{n}$ is positive at each of its critical points, the absolute minimum of $T_{n}(x)$ must be a positive real number.

Thus $T_{n}(x)$ has no real roots when $n$ is even.
Suppose that $n$ is an odd integer. Since $T_{n}{ }^{\prime}(x)=T_{n-1}(x)$ and $n-1$ is even, then $T_{n}(x)$ has no critical points as shown above. So $T_{n}(x)$ is an increasing odd degree polynomial. Therefore, there is exactly one real root of $T_{n}(x)$ when $n$ is odd.

Also solved by Brian Bradie, Christopher Newport University, Newport News, VA; Brent Dozier, North Carolina Wesleyan College, Rocky Mount, NC; Avinaba Majumdar, Bandel, India; Henry Ricardo, Westchester Area Math Circle, Purchase, NY; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Daniel Văcaru, "Maria Teiuleanu" National Economic College, Pitesti, Romania; and the proposer.

Problem 837. Proposed by Pedro H.O. Pantoja, University of Campina Grande, Brazil.

Evaluate $\lim _{n \rightarrow \infty} \int_{-1}^{1}\left(x^{2 n+1}+\frac{1}{x^{2 n+1}}\right) \ln \left(1+e^{n x}\right) d x$.
Solution by Brent Dozier, North Carolina Wesleyan College, Rocky Mount, $N C$.

$$
\text { Let } \begin{aligned}
& f_{n}(x)=\left(x^{2 n+1}\right.\left.+\frac{1}{x^{2 n+1}}\right) \ln \left(1+e^{n x}\right) \text {. Then } \\
& \qquad \begin{aligned}
f_{n}(-x) & =-\left(x^{2 n+1}+\frac{1}{x^{2 n+1}}\right) \ln \left(1+e^{-n x}\right) \\
& =-\left(x^{2 n+1}+\frac{1}{x^{2 n+1}}\right) \ln \frac{1+e^{n x}}{e^{n x}} \\
& =-\left(x^{2 n+1}+\frac{1}{x^{2 n+1}}\right)\left(\ln \left(1+e^{n x}\right)-n x\right) \\
& =-f_{n}(x)+n\left(x^{2 n+2}+x^{-2 n}\right)
\end{aligned}
\end{aligned}
$$

Therefore $f_{n}(x)=-f_{n}(-x)+n\left(x^{2 n+2}+x^{-2 n}\right)$. Integrating

$$
\begin{aligned}
\int_{-1}^{1} f_{n}(x) d x & =-\int_{-1}^{1} f_{n}(-x) d x+n \int_{-1}^{1}\left(x^{2 n+2}+x^{-2 n}\right) d x \\
& =-\int_{-1}^{1} f_{n}(x) d x+n \int_{-1}^{1}\left(x^{2 n+2}+x^{-2 n}\right) d x
\end{aligned}
$$

which gives

$$
\int_{-1}^{1} f_{n}(x) d x=\frac{n}{2} \int_{-1}^{1}\left(x^{2 n+2}+x^{-2 n}\right) d x=n\left(\frac{1}{2 n+3}+\frac{1}{1-2 n}\right) \rightarrow 0
$$

as $n \rightarrow \infty$. Therefore $\lim _{n \rightarrow \infty} \int_{-1}^{1}\left(x^{2 n+1}+\frac{1}{x^{2 n+1}}\right) \ln \left(1+e^{n x}\right) d x=0$.
Also solved by Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; and the proposer.

Problem 838. Proposed by Mathew Cropper, Eastern Kentucky University, Richmond, KY.

Let $A_{n}$ be the number of $n$-bit strings of zeros and ones that contain at least one sequence of three consecutive ones (111) and no sequence of four or more consecutive ones. The sequence starts

$$
A_{1}=0, A_{2}=0, A_{3}=1, A_{4}=2
$$

Using the well-known tribonacci sequence

$$
T_{0}=0, T_{1}=0, T_{2}=1
$$

which counts the number of $(n-3)$-bit strings that contain NO sequence of three consecutive ones, develop a recursive formula for $A_{n}$ and use it to compute $A_{15}$.

## Solution by the proposer.

The recurrence desired is $A_{n}=A_{n-1}+A_{n-2}+A_{n-3}+A_{n-4}+T_{n-1}$. This comes from:
adding a leading 0 to any $(n-1)$-bit string counted by $A_{n-1}$, or
adding a leading 10 to any $(n-2)$-bit string counted by $A_{n-2}$, or
adding a leading 110 to any $(n-3)$-bit string counted by $A_{n-3}$, or
adding a leading 1110 to any $(n-4)$-bit string counted by $A_{n-4}$, or
adding a leading 1110 to any $(n-4)$-bit string which has NO sequence of three consecutive ones.
Since $n-4=(n-1)-3$, the number of such $(n-4)$-bit strings is counted by $T_{n-1}$.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathrm{~A}_{n}$ | 0 | 0 | 0 | 1 | 2 | 5 | 12 | 27 | 59 | 127 | 269 | 563 | 1167 | 2400 | 4903 | 9960 |
| $\mathrm{~T}_{n}$ | 0 | 0 | 1 | 1 | 2 | 4 | 7 | 13 | 24 | 81 | 149 | 274 | 504 | 927 | 1705 | 3136 |

So $A_{15}=9960$.
Also partially solved by Ioannis Sfikas, National and Kapodistrian Uni-
versity of Athens, Greece.
Problem 839. Proposed by the editor.
A recurrence is defined in the following way: $c_{1}=3, c_{n}=4+\sum_{i=1}^{n-1} c_{i}$ for all $n \geqslant 2$. Find a formula for $c_{n}$ for $n \geqslant 2$ that just involves $n$.

