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Classification of Consonance in Generalized Tonal Systems

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Abstract

In this paper, we expand on Carmen Weddell's work in describing generalized tonal systems. We work towards developing a framework in which we can actually compose music, by beginning to develop generalized counterpoint. In particular, we present an algorithm for classification of consonance of tones in a generalized tonal system.

1. Introduction, Background, and Motivation

Western music utilizes a twelve-tone system, in which each octave has twelve semitones[1]. Carmen Weddell's work utilized group theory to describe equal-temperament tonal systems using more than twelve semitones for an octave [1]. Important results from Weddell's work included choosing octaves to be $N = 8k + 4$ semitones (for natural numbers k at least 1), definition of the generalized dominant and subdominant[2] as the intervals of $(N/2) + 1$ and $(N/2) - 1$, respectively. Throughout this work, we will assume a basic familiarity with Weddell's framework and results and the appropriate music theory and group theory. Weddell's work developed the foundations of generalized tonal systems, and even went so far as to describe generalized chords and harmonic progressions. However, no mention of counterpoint was made. Counterpoint is a set of rules describing a certain style of Baroque music epitomized by the works of J.S. Bach. In contrapuntal music, independent melodic lines are prescribed to move in certain ways relative to each other, such that acceptable harmonic

[1] The term "twelve-tone" is thus a slight misnomer, as each octave contains twelve semitones. The use of "twelve-tone" in this work should mean only that the octave has twelve semitones, and should not be confused with other meanings.

[2] Referred to by Weddell as the generalized fifth/fourth, respectively.

motion occurs.

In counterpoint, the idea of consonance (informally, the degree to which two pitches played simultaneously sound harmonic or discordant) is an important one. By developing a way to classify the consonance of intervals in a generalized tonal system, we can progress towards a rigorously defined model in which music can be written. To describe consonance, a property of intervals, we must first define intervals.

An interval is the distance between two pitches, informally defined as the ratio or logarithmic difference of their frequencies. In Western music theory, a pitch p has a frequency (in Hz) of the form $440 \times \sqrt[12]{2}^n$, where the integer n represents the distance in semitones from the base pitch of A440 (440 Hz). (For context: an ordinary piano has $n \in [-48, 39]$, ranging from 4 octaves below A440 to 3 octaves and 3 semitones above.) An analysis of this equation is given by Stewart [2, p. 32]. The interval between two pitches $p_0 = 440 \times \sqrt[12]{2}^{n_0}$ and $p_1 = 440 \times \sqrt[12]{2}^{n_1}$ is defined to be $(n_0 - n_1)$ mod 12, in units of semitones. (Some areas of music theory ignore the modulus to consider intervals larger than an octave.)

In traditional music theory, intervals are classified as perfect (extremely consonant), consonant, or dissonant, based on how pleasing these intervals sound to the human ear. Consonance in standard music theory is a result of the underlying ratios that compose different intervals—simpler ratios sound more pleasing. In standard music theory, classification of intervals is as follows:

- Perfect (perfectly consonant) intervals are the perfect unison (0 semitones difference, ratio between the frequencies of 1), octave (12 semitones difference, frequency ratio of 2 considered to be "enharmonic", i.e. harmonically identical for most purposes with the perfect unison), perfect fourth (5 semitones difference, also referred to as the subdominant tone), and perfect fifth (7 semitones difference, also referred to as the dominant tone). These intervals are consonant enough to have a "perfect" quality to them.

- Consonant intervals are the major/minor pairs closest to the central perfect intervals: the major/minor third (4 and 3 semitones difference, respectively) and the major/minor sixth (9 and 8 semitones difference, respectively). While these tones are not as consonant as perfect intervals, they are still consonant and pleasing to the ear.

- Dissonant intervals are the remaining tones: the major/minor second (2 and 1 semitones difference, respectively), the major/minor seventh (11 and 10 semitones difference, respectively), and the tritone (6 semitones difference, the interval between the perfect fourth and perfect fifth).

This is summarized graphically in Table 1, with P representing perfectly consonant intervals, C representing the consonant intervals, and D representing the dissonant intervals.

Semitone	0	1	2	3	4	5	6	7	8	9	10	11
Consonance	P	D	D	C	C	P	D	P	C	C	D	D

Table 1: Consonances in the twelve-tone system

Counterpoint is highly dependent on consonances, with many of the rules of counterpoint being dependent on the consonances of the different melodic lines at a given point in time. (For example, the first and last notes of a piece should be a consonant interval that is not a perfect fourth.) By developing a model of consonance in a generalized tonal system, we can create a foundation on which to describe counterpoint in a generalized tonal system, and work towards a full framework of music in a generalized tonal system.

2. Criteria and Observations

To begin classifying intervals as consonant, dissonant, or perfect, our strategy is to observe traits in the twelve-tone system that are distinctive, important, and desirable. Further, these traits should be describable in abstract terms not tied to the 12-tone system. The noted traits are:

- The unison has perfect consonance.
- The "central region of perfection", i.e. the set of intervals comprised of the subdominant, augmented fourth, and dominant, has a pattern of consonance (two perfect intervals enclosing a dissonant interval, with the dissonant referred to as the tritone). This region, the tritone in particular, serves as a natural halfway point.
- The intervals on either side of the central region of perfection are consonant.
- The perfect unison/octave (enharmonic pitches) have dissonant tones on either side. (The word "octave" is highly context-dependent; it can mean either an interval enharmonic with the perfect unison, or the whole set of intervals.)
- Intervals not perfect or the tritone should be arranged as adjacent pairs, with the larger of the two being referred to as the major interval and the smaller of the two being referred to as the minor interval. Minor intervals should have the same consonance as their corresponding major interval, and vice versa.

- Consonant and dissonant major/minor pairs (as formed from the above point) alternate. For simplicity, we will say that consonant and dissonant are opposite terms.
- Any interval has the same consonance as its inverse. The inverse of an interval, in correspondence with an interval's group-theoretic representation, is the interval such that the sum of the intervals is a full octave, enharmonic with the unison. (For simplicity, we consider the unison to be its own inverse.)
- There should be an equal amount of consonant and dissonant intervals not equal to the unison, subdominant, tritone, or dominant, and equal amounts of consonant and dissonant tones greater than and less than the tritone.

3. Algorithm and Justification

We now can present the algorithm (next page) for classifying tones as consonant, dissonant, or perfect. An implementation in C++ is provided at https://gitlab.com/jtakeshi/consonances. Because the workings and correctness may not immediately be apparent, a qualitative analysis will follow.

As noted above, we assume basic knowledge on the reader's part of the work of Weddell, specifically of the natural correspondence between pitches in a N-tone system and the elements of $\mathbb{Z}_N$. Without this, the casual use of mathematical operations (addition, modulus, etc.) on harmonic intervals may be difficult to follow.

3.1 Algorithm

The algorithm takes as input a natural number N of the form $8k + 4$, where $k \geq 1$ and $k \in \mathbb{N}$. It returns a list of the classifications (Perfect, Consonant, or Dissonant) for the intervals of size 0 to $N - 1$ in a generalized tonal system of size N. Array indexing is zero-based, reflecting the context (and making implementation in most programming languages easier).

The workings, while difficult to see at a first glance, are simple: after classifying the unison and central region, the algorithm will iterate over each interval of i semitones, such that i ranges from 0 to $N-1$ semitones. (While iterating in this manner is not strictly necessary, it is easiest to understand and easier to implement iterating i upwards from 0 to $N - 1$.) At each interval not yet classified, the interval is classified based on its distance from the unison/octave and whether it is closer to a unison or octave, i.e. if the interval is larger than the tritone. (The tritone is

Algorithm 1 Classification of Consonances in Generalized Tonal Systems

procedure Consonances(N)
    $vals \leftarrow$ array with N elements, each element initialized to be Unclassified
    $vals[0] \leftarrow$ Perfect
    $vals[\frac{N}{2} - 1] \leftarrow$ Perfect
    $vals[\frac{N}{2} + 1] \leftarrow$ Perfect
    $vals[\frac{N}{2}] \leftarrow$ Dissonant
    for $i \in \mathbb{Z}_N$ **do**
        if $vals[i] =$ Unclassified **then**
            $k \leftarrow i$ modulo 4
            if $i < \frac{N}{2}$ **then**
                if $k \in \{1, 2\}$ **then**
                    $vals[i] \leftarrow$ Dissonant
                else
                    $vals[i] \leftarrow$ Consonant
                end if
            else
                if $k \in \{2, 3\}$ **then**
                    $vals[i] \leftarrow$ Dissonant
                else
                    $vals[i] \leftarrow$ Consonant
                end if
            end if
        end if
    end for
    return $vals$
end procedure

defined to be the interval of $\frac{N}{2}$ semitones.) The unison is ignored, as it is already classified as perfect. The next interval after the unison, the minor second ($i = 1$) is classified as dissonant. Pitches not perfect or the tritone come in major/minor pairs, so the next highest tone, ($i = 2$) is the major second, and is assigned a consonance of dissonant, in accordance with its corresponding minor interval. The next two intervals comprising the major/minor third ($i = 4, 3$) are then given the opposite consonance of the preceding major/minor pair, and classified as consonant. At this point, the first half of the octave in a twelve-tone system has been fully classified. For larger systems, the process continues, assigning each major/minor pair a consonance the opposite of the previous pair, until the central region is reached.

At this point, the lower half of the octave has been finished, along with the interval $i = \frac{N}{2} + 1$, the generalized dominant. At this point, the algorithm results in the upper half of the octave being classified in a similar but different manner: the upper half of the octave will be a mirror image of the lower half. Musicians may find it helpful to think of this part of the process as progressing downwards from the octave instead of moving upwards from the middle. No matter how one approaches the process mentally (upwards from the unison, downwards from the octave, outwards from the central region), the result is the same.

4. Consequences

In this section we present theorems and lemmas that collectively show that the algorithm presented in Section 3.1 does classify the intervals of a generalized tonal system, such that the criteria observed earlier are fulfilled. The result will be that the abstract, notable traits of a tonal system described in Section 2 are shown to be consequences of the algorithm. For each of these, we take as universal assumptions that N is of the form $8k + 4$ for $k \in \mathbb{Z}$ and $k \geq 1$, and that every interval $i \in \mathbb{Z}_N$ has been classified in accordance with the above algorithm. Mathematical operations and relations on intervals are taken to have the same meaning as their corresponding element of $\mathbb{Z}_N$. The reader should refer frequently to the algorithm's specification while reading these proofs, as it will make the reasoning behind the assertions made casewise more clear.

Lemma 1 *The unison, generalized subdominant, and generalized dominant are all classified as perfect. Additionally, the tritone is classified as dissonant.*

Proof. This is trivially seen by observing the steps of the algorithm preceding the iterative portion: the intervals $0, \frac{N}{2} - 1$, and $\frac{N}{2} + 1$ are classified as perfect, and the interval $\frac{N}{2}$ is classified as dissonant. ■

Theorem 2 *The intervals on either side of the central region of perfection are consonant.*

Proof. The intervals in question are $(\frac{N}{2} - 1) - 1$ and $(\frac{N}{2} + 1) + 1$, better written as $\frac{N}{2} - 2$ and $\frac{N}{2} + 2$. Recalling that $N = 8k + 4$, we again rewrite these tones as $(4k + 2) - 2$ and $(4k + 2) + 2$, and simplify these to $4k$ and $4k + 4$. The interval $4k$ is less than $\frac{N}{2}$, not 0, and $4k \equiv 0 \bmod 4$, so $4k$ is classified as consonant. The interval $4k + 4$ is greater than $\frac{N}{2}$, and $4k + 4 \equiv 0 \bmod 4$, so $4k + 4$ is classified as consonant. ■

Theorem 3 *The intervals directly adjacent to the unison modulo N (i.e. the intervals 1 and $N - 1$) are dissonant.*

Proof. The interval 1 is less than $\frac{N}{2}$, and $1 \equiv 1 \bmod 4$, so 1 is classified as dissonant. The interval $N - 1$ is greater than $\frac{N}{2}$, and $N - 1 \equiv 3 \bmod 4$, so $N - 1$ is classified as dissonant. ■

The musician will note that intervals in the twelve-tone system occur in major-minor adjacent pairs with the same consonance (e.g. the major and minor third are one semitone removed, and have the same consonance).

Theorem 4 *Intervals not the perfect unison, generalized subdominant, generalized tritone, or generalized dominant (the four excepted intervals) occur in adjacent pairs having the same consonance.*

Proof. This can be seen by observing how intervals are assigned. Suppose i is an interval not equal to any of the four excepted intervals. We then wish to show that for each interval i where $i \notin \{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$, there exists a unique interval j such that $|i - j| = 1$ (i.e. i and j are adjacent and not equal), i and j have the same consonance, and j is also not equal to any of the four excepted intervals (the set $\{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$). We let $k = i \bmod 4$, and proceed to find j by cases:

- If $i < \frac{N}{2}$, we note that $i \neq 0$ and $i \neq \frac{N}{2} - 1$.
  - If $k = 0$, then we choose $j = i - 1$. (i and j are consonant.)
  - If $k = 1$, then we choose $j = i + 1$. (i and j are dissonant.)
  - If $k = 2$, then we choose $j = i - 1$. (i and j are dissonant.)
  - If $k = 3$, then we choose $j = i + 1$. (i and j are consonant.)
- On the other hand, if $i > \frac{N}{2}$, we note that $i \neq \frac{N}{2}$ and $i \neq \frac{N}{2} + 1$.

- If $k = 0$, then we choose $j = i + 1$. (i and j are consonant.)
- If $k = 1$, then we choose $j = i - 1$. (i and j are consonant.)
- If $k = 2$, then we choose $j = i + 1$. (i and j are dissonant.)
- If $k = 3$, then we choose $j = i - 1$. (i and j are dissonant.)

