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Why Do We Use RSA?

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Abstract

The RSA encryption algorithm was created in the late 1970's and has been in use since. This paper explores the implementation of RSA using an example. In addition we explore a few different attacks with a focus on the generic break which is factoring a large integer n.

1. Introduction

In [5], Diffie and Hellman described how to build a machine that would break the National Bureau of Standards (NBS) encryption standard which was being proposed for use at the time. They estimated the cost at $20 million with a decrease over 10 years to just $200,000, and the computation time was given as 12 hours per cipher.

The next year, the duo published *New Directions in Cryptography* [6] in which they outlined the 4 main components needed for a public key cryptosystem, which are paraphrased below:

1. Every encrypted message can be decrypted,
2. It is computationally easy to compute the encryption key from the decryption key,
3. It is computationally easy to encrypt and decrypt a message, and
4. It is computationally infeasible to compute the decryption key from the encryption key.

Later that year, Rivest, Shamir, and Adleman answered the call for this cryptosystem and published *A Method for Obtaining Digital Signatures and Pubic-Key Cryptosystems* [14] which lays out what is now know as the RSA encryption algorithm. During our paper we will explore this algorithm with a simplified example and examine why RSA holds for the first three points. We will finish with a look at why RSA is believed to hold for point four and, more specifically, we will look at a way to factor a large integer n.

2. Definitions

The following definitions were found in [7], [8], and [11] and will be used throughout this paper.

1. For any positive integer n the **ring of integers modulo** n, denoted $\mathbb{Z}_n$, is the set $\{z \in \mathbb{Z} | 0 \leq z < n\}$.
2. The **group of units** for a positive integer n, denoted by $U(n)$, is the set $\{z \in \mathbb{Z}_n | \gcd(n, z) = 1\}$.
3. **Euler's totient function** for a positive integer n, denoted $\phi(n)$, is the number of integers less than n and greater than 0 that are relatively prime to n.
4. For a set $\mathcal{B}$ of primes, a positive integer is $\mathcal{B}$**-smooth** if it can be written as the product of primes in $\mathcal{B}$.
5. For any integer n and prime p the **Legendre symbol**, denoted $\left(\frac{n}{p}\right)$, has a value of 1 if n not a multiple of p and is a quadratic residue of p, meaning there exists some $c \in \mathbb{Z}$ such that $c^2 \equiv n \pmod{p}$. The value is -1 if n is not a multiple of p and is a quadratic non-residue of p and equal to 0 if $p|n$.

3. How RSA works, with example

Ultimately, we are looking for a public encryption key $< n, e >$ which will be distributed so others may encrypt a message. Additionally, we want the only person able to decrypt the message to be someone who is in possession of the private decryption key $< n, d >$.

The first step for RSA is to pick two large prime numbers p and q; for our example we will let $p = 5077$ and $q = 5783$. Since we are using such small numbers we used the Sieve of Eratosthenes, outlined in [9], to determine these numbers are prime. However, there are more efficient ways to determine primality in larger numbers; some are described in [12].

Next we compute

$$
\begin{aligned}
n &= pq \\
n &= 5077 \cdot 5783 \\
n &= 29360291
\end{aligned}
$$

This tells us the integers we will use to encipher our message are in $\mathbb{Z}_{29360291}$, which we will refer to as $\mathbb{Z}_n$ throughout the example.

Next we need our encryption and decryption keys and these numbers will come from the $U(\phi(n))$. The following is a well known lemma about Euler's totient function, we include the proof for completeness.

Lemma 1 *Given $n = pq$ where p, q are prime, $\phi(n) = (p-1)(q-1)$*

Proof. Since p and q are prime, we know that $\phi(p) = p - 1$ and $\phi(q) = q - 1$ because every positive integer less than them is relatively prime to them.
We know the number of integers less than n and greater than 0 is $n - 1$ so we will begin there and remove all numbers that are not relatively prime to p or q since this would be the only way to not be relatively prime to n. Consider the sequence $p, 2p, 3p, \ldots, p(q-1)$, which are all less than n but not relatively prime to n so we must remove $q - 1$ numbers. Similarly, we examine $q, 2q, 3q, \ldots, q(p-1)$ and see we must also remove $p - 1$ numbers.
$\phi(n) = n - 1 - [(p-1) + (q-1)]$
$\phi(n) = n - 1 - p + 1 - q + 1$
$\phi(n) = pq - p - q + 1$
$\phi(n) = (p-1)(q-1)$
Thus the lemma is true. ■

Since n is the product of two primes, by Lemma 1 we know

$$
\begin{aligned}
\phi(n) &= (p-1)(q-1) \\
&= 5076 \cdot 5782 \\
&= 29349432
\end{aligned}
$$

Next, we need to choose an encrpytion key $e \in U(\phi(n))$; we choose $e = 4441$. When choosing e for a real implementation we must take care because we want e's binary expansion to have few ones (a low Hamming weight); this makes encryption faster as described in [16]. Also, we don't want it too small because it opens the algorithm to certain attacks outlined in [3].

Now we need to find d such that $e \cdot d \equiv 1 \pmod{\phi(n)}$ and, since we choose e from $U(\phi(n))$, we are assured that the $\gcd(\phi(n), e) = 1$, which means we can use the Extended Euclidean Algorithm (EEA) to write $1 = a\phi(n) + de$ as explained in [8] to find d.

Using this, we find $1 = 332\phi(n) - 2194103e$ which gives us:

$$d = -2194103 \mod \phi(n)$$
$$d = 27155329$$

Now we have almost all of the components needed to encrypt a message.

The last thing we need is a message (M) and that message needs to be converted into integers so we can use the algorithm and turn it into a cyphertext (C). To get the letters into numbers we will use the American Standard Code for Information Interchange (ASCII).

$$\text{Mtext : Hello World!}$$
$$\text{Mnums : 072/101/108/108/111/032/087/111/114/108/100/033}$$

Before we take these numbers through the algorithm, we need to think about how to group them. The easiest would be to just encode each letter individually, however, this is open to a frequency analysis attack, given in [8], which is fairly quick so we don't want to do that. For this example we will group by two letters so we stay in $\mathbb{Z}_n$ but, this approach is only marginally better than individual letters. To be truly effective the message needs to be padded with some randomness before encryption, as explained in [2].

M grouped: 072101/108108/111032/087111/114108/100033

$m_1 = 72101$, $m_2 = 108108$ $m_3 = 111032$, $m_4 = 87111$, $m_5 = 114108$, $m_6 = 100033$

To find the ciphertext we use $C = M^e \mod n$. There are a few different ways to do modulo powers quickly as shown in [16] but we used the Fast Power Algorithm (FPA) found in [8]. We will walk through the FPA to get c_1 where

$$c_1 = 72101^{4441} \mod 29360291$$

The first step is the rewrite the exponent in binary:

$$(4441)_{10} = (1000101011001)_2$$

We then create a table with the base number (72101) raised to powers of 2 $\mod 29360291$.

i	1	2	3	4	5	6
$72101^{2^i} \mod 29360291$	1782694	10639505	24290160	1733857	1180377	26613015
i	7	8	9	10	11	12
$72101^{2^i} \mod 29360291$	22214261	7190711	4286294	26080313	28491973	3876244

Table 1

Using this and properties of exponents we can rewrite c_1 as:

$$\begin{aligned} c_1 &= 72101^{(1000101011001)_2} \mod 29360291 \\ &= 72101^{2^{12}+2^8+2^6+2^4+2^3+2^0} \mod 29360291 \\ &= 3876244 \cdot 7190711 \cdot 26613015 \cdot 1733857 \\ &\quad \cdot 24290160 \cdot 72101 \mod 29360291 \\ &= 4171307 \end{aligned}$$

We use the same process to get $c_2, \ldots, c_6$

$$\begin{aligned} c_2 &= 108108^{4441} \mod 29360291 = 23892026 \\ c_3 &= 111032^{4441} \mod 29360291 = 20217151 \\ c_4 &= 87111^{4441} \mod 29360291 = 13016824 \\ c_5 &= 114108^{4441} \mod 29360291 = 10672456 \\ c_6 &= 100033^{4441} \mod 29360291 = 10797272 \end{aligned}$$

C = 4171307/23892026/10052990/26531317/22245108

Once the message is encrypted, it is sent to its intended recipient who is in possession of $< n, d >$ and can decrypt the message with the following $M = C^d \mod n$.

$$\begin{aligned} m_1 &= 4171307^{27155329} \mod 29360291 = 72101 \\ m_2 &= 23892026^{27155329} \mod 29360291 = 108108 \\ m_3 &= 20217151^{27155329} \mod 29360291 = 111032 \\ m_4 &= 13016824^{27155329} \mod 29360291 = 87111 \\ m_5 &= 10672456^{27155329} \mod 29360291 = 114108 \\ m_6 &= 10797272^{27155329} \mod 29360291 = 100033 \end{aligned}$$

4. How RSA makes a good public system

Now that we have a better understanding of how RSA works and the notation used we are going to show that it follows the first three points to be a good public-key cryptosystem. That is, if E is the encryption system, D is the decryption system, m is the plain text and c is the cipher text then we have:

1. $D(E(m)) = m$
2. Given $< n, d >$ finding $< n, e >$ is easy
3. Evaluating $E(m)$ and $D(c)$ is computationally easy

Point one is proven for all cases in [14]. We include a proof for the special case when m is relatively prime to n, meaning $m \in U(n)$ which has order $\phi(n)$ from [7]. Recall $c := E(m) = m^e \pmod{n}$, $D = c^d \mod n$, and d, e were chosen such that $ed \equiv 1 \mod \phi(n)$ meaning we can write $ed = 1 + k\phi(n)$ where $k \in \mathbb{Z}$.

$$\begin{aligned} D(E(m)) &\equiv (D(m^e)) \pmod{n} \\ &\equiv (m^e)^d \pmod{n} \\ &\equiv m^{ed} \pmod{n} \\ &\equiv m^{1+k\phi(n)} \pmod{n} \\ &\equiv m \cdot 1 \pmod{n} \end{aligned}$$

The last by the fact that any element raised to the group order is identity [7]. Thus $D(E(m)) = m$.

For points two and three, we will take "easy" to mean it can be done with a finite number of steps in a reasonable amount of time. These points are left with a bit of leeway since certain applications will accept a longer run time in order to get more security by using bigger numbers.

From Lamé [10] we know that the number of operations required to find $\gcd(u, v)$, with v being the smaller number, is $5 \lfloor \log v \rfloor + 5$. Since RSA is generally implemented with $p, q \approx 10^{100}$ we will assume this size for our input v (so that we cover the worst case scenario for the size of e).

$5 \lfloor \log 10^{100} \rfloor + 5 = 5(100) + 5 = 505$

So it will take roughly 500 steps to find the $\gcd(e, \phi(n))$ which is how we find $< n, e >$ given $< n, d >$. Since computer speeds are reported in one or more giga-hertz, this step will take less than a second.

For point three, we will look specifically at the run time of FPA, which is not the fastest known algorithm for quickly computing powers modulo n but, it works well enough for normal ($n \approx 10^{200}$) RSA implementations.

The number of modulo multiplications needed for this algorithm is given in *An Introduction to Mathematical Cryptography* [8] as $2\log_2(A)$ where A is the power being raised to; we will again use a worst case for $e \approx 10^{100}$. The number of modulo multiplications required is $2\log_2(10^{100}) \approx 600$, which can be done quickly given computer processors of today.

5. The Fourth Point

The biggest point in any cryptosystem is that knowledge of the encryption key should not allow someone to find the decryption key (point four), which RSA is believed to satisfy. In the past forty years, there have been multiple attacks on RSA yet none has fully prevailed. What we have learned from these attacks are ways in which RSA should not be implemented.

For instance in 1997 Coppersmith published a paper that showed how to find small roots of polynomial equations, and this algorithm was then used to break RSA given a small encryption key[4]. Additionally, if a quarter of the bits from d are know and $e < \sqrt{n}$ it is possible to reconstruct the rest of d [3].

In 2003, Rivest and Kaliski published a paper that discusses the RSA Problem (finding M given C, n, and e) [13]. In it they explore the bit security of RSA as well as its security against a chosen ciphertext attack. At the time neither offered a generic break for RSA.

It is believed that no known attack can break RSA generically beyond finding an efficient way to factor n [1].

6. Breaking RSA by Factoring

Since factoring is the method that could break RSA for any situation, it is the one we will focus on here. To break our example from earlier, we will use the Quadratic Sieve factorization method, which is one of several well known factoring methods. This method will work for us but for numbers above 120 digits it takes too long so is no longer a viable option for factoring [11].

The goal of the method is to find a non-trivial x, y such that $x^2 \equiv y^2 \pmod{n}$ and $x \not\equiv \pm y \pmod{n}$. The algorithm as described in [15] is outlined in the following steps, which we will explore in more detail as we work through factoring n:

1. Find a set of z_i such that $z_i^2 - n$ is $\mathcal{B}$-smooth
2. Create vectors whose values are the powers of the $\mathcal{B}$-smooth numbers found in step 1 modulo 2 and find a subset that's linearly dependent
3. Construct x and y from the dependent vectors in step 3
4. Compute $\gcd(x-y, n)$ to find a factor of n then divide

To find our $\mathcal{B}$-smooth numbers, we need a factor base $\mathcal{B}$ which is a set of primes less than a specific integer B, an optimal value of which will be given later. Additionally for every $p \in \mathcal{B}$ we need $\left(\frac{n}{p}\right) = 1$. The method used for evaluating is described in [8]. To understand this condition consider the equation we are solving:

$$z^2 - n = p_1 p_2 \dots p_i$$
$$z^2 = p_1 p_2 \dots p_i + n$$
$$z^2 \equiv n \pmod{p_1}$$

Which tells us n must be a quadratic residue of p_1. Following the same argument, we seen that n must be in the quadratic residue for each p_i.

If we look at this another way, we see that given any factor base $\mathcal{B}$ we can find infinitely many numbers that do not produce $\mathcal{B}$-smooth numbers.

Theorem 1 *Given a factor base $\mathcal{B}$ of odd primes there exists infinitely many positive integers n such that for all $z \in \mathbb{Z}$ with $z \geq \lceil n^{1/2} \rceil$, $z^2 - n$ is not $\mathcal{B}$-smooth.*

Proof. Consider $\mathcal{B} = \{p_1, p_2, \dots, p_k\}$, given a specific $p_i \in \mathcal{B}$ where $1 \leq i \leq k$, from [17] we know we can find an integer not in the quadratic residue of $\mathbb{Z}_{p_i}$. Meaning $x_i \not\equiv a^2 \pmod{p_i} \forall a \in \mathbb{Z}_{p_i}$. We can then find such values for every $p_i \in \mathcal{B}$.

By using the Chinese Remainder Theorem which is stated in [8] as, given $x \equiv b \pmod{m}$ and $x \equiv a \pmod{n}$ where m, n are relatively prime, there exists a unique solution c such that $x \equiv c \pmod{mn}$. Using this, we can build a number x such that $x \equiv x_i \pmod{p_i} \forall p_i \in \mathcal{B}$ therefor $x \not\equiv a^2 \pmod{p_1 \dots p_k} \forall a \in \mathbb{Z}_{p_1 \dots p_k}$.