Solution by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The solution is $c_{n}=7 \cdot 2^{n-2}$ for all $n \geq 2$. It is enough to see that for for $n>2, c_{n}=2 c_{n-1}$. This comes from

$$
c_{n}=4+\sum_{i=1}^{n-1} c_{i}=4+\sum_{i=1}^{n-2} c_{i}+c_{n-1}=2 c_{n-1} .
$$

Also solved by Brian Beasley, Presbyterian College, Clinton, SC; Carl Libis, Columbia Southern University, Orange Beach, AL; Corneliu ManescuAvram, Ploiesti, Romania; Henry Ricardo, Westchester Area Math Circle, Purchase, NY; Ioannis Sfikas, National and Kapodistrian University of Athens, Greece; Bill Yankosky, North Carolina Wesleyan College, Rocky Mount, NC; and the proposer.

## Kappa Mu Epsilon News

Edited by Cynthia Huffman, Historian

Updated information as of January 2020
News of chapter activities and other noteworthy KME events should be sent to

Cynthia Huffman, KME Historian
Pittsburg State University
Mathematics Department
1171701 S. Broadway
Pittsburg, KS 66762
or to
cjhuffman@pittstate.edu

## Chapter News

AL Beta - University of North Alabama<br>AL Beta - University of North Alabama<br>New Initiates - Sylver Carter, Walker Ericsson, Cyndra Graves, Chase Holcombe, Harris Kain, Bella Martinez, Olivia McGriff, Joshua Morgan, Jacob Morris, Areanna Orozco, Molly Palmer, Jeanette Pina, Jessa Plunket, Kevin Saint, Lucas Scott, Ethan Sutherland, and Sara Woodley.

## AL Gamma - University of Montevallo

Corresponding Secretary - Scott Varagona; 10 New Members; 699 Total Members

New Initiates - Autumn Bruncz, Rachel Cox, Johnathan Ridley Herron, Sumer Hudson, Krenar Krasniqi, William Lowery, Milan Ludlage, Ashlynn Partridge, Victoria Evelina Teran, and Alexander Weldon.

## AL Theta - Jacksonville State University <br> Chapter President - Marcus Shell; 50 Current Members

Other Fall 2019 Officers: Ben Junkins, Vice President; Sabin Banjara, Secretary; LeeAnne Powell, Treasurer; and Dr. David Dempsey, Corresponding Secretary and Faculty Sponsor
The Alabama Theta chapter met biweekly during Fall 2019. Meetings included business (T-shirt and fundraiser ideas), card \& board games, as well as pizza and snacks. Early in the semester, students organized weekly Homework Nights, reserving a classroom for students to get together to study, help each other out, and hold each other accountable; snacks were provided. Our now-traditional "MathCon" (Nov. 1) saw several students dress up as math-related characters, including one near-clone of our pro-
fessors. In December, we hosted an end-of-semester holiday party, including a "Dirty Santa" gift exchange.

## AR Beta - Henderson State University

Chapter President - Bryan Neal; 8 New Members; 56 Total Members
Other Fall 2019 Officers: Kelli LaRue, Vice President; Dallas Crumby, Secretary; Kayla Earnest, Treasurer; Dr. Fred Worth, Corresponding Secretary; and Dr. Carolyn Eoff, Faculty Sponsor
New Initiates - Nadia Ballarin, Carlos Barbosa, Dallas Crumley, Gillian Garamone, Kelli LaRue, C. Brett Little, Nestor Molina, and Jacquelyn Mosely.
CT Beta - Eastern Connecticut State University
Corresponding Secretary and Faculty Sponsor - Dr. Mehdi Khorami; 501
Current Members
CT Gamma - Central Connecticut State University
Chapter President - Nicholas Sabia; 70 Total Members
Other Fall 2019 Officers: Jonathan Maldonado, Vice President; Alyssa
Mercaldi, Secretary; Sabrina Doolgar, Treasurer; Dr. Leah Frazee, Corresponding Secretary; and Dr. Marian Anton, Faculty Sponsor
FL Beta - Florida Southern College
Corresponding Secretary - Dr. Susan Serrano; 8 New Members; 455 Total Members
New Initiates - Braden Arango, Jeffrey Bindeman, Andrew Boesenberg, Alejandra Brewer, Jacqueline Carlton, Ashlee Carnahan, Brittany Drummond, Zachary Fralish, Alexandra Garcia, Alexis Hall, Samantha Hamontree, Kathryn Hoffman, Allie Johnson, Kelly Kramer, Amanda Koski, Risley Mabile, Lillian Mulligan, Brian Roney, John Rosario, Casey Selzak, Anthony Stefan, Katherine Tragakis, and John White.