In each case, we have chosen j to be one semitone removed from i. It can be verified that j will have the same consonance as i. We now verify that this selection of j also results in j not being one of the four excepted intervals:

For j to be one of the four excepted intervals and be one semitone removed from i, then i must be one semitone removed from one of the four excepted intervals (i is also assumed to not be equal to any of the four excepted intervals). Then $i \in \{1, \frac{N}{2} - 2, \frac{N}{2} + 2, N - 1\}$. (We consider $N - 1$ because it is a semitone removed from the unison, modulo N.) We again proceed by cases:

- If $i = 1$, then $i < \frac{N}{2}$ and $r = 1$, so $j = i + 1 = 2 \notin \{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$.
- If $i = \frac{N}{2} - 2$, then $i < \frac{N}{2}$ and $r = 0$, so $j = i - 1 = \frac{N}{2} - 3 \notin \{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$.
- If $i = \frac{N}{2} + 2$, then $i > \frac{N}{2}$ and $r = 2$, so $j = i + 1 = \frac{N}{2} + 3 \notin \{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$.
- If $i = N - 1$, then $i > \frac{N}{2}$ and $r = 3$, so $j = i - 1 = N - 2 \notin \{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$.

The assertion that $j \notin \{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$ in each case is clear by the condition that $N \geq 12$. (N would have to be 2, 4, or 6 for j to be in $\{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$).

In each case, $|i - j| = 1$ forces either $j = i + 1$ or $j = i - 1$. The reader can verify that in each choice of j, the other choice results in a pair of intervals that do not share the same consonance. This uniqueness is what allows us to state that intervals sharing the same consonance occur in pairs.

We have thus proven that for each $i \notin \{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$, there exists $j \notin \{0, \frac{N}{2} - 1, \frac{N}{2}, \frac{N}{2} + 1\}$ such that $|i - j| = 1$ and i and j have the same consonance. ■

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These pairs of intervals with the same consonance are referred to as major/minor pairs, as mentioned above.

Theorem 5 *Consonant and dissonant major/minor pairs alternate.*

Proof. As a consequence of the previous theorem, this reduces to showing that for all intervals i where $i \notin \{0, \frac{N}{2}-1, \frac{N}{2}+1, N-1\} \cup \{1, \frac{N}{2}-2, \frac{N}{2}+2\}$ (the excepted four and their neighbors), there exists a unique interval j such that $|i-j|=1$, $j \notin \{0, \frac{N}{2}-1, \frac{N}{2}, \frac{N}{2}+1\}$, and the consonances of i and j are opposite. (We restrict i as such because when i is adjacent to one of the four excepted intervals, we want to instead consider the major/minor partner of i.) We again let $k = i \bmod 4$, and proceed by cases:

- If $i < \frac{N}{2}$:
  - If $k=0$, then i is consonant. We choose $j=i+1$, so that j is dissonant.
  - If $k=1$, then i is dissonant. We choose $j=i-1$, so that j is consonant.
  - If $k=2$, then i is dissonant. We choose $j=i+1$, so that j is consonant.
  - If $k=3$, then i is consonant. We choose $j=i-1$, so that j is dissonant.
- On the other hand, if $i > \frac{N}{2}$:
  - If $k=0$, then i is consonant. We choose $j=i-1$, so that j is dissonant.
  - If $k=1$, then i is consonant. We choose $j=i+1$, so that j is dissonant.
  - If $k=2$, then i is dissonant. We choose $j=i-1$, so that j is consonant.
  - If $k=3$, then i is dissonant. We choose $j=i+1$, so that j is consonant.

By choosing j in such a way, we ensure that j and i have opposite consonances and that $|i-j|=1$. From the previous theorem, we know that each considered interval will have one neighbor sharing its consonance, and its other neighbor will have the opposite consonance. Because
$i \notin \{1, \frac{N}{2}-2, \frac{N}{2}+2\}$ and $|i-j|=1$, $j \notin \{0, \frac{N}{2}-1, \frac{N}{2}, \frac{N}{2}+1\}$. ■

Remark 6 *One should note that as a result of the structure of the algorithm, the choices of j in the two above proofs are necessarily unique. To uphold the condition that $|i-j|=1$, there are exactly two choices for j: j must be equal to $i+1$ or $i-1$. The reader can verify in each case that one* *choice of j would lead to at least one of the desired conditions for j not being fulfilled, which leaves the other choice as the correct one to prove the theorem.*

Lemma 7 *In any generalized tonal system, there exist an equal amount of consonant and dissonant intervals, excepting the intervals* $0, \frac{N}{2}-1, \frac{N}{2}, \frac{N}{2}+1$.

Proof. This follows from the above two theorems. ■

Theorem 8 *There exist equal amounts of consonant and dissonant intervals greater and less than* $\frac{N}{2}$*, excepting the intervals* $0, \frac{N}{2}-1, \frac{N}{2}+1$.

Proof. We note that the octave contains the $8k+4$ intervals $\{0, 1, \ldots, N-2, N-1\}$. By excepting $0, \frac{N}{2}-1, \frac{N}{2}, \frac{N}{2}+1$, we see that there are $8k$ intervals remaining. Half of these are greater than $\frac{N}{2}$, and half are less than $\frac{N}{2}$, so each half of the octave contains $4k$ intervals. Because each iteration of 4 steps of the iterative portion of the algorithm assigns two intervals as consonant and two as dissonant, we see that the algorithm assigns an equal amount of consonant and dissonant intervals to either half of the octave. ■

Remark 9 *The above theorem's line of reasoning can also be extended into another proof of the preceding lemma.*

Theorem 10 *Any interval has the same consonance as its inverse.*

Proof. Let i be an interval, and k its residue modulo 4. We let i^{-1} be the additive inverse of $i \in \mathbb{Z}_N$, fulfilling the condition that $i+i^{-1} \equiv 0 \bmod N$. Noting that the residue s of i^{-1} is equal to $(4-k) \bmod N$, we proceed by cases:

- If $i = 0$ or $i = \frac{N}{2}$, then $i = i^{-1}$, and it is trivially true that i has the same consonance as its inverse.
- If $i = \frac{N}{2} - 1$ (i.e. i is the subdominant interval), then we have $i^{-1} = \frac{N}{2} + 1$ (the dominant interval). Similarly, if $i = \frac{N}{2} + 1$, then $i^{-1} = \frac{N}{2} - 1$. Both $\frac{N}{2} - 1$ and $\frac{N}{2} + 1$ are perfectly consonant.
- If $i < \frac{N}{2}$, then we proceed by cases:
  - If $k = 0$, then i is consonant. Then $s = 0$ and $i^{-1} > \frac{N}{2}$, so i^{-1} is also consonant.

  - If $k = 1$, then i is dissonant. Then $s = 3$ and $i^{-1} > \frac{N}{2}$, so i^{-1} is also dissonant.
  - If $k = 2$, then i is dissonant. Then $s = 2$ and $i^{-1} > \frac{N}{2}$, so i^{-1} is also dissonant.
  - If $k = 3$, then i is consonant. Then $s = 1$ and $i^{-1} > \frac{N}{2}$, so i^{-1} is also consonant.

- On the other hand, if $i > \frac{N}{2}$, then we again proceed by cases:
  - If $k = 0$, then i is consonant. Then $s = 0$ and $i^{-1} < \frac{N}{2}$, so i^{-1} is also consonant.
  - If $k = 1$, then i is consonant. Then $s = 3$ and $i^{-1} < \frac{N}{2}$, so i^{-1} is also consonant.
  - If $k = 2$, then i is dissonant. Then $s = 2$ and $i^{-1} < \frac{N}{2}$, so i^{-1} is also dissonant.
  - If $k = 3$, then i is dissonant. Then $s = 1$ and $i^{-1} < \frac{N}{2}$, so i^{-1} is also dissonant.

■

Lemma 11 *A major/minor pair of intervals will have a major/minor pair of intervals as their inverses, and will share the same consonances.*

Proof. This follows from the above theorems. ■

Lemma 12 *If i and j are two intervals whose residues modulo 4 are both zero, then their sum is a consonant interval or equivalent to* 0 *mod* N.

Proof. If $i \equiv 0 \bmod 4$ and $j \equiv 0 \bmod 4$ then $i + j \equiv 0 \bmod 4$. By observing the algorithm, it is clear that an interval with a residue of zero modulo 4 will always be either classified as consonant or be equivalent to the unison/octave (0 mod N). ■

Remark 13 *Musicians will note that this lemma shows that every interval that is a sum of any number of major thirds is consonant or perfect. This is certainly true in the twelve-tone system, although not very interesting, as the only intervals to consider are the unison, major third, minor sixth, and octave.*

5. Example: Consonance of tones in a twenty-tone system

We will demonstrate this procedure with the twenty-tone system. As an aside, the reader will note that one can determine the consonance of intervals either by directly applying the algorithm, or by applying the above theorems and lemmas. We will use the second strategy, as it is more informative. To start, we classify the intervals $\{0, N/2-1, N/2+1\}$ as perfect: in a 20 - tone system, these are the intervals $\{0, 9, 11\}$. Additionally, the tritone N/2 (the interval of 10 semitones in a 20-tone system) is dissonant. We then progress up the octave from the unison to the subdominant:

- The interval 1 is adjacent to the unison, so it is classified as dissonant.
- The interval 2 is the corresponding major interval to the interval 1, so it is also dissonant.
- The major/minor pair $4, 3$ is adjacent to the dissonant major/minor pair $2, 1$, so 3 and 4 are both classified as consonant.
- The major/minor pair $6, 5$ is adjacent to the consonant major/minor pair $4, 3$, so 5 and 6 are both classified as dissonant.
- The major/minor pair $8, 7$ is adjacent to the dissonant major/minor pair $6, 5$, so 7 and 8 are both classified as consonant. This could also have been done by noting that 8 is one less than 9, the subdominant, so 8 and also 7 will be consonant.

At this point, we have classified the lower half of the octave. There are now multiple possible ways to proceed. One way would be to proceed in the same way as above, by moving up from the dominant or down from the octave, and giving adjacent major/minor pairs consonances alternating between consonant and dissonant. Because this method has already been illustrated, we will instead use the fact that inverses of major/minor pairs have the same consonance:

- The major/minor pair $19, 18$ are inverses of the major/minor pair $2, 1$, so $19, 18$ are also dissonant.
- The major/minor pair $17, 16$ are inverses of the major/minor pair $4, 3$, so $17, 16$ are also consonant.
- The major/minor pair $15, 14$ are inverses of the major/minor pair $6, 5$, so $15, 14$ are also dissonant.
- The major/minor pair $13, 12$ are inverses of the major/minor pair $8, 7$, so $13, 12$ are also consonant.

We have classified the tones 0 through 19 as follows:

- Consonant: $\{3, 4, 7, 8, 12, 13, 16, 17\}$
- Dissonant: $\{1, 2, 5, 6, 10, 14, 15, 18, 19\}$
- Perfect: $\{0, 9, 11\}$

The structure of the twenty-tone system is shown graphically in Table 2. The reader can verify that this assignment of consonances satisfies the results in Section 6.

Semitone	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Consonance	P	D	D	C	C	D	D	C	C	P	D	P	C	C	D	D	C	C	D	D

Table 2: Consonances in the twenty-tone system

6. Future Goals

In future research, goals include a more careful definition of interval equality, rigorous statement of rules of counterpoint (in both twelve-tone and generalized systems), and seeking deeper mathematical relationships in the study of generalized tonal systems. The overarching goal is to develop a framework for generalized tonal systems in which well-formed music can be composed (by human or computer musicians).

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Concerning the Growth of Minecraft Animal Populations

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Abstract

Several theorems are proved concerning the asymptotic nature of the growth of animal populations in the popular computer game Minecraft. These results are then generalized to different kinds of populations.

Introduction

Many will be familiar with the popular video game, Minecraft. Minecraft is a sandbox-type game in which the player can gather blocks and resources, build with these blocks, explore and generate new terrain, fight enemies, and engage in a host of other activities. The player can also tame and breed animals. The animals in Minecraft are asexual. That is, there are no male or female animals. There are simply cows, pigs, sheep, etc. To breed these animals, one needs the right breeding food. Cows, for instance, require wheat. To breed two cows and create a new baby cow, the player feeds a piece of wheat to each cow, causing red hearts to appear above each. Soon after, a single baby cow is born which can then be bred with another member of the population after a 20 minute maturation cycle. Minecraft animals do not age and hence will not die unless killed by the player or by some other factor.

The goal of this chapter is to find an effective method of calculating the number of animals in an animal population after n breeding cycles. By a *breeding cycle*, we mean that we breed together as many animals as possible (there is potentially one animal not bred if the population size is odd) and then wait for all baby animals to mature into breedable animals.

In Section 1, we derive an intuitive asymptotic relationship for the growth rate of a Minecraft animal population. In Section 2, we use this relation to derive an exact formula for the population size p_n after n breeding cycles. In Section 3, we present a generalization of the results in Section 2.

As a note on terminology, an *animal population* is taken to mean a population consisting of only a single kind of animal.

1. Asymptotic Relation

Suppose we have an initial animal population of size $p_0 \geq 2$. The population size p_n after n breeding cycles is given by the recurrence relation

$$p_n = p_{n-1} + \left\lfloor \frac{p_{n-1}}{2} \right\rfloor, \qquad n \geq 1,$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function. Using this relation, we make the following definition:

Definition. Let $p_0 \in \mathbb{N}$ with $p_0 \geq 2$; let (p_n) be the sequence defined by

$$p_n = \begin{cases} p_0 & n = 0, \\ p_{n-1} + \left\lfloor \frac{p_{n-1}}{2} \right\rfloor & n \geq 1. \end{cases}$$

The sequence (p_n) is called the *p-sequence with parameter* p_0.