With this x we can continually add $p_1 \dots p_k$ to get infinitely many solutions that will not work. ■

The optimal value of B as found in [11] is given by

$$exp\left(\frac{1}{2}\sqrt{\ln(n)\ln(\ln(n))}\right).$$

In our case for $n = 29360291$ we get B ≈ 33. We start with the full set of primes $\mathcal{B} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$ then remove any primes such that $\left(\frac{n}{p}\right) \neq 1$. We used quadratic reciprocity rules from [8] to rule out the number 3 and are left with:

$$\mathcal{B} = \{2, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$$

Since we are looking for numbers of the form $z^2 - n$ we will start with $z = \lceil n^{1/2} \rceil = 5419$, keeping in mind we only want to keep values that can be written as a product of primes in $\mathcal{B}$. We divided each number by all the primes in the factor base and any number that was reduced to 1 was kept as $\mathcal{B}$-smooth, all others were discarded. In Table 2, we show the first ten elements to demonstrate how the sieving works and the remaining were checked before inclusion.

z	$z^2 - n$	after prime division	factorization, if $\mathcal{B}$-smooth	power vector mod 2
5419	5270	1	$2^1 5^1 7^0 11^0 13^0 17^1 19^0 23^0 29^0 31^1$	(1, 1, 0, 0, 0, 1, 0, 0, 0, 1)
5420	16109	16109		
5421	26950	1	$2^1 5^2 7^2 11^1 13^0 17^0 19^0 23^0 29^0 31^0$	(1, 0, 0, 1, 0, 0, 0, 0, 0, 0)
5422	37793	5399		
5423	48638	24319		
5424	59485	11897		
5425	70334	139		
5426	81185	1249		
5427	92038	2707		
5428	102893	14699		
5429	113750	1	$2^1 5^4 7^1 11^0 13^1 17^0 19^0 23^0 29^0 31^0$	(1, 0, 1, 0, 1, 0, 0, 0, 0, 0)
5431	135470	1	$2^1 5^1 7^0 11^0 13^0 17^0 19^1 23^1 29^0 31^1$	(1, 1, 0, 0, 0, 0, 1, 1, 0, 1)
5436	189805	1	$2^0 5^1 7^1 11^1 13^0 17^1 19^0 23^0 29^1 31^0$	(0, 1, 1, 1, 0, 1, 0, 0, 1, 0)
5481	681070	1	$2^1 5^1 7^0 11^0 13^3 17^0 19^0 23^0 29^0 31^1$	(1, 1, 0, 0, 1, 0, 0, 0, 0, 1)
5494	823745	1	$2^0 5^1 7^0 11^0 13^1 17^0 19^1 23^1 29^1 31^0$	(0, 1, 0, 0, 1, 0, 1, 1, 1, 0)
5504	933725	1	$2^0 5^2 7^0 11^0 13^3 17^1 19^0 23^0 29^0 31^0$	(0, 0, 0, 0, 1, 1, 0, 0, 0, 0)
5546	1397825	1	$2^0 5^2 7^0 11^1 13^1 17^1 19^0 23^1 29^0 31^0$	(0, 0, 0, 1, 1, 1, 0, 1, 0, 0)
5555	1497734	1	$2^1 5^0 7^2 11^0 13^0 17^1 19^0 23^0 29^1 31^1$	(1, 0, 0, 0, 0, 1, 0, 0, 1, 1)
5729	3461150	1	$2^1 5^2 7^1 11^1 13^0 17^0 19^0 23^0 29^1 31^1$	(1, 0, 1, 1, 0, 0, 0, 0, 1, 1)

Table 2

Since we are looking to find a subset of column 5 in Table 2 that are dependent vectors we may stop finding $\mathcal{B}$-smooth numbers when we have one more vector than the size of our set. Now we will construct a matrix A whose columns are indexed by the power vectors mod 2 for each $\mathcal{B}$-smooth $z^2 - n$, and the rows are indexed by each prime in our set, i.e. row 1 will contain the power of 2 for $5270, 26950, \ldots, 3461150$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

One solution for the equation $Aw = 0$, using modulo 2 arithmetic, is $w = (1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 1)^T$, which tells us the number pairs we want to use to build x, y are

$$U = \{(5419, 5270), (5431, 135470), (5436, 189805), (5494, 823745), (5504, 933725), (5555, 1497734), (5729, 3421150)\}$$

To build x we simply multiply all the $z's$ that gave us a value in U and then take that number modulo n, and we get

$$x = (5419 \cdot 5431 \cdot 5436 \cdot 5494 \cdot 5504 \cdot 5555 \cdot 5729) \mod n = 10712445$$

For y we take each prime in $\mathcal{B}$ and raise it to half the sum of its power for every element in U. Consider 2, we look at Table 2 column 4 and add up all the powers of 2 corresponding to the numbers in U we want $(1 + 1 + 0 + 0 + 0 + 1 + 1 = 4)$ then take half $y = 2^2 \cdots$. The process continues for all primes, and we are left with

$$y = (2^2 \cdot 5^4 \cdot 7^2 \cdot 11^1 \cdot 13^2 \cdot 17^2 \cdot 19^1 \cdot 23^1 \cdot 29^2 \cdot 31^2) \mod n = 1264148$$

Clearly $x \not\equiv \pm y \pmod n$, so we just need to check that $x^2 \equiv y^2 \pmod n$.

$$x^2 \mod n = 10712245^2 \mod n = 18887065$$
$$y^2 \mod n = 1264148^2 \mod n = 18887065$$

This means we have found our x and y, now we will take $\gcd(x - y, n)$

$$\gcd(9448291, 29360291)$$

$$\begin{aligned}
9448297 &= 0(29360291) + 9448297\\
29360291 &= 3(9448297) + 1015400\\
9448297 &= 9(1015400) + 309697\\
1015400 &= 3(309697) + 86309\\
309697 &= 3(86309) + 50770\\
86309 &= 1(50770) + 35539\\
50770 &= 1(35539) + 15231\\
35539 &= 2(15231) + 5077\\
15231 &= 3(5077) + 0
\end{aligned}$$

For the final step we divide n by the $\gcd$ found above and get

$$\frac{29360291}{5077} = 5783$$

$$n = 5077 \cdot 5783$$

Once we have factored n, we simply follow the steps for the EEA to find d from e and we have broken the encryption.

We were able to break our example by factoring. However, we knew from the beginning that our example would not be secure by prior research. What we have shown is that if RSA is implemented correctly, it follows the first three points needed to be a useful secure public-key cryptosystem. Additionally, since it has been in use for 40 years and there is still not an efficient attack against it RSA, is commonly believed to hold for point four, although it has never been proven.

References

[1] D. Aggarwal, U. Maurer, Breaking RSA Generically Is Equivalent to Factoring, IEEE: Transactions on Information Theory. 62(11), Nov. 2016.

[2] M. Bellare, P. Rogway, Optimal asymmetric encryption. Eurocrypt ‘94 950, pp 92-111, 1994.

[3] Boneh, D. Twenty Years of Attacks on the RSA Cryptosystem, Notices Of The American Mathematical Society, 46(2) pp. 203 - 209, Jan. 1999.

[4] D. Coppersmith, Small solutions to polynomial equations, and low exponent RSA vulnerabilities, Journal of Cryptography. 10 pp 233-269, 1997.

[5] W. Diffie and M. Hellman, Exhaustive Cryptanalysis of the NBS Data Encryption Standard, Computer. 10(6) pp. 74-84, 1977.

[6] W. Diffie and M. Hellman, New Directions in Cryptography, IEEE Transactions on Information Theory. IT-22(6) pp.644 - 654, 1976.

[7] Joseph Gallian, Contemporary Abstract Algebra, Brooks/Cole, Cengage Learning. 2013.

[8] J. Hoffstein, J. Pipher, J. Silverman, An Introduction to Mathematical Cryptography, Springer. 2010.

[9] M. Kaczorowski, The Evolution of Algorithms to find Prime Numbers, McGill Science Undergraduate Research Journal. 4(1) pp. 36-38, 2009.

[10] G. Lamé, Note sur la limite du nombre des divisions dans la recherche du plus grand commun diviseur entre deux nombres entiers. Comptes Rendus Acad. Sci. 19: 867–870, 1844.

[11] C. Pomerance, Smooth numbers and the quadratic sieve, MSRI Publications, 44, 2008.

[12] René Schoof, Four primality testing algorithms, http://arxiv.org/abs/0801.3840.

[13] R. Rivest, B. Kaliski, RSA Problem, https://people.csail.mit.edu/rivest/RivestKaliski-RSAProblem.pdf. Dec, 2003

[14] R.L Rivest, A. Shamir, and L. Adleman, A Method for Obtaining Digital Signature and Public-Key Cryptosystems, Communication of the ACM. pp. 120-126, 1978.

[15] J. Smith, The Sieve Re-Imagined: Integer Factorization Methods, University of Waterloo, 2012.

[16] Sridevi and Manajaih, Modular Arithmetic in RSA Cryptography, International Journal of Advanced Computer Research. 4(17) pp. 973 - 979, Dec. 2014.

[17] A. Witno, Theory of Numbers, BookSurge Publishing. 2008.

Generalized Kasner Polygons

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Abstract

If a polygon P' with k sides, a k-gon, is given, J. Douglas [1] and E. Kasner [4] determined when it was possible to find a k-gon (or many such k-gons) P such that the vertices of P' are the midpoints of the sides of polygon P. We refer to these results as Kasner's Theorem.The key distinguishing factor in the existence and uniqueness of such a polygon P, described in Kasner's Theorem, is whether k is odd or even. In this paper, we generalize this theorem by now considering polygons P such that the vertices of P' divide the sides of P in the common ratio $t : 1 - t$ for a fixed $0 < t < 1$ (So Kasner's Theorem is the special case where $t = 1/2$.). We show that for $t \neq 1/2$ there is always a unique such polygon P, so it was discovered that k being even or odd is only a distinguishing factor for $t = 1/2$. It is then shown how to actually construct the polygon P that has this property for any fixed t. Finally, it is shown that a sequence of k-gons, $\left\{P^{(0)}, P^{(1)}, P^{(2)}, \cdots\right\}$, where the vertices of $P^{(i+1)}$ divide the sides of $P^{(i)}$ in the ratio $t : 1 - t$ for a fixed $0 < t < 1$, converges to the centroid of $P^{(0)}$.

1. Introduction

Some interesting theorems in elementary plane geometry concerning polygons are those which indicate that the parity of the number of sides is the deciding factor on whether or not particular results hold. One such problem, that is stated in the complex plane, was studied by J. Douglas [1] and E. Kasner [4].

Let $P = (z_1, z_2, \cdots, z_k)$ be a closed k-gon and let $P' = (z'_1, z'_2, \cdots, z'_k)$ be the k-gon whose vertices are the midpoints of successive sides of P.

More precisely, define

$$z'_i = \frac{1}{2}(z_i + z_{i+1}) \tag{1}$$

for $i = 1, \cdots, k$ and $z_{k+1} = z_1$.

Problem 1 (Kasner) *If the midpoint polygon P' is given, can the original polygon P be constructed?*

As in [6, p. 91], we will call P' the *Kasner polygon* of P . That is, if we begin with the polygon P', can we construct a polygon P such that P' is the midpoint polygon of P ? Construct, in this case, means in the classical sense with straightedge and compass. In 1903, Kasner [4] found conditions for when the given polygon P' had vertices that were the midpoints of successive sides of another polygon P. Kasner not only gave the conditions for when P could be constructed, but also when P was unique and, in all cases, how to perform the actual construction. These results are stated in detail in the statement of Kasner's Theorem 3 in Section 2. Familiar examples with the triangle and quadrilateral are also given in Section 2, and these illustrate how the parity of k affects the results.

In this paper, the same problem will be answered for the polygon $P' = (z'_1, z'_2, \cdots, z'_k)$ where

$$z'_i = (1-t)z_i + tz_{i+1} \tag{2}$$

for a fixed t with $0 < t < 1$, $i = 1, \cdots, k$ and $z_{k+1} = z_1$. We call P' the *generalized Kasner polygon* of P .

Problem 2 (Generalization of Problem 1) *If a generalized Kasner polygon P' is given, can the original polygon P be constructed?*

Each z'_i divides the side $z_i z_{i+1}$ of P in the common ratio $t : 1 - t$, and Kasner answered this question for the case where $t = 1/2$. The interesting part of the solution to Problem 2, which is stated in Theorem 4, is that it does not depend on the parity of k as much as it depends on the value of t. The solution to Problem 1, which is Kasner's Theorem 3, the parity of k is the key distinguishing factor. Section 4 demonstrates the method of constructing the polygon(s) P, when possible, from the polygon P'. The ability to construct P with straightedge and compass directly corresponds with t being a constructible number.

In Section 5, we then look at the sequence of generalized Kasner polygons

$$P = P^{(0)}, P' = P^{(1)}, P^{(2)}, P^{(3)}, \cdots, P^{(n)}, \cdots \tag{3}$$

where $P^{(n)}$ is defined to be a generalized Kasner polygon of $P^{(n-1)}$ for all $n \geq 1$. It is shown that this sequence, regardless of parity or the value of t, will always converge to the centroid of the original polygon P, see Theorem 7 and Figure 3.

2. Kasner's Theorem

Kasner showed in [4] that the answer to Problem 1 depends on the parity of k. In 1936, in [5], Kasner showed that the number of different polygons P which produce a given P' also depends on the parity of k. Furthermore, Kasner [5] gives a method to construct, in the classical geometry sense, such a polygon P (when one exists). We will discuss the construction method in Section 4. We state the cumulative results of these two papers concerning this problem as:

Theorem 3 (Kasner's Theorem) *Kasner Let $P' = (z_1', z_2', \cdots, z_k')$ be the k-gon whose vertices are the midpoints of successive sides of the closed k-gon $P = (z_1, z_2, \cdots, z_k)$ in the plane, i.e. $z_i' = (1/2)(z_i + z_{i+1})$. Assume P' is given. Then*

1. *If k is odd, then P can be determined uniquely.*
2. *If $k = 2q$ is even, then P can be determined if and only if the polygon P' is such that*
$$z_1' + z_3' + \cdots + z_{2q-1}' = z_2' + z_4' + \cdots + z_{2q}'. \tag{4}$$
Furthermore, when P can be determined, there are infinitely many such polygons P.

Note that, geometrically, the relation in (4) means that the vertices of P' having even subscripts and those having odd subscripts have the same centroid.

Kasner's results are derived primarily using group theory utilizing the affine geometry of polygons. Douglas, in [1], invoked algebra in the complex plane to obtain Kasner's results as well as other related results. Algebraically, it is a matter of solving the k equations given in (1) for the unknowns $z_1, z_2, \cdots, z_k$ and determining whether this solution is unique or not. However, to construct the polygon(s), when possible, it is necessary to look at the geometry. The complete proof of this theorem can be found in [4], but here we will demonstrate and distinguish between the two cases by using the two most common polygons, the triangle ($k = 3$) and the quadrilateral ($k = 4$).

Example 1: Triangle The most familiar example is where P is a tri-

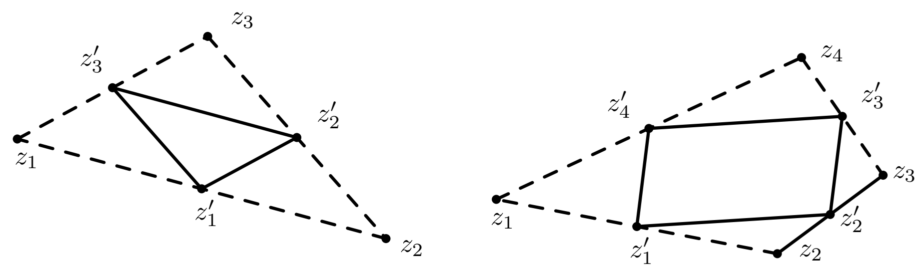


Figure 1: Triangle and Quadrilateral, $t = 1/2$

angle and P' is its medial triangle. Geometrically, it is easy to verify that P can be uniquely recovered because the sides of P must be parallel to those of its medial triangle P'. Since the parallels through the vertices of P' that are parallel to the opposite side are unique, the vertices of P are uniquely determined. This method of constructing parallels does not, however, work for odd k larger than 3. Algebraically, we can also see that the vertex z_1 (and, therefore, z_2 and z_3) is uniquely determined by solving the three equations in (1). The succession of equations gives

$$z_2 = 2z'_1 - z_1,\ z_1 - z_3 = 2z'_1 - 2z'_2,\ z_1 + z_3 = 2z'_3,$$

and adding the last two equations gives the unique solution

$$z_1 = z'_1 - z'_2 + z'_3. \tag{5}$$

From equation (5), it is readily seen that z_1 is the fourth vertex of the parallelogram with vertices z_1, z'_1, z'_2, z'_3, as seen in the triangle of Figure 1.