## FL Delta - Embry-Riddle Aeronautical University

Chapter President-Andrew A. McClary; 84 Total Members
Other Fall 2019 Officers: Hayley Lewis, Vice President; Mariah Marin, Secretary; Taylor Stark, Treasurer; and Dr. Sirani M. Perera, Corresponding Secretary and Faculty Sponsor
In addition to our usual meetings, we went on a tour of the UF Health Proton Institute, in Jacksonville FL and learned about real world applications of mathematics and how a Hydrogen ion is able to create a pinpointed high dose of radiation that can be used in the fight against cancer.

## FL Gamma - Southeastern University

Corresponding Secretary - Dr. Berhane Ghaim; 3 New Members; 63 Total Members
New Initiates - Elizabeth Bernatowicz, Kaitlyn Brett, and Marilyn Ikahane.

## GA Gamma - Piedmont College

Chapter President - Rebecca Bowen; 31 Current Members
Other Fall 2019 Officers: Hope Menzel, Corresponding Secretary and

## Faculty Sponsor

## IA Alpha - University of Northern Iowa

Chapter President - Jaclyn Miller; 25 Current Members; 7 New Members Other Fall 2019 Officers: Mariah Piippo, Vice President; Rachel Liercke, Secretary; Stephanie Peiffer, Treasurer; and Mark D. Ecker, Corresponding Secretary and Faculty Sponsor
Our first fall KME meeting was held on October 9, 2019 in Wright Hall where student member Rachel Liercke presented "Iowa Workforce Development Laborshed Analysis". Student member Mariah Piippo presented her paper, entitled "2-Dimensional Crystallographic Groups" at our second meeting on November 13, 2019. Student member Brynn Harberts addressed the John Cross Fall KME Banquet with "Statistical Analysis of College Volleyball Teams". Our banquet was held at Peppers restaurant in Cedar Falls on December 11, 2019 where seven new members were initiated.

## IA Gamma - Morningside College

Chapter President - Billy Salber; 437 Total Members; 4 New Members
Other Fall 2017 Officers: ETCOther Fall 2019 Officers: Krista Hogstad, Vice President; Samantha Anderson, Secretary; David Swerev, Treasurer; and Mitchel T. Keller, Corresponding Secretary and Faculty Sponsor
New Initiates - Anthony Glackin, Usame Suud, Mitchell Fulton, and Ethan Wyant.

## IA Delta - Wartburg College

Corresponding Secretary - Dr. Brian Birgen; 12 New Members; 756 Total Members
New Initiates - Takeaki Doi, Olivia J Klaas, Darby M Kramer, Emily L Leonhart, Sabah S
Munir, Bailey L Naig, Rachel S Ndjuluwa, Erica J Rittgers, Bridget S Schaufenbuel, Justin M Schoppe, Ali Williams Perez, and Dr. Cristian Allen.

## IL Zeta - Dominican University

Corresponding Secretary - Aliza Steurer and Mihaela Blanariu; 21 Current Members
Other Fall 2017 Officers: ETC
In spring 2019, two new members were initiated. Dr. Amanda Harsy, Assistant Professor of Mathematics at Lewis University in Romeoville, Illinois, will give a talk at the Dominican University KME initiation ceremony on April 23, 2020 at 6 p.m. Dominican University is located in River Forest, IL. The entire Dominican University community, as well as folks from nearby schools, are welcome. Food will be served.

## KS Alpha - Pittsburg State University

Faculty Sponsor - Dr. Scott Thuong; 8 New Members, 2144 Total Members
The KS Alpha chapter will be hosting the North Central KME Regional

Convention on April 17-18, 2020.
New Initiates - Sarah Case, Andrew Chesney, Tyler Clark, Rylee Dennis, Sloan Geddry, Skyler Hausback, Morgan Panovich, and Morgan Singletary.
KS Beta - Emporia State University
Chapter President - Katherine Beckley; 32 Current Members; 7 New Members
Other Fall 2019 Officers: Alec Bergeron, Vice President; Elisabeth Evans, Secretary; Amber Innes, Treasurer; Tom Mahoney, Corresponding Secretary; and Brian Hollenbeck, Faculty Sponsor

## KS Delta - Washburn University

Chapter President - Jacob Talkin; 20 Current Members
Other Fall 2019 Officers: Abby Beliel, Vice President;Madison Henley, Secretary; Mary Greene, Treasurer; and Kevin Charlwood, Corresponding Secretary and Faculty Sponsor
The Kansas Delta chapter of KME met with our math club for four luncheon meetings during fall 2019 to hear speakers from Security Benefit, SE2 and Megan Jones Advisory Group (financial services). Dr. Charlwood also gave a presentation on how to solve cubic equations in radical form.

## KY Beta - University of the Cumberlands

Corresponding Secretary - Dr. Jonathan Ramey; 15 New Members; 264 Total Members
Other Fall 2017 Officers: ETC
New Initiates - Ethan F. Brown, Amber Bunch, Alexander G. Franklin, Cortina L. Hall, Bradley Karr, Stuart Christopher Lockhart, Matthew Maher, Rachel M. Pingleton, Patrick R. Rowe, Joshua Ramsey, David Andrew Tarrence, Hannah Spangler, Deborah Wilkerson, Jon Kenyon Wilson, and YuChen Wu.