Clearly we would prefer a method of calculating the population size that is not recursive. Moreover, we would like this to be in closed-form. In fact, there is an exact, closed-form formula for p_n. We begin with several lemmas, the fist of which will allow us to make certain inequalities strict later on in the proof of our main results.

Lemma 1 *Let (p_n) be a p-sequence. For every n, there exists $k \geq 1$ such that p_n and p_{n+k} have opposite parity.*

Proof. Let $n, n' \in \mathbb{N}$ such that p_n is even and $p_{n'}$ is odd. Then $p_n = q\left(2^k\right)$ for some $k \geq 1$ and odd q, and $p_{n'} = q'\left(2^{k'}\right) + 1$ for some $k' \geq 1$ and odd q'. We have

$$\begin{aligned} p_{n+1} &= 3q\left(2^{k-1}\right), & p_{n'+1} &= 3q'\left(2^{k'-1}\right) + 1, \\ p_{n+2} &= 9q\left(2^{k-2}\right), & p_{n'+2} &= 9q'\left(2^{k'-2}\right) + 1, \\ &\vdots & &\vdots \\ p_{n+i} &= 3^i q\left(2^{k-i}\right), & p_{n'+i'} &= 3^{i'} q'\left(2^{k'-i'}\right) + 1, \\ &\vdots & &\vdots \\ p_{n+k} &= 3^k q, & p_{n'+k'} &= 3^{k'} q' + 1. \end{aligned}$$

This completes the proof since p_{n+k} is odd and $p_{n'+k'}$ is even. ■

Corollary 2 *Every p-sequence contains infinitely many even and odd terms.*

Lemma 3 *Let (p_n) be a p-sequence with parameter p_0.*

1. *For all n,*
$$p_0\left(\frac{3}{2}\right)^n - p_n \leq \left(\frac{3}{2}\right)^n - 1. \tag{1}$$
2. *There exists $N \geq 1$ such that this inequality is strict for all $n \geq N$.*

Proof. To prove the first statement, note that for all n we have
$$p_n\left(\frac{3}{2}\right) - \frac{1}{2} \leq p_{n+1} \leq p_n\left(\frac{3}{2}\right).$$
To prove (1) we proceed by induction on n. The case of $n = 0$ yields equality. Supposing (1) holds for some arbitrary n, we have
$$\begin{aligned} p_0\left(\frac{3}{2}\right)^{n+1} - p_{n+1} &\leq p_0\left(\frac{3}{2}\right)^{n+1} - \left(p_n\left(\frac{3}{2}\right) - \frac{1}{2}\right) \\ &= \frac{3}{2}\left(p_0\left(\frac{3}{2}\right)^n - p_n\right) + \frac{1}{2} \qquad (2) \\ &\leq \frac{3}{2}\left(\left(\frac{3}{2}\right)^n - 1\right) + \frac{1}{2} = \left(\frac{3}{2}\right)^{n+1} - 1. \end{aligned}$$
Thus by the induction hypothesis, (1) must hold for all $n \geq 0$. This proves the first statement.

We now prove the second statement. From (2), it is clear that if the inequality (1) is strict for some N, then it must also be strict for $N + 1$. Thus, it suffices to show that such an N exists for all p_0. We have two cases. If p_0 is even, then $p_1 = p_0\left(\frac{3}{2}\right)$ and the inequality is strict for $n = 1$ since $p_0\left(\frac{3}{2}\right) - p_1 = 0 < \frac{1}{2}$. In this case, set $N = 1$. If p_0 is odd, then by Lemma 1 there exists k such that p_k is even. Hence $p_{k+1} = p_k\left(\frac{3}{2}\right)$. Using the same argument as in (2), we have
$$p_0\left(\frac{3}{2}\right)^{k+1} - p_{k+1} < p_0\left(\frac{3}{2}\right)^{k+1} - p_k\left(\frac{3}{2}\right) + \frac{1}{2} \leq \left(\frac{3}{2}\right)^{k+1} - 1.$$
Thus if p_0 is odd, set $N = k + 1$. This completes the proof. ■

Theorem 4 *Let p_n be a p-sequence with parameter p_0. Then there exists a constant c with $p_0 - 1 < c < p_0$ such that*

$$\lim_{n\to\infty} \frac{p_n}{\left(\frac{3}{2}\right)^n} = c. \tag{3}$$

Proof. By Lemma 3, there exists N such that

$$p_0 \left(\frac{3}{2}\right)^n - p_n < \left(\frac{3}{2}\right)^n - 1$$

for all $n \geq N$. Hence

$$\frac{p_N}{\left(\frac{3}{2}\right)^N} > p_0 - 1 + \frac{1}{\left(\frac{3}{2}\right)^N}.$$

Define δ to be half the distance between $\frac{p_N}{(3/2)^N}$ and $p_0 - 1 + \frac{1}{(3/2)^N}$ in the above inequality. In other words, $\delta := \frac{1}{2}\left(\frac{p_N}{\left(\frac{3}{2}\right)^N} - p_0 + 1 - \frac{1}{\left(\frac{3}{2}\right)^N}\right)$. Then

$$\frac{p_N}{\left(\frac{3}{2}\right)^N} > p_0 - 1 + \delta + \frac{1}{\left(\frac{3}{2}\right)^N}.$$

Note that for any $x \in \mathbb{R}$, if $\frac{p_n}{\left(\frac{3}{2}\right)^n} > x + \frac{1}{\left(\frac{3}{2}\right)^n}$, then

$$\begin{aligned} \frac{p_{n+1}}{\left(\frac{3}{2}\right)^{n+1}} &\geq \frac{p_n\left(\frac{3}{2}\right) - \frac{1}{2}}{\left(\frac{3}{2}\right)^{n+1}} = \frac{p_n}{\left(\frac{3}{2}\right)^n} - \frac{1}{2\left(\frac{3}{2}\right)^{n+1}} \\ &> x + \frac{1}{\left(\frac{3}{2}\right)^n} - \frac{1}{2\left(\frac{3}{2}\right)^{n+1}} = x + \frac{1}{\left(\frac{3}{2}\right)^{n+1}}. \end{aligned} \tag{4}$$

Setting $x = p_0 - 1 + \delta$, it follows from (4) and induction that

$$\frac{p_n}{\left(\frac{3}{2}\right)^n} > p_0 - 1 + \delta + \frac{1}{\left(\frac{3}{2}\right)^n} > p_0 - 1 + \delta$$

for all $n \geq N$.

The sequence $\frac{p_n}{\left(\frac{3}{2}\right)^n}$ is decreasing since

$$\frac{p_{n+1}}{\left(\frac{3}{2}\right)^{n+1}} = \frac{p_n + \left\lfloor \frac{p_n}{2} \right\rfloor}{\left(\frac{3}{2}\right)^{n+1}} \leq \frac{p_n\left(\frac{3}{2}\right)}{\left(\frac{3}{2}\right)^{n+1}} = \frac{p_n}{\left(\frac{3}{2}\right)^n}.$$

Thus by the Monotone Convergence Theorem, there exists a constant $c \geq p_0 - 1 + \delta$ such that

$$\lim_{n\to\infty} \frac{p_n}{\left(\frac{3}{2}\right)^n} = c.$$

Since $\frac{p_n}{\left(\frac{3}{2}\right)^n}$ is decreasing, and since $\delta > 0$, we have $p_0 - 1 < c < p_0$. ■

2. Population Approximation

Theorem 5 *Let (p_n) be a p-sequence. Then*

$$0 < p_n - c\left(\frac{3}{2}\right)^n < 1 \tag{5}$$

for all $n \geq 0$.

Proof. By Corollary 2, there exists a subsequence (p_{n_j}) of (p_n) such that p_{n_j} is odd for all n_j. Thus $p_{n_j+1} = p_{n_j}\left(\frac{3}{2}\right) - \frac{1}{2}$. Since $\frac{p_n}{(3/2)^n}$ is a decreasing sequence, we have

$$\frac{p_{n_j}}{\left(\frac{3}{2}\right)^{n_j}} = \frac{p_{n_j}\left(\frac{3}{2}\right)}{\left(\frac{3}{2}\right)^{n_j+1}} > \frac{p_{n_j}\left(\frac{3}{2}\right) - \frac{1}{2}}{\left(\frac{3}{2}\right)^{n_j+1}} = \frac{p_{n_j+1}}{\left(\frac{3}{2}\right)^{n_j+1}} \geq \frac{p_{n_{j+1}}}{\left(\frac{3}{2}\right)^{n_{j+1}}}.$$

Hence the subsequence $\frac{p_{n_j}}{(3/2)^{n_j}}$ is strictly decreasing. Since $\frac{p_n}{(3/2)^n}$ is monotonic and has a strictly decreasing subsequence, we must have $\frac{p_n}{(3/2)^n} > c$ for all n. Thus $p_n - c\left(\frac{3}{2}\right)^n > 0$.

Since $\lim \frac{p_n}{(3/2)^n} = c$ by Theorem 4, we also have $\lim \frac{p_n - 1}{(3/2)^n} = c$. In order to show that $p_n - c\left(\frac{3}{2}\right)^n < 1$, we show that the sequence $\frac{p_n-1}{(3/2)^n}$ is both increasing and has a strictly increasing subsequence. We have

$$\frac{p_{n+1} - 1}{\left(\frac{3}{2}\right)^{n+1}} \geq \frac{p_n\left(\frac{3}{2}\right) - \frac{1}{2} - 1}{\left(\frac{3}{2}\right)^{n+1}} = \frac{p_n - 1}{\left(\frac{3}{2}\right)^n}.$$

Hence $\frac{p_n-1}{(3/2)^n}$ is increasing. Again by Corollary 2, there exists a subsequence (p_{n_l}) of (p_n) such that p_{n_l} is even for all n_l. Thus $p_{n_l+1} = p_{n_l}\left(\frac{3}{2}\right)$. Since $\frac{p_n-1}{(3/2)^n}$ is increasing, we have

$$\frac{p_{n_l} - 1}{\left(\frac{3}{2}\right)^{n_l}} = \frac{p_{n_l}\left(\frac{3}{2}\right) - \frac{3}{2}}{\left(\frac{3}{2}\right)^{n_l+1}} = \frac{p_{n_l+1} - \frac{3}{2}}{\left(\frac{3}{2}\right)^{n_l+1}} < \frac{p_{n_l+1} - 1}{\left(\frac{3}{2}\right)^{n_l+1}} \leq \frac{p_{n_{l+1}} - 1}{\left(\frac{3}{2}\right)^{n_{l+1}}}.$$

Hence the subsequence $\frac{p_{n_l}-1}{(3/2)^{n_l}}$ is strictly increasing. Since $\frac{p_n-1}{(3/2)^n}$ is monotonic and has a strictly increasing subsequence, we must have $\frac{p_n-1}{(3/2)^n} < c$ for all n. Thus $p_n - c\left(\frac{3}{2}\right)^n < 1$, which proves the theorem. ■

Letting $\lceil k \rceil$ denote the smallest integer greater than or equal to k (that is, k rounded up if k is not an integer), we can now give an exact formula for p_n.

Corollary 6 *Let (p_n) be a p-sequence with parameter p_0. For all n, we have*

$$p_n = \left\lceil c\left(\frac{3}{2}\right)^n \right\rceil .$$

Remark. Although there exists a closed form expression for p_n involving c, it is *not* known (at the time of this writing) if there exists a closed form expression for c. Although c may be calculated to arbitrary precision using known terms of (p_n), the exact value of c remains unknown. As well, the questions of the irrationality and transcendence of c remain open.

3. Generalization to Other p-sequences

We move now to a generalization of the above results. Consider a population in which a offspring are produced by b parents. This yields the following generalized definition for p-sequences.

Definition. Let $a, b \in \mathbb{N}$ with $b \geq 2$; let $p \in \mathbb{N}$ with $p \geq b$; let $P_{a,b}(n)$ be the sequence defined by

$$P_{a,b}(n) = \begin{cases} p & n = 0, \\ P_{a,b}(n-1) + a\left\lfloor \frac{P_{a,b}(n-1)}{b} \right\rfloor & n \geq 1. \end{cases}$$

The sequence $P_{a,b}(n)$ is said to be the *p-sequence with parameters* a, b and p. The p-sequence examined in Sections 1 and 2 can thus be written as

$P_{1,2}(n)$. What we shall see is that the same results as before, specifically Theorem 5, hold for the generalized case. Unfortunately, Corollary 6 does not hold in general. Instead, we prove that there exists a constant c such that

$$0 < P_{a,b}(n) - c\left(\frac{a+b}{b}\right)^n < b - 1.$$

We follow the same approach as in Sections 1 and 2. We begin our generalization by treating the case of $b \mid a$, as this permits a simple closed form expression.

Before treating this case though, let us first introduce some convenient notation. Let $r(x, y)$ denote the remainder of x on division by y. Thus $r(8, 3) = 2$ and $r(12, 9) = 3$. It is clear then that $r(a, b) < b$; we shall make much use of this inequality. Also, we let (a, b) denote the greatest common divisor of a and b.

