THE PENTAGON
A Mathematics Magazine for Students

Volume 72 Number 1
Fall 2012

## Contents

Oddities in $C \times C$ ..... 3
Kimberly Lukens
Patterns within the Final Digits of the Fibonacci Numbers ..... 15
Joseph Shoulak
Generalizations and Applications of Problems 682 and 694 ..... 21
D. M. Bătinetu-Giurgiu and Neculai Stanciu
On the Theory of Numbers: A Paper by Evariste Galois ..... 34
Emilie Huffinan
The Problem Corner ..... 43
Kappa Mu Epsilon News ..... 57
Kappa Mu Epsilon National Officers ..... 80
Active Chapters of Kappa Mu Epsilon ..... 81
(c) 2012 by Kappa Mu Epsilon (http://www.kappamuepsilon.org). All rights reserved. General permission is granted to KME members for noncommercial reproduction in limited quantities of individual articles, in whole or in part, provided complete reference is given as to the source. Typeset in Scientific WorkPlace. Printed in the United States of America.

The Pentagon (ISSN 0031-4870) is published semiannually in December and May by Kappa Mu Epsilon. No responsibility is assumed for opinions expressed by individual authors. Papers written by undergraduate mathematics students for undergraduate mathematics students are solicited. Papers written by graduate students or faculty will be considered on a space-available basis. Submissions should be made by means of an attachment to an e-mail sent to the editor. Either a TeX file or Word document is acceptable. An additional copy of the article as a pdf file is desirable. Standard notational conventions should be respected. Graphs, tables, or other materials taken from copyrighted works MUST be accompanied by an appropriate release form the copyright holder permitting their further reproduction. Student authors should include the names and addresses of their faculty advisors. Contributors to The Problem Corner or Kappa Mu Epsilon News are invited to correspond directly with the appropriate Associate Editor.

## Editor:

Chip Curtis
Department of Mathematics
Missouri Southern State University
3950 E Newman Road
Joplin, MO 64801-1595
curtis-c@mssu.edu

## Business Manager:

Don Tosh
Department of Science and Technology
Evangel University
1111 N. Glenstone Ave.
Springfield, MO 65802-2191
toshd@evangel.edu

## Associate Editors:

The Problem Corner: Pat Costello
Department of Math. and Statistics
Eastern Kentucky University
521 Lancaster Avenue
Richmond, KY 40475-3102
pat.costello@eku.edu

Kappa Mu Epsilon News: Peter Skoner
Department of Mathematics
Saint Francis University
117 Evergreen Drive
Loretto, PA 15940
pskoner@francis.edu

As of Volume 70, Number 1, (Fall 2010), The Pentagon will be available only in electronic format. A very limited number of hard copies will be printed to honor pre-existing subscriptions, but new subscriptions will no longer be accepted. Hard copies of current and back issues may be available on a per issue basis. For information regarding hard copy pricing and availability, contact the business manager.

The Pentagon will be available free for viewing and downloading in pdf formatat:http://www.kappamuepsilon.org/pages/Pentagon. php. Or you can go to the official KME website at http://www. kappamuepsilon.org/ and follow the links.

## Oddities in $\boldsymbol{C} \times \boldsymbol{C}$

Kimberly Lukens, student

TN Gamma
Union University
Jackson, TN 38305

Presented at the 2012 North Central Regional Convention

## 1. Introduction

We begin by looking at two theorems valid in the standard Euclidean plane $\mathbb{R} \times \mathbb{R}$ :

Theorem 1 (Gallai 1944) Given $n \geq 3$ points in the plane, not all collinear, there exists a line connecting exactly two of the points. Such a line is called an ordinary line.

Theorem 2 (Ungar 1982) Given $n \geq 3$ points in the plane, not all collinear, there are at least $n-1$ distinct slopes among the joining lines, where equality is possible if and only if $n$ is odd and $n \geq 5$.

The proofs of Theorems 1 [3] and 2 [7] rely on the fact that the base field, $\mathbb{R}$, is an ordered field. As such, the theorems do not easily generalize to planes with different coordinate fields. For instance, Theorem 2 fails over finite fields of the form $\mathbb{Z} \bmod p\left(\right.$ denoted $\left.\mathbb{Z}_{p}\right)$, where $p$ is a prime. An example can be seen in Figure 1, representing the plane $\mathbb{Z}_{3} \times \mathbb{Z}_{3}$. Taking the entire plane as our set of points, we have $n=9$ points but only 4 distinct slopes, namely, $0, \infty, 1$, and 2 . Additionally, Theorem 1 also fails over $\mathbb{Z}_{3} \times \mathbb{Z}_{3}$. There is no ordinary line in this plane; all lines are incident on exactly three points.


Figure $1: \mathbb{Z}_{3} \times \mathbb{Z}_{3}$ and its parallel classes
Though these theorems definitely fail in the case of finite base fields, that failure does not prove or disprove the case of $\mathbb{C} \times \mathbb{C}$, a 'plane' with an infinite, non-ordered base field. If it could be proven that Theorems 1 and 2 also hold in $\mathbb{C} \times \mathbb{C}$, this would not only expand the reach of these theorems but also, since $\mathbb{R} \subset \mathbb{C}$, provide a new proof for the real plane. Unfortunately, work this past summer revealed this impossible, as both theorems fail over $\mathbb{C} \times \mathbb{C}$. A new conjecture with a stronger hypothesis, however, still holds promise.

Conjecture 1 Given $n \geq 3$ points in $\mathbb{C} \times \mathbb{C}$ with no three collinear, the joining lines determine at least $n$ distinct slopes.

The remainder of this paper details the work leading to this conjecture and its attempted proof in addition to outlining the failures of the other two theorems.

A word of notation: we shall refer to $\mathbb{C} \times \mathbb{C}$ as both $\mathbb{C}^{2}$ and the complex plane.

## 2. Failure of Theorem 1 in the Complex Plane

The failure of this theorem was fairly immediate upon the realization that a point configuration similar to the one seen in Figure 1 exists in the complex plane. A two-dimensional representation of it can be seen in Figure 2(b). This graph possesses the same point-line incidence as $\mathbb{Z}_{3} \times \mathbb{Z}_{3}$ but not the same parallel classes. In fact, this point configuration determines eight distinct slopes (repeated values listed first): 0 , undefined, $-\frac{1}{2}+i \frac{\sqrt{3}}{2}$, $\frac{1}{2}+i \frac{\sqrt{3}}{2}, 1,-1, i \sqrt{3}$, and $-i \frac{\sqrt{3}}{3}$. Thus, we have not also disproved Theorem 2 .