## MA Alpha - Assumption College

Corresponding Secretary - Dr. Joseph Alfano; 13 New Members; 345 Total Members
New Initiates - Samantha H Bengiovanni, Jordan M Burt, Zachary W Durand, Callie A Dwyer, Catherine A Harvey, Caroline R James, Sarah E Keohane, Brooke J Mullen, Sheila R Orlando, Miranda Page, Matthew W Pugliese, Gianna Rousseau, and Michaela F Smith.
MD Alpha - Notre Dame of Maryland University
Chapter President - Amanda Ashton; 9 Current Members
Other Fall 2019 Officers: Hannah Campbell, Vice President; Emily Garzon, Secretary; Aisha Aizhar, Treasurer; and Charles Buehrle, Corresponding Secretary and Faculty Sponsor
Flyers from MD Alpha events are below:


## MA Beta - Assumption College

Corresponding Secretary - Spencer Hamblen; 10 New Members; 436 Total Members
New Initiates - Lucas Anthony, Shannon Bernier, Blair Boyle, Nicholas Cummings, Moira DiGiacomantonio, SoYoung Jeon, Andrew Murphy, Greta Ouimette, Luke Shuck, and Ashley Wright.

## MD Delta - Frostburg State University

Chapter President - Jordan Thomas; 19 Current Members
Other Fall 2019 Officers: Katelynn Suesse, Vice President; Bailey Brewer, Secretary; Chad Shumaker, Treasurer; Mark Hughes, Corresponding Secretary and Faculty Sponsor; and Frank Barnet, Faculty Sponsor
Maryland Delta Chapter held monthly meetings during the fall semester. Each meeting featured puzzles, games and pizza. Though non-members have always been welcome to attend meetings, our current group of officers decided to focus on getting the word out about KME. The result was excellent in that several non-members attended meetings and we expect most of them to join us officially at the Induction Ceremony to be held in the spring of 2020. Our bake sale in October was the most successful one that we ever had. Chapter members represented the Mathematics Department at the university's Majors Fair held in November. Congratulations and best wishes to graduating Maryland Delta members Ryan Miller and Matt Beall.

## MD Epsilon - Stevenson University

Chapter President - Alayna Roesener; 161 Current Members; 10 New Members
Other Fall 2019 Officers: William Heidel, Vice President; Darian Hileman, Secretary; Katlyn Leftridge, Treasurer; Benjamin Wilson, Corre-
sponding Secretary and Faculty Sponsor
Our chapter of KME sponsored our Hurricane Relief bake sale for Hurricane Dorian in which we raised $\$ 500$.
MI Beta - Central Michigan University
Chapter President - Natalie DeVos; 15 Current Members; 0 New Members
Other Fall 2019 Officers: Austin Konkel, Vice President; Emily Naegelin, Secretary; Evan Miller, Treasurer; and Dr. Ben Salisbury, Corresponding Secretary and Faculty Sponsor
KME participated in the CMU MainStage event at the start of Fall 2019. The first meeting of the semester was held on September 3, and featured several ice breakers and opportunities for new and old members to become acquainted. KME held a book sale in from September 16 until September 18 to raise funds for their activities. On September 14, KME Secretary Emily Naegelin used simple crafts to demonstrate the mathematical patterns in nature. On October 1, KME Vice President Austin Konkel and Treasurer Evan Miller presented on their results from summer research and discussed their experience with research in mathematics. On October 15, KME invited guest speaker Professor Emeritus Robert Chaffer to give a talk about his artwork and the mathematical ideas he uses to create his pieces. On October 29, the meeting was devoted to the interests of the EBoard. Each officer gave a brief presentation on a mathematical idea that intrigues them. With the semester winding down, KME held a game night on November 12. The game was mathematical Jeopardy! Math-a-palooza was co-organized by KME and the AMS Graduate Student Chapter. The Fall 2019 event was held on December 6.

## MI Delta - Hillsdale College

Chapter President - Benjamin Becker; 47 Current Members; 10 New Members
Other Fall 2019 Officers: Emma Clifton, Vice President; Olivia Mulley, Secretary; Nicholas West, Treasurer; and Dr. Kevin Gerstle, Corresponding Secretary and Faculty Sponsor
The Michigan Delta chapter initiated 10 new members in the Fall 2019 semester on October 24. Dr. Ryan Hutchinson gave an accompanying talk "Principal Component Analysis: An Application of Matrix Algebra in Data Analysis."