Theorem 7 *Let $P_{a,b}(n)$ be a p-sequence with parameters a, b and p and suppose $b \mid a$. Then for all n, we have*

$$P_{a,b}(n) = b\left(\frac{a}{b} + 1\right)^n \left\lfloor \frac{p}{b} \right\rfloor + r(p, b). \tag{6}$$

Proof. We proceed by induction on n. For $n = 0$, note that $p = b \left\lfloor \frac{p}{b} \right\rfloor + r(p, b)$. Supposing that (6) holds for some arbitrary $n \geq 0$, we have

$$\begin{aligned} P_{a,b}(n+1) &= P_{a,b}(n) + a \left\lfloor \frac{P_{a,b}(n)}{b} \right\rfloor \\ &= b \left(\frac{a}{b}+1\right)^n \left\lfloor \frac{p}{b} \right\rfloor + r(p,b) + a \left\lfloor \frac{b \left(\frac{a}{b}+1\right)^n \left\lfloor \frac{p}{b} \right\rfloor + r(p,b)}{b} \right\rfloor \\ &= b \left(\frac{a}{b}+1\right)^n \left\lfloor \frac{p}{b} \right\rfloor + r(p,b) + a \left\lfloor \left(\frac{a}{b}+1\right)^n \left\lfloor \frac{p}{b} \right\rfloor + \frac{r(p,b)}{b} \right\rfloor . \end{aligned} \tag{7}$$

Since $\left(\frac{a}{b}+1\right)^n \left\lfloor \frac{p}{b} \right\rfloor$ is an integer, and since $r(p, b) < b$, we have $\frac{r(p,b)}{b} < 1$ and the last line of (7) becomes

$$\begin{aligned} P_{a,b}(n+1) &= b \left(\frac{a}{b}+1\right)^n \left\lfloor \frac{p}{b} \right\rfloor + r(p,b) + a \left(\frac{a}{b}+1\right)^n \left\lfloor \frac{p}{b} \right\rfloor \\ &= b \left(\frac{a}{b}+1\right)^{n+1} \left\lfloor \frac{p}{b} \right\rfloor + r(p,b). \end{aligned}$$

Thus (6) holds for all $n \geq 0$. ■

Having treated this case, it shall be assumed henceforth that $b \nmid a$. The proof of our next lemma makes use of several divisibility properties of the integers. We state these properties now without proof, as they are basic results in number theory.[1]

Property 8 *Let a, b be integers with $(a, b) = 1$. Then $\left((a+b)^k, b\right) = 1$ for all $k \geq 1$.*

Property 9 *Let a, b, c be integers with $(a, b) = 1$. If $a \mid bc$, then $a \mid c$.*

We now prove our next lemma which contains two statements, both of which are proven in a nearly identical manner. As such, we prove only the first statement and leave the minor details of the proof of the second to the reader.

Lemma 10 *Let $P_{a,b}(n)$ be a p-sequence with parameters a, b and p. For every n,*

1. *There exists an integer $M \geq 0$ such that $r(P_{a,b}(n + M), b) < b - 1$;*
2. *There exists an integer $M' \geq 0$ such that $r(P_{a,b}(n + M'), b) > 0$.*

Proof. We prove the first statement. If $r(n, b) \neq b-1$, then $r(n, b) < b-1$ and we set $M = 0$. Otherwise, let k be the smallest nonzero power of b

[1] The proofs of these properties can be found in the Appendix.

that divides $P_{a,b}(n)-(b-1)$. That is, $k \geq 1$ and $b^k \mid P_{a,b}(n)-(b-1)$. Such a k exists since $P_{a,b}(n) \geq b$. Then $P_{a,b}(n) = qb^k + (b-1)$ for some q. Since k is minimal, $b \nmid q$. Let $d = (a,b)$. Then $a = dt_a$ and $b = dt_b$ for some t_a and t_b with $(t_a, t_b) = 1$. Thus we have

$$\begin{aligned}
P_{a,b}(n) &= qb^k + (b-1).\\
P_{a,b}(n+1) &= qb^k + (b-1) + aqb^{k-1}.\\
&= q(dt_b)^k + (b-1) + dt_a q(dt_b)^{k-1}.\\
&= qd^k(t_a+t_b)t_b^{k-1} + (b-1).\\
P_{a,b}(n+2) &= qd^k(t_a+t_b)^2 t_b^{k-2} + (b-1).\\
&\vdots\\
P_{a,b}(n+i) &= qd^k(t_a+t_b)^i t_b^{k-i} + (b-1).\\
&\vdots\\
P_{a,b}(n+k) &= qd^k(t_a+t_b)^k + (b-1).
\end{aligned}$$

Since t_a and t_b are relatively prime, $\big((t_a+t_b)^k, t_b\big) = 1$ by Property 8. If $t_b \nmid qd^{k-1}$, then $t_b \nmid qd^{k-1}(t_a+t_b)^k$ by contraposition of Property 9. Hence $b \nmid qd^k(t_a+t_b)^k$ and thus $r(qd^k(t_a+t_b)^k, b) \neq 0$. Therefore $r(P_{a,b}(n+k), b) \neq b-1$. In this case, set $M = k$. Otherwise, let j be the largest power of t_b that divides qd^{k-1}. That is,

$$t_b^j \mid qd^{k-1} \qquad \text{and} \qquad t_b^{j+1} \nmid qd^{k-1}.$$

Then we have

$$\begin{aligned}
P_{a,b}(n+k) &= qd^k(t_a+t_b)^k + (b-1).\\
P_{a,b}(n+k+1) &= \left(1+\frac{t_a}{t_b}\right) qd^k(t_a+t_b)^k + (b-1).\\
&\vdots\\
P_{a,b}(n+k+i) &= \left(1+\frac{t_a}{t_b}\right)^i qd^k(t_a+t_b)^k + (b-1).\\
&\vdots\\
P_{a,b}(n+k+j) &= \left(1+\frac{t_a}{t_b}\right)^j qd^k(t_a+t_b)^k + (b-1).
\end{aligned}$$

Since j is maximal, $t_b \nmid \left(\frac{t_a}{t_b}\right)^j qd^{k-1}$. Since $(t_b, (t_a+t_b)^k) = 1$, we have also that $t_b \nmid \left(\frac{t_a}{t_b}\right)^j qd^{k-1}(t_a+t_b)^k$ and thus $b \nmid \left(\frac{t_a}{t_b}\right)^j qd^k(t_a+t_b)^k$. Since

b divides $\left(\frac{t_a}{t_b}\right)^i qd^k(t_a+t_b)^k$ for each $i<j$, we must have

$$b \nmid qd^k(t_a+t_b)^k\left(\sum_{i=0}^{j}\binom{j}{i}\left(\frac{t_a}{t_b}\right)\right)=\left(1+\frac{t_a}{t_b}\right)^j qd^k(t_a+t_b)^k.$$

In other words, b divides each term in the product of $qd^k(t_a+t_b)^k$ and the binomial expansion of $\left(1+\frac{t_a}{t_b}\right)^j$ except the single term $\left(\frac{t_a}{t_b}\right)^j qd^k(t_a+t_b)^k$, so that b does not divide the whole product $\left(1+\frac{t_a}{t_b}\right)^j qd^k(t_a+t_b)^k$. Thus $r(P_{a,b}(n+k+j),b)\neq b-1$. In this case, set $M=k+j$. This completes the proof of the first statement. The proof of the second statement is nearly identical, as one need only replace each $(b-1)$ with 0 in the above equations and make the appropriate modifications. ■

Corollary 11 *Every p-sequence contains infinitely many terms $P_{a,b}(n)$ and $P_{a,b}(m)$ such that $r(P_{a,b}(n),b)<b-1$ and $r(P_{a,b}(m),b)>0$.*

Remark. Throughout the remainder of this chapter, we shall make abundant use of the quantities $\frac{a+b}{b}$ and $\left(\frac{a+b}{b}\right)^n$. In order to avoid cumbersome notation, we define the $\beta=\frac{a+b}{b}$.

Lemma 12 *Let $P_{a,b}(n)$ be p-sequence with parameters a,b and p.*

1. *For all $n\geq 0$,*
$$p\beta^n - P_{a,b}(n)\leq (b-1)\left(\beta^n-1\right). \tag{8}$$
2. *There exists $N\geq 1$ such that this inequality is strict for all $n\geq N$.*

Proof. For all positive integers a and b, the difference $\frac{a}{b}-\left\lfloor\frac{a}{b}\right\rfloor$ is at most $\frac{b-1}{b}$. Hence $\left\lfloor\frac{a}{b}\right\rfloor\geq\frac{a}{b}-\frac{b-1}{b}$ so that

$$\begin{aligned}
P_{a,b}(n+1) &= P_{a,b}(n)+a\left\lfloor\frac{P_{a,b}(n)}{b}\right\rfloor\\
&\geq P_{a,b}(n)+a\left(\frac{P_{a,b}(n)}{b}-\frac{b-1}{b}\right)\\
&=\beta P_{a,b}(n)-a\left(\frac{b-1}{b}\right).
\end{aligned}$$

When $P_{a,b}(n)$ is divisible by b, we have $P_{a,b}(n+1)=\beta P_{a,b}(n)$. Hence for all n,

$$\beta P_{a,b}(n)-a\left(\frac{b-1}{b}\right)\leq P_{a,b}(n+1)\leq\beta P_{a,b}(n).$$

To prove the first statement, we proceed by induction on n. As in Lemma 3, the case of $n = 0$ yields equality. Supposing (8) holds for some arbitrary $n \geq 1$,

$$\begin{aligned} p\beta^{n+1} &- P_{a,b}(n+1) \\ &\leq p\beta^{n+1} - \left(\left(\frac{a+b}{b}\right) P_{a,b}(n) - a\left(\frac{b-1}{b}\right)\right) \\ &= \beta\left(p\beta^{n} - P_{a,b}(n)\right) + a\left(\frac{b-1}{b}\right) \qquad (9) \\ &\leq \beta(b-1)\left(\beta^{n} - 1\right) + a\left(\frac{b-1}{b}\right) \\ &= (b-1)\left(\beta^{n+1} - 1\right). \end{aligned}$$

Thus by the induction hypothesis, (8) holds for all $n \geq 1$. This proves the first statement.

We now prove the second statement. From (9), it is clear that if the inequality (8) is strict for some N, then it must also be strict for $N+1$. Thus, it suffices to show that there exists such an N for all a, b, and p. If $r(p, b) < b - 1$, then $p\beta - P_{a,b}(1) < a\left(\frac{b-1}{b}\right) = (b-1)(\beta - 1)$. In this case, set $N = 1$. Otherwise, we have by Lemma 10 that there exists M such that $r(P_{a,b}(M), b) < b - 1$. Substituting $n = M$ into (9) yields

$$\begin{aligned} p\beta^{M+1} &- P_{a,b}(M+1) \\ &< p\beta^{M+1} - \left(\beta P_{a,b}(M) - a\left(\frac{b-1}{b}\right)\right) \\ &\leq (b-1)\left(\beta^{M+1} - 1\right). \end{aligned}$$

In this case, set $N = M + 1$. This completes the proof. ■

Theorem 13 *Let $P_{a,b}(n)$ be a p-sequence with parameters a, b and p. Then there exists a constant c with $p - b + 1 < c < p$ such that*

$$\lim_{n\to\infty} \frac{P_{a,b}(n)}{\beta^{n}} = c.$$

Proof. By Lemma 12, there exists N such that

$$p\beta^{n} - P_{a,b}(n) < (b-1)\left(\beta^{n} - 1\right)$$

for all $n \geq N$. Hence

$$\frac{P_{a,b}(N)}{\beta^{N}} > p - b + 1 + \frac{b-1}{\beta^{N}}.$$

Define δ to be half the distance between $\frac{P_{a,b}(N)}{\beta^{N}}$ and $p - b + 1 + \frac{b-1}{\beta^{N}}$ in the above inequality. In other words,

$$\delta := \frac{1}{2}\left(\frac{P_{a,b}(N)}{\beta^N} - p + b - 1 - \frac{b-1}{\beta^N}\right).$$

Hence we have

$$\frac{P_{a,b}(N)}{\beta^N} > p - b + 1 + \delta + \frac{b-1}{\beta^N}.$$

For any $x \in \mathbb{R}$, if $\frac{P_{a,b}(n)}{\beta^n} > x + \frac{b-1}{\beta^n}$, then

$$\begin{aligned} \frac{P_{a,b}(n+1)}{\beta^{n+1}} &\geq \frac{\beta P_{a,b}(n) - a\left(\frac{b-1}{b}\right)}{\beta^{n+1}} = \frac{P_{a,b}(n)}{\beta^n} - \frac{a\left(\frac{b-1}{b}\right)}{\beta^{n+1}} \\ &> x + \frac{b-1}{\beta^n} - \frac{a\left(\frac{b-1}{b}\right)}{\beta^{n+1}} = x + \frac{b-1}{\beta^{n+1}}. \end{aligned} \tag{10}$$

Setting $x = p - b + 1 + \delta$, it follows from (10) and induction that

$$\frac{P_{a,b}(n)}{\beta^n} > p - b + 1 + \delta + \frac{b-1}{\beta^n} > p - b + 1 + \delta$$

for all $n \geq N$.

The sequence $\frac{P_{a,b}(n)}{\beta^n}$ is decreasing since

$$\frac{P_{a,b}(n+1)}{\beta^{n+1}} \leq \frac{\beta P_{a,b}(n)}{\beta^{n+1}} = \frac{P_{a,b}(n)}{\beta^n}.$$

Thus by the Monotone Convergence Theorem, there exists a constant $c \geq p - b + 1 + \delta$ such that

$$\lim_{n\to\infty} \frac{P_{a,b}(n)}{\beta^n} = c.$$

Since $\frac{P_{a,b}(n)}{\beta^n}$ is decreasing, and since $\delta > 0$, we have $p - b + 1 < c < p$. ∎

Theorem 14 *Let $P_{a,b}(n)$ be a p-sequence with parameters a, b and p. Then*

$$0 < P_{a,b}(n) - c\beta^n < b - 1 \tag{11}$$

for all $n \geq 0$.