Example 2: Quadrilateral In the case of the quadrilateral it is well known that the midpoint quadrilateral P' of any quadrilateral P is a parallelogram, so the only polygons P' that can be pre-assigned are parallelograms. Once this is established, we then want to know how many polygons P exist such that P' is its Kasner polygon, and how to construct them. Begin with an *arbitrary* point z_1 in the plane (not coinciding with any vertex of P') and locate z_2 so that z'_1 is the midpoint of segment z_1z_2 (see Figure 1). This means, from (1), that $z_2 = 2z'_1 - z_1$. Similarly, using (1) to locate z_3 and z_4, we have

$$z_1 + z_4 = 2(z'_3 - z'_2 + z'_1). \tag{6}$$

To determine if P has P' as its Kasner quadrilateral, it needs to be determined if z'_4 is the midpoint of segment z_4z_1, i.e., $z_1 + z_4 = 2z'_4$. Combining this with (6), the condition that is necessary is that $z'_2 + z'_4 = z'_1 + z'_3$. This is equivalent to the diagonals of P' bisecting each other and, therefore, P' must be a parallelogram. Since the choice of z_1 was arbitrary, there are infinitely many polygons P that have any given parallelogram P' as its midpoint polygon. (See Figure 1)

3. A Generalization

In this section, we consider the natural extension to Kasner's Theorem 3 and answer Problem 2, which was discussed in Section 1. In particular, we no longer insist that the points of P' be chosen as midpoints, but rather the vertices of P' divide the corresponding side of P in the ratio $t : 1 - t$ for arbitrary fixed $0 < t < 1$. Again, the interesting part of this result is that for $t \neq 1/2$, the parity of k does not matter.

Theorem 4 *Let $P = (z_1, z_2, \cdots, z_k)$ be a closed k-gon in the plane. Let $P' = (z'_1, z'_2, \cdots, z'_k)$ be the associated k-gon whose vertices are defined as*

$$z'_i = (1 - t)z_i + tz_{i+1} \tag{7}$$

for some fixed t, with $0 < t < 1$, $i = 1, \cdots, k$ and $z_{k+1} = z_1$. Assume P' is given. Then

1. *if $t \neq 1/2$, P can be determined uniquely.*
2. *if $t = 1/2$, and*

   a. *k is odd, P can be determined uniquely.*

   b. *$k = 2q$ is even, P exists if and only if*
   $$z'_1 + z'_3 + \cdots + z'_{2q-1} = z'_2 + z'_4 + \cdots + z'_{2q}.$$
   If P exists then there are infinitely many such polygons P .

Furthermore, in all cases, if such a polygon P exists, it can be constructed in the classical geometry sense precisely when t is a constructible number.

Proof. This proof is simply a matter of solving the k equations in (7) for the k unknowns $z_1, z_2, \cdots, z_k$. Let $P = [z_1, z_2, \cdots, z_k]^T$ and $P' = [z'_1, z'_2, \cdots, z'_k]^T$ be column vectors in $\mathbb{C}^k$ (the k-dimensional vector space of all k-tuples of complex numbers) and then this system of equations can be written $P' = MP$ where

$$M = \begin{bmatrix} 1-t & t & 0 & \cdots & 0 & 0 \\ 0 & 1-t & t & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1-t & t \\ t & 0 & \cdots & \cdots & 0 & 1-t \end{bmatrix}_{k\times k} . \tag{8}$$

Now, by using the cofactor expansion along the first column,

$$\det(M) = (1-t)^k + (-1)^{k-1}t^k. \tag{9}$$

If k is odd, then $\det(M) = (1-t)^k + t^k > 0$ for all t. Thus, the system has a unique solution and the polygon P can be determined uniquely when k is odd. For $t = 1/2$, this is part (1) of Kasner's Theorem 3 and can be found in [5].

If k is even, then $\det(M) = (1-t)^k - t^k$, which equals zero only when $t = 1/2$. Thus, the system has a unique solution for all $t \neq 1/2$ and, therefore, the polygon P can be determined uniquely when k is even (and $t \neq 1/2$). For $t = 1/2$, this is part (2) of Kasner's Theorem 3 and the proof can be found in [5]. ■

The existence of the polygons P was easily answered using algebra, and next in Section 4 we discuss the method of actually constructing the polygons when they exist.

Construction Method

In the cases where P exists, as stated in Theorem 4, we will now describe how the vertices of P can be obtained geometrically from the given vertices of P', where P' is the generalized Kasner polygon of P. In Examples 1 and 2 in Section 2, it was only necessary to perform the construction so that each z_i' was the midpoint of the segment $z_i z_{i+1}$, but in the generalized case, the construction of each z_i is described in terms of a dilation. This method will be justified and the actual steps with illustrations of the construction will be given at the end of the section.

Recall that the image, z', of z under a *dilation* centered at z_0 with real constant of proportionality (ratio) r is given by

$$z' = (1-r)z_0 + rz. \tag{10}$$

Considering z_0, z and z' as vertices, this means z_0, z and z' are collinear with z between z_0 and z' if $r > 1$, z' between z_0 and z if $0 < r < 1$ and z_0 between z' and z if $r < 0$. Note that by solving equation (10) for z , we see that the inverse of this dilation is the dilation centered at z_0 with constant of proportionality $1/r$.

Comparing the equations (7) and (10), the vertex z_i' of P' can be described as the image of z_{i+1} under a dilation centered at z_i with ratio t. We locate the vertex z_1 of P, and thus all vertices of P, by constructing a sequence of points defined by the inverse of this dilation.

More precisely, for an arbitrary point z in the plane, define $z^{(1)}$ as the image of z_1' under the dilation centered at z with ratio $1/t$. Similarly, define $z^{(i)}$ as the image of z_i' under the dilation centered at $z^{(i-1)}$ with ratio $1/t$ for $i = 2, 3, \cdots, k$. This system of equations can be written as

$$\begin{array}{lclcl} (1-t)z & + & tz^{(1)} & = & z_1' \\ (1-t)z^{(1)} & + & tz^{(2)} & = & z_2' \\ & \vdots & & & \\ (1-t)z^{(k-1)} & + & tz^{(k)} & = & z_k'. \end{array} \tag{11}$$

Utilizing matrices and vectors as in the proof of Theorem 4, let $P = [z_1, z_2, \cdots, z_k]^T$ and $P' = [z_1', z_2', \cdots, z_k']^T$ be column vectors in $\mathbb{C}^k$, and $Q = [z, z^{(1)}, z^{(2)}, \cdots, z^{(k)}]^T$ in $\mathbb{C}^{k+1}$. Then this system of equations (11) can be written $AQ = P'$ where

$$A := \begin{bmatrix} 1-t & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-t & t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1-t & t & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1-t & t \end{bmatrix}_{k\times(k+1)}.$$

But, $P' = MP$, where M is the matrix in (8). Therefore, we need to solve

$$AQ = MP$$

for P.

Case 1: If k is odd or $t \neq 1/2$, then M is invertible and P can be recovered from Q since $M^{-1}AQ = P$. Actually, it is only necessary to find the single vertex z_1 of P from the arbitrary point z, so only the first row of M^{-1} is needed and this is the vector $M_1 = \frac{1}{\det M}[\sigma_0, \sigma_1, \cdots, \sigma_{k-1}]$ where $\sigma_i = (-1)^i t^i (1-t)^{k-i-1}$ for $i = 0, 1, \cdots, k-1$. Notice that

$t\sigma_i + (1-t)\sigma_{i+1} = 0$ for each i, so the first row of $M^{-1}A$ is the vector $M_1A = [(1-t)\sigma_0, 0, 0, \cdots, 0, 0, t\sigma_{k-1}]$. Since $\sigma_0 = (1-t)^{k-1}$ and $\sigma_{k-1} = (-1)^{k-1}t^{k-1}$, the first row of P is $M_1AQ = z_1$, which can be written

$$\frac{1}{\text{det}(M)}[(1-t)^k z + (-1)^{k-1}t^k z^{(k)}] = z_1. \tag{12}$$

Geometrically, comparing equations (12) and (10), the vertex z_1 of P is the image of $z^{(k)}$ under the dilation centered at z with ratio $r = \dfrac{(-1)^{k-1}t^k}{\text{det}(M)}$. As a consequence, then, if k is odd then $0 < r < 1$ and vertex z_1 is between vertices z and $z^{(k)}$. However, if k is even then $r < 0$ and vertex z is between vertices z_1 and $z^{(k)}$. Again, now that z_1 has been uniquely determined, the remaining vertices z_i of P can be found successively as the image of z'_{i-1} under the dilation centered at z_{i-1} with ratio $1/t$. See Figures (2a,c,d).

Case 2: If k is even and $t = 1/2$, then, as is shown in [5], the condition (4) given in Kasner's Theorem 3 being satisfied by P' will result in $z^{(k)} = z$ for any choice of z. Therefore, P' is the generalized Kasner polygon for the *closed* polygon $Q = (z, z^{(1)}, z^{(2)}, \cdots, z^{(k-1)})$ for arbitrary z. See Figure (2b).

Thus, as long as the constant t is a constructible number, then $1/t$ and r are also constructible numbers and so the dilations can be accomplished by straightedge and compass.

These constructions are quite easy with a dynamic geometry software program. With these you can construct a variable ratio t to investigate the difference between Case 1 and Case 2.

Construction 5 (k is odd or $t \neq 1/2$)

1. Construct any polygon $P' = (z_1, z_2, \cdots, z_k)$.
2. Construct any value of $0 < t < 1$. ($t \neq 1/2$ if k is even.)
3. Construct the value $1/t$, det$M$ (see equation (9)), and the value $r = \dfrac{(-1)^{k-1}t^k}{\text{det}M}$.
4. Choose any point z in the plane that does not coincide with a vertex of P'.
5. Construct $z^{(1)}$ as the image of z'_1 under the dilation centered at z with ratio $1/t$. Construct in succession, $z^{(2)}, \cdots z^{(k)}$, such that $z^{(i)}$ is the

image of z'_i under the dilation centered at $z^{(i-1)}$ with ratio $1/t$.

6. Construct the image of $z^{(k)}$ under the dilation centered at z with ratio $r = \dfrac{(-1)^{k-1}t^k}{\det M}$. This point is the vertex z_1 of the polygon P.

7. Construct, in succession, $z_2, \cdots, z_k$, such that z_i is the image of z'_i under the dilation centered at z_{i-1} with ratio $1/t$. You now have $P = (z_1, z_2, \cdots, z_k)$ where P' is the generalized Kasner polygon of P. See Figures 2a,c,d.

Construction 6 (**k is even** $t = 1/2$)

1. Construct a polygon $P' = (z'_1, z'_2, \cdots, z'_k)$, such that equation (4) is satisfied. The simplest way to do this is to begin with a closed k-gon $P = (z_1, z_2, \cdots, z_k)$ and construct the polygon $P' = (z'_1, z'_2, \cdots, z'_k)$ such that z'_i is the midpoint of the segment $z_i z_{i+1}$. Now you can be sure that such a P exists, and thus equation (4) is satisfied. Now erase(hide) the vertices of P.

2. Perform Step 4 and Step 5 from construction 5 . Notice that $z^{(k)} = z$ and that $Q = (z, z^{(1)}, z^{(2)}, \cdots, z^{(k-1)})$ is such that P' is the generalized Kasner polygon of Q. See Figure 2b.

Converging Kasner Polygons

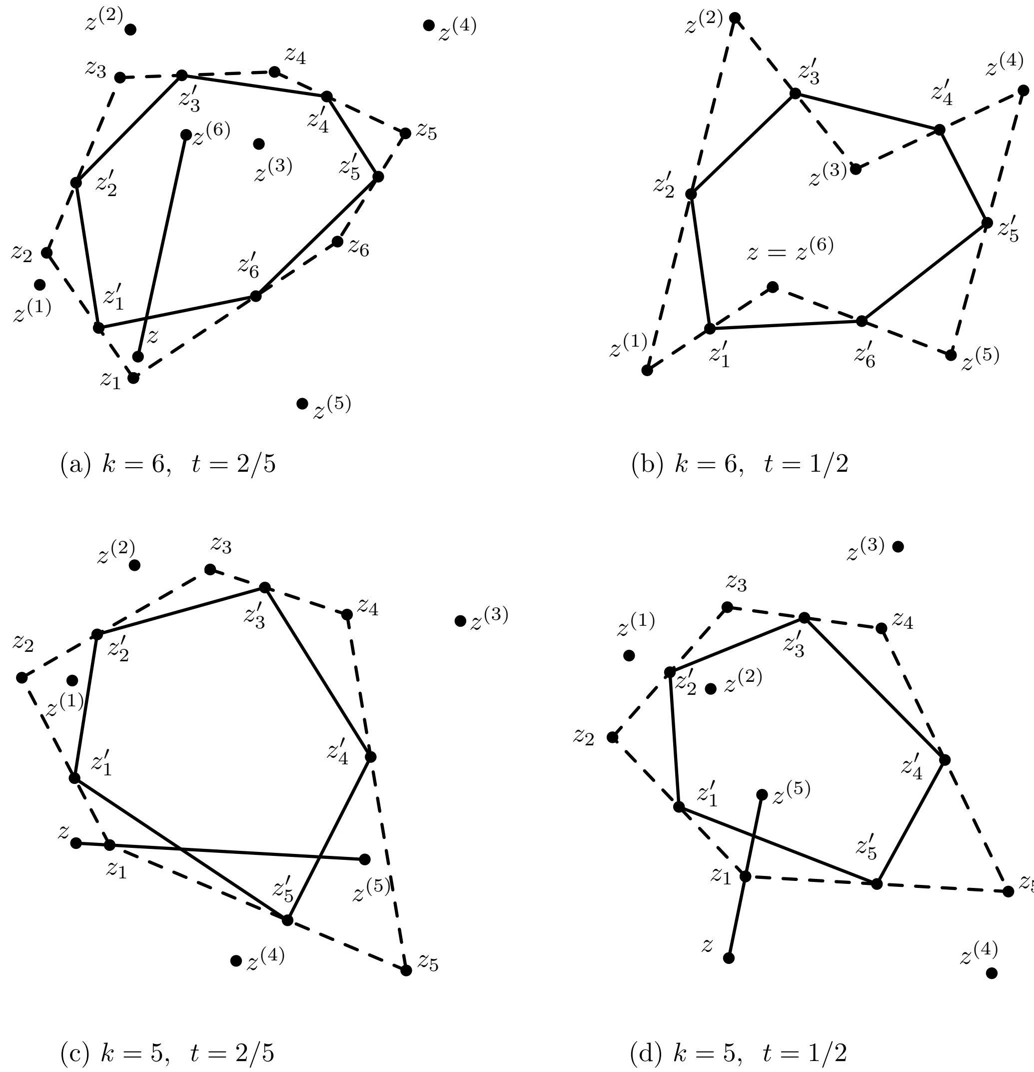
In [6, p. 91], the *n-th order Kasner polygon of P* is defined to be $P^{(n)} = M^n P$ where M is defined in (8) with $t = 1/2$. For example, the midpoint polygon P', as defined in (1) , is the first order Kasner polygon of P and is now denoted by $P^{(1)} = P' = MP$, and we continue by letting $P^{(2)} = MP^{(1)} = M^2 P$ be the midpoint polygon of $P^{(1)}$, etc. So, in a similar fashion, we will let the *generalized n-th order Kasner polygon of P* be defined the same way,

$$P^{(n)} = M^n P,$$

but M, as defined in (8), will have $0 < t < 1$ an arbitrary real number.