## MO Beta - University of Central Missouri

Corresponding Secretary - Rhonda McKee; 7 New Members; 1519 New Memberss
New Initiates - Georgiana Bray, Peter Gossell, Ryan Naugher, Victor Everett Ortiz, Abigail

Stevens, Bo Varvil, and Briana Ward.
MO Theta - Evangel University
Chapter President - Heather Culbertson; 13 Current Members
Other Fall 2019 Officers: Jacob Crews, Vice President; and Don Tosh, Corresponding Secretary and Faculty Sponsor
Meetings were held monthly. In December we held a pasta party at the home of Daniel Bowerman.
MO Lambda - Missouri Western State University
Corresponding Secretary - Dr. Steve Klassen; 6 New Members; 369 Total Members
New Initiates - Nathanial Jelinek, Nicholas Kempf, Randy Rouse, Kaitlyn Schildknecht, Cecilia Tackett, and Kady Vandendaele.
MO Nu - Columbia College
Corresponding Secretary - Kenny Felts; 5 Current Members
MS Gamma - University of Southern Mississippi
Chapter President - Yumi Maharjan; 18 Current Members
Other Fall 2019 Officers: Hamas Tahir, Vice President; Gokul Bhusal, Secretary; Amit Tripathi, Treasurer; Zhifu Xie, Corresponding Secretary; and Ana Wan, Faculty Sponsor
The Chapter of Mississippi Gamma organized a movie night for all students who are interested in mathematics. Three members volunteered in the Golden Eagle Day at the University of Southern Mississippi to help school recruit students from high school visitors.
MS Delta - William Carey University
Corresponding Secretary - Janie Bower; 12 New Members, 214 Total Members
New Initiates - Caitlyn Castille, Nelson Conley, Kara Crosby, Anthony Jones, Halethe Jones, Jenna Lee, Abby Odom, Hunter Phelps, Jakolbia Shipmon, Mallory Smith, Ashlyn Stringfellow, and Mallory Thompson.
MS Epsilon - Delta State University
Corresponding Secretary - Lee Virden; 4 New Members; 117 Total Members
New Initiates - Virginia Baker, James Walker Dean, Allison Duthu, and Ida B. Nielsen.
NE Beta - University of Nebraska Kearney
Chapter President - Tiffany Collins; 5 New Members; 927 Total Memers
Other Fall 2019 Officers: Joshua Garcia, Vice President; Evan Olson, Secretary; and Julie Kent, Treasurer; and Dr. Katherine Kime, Corresponding Secretary and Faculty Sponsor
KME member Ryan Clark was the Fall 2019 Commencement Speaker in December. There were approximately 360 graduates, and hundreds of family members in attendance in the large arena. Ryan is interested in
graduate school in finance. Also, KME member Alex Sellers was one of the December graduates. Meetings were well attended this semester. A major topic was determining a fund raiser for travel to the Regional Convention at Pittsburg State in April 2020 (In January, the decision was made to sell cookie dough from Eileen's, a local shop). Several KME members serve as tutors in the Learning Commons.
New Initiates - - Paige Arnold, Lena Janssen, Amanda Larson, Carli Pofahl, and Alex Sellers.

## NE Delta - Nebraska Wesleyan University

Chapter President - Drew Damme; 18 Current Memers
Other Fall 2019 Officers: Alex Kerr, Vice President; Samantha Wright, Secretary/ Treasurer; and Dr. Melissa Erdmann, Corresponding Secretary and Faculty Sponsor
In December, we had our joint holiday party with the Physics Club. One professor made chili, and all of the faculty brought sides. Physics and mathematics carols were enjoyed by all. Earlier in the semester we had a BINGO event and a problem solving event that were well-attended.
NY Kappa - Pace University
Corresponding Secretary - Shamita Dutta Gupta; 1 New Member; 397 Total Members
New Initiates - Jian Tong Liu.

## NY Omicron - St. Joseph's College

Chapter President - Christiana R. Morante; 15 Current Members Other Fall 2019 Officers: Frank D. Loglisci, Vice President; Abbey V. Knowles, Secretary; Scott T. McDonald, Treasurer; Dr. Elana Reiser, Corresponding Secretary; and Dr. Donna Pirich, Faculty Sponsor
The New York Omicron chapter of KME held our annual Cookie Dough and Popcorn Fundraiser in October, from which we raised money to buy toys to be given as Christmas gifts to children who have been affected by domestic violence. Shopping took place in November and December. The toys will be given out as part of KME-SJC graduate Assemblyman Doug Smith's eighth annual Holiday Toy Drive. Doug accepted the gifts at our annual Christmas Toy Drive Celebration on December 12th. Photos are included below. In addition, our KME members hosted our Math Clinic on Saturday mornings throughout the Fall 2019 semester. The Math Clinic offers free tutoring in mathematics to local high school students. (NY Omicron pictures are below.)


PA Delta - Marywood University
Corresponding Secretary - Dr. Dhanapati Adhikari; 4 New Members; 306
Total Members
New Initiates - Zachary Beja, Kimberly Sandone-Lee, Mikayla Nardone, and Samantha Wigley.
PA Eta - Grove City College
Corresponding Secretary - Dale L. McIntyre; 11 New Members; 852 Total Members
New Initiates - Ethan Greenly, Nicholas Grube, Jared Kettinger, Melissa Martin, Courtney Mattey, Caleb Miller, Corrine Mummau, Micah Nelson, Isabella Patnode, Alan Potok, and Elise Wiggins.

## PA Gamma - Waynesburg University

Corresponding Secretary - James R. Bush; 3 New Members; 545 Total

## Members

New Initiates - Teagan Rae Jenner, Courtney Lynn Syfert, and Carly Breach.