Proof. By Corollary 11, there exists a subsequence $P_{a,b}(n_j)$ such that $r(P_{a,b}(n_j), b) > 0$ for all n_j. Hence $\beta P_{a,b}(n_j) > P_{a,b}(n_j + 1)$. Since $\frac{P_{a,b}(n)}{\beta^n}$ is a decreasing sequence, we have

$$\frac{P_{a,b}(n_j)}{\beta^n} = \frac{\beta P_{a,b}(n_j)}{\beta^{n_j+1}} > \frac{P_{a,b}(n_j+1)}{\beta^{n_j+1}} \geq \frac{P_{a,b}(n_{j+1})}{\beta^{n_{j+1}}}.$$

Hence the subsequence $\frac{P_{a,b}(n_j)}{\beta^{n_j}}$ is strictly decreasing. Since $\frac{P_{a,b}(n)}{\beta^n}$ is monotonic and has a strictly decreasing subsequence, we must have $\frac{P_{a,b}(n)}{\beta^n} > c$ for all n. Thus $P_{a,b}(n) - c\beta^n > 0$.

Since $\lim \frac{P_{a,b}(n)}{\beta^n} = c$ by Theorem 13, we also have

$$\lim \frac{P_{a,b}(n) - (b-1)}{\beta^n} = c.$$

In order to show that $P_{a,b}(n) - c\beta^n < b - 1$, we show that the sequence $\frac{P_{a,b}(n)-(b-1)}{\beta^n}$ is both increasing and has a strictly increasing subsequence. We have

$$\begin{aligned} \frac{P_{a,b}(n+1) - (b-1)}{\beta^{n+1}} &\geq \frac{P_{a,b}(n)\beta - a\left(\frac{b-1}{b}\right) - (b-1)}{\beta^{n+1}} \\ &= \frac{P_{a,b}(n)\beta - (b-1)(\frac{a}{b}+1)}{\beta^{n+1}} = \frac{P_{a,b}(n) - (b-1)}{\beta^n}. \end{aligned}$$

By Corollary 11, there exists a subsequence $P_{a,b}(n_l)$ such that $r(P_{a,b}(n_l), b) < b - 1$ for all n_l. Hence $P_{a,b}(n_l)\beta - a\left(\frac{b-1}{b}\right) < P_{a,b}(n_l + 1)$. Since $\frac{P_{a,b}(n)}{\beta^n}$ is an increasing sequence, we have

$$\begin{aligned} \frac{P_{a,b}(n_l) - (b-1)}{\beta^{n_l}} &= \frac{P_{a,b}(n_l)\beta - (b-1)\beta}{\beta^{n_l+1}} = \frac{P_{a,b}(n_l)\beta - a\left(\frac{b-1}{b}\right) - (b-1)}{\beta^{n_l+1}} \\ &< \frac{P_{a,b}(n_l+1) - (b-1)}{\beta^{n_l+1}} \leq \frac{P_{a,b}(n_{l+1}) - (b-1)}{\beta^{n_{l+1}}}. \end{aligned}$$

Hence the subsequence $\frac{P_{a,b}(n_l)-(b-1)}{\beta^{n_l}}$ is strictly increasing. Since $\frac{P_{a,b}(n)}{\beta^n}$ is monotonic and has a strictly increasing subsequence, we must have $\frac{P_{a,b}(n)-(b-1)}{\beta^n} < c$ for all n. Thus $P_{a,b}(n) - c\beta^n < b - 1$. ■

Appendix

Proof of Property 8

Proof. Since $(a, b) = 1$, there exist distinct integers x and y such that $ax + by = 1$. We have

$$ax + by = ax + b(y + x - x) = (a + b)x + b(y - x) = 1$$

so that $(a + b, b) = 1$. Thus $a + b$ and b share no prime factors. Since $a + b$ and $(a + b)^k$ have the same prime factors for all $k \geq 1$, $(a + b)^k$ and b share no prime factors. Therefore $\big((a + b)^k, b\big) = 1$ for all $k \geq 1$. ■

Proof of Property 9

Proof. Since $(a, b) = 1$, a and b share no prime factors. Since $a \mid bc$, a divides b or c. Since a and b share no prime factors, a cannot divide b. Thus a divides c. ■

The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before March 15, 2018. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring 2018 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051).

NEW PROBLEMS 798-807

Problem 798. *Proposed by the editor.*

In 2002, Britney Gallivan (high school junior) found a formula for paper folding and managed to do 12 folds of a long sheet of toilet paper. She found that

$$L = \frac{\pi t}{6}\left(2^n + 4\right)\left(2^n - 1\right)$$

where t represents the thickness of the material to be folded, L is the length of the paper to be folded and n is the number of folds desired (in only one direction). Suppose you tape together sheets of standard 8.5” x 11” copier paper (thickness .0035”) end to end, how many sheets would be needed to be able to fold the long taped sheet 14 times?

Problem 799. *Proposed by Daniel Sitaru, “Theodor Costescu” National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a, b, c \in (0, 2]$ then

$$3\sqrt{2} \leq \sum \frac{b(\sqrt{a} + \sqrt{2-a})}{c} \leq 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right).$$

Problem 800. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a \in \mathbb{R}$,then

$$\int_{a+3}^{a+5} \ln(1+e^x)dx + \int_{a+6}^{a+8} \ln(1+e^x)dx \leq \int_{a}^{a+2} \ln(1+e^x)dx + \int_{a+9}^{a+11} \ln(1+e^x)dx.$$

Problem 801. *Proposed by Jose Luis Diaz-Barrero, Barcelona Tech-UPC, Barcelona, Spain.*

Compute

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} \frac{k^2+n^2}{1+2\sqrt{\frac{k^2+n^2+n^3}{n^3}}}.$$

Problem 802. *Proposed by Jose Luis Diaz-Barrero, Barcelona Tech-UPC, Barcelona, Spain.*

Let $n \geq 1$ be an integer. Prove that

$$\sqrt[n]{\prod_{k=1}^{n} F_{k+1}} \geq \frac{1}{2}\left(\sqrt[n]{\prod_{k=1}^{n} F_k} + \sqrt[n]{\prod_{k=1}^{n} L_k}\right),$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ and by $L_1 = 1$, $L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

Problem 803. *Proposed by Ovidiu Furdui and Alina Sintamarian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Calculate

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{H_{n+m}}{n(n+m)^2}$$

where $H_n = 1 + 1/2 + \cdots + 1/n$ denotes the n^{th} harmonic number.

Problem 804. *Proposed by D.M. Batinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Compute the following limit

$$\lim_{n\to\infty} \frac{\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}}{\sqrt[n+1]{(2n+1)!!}}.$$

Problem 805. *Proposed by D.M. Batinetu–Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Let $z_k = x_k + iy_k$ be a complex number where $k \in \{1, 2, \ldots, n\}$. Prove that

$$\sum_{k=1}^{n} \sqrt{{x_k}^4 + {y_{n-k+1}}^4} \geq \frac{\sqrt{2}}{2} \sum_{k=1}^{n} |z_k|^2.$$

Problem 806. *Proposed by Marius Dragan, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

If $a_1, a_2, \ldots, a_n > 0$ are such that $\sum\limits_{k=1}^{n} a_k = 1$, then prove that

$$(1+1/a_2)^{na_1^2}(1+1/a_3)^{na_2^2}...(1+1/a_n)^{na_{n-1}^2}(1+1/a_1)^{na_n^2} \geq n+1.$$

Problem 807. *Proposed by Titu Zvonaru, Comanesti, Romania.*

If A, B, and C are the angles of a triangle and $\alpha = A/2, \beta = B/2$, and $\gamma = C/2$, prove that

$$\sqrt{6(1+\cos A\cos B\cos C) - 2\sin\alpha\sin\beta\sin\gamma(1-8\sin\alpha\sin\beta\sin\gamma)} \\ \geq 4\cos\alpha\cos\beta\cos\gamma.$$

SOLUTIONS TO PROBLEMS 780-788

Problem 780. *Proposed by Daniel Sitaru, Colegiul National Economic Theodor Costescu, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a, b, c \in [1, \infty)$, then $ab + bc + ca \geq 3 + 2\ln(a^b b^c c^a)$.

Solution *by Richdad Phuc, University of Sciences, Hanoi, Vietnam*

We have

$LHS - RHS = b(a - 2\ln a) + c(b - 2\ln b) + a(c - 2\ln c) - 3$

$= (b/a)a(a - 2\ln a) + (c/b)b(b - 2\ln b) + (a/c)c(c - 2\ln c) - 3.$

Let $f(x) = x(x - 2\ln x)$ for $x \geq 1$. The derivative is $f'(x) = 2x - 2\ln x - 2$ and $f''(x) = 2 - 2/x \geq 0$ for all $x \geq 1$, so $f'(x) \geq f'(1) = 0$. This means that $f(x)$ is strictly increasing on $[0, 1)$. Then $f(x) \geq f(1)$ for all $x \geq 1$. Hence $a(a-2\ln a) \geq 1$, $b(b-2\ln b) \geq 1$, and $c(c-2\ln c) \geq 1$. Then

$$LHS - RHS \geq (b/a) + (c/b) + (a/c) - 3$$

so $LHS - RHS \geq 0$ by the AM-GM inequality. Equality holds if $a = b = c = 1$.

Also solved by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Henry Ricardo, New York Math Circle, NY; and the proposer.

Problem 781. *Proposed by Daniel Sitaru, Colegiul National Economic Theodor Costescu, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a, b, c \in (0, \infty)$, then

$$\sum a\sqrt{\frac{(b^4 + c^4)}{2}} \geq a^2(b + c) + b^2(a + c) + c^2(a + b) - 3abc.$$

Solution *by the proposer.*

We prove that if $x, y \in (0, \infty)$, then

$$x + y - \sqrt{xy} \geq \sqrt{\frac{x^2 + y^2}{2}}. \tag{1}$$

We denote $u = \sqrt{\frac{x^2+y^2}{2}}$, which means $2u^2 = x^2 + y^2$, and let $v = \sqrt{xy}$ so $v^2 = xy$. With these notations, we have

$$2u^2 + 2v^2 = x^2 + 2xy + y^2 = (x + y)^2.$$

We can rewrite (1) as $x+y-v \geq u$ or

$$\begin{aligned}(x+y)^2 &\geq (u+v)^2\\ &\Leftrightarrow 2u^2+2v^2-u^2-v^2-2uv \geq 0\\ &\Leftrightarrow (u-v)^2 \geq 0.\end{aligned}$$

Now replace x with x/y and y with y/x in (1) to get

$$\begin{aligned}\frac{x}{y}+\frac{y}{x} &\geq \sqrt{\frac{(x/y)^2+(y/x)^2}{2}}+\sqrt{\frac{x}{y}\cdot\frac{y}{x}}\\ &\Leftrightarrow \frac{x^2+y^2}{xy} \geq \frac{1}{xy}\sqrt{\frac{x^4+y^4}{2}}+1\\ &\Leftrightarrow x^2+y^2 \geq \sqrt{\frac{x^4+y^4}{2}}+xy.\end{aligned}$$

For $x=a$ and $y=b$ and multiplying by c we have

$$a^2c+b^2c \geq c\sqrt{\frac{a^4+b^4}{2}}+abc.$$

Analogously,

$$b^2a+c^2a \geq a\sqrt{\frac{b^4+c^4}{2}}+abc$$

and

$$c^2b+a^2b \geq b\sqrt{\frac{c^4+a^4}{2}}+abc.$$

Adding the last three inequalities gives the desired result.

Also solved by Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Soumitra Moukherjee, Scottish Church College, Chandar Nagore, India; Ravi Prakash, Oxford University Press, New Delhi, India.

Problem 782. *Proposed by Jose Luis Diaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*

Let a, b, c be the lengths of the sides of triangle *ABC* and m_a, m_b, and m_c the lengths of its medians. Prove that

$$\frac{2^{m_a}+2^{m_b}+2^{m_c}}{2^a+2^b+2^c} < 1.$$

Solution *by Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George*

Emil Palade" School, Buzau, Romania.

Since $m_c < \frac{a+b}{2}$, by the AM-GM inequality we have

$$2^a + 2^b \geq 2\sqrt{2^a 2^b} = 2 \cdot 2^{\frac{a+b}{2}} > 2 \cdot 2^{m_c}.$$

Writing the other two similar inequalities and adding all three gives the desired result.

Also solved by Madison Estabrook, Missouri State University, Springfield, MO; Rovsen Pirkulyev, Baku State University, Sumgait, Azerbaidjian; and the proposer.

Problem 783. *Proposed by Jose Luis Diaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*

Find all real solutions to the following system of equations:

$$x^3 + 2x + y = 9 + 3x^2$$
$$3y^2 + 6y + z = 21 + 9y^2$$
$$5z^3 + 10z + x = 33 + 15z^2.$$

Solution *by the proposer.*

We can rewrite the system as

$$3 - y = x^3 - 3x^2 + 2x - 6$$
$$3 - z = 3(y^3 - 3y^2 + 2y - 6)$$
$$3 - x = 5(z^3 - 3z^2 + 2z - 6).$$

Since $t^3 - 3t^2 + 2t - 6 = (t-3)(t^2+2)$, we have

$$3 - y = (x-3)(x^2+2)$$
$$3 - z = 3(y-3)(y^2+2)$$
$$3 - x = 5(z-3)(z^2+2).$$

Multiplying these together gives

$$-(x-3)(y-3)(z-3) = 15(x-3)(y-3)(z-3)(x^2+2)(y^2+2)(z^2+2).$$

From this we get

$$0 = (x-3)(y-3)(z-3)(15(x^2+2)(y^2+2)(z^2+2)+1).$$

Since the last factor above is positive, either $x = 3$, $y = 3$ or $z = 3$. If we assume that $x = 3$, the first equation says $y = 3$. Substituting this into the second equation implies that $z = 3$. The same occurs for starting with $y = 3$ or $z = 3$. So $x = y = z = 3$ is the only real solution.

Also solved by Soumava Chakraborty, Softweb Technologies, Kolkota, India.