Theorem 7 *If $P^{(n)}$ is the generalized n-th order Kasner Polygon of P, then $\lim_{n\to\infty} P^{(n)} = z^*$, where $z^* = \frac{1}{k}(z_1 + z_2 + \cdots + z_k)$ is the centroid of the polygon P , regardless of the value of $0 < t < 1$ or the parity of k.*

Proof. The matrix M in (8) of this sequence $P^{(1)}, P^{(2)}, \cdots, P^{(n)}$ with $P^{(n)} = M^n P$ is a Markov chain with M the transition matrix. The reader can verify that M is a stochastic matrix, as it has non-negative entries and



(a) $k = 6, \;\; t = 2/5$ (b) $k = 6, \;\; t = 1/2$

(c) $k = 5, \;\; t = 2/5$ (d) $k = 5, \;\; t = 1/2$

Figure 2: Constructions

the entries in each column sum to $(1 - t) + t = 1$. Furthermore, M is a regular stochastic matrix since it can be verified that M^{k-1} has all positive entries. With these properties, it is known (see a linear algebra text such as [2]) that $\lambda = 1$ is a dominant eigenvalue and

$$\lim_{n\to\infty} M^n = V,$$

where V is the stochastic matrix with identical columns, each being the

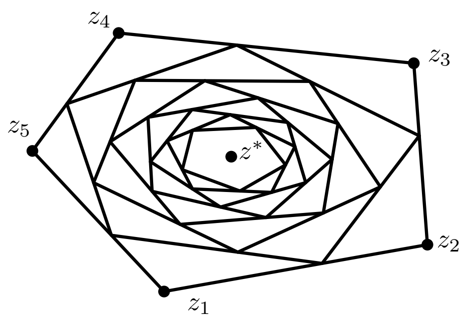


Figure 3: $k = 5$, $t = 2/5$

eigenvector corresponding to $\lambda = 1$. These general results for regular stochastic matrices will be partially verified directly for the matrix M in (8) to complete the proof.

The fact that $\lambda = 1$ is a dominant eigenvalue can be verified directly for the matrix M in (8) since M is also a *circulant* matrix as each row is the previous row shifted to the right one position with the element of that row in the last column wrapping around to the first column. For these matrices (see [3]), the eigenvalues and corresponding eigenvectors are known. In general, for a $k \times k$ circulant matrix the k linearly independent eigenvectors are

$$v_m = [1, \zeta_m, \zeta_m^2, \cdots, \zeta_m^{k-1}]^T, \tag{13}$$

where $\zeta_m = \exp(2\pi i m/k)$, for $m = 0, 1, 2, \cdots, k-1$, are the k-th roots of unity. The eigenvalues, with λ_m corresponding to the eigenvetor v_m, are given by

$$\lambda_m = a_0 + a_1\zeta_m + a_2\zeta_m^2 + \cdots a_k\zeta_m^{k-1},$$

where $[a_0 \;\; a_1 \;\; a_2 \cdots a_k]$ is the first row of the circulant matrix. So, for our circulant matrix M in (8), we have eigenvalues

$$\lambda_m = (1-t) + t\zeta_m.$$

Since the k eigenvectors $v_0, \cdots v_{k-1}$ are linearly independent, they form a basis for the vector space $\mathbb{C}^k$. Therefore, the vector P satisfies

$$P = [z_1, z_2, \cdots, z_k]^T = c_0v_0 + c_1v_1 + \cdots + c_{k-1}v_{k-1},$$

for some $c_i \in \mathbb{C}$, $i = 0, 1, \cdots, k-1$. Since λ^n is an eigenvalue for M^n, we have

$$M^n P = P^{(n)} = c_0\lambda_0^n v_0 + c_1\lambda_1^n v_1 + \cdots + c_{k-1}\lambda_{k-1}^n v_{k-1}. \tag{14}$$

Now $\lambda_0 = 1$, since $\zeta_0 = 1$, and $||\lambda_m|| = ||(1-t) + t\zeta_m|| < ||1 - t|| + ||t\zeta_m|| = 1 - t + t = 1$ for all $m = 1, \cdots, k-1$. Notice that we get strict inequality because equality holds if and only if $\frac{t}{1-t}\zeta_m$ is a positive real number, but this is only the case when $\zeta_m = 1$, i.e when $m = 0$. Therefore, $\lambda = 1$ is dominant and λ_m^n converges to 0 for $m = 1, \cdots, k-1$. Therefore, from (14), $P^{(n)} = M^n P$ converges to the steady-state vector $c_0 v_0 = [c_0, c_0, \cdots, c_0]^T$ since $v_0 = [1, 1, \cdots, 1]^T$from 13 above. As discussed earlier in the proof, M^n converges to the stochastic matrix $V = [v_0, v_0, \cdots, v_0]$, where $v_0 = \frac{1}{k}[1, 1, \cdots, 1]^T$ (the entries of each column of a stochastic matrix sum to 1). Now $VP = c_0 v_0$, which can be written

$$\frac{1}{k}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_k \end{bmatrix} = \begin{bmatrix} c_0 \\ c_0 \\ c_0 \\ \vdots \\ c_0 \end{bmatrix},$$

which shows that $c_0 = \frac{1}{k}(z_1 + z_2 + \cdots + z_k)$. Therefore, $P^{(n)}$ converges to the centroid of P. ■

References

[1] J. Douglas, *Geometry of polygons in the complex plane*, J. Math. Phys.,19 (1940) pp. 93-130.

[2] S. Friedberg, A. Insel and L. Spence, *Linear Algebra, 4th edition*, Upper Saddle River, NJ: Prentice Hall, (2003), pp. 283-307.

[3] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products, 6th ed.* San Diego, CA: Academic Press, (2000), pp. 1111-1112.

[4] E. Kasner, *The group generated by central symmetries, with application to polygons*, Amer. Math. Monthly, vol. 10, issue 3 (1903) pp. 57-63.

[5] E. Kasner and G. Comenetz, *Groups of multipoint transformations with application to polygons*, Scripta Math., vol. 4 (1936) pp. 37-49.

[6] I. Schoenberg, *Mathematical Time Exposures*, MAA, (1982), pp. 60-77,91-102.

The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before March 15, 2019. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring 2019 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051).

NEW PROBLEMS 820 - 828

Problem 820. *Proposed by the editor*

Find a 4-digit positive integer $N = abcd$ which is divisible by 11 and $N/11 \ = b^2 + c^2 + d^2$.

Problem 821. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a, b, c \in \mathbb{R}$ then $4 \sum\limits_{cyclic} a|b(1-b^2)| \leqslant \sum\limits_{cyclic} a(1+b^2)^2$.

Problem 822. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that in any acute-angled ΔABC you have

$$2\sum_{cyclic} \tan^3 A \geqslant \sum_{cyclic} \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + 3(\tan A + \tan B + \tan C).$$

Problem 823. *Proposed by Pedro H.O. Pantoja, University of Campina Grande, Brazil.*

Let x, y, z be positive real numbers. Prove that

$$\frac{1}{xy+yz+zx} \leqslant \frac{3x}{(y+2z)^3} + \frac{3y}{(z+2x)^3} + \frac{3z}{(x+2y)^3} \leqslant \frac{x^3y+y^3z+z^3x}{(3xyz)^2}.$$

Problem 824. *Proposed by Pedro H.O. Pantoja, University of Campina Grande, Brazil.*

Find all positive integers a, b, c where a and b are prime numbers with $a \not\equiv 0 \pmod c$ such that $51a + 7ab + bc^2 = abc^2$.

Problem 825. *Proposed by Ovidiu Furdui and Alina Sintamarian, Technical University of Cluj–Napoca, Cluj–Napoca, Romania.*

Let $k \geqslant 0$ be an integer. Calculate

$$\sum_{n=1}^{\infty} \left[\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots \right) - \frac{1}{n+k} \right].$$

Problem 826. *Proposed by D.M. Batinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Let F_n and L_n be the nth Fibonacci and Lucas numbers defined by $F_1 = F_2 = 1$ and $F_n = F_{n+1} + F_{n+2}$for $n \geqslant 3$ and by $L_1 = 1$, $L_2 = 3$ and $L_n = L_{n+1} + L_{n+2}$ for $n \geqslant 3$. Let k be a positive integer and

$$F(k) = \begin{pmatrix} F_k^2 & F_{k+1}^2 \\ F_{k+1}^2 & F_k^2 \end{pmatrix} \begin{pmatrix} L_{k+1} & L_k \\ L_k & L_{k+1} \end{pmatrix}.$$

Evaluate $\prod_{k=1}^{n} F(k)$ as a multiple of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Problem 827. *Proposed by D.M. Batinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Let (a_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{a_n}{n!} = a > 0$. Find

$$\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}} - \frac{n^2}{\sqrt[n]{a_n}} \right).$$

Problem 828. *Proposed by D.M. Batinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Determine all injective functions $f : \mathbb{R} \to \mathbb{R}$ with $f(0) \neq 1/b$ and

$$f(f(x)y^3) + ax^9y^9 = bf(x^3)f(y^3)$$

for all $x, y \in \mathbb{R}$, where $a > 0$, $b > 0$.

SOLUTIONS TO PROBLEMS 798 - 807

Problem 798. *Proposed by the editor.*

In 2002, Britney Gallivan (high school junior) found a formula for paper folding and managed to do 12 folds of a long sheet of toilet paper. She found that

$$L = \frac{\pi t}{6}\left(2^n + 4\right)\left(2^n - 1\right)$$

where t represents the thickness of the material to be folded, L is the length of the paper to be folded and n is the number of folds desired (in only one direction). Suppose you tape together sheets of standard 8.5" x 11" copier paper (thickness .0035") end to end, how many sheets would be needed to be able to fold the long taped sheet 14 times?

Solution *by the proposer*

According to the formula, the length needed would be $L = \frac{.0035\pi}{6}(2^{14}+4)(2^{14}-1) = 2952142.408''$. Dividing this by the 11" for each sheet of copier paper, we get 268376.58 sheets. Since the number of sheets must be an integer, you would need 268377 sheets.

Problem 799. *Proposed by Daniel Sitaru, "Theodor Costescu" National*

Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.

Prove that if $a, b, c \in (0, 2]$ then

$$3\sqrt{2} \leq \sum \frac{b(\sqrt{a}+\sqrt{2-a})}{c} \leq 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Solution *by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.*

Elementary calculus shows that on the interval (0,2], the function $f(x) = \sqrt{x}+\sqrt{2-x}$ attains a maximum of 2 at $x = 1$ and a minimum of $\sqrt{2}$ at $x = 2$. Therefore

$$\sum_{cyclic} \frac{b(\sqrt{a}+\sqrt{2-a}}{c} \leqslant 2\sum_{cyclic} \frac{b}{c} = 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Similarly, the AM–GM inequality yields

$$\sum_{cyclic} \frac{b(\sqrt{a}+\sqrt{2-a}}{c} \geqslant \sqrt{2} * 3\sqrt[3]{\frac{b}{c} * \frac{c}{a} * \frac{a}{b}} = 3\sqrt{2}.$$

Equality holds on the left side of the original inequality if and only if $a = b = c = 2$ on the right side if and only if $a = b = c = 1$.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Ravi Prakash, Oxford University Press, New Delhi, India; Soumava Chakraborty, Softweb Technologies, Kolkata, India; Myagmarsuren Yadamsuren, Ulanbataar University, Mongolia; and Almas Bebirov, Baku State University, Lerik, Azerbaijan; and the proposer.

Problem 800. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a \in \mathbb{R}$,then

$$\int_{a+3}^{a+5} \ln(1+e^x)dx + \int_{a+6}^{a+8} \ln(1+e^x)dx \leq \int_{a}^{a+2} \ln(1+e^x)dx + \int_{a+9}^{a+11} \ln(1+e^x)dx.$$

Solution *by Angel Plaza, Department of Mathematics, Universidad de Las*

Palmas de Gran Canaria, Spain.

Notice that by a change of variables

$$\int_{a+3}^{a+5} \ln(1+e^x)dx = \int_a^{a+2} \ln(1+e^{x-3})dx,$$

$$\int_a^{a+2} \ln(1+e^{x-3})dx + \int_a^{a+2} \ln(1+e^{x-6})dx$$
$$\leqslant \int_a^{a+2} \ln(1+e^x)dx + \int_a^{a+2} \ln(1+e^{x-9})dx.$$

Using the properties of the natural logarithm, we get

$$\int_a^{a+2} \ln\left(\frac{1+e^{x-3}+e^{x-6}+e^{2x-9}}{1+e^x+e^{x-9}+e^{2x-9}}\right)dx \leqslant 0.$$

The result will follow by showing that $e^{x-3}+e^{x-6} \leqslant e^x + e^{x-9}$ for all x. This is equivalent to $\frac{e^{x-6}-e^{x-9}}{3} \leqslant \frac{e^x-e^{x-3}}{3}$ and by Lagrange's Mean Value Theorem the left side is equal to e^μ for some μ in $(x-9, x-6)$ and the right side is equal to e^η for some η in $(x-3, x)$. Since the function e^x is increasing, the inequality is true.

Also solved by Chris Kyriazis, Second High School of Santa Barbara, Athens, Greece; Madan Mastermind, (student) Varanasi-Indian Institute of Technologies, India; Rovsen Pirguliyev, Sumgait, Azerbaidian; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; and the proposer.

Problem 801. *Proposed by Jose Luis Diaz-Barrero, Barcelona Tech-UPC, Barcelona, Spain.*

Compute

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \frac{k^2+n^2}{1+2\sqrt{\frac{k^2+n^2+n^3}{n^3}}}.$$

Solution *by the Missouri State Problem Solving Group, Missouri State*

University, Springfield, MO.

We have

$$\frac{1}{1+2\sqrt{\frac{2n^2+n^3}{n^3}}} \leqslant \frac{1}{1+2\sqrt{\frac{k^2+n^2+n^3}{n^3}}} \leqslant \frac{1}{1+2\sqrt{\frac{n^2+n^3}{n^3}}}$$

and

$$\lim_{n\to\infty} \frac{1}{1+2\sqrt{\frac{2n^2+n^3}{n^3}}} = \frac{1}{3} = \lim_{n\to\infty} \frac{1}{1+2\sqrt{\frac{n^2+n^3}{n^3}}}.$$

We also have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n}(k^2+n^2) &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\left(\frac{k}{n}\right)^2+1\right)\\ &= \int_0^1 x^2+1\ dx\ (Rieman\ sums)\ = 4/3.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{n^3}\sum_{k=1}^{n}\frac{k^2+n^2}{1+2\sqrt{\frac{2n^2+n^3}{n^3}}} &\leqslant \frac{1}{n^3}\sum_{k=1}^{n}\frac{k^2+n^2}{1+2\sqrt{\frac{k^2+n^2+n^3}{n^3}}}\\ &\leqslant \frac{1}{n^3}\sum_{k=1}^{n}\frac{k^2+n^2}{1+2\sqrt{\frac{n^2+n^3}{n^3}}}.\end{aligned}$$

Now

$$\lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n}\frac{k^2+n^2}{1+2\sqrt{\frac{2n^2+n^3}{n^3}}} = \frac{4}{3}*\frac{1}{3} = \frac{4}{9}$$

and

$$\lim_{n\to\infty} \frac{1}{n^3}\sum_{k=1}^{n}\frac{k^2+n^2}{1+2\sqrt{\frac{n^2+n^3}{n^3}}} = \frac{4}{3}*\frac{1}{3} = \frac{4}{9}.$$

So the original limit is $4/9$.