This counterexample can be found in the following way: first, set the coordinate values of the lower-left points as $(0,0),(0,1),(1,0)$, and $(1,1)$. Then, assign the remaining points coordinates based on the previously chosen ones and obvious collinearity requirements (see Figure 2a). This leaves five unknowns. To solve for these remaining variables, equations of the form

$$
\left|\begin{array}{lll}
x_{j} & y_{j} & 1  \tag{1}\\
x_{k} & y_{k} & 1 \\
x_{\ell} & y_{\ell} & 1
\end{array}\right|=0
$$

prove useful, as this determinant evaluating to zero is a necessary and sufficient condition for the points $j, k$, and $\ell$ to be collinear. Twelve equations of this form may be created since we are assuming that each of the twelve lines in Figure 2a consists of three collinear points. Simultaneously solving these equations over $\mathbb{C}$ (using a system such as Mathematica) yields the solution seen in Figure 2b.


Figure 2: A counterexample to Theorem 1 in the complex plane.

## 3. Failure of Theorem $\mathbf{2}$ in the Complex Plane

Initial attempts to extend Theorem 2 to the complex plane used a direct approach that proved cumbersome yet held some promise, at least for small values of $n$. In fact, we can show that the theorem holds for $n \leq 5$.

Theorem 3 Given $n=3$, 4, or 5 points in the complex plane, not all on a line, the joining lines determine at least $n-1$ distinct slopes.

Proof. The case of $n=3$ is trivial; we shall always have exactly three distinct slopes among a noncollinear configuration of three points. We now look at the case of $n=4$.

Let $n=4$ points in the complex plane be given, not all collinear. Define the coordinate system as follows: Translate the axes so that one point lies at $(0,0)$; call it $p_{1}$. Define the coordinate system such that a second point, $p_{2}$ lies at $(1,0)$. Of the remaining two points, at least one of them is not collinear with $p_{1}$ and $p_{2}$. Call it $p_{3}$ and give it coordinates $(0,1)$. Here, it may seem strange to say that we can pick these points to be specifically $(0,0),(1,0)$, and $(0,1)$, but since these are three noncollinear points in $\mathbb{C} \times \mathbb{C}$, the vectors $\mathbf{p}_{1} \mathbf{p}_{2}$ and $\mathbf{p}_{1} \mathbf{p}_{3}$ are linearly independent and span $\mathbb{C}^{2}$. All points in $\mathbb{C}^{2}$ can be assigned coordinates with respect to this basis; furthermore, distinctness of slopes and parallelism are conserved. Finally, label the last point $p_{4}$ and give it the coordinates $(a, b)$ with respect to this basis, where $a, b \in \mathbb{C}$ and $(a, b) \neq(0,0),(1,0),(0,1)$. Then there are six joining line segments between the four points that, in this coordinate system, have slopes:

$$
\begin{aligned}
p_{1} p_{2} & =0 \\
p_{1} p_{3} & =\infty \\
p_{2} p_{3} & =-1 \\
p_{1} p_{4} & =\frac{b}{a} \\
p_{2} p_{4} & =\frac{b}{a-1} \\
p_{3} p_{4} & =\frac{b-1}{a}
\end{aligned}
$$

Thus, there are at least three distinct slopes, namely, $0, \infty$, and -1 . We now examine all possible values of $a$ and $b$ to show that a fourth always exists.

Case 1 Suppose $a=0, b \in \mathbb{C} \backslash\{0,1\}$. Then $p_{2} p_{4}=-b$ is distinct.
Case 2 Let $a=1$. We have two subcases.

- Let $b \in \mathbb{C} \backslash\{0,-1\}$. Then $p_{1} p_{4}=b$ is distinct.
- Suppose $b=-1$. Then $p_{3} p_{4}=-2$.

Case 3 Let $b=0, a \in \mathbb{C} \backslash\{0,1\}$. Then $p_{3} p_{4}=\frac{-1}{a}$ is distinct.
Case 4 Suppose $b=1$. There are two subcases.

- Let $a \in \mathbb{C} \backslash\{0,-1\}$. Then $p_{1} p_{4}=\frac{1}{a}$ is distinct.
- Let $a=-1$. Then $p_{2} p_{4}=\frac{-1}{2}$.

Case 5 Let $a \in \mathbb{C} \backslash\{0,1\}, b \in \mathbb{C} \backslash\{0,1\}$. We have two subcases.

- Suppose $b \neq-a$. Then $p_{1} p_{4}=\frac{b}{a}$ is distinct.
- Let $b=-a$. Then $p_{3} p_{4}=\frac{b-1}{a}=\frac{-(a+1)}{a}$ is distinct.

This completes the case of $n=4$. The case of $n=5$ follows immediately, since any set of five points in the complex plane contains a four-point subset and, thus, at least $5-1=4$ distinct slopes.

While this proof may make the extension of Theorem 2 to $\mathbb{C}^{2}$ seem plausible, its technique does not generalize well to larger values of $n$, as the number of cases to examine grows too quickly. Furthermore, Jamison showed in [5] that Theorem 2 fails over the complex plane. In his paper, he details an infinite family of point arrangements in $\mathbb{C}^{2}$ called trimodular arrays, $T M(r)$, that are defined as follows:

$$
\begin{align*}
T M(r)= & \left\{(0,0),\left(0, \omega^{k}\right),\left(\omega^{k}, 0\right): k=1, \ldots, r ;\right.  \tag{2}\\
& \left.\omega \text { is a primitive } r^{\text {th }} \text { root of unity }\right\} .
\end{align*}
$$

Now this set contains $n=2 r+1$ points but only $r+2=\frac{n+3}{2}$ distinct slopes, which is less than the desired minimum of $n-1$ distinct slopes when $n>5$ (Equality holds when $n=5$ ). A two-dimensional example of the trimodular array $T M(4)$ can be seen in Figure 3. This trimodular array contains nine points in $\mathbb{C}^{2}$ but only six distinct slopes, namely, $0, \infty,-1,1,-i$, and $i$.


Figure 3: $T M(4)$
Additionally in [5], Jamison made the following conjecture:
Conjecture 2 (Jamison 1985) Any set $X$ of $n$ noncollinear points in the complex plane determines at least $\frac{n+3}{2}$ slopes, with equality if and only if $X$ is affinely equivalent to $T M(r)$ for some $r$.

Instead of looking to prove or disprove this weaker hypothesis, we now focus our attention on the stronger hypothesis of Conjecture 1.