Problem 784. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, Neculai Stanciu, "George Emil Palade", Buzau, Romania.*

Prove that in any triangle *ABC* with $BC = a, CA = b, AB = c$ and area *F*, the following inequalities are true.

$$(b^2+c^2)\sin\frac{A}{2} + (c^2+a^2)\sin\frac{B}{2} + (a^2+b^2)\sin\frac{C}{2} \geq 4\sqrt{3}F,$$
$$ab(1+\sin^2\frac{C}{2}) + bc(1+\sin^2\frac{A}{2}) + ca(1+\sin^2\frac{B}{2}) \geq 4\sqrt{3}F.$$

Solution *by Ioan Viorel Codreanu, Satulung, Maramures. Romania.*

We have

$$(b^2+c^2)\sin\frac{A}{2} \geq 2bc\sin\frac{A}{2} = \frac{bc\sin A}{\cos\frac{A}{2}} = \frac{2F}{\cos\frac{A}{2}} = 2F\sec\frac{A}{2}.$$

Similarly, $(c^2+a^2)\sin\frac{B}{2} \geq 2F\sec\frac{B}{2}$ and $(a^2+b^2)\sin\frac{C}{2} \geq 2F\sec\frac{C}{2}$. Then $\sum(b^2+c^2)\sin\frac{A}{2} \geq 2F\sum\sec\frac{A}{2}$. Using Jensen's Inequality and that $f(x) = \sec x$ on $(0, \pi/2)$ is a convex function, we get $\sum\sec\frac{A}{2} \geq 3\sec\frac{\sum A}{6} = 2\sqrt{3}$. Thus $\sum(b^2+c^2)\sin\frac{A}{2} \geq 4\sqrt{3}F$. Next

$$ab(1+\sin^2\frac{C}{2}) \geq 2ab\sin\frac{C}{2} = \frac{ab\sin C}{\cos\frac{C}{2}} = 2F\sec\frac{C}{2}.$$

Similarly, $bc(1+\sin^2\frac{A}{2}) \geq 2F\sec\frac{A}{2}$ and $ca(1+\sin^2\frac{B}{2}) \geq 2F\sec\frac{B}{2}$. Then

$$\sum ab(1+\sin^2\frac{C}{2}) \geq 2F\sum\sec\frac{A}{2} \geq 4\sqrt{3}F.$$

Also solved by Soumava Chakraborty, Softweb Technologies, Kolkota, India; Ravi Prakash, Oxford University Press, New Delhi, India; Soumitra Moukherjee, Scottish Church College, Chandar Nagore, India; and the proposer.

Problem 785. *Proposed by Iuliana Trasca, Scornicesti, Romania.*

Show that if $x, y, z > 0, then$

$$\frac{x^6z^3 + y^6x^3 + z^6y^3}{x^2y^2z^2} \geq \frac{x^3+y^3+z^3+3xyz}{2}.$$

Solution *by Soumava Chakraborty, Softweb Technologies, Kolkota, India.*

The inequality is equivalent to

$$2\left(x^6z^3+y^6x^3+z^6y^3\right)\geq x^5y^2z^2+y^5z^2x^2+z^5x^2y^2+3x^3y^3z^3.$$

Using the AM-GM inequality, we have

$$\begin{aligned}x^6z^3+x^6z^3+x^3y^6 &\geq 3x^5y^2z^2\\ y^6x^3+y^6x^3+y^3z^6 &\geq 3y^5z^2x^2\\ z^6y^3+z^6y^3+z^3x^6 &\geq 3z^5x^2y^2.\end{aligned}$$

Adding these gives

$$3\left(x^6z^3+y^6x^3+z^6y^3\right)\ \geq 3\left(x^5y^2z^2+y^5z^2x^2+z^5x^2y^2\right).$$

Dividing by 3 yields

$$x^6z^3+y^6x^3+z^6y^3\ \geq\ x^5y^2z^2+y^5z^2x^2+z^5x^2y^2.$$

The AM-GM inequality also gives us

$$x^6z^3+y^6x^3+z^6y^3\ \geq 3x^3y^3z^3.$$

Summing the previous two inequalities gives the inequality that is equivalent to the one of the problem.

Also solved by Soumitra Moukherjee, Scottish Church College, Chandar Nagore, India; Ioan Viorel Codreanu, Satulung, Maramures, Romania; Titu Zvonaru, Comanesti, Romania; and the proposer.

Problem 786. *Proposed by Thomas Chu, Macomb, Illinois.*

If $x,y,z>1$, then

$$(x^2+y^2+z^2)(x+y+z)+x^3+y^3+z^3>4xy+4xz+4yz.$$

Solution *by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.*

By changing variables $x=1+a$, $y=1+b$ and $z=1+c$, the problem reads as: Prove that if $a,b,c>0$, then

$$\begin{aligned}&\left(\sum(1+a)^2\right)\left(3+\sum a\right)+\sum(1+a)^3\\ &\qquad >4(1+a)(1+b)+4(1+b)(1+c)+4(1+c)(1+a).\end{aligned}$$

Expanding the right-hand side and left-hand sides, we get

$LHS = 12{+}12\,(a+b+c){+}4\,(ab+ac+bc){+}\ 8\sum a^2{+}\sum a^2b{+}2\sum a^3$

and

$$RHS = 12 + 8\,(a+b+c) + 4\,(ab+ac+bc)\,.$$

We can clearly see that the $LHS > RHS$ when $a, b, c > 0$.

Also solved by Anas Adlany (student), Omar Ben Abdelaziz University, El Jadida, Morroco; Myagmarsuren Yadamsuren, Ulanbataar University, Ulanbataar, Mongolia; Soumava Chakraborty, Softweb Technologies, Kolkota, India; and the proposer.

Problem 787. *Proposed by the editor.*

Mike buys some pants and shorts at the Great Pants Store. Mike buys shorts that cost \$11each and pants that cost \$14 each. His total before taxes is \$283. How many shorts and how many pants did Mike buy?

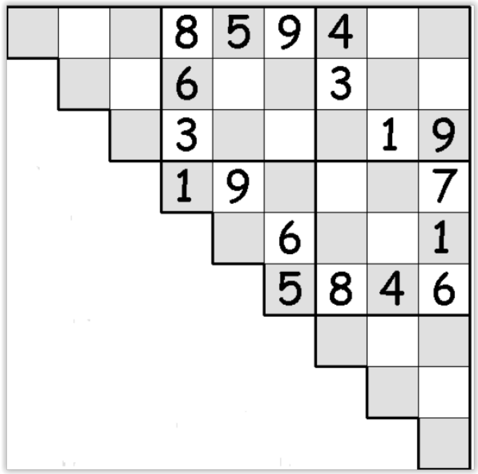
Solution *by by Robert Bailey (former KME national President 2001-2005), Niagara University, NY.*

Let x = number of shorts and y = number of pants. We have $11x + 14y = 283$ which is a linear Diophantine equation in two variables. Then $14y = 28311x$ which is equivalent to $14y \equiv 283(\text{mod}11)$ or $3y \equiv 8(\text{mod}11)$ or $3y \equiv -3(\text{mod}11)$. Since 3 is relatively prime to 11, we get $y \equiv -1(\text{mod}11)$. This means $y = 10, 21, 32,$. The only value for y that causes x to be positive in the equation $11x + 14y = 283$ is $y = 10$ in which case $x = 13$.

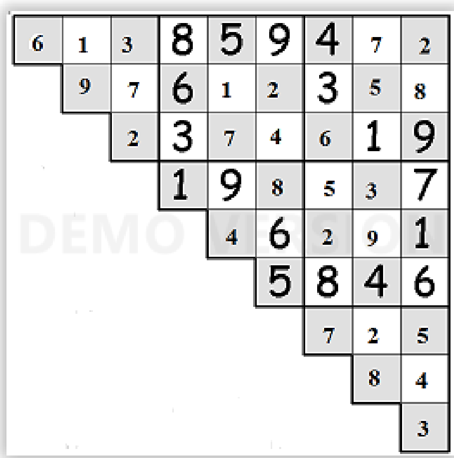
Also solved by Michael Bhujel, Bobbie Legg, Katie Tyson (students), and Bill Yankosky, North Carolina Wesleyan College, Rocky Mount, NC; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; and the proposer.

Problem 788. *Proposed by George Heineman, Worcester Polytechnic Institute, Worcester, MA.*

A Sujiken$^{\text{TM}}$ puzzle has a triangular grid of cells containing digits from 1 to 9. You must place a digit in each of the empty cells with the constraint that no digit can repeat in any row, column, or diagonal. Additionally, no digit can repeat in the 3x3 large squares withthick borders or the three triangular regions with thick borders. The puzzle below is of intermediate difficulty.



Solution



Solved by Jamie Farrar, Destinee Fisher, Nicole Kettle, Courtney Lush (students), Ed Wilson (retired faculty), Eastern Kentucky University, Richmond, KY; Jeremiah Bartz, University of North Dakota, Grand Forks, ND; Katie Tyson (student), Gail Stafford, Carol Lawrence, Bill Yankosky, North Carolina Wesleyan College, Rocky Mount, NC.

Kappa Mu Epsilon News

Edited by Peter Skoner, Historian
Updated information as of June 2017

Another Historian was elected in April, so news of chapter activities and other noteworthy KME events should now be sent to

Cynthia Huffman, KME Historian
Pittsburg State University
Mathematics Department
117 1701 S. Broadway
Pittsburg, KS 66762
or to
cjhuffman@pittstate.edu

KAPPA MU EPSILON
Installation Report
Connecticut Gamma, Central Connecticut University
New Britain, Connecticut

The installation of the Connecticut Gamma Chapter of Kappa Mu Epsilon was held in the Connecticut Room of Memorial Hall on the campus of Central Connecticut State University on Friday, March 24, 2017, at 5:30 PM.

Faculty sponsor, Professor Marian Anton organized the event, and corresponding secretary Professor Mihai Bailesteanu opened the event with a welcome and led the events. Dean Faris Malhas offered a formal university welcome and opening remarks. An elegant dinner was followed by an interesting talk by Professor Rachel Schwell, who presented on "What Topology Tells Us about the Nash Equilibrium." The initiation and installation ceremony was led by the installing officer, KME National Historian Peter Skoner.

Participating in the ceremony were the charter officers: Zoe Anne Cramer, President; Nilay Nitin Bhatt, Vice President, Rina Saliu; Recording Secretary; and Cassady Brooke Zipkin, Treasurer. Each officer was charged with the responsibilities of the office, and each chose to accept those responsibilities. After Secretary Saliu completed describing KME's crest, the organization was declared to be the Connecticut Gamma Chapter of Kappa Mu Epsilon and the chapter's charter was presented to chapter president Cramer.

In addition to the faculty mentioned, the other faculty charter members of Connecticut Gamma include Professors Nelson Castaneda and Frederic Latour. The other student charter members of Connecticut Gamma are Olivia Baillargeon, Jeffrey Blankenship, Rong Chen, Luke D'Ascoli, Natalie Decker, Patrick Dzioba, Robert Johnston III, Heath Loder, Lilia Miller, Nicholas Pipino, Andrew Pelletier, Michael Quinonez, Landon Renzullo, Ryan Schmidt, Patryk Stolarz, Damian Szarwacki, David Thorne, Pedro Urbina, and Peter Woolard. Each initiate was invited to sign the Connecticut Gamma Chapter Roll, and was presented with a certificate, membership card, KME brochure, a program, and a KME jewelry pin. Professor Skoner offered remarks about the history of Kappa Mu Epsilon and best wishes to the charter and future members of Connecticut Gamma. A total of 68 people attended.

Connecticut Gamma

KAPPA MU EPSILON
Installation Report
Georgia Eta, Atlanta Metropolitan State College
Atlanta, Georgia

The installation of the Georgia Eta Chapter of Kappa Mu Epsilon was held at 2:00 P.M. on Tuesday, January 31, 2017, in the Edwin Thompson Student Activity Center Conference Room on the campus of Atlanta Metropolitan State College in Atlanta, GA. Dr. Bryan Mitchell, Dean of the Division of Science, Math, and Health Professions, served as master of ceremonies and conductor. Dr. Michael Heard, Vice President for Academic Affairs, welcomed those attending. Dr. David Dempsey, KME National Treasurer, served as the installing officer after giving a short talk on "A Brief History of ~~Time~~ ... Functions." After the installation ceremony, Dr. Gary McGaha, President of Atlanta Metropolitan State College, gave some remarks and congratulated the initiates. The following charter members were initiated during the installation:

Faculty	Students
Bassam Abduliatif	Malik Burton
Gyuheui Choi	Toni Byrd
Shreyas Desai	Simone Clark
James D. Dowdell	Joseph Daniels
Anthonia Ekwuocha	Rama Hawai
Raghu Gompa	Bruce McNeal
Jackson Henry	Lanessa Northcut
Mulugeta Markos	Leia Singh
Joseph Patterson	Devante Singletary
Pitso Senalte	
Noel Whelchel	
Dongwook Kim	
Kwan Lam	

The first officers of the Georgia Eta chapter were installed and are as follows: Leia Singh, president; Devante Singletary, vice president; Joseph Daniels, secretary; Lanessa Northcut, treasurer; Dr. Mulugeta Markos, corresponding secretary; and Dr. Kwan Lam, faculty sponsor. The afternoon concluded with a reception and refreshments. About 40 people attended the event.

Georgia Eta

Chapter News

AL Zeta – Birmingham-Southern College
Chapter President – Adam Pratt; 31 Current Members
Other Fall 2016 Officers: Marjory Day, Vice President; Mary-Stewart Wachter, Secretary; Cynthia Kagambirwa, Treasurer; and Maria Stadnik, Corresponding Secretary and Faculty Sponsor
This fall our KME colloquium was given on October 18 by Dr. Joshua Zelinsky of Birmingham-Southern College. Dr. Zelinsky gave an interesting talk entitled "The ABC Conjecture, Mason-Stothers, and Fermat's Last Theorem."