## PA Iota - Shippensburg University

Corresponding Secretary - Paul Taylor; 7 New Members; 761 Total Members

New Initiates - Zachary Amisano, Conner Chapman, Crystal Evans, Rebecca Feaser, Krista Moll, Josue Murillo, and Sumer Rininger.

## PA Kappa - Holy Family University

Chapter Presidents - Melissa Cahill; 4 Current Members
Other Fall 2019 Officer: Sister Marcella Louise Wallowicz CSFN, PhD, Corresponding Secretary and Faculty Sponsor
The Chapter did not conduct any formal Fall activities. One inductee for the Spring semester completed his pre-induction service project: 10 hours of volunteer math tutoring in our Center for Academic Enhancement. Two other potential inductees will collaborate on a Spring project.

## PA Lambda - Bloomsburg University

Corresponding Secretary - Dr. Eric B. Kahn; 6 New Members; 759 Total Members

New Initiates - Caleb Beard, Dario D'Amato, Brianna Denniston, Chandler Hughes, Schyler Kelsch, and Connor Landis.

## PA Rho - Thiel College

Chapter President - Breanna Mesich; 2 New Members; 132 Total Members
Other Fall 2019 Officers: Taylor Guth, Vice President; Courtney Harriman, Secretary; Emily Groves, Treasurer; Russ Richins, Corresponding Secretary; and Dr. Jie Wu, Faculty Sponsor
This semester we had several meetings as well as our usual "Challenge 24 " activity, which also doubles as a food drive for the local food bank.
New Initiates - Emily Groves and Macy Siefert.

## PA Sigma - Lycoming College

Corresponding Secretary - Dr. Christopher Reed; 13 New Members; 133 Total Members
New Initiates - John Balas, Madison Brown, Shannon Coriddi, Mackaella Goodwin, Narshini Gunputh, Kelly Hoffman, Keely Laidacker, Maya Merhi, Elena Pikounis, Ansharah Saib, Jeniffer Schwartz, Sheila Whitman, and Nathaniel Wilston.

## RI Beta - Bryant University

Chapter President - Christopher Ethier; 16 Current Members
Other Fall 2019 Officers: Constance Tang, Vice President; Alexandra
Sherman, Secretary; Liam Mahler, Treasurer; Professor John Quinn, Corresponding Secretary; and Professor Alan Olinsky, Faculty Sponsor We are planning to host the New England KME Conference in spring 2022.

We also hope to send some students to the regional conference at Molloy College during the spring 2020.
TN Gamma - Union University
Chapter President - Jenna Dula
Other Fall 2019 Officers: Josie Carrier, Vice President; Ainsley Duncan, Secretary and Treasurer; John Mayer, Webmaster and Historian; Bryan Dawson, Corresponding Secretary; and Matt Lunsford, Faculty Sponsor
TX Kappa - University of Mary Hardin-Baylor
Corresponding Secretary - Dr. Peter H. Chen; 7 New Members; 274 Total Members
New Initiates - Jacob Fitzwater, Mariah Harris, Jenica McGill, Maricela Ramirez, Jonathan Rosales, Ashlyn Strittmatter, and Ashley Winkle.
TX Lambda - Trinity University
Corresponding Secretary - Dr. Hoa Nguyen; 10 New Members; 290 Total Members
New Initiates - Thomas Baer, Melinda Benavides, Alyson Laskowski, David Migl, Emi Mondragon, Michelle Nguyen, Regan Ramirez, Nathan Richter, Derek Weix, and Jasmine Yang.
VA Beta - Radford University
Chapter President - Morgyn Church; 4 New Members, 576 Total Members Other Fall 2019 Officer: Eric P. Choate, Corresponding Secretary and Faculty Sponsor
New Initiates - Ameen Ahmed, Sara Church, Katherine Mankowski, and Winston Smith.

## VA Delta - Marymount University

Corresponding Secretary - Jacquelyn Rische; 4 New Members; 51 Total Members
New Initiates - Jennifer Martin, Jasmine Roy, Joseph Scafetta, and Matthew Schneider.

# Active Chapters of Kappa Mu Epsilon 

## Listed by date of installation

Chapter
OK Alpha
IA Alpha
KS Alpha
MO Alpha
MS Alpha
NE Alpha
KS Beta
AL Alpha
NM Alpha
IL Beta
AL Beta
AL Gamma
OH Alpha
MI Alpha
MO Beta
TX Alpha
KS Gamma
IA Beta
TN Alpha
MI Beta
NJ Beta
IL Delta
KS Delta
MO Gamma
TX Gamma
WI Alpha
OH Gamma
MO Epsilon
MS Gamma
IN Alpha
PA Alpha
IN Beta
KS Epsilon
PA Beta
VA Alpha
IN Gamma
CA Gamma
TN Beta
PA Gamma
VA Beta
NE Beta
IN Delta
OH Epsilon
MO Zeta
Ma