## 4. Attempts at Conjecture 1

Though Theorem 2 as stated does not generalize to the complex plane, there is still a chance that Conjecture 1 holds. Adopting the stronger hypothesis of no three points collinear yields a new conclusion with a greater possibility of generalization, as seen in the following theorem:

Theorem 4 Given $n \geq 3$ points in the Euclidean plane with no three collinear, the joining lines determine at least $n$ distinct slopes. ${ }^{1}$

Proof. We have two cases. First, suppose $n$ is even. Then, by Theorem 2, we must have at least $n$ distinct slopes (see [7] for proof). Now, suppose $n$

[^0]is odd. Then there are
$$
\binom{n}{2}=\frac{n(n-1)}{2}
$$
joining lines among the points. However, we can have at most
$$
\left\lfloor\frac{n}{2}\right\rfloor=\frac{(n-1)}{2}
$$
lines with a given slope since no three points are collinear. Hence, there must be at least $n$ distinct slopes.

The combinatorial approach used in the latter half of this proof is equally valid in the complex plane; therefore, we can assert that, given an odd number $n \geq 3$ points in $\mathbb{C}^{2}$, there are at least $n$ distinct slopes. Unfortunately, this approach only guarantees $n-1$ distinct slopes when $n$ is even. Thus, in order to prove Conjecture 1, we must show that any even number of points $n$ only possessing $n-1$ slopes must contain a collection of three or more collinear points. We now detail a computational approach to solving this problem for a fixed $n$ using Gröbner bases and Mathematica.

First, we wish to determine all theoretically possible configurations of $n=2 k$ points in $\mathbb{C}^{2}$ with no three collinear and only $n-1=2 k-1$ slopes. With this end in mind, we momentarily digress to pick up some tools from graph theory. Using the proper terminology, our $n$ points are now $n$ vertices, and the lines between them are edges. Also, since we want to examine all slopes present, we focus our attention on graphs where each vertex is connected to all others via exactly one edge. Such a graph is called a complete graph, and the complete graph on $n$ vertices is denoted $K_{n}$.

Next, we group these edges into sets, where each set represents a distinct slope. Since we previously specified that no three points are collinear, two edges in a set cannot share a vertex. Thus, these sets are called matchings. Furthermore, they are perfect matchings or one-factors since for any given set, each vertex is touched by exactly one edge in that set (an immediate consequence of partitioning the edges of $K_{2 k}$ into $2 k-1$ matchings). Together, all these one-factors constitute a one-factorization on $K_{2 k}$. An example of a one-factorization can be seen in Figure 4, where one-factors are denoted by line style.


Figure 4: A one-factorization of $K_{6}$
Returning to the problem at hand, we see that any configuration of $2 k$ points in $\mathbb{C}^{2}$ with no three collinear and only $2 k-1$ slopes corresponds to a one-factorization of $K_{2 k}$, with each one-factor corresponding to lines of the same slope. Thus, we use one-factorizations as our starting point for determining all theoretically possible configurations in $\mathbb{C}^{2}$. When $n=4$ or $n=6$, there is only one one-factorization of $K_{n}$ up to isomorphism, and it is fairly simple-especially if employing the forthcoming methodto show that forcing parallelism among the edges/lines in each one-factor forces all points collinear. However, $K_{8}$ has six non-isomorphic onefactorizations to check. Using the one-factors listed in [2], we now check two of $K_{8}$ 's one-factorizations while detailing the Gröbner basis method. The other four one-factorizations are similar and omitted for brevity.

First, defining two functions in Mathematica simplifies the process. These functions can be seen in Figure 5. The x and y vectors list the arbitrary point labels to simplify function input. The par function arises from the equation

$$
\begin{equation*}
\left(y_{j}-y_{k}\right)\left(x_{\ell}-x_{m}\right)-\left(y_{\ell}-y_{m}\right)\left(x_{j}-x_{k}\right)=0, \tag{3}
\end{equation*}
$$

an equation that forces the line through points $j$ and $k$ parallel to the line through $\ell$ and $m$, and the function takes the numbered labels from the points as arguments. For instance, par $[1,2,3,4]==0$ forces the line through $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ parallel to the line through $\left(x_{3}, y_{3}\right)$ and $\left(x_{4}, y_{4}\right)$. The coll function arises from Equation (1) and works in a similar fashion as par.

```
x:= {x1, x2, x3, x4, x5, x6, x7, x8}
y := {y1, y2, y3, y4, y5, y6, y7, y8}
par[j_, k_, l_, m_] := (x[[j]]-x[[k]]) (y[[l]]-y[[m]])-(x[[l]]-x[[m]]) (y[[j]]-y[[k]])
coll[j_, k_, 1_] := Det[[\begin{array}{llll}{x[[j]]}&{y[[j]] 1}&{1}\\{x[[k]]}&{y[[k]]}&{1}\\{x[[l]]}&{y[[l]]}&{1}\end{array})]
```

Figure 5: Function Definitions

Next, we create a set of parallelism equations that arises from a onefactorization of $K_{8}$ and calculate its Gröbner basis, as seen in Figure 6. We do this using the built-in Mathematica function GroebnerBasis, using as inputs the set of parallelism equations and the set of variables. By calculating the Gröbner basis, we obtain a 'simpler' set of polynomials than the original set of equations, i.e. one with greater structural information, but the new set of polynomials has the same shared root set as the original. Thus, it makes subsequent calculations more manageable. For more information on the Gröbner basis, see [8].

```
g1 := GroebnerBasis [{par[1, 5, 2, 6], par[1, 5, 3, 7], par[1, 5, 4, 8],
    par[1,6,2,5], par[1,6,3,4], par[1,6,7, 8], par[1, 7, 2, 4],
    par[1, 7, 3, 5], par[1, 7, 6, 8], par[1, 8, 2, 3], par[1, 8, 4, 5], par[1, 8, 6, 7],
    par[1, 2, 3, 8], par[1, 2, 4, 7], par[1, 2, 5, 6], par[1, 3, 2, 8], par[1, 3, 4, 6],
    par[1, 3, 5, 7], par[1, 4, 2, 7], par[1, 4, 3, 6], par[1, 4, 5, 8]},
{x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4, y5, y6, y7, y8}]
```