Also solved by Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain; and the proposer.

Problem 802. *Proposed by Jose Luis Diaz-Barrero, Barcelona Tech-UPC, Barcelona, Spain.*

Let $n \geq 1$ be an integer. Prove that

$$\sqrt[n]{\prod_{k=1}^{n} F_{k+1}} \geq \frac{1}{2}\left(\sqrt[n]{\prod_{k=1}^{n} F_k} + \sqrt[n]{\prod_{k=1}^{n} L_k}\right),$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ and by $L_1 = 1$, $L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

Solution *by the proposer*

The inequality claimed is equivalent to

$$\sqrt[n]{\prod_{k=1}^{n} 2F_{k+1}} \geqslant \sqrt[n]{\prod_{k=1}^{n} F_k} + \sqrt[n]{\prod_{k=1}^{n} L_k}.$$

Since 2F$_{n+1} = F_n + L_n$, which can easily be proved by induction, the above inequality becomes

$$\sqrt[n]{\prod_{k=1}^{n} (F_k + L_k)} \geqslant \sqrt[n]{\prod_{k=1}^{n} F_k} + \sqrt[n]{\prod_{k=1}^{n} L_k}. \tag{1}$$

The AM-GM inequality says

$$\frac{1}{n}\sum_{k=1}^{n} \frac{F_k}{F_k + L_k} \geqslant \left(\prod_{k=1}^{n} \frac{F_k}{F_k + L_k}\right)^{1/n}$$

and

$$\frac{1}{n}\sum_{k=1}^{n} \frac{L_k}{F_k + L_k} \geqslant \left(\prod_{k=1}^{n} \frac{L_k}{F_k + L_k}\right)^{1/n}.$$

Adding up the above inequalities yields

$$1 \geqslant \frac{\sqrt[n]{\prod\limits_{k=1}^{n} F_k} + \sqrt[n]{\prod\limits_{k=1}^{n} L_k}}{\sqrt[n]{\prod\limits_{k=1}^{n} (F_k + L_k)}}.$$

This gives inequality (1) with equality when $n = 1$.

Also solved by Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain; and Ioannis D. Sfikas, National and

Kapodistrian University of Athens, Greece.

Problem 803. *Proposed by Ovidiu Furdui and Alina Sintamarian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{H_{n+m}}{n(n+m)^2}$$

where $H_n = 1 + 1/2 + \cdots + 1/n$ denotes the n^{th} harmonic number.

Solution *by Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece.*

Let I be the given double sum. Setting $k = n + m$ and rearranging the order of summation gives $I = \sum\limits_{k=2}^{\infty} \frac{H_k H_{k-1}}{k^2}$. The harmonic numbers satisfy the recurrence equation $H_k = 1/k + H_{k-1}$. So the numerator becomes $H_k H_{k-1} = H_k^2 - \frac{H_k}{k}$ and

$$I = \sum_{k=2}^{\infty}\left(\frac{H_k^2}{k^2} - \frac{H_k}{k^3}\right) = \sum_{k=2}^{\infty}\frac{H_k^2}{k^2} - \sum_{k=2}^{\infty}\frac{H_k}{k^3}\,.$$

Now, Euler found that

$$\sum_{k=1}^{\infty}\frac{H_k}{k^a} = \left(1+\frac{a}{2}\right)\zeta(a+1) - \frac{1}{2}\sum_{p=1}^{a-2}\zeta(p+1)\zeta(a-p)$$

with ζ the zeta function and $a \geqslant 2$. Then

$$\sum_{k=2}^{\infty}\frac{H_k}{k^3} = \sum_{k=1}^{\infty}\frac{H_k}{k^3} - H_1 = \frac{\pi^4}{72} - 1.$$

Borwein and Borwein (1995) proved that

$$\sum_{k=1}^{\infty}\frac{H_k}{k^2} = \frac{17\pi^4}{360}\,.$$

Finally,

$$I = \sum_{k=2}^{\infty}\frac{H_k^2}{k^2} - \sum_{k=2}^{\infty}\frac{H_k}{k^3} = \left(\frac{17\pi^4}{360} - 1\right) - \left(\frac{\pi^4}{72} - 1\right) = \frac{\pi^4}{30}.$$

Also solved by the Missouri State Problem Solving Group, Missouri State University, Springfield, MO; and the proposers.

Problem 804. *Proposed by D.M. Batinetu–Giurgiu, "Matei Basarab"*

National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.

Compute the following limit

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}}.$$

Solution *by Shafiqur Rahman, Bangladesh.*

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty} \frac{\sqrt[n]{\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{n^n}}}{\sqrt[n]{\frac{(2n-1)!!}{n^n}}} * \lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{\sqrt[n+1]{(2n+1)!!}}$$

$$= \lim_{n\to\infty} \frac{\frac{\sqrt[n+1]{(n+1)!}}{(n+1)^{n+1}} n^n}{\frac{2n+1}{(n+1)^{n+1}} n^n} = \lim_{n\to\infty} \frac{n+1}{2n+1} * \lim_{n\to\infty} \sqrt[n+1]{\frac{(n+1)!}{(n+1)^{n+1}}}$$

$$== \frac{1}{2} \lim_{n\to\infty} \frac{n+2}{(n+2)^{n+2}} (n+1)^{n+1} = \frac{1}{2} \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \frac{1}{2e}.$$

Also solved by Ravi Prakash, New Delhi, India; Soumitra Mandal, Chandar Nagore, India; Marian Ursarescu, Romania; Angel Plaza, Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain; the Missouri State Problem Solving Group, Missouri State University, Springfield, MO; and the proposers.

Problem 805. *Proposed by D.M. Batinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Let $z_k = x_k + iy_k$ be a complex number where $k \in \{1, 2, \ldots, n\}$. Prove that

$$\sum_{k=1}^{n} \sqrt{{x_k}^4 + {y_{n-k+1}}^4} \geq \frac{\sqrt{2}}{2} \sum_{k=1}^{n} |z_k|^2.$$

Solution *by Ravi Prakash, New Delhi, India.*

We have $(x^2+y^2)^2 \leqslant 2(x^4+y^4)$. Thus

$$\sqrt{2}\sqrt{x_k^4+y_{n-k+1}^4} \geqslant x_k^2+y_{n-k+1}^2$$
$$\Rightarrow \sqrt{2}\sum_{k=1}^{n}\sqrt{x_k^4+y_{n-k+1}^4}$$
$$\geqslant \sum_{k=1}^{n}(x_k^2+y_{n-k+1}^2) = \sum_{k=1}^{n}x_k^2+\sum_{k=1}^{n}y_k^2 = \sum_{k=1}^{n}|z_k|^2.$$

This gives the desired inequality.

Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania; Ioannis D. Sfikas, National and Kapodistrian University of Athens, Greece; Myagmarsuren Yadamsuren, Darkhan, Mongolia; and the proposers.

Problem 806. *Proposed by Marius Dragan, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

If $a_1, a_2, \ldots, a_n > 0$ are such that $\sum\limits_{k=1}^{n} a_k = 1$, then prove that

$$(1+1/a_2)^{na_1^2}(1+1/a_3)^{na_2^2}...(1+1/a_n)^{na_{n-1}^2}(1+1/a_1)^{na_n^2} \geq n+1.$$

Solution *by the proposers*

By Bergstrom' inequality, we obtain

$$\frac{a_1^2}{x+a_2}+\frac{a_2^2}{x+a_3}+\cdots+\frac{a_n^2}{x+a_1} \geqslant \frac{(a_1+a_2+\cdots+a_n)^2}{(nx+a_1+a_2+\cdots+a_n)}$$
$$\Leftrightarrow \frac{a_1^2}{x+a_2}+\frac{a_2^2}{x+a_3}+\cdots+\frac{a_n^2}{x+a_1} \geqslant \frac{1}{n\left(x+\frac{1}{n}\right)}.$$

If we integrate this with respect to x on [0,1], we deduce

$$a_1^2\ln\left(1+\frac{1}{a_2}\right)+a_2^2\ln\left(1+\frac{1}{a_3}\right)+\cdots+a_n^2\ln\left(1+\frac{1}{a_1}\right) \geqslant \frac{1}{n}\ln(n+1),$$

Which is equivalent to the desired inequality.

Problem 807. *Proposed by Titu Zvonaru, Comanesti, Romania.*

If A, B, and C are the angles of a triangle and $\alpha = A/2, \beta = B/2$, and $\gamma = C/2$, prove that

$$\sqrt{6(1+\cos A\cos B\cos C)-2\sin\alpha\sin\beta\sin\gamma(1-8\sin\alpha\sin\beta\sin\gamma)}$$

$$\geq 4\cos\alpha\cos\beta\cos\gamma.$$

Solution *by by Ioan Viorel Codreanu, Satulung, Maramures, Romania.*

Using the well-known equalities $\prod \cos A = \frac{s^2-(2R+r)^2}{4R^2}$, $\prod \sin \frac{A}{2} = \frac{r}{4R}$, and $\prod \cos \frac{A}{2} = \frac{s}{4R}$, the inequality is equivalent to

$$6\left(1+\frac{s^2-(2R+r)^2}{4R^2}\right) - 2 * \frac{r}{4R}\left(1 - 8 * \frac{r}{4R}\right) \geqslant 16 * \frac{s^2}{16R^2}$$

or $s^2 \geqslant 13Rr + r^2$. Using the Gerretsen Inequality $s^2 \geqslant 16Rr - 5r^2$ we need only prove that $16Rr - 5r^2 \geqslant 13Rr + r^2$ or $R \geqslant 2r$, which is the Euler inequality..

Also solved by the proposer.

KAPPA MU EPSILON

Report of the KME South Eastern Regional Convention

Friday, April 6, 2018 - Saturday, April 7, 2018

Hosted by GA Zeta at Georgia Gwinnett College, Lawrenceville, GA

Submitted by Jamye Curry, South Eastern Regional Director

The South Eastern Regional Convention was hosted by the Georgia Zeta Chapter of the Kappa Mu Epsilon National Mathematics Honor Society at Georgia Gwinnett College (GGC) in Lawrenceville, Georgia.

The meeting began Friday evening on April 7, at 5:30 P.M. starting with registration and greetings by student volunteers (Carmen Roland and Antoinette Miezan), and faculty volunteer (Tonya DeGeorge), followed by a social event at 6:30 PM, which included a math movie.

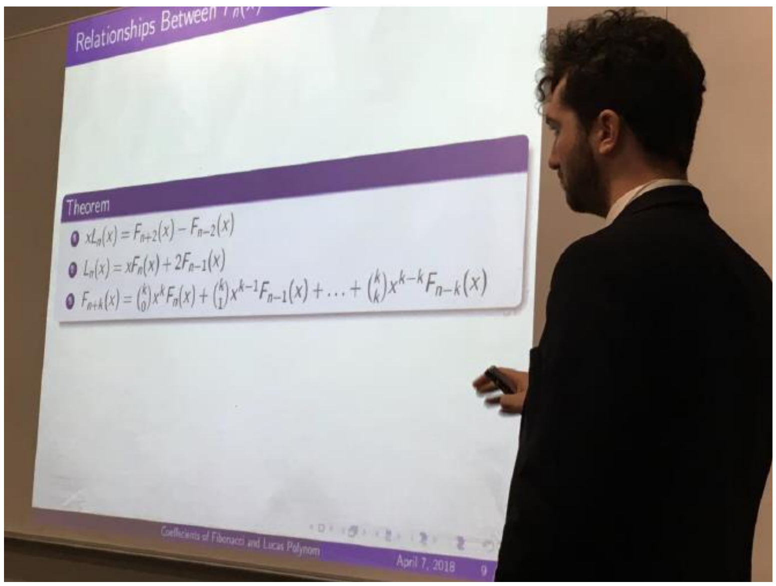
On Saturday morning, the meeting continued at 8:00 AM with a continental breakfast, followed by a very inspiring welcome address by Dr. Aris Winger, mathematics professor of GGC, which set the tone for the friendships and the contacts that the student and faculty participants developed during the event. Afterwards, there were five student presentations of three separate chapters, with a keynote address given by Dr. Livinus Uko, Faculty Co-Sponsor of GA Zeta. (Please see the outline of the program attached with this report for more details on presenters, and titles and abstracts of each presentation.)

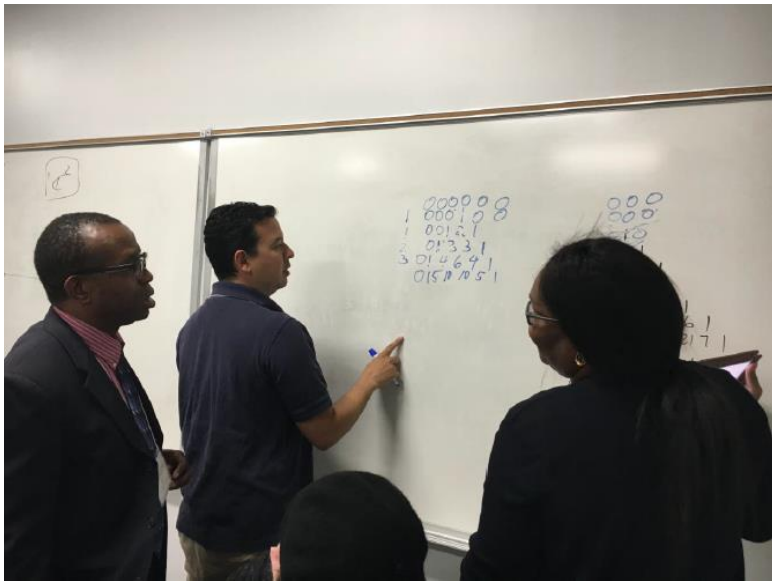
The GGC faculty participants served as the Judging Committee of the student presentations, which included Dr. JC Price, Dr. Daniel Pinzon, Paula Krone, and Tonya DeGeorge. Each student presenter received an award certificate coordinated by Dr. Jennifer Sinclair, Faculty Co-Sponsor of GA Zeta.

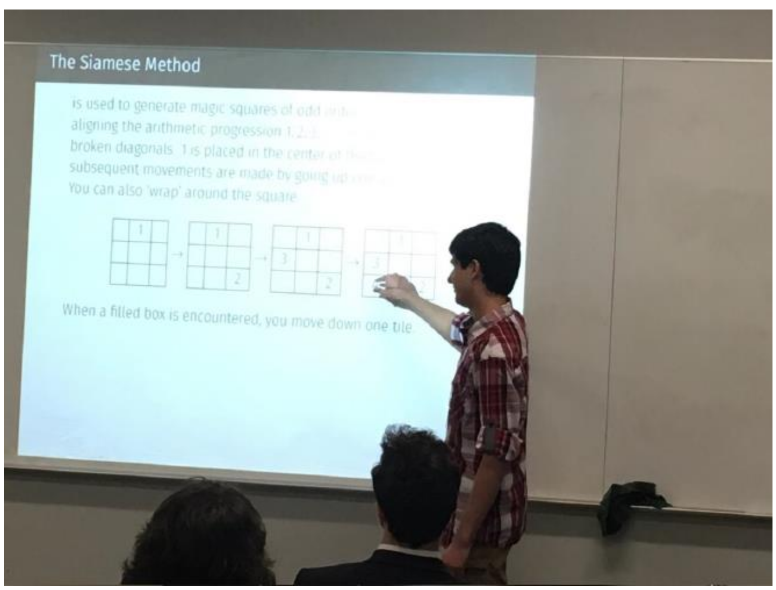
The table below includes the attendance by chapter.