| Installation Date |  |
| :---: | :---: |
| Northeastern State University, Tahlequah | 18 Apr 1931 |
| University of Northern Iowa, Cedar Falls | 27 May 1931 |
| Pittsburg State University, Pittsburg | 30 Jan 1932 |
| Missouri State University, Springfield | 20 May 1932 |
| Mississippi University for Women, Columbus | 30 May 1932 |
| Wayne State College, Wayne | 17 Jan 1933 |
| Emporia State University, Emporia | 12 May 1934 |
| Athens State University, Athens | 5 Mar 1935 |
| University of New Mexico, Albuquerque | 28 Mar 1935 |
| Eastern Illinois University, Charleston | 11 Apr 1935 |
| University of North Alabama, Florence | 20 May 1935 |
| University of Montevallo, Montevallo | 24 Apr 1937 |
| Bowling Green State University, Bowling Green | 24 Apr 1937 |
| Albion College, Albion | 29 May 1937 |
| University of Central Missouri, Warrensburg | 10 Jun 1938 |
| Texas Tech University, Lubbock | 10 May 1940 |
| Benedictine College, Atchison | 26 May 1940 |
| Drake University, Des Moines | 27 May 1940 |
| Tennessee Technological University, Cookeville | 5 Jun 1941 |
| Central Michigan University, Mount Pleasant | 25 Apr 1942 |
| Montclair State University, Upper Montclair | 21 Apr 1944 |
| University of St. Francis, Joliet | 21 May 1945 |
| Washburn University, Topeka | 29 Mar 1947 |
| William Jewell College, Liberty | 7 May 1947 |
| Texas Woman's University, Denton | 7 May 1947 |
| Mount Mary College, Milwaukee | 11 May 1947 |
| Baldwin-Wallace College, Berea | 6 Jun 1947 |
| Central Methodist College, Fayette | 18 May 1949 |
| University of Southern Mississippi, Hattiesburg | 21 May 1949 |
| Manchester College, North Manchester | 16 May 1950 |
| Westminster College, New Wilmington | 17 May 1950 |
| Butler University, Indianapolis | 16 May 1952 |
| Fort Hays State University, Hays | 6 Dec 1952 |
| LaSalle University, Philadelphia | 19 May 1953 |
| Virginia State University, Petersburg | 29 Jan 1955 |
| Anderson University, Anderson | 5 Apr 1957 |
| California Polytechnic State University, San Luis Obispo | 23 May 1958 |
| East Tennessee State University, Johnson City | 22 May 1959 |
| Waynesburg College, Waynesburg | 23 May 1959 |
| Radford University, Radford | 12 Nov 1959 |
| University of Nebraska-Kearney, Kearney | 11 Dec 1959 |
| University of Evansville, Evansville | 27 May 1960 |
| Marietta College, Marietta | 29 Oct 1960 |
| University of Missouri-Rolla, Rolla | 19 May 1961 |

NE Gamma
MD Alpha
CA Delta
PA Delta
PA Epsilon
AL Epsilon
PA Zeta
TN Gamma
IA Gamma
MD Beta
IL Zeta
SC Beta
PA Eta
NY Eta
MA Alpha
MO Eta
IL Eta
OH Zeta
PA Theta
PA Iota
MS Delta
MO Theta
PA Kappa
CO Beta
KY Alpha
TN Delta
NY Iota
SC Gamma
IA Delta
PA Lambda
OK Gamma
NY Kappa
TX Eta
MO Iota
GA Alpha
WV Alpha
FL Beta
WI Gamma
MD Delta
IL Theta
PA Mu
AL Zeta
CT Beta
NY Lambda
MO Kappa
CO Gamma
NE Delta
TX Iota
PA Nu
VA Gamma
Mama

| Chadron State College, Chadron | 19 May 1962 |
| :---: | :---: |
| College of Notre Dame of Maryland, Baltimore | 22 May 1963 |
| California State Polytechnic University, Pomona | 5 Nov 1964 |
| Marywood University, Scranton | 8 Nov 1964 |
| Kutztown University of Pennsylvania, Kutztown | 3 Apr 1965 |
| Huntingdon College, Montgomery | 15 Apr 1965 |
| Indiana University of Pennsylvania, Indiana | 6 May 1965 |
| Union University, Jackson | 24 May 1965 |
| Morningside College, Sioux City | 25 May 1965 |
| McDaniel College, Westminster | 30 May 1965 |
| Dominican University, River Forest | 26 Feb 1967 |
| South Carolina State College, Orangeburg | 6 May 1967 |
| Grove City College, Grove City | 13 May 1967 |
| Niagara University, Niagara University | 18 May 1968 |
| Assumption College, Worcester | 19 Nov 1968 |
| Truman State University, Kirksville | 7 Dec 1968 |
| Western Illinois University, Macomb | 9 May 1969 |
| Muskingum College, New Concord | 17 May 1969 |
| Susquehanna University, Selinsgrove | 26 May 1969 |
| Shippensburg University of Pennsylvania, Shippensburg | 1 Nov 1969 |
| William Carey College, Hattiesburg | 17 Dec 1970 |
| Evangel University, Springfield | 12 Jan 1971 |
| Holy Family College, Philadelphia | 23 Jan 1971 |
| Colorado School of Mines, Golden | 4 Mar 1971 |
| Eastern Kentucky University, Richmond | 27 Mar 1971 |
| Carson-Newman College, Jefferson City | 15 May 1971 |
| Wagner College, Staten Island | 19 May 1971 |
| Winthrop University, Rock Hill | 3 Nov 1972 |
| Wartburg College, Waverly | 6 Apr 1973 |
| Bloomsburg University of Pennsylvania, Bloomsburg | 17 Oct 1973 |
| Southwestern Oklahoma State University, Weatherford | 1 May 1973 |
| Pace University, New York | 24 Apr 1974 |
| Hardin-Simmons University, Abilene | 3 May 1975 |
| Missouri Southern State University, Joplin | 8 May 1975 |
| State University of West Georgia, Carrollton | 21 May 1975 |
| Bethany College, Bethany | 21 May 1975 |
| Florida Southern College, Lakeland | 31 Oct 1976 |
| University of Wisconsin-Eau Claire, Eau Claire | 4 Feb 1978 |
| Frostburg State University, Frostburg | 17 Sep 1978 |
| Benedictine University, Lisle | 18 May 1979 |
| St. Francis University, Loretto | 14 Sep 1979 |
| Birmingham-Southern College, Birmingham | 18 Feb 1981 |
| Eastern Connecticut State University, Willimantic | 2 May 1981 |
| C.W. Post Campus of Long Island University, Brookville | 2 May 1983 |
| Drury University, Springfield | 30 Nov 1984 |
| Fort Lewis College, Durango | 29 Mar 1985 |
| Nebraska Wesleyan University, Lincoln | 18 Apr 1986 |
| McMurry University, Abilene | 25 Apr 1987 |
| Ursinus College, Collegeville | 28 Apr 1987 |
| Liberty University, Lynchburg | 30 Apr 1987 |