Figure 6: Calculating the Gröbner Basis

We now wish to see if collinearity necessarily follows as a consequence of our chosen parallel classes. If the determinant in Equation (1) can be rewritten as a linear combination of the Gröbner basis polynomials, then when the latter polynomials evaluate to zero, the same must hold for the determinant. Such is accomplished using the Mathematica function PolynomialReduce as seen in Figure 7. Here, PolynomialReduce takes the collinearity function as first input and reduces it with respect to the second input, the Gröbner basis $g 1$. The function then outputs a list of coefficients a and the remainder b. Note that, as listed in Mathematica documentation [6], reducing with respect to a set of Gröbner basis polynomials ensures that b is not only minimal but also unique. In Figure 7, the remainder is always zero, meaning that all points are collinear with points 1 and 2 . Thus, we may stop at this point and assert that the only configuration of eight points in $\mathbb{C}^{2}$ satisfying the slope sets dictated by this one-factorization is a straight line.

```
    For [p = 3, p s 8, p ++,
    Print [{a, b} = PolynomialReduce [coll[p, 1, 2],
        g1, {x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4, y5, y6, y7, y8}]
        ]
    ]
{{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1, 1}, 0}
{{0,0,0,0,0,0,0,0,0,0,0,0,0, 1,0,0,0,0, -1,0, 1},0}
{{0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,-1,0,0,1},0}
{{0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0, -1,0,0,0, 1},0}
{{0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,-1,0,0,0,0,1},0}
{{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1},0}
```

Figure 7: Checking for Collinearity

This first one-factorization of $K_{8}$ proved very simple to check, but the second one, as seen in Figure 8, requires an extra step. Going through the same process as the first, we find that all remainders b are nonzero; however, this does not necessarily mean that the points are not collinear. Take, for instance, the polynomials $2 x+3$ and $4 x^{2}+12 x+9$. If we reduce the first polynomial with respect to the second, we obtain a nonzero remainder, yet $2 x+3$ must be zero when $4 x^{2}+12 x+9=0$, since $(2 x+3)^{2}=4 x^{2}+12 x+9$. In a similar way, if we obtain nonzero remainders from PolynomialReduce, we then must check powers of the determinant. The collinearity functions on their own cannot be written as linear combinations of the polynomials in g 2 , but by squaring them, we obtain the zero remainders seen in Figure 9. Once again, it is impossible to have slope sets resembling this one-factorization in $\mathbb{C}^{2}$ without having all points collinear.

```
g2 := GroebnerBasis [{par[1, 5, 2, 6], par[1, 5, 3, 7], par[1, 5, 4, 8],
    par[1,6,2,3], par[1,6,4,5], par[1,6,7,8], par[1, 7, 2, 8],
    par[1, 7, 3, 5], par[1, 7, 4, 6], par[1, 8, 2,5], par[1, 8, 3, 6], par[1, 8, 4, 7],
    par[1, 2, 3,4], par[1, 2, 5, 8], par[1, 2,6,7], par[1, 3, 2,4], par[1, 3, 5, 7],
    par[1, 3, 6, 8], par[1, 4, 2, 7], par[1, 4, 3, 8], par[1, 4, 5, 6]},
    {x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4, y5, y6, y7, y8}]
```

Figure 8: Gröbner Basis Calculation on Another One-Factorization

This concludes our discussion of the basic computational approach to verifying Conjecture 1 . Based on all calculations performed in this manner plus the previous result for odd $n$, we can state that Conjecture 1 holds when $n \leq 9$. Unfortunately, multiple challenges bar the way for con-
tinued implementation of this method for larger $n$. First of all, the number of non-isomorphic one-factorizations increases rapidly as $n$ increases; there are 396 non-isomorphic one-factorizations of $K_{10}$ and 526, 915, 620 of $K_{12}$ [1]. Even with high-end computing power it will take substantial time and programming to check all these possibilities. Secondly, the limits of Mathematica will likely be reached with continued testing. The documentation for GroebnerBasis [4] concedes that only a semi-reduced Gröbner basis is calculated, and the basis is calculated over the field of rationals, not complexes. Though these limitations did not significantly affect results for small $n$, they still have the potential to greatly increase computation time as $n$ increases. Yet despite these hindrances, continued computer implementation to check larger values of $n$ still is a valid pursuit. Such investigation will either strengthen Conjecture 1 or uncover a counterexample, both helpful to the problem at hand.

```
            For [p=3,p s 8, p++,
            Print [{a,b} = PolynomialReduce [(coll [p, 1, 2]) 2,
                g2, {x1, x2, x3, x4, x5, x6, x7, x8, y1, y2, y3, y4, y5, y6, y7, y8}]
        ]
            ]
{ {0, 0, 0, 0, x1 - x2, x1 + x < - 2 x8, -2 x1 + x2 + x3, 0, 0, 0, 0, 0, 0, -2 x1 + x2 + x8,
    x1-x3, 2 x1 - x2 - x8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2 x1 y1 - 2 x2 y1 + 2 x y y1 -
        2 x8 y1 + 2 x1 y2 - x3 y2 - x8 y2 - 2 x1 y3 + x2 y y + x8 y % - 2 x1 y8 + x2 y8 - x3 y8 + 2 x8 y8,
    0,0,0,0, -x3 y1 + x8 y1 - 2 x1 y2 + 2 x8 y2 + x1 y3 -x8 y3 + x1 y8 + x3 y8 - 2 x8 y8,
    -x2 y1 + 2 x x y1 - x8 y1 + x1 y2 - x8 y2 - x1 y8 + x2 y8 - 2 x ( y y + 2 x8 y8), 0)
{{0,0,0, -x1 + x2, 0, x1-x4, 2 x1- 3 x2 + 3 x4-2 x8, 0, 0, 0, 0, 0, - 2 x1 + x2 + x8,
    0, x1-x4, 2 x1-x2-x8,0,0,0,0,0,0,0,0,0,0,0,0,0, 2 x1 y1-2 x2 y1 + 2 x4 y1 -
        2 x8 y1 + 2 x1 y2 - x4 y2 - x8 y2 - 2 x1 y4 + x2 y4 + x8 y4 - 2 x1 y8 + x2 y8 - x4 y8 + 2 x8 y8,
    0, 0, 0, 0, -x4 y1 + x8 y1 - 2 x1 y2 + 2 x8 y2 + x1 y4 - x8 y4 + x1 y8 + x4 y8 - 2 x8 y8, 0,
    -x2 y1 + 2 x4 y1 - x8 y1 + x1 y2 - x8 y % - x1 y 8 + x2 y8 - 2 x4 y8 + 2 x8 y8}, 0)
{{0, 0, -x1 + x2, 0, 0, x1 - x5, 2 x1 - 3 x2 + 3 x5 - 2 x8, 0, 0, 0, 0, x1 - x2, 0, 0, -x1 + x5,
    -2 x1 + 3 x2 - 3 x5 + 2 x8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2 x1 y1 - 2 x2 y1 + 2 x5 y1 -
    2 x8 y1 + 2 x1 y2 - x5 y2 - x8 y2 - 2 x1 y5 + x2 y5 + x8 y5 - 2 x1 y8 + x2 y8 - x5 y % + 2 x8 y8,
    0,0, 0, 0, -x5 y1 + x8 y1 - 2 x1 y2 + 2 x8 y2 + x1 y5 - x8 y5 + x1 y8 + x5 y8 - 2 x8 y8, 0,
    0, -x2 y1 + 2 x5 y1 - x8 y1 + x1 y2 - x8 y2 - x1 y8 + x2 y8 - 2 x5 y8 + 2 x8 y8}, 0}
{{6 x1-3 x2 - 3 x8, 5 x1 - 3 x2 - 2 x8, - 2 x1 + x2 + x8, 0, 0,
    -3 x1 - x5 + 3 x6 + 3 x7 - 2 x8, -6 x1 + 5 x2 - 3 x6 + 4 x8, 0, 0, -4 x1 + 2 x2 + 2 x8,
    -x7 y2 + x8 y2 - 2 x1 y7 + x2 y7 + x8 y7 + 2 x1 y8 - x2 y8 + x7 y8 - 2 x8 y8, 0, 0, 0,
    x1-2x7 + x8, 2 x1-x2-x8,0,0,0,0,0,0,0,0,0,0,0,2 x1 y1-2 x2 y1 + 2 x6 y1-
        2 x8 y1 + 2 x1 y2 - x6 y2 - x8 y2 - 2 x1 y6 + x2 y6 + x8 y6 - 2 x1 y8 + x2 y8 - x6 y8 + 2 x8 y8,
    0, 0, 0, 0, -x6 y1 + x8 y1 - 2 x1 y2 + 2 x8 y2 + x1 y6 - x8 y6 + x1 y8 + x6 y8 - 2 x8 y8, 0,
    0, 0, -x2 y1 + 2 x6 y1 - x8 y1 + x1 y2 - x8 y2 - x1 y8 + x2 y8 - 2 x6 y8 + 2 x8 y8 , 0}
{{3 x1-x2 - 2 x8, 0, 0, 0, 0, -3 x1 + x7 + 2 x8, -2 x1 + x2 - x7 + 2 x8, 0, 0, - 2 x1 + x2 + x8,
    0, 0, 0, 0, x1-x7, 2 x1 - x2 - x8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2 x1 y1 - 2 x2 y1 + 2 x7 y1 -
        2 x8 y1 + 2 x1 y2 - x7 y y - x8 y2 - 2 x1 y7 + x2 y7 + x8 y7 - 2 x1 y8 + x2 y8 - x7 y % + 2 x8 y8,
    0,0,0,0,-x7 y1 + x8 y1 - 2 x1 y2 + 2 x % y 2 + x1 y7 - x 8 y 7 + x1 y8 + x7 y8 - 2 x8 y8, 0, 0,
    0, 0, -x2 y1 + 2 x7 y1 - x8 y1 + x1 y2 - x8 y2 - x1 y8 + x2 y8 - 2 x7 y8 + 2 x8 y8}, 0}
{{0,0,0,0,0, x1-x8,-x2 + x8, 0, 0, 0, 0, 0, 0, 0, -x1 + x8, x2 - x8, 0, 0, 0, 0, 0, 0,
    0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -x2 y1 + x8 y1 +x1 y2 - x8 y2 - x1 y8 + x2 y8}, 0} 
```