AL Eta – The University of West Alabama
Corresponding Secretary – Hazel Truelove; 14 New Members
New Initiates – Brady Badon, Kaylee Ryan Banister, Katelynn Carlson, Jaamal Elkins, Chicko Jones, Jacob Adam Ramsey, Kirsten Jan Reilly, Jallena Roberts, Yana Alexis Rodgers, William Waylong Rowell, Timothy Wayne Watson, Keith Watson, Lane Reid Weaver, and KelviNeisha Williams.

AL Theta – Jacksonville State University
Chapter President – Daniel Miradakis; 50 Current Members
Other Fall 2016 Officers: Other Fall 2016 Officers: Timothy Garrett, Vice President; Jasmine Beaudette, Secretary; James Thompson, Treasurer; and Dr. David Dempsey, Corresponding Secretary and Faculty Sponsor
The Alabama Theta chapter met biweekly during Fall 2016 and had at least monthly events. September's event was an outing for dinner and the musical "The Addams Family," in which our chapter secretary, Jasmine Beaudette, played a role. On October 21, we held our annual Halloween Party, complete with math-themed costumes and pumpkin carving. On December 2, we combined our Christmas/End-of-the-Semester Party with Psi Chi (psychology honor society). Our annual Spring Initiation Ceremony is planned for March 3, 2017.

AR Beta – Henderson State University
Chapter President – Jacob Woodall; 39 Current Members
Other Fall 2016 Officers: Zach Winfield, Vice President; Jacob Woodall, Secretary; Carmen Wise, Treasurer; Fred Worth, Corresponding Secretary; and Carolyn Eoff, Faculty Sponsor
Six new members were initiated in a ceremony in April; the other main activity of HSU's KME chapter during 2016 was a Pi-day celebration, coordinated with the HSU Math-Stat club.

CA Eta – Fresno Pacific University
Chapter President – Elaine Draper; 33 Current Members
Other Fall 2016 Officers: Kimberlie Raulino, Vice President and Trea-

surer; Terence Yi, Corresponding Secretary; and Ron Pratt, Faculty Sponsor

CT Beta – Eastern Connecticut State University

Corresponding Secretary and Faculty Sponsor – Mehdi Khorami; 462 Current Members

FL Delta – Embry Riddle Aeronautical University

Corresponding Secretary – Jayathi Raghavan; 14 New Members

New Initiates – James Bukowski, Naia Butler-Craig, Benjamin Button-Edelson, Janice D. Cabrera, Alexander Paul Donato, Steve Gulliksen, Wanjiku Kanjumba, Brooke Linendoll, Meghan Ray, Daniel Silverio, Taylor Stark, Sandra Pamela Torrez, Katelyn Wentworth, and Anissa Zacharias.

GA Zeta – Georgia Gwinnett College

Chapter President – Shahriyar Roshan Zamir; 44 Current Members

Other Fall 2016 Officers: Bess Burnett, Vice President; Heather McAfee, Secretary; Antoinette Miezan, Treasurer; Dr. Jamye Curry, Corresponding Secretary; and Drs. Jenny Sinclair and Livy Uko, Faculty Sponsors

HI Alpha – Hawaii Pacific University

Chapter President – Dyon Buitenkamp; 15 Current Members; 6 New Members

Other Fall 2016 Officers: Tara Davis, Corresponding Secretary and Faculty Sponsor

We had an initiation dinner in November.

New Initiates – Kristofer Francis Caluya, Tram Hoang-Nguyen, Pancy T. Lwin, Monica Parrish, Dylan West-Von Sonn, and Taylor Suzanne Tuleja.

IA Alpha – University of Northern Iowa

Chapter President – Toby Maggert; 26 Current Members; 5 New Members

Other Fall 2016 Officers: Julie Kirkpatrick, Vice President; Destiny Leitz, Secretary; Jake Weber, Treasurer; and Mark D. Ecker, Corresponding Secretary and Faculty Sponsor

Our first fall KME meeting was held on September 28, 2016 at Professor Doug Mupasiri's house where student member Julie Kirkpatrick presented her paper entitled "Knot Theory." Student member Heather Bavido presented her paper entitled "3D Modelling and Printing for Math" at our second meeting on November 2, 2016 at Professor Mark Ecker's home. Morgan Bigbee addressed the fall initiation banquet with "St. Louis Cardinals Run Differential." Our fall banquet was held at Peppers restaurant in Cedar Falls on December 7, 2016, where five new members were initiated.

New Initiates – Nathan Flaherty, Nathan Matthes, Duece Phaly, Taryn Vanryswyk, and

Macey Winter.

IL Zeta – Dominican University

Corresponding Secretary – Aliza Steurer; 19 Current Members

We are planning a spring semester imitation ceremony.

KS Beta – Emporia State University

Chapter President – Brian Mosier; 44 Current Members

Other Fall 2016 Officers: Michelle Foster, Vice President; Kandace Miller, Secretary; Dallas Shafer, Treasurer; Tom Mahoney, Corresponding Secretary; and Brian Hollenbeck, Faculty Sponsor

The Kansas Beta chapter recently opened a KME Lounge for mathematics students to come to for studying or relaxing. We visited the Linda Hall Library in Kansas City and participated in an Escape Room challenge. The chapter also constructed their own Escape Room experience to challenge our Intro to Math course.

KS Delta – Washburn University

Chapter President – Katelyn Rollins; 20 Current Members

Other Fall 2016 Officers: Leanna Willer, Vice President; Katelyn Skillingstad, Secretary; Taylor Balsmeier, Treasurer; and Kevin Charlwood, Corresponding Secretary and Faculty Sponsor

We met four times over lunch this fall with our math club, Club Mathematica, and had speakers on three occasions. Our Math Club adviser, Jason Shaw, spoke on several of Ramanujan's contributions to mathematics. John Blocher from Security Benefit Group spoke on the actuarial profession and gave advice to students on resume preparation and interviewing skills. He also textitasized the need for students to take actuarial exams earlier in their academic careers. Bill Gahnstrom in our department gave a presentation on how to use a slide rule.

KY Beta – University of the Cumberlands

Chapter President – Daniel Enge; 16 Current Members

Other Fall 2016 Officers: James MacPherson, Vice President; Dage Spinning, Secretary; McKenzie Wheeler, Treasurer; Dr. Reid Davis, Corresponding Secretary; and Dr. Jonathan Ramey, Faculty Sponsor

Along with the Mathematics and Physics Club and Sigma Pi Sigma, the chapter had a picnic at Briar Creek Park on October 11. On December 10, the entire department, including the Kentucky Beta chapter, had a Christmas party with about 22 people in attendance.

MD Alpha – Notre Dame of Maryland University

Chapter President – Kristin Kneller; 14 Current Members

Other Fall 2016 Officers: Stephanie Roche, Vice President; Margaret Pederson, Secretary; Fareeha Syed, Treasurer; and Charles Buehrle, Corre-

sponding Secretary and Faculty Sponsor

MD Alpha hosted a KME cookie social on Wednesday October 4th.

MD Beta – McDaniel College

Corresponding Secretary – Spencer Hamblen; 12 New Members

New Initiates – Joshua Bussiere, Yann Wendeu (Steve) Foyet, Madison Gamble, Elias Jaffe, Vi Lam, Grace Lyons, Ann Marshall, Matthew Meagher, Noel Nunnermacker, Kenneth Porter, Samantha Smith, and Chung Truong.

MD Delta – Frostburg State University

Chapter President – Rebecca Lee; 18 Current Members

Other Fall 2016 Officers: Jimmy West, Vice President; Emma Siebert, Treasurer; Mark Hughes, Corresponding Secretary and Faculty Sponsor; and Frank Barnet, Faculty Sponsor

The Maryland Delta Chapter held monthly meetings throughout the fall semester. Each meeting featured pizza and fun videos on a variety of mathematical topics. Chapter members also took part in some mathematics department activities, in particular, a movie night featuring a film on H. S. M. Coxeter and a presentation from a visiting lecturer on using game theory to develop strategies in playing basketball. Chapter members also participated in the university's annual Majors Fair.

MI Delta – Hillsdale College

Chapter President – Hannah Andrews; 37 Current Members; 16 New Members

Other Fall 2016 Officers: Linnet Mbogo, Vice President; Nathanael Meadowcroft, Secretary; Tanner Orion Wright, Treasurer; and Dr. David Gaebler, Corresponding Secretary and Faculty Sponsor

The Michigan Delta chapter welcomed three news officers at the beginning of the 2016-2017 school year. Due to a surge in the number of eligible candidates, reflecting a trend in the college toward more math majors and more non-majors taking advanced math classes, we have switched to holding initiation every semester rather than once a year. We continue to offer 20-minute mini-math-talks at the initiation ceremony, and more treats afterward than we can eat. In addition to initiation, we held a social "Math Jeopardy" event which drew close to 20 people for a fun and intense evening of trivia.

New Initiates – Shelby Bargenquast, Rebecca Carlson, Adrienne Carrier, Conner Dwinell, Curtis Fullom, Madeline Greb, Nathan Hollern, Jean Pendergrass, Christopher Pudenz, Amanda Reagle, Thomas Reusser, Laura Salo, Rose Schweizer, Abigail Trouwborst, Gill West, and Conor Woodfin.

MO Alpha – Missouri State University

Chapter President – Paige Buchmueller; 40 Current Members; 5 New

Members
Other Fall 2016 Officers: Ashley Kingston, Vice President; Rebecca Crow, Secretary; Sara Jones, Treasurer; and Jorge Rebaza, Corresponding Secretary and Faculty Sponsor

1. As in every semester, we had three seminars:
   - Seminar 1: Monday September 26th. Dr. Peter Plavchan, faculty member in the Physics and Astronomy Department at MSU, presented the talk "Astrostatistics: The Intersection of Astronomy and Mathematics." Pizza and soda were served.
   - Seminar 2: Monday October 25th. Meagan Leppien, actuarial analyst at American National Property and Casualty Company, presented the talk "Careers in Actuarial Science: From College to the Workplace." Pizza and soda were served. At this event, we also initiated 5 new KME members.
   - Seminar 3: Tuesday November 15th.Two students from the Senior Seminar class (MTH 497) presented their papers: "Exploring NFL Player Ranking with Mathematics", by Victoria Hagan, and "Mastermind Decision Rule" by Michelle Pellegrino. Pizza and soda were served.
2. As in every fall semester, we organized a picnic on Wednesday September 7th, starting at 5PM at Phelps Grove Park. As usual, we had a great turnout of students, faculty, and their families!
3. We also had an end-of-semester party on Thursday December 8th, the last day of classes. We had lots of games music, food, drinks, and desserts. Students from other student organizations in the College participated at this event.

New Initiates – Joshua Gooch, Madison Jones, Kendra Larsen, Mengqing Qin, and Adam Somers.

MO Beta – University of Central Missouri

Chapter President – Madison Ultican; 26 Current Members
Other Fall 2016 Officers: Christina Duerr, Vice President; Aaron Butz, Secretary; Nicholas Purcell, Treasurer; Ashley Beard, Historian; Rhonda McKee, Corresponding Secretary; Rhonda McKee, Steve Shattuck and Nicholas Baeth, Faculty Sponsors

The Missouri Beta chapter of KME met monthly during the fall 2016 semester. One of our favorite meetings included a game of Math Jeopardy. We are looking forward to the National Convention in April!

MO Eta – Truman State University

Corresponding Secretary – David Garth; 8 New Members

New Initiates – Christian Burton, Lucas Doherty, Erin Leventhal, Rachel Miller, Brandon

Mueller, Megan Perry, Allison Smith, and Andrew Wolf.

MO Theta – Evangel University

Chapter President – Kevin Grimes; 16 Current Members

Other Fall 2016 Officers: Samantha Orr, Vice President; and Don Tosh, Corresponding Secretary and Faculty Sponsor

Meetings were held monthly. In December we held an ice cream social at the home of Don Tosh.

MO Nu – Columbia College

Corresponding Secretary – Kenny Felts; 10 Current Members

MS Alpha – Mississippi University for Women

Chapter President and Treasurer – Ciara Peoples; 10 Current Members; 2 New Members

Other Fall 2016 Officers: Sugam Bhattarai, Vice President; Aastha Ghimire, Secretary; Dr. Joshua Hanes, Corresponding Secretary and Faculty Sponsor

In the fall semester we initiated two new members and we gathered supplies to put together boxes for Operation Christmas Child.

New Initiates – Aisha Ghirmire and Sweyaksha Srestha.

MS Delta – William Carey University

Corresponding Secretary – Janie Bower; 5 New Members

New Initiates – Pankaj Bhatta, Ashleigh Jones, Brent Manint, Taylor McCollister, and Gretchen Waters.

NC Zeta – Catawba College

Chapter President – Declan Stinson; 19 Current Members; 8 New Members

Other Fall 2016 Officers: Alicia Richards, Vice President; Dominique Karriker, Secretary; Christian Watts, Treasurer; and Doug Brown, Corresponding Secretary and Faculty Sponsor

New Initiates – Kerry Aitken, Cody Bennett, Avery Denton, Dagur Ebenezersson, Marcia-Mariel Erhart, Matthew Hefner, Dr. Karen Lucas, and Erin Moore.

NE Delta – Nebraska Wesleyan University

Chapter President – Karissa Vandenberg; 10 Current Members

Other Fall 2016 Officers: Madison Montgomery, Vice President; Will Reimer, Secretary and Treasurer; and Melissa Erdmann, Corresponding Secretary and Faculty Sponsor

The Nebraska Delta KME chapter had a great autumn! We had a beginning-of-the-year picnic and game night, a career panel, a math movie night, and a joint holiday party with the math/physics club, complete with math/-

physics carols.