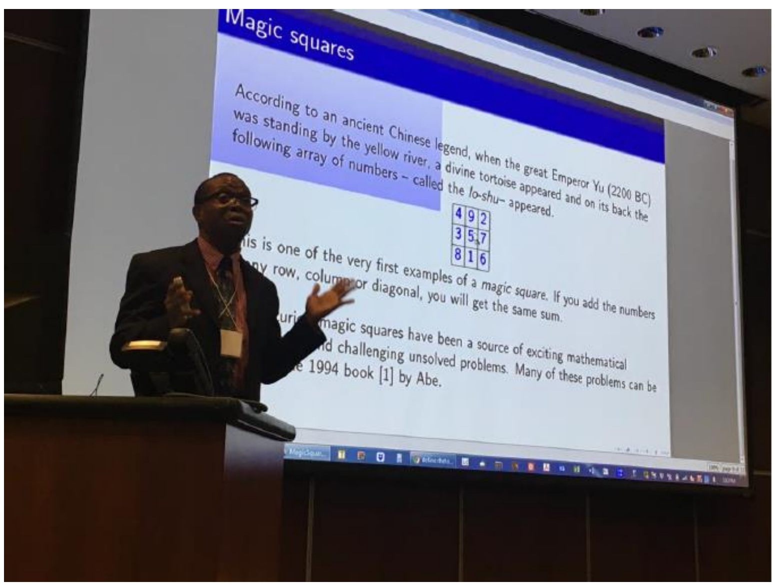
Chapter	School	Faculty	Student(s)
Alabama Beta	University of North Alabama	2	2
Alabama Theta	Jacksonville State University	1	7
Florida Delta	Embry-Riddle Aeronautical University	0	1
Georgia Zeta	Georgia Gwinnett College	7	5
North Carolina Zeta	Catawba College	3	1
Five Chapters Attending		13	16

In conclusion, it was a very successful meeting as everyone enjoyed the social event on Friday, and enjoyed breakfast, lunch and snacks throughout the day on Saturday, while collaborating and discussing mathematics.
At the closing, there were discussions about the opportunity to consider hosting the next South Eastern Regional Convention in 2020, and the meeting concluded at 2:00 P.M. on Saturday afternoon. Thank you GA Zeta for hosting the meeting, and to each chapter for attending!






The Siamese Method
When a filled box is encountered, you move down one tile.


Magic squares
According to an ancient Chinese legend, when the great Emperor Yu (2200 BC) was standing by the yellow river, a divine tortoise appeared and on its back the following array of numbers – called the lo-shu– appeared.
4 9 2
3 5 7
8 1 6
is one of the very first examples of a magic square. If you add the numbers
row, or diagonal, you will get the same sum.
magic squares have been a source of exciting mathematical
challenging unsolved problems. Many of these problems can be
1994 book [1] by Abe.

KAPPA MU EPSILON
Report of the KME Great Lakes Regional Convention
Saturday, April 7, 2018
Hosted by WV Alpha and WV Beta at Wheeling Jesuit University;
Wheeling, WV
Submitted by Pete Skoner, Great Lakes Regional Director

This Great Lakes Regional Convention was hosted by the West Virginia Alpha (Bethany College) and West Virginia Beta (Wheeling Jesuit University) chapters of the Kappa Mu Epsilon National Mathematics Honor Society at Wheeling Jesuit University in Wheeling, West Virginia.

The meeting began on the morning of Saturday, April 7, at 9:00 A.M. after registration and a continental breakfast with welcomes from both professors Adam Fletcher from WV Alpha and Marc Brodie from WV Beta. Regional director Pete Skoner then offered opening remarks including a review of the schedule, KME and Great Lakes history, and general KME announcements.

The keynote address followed. Pennsylvania Mu student Vanessa Valovage provided the introduction to the keynote speaker, professor Brendon LaBuz, also from Pennsylvania Mu. His interesting talk was titled "Connectedness and Exotic Spaces." He laid a foundation with multivariable calculus to describe connectedness and path connectedness leading to exotic spaces including the Alexander horned sphere, the Hawaiian earring, and the Griffiths twin cone.

Five student presentations followed with three before lunch and two after. The presentations are listed below in the order presented.

1. *Elliptic Curve Cryptography*, Alissa Whiteley, Maryland Delta, Frostburg State University

2. *And It Goes Like This*, the Third, the Fifth, Alyssa Smydo, West Virginia Alpha, Bethany College

3. *Tessellations, Emma Seibert, Maryland Delta*, Frostburg State University

4. *Sudoku Solutions in Mupad Using Gröbner Basis*, AJ Zelenky, Pennsylvania Mu, Saint Francis University

5. *Euler's Polyhedral Formula*, Sarah Sparks, Maryland Delta, Frostburg State University

The Awards Committee was chaired by professor Marc Brodie of WV Beta, and included the following other members: professor Marc Michael, MD Delta; Demetrick McDonald, MD Delta; Rachel Gruber, PA Mu; and Austin Paul-Orecchio, WV Alpha. After all the presentations and a meeting of the Awards Committee, Demetrick explained that the committee considered all the presentations as exceptional and rated the presentations by Sarah Sparks of MD Delta and AJ Zelenky of PA Mu as the best. Each student presenter received a cash award and a certificate, with larger cash awards for the two voted best.

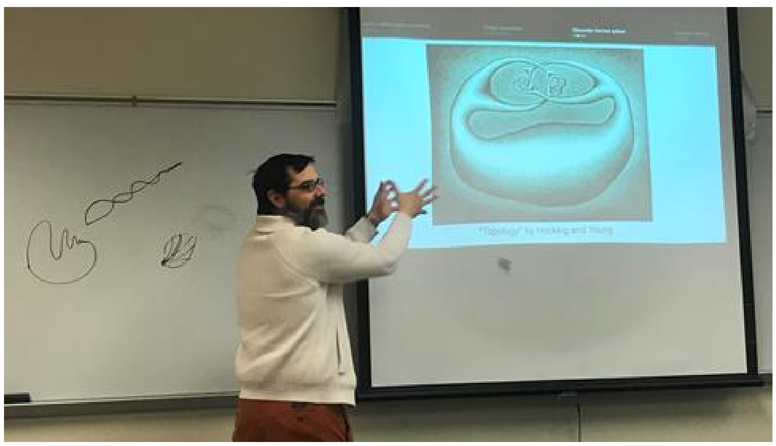
The table below includes the attendance by chapter.

Chapter	School	Faculty	Student(s)
Maryland Delta	Frostburg State University	3	5
Pennsylvania Mu	Saint Francis University	2	3
West Virginia Alpha	Bethany College	2	3
West Virginia Beta	Wheeling Jesuit University	1	0
Four Chapters Attending		8	11

In conclusion, it was a very productive meeting for the four chapters attending with collaboration in planning and coordinating the meeting, an excellent keynote address from professor LaBuz, five outstanding student presentations, student involvement in introducing speakers and serving on the awards committee, robust mathematics conversations, the opportunity to renew friendships and make new friends, and sharing good fellowship. Facebook and Twitter were both used to share meeting highlights.

At the closing, there were discussions about the opportunity to consider hosting the 2019 national convention, and the next Great Lakes regional convention in 2020. The meeting concluded at 2:30 P.M. Thank you to WV Alpha and Beta for hosting the meeting, and to all four chapters for attending!


FROSTBURG


"Topology" by Hocking and Young

KAPPA MU EPSILON
Report of the KME New England Regional Convention
Friday, April 7, 2018
Hosted by New York Omicron at St. Joseph's College, Long Island, NY
Submitted by Donna Marie Pirich, New England Regional Director

The New England Regional Convention was hosted by New York Omicron (St. Joseph's College) chapter of the Kappa Mu Epsilon National Mathematics Honor Society at St. Joseph's College on Long Island, New York.

Everyone enjoyed breakfast, lunch, cake, and snacks throughout the day, as well as the souvenir gifts they were presented with at the Welcome Table upon arrival.

The keynote address, *"The Math and Psychology Behind Rock, Paper, Scissors"* given by Dr. Elana Resiser, Corresponding Secretary of New York Omicron, was followed by a very enjoyable RPS tournament. Robert Dilworth, President of York Omicron, gave a welcome address, and spoke about the many activities of the host chapter, New York Omicron, that support both the purposes of Kappa Mu Epsilon and the mission of St. Joseph's College. Some of these activities include the Saturday Morning Math Clinic which offers free math tutoring to local high schools, as well as fundraising activities to support an annual Christmas Toy Drive and Easter Basket Drive for local children who have been impacted by domestic violence. Dr. Donna Marie Pirich, New England Regional Director, and New York Omicron Faculty Sponsor planned and coordinated the convention and presented award certificates at the end of the day.

Nine students presented, five before lunch and four after. The presentations are listed in the order in which they were presented.

1. *Polynomial Division and Gröbner Bases*, Christopher Guevara, Long Island University
2. *Square-wheeled Bicycles*, Bushra Kazmi, Long Island University
3. *Constructing Chords Between Concentric Circles*, Sam McCrosson, Long Island University
4. *Credit Card Default Prediction Using SAS Enterprise Miner: A Comparative Study of Taiwanese Companies*, Eleni Diakolambrianos, Bryant University
5. P*redicting Airline Satisfaction Using Machine Learning Algorithms*,

Elise Chen, Bryant University

6. *Pricing of Cryptocurrency: A Data Analytics Study*, Matthew Picard, Bryant University

7. *Girls and Women in STEM: Understanding the Relationship Between Influence and Career Participation*, Noel King, Molloy College

8. *Where is the Math in Abstract Algebra?*, Chelsea Brandimarte, Kayla Gill, and Britannie Naughton, Molloy College

9. *iPads, SMART Boards, and Mathematical Achievement: What's the Deal?*, By Stephen Bates, St. Joseph's College

Chapter	School	Faculty	Student(s)
Connecticut Gamma	Central Connecticut State University	1	5
New York Lambda	Long Island University	3	17
New York Omicron	St. Joseph's College	6	16
New York Rho	Molloy College	1	4
Rhode Island Beta	Bryant University	1	3
Five Chapters Attending		12	45


St.Joseph's College
NEW YORK
OMICRON CHAPTER
KAPPA MU EPSILON
NATIONAL MATHEMATICS
HONOR SOCIETY


EXIT


St Joseph's College
NEW YORK

The abstracts of the talks, a copy of the convention program and additional photographs can be found at: https://dpirich.wixsite.com/kmeneregional2018.

Kappa Mu Epsilon News

Edited by Cynthia Huffman, Historian
Updated information as of January 2018

News of chapter activities and other noteworthy KME events should be sent to

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KAPPA MU EPSILON
Installation Report
Kansas Eta, Sterling College
Sterling, Kansas

The Kansas Eta Chapter of Kappa Mu Epsilon was installed at 7:00 p.m. on November 30, 2017 at a ceremony held on the campus of Sterling College in Sterling, Kansas. The meeting was conducted by Dr. Daniel Julich. KME national president Dr. Brian Hollenbeck served as the installing officer. Seven students and two faculty members were initiated as the charter members of the Kansas Eta Chapter. They are: Assistant Professors of Mathematics Pete Kosek and Amy Kosek, Tiana Bonde, Lucas Garrett, Aaron Pinkerton, Shelby Stowe, Jenna Weimer, Mikaela Wells, and Zachary Williams. The first officers of the chapter were installed: Shelby Stowe, President; Mikaela Wells, Vice President; Jenna Weimer, Secretary; Aaron Pinkerton, Treasurer; Amy Kosek, Corresponding Secretary; and Pete Kosek, faculty sponsor. Following the installation ceremony, Dr.

Hollenbeck presented a talk titled, Mathematical Illusions, and Pete Kosek concluded the ceremony in prayer. Several friends and family members of the initiates attended the event. Below is a picture of Dr. Hollenbeck and the charter members of Kansas Eta.

Chapter News

AL Gamma – University of Montevallo

Corresponding Secretary – Scott Varagona; 8 New Members

New Initiates – Elizabeth Blue, James Boynton, William Carman, Melina Carranza Cortes, Noah Estus, Tatum Etten-Bohm, Evelyn Myers, and Jonathan Torres.

AL Theta – Jacksonville State University

Chapter President – Daniel Miradakis; 50 Current Members

Other Fall 2017 Officers: Blake Jackson, Vice President; Becky Peters, Secretary; Holly Sparkman, Treasurer; and Dr. David Dempsey, Corresponding Secretary and Faculty Sponsor

The Alabama Theta chapter met biweekly during Fall 2017 and had at least monthly events. September's event was an outing for dinner and the musical "Young Frankenstein," in which KME member Jasmine Beaudette played a role. On October 27, we held our annual Halloween Party, complete with math-themed costumes and pumpkin carving. On December 1, we held our Christmas/End-of-the-Semester Party, complete with games and "Dirty Santa" gift swamp. Our annual Spring Initiation Ceremony is planned for March 30, 2018.

AR Beta – Henderson State University

Corresponding Secretary – Dr. Fred Worth; 3 New Members

New Initiates – N. Ryen Gillespie, Benjamin Hynds, and Colton B. Henard.

CO Delta – Colorado Mesa University

Corresponding Secretary – Erik Packard; 13 New Members

New Initiates – Jeri Abegglen, Bret Brouse, Emilee Castleton, Saige Dacuycuy, Erika Equinca, Steven Hale, Scott Jackson, Brittany Kelleher, Jeremiah Moskal, Kayla Murphy, Seth Richard, Alexandra Tennant, and Austin Zanoni.

CT Beta – Eastern Connecticut State University

Corresponding Secretary and Faculty Sponsor – Dr. Mehdi Khorami; 475 Current Members

CT Gamma – Central Connecticut State University

Chapter President – Rina Saliu; 53 Current Members; 25 New Members

Other Fall 2017 Officers: Nilay Nitin Bhatt, Vice President; Luke Robert D'Ascoli, Secretary; Damian Szarwacki, Treasurer; Mihai Bailesteanu, Corresponding Secretary; and Marian F. Anton, Faculty Sponsor

The second annual initiation ceremony of the Connecticut Gamma Chapter of Kappa Mu Epsilon was held in the Connecticut Room of Memorial Hall on the campus of Central Connecticut State University on Friday, November 17, 2017, at 5:30 PM. Faculty sponsor, Professor Marian Anton organized the event, and corresponding secretary Professor Mihai Bailesteanu opened the event with a welcome and led the events. Dr. Robin Kalder, the chair elect of the Department of Mathematical Sciences gave some opening remarks, while Dean Zdzislaw Kremens, offered a university welcome. An elegant dinner was followed by Professor Mihai Bailesteanu, who presented a talk on "The Mathematics behind Gerrymandering." The initiation ceremony was led by Dr. Marian Anton, who is the corresponding secretary. Participating in the ceremony were the chapter officers: Rina Saliu, President; Nilay Nitin Bhatt, Vice President; Luke Robert D'Ascoli, Recording Secretary; and Damian Szarwacki, Treasurer. Each officer was charged with the responsibilities of the office, and each chose to accept those responsibilities. After Secretary D'Ascoli completed describing KME's crest, the initiation ceremony commenced. Two members of the faculty became members of Connecticut Gamma, Professors Philip Halloran and Shelly Jones, while 23 students became new members: Kaleab Admassu, Khaled Alsaqri, Nicholas Domenic Daddona, Sabrina Doolgar, Justin Tyler Fagnoni, Brandon T. Fennell, Rebecca Marie Fontaine, John Fox, Luke Daniel Graff, Alexander Konefal, Matthew Konnik, Jason Madej, Cierra McGowan, Alyssa R. Mercaldi, Kevin John Moore, Karolina Aleksandra Osowiecka, Christopher P. Oville, Kathryn Ellen Rohner, Nicholas S. Tseka, William A. Tuxbury, Ryan Zimmer and Gabriel Yawin. Each initiate was invited to sign the Connecticut Gamma Chapter Roll, and was presented with a certificate, membership card, KME brochure, a program, and a KME jewelry pin. Dr. Mihai Bailesteanu offered closing remarks. A total of 60 people were in attendance.