Figure 9: Modified Collinearity Check

## 5. Conclusion

As demonstrated throughout the body of this paper, the differences between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{C} \times \mathbb{C}$ were much greater than originally anticipated. Theorems 1 and 2 rely pivotally on the total ordering of the real numbers and, as such, cannot be fully realized in the complex plane. Nonetheless, a geometrical connection between $\mathbb{R} \times \mathbb{R}$ and $\mathbb{C} \times \mathbb{C}$ may still exist should Conjecture 1 hold true. Continued research into that conjecture, computational or otherwise, should be seen as a step toward greater understanding of the two fields and their relationships.

Acknowledgement. This research was initiated during the University of Wyoming Mathematics REU and partially supported by NSF DMS grant no. 0755450. Special thanks to the REU director Dr. Bryan Shader and advising professor Dr. Eric Moorhouse, both of the University of Wyoming. Also thanks to graduate student advisor Reshmi Nair of the University of Wyoming and undergraduate collaborators: Sarah Terrill, Fort Lewis College, and Ace White, Whittier College. Final thanks to Dr. Bryan Dawson of Union University for mentoring during the writing process.

## References

[1] C. J. Colbourn and J. H. Dinitz, Ed., The CRC Handbook of Combinatorical Designs, CRC Press, 1996.
[2] L. E. Dickson, and F. H. Safford, "Group Theory." American Mathematical Monthly 13.6 (1906): 150-151. JSTOR. 22 June 2011.
[3] P. Erdős and R. Steinberg, "Problem 4065," American Mathematical Monthly 51.3 (1944): 169-171. JSTOR. 6 Feb. 2012.
[4] "GroebnerBasis," Wolfram Mathematica Documentation Center. Wolfram Research, Inc., 2011. 24 Sept. 2011.
[5] R. E. Jamison, "A Survey of the Slope Problem," Annals of the New York Academy of Sciences 440 (1985): 34-51.
[6] "PolynomialReduce," Wolfram Mathematica Documentation Center, Wolfram Research, Inc., 2011. 24 Sept. 2011.
[7] P. Ungar, "2N Noncollinear Points Determine at least 2N Directions," Journal of Combinatorial Theory Series A. 33 (1982): 343-347.
[8] E. Weisstein, "Gröbner Basis," Wolfram Mathworld, Wolfram Research, Inc., 2012. 1 Oct. 2012.

# Patterns within the Final Digits of the Fibonacci Numbers 

Joseph Shoulak

## 1. Introduction

In this essay, I will review many interesting patterns in the Fibonacci numbers, modulo 10 . Their elegance and how intertwined they are, with no seeming need for them to behave this way, is mind-blowing.

A key result is the following.

Theorem 5 For all integers $n$,

$$
F_{n} \bmod 10=F_{n+60} \bmod 10
$$

This is a statement that the sequence of the final digits of Fibonacci numbers has a period of 60 , which can be easily shown. [1]

Proof. We note that

$$
\begin{aligned}
& F_{58}=591286729879 \equiv 9 \bmod 10 \\
& F_{59}=956722026041 \equiv 1 \bmod 10 \\
& F_{60}=1548008755920 \equiv 0 \bmod 10 .
\end{aligned}
$$

Since $9+1 \equiv 0 \bmod 10,1+0 \equiv 1 \bmod 10$, and $0+1 \equiv 1 \bmod 10$, it is thus inevitable that 0 will emerge, then two consecutive 1 s , which are equal to $F_{1}, F_{2}$. By starting with 1,1 , we will eventually reach $9,1,0$, which starts it all over again. This proves that the Fibonacci numbers mod 10 have a periodicity of 60 , as stated above.
2. General Statements about $F_{n} \bmod 10, n \in[1,60]$

In this section we make some observations about patterns in the ending digits of the Fibonacci numbers.