NJ Epsilon – New Jersey City University

Corresponding Secretary – Beimnet Teclezghi; 23 New Members

New Initiates – Gabriella Ariemma, Sofiane Boudib, Gunhan Caglayan, Geomara Cando, Manuel Dones, Caitlin Dugan, Andrea Gloetzer, Leslie Gomez, Camila Guerrero, Mahamoud Hassan, Zined Hassoune, Ivana Lopa, Valerie Nigrelli, Daniell Olivera, Krupa Patel, Jose Pedroza, Abdurrahaman Pllana, Laura Pojero, Zanib Saeed, Rebecca Semeniak, Irene Umana, Irley Vallejo, and Fatima Yusuf.

NY Omicron – St. Joseph's College

Chapter President – Ryan Stephens; 25 Current Members

Other Fall 2016 Officers: Michael Mirrione, Vice President; Melissa De Jesus, Secretary; Angela Vetere, Treasurer; Dr. Elana Reiser, Corresponding Secretary; and Dr. Donna Pirich, Faculty Sponsor

We ran a Christmas toy drive and donated to a local charity. Our members volunteered their time to tutor in our math clinic for local high school students.

OH Gamma – Baldwin Wallace University

Chapter President – Corrinne Horvath; 60 Current Members

Other Fall 2016 Officers: Natalie Castragano, Vice President; Stephen Osborn, Secretary; and David Calvis, Corresponding Secretary and Faculty Sponsor

OH Zeta – Muskingum University

Corresponding Secretary – Richard Daquila; 4 New Members

New Initiates – Stephanie Clark, Tyler Miller, Matthew Nardi, and Jacob Shoup.

OH Theta – Capital University

Chapter President – Julia Kunkel; 14 Current Members

Other Fall 2016 Officers: Jennie White, Vice President; Nick Hernandez, Secretary; Jack Gorden, Treasurer; Paula Federico, Corresponding Secretary; and Jonathan Stadler, Faculty Sponsor

OK Epsilon – Oklahoma Christian University

Corresponding Secretary – Jennifer Bryan; 17 New Members

New Initiates – Drew Bellcock, Josh Bilello, Laura Blair, Seth Brown, Jason Brunner, Ines Dushime, Kaylee Eubank, Aubrey Gonzalez, Michael Harlan, Brennym Kaelin, Kristen Lindsey, David Lopez, Hannah McKenzie, Brayden Reiter, Haylie Ritchie, Anna Taylor, and Grant Tucker.

PA Lambda – Bloomsburg University of Pennsylvania

Corresponding Secretary – Elizabeth Mauch; 6 New Members

New Initiates – Kate Cossitor, Brianna Hendrickson, Jonathan Piperato, Chase Sakitis, Marissa Shelhammer, and Derek Stahl.

PA Mu – Saint Francis University

Corresponding Secretary – Pete Skoner: 39 Current Members

Other Fall 2016 Officers: Brendon LaBuz, Faculty Sponsor

PA Rho – Thiel College

Chapter President – Julia Fink; 8 Current Members

Other Fall 2016 Officers: Jennifer Rickens, Vice President; Amanda Dobi, Secretary; Jesse Sealand, Treasurer; Russell Richins, Corresponding Secretary; and Dr. Jie Wu, Faculty Sponsor

In the fall, we hosted a Challenge 24 competition with donation for the Good Shepherd of Grenville and we also initiated a Math Zone drop-in tutoring.

PA Sigma – Lycoming College

Chapter President – Rachel Duncan; 20 Current Members; 7 New Members

Other Fall 2016 Officers: Bethany Hipple, Vice President; Coral Chiaretti, Secretary; Amanda MacTarnaghan, Treasurer; and Santu de Silva, Corresponding Secretary.

The Fall of 2016 saw a significant increase in Pennsylvania Sigma activity. The following were more in the line of social activities: Lottery Ticket Tree Fundraiser – October 17th-21st, 2016; Ard's Farm Trip and Dinner – Saturday, October 22nd, 2016 –a Halloween visit to a pumpkin farm, followed by dinner; Peer Interview Program – Thursday, November 3rd, 2016 –practice interviews to help graduating seniors prepare for their employment search; Moe's Southwest Grill Fundraiser – Friday, November 11th, 2016. Unlike the funds allocated to honor societies by the college, money raised by the students themselves may be kept for use in future years. In addition, we had a visit and presentations by a successful alumna who worked for the NSA, describing the work environment, and what it was like to work on certain sorts of projects. And these were service activities: KME members prepared materials, and held classes to help students certifying in education prepare for the State Board examinations in Mathematics. The student officers this year have been very pro-active, and there is reason to believe that this level of activity will continue.

New Initiates – Dr. Andrew Brandon, Coral Chiaretti, Katherine Cleland, Ian Fairclough, Bethany Hipple, Amanda McTarnaghan, and Kimberly Perotta.

RI Beta – Bryant University

Chapter President – William Kelley; 25 Current Members

Other Fall 2016 Officers: Emma Wieduwilt, Vice President; Nathaniel Morgan, Secretary; Owen Wrinn, Treasurer; John Quinn, Corresponding

Secretary; and Alan Olinsky, Faculty Sponsor

We met with our student executive board to do early planning for the initiation ceremony for new member which will be held during the spring 2017. We also made a presentation of the KME Honor Society to our mathematics majors during the Actuarial Association meeting on November 16 2016.

SC Gamma – Winthrop University

Chapter President – MaLyn Lawhorn; 14 Current Members; 8 New Members

Other Fall 2016 Officers: Alison Tighe, Vice President; Genia Kennedy, Secretary; Jean Wolfe, Treasurer; and Jessie Hamm, Corresponding Secretary and Faculty Sponsor

Winthrop University's chapter of Kappa Mu Epsilon initiated 8 new members this fall. We have had several meetings throughout the year. Last weekend we went to dinner and the movie Hidden Figures. We are planning a community outreach event for April. For this event we are hosting a fun "Math Day" for a local elementary school. We are also planning to celebrate Math Awareness Month in April by hosting a variety of fun math activities on campus. We have a busy spring ahead of us!

New Initiates – Colin Frazier, Genia Kennedy, Christina Knight, Sydney McCall, Justin McCullough, Victoria Nidiffer, Christina Sadak, and Jean Wolfe.

SC Epsilon – Francis Marion University

Corresponding Secretary – Damon Scott; 5 New Members

New Initiates – Amy N. Benton, Chase Covington, April Garrity, Alexis K. Glover, and Nicholas Tomlinson.

TN Beta – East Tennessee State University

Corresponding Secretary – Bob Gardner; 12 New Members

New Initiates – Devanshu Agrawal, Tiffany Blevins, Kendra Disney, Kaeli Gardner, Sydney Gardner, Jessica Lang, Miranda Lawhorn, Macon Magno, Ashton Morelock, Kyle Murphy, Natalie Murray, and Logan Norton.

TN Gamma – Union University

Chapter President – Rachel Brewer

Other Fall 2016 Officers: Joshua Stucky, Vice President; Amy Murdaugh, Secretary and Treasurer; Andrew Edmiston, Webmaster and Historian; Bryan Dawson, Corresponding Secretary; and Matt Lunsford, Faculty Sponsor

TN Gamma held its annual picnic on September 23.

TX Iota – McMurry University

Corresponding Secretary – Dr. Kelly L. McCoun; 10 New Members

New Initiates – Blake Cochran, Nicholas Conklin, Nathan Elders, Derek Gainey, Gladys

Hinestroza, Rodney Jones, Ryan Pittman, Kandi Rose, Curtis Summers, and Shantel Thomas.

TX Lambda – Trinity University

Chapter President – Zach Tuten; 249 Current Members

Other Fall 2016 Officers: Shelby Luikartt, Vice President; David Stroud, Secretary; and Dr. Hoa Nguyen, Corresponding Secretary and Faculty Sponsor

VA Delta – Marymount University

Chapter President – Bernadette Wunderly; 40 Current Members

Other Fall 2016 Officers: Kayla Baughman, Vice President; Nicole Ferree, Secretary; Katherine Martin, Treasurer; Will Heuett, Corresponding Secretary and Faculty Sponsor

WV Alpha – Bethany College

Chapter President – Alyssa K. Smydo; 7 Current Members

Other Fall 2016 Officers: Robert A. Murano, Vice President; and Adam C. Fletcher, Corresponding Secretary and Faculty Sponsor

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 Apr 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
MS Beta	Mississippi State University, Mississippi State	14 Dec 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
AL Alpha	Athens State University, Athens	5 Mar 1935
NM Alpha	University of New Mexico, Albuquerque	28 Mar 1935
IL Beta	Eastern Illinois University, Charleston	11 Apr 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 Apr 1937
OH Alpha	Bowling Green State University, Bowling Green	24 Apr 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	University of Central Missouri, Warrensburg	10 Jun 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 Jun 1941
MI Beta	Central Michigan University, Mount Pleasant	25 Apr 1942
NJ Beta	Montclair State University, Upper Montclair	21 Apr 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 Mar 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 Jun 1947
CO Alpha	Colorado State University, Fort Collins	16 May 1948
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 Apr 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960

OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 Apr 1965
AL Epsilon	Huntingdon College, Montgomery	15 Apr 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
AR Alpha	Arkansas State University, Jonesboro	21 May 1965
TN Gamma	Union University, Jackson	24 May 1965
WI Beta	University of Wisconsin—River Falls, River Falls	25 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	McDaniel College, Westminster	30 May 1965
IL Zeta	Dominican University, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 Mar 1971
KY Alpha	Eastern Kentucky University, Richmond	27 Mar 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 Apr 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 Apr 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State University, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sep 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis University, Loretto	14 Sep 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury University, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 Mar 1985

NE Delta	Nebraska Wesleyan University, Lincoln	18 Apr 1986
TX Iota	McMurry University, Abilene	25 Apr 1987
PA Nu	Ursinus College, Collegeville	28 Apr 1987
VA Gamma	Liberty University, Lynchburg	30 Apr 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 Apr 1990
CO Delta	Mesa State College, Grand Junction	27 Apr 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 Apr 1991
SD Alpha	Northern State University, Aberdeen	3 May 1992
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 Mar 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 Apr 1997
MI Delta	Hillsdale College, Hillsdale	30 Apr 1997
MI Epsilon	Kettering University, Flint	28 Mar 1998
KS Zeta	Southwestern College, Winfield	14 Apr 1998
TN Epsilon	Bethel College, McKenzie	16 Apr 1998
MO Mu	Harris-Stowe College, St. Louis	25 Apr 1998
GA Beta	Georgia College and State University, Milledgeville	25 Apr 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
NY Xi	Buffalo State College, Buffalo	12 May 1998
NC Delta	High Point University, High Point	24 Mar 1999
PA Pi	Slippery Rock University, Slippery Rock	19 Apr 1999
TX Lambda	Trinity University, San Antonio	22 Nov 1999
GA Gamma	Piedmont College, Demorest	7 Apr 2000
LA Delta	University of Louisiana, Monroe	11 Feb 2001
GA Delta	Berry College, Mount Berry	21 Apr 2001
TX Mu	Schreiner University, Kerrville	28 Apr 2001
NJ Gamma	Monmouth University, West Long Branch	21 Apr 2002
CA Epsilon	California Baptist University, Riverside	21 Apr 2003
PA Rho	Thiel College, Greenville	13 Feb 2004
VA Delta	Marymount University, Arlington	26 Mar 2004
NY Omicron	St. Joseph's College, Patchogue	1 May 2004
IL Iota	Lewis University, Romeoville	26 Feb 2005
WV Beta	Wheeling Jesuit University, Wheeling	11 Mar 2005
SC Epsilon	Francis Marion University, Florence	18 Mar 2005
PA Sigma	Lycoming College, Williamsport	1 Apr 2005
MO Nu	Columbia College, Columbia	29 Apr 2005
MD Epsilon	Stevenson University, Stevenson	3 Dec 2005
NJ Delta	Centenary College, Hackettstown	1 Dec 2006
NY Pi	Mount Saint Mary College, Newburgh	20 Mar 2007
OK Epsilon	Oklahoma Christian University, Oklahoma City	20 Apr 2007
HA Alpha	Hawaii Pacific University, Waipahu	22 Oct 2007
NC Epsilon	North Carolina Wesleyan College, Rocky Mount	24 Mar 2008

CA Zeta	Simpson University, Redding	4 Apr 2009
NY Rho	Molloy College, Rockville Center	21 Apr 2009
NC Zeta	Catawba College, Salisbury	17 Sep 2009
RI Alpha	Roger Williams University, Bristol	13 Nov 2009
NJ Epsilon	New Jersey City University, Jersey City	22 Feb 2010
NC Eta	Johnson C. Smith University, Charlotte	18 Mar 2010
AL Theta	Jacksonville State University, Jacksonville	29 Mar 2010
GA Epsilon	Wesleyan College, Macon	30 Mar 2010
FL Gamma	Southeastern University, Lakeland	31 Mar 2010
MA Beta	Stonehill College, Easton	8 Apr 2011
AR Beta	Henderson State University, Arkadelphia	10 Oct 2011
PA Tau	DeSales University, Center Valley	29 Apr 2012
TN Zeta	Lee University, Cleveland	5 Nov 2012
RI Beta	Bryant University, Smithfield	3 Apr 2013
SD Beta	Black Hills State University, Spearfish	20 Sept 2013
FL Delta	Embry-Riddle Aeronautical University, Daytona Beach	22 Apr 2014
IA Epsilon	Central College, Pella	30 Apr 2014
CA Eta	Fresno Pacific University, Fresno	24 Mar 2015
OH Theta	Capital University, Bexley	24 Apr 2015
GA Zeta	Georgia Gwinnett College, Lawrenceville	28 Apr 2015
MO Xi	William Woods University, Fulton	17 Feb 2016
IL Kappa	Aurora University, Aurora	3 May 2016
GA Eta	Atlanta Metropolitan University, Atlanta	1 January 2017
CT Gamma	Central Connecticut University, New Britan	24 March 2017