NY Mu
OH Eta
OK Delta
CO Delta
PA Xi
MO Lambda
TX Kappa
SC Delta
NY Nu
NH Alpha
LA Gamma
KY Beta
MS Epsilon
PA Omicron
MI Delta
MI Epsilon
MO Mu
GA Beta
AL Eta
PA Pi
TX Lambda
GA Gamma
LA Delta
GA Delta
TX Mu
CA Epsilon
PA Rho
VA Delta
NY Omicron
IL Iota
WV Beta
SC Epsilon
PA Sigma
MO Nu
MD Epsilon
NJ Delta
NY Pi
OK Epsilon
HA Alpha
NC Epsilon
NY Rho
NC Zeta
RI Alpha
NJ Epsilon
NC Eta
AL Theta
GA Epsilon
FL Gamma
MA Beta
AR Beta
Ma


14 May 1987
15 Dec 1987
10 Apr 1990
27 Apr 1990
30 Oct 1990
10 Feb 1991
21 Feb 1991
28 Apr 1991
14 May 1992
16 Feb 1993
24 Mar 1993
3 May 1993
19 Nov 1994
10 Apr 1997
30 Apr 1997
28 Mar 1998
25 Apr 1998
25 Apr 1998
4 May 1998
19 Apr 1999
22 Nov 1999
7 Apr 2000
11 Feb 2001
21 Apr 2001
28 Apr 2001
21 Apr 2003
13 Feb 2004
26 Mar 2004
1 May 2004
26 Feb 2005
11 Mar 2005
18 Mar 2005
1 Apr 2005
29 Apr 2005
3 Dec 2005
1 Dec 2006
20 Mar 2007
20 Apr 2007
22 Oct 2007
24 Mar 2008
21 Apr 2009
17 Sep 2009
13 Nov 2009
22 Feb 2010
18 Mar 2010
29 Mar 2010
30 Mar 2010
31 Mar 2010
8 Apr 2011
10 Oct 2011

| PA Tau | DeSales University, Center Valley | 29 Apr 2012 |
| :---: | :---: | :---: |
| TN Zeta | Lee University, Cleveland | 5 Nov 2012 |
| RI Beta | Bryant University, Smithfield | 3 Apr 2013 |
| SD Beta | Black Hills State University, Spearfish | 20 Sept 2013 |
| FL Delta | Embry-Riddle Aeronautical University, Daytona Beach | 22 Apr 2014 |
| IA Epsilon | Central College, Pella | 30 Apr 2014 |
| CA Eta | Fresno Pacific University, Fresno | 24 Mar 2015 |
| OH Theta | Capital University, Bexley | 24 Apr 2015 |
| GA Zeta | Georgia Gwinnett College, Lawrenceville | 28 Apr 2015 |
| MO Xi | William Woods University, Fulton | 17 Feb 2016 |
| IL Kappa | Aurora University, Aurora | 3 May 2016 |
| GA Eta | Atlanta Metropolitan University, Atlanta | 1 Jan 2017 |
| CT Gamma | Central Connecticut University, New Britan | 24 Mar 2017 |
| KS Eta | Sterling College, Sterling | 30 Nov 2017 |
| NY Sigma | College of Mount Saint Vincent, The Bronx | 4 Apr 2018 |
| PA Upsilon | Seton Hill University, Greensburg | 5 May 2018 |
| KY Gamma | Bellarmine University, Louisville | 23 April 2019 |


[^0]:    1 For examples of such tableaux relating to matrix representations of the symmetric groups, see for instance [12]; for many examples arising in Lie theory, see [9].