- An interesting fact in this periodicity 60 pattern is that there are exactly 40 odd numbers and 20 even numbers. One way to make sense of this is to consider the Fibonacci sequence mod 2, i.e., look at evens and odds. Since $1+1 \equiv 0 \bmod 2,1+0 \equiv 1 \bmod 2$, and $0+1 \equiv 1 \bmod 2$, the terms of the Fibonacci sequence $\bmod 2$ are

$$
1,1,0,1,1,0,1,1,0, \ldots,
$$

which is periodic with period 3 . Thus, $2 / 3$ of the Fibonacci numbers are odd. The same argument illustrates that there will never be two consecutive even Fibonacci numbers:

- What is even more interesting is the count of each number mod 10 . Writing

$$
f_{n}=F_{n} \bmod 10,
$$

we have

| $f_{n}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| occurrences for $1 \leq n \leq 60$ | 4 | 8 | 4 | 8 | 4 | 8 | 4 | 8 | 4 | 8 |

This can be summed up as:

$$
\operatorname{count}\left(f_{n}\right), n \in[1,60]=\left\{\begin{array}{ccc}
4 & \text { if } & f_{n} \text { is even } \\
8 & \text { if } & f_{n} \text { is odd }
\end{array}\right.
$$

- As a further observation, in the list of Fibonacci numbers $\bmod 10$, there will never be two consecutive numbers unless there is a 0 before them. This is apparent since, if $F_{n+1} \equiv F_{n} \bmod 10$, then $F_{n-1} \equiv F_{n+1}-$ $F_{n} \equiv 0 \bmod 10$.


## 3. Analysis of the Rows and Columns of the final Fibonacci Digits laid out in a $6 \times 10$ grid, descending

To more readily see the patterns in the first 60 terms of $f_{n}$, we view these numbers in a grid.

| $\mathbf{1}$ | 9 | 6 | $\mathbf{9}$ | 1 | 4 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $\mathbf{1}$ | 4 | 1 | $\mathbf{9}$ | 6 | 9 |
| 2 | 3 | 7 | 8 | 7 | 3 |
| 3 | $\mathbf{7}$ | 8 | 7 | $\mathbf{3}$ | 2 |
| $\mathbf{5}$ | $\mathbf{0}$ | $\mathbf{5}$ | $\mathbf{5}$ | $\mathbf{0}$ | $\mathbf{5}$ |
| 8 | $\mathbf{7}$ | 3 | 2 | $\mathbf{3}$ | 7 |
| 3 | $\mathbf{7}$ | 8 | 7 | $\mathbf{3}$ | 2 |
| 1 | 4 | 1 | 9 | 6 | 9 |
| 4 | 1 | $\mathbf{9}$ | 6 | 9 | $\mathbf{1}$ |
| $\mathbf{5}$ | $\mathbf{5}$ | $\mathbf{0}$ | $\mathbf{5}$ | $\mathbf{5}$ | $\mathbf{0}$ |

Following the grid down one column after another, the first two apparent patterns are $n, 0, n, n$ occurring vertically, and 5,5,0 recurring horizontally. The first of these follows readily from the recursive formula for the Fibonacci sequence.: If $f_{n} \equiv 0$, then $f_{n+1} \equiv f_{n-1}+f_{n} \equiv f_{n-1}$, and $f_{n+2} \equiv f_{n}+f_{n+1} \equiv f_{n+1}$.

It can also be noticed that the 5 s and 0 s are isolated to rows 5 and 10 . Those rows only contain 5 and 0 , and 5 and 0 are only contained in those rows. The next thing to discover is that row 2 is just row 1 shifted, row 4 is just row 3 shifted, row 7 is just row 6 shifted, row 9 is just row 8 shifted, and row 10 is just row 5 shifted. I will now demonstrate this by italicizing and boldfacing different parts to show its repetitions:

| $\mathbf{1}$ | $\mathbf{9}$ | $\mathbf{6}$ | 9 | 1 | 4 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 4 | $\mathbf{1}$ | $\mathbf{9}$ | $\mathbf{6}$ | 9 |
| $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{7}$ | 8 | 7 | 3 |
| $\mathbf{3}$ | $\mathbf{7}$ | 8 | 7 | 3 | $\mathbf{2}$ |
| $\mathbf{5}$ | $\mathbf{0}$ | $\mathbf{5}$ | 5 | 0 | 5 |
| $\mathbf{8}$ | $\mathbf{7}$ | $\mathbf{3}$ | $\mathbf{2}$ | 3 | 7 |
| 3 | 7 | $\mathbf{8}$ | $\mathbf{7}$ | $\mathbf{3}$ | $\mathbf{2}$ |
| $\mathbf{1}$ | $\mathbf{4}$ | $\mathbf{1}$ | 9 | 6 | 9 |
| $\mathbf{4}$ | $\mathbf{1}$ | 9 | 6 | 9 | $\mathbf{1}$ |
| $\mathbf{5}$ | $\mathbf{5}$ | $\mathbf{0}$ | $\mathbf{5}$ | 5 | 0 |

## 4. Analysis of the Rows and Columns of the final Fibonacci Digits laid out in a $12 \times 5$ grid, descending

To see additional patterns in the sequence $\left\{f_{n}\right\}$, we display the first 60 values in 12 descending columns of 5:

| 1 | 8 | 9 | 7 | 6 | 3 | 9 | 2 | 1 | 3 | 4 | 7 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 3 | 4 | 7 | 1 | 8 | 9 | 7 | 6 | 3 | 9 | 2 |
| 2 | 1 | 3 | 4 | 7 | 1 | 8 | 9 | 7 | 6 | 3 | 9 |
| 3 | 4 | 7 | 1 | 8 | 9 | 7 | 6 | 3 | 9 | 2 | 1 |
| 5 | 5 | 0 | 5 | 5 | 0 | 5 | 5 | 0 | 5 | 5 | 0 |

- First, let us focus on the implications of row 5:

$$
f_{5 n}=\left\{\begin{array}{llc}
5 & \text { if } & n=3 m \pm 1 \\
0 & \text { if } & n=3 m
\end{array}\right.
$$

One can also notice that 5 s and 0 s do not occur outside of row 5 . Using matrix notation, we can say

$$
f_{m n} \in\left\{\begin{array}{ll}
\{0,5\} & \text { if } m=5 \\
\{1,2,3,4,6,7,8,9\} & \text { if } m \neq 5
\end{array} .\right.
$$