New Initiates – Kaleab Admassu, Khaled Alsaqri, Nicholas Domenic Daddona, Sabrina Doolgar, Justin Tyler Fagnoni, Brandon T. Fennell, Rebecca Marie Fontaine, John Fox, Luke Daniel Graff, Philip Halloran, Shelly Jones, Shelby Kelley, Alexander Konefal, Matthew Konnik, Jason Madej, Cierra McGowan, Alyssa R. Mercaldi, Kevin John Moore, Karolina Aleksandra Osowiecka, Christopher P. Oville, Kathryn Ellen Rohner, Nicholas S. Tseka, William A. Tuxbury, Gabriel Yawin, and Ryan Zimmer.

GA Eta – Atlanta Metropolitan State College

Chapter President – Leia Singh; 22 Current Members

Other Fall 2017 Officers: Devante Singletary, Vice President; Joseph Daniels, Secretary; Lanessa Northcut, Treasurer; Mulugeta Markos, Corresponding Secretary; and Kwan Lam, Faculty Sponsors

HI Alpha – Hawaii Pacific University

Chapter President – Andreas Koutsogiannis; 10 Current Members; 2 New Members

Other Fall 2017 Officers: Tara Davis, Corresponding Secretary and Faculty Sponsor

The Hawaii Alpha chapter had an induction dinner on November 16, 2017.

IA Alpha – University of Northern Iowa

Chapter President – Jake Weber; 25 Current Members; 2 New Members

Other Fall 2017 Officers: Maria Ahrens, Vice President; Destiny Leitz, Secretary; Nicole Mlodzik, Treasurer; and Mark D. Ecker, Corresponding Secretary and Faculty Sponsor

Our first fall KME meeting was held on October 5, 2017 at Professor Mark Ecker's house where retired John Deere engineer James Stevenson presented "Applying Mathematics - 50 Years." Graduate student Jason Shultz presented his paper entitled "Confidence Intervals for the Coefficient of Variation: A Bootstrapping Approach" at our second meeting on November 9, 2017 at Professor Doug Mupasiri's home. Student member Nicole Mlodzik addressed the fall initiation banquet with "Resale Value of Individual Sets of Magic the Gathering." Our fall banquet was held at Peppers restaurant in Cedar Falls on December 5, 2017, where two new members were initiated.

IA Gamma – Morningside College

Corresponding Secretary – Chris Spicer; 12 New Members

New Initiates – Samantha Anderson, Emily Bock, Logan Buth, Kevin Hancock, Clare Kortlever, Jason Latta, Jacob Miller, Melissa Pauley, Billy Salber, David Swerev, Torie Sykes, and Alex Watkins.

IL Zeta – Dominican University

Corresponding Secretary – Aliza Steurer; 18 Current Members; 4 New Members

IN Alpha – Manchester University

Corresponding Secretary – Jim Brumbaugh-Smith; 22 New Members

New Initiates – Nebiyu Alemu, Alivia Banks, Thomas Dean, Blaine Dean-Hamilton, Nathan

Frantz, Frehiwot Gudeta, Mitchell Jenkins, Tyler Klein, Caleb Leininger, Micah Leininger, Karmen Marquardt, Robin Mitchell, Joshua McCoy, Ryan Morley, Benjamin Niederhelman, Curtis Nordmann, Emily Pleadwell, Sean Rizvic, Alex Spencer, Joseph Swartz, Kidist Tessema, and Kabrina Hydre Yusoff.

KS Alpha – Pittsburg State University

Chapter President – Georgette Searan; 30 Current Members; 13 New Members

Other Fall 2017 Officers: Jordan Bailey, Vice President; Zach Bowen, Secretary; Peyton Burlingame, Treasurer; Dr. Tim Flood, Corresponding Secretary; and Dr. Scott Van Thuong, Faculty Sponsor

The Kansas Alpha Chapter continued to remain active with a lively Fall 2017 semester. We had meetings in September, October, and November. Our KME members presented on coloring n-dimensional Euclidean space, constructing Möbius bands from fruit roll ups, and Archimedes' Cattle Problem. Spirited conversation over pizza and beverages followed these presentations. In December, Dr. Leah Childers and her family hosted a wonderful Christmas party at their home, thus concluding the semester.

KS Beta – Emporia State University

Chapter President – Paige Nurnberg; 41 Current Members; 10 New Members

Other Fall 2017 Officers: CeJae Jordan, Vice President; Henry Weiner, Secretary; Anthony Gagliano, Treasurer; Tom Mahoney, Corresponding Secretary; and Brian Hollenbeck, Faculty Sponsor

NEWS ITEMSThe Kansas Beta chapter took a trip to the Linda Hall Library at the University of Missouri - Kansas City to visit their collection of old and rare books of mathematics. The KME Lounge sees regular use for KME and faculty game nights and for exploratory undergraduate seminars.

New Initiates – Adelaide Akers, Amy Cox, Geoffrey Critzer, Kaitlin Ferman, Shibo Gao, Emma Griem, Natalie Leffler, Montan Loibl, Terry Salava, and Bekah Selby.

KS Delta – Washburn University

Chapter President – Katelyn Skillingstad; 22 Current Members

Other Fall 2017 Officers: Kaitlyn Jones, Vice President; Alex Rottinghaus, Secretary; Laura Crosswhite, Treasurer; and Kevin Charlwood, Corresponding Secretary and Faculty Sponsor

The Kansas Delta chapter of KME met with our math club for three lunchtime meetings in the Fall 2017 semester. All three meetings featured speakers, two from local firms who gave presentations about their work in statistics at their companies, and one presentation from a faculty member from our department.

KS Eta – Sterling College

Chapter President – Shelby Stowe; 10 Current Members; 10 New Members
Other Fall 2017 Officers: Mikaela Wells, Vice President; Jenna Weimer, Secretary; Aaron Pinkerton, Treasurer; Amy Kosek, Corresponding Secretary; and Pete Kosek, Faculty Sponsor

The Kansas Eta Chapter was installed at 7:00 pm on November 30, 2017. There were seven students and two faculty members who were initiated as the charter members of Kansas Eta. After the ceremony, KME President, Dr. Brian Hollenbeck, presented a talk titled, Mathematical Illusions, which was widely appreciated by the charter members in addition to the friends, family, and other Sterling College students, professors, and administrators in attendance.

MD Alpha – Notre Dame of Maryland University

Chapter President – Chinwendu Nwokeabia; 13 Current Members
Other Fall 2017 Officers: Bhavya Bhardwaj, Vice President; Justice Walrath, Secretary; Hannah Woodworth, Treasurer; and Charles Buehrle, Corresponding Secretary and Faculty Sponsor

MD Alpha hosted a KME cookie social on Wednesday October 4^{th}.

MD Delta – Frostburg State University

Chapter President – Sarah Sparks; 17 Current Members
Other Fall 2017 Officers: Demetrick McDonald, Vice President; Zach Kline, Secretary; Emma Siebert, Treasurer; Mark Hughes, Corresponding Secretary and Faculty Sponsor; and Frank Barnet, Faculty Sponsor

The Maryland Delta Chapter had its first meeting in September featuring pizza and some fun with a Sudoku contest. Our next activity was a lecture on November 1 from an FSU alumnus Dr. Brendon LaBuz who is currently a professor of mathematics at St. Francis University in Pennsylvania. His lecture involved some new ideas on the axioms of geometry. In mid-November we had a very successful pancake and bake sale. Throughout the semester, Calculus II tutoring was provided on a walk-in basis on Wednesday evenings by chapter vice president Demetrick McDonald and chapter member Devon Zollinhofer.

MD Epsilon – Stevenson University

Chapter President – Rebecca Gruver; 13 Current Members
Other Fall 2017 Officers: Courtney Hohn, Vice President; Alli Culp, Secretary; Jess Rega, Treasurer; and Benjamin Wilson, Corresponding Secretary and Faculty Sponsor.

This Fall, KME at Stevenson took part in and ran several events including a day of service at the University called Mustangs Make a Difference Day at which KME students facilitated the construction of tangram sets to be donated to a local elementary school, a bake sale for Hurricane Maria relief at which over $1000 was raised, and an event for Maryland STEM Festival called the Mathstravaganza which included several math themed exhibits, demonstrations, and games open to the general public.

MI Alpha – Albion College

Corresponding Secretary – Mark Bollman; 5 New Members

New Initiates – Jodie Bosheers, Henry Carnick, Antoniu Fodor, Nicholas Rafaill, and Oana Vesa.

MI Beta – Central Michigan University

Chapter President – Yejean Han; 12 New Members

Other Fall 2017 Officers: Savannah Swiatlowski, Vice President; Natalie DeVos, Secretary; Jared Williams, Treasurer; and Ben Salisbury, Faculty Sponsor

During the Fall 2017 semester, our chapter held six meetings. The meetings varied from expository mathematical talks to mathematical games and challenges. One meeting was themed "undergraduate research," and two of our chapter officers detailed their experience in research, which included an NSF-funded Research Experience for Undergraduates. The chapter hosted a panel event focused on the job possibilities for a student with a degree in mathematics; this is an event that the chapter hopes will continue annually. There were also two special events hosted by our KME chapter: a book sale (which took place from November 13-15) and a tutoring event before final exams entitled "Math-a-palooza."

New Initiates – Richard Clark , Eric Garinger, Abdulaziz Ghandorah, Garrett Goniea, Courtney Guthrie, Yejean Han, Christopher Hebert, Maxwell Hogue, Richard Sivak, Nathan Stempky, Savannah Swiatlowski, and Jared Williams.

MI Delta – Hillsdale College

Chapter President – Tanner Orion Wright; 45Current Members; 14 New Members

Other Fall 2017 Officers: Abigail Trouwborst, Vice President; Gill West, Secretary; Emma Clifton, Treasurer; and Dr. David Gaebler, Corresponding Secretary and Faculty Sponsor

The Michigan Delta chapter welcomed three new officers at the beginning of the 2017-2018 academic year. We hosted three events during the semester. First, the initiation ceremony on October 19 inducted 14 new members into the chapter, and also featured a math talk by Dr. Sam Webster on Euler's theory and practice of infinite series. Next, KME hosted a panel discussion with three faculty members on topics related to mathematics graduate school, which drew about 15 students. Finally, the KME officers and a couple of interested friends gathered for a lesson in the world's greatest card game (no offense to any Euchre fans) from bridge aficionado Prof. Jonathan Gregg.

New Initiates – Audrey Cooley, Kathryn Duhadway, Alexandra Howell, Nathaniel Jones, Hope Jonker, Drew Lieske, Matt Nolan, Joel Pietila, Devon Poage, Hannah Stevenson, Colleen Trainor, Tavia Vitkauskas, Nathan Wamsley, and Myrica Wildes.

MO Alpha – Missouri State University

Chapter President – Rebecca Crow; 30 Current Members; 1 New Member
Other Fall 2017 Officers: Adam Somers, Vice President; Caleb Marshall, Vice President Elect: Symantha Campbell, Secretary; Nicholas Delamora, Public Relations; Jacob Miles, General and Corresponding Secretary; and Dr. Steven Senger, Faculty Sponsor

Missouri Alpha began Fall 2017 with a new faculty sponsor, Dr. Steven Senger, who has been a fun yet responsible addition to our Chapter. Our first events were the student exposition with the College of Natural and Applied Sciences, as well as the KME Picnic, at which students enjoyed food, fun games, and recognition for scholarships from the mathematics department. Our Chapter had three meetings throughout the semester. We were first visited by Dr. Erin Buchanan, who presented on statistical bias in effect sizes for ANOVAs. Secondly, Dr. Steven Senger and Dr. Matthew Wright presented and gave advice with regards to undergraduate research and planning for graduate school. Lastly, several seniors in the Math Department presented their semester research projects on a broad range of topics.

MO Beta – University of Central Missouri

Chapter President – Christina Duerr; 25 Current Members; 7 New Members
Other Fall 2017 Officers: Mackenzie Snyder, Vice President; Matthew Enlow, Secretary; Juliana Ortiz, Treasurer; Daniel Akin, Historian; Rhonda McKee, Corresponding Secretary; Steve Shattuck and Nicholas Baeth, Faculty Sponsors

Missouri Beta chapter held monthly meetings during the Fall 2017 semester. Each meeting includes a mathematical speaker or activity as well as food and fellowship. At the first meeting of the semester, three students presented information about their summer internships. Other topics for the semester included The Great Pi *vs* e Debate video and a rousing game of mathematical Jeopardy. In October, we held our semi-annual book sale to raise funds for meetings and travel to conventions.

New Initiates – Yah Lee Chua, Yah See Chua, Morgan Jorgenson, Salvation Lee, Parker McBride, Zoe Mitchell, and John Derek Noe.

MO Eta – Truman State University

Chapter President – Jennie Huynh; 17 Current Members; 7 New Members
Other Fall 2017 Officers: Kimberlyn Eversman, Vice President; Margaret

Tauser, Secretary; Elizabeth Hale, Treasurer; Andrew Wolf, Historian; Dr. David Garth, Corresponding Secretary and Faculty Sponsor

In our first meeting of the Fall 2017 semester, we introduced Kappa Mu Epsilon to new members by explaining initiation requirements, future meeting dates and topics, and upcoming events. We held a meeting that covered career preparation and encouraged students to attend our university's career fair. The next meeting explored summer opportunities where students shared experience and advice on internships and REUs. Our most popular meeting is registration when students help each other pick classes and plan their schedules. We also cover capstones and graduate school to keep seniors informed. To end the semester, our final meeting is a game night where members play board games together. This semester we brought back an old tradition of having KME dinner following meetings. After our meetings are adjourned, members go to have dinner together in the Student Union Building on campus. Every semester we hold an event called Lunch with Professors where KME reserves a conference room in the math department and invite math students and faculty to enjoy lunch together. This has proven a great way for members to make connections with professors and other math students. Every fall it is tradition for the Mo Eta chapter to hold a bonfire at the university farm. Math students and faculty are welcome to join and eat s'mores. Other events we held included a potluck, a movie night, and hanging out on the quad. We assisted a biology organization with their event Science on Saturday where 2nd through 5th graders visited our university to take classes for one day. Our members taught a class titled Mathemagic where students learned cool math problems, math card tricks, and puzzles. For the second year, we implemented a "big-little" mentor system. This is where upperclassmen, or bigs, are paired with underclassmen, or littles. This was introduced in Fall 2016 and proved to be successful and increase member retention. This system has helped freshmen and sophomores gain advice and knowledge from juniors and seniors.

MO Theta – Evangel University

Chapter President – Brianna Maybee; 14 Current Members

Other Fall 2017 Officers: Brittany Knowlton, Vice President; and Don Tosh, Corresponding Secretary and Faculty Sponsor

Meetings were held monthly. In December we held a pizza party at the home of Dianne Twigger.

MO Lambda – Missouri Western State University

Corresponding Secretary – Dr. Steve Klassen; 4 New Members

New Initiates – Calan Hensley, Brooke Kessler, Cody Osmon, and Tosha Wilson.

MS Alpha – Mississippi University for Women

Chapter President – Aastha Ghimire; 15 Current Members; 1 New Member

Other Fall 2017 Officers: Sugam Bhattarai, Vice President; Sagarina Thapa, Secretary; Dr. Joshua Hanes, Corresponding Secretary and Faculty Sponsor

In the fall semester we inducted one new member and, we gathered supplies and put together 7 boxes for Operation Christmas Child.