- Also, the row shifts from earlier are still here, just in an even more elegant form:

| $\mathbf{1}$ | $\mathbf{8}$ | $\mathbf{9}$ | $\mathbf{7}$ | $\mathbf{6}$ | $\mathbf{3}$ | 9 | 2 | 1 | 3 | 4 | 7 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 3 | 4 | 7 | $\mathbf{1}$ | $\mathbf{8}$ | $\mathbf{9}$ | $\mathbf{7}$ | $\mathbf{6}$ | $\mathbf{3}$ | 9 | 2 |
| 2 | 1 | 3 | 4 | 7 | $\mathbf{1}$ | $\mathbf{8}$ | $\mathbf{9}$ | $\mathbf{7}$ | $\mathbf{6}$ | $\mathbf{3}$ | 9 |
| 3 | 4 | 7 | $\mathbf{1}$ | $\mathbf{8}$ | $\mathbf{9}$ | $\mathbf{7}$ | $\mathbf{6}$ | $\mathbf{3}$ | 9 | 2 | 1 |
| 5 | 5 | 0 | 5 | 5 | 0 | 5 | 5 | 0 | 5 | 5 | 0 |

It's the same row, repeated four times, just shifted!

- Another interesting thing is that each row also follows the column rule of $f_{n}=f_{n-1}+f_{n-2}$. For instance, in matrix notation,

$$
\begin{aligned}
f_{11}+f_{12} & =1+8 \equiv 9=f_{13} \bmod 10 \\
f_{12}+f_{13} & =8+9 \equiv 7=f_{14} \bmod 10
\end{aligned}
$$

- The next pattern I discovered is the fact that the diagonals from upper left to lower right are in the form: $X, Y, Y, X, Z ; Z \in\{0,5\}$.

| $\mathbf{1}$ | 8 | $\mathbf{9}$ | 7 | $\mathbf{6}$ | 3 | $\mathbf{9}$ | 2 | $\mathbf{1}$ | 3 | $\mathbf{4}$ | 7 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | $\mathbf{3}$ | 4 | $\mathbf{7}$ | 1 | $\mathbf{8}$ | 9 | $\mathbf{7}$ | 6 | $\mathbf{3}$ | 9 | 2 |
| $\mathbf{2}$ | 1 | $\mathbf{3}$ | 4 | $\mathbf{7}$ | 1 | $\mathbf{8}$ | 9 | $\mathbf{7}$ | 6 | $\mathbf{3}$ | 9 |
| 3 | $\mathbf{4}$ | 7 | $\mathbf{1}$ | 8 | $\mathbf{9}$ | 7 | $\mathbf{6}$ | 3 | $\mathbf{9}$ | 2 | $\mathbf{1}$ |
| 5 | 5 | 0 | 5 | 5 | 0 | 5 | 5 | 0 | 5 | 5 | 0 |

In matrix notation,
$\left(f_{1, n}, f_{2, n+1}, f_{3, n+2}, f_{4, n+3}, f_{5, n+4}\right)=(X, Y, Y, X, Z), Z \in\{0,5\}$.

- Next come the diagonals from the opposite direction. We have

$$
\begin{equation*}
f_{1, n}+f_{2, n-1} \equiv f_{3, n-2} \bmod 10 \tag{4}
\end{equation*}
$$

starting at row one, and

$$
\begin{equation*}
f_{2, n}-f_{3, n-1} \equiv f_{4, n-2} \bmod 10 \tag{5}
\end{equation*}
$$

when starting at row two. Substituting, we have

$$
f_{2, n}-\left(f_{1, n+1}+f_{2, n}\right) \equiv f_{4, n-2} \bmod 10,
$$

which simplifies to

$$
-f_{1, n} \equiv f_{4, n-3} \bmod 10
$$

or

$$
\begin{equation*}
f_{1, n}+f_{4, n-3}=10 . \tag{6}
\end{equation*}
$$

For the sums of the numbers in between, which have the form $f_{2, n}+$ $f_{3, n-1}$, we have

$$
\begin{equation*}
f_{2, n}+f_{3, n-1} \in\{5,10,15\} . \tag{7}
\end{equation*}
$$

In fact, we see all of

$$
\{5,15\} \times\{5,15\}
$$

$(5,5),(5,15),(15,15),(15,5)$ in that order. The sum $\left(f_{2, n}+f_{3, n-1}\right) \bmod 10$ yields

$$
5,5,0,5,5,0,5,5,0,5,5,0 .
$$

Also, the 0 s occur when $f_{5, n-1}=0$. This entire pattern can be summed up as follows:

$$
f_{2, n}+f_{3, n-1}=\left\{\begin{array}{ccc}
10 \pm 5 & \text { if } & f_{5, n-1} \neq 0  \tag{8}\\
10 & \text { if } & f_{5, n-1}=0
\end{array}\right.
$$

or, since

$$
\begin{align*}
f_{5, n} & =\left\{\begin{array}{llc}
5 & \text { if } & n=3 m \pm 1 \\
0 & \text { if } & n=3 m
\end{array}\right.  \tag{9}\\
f_{2, n}+f_{3, n-1} & =\left\{\begin{array}{ccc}
10 \pm 5 & \text { if } & n=3 m \pm 1 \\
10 & \text { if } & n=3 m
\end{array}\right. \tag{10}
\end{align*}
$$

## 5. Conclusion

There are mysteries and miracles in the Fibonacci numbers. If someone had told me to find a series of numbers that fit all the patterns I have listed, I would have given up almost immediately. I find it ironically amusing that, of course, it's the Fibonacci sequence that holds these patterns. It seems like it answers most of our questions. Even so, I hope we never find every answer, because finding the answers is so fun. I don't want to run out of things to learn. I want to find the next answer, but never the last. I hope you enjoyed discovering these interesting patterns within the patterns of the Fibonacci Sequence. Dig deeper, my friends.

Thank you.

## References

[1] A. S. Poamentier and I. Lehmann, The Fabulous Fibonacci Numbers, Prometheus Books, 2007, p. 32.

# Generalizations and Applications of Problems 682 and 694 

D.M. Bătineţu-Giurgiu<br>'Matei Basarab’ National College, Bucharest, Romania<br>and<br>Neculai Stanciu<br>‘George Emil Palade’ General School, Buzău, Romania

## 1. Introduction

In [1], José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain, proposed

Problem 682. Let $a, b, c$ be three positive numbers such that

$$
a^{2}+b^{2}+c^{2}=1
$$

Prove that

$$
\left[\frac{1}{a^{3}(b+c)^{5}}+\frac{1}{b^{3}(c+a)^{5}}+\frac{1}{c^{3}(a+b)^{5}}\right]^{1 / 5} \geq \frac{3}{2}
$$

In [2], Díaz-Barrero proposed
Problem 694. Let $a, b, c$ be positive real numbers such that

$$
a+b+c=1 .
$$

Prove that

$$
\frac{a^{3}}{a+b}+\frac{b^{3}}{b+c}+\frac{c^{3}}{c+a} \geq \frac{1}{6} .
$$

Our aim in this note is to generalize the results of these two problems and give some of their applications.