MS Gamma – University of Southern Mississippi

Chapter President – Jesse Robinson; 11 Current Members

Other Fall 2017 Officers: Nya Williams, Vice President; Andrew Giovengo, Treasurer; Zhifu Xie, Corresponding Secretary; and Ana Wan, Faculty Sponsor

In the Fall 2017, the MS Gamma Chapter of Kappa Mu Epsilon at The University of Southern Mississippi elected its new officers and passed its new chapter bylaws.

NE Beta – University of Nebraska Kearney

Chapter President – Candy Smith; 12 Current Members; 1 New Member

Other Fall 2017 Officers: Shelby Study, Vice President; Alexis Stockton, Secretary; and Stephanie Stayden, Treasurer; and Dr. Katherine Kime, Corresponding Secretary and Faculty Sponsor

In November 2017, we had one initiate, Kaitlynn Thomas. She is a UNK volleyball player. We invited faculty and their families to the initiation and had a potluck and game night. Numerous children were in attendance. Over the course of the semester, we discussed inviting a speaker and decided on Dr. Daniel Mowrey. He received a bachelor's from UNK and went on to a doctorate in statistics and a career as a statistician at Eli Lilly. His talk is planned for March, 2018.

New Initiates – Kaitlynn Thomas.

NE Delta – Nebraska Wesleyan University

Chapter President – Carter Lyons; 19 Current Members

Other Fall 2017 Officers: Mackenzie Maschka, Vice President; Trevor McKeown, Secretary and Treasurer; and Melissa Erdmann, Corresponding Secretary and Faculty Sponsor

We have had an active and successful fall. In September we had a joint picnic with the Physics Club, and in December we had a joint holiday party with chili. Both events were attended by about 30 including students and faculty and their families. In September a panel of 4 students spoke about their summer research/travel experiences. We had a math graduate student panel made up of 3 students from the University of Nebraska in October. In November Dr. Melissa Erdmann gave a math talk on the imaging technique of Electrical Impedance Tomography; the research team she was privileged to be a part of does lung imaging with EIT.

NY Lambda – LIU Post

Chapter President – Christopher Guevara

Other Fall 2017 Officers: Michelle Arenella, Vice President; Ahok Prabhu, Secretary; Clifford Clark, Treasurer; and Dr. Corbett Redden, Corresponding Secretary and Faculty Sponsor

NY Omicron – St. Joseph's College

Chapter President – Robert Dilworth; 26 Current Members

Other Fall 2017 Officers: Kathrine Miller, Vice President; Frank Loglisci, Secretary; Christiana Morante, Treasurer; Dr. Elana Reiser, Corresponding Secretary; and Dr. Donna Pirich, Faculty Sponsor

On Giving Tuesday, November 28th, the New York Omicron Chapter of Kappa Mu Epsilon National Mathematics Honor Society (KME) held its annual Christmas Toy Drive Celebration. NYS Assemblyman Al Graf and his Chief of Staff, KME SJC alumnus Doug Smith, were on hand to accept our donation of toys in support of Assemblyman Graf's Annual Holiday Toy Drive. KME fundraising efforts enabled our students to purchase toys which will be distributed by Assemblyman Graf to Victims Information Bureau of Suffolk (VIBS). According to Assemblyman Graf, "Every year we reach out to victims of domestic violence who have fled their homes with nothing but their children and the clothes on their backs. These parents and children have suffered unconscionable acts of violence; so for all of us to ensure they have a joyous Christmas season is what makes our community so great. I'd also like to extend a special thank you to St. Joseph's College Kappa Mu Epsilon Mathematics Honor Society, which donated and held a fundraiser to help purchase toys." Chapter President Robert Dilworth, Vice-President Kathrine Miller, Secretary Frank Loglisci, Treasurer Christiana Morante, Corresponding Secretary Dr. Elana Reiser, and Faculty Sponsor / New England Regional Director Dr. Donna Marie Pirich were awarded New York State Assembly Citations of Appreciation at the celebration. Plans are now in the works for the group's Annual Easter Basket Drive, in Spring 2018. A Chipotle fundraiser will help support our efforts. Chipotle in Selden (located at 261 Middle Country Road) will donate 50% of all cash receipts accompanied by the event flyer on Monday, December 11th between 4:00 PM and 8:00 PM. (Pictures are below.)

 

In addition, the New York Omicron Chapter of KME will be hosting the 2018 New England Regional Convention on the Patchogue campus of SJC on Saturday, April 7th, 2018. We are pleased to announce that Dr. Elana Reiser will be our keynote speaker, and will present "The Math and Psychology Behind Rock, Paper, Scissors." The deadline for abstracts and letters of support for consideration is February 1, 2018. Further information about the convention can be found on the convention Web page (https://dpirich.wixsite.com/kmeneregional2018).

OH Theta – Capital University

Chapter President – Nick Hernandez; 17 Current Members; 11 New Members

Other Fall 2017 Officers: Sage Conger, Vice President; Tiffany Kempthorne, Secretary; Lisa Lotz, Treasurer; Paula Federico, Corresponding Secretary; and Jonathan Stadler, Faculty Sponsor

During the semester, our members, led by Nick Hernandez, worked on creating a constitution for our chapter. After that, Nick submitted all the required documentation to have our chapter recognized as a student organization. Our chapter was recently recognized as a campus student organization.

PA Eta – Grove City College

Corresponding Secretary – Dale L. McIntyre; 17 New Members

New Initiates – Michael Augspurger, Anna Brinling, Hannah Dunmire, Samantha Edinger, Rachel Falk, Laura Harms, Lindsey Harrington, Esther James, Wei-En Lu, Daniel Maienshein, Allison Miller, Nate Shackleton, Bethany Stobie, Kate (Kathryn) Storm, Joe Swanson, Justin Sybrandt, and Jonathan Worobey.

PA Iota – Shippensburg University

Corresponding Secretary – Paul Taylor; 5 New Members

New Initiates – Chase Angle, Dylan Herb, Vincent Preli, Daniel Rosenobey, and Kaitlyn

Shultz.

PA Mu – Saint Francis University

Chapter President – David Madl: 59 Current Members

Other Fall 2017 Officers: Staci Shoemaker, Vice President; Nicholas Frank, Secretary; Hannah Boyd, Treasurer; Pete Skoner, Corresponding Secretary; and Brendon LaBuz, Faculty Sponsor

PA Pi – Slippery Rock University

Corresponding Secretary – Elise Grabner; 7 New Members

New Initiates – Alexander Barclay, Quentin Donofrio, Gennifer Elise Farrell, Timothy Samec, Eric VonKaenel, Melinda Whelan, and John Yannotty.

PA Rho – Thiel College

Chapter President – Rebecca Adams; 8 Current Members

Other Fall 2017 Officers: Sarah McConnell, Vice President; Joshua Evjene, Secretary; John Thiel, Treasurer; Russ Richins, Corresponding Secretary; and Dr. Jie Wu, Faculty Sponsor

Our KME chapter participated in our department potluck dinner in October. In November, KME held their annual Challenge 24 Competition. Students donate $1 or a food item to the local food bank in return for a chance to compete in the Challenge 24 tournament. The winner was awarded a gift card.

RI Beta – Bryant University

Chapter President – Nathaniel Morgan; 40 Current Members

Other Fall 2017 Officers: Owen Wrinn, Vice President; Tyler Talbot, Secretary; Danica Butler, Treasurer; John Quinn, Corresponding Secretary; and Alan Olinsky, Faculty Sponsor

We are currently planning on having a faculty member and several students to attend and present at the 2018 KME New England Regional Convention, hosted by New York Omicron, at St. Joseph's College in Patchogue, NY in April.

SC Gamma – Winthrop University

Chapter President – Colin Frazier; 17 Current Members; 8 New Members

Other Fall 2017 Officers: Christy Knight, Vice President; Sydney McCall, Secretary; Victoria Nidiffer, Treasurer; and Jessie Hamm, Corresponding Secretary and Faculty Sponsor

This fall semester our chapter of Kappa Mu Epsilon initiated 8 new members! Below is a picture of our new members along with our faculty advisor and officers at the induction ceremony.

Our service project in the fall was helping with a Halloween event organized by Winthrop University. We manned two tables of math and Halloween themed activities for kids in the Rock Hill area. Our spring service project will be on March 2. We will be organizing a Math Day for a local elementary school. Over 200 students ages 4-11 will participate in a variety of fun mathematical activities! It should be exciting! We will also be participating in a number of activities for Math Awareness Month in April and we look forward to that as well!

New Initiates – James Camp, Andrew Davis, Leyia Grant, Zchimon Herndon, Mary McBride, Angel Nesbitt, Emily Sparrow, and Jessica Stevens.

TN Gamma – Union University

Chapter President – Amy Murdaugh

Other Fall 2017 Officers: Andrew Edmiston, Vice President; Yoo Jin (Ashley) Moon, Secretary and Treasurer; Cole Le Mahieu, Webmaster and Historian; Bryan Dawson, Corresponding Secretary; and Matt Lunsford, Faculty Sponsor

Our chapter sponsored a back-to-school pizza party in September.

TX Lambda – Trinity University

Chapter President – Eliza Wright; 263 Current Members

Other Fall 2017 Officers: Danielle King, Vice President; Shelby Luikart, Secretary and Treasurer; and Dr. Hoa Nguyen, Corresponding Secretary *and Faculty Sponsor*

VA Alpha – Virginia State University

Corresponding Secretary – Andrew Wynn; 42 New Members

New Initiates – Pres. Makola Abdullah, Ph.D., Cheryl Adeyemi, Ph.D., Latasha Bagby, Queyera Bagley, Danielle Bryan, Alexandria Burke, Brandice Canty, Ty-Li Brickus Clark, Morgan DiBello, Mikela Dockery, William Fike, James Finnie, Brae Fletcher, Monique Gary, Joel Goddot, Anthoneya Hodges, Erick Huston, Eric Hutcherson, Jasmine C. Jackson, N'Dea Jackson, Janay Joseph, Christina King, Ashley Langford, Brian Lewis, Eder Lopez, Troy Lyne, Ayana Lyttle, Brandi Massey, Devon McKiver, Chelsea Mitchell, Denford Moore, Briana Peebles, Marco Ragland, Sandra Richardson, Ph.D., Chad Sadler, Jasper Short, III, Danelle Singer, Jamon Smith, Joshua Sylve, Alexzander Walker, Aman Williams, and Robert Wieman, Ph.D.

VA Beta – Radford University

Chapter President – Cameron Leo; 11 Current Members

Other Fall 2017 Officers: Eric P. Choate, Corresponding Secretary and Faculty Sponsor

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 Apr 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
AL Alpha	Athens State University, Athens	5 Mar 1935
NM Alpha	University of New Mexico, Albuquerque	28 Mar 1935
IL Beta	Eastern Illinois University, Charleston	11 Apr 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 Apr 1937
OH Alpha	Bowling Green State University, Bowling Green	24 Apr 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	University of Central Missouri, Warrensburg	10 Jun 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 Jun 1941
MI Beta	Central Michigan University, Mount Pleasant	25 Apr 1942
NJ Beta	Montclair State University, Upper Montclair	21 Apr 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 Mar 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 Jun 1947
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 Apr 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961

NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 Apr 1965
AL Epsilon	Huntingdon College, Montgomery	15 Apr 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
TN Gamma	Union University, Jackson	24 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	McDaniel College, Westminster	30 May 1965
IL Zeta	Dominican University, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 Mar 1971
KY Alpha	Eastern Kentucky University, Richmond	27 Mar 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 Apr 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 Apr 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State University, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sep 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis University, Loretto	14 Sep 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury University, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 Mar 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 Apr 1986
TX Iota	McMurry University, Abilene	25 Apr 1987
PA Nu	Ursinus College, Collegeville	28 Apr 1987
VA Gamma	Liberty University, Lynchburg	30 Apr 1987

NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 Apr 1990
CO Delta	Mesa State College, Grand Junction	27 Apr 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 Apr 1991
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 Mar 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 Apr 1997
MI Delta	Hillsdale College, Hillsdale	30 Apr 1997
MI Epsilon	Kettering University, Flint	28 Mar 1998
MO Mu	Harris-Stowe College, St. Louis	25 Apr 1998
GA Beta	Georgia College and State University, Milledgeville	25 Apr 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
PA Pi	Slippery Rock University, Slippery Rock	19 Apr 1999
TX Lambda	Trinity University, San Antonio	22 Nov 1999
GA Gamma	Piedmont College, Demorest	7 Apr 2000
LA Delta	University of Louisiana, Monroe	11 Feb 2001
GA Delta	Berry College, Mount Berry	21 Apr 2001
TX Mu	Schreiner University, Kerrville	28 Apr 2001
CA Epsilon	California Baptist University, Riverside	21 Apr 2003
PA Rho	Thiel College, Greenville	13 Feb 2004
VA Delta	Marymount University, Arlington	26 Mar 2004
NY Omicron	St. Joseph's College, Patchogue	1 May 2004
IL Iota	Lewis University, Romeoville	26 Feb 2005
WV Beta	Wheeling Jesuit University, Wheeling	11 Mar 2005
SC Epsilon	Francis Marion University, Florence	18 Mar 2005
PA Sigma	Lycoming College, Williamsport	1 Apr 2005
MO Nu	Columbia College, Columbia	29 Apr 2005
MD Epsilon	Stevenson University, Stevenson	3 Dec 2005
NJ Delta	Centenary College, Hackettstown	1 Dec 2006
NY Pi	Mount Saint Mary College, Newburgh	20 Mar 2007
OK Epsilon	Oklahoma Christian University, Oklahoma City	20 Apr 2007
HA Alpha	Hawaii Pacific University, Waipahu	22 Oct 2007
NC Epsilon	North Carolina Wesleyan College, Rocky Mount	24 Mar 2008
NY Rho	Molloy College, Rockville Center	21 Apr 2009
NC Zeta	Catawba College, Salisbury	17 Sep 2009
RI Alpha	Roger Williams University, Bristol	13 Nov 2009
NJ Epsilon	New Jersey City University, Jersey City	22 Feb 2010
NC Eta	Johnson C. Smith University, Charlotte	18 Mar 2010
AL Theta	Jacksonville State University, Jacksonville	29 Mar 2010
GA Epsilon	Wesleyan College, Macon	30 Mar 2010
FL Gamma	Southeastern University, Lakeland	31 Mar 2010
MA Beta	Stonehill College, Easton	8 Apr 2011
AR Beta	Henderson State University, Arkadelphia	10 Oct 2011

PA Tau	DeSales University, Center Valley	29 Apr 2012
TN Zeta	Lee University, Cleveland	5 Nov 2012
RI Beta	Bryant University, Smithfield	3 Apr 2013
SD Beta	Black Hills State University, Spearfish	20 Sept 2013
FL Delta	Embry-Riddle Aeronautical University, Daytona Beach	22 Apr 2014
IA Epsilon	Central College, Pella	30 Apr 2014
CA Eta	Fresno Pacific University, Fresno	24 Mar 2015
OH Theta	Capital University, Bexley	24 Apr 2015
GA Zeta	Georgia Gwinnett College, Lawrenceville	28 Apr 2015
MO Xi	William Woods University, Fulton	17 Feb 2016
IL Kappa	Aurora University, Aurora	3 May 2016
GA Eta	Atlanta Metropolitan University, Atlanta	1 Jan 2017
CT Gamma	Central Connecticut University, New Britan	24 Mar 2017
KS Eta	Sterling College, Sterling	30 Nov 2017
NY Sigma	College of Mount Saint Vincent, The Bronx	4 Apr 2018
PA Upsilon	Seton Hill University, Greensburg	5 May 2018