## 2. Main results

Proposition 1 If $\alpha, \beta, \gamma, x, y>0$ and $m \in[2, \infty)$, then

$$
\begin{align*}
& \frac{1}{\alpha^{m-2}(x \beta+y \gamma)^{m}}+\frac{1}{\beta^{m-2}(x \gamma+y \alpha)^{m}}+\frac{1}{\gamma^{m-2}(x \alpha+y \beta)^{m}} \\
\geq & \frac{3^{m}}{(x+y)^{m}\left(\alpha^{2}+\beta^{2}+\gamma^{2}\right)^{m-1}} . \tag{11}
\end{align*}
$$

Proof. We have

$$
\begin{align*}
S & =\sum_{\text {cyc }} \frac{1}{\alpha^{m-2}(x \beta+y \gamma)^{m}}=\sum_{\text {cyc }} \frac{\alpha^{2}}{\alpha^{m}(x \beta+y \gamma)^{m}} \\
& \Longleftrightarrow \frac{S}{\alpha^{2}+\beta^{2}+\gamma^{2}}=\sum_{\text {cyc }} \frac{\alpha^{2}}{\alpha^{2}+\beta^{2}+\gamma^{2}} \cdot \frac{1}{[\alpha(x \beta+y \gamma)]^{m}} . \tag{12}
\end{align*}
$$

We consider the function $f:(0, \infty) \rightarrow(0, \infty)$ defined by

$$
f(x)=x^{m} .
$$

Since $f^{\prime}(x)=m x^{m-1}$ and $f^{\prime \prime}(x)=m(m-1) x^{m-2}>0$ for all $x \in$ $(0, \infty)$, we deduce that $f$ in convex on $(0, \infty)$, so by Jensen's inequality
$\lambda_{1} f\left(x_{1}\right)+\lambda_{2} f\left(x_{2}\right)+\lambda_{3} f\left(x_{3}\right) \geq\left(\lambda_{1}+\lambda_{2}+\lambda_{3}\right) f\left(\lambda_{1} x_{1}+\lambda_{2} x_{2}+\lambda_{3} x_{3}\right)$
for all positive $\lambda_{k}$ and $x_{k}$. Taking in (3)

$$
\begin{align*}
& \lambda_{1}=\frac{\alpha^{2}}{\alpha^{2}+\beta^{2}+\gamma^{2}}, \lambda_{2}=\frac{\beta^{2}}{\alpha^{2}+\beta^{2}+\gamma^{2}}, \lambda_{3}=\frac{\gamma^{2}}{\alpha^{2}+\beta^{2}+\gamma^{2}}  \tag{13}\\
& x_{1}=\frac{1}{\alpha(x \beta+y \gamma)}, x_{2}=\frac{1}{\beta(x \gamma+y \alpha)}, x_{3}=\frac{1}{\gamma(x \alpha+y \beta)},
\end{align*}
$$

we have

$$
\begin{align*}
& \sum_{\text {cyc }} \frac{\alpha^{2}}{\alpha^{2}+\beta^{2}+\gamma^{2}} \cdot \frac{1}{[\alpha(x \beta+y \gamma)]^{m}} \\
\geq & \left(\sum_{\text {cyc }} \frac{\alpha^{2}}{\alpha^{2}+\beta^{2}+\gamma^{2}} \cdot \frac{1}{\alpha(x \beta+y \gamma)}\right)^{m}  \tag{14}\\
= & \frac{1}{\left(\alpha^{2}+\beta^{2}+\gamma^{2}\right)^{m}}\left(\frac{\alpha}{x \beta+y \gamma}+\frac{\beta}{x \gamma+y \alpha}+\frac{\gamma}{x \alpha+y \beta}\right)^{m} .
\end{align*}
$$

By Bergström's inequality,

$$
\sum_{\mathrm{cyc}} \frac{\alpha}{x \beta+y \gamma}=\sum_{\mathrm{cyc}} \frac{\alpha^{2}}{\alpha \beta x+\alpha \gamma y} \geq \frac{(\alpha+\beta+\gamma)^{2}}{(x+y)(\alpha \beta+\beta \gamma+\gamma \alpha)} \geq \frac{3}{x+y}
$$

so that by (2) and (4),

$$
S \geq \frac{3^{m}}{\left(\alpha^{2}+\beta^{2}+\gamma^{2}\right)^{m-1}(x+y)^{m}}
$$

and we are done.
Observation 1.1. For $\alpha=a, \beta=b, \gamma=c$, and $m=5$, we obtain the result of problem 682.

Proposition 2 Suppose $n \geq 2$ is a positive integer; $\alpha \geq 0 ; \beta, \gamma, \delta>0$; and $x_{k}>0$ for $1 \leq k \leq n$. Set $X_{n}=\sum_{k=1}^{n} x_{k}$, and assume $\gamma X_{n}>$ $\delta \max _{1 \leq k \leq n} x_{k}$. Then

$$
\begin{equation*}
\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}} \geq \frac{(\alpha n+\beta) n}{\gamma n-\delta} \tag{15}
\end{equation*}
$$

We give four proofs of Proposition 2.
Proof 2.1. In fact (5) is a generalization of Nesbitt's inequality:

$$
\begin{equation*}
\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geq \frac{3}{2} \text { for all } x, y, z>0 . \tag{16}
\end{equation*}
$$

We have that:

$$
\begin{gathered}
U_{n}=\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}} \\
\Longleftrightarrow \delta U_{n}=\sum_{k=1}^{n} \frac{\alpha \delta X_{n}+\beta \delta x_{k}}{\gamma X_{n}-\delta x_{k}} \\
\Longleftrightarrow \delta U_{n}+n \beta=\sum_{k=1}^{n}\left(\frac{\alpha \delta X_{n}+\beta \delta x_{k}}{\gamma X_{n}-\delta x_{k}}+\beta\right) \\
=(x \delta+\beta \gamma) X_{n} \cdot \sum_{k=1}^{n} \frac{1}{\gamma X_{n}-\delta x_{k}}
\end{gathered}
$$

$$
\begin{aligned}
& \geq(\alpha \delta+\beta \gamma) X_{n} \cdot \frac{n^{2}}{\sum_{k=1}^{n}\left(\gamma X_{n}-\delta x_{k}\right)} \\
& =(\alpha \delta+\beta \gamma) X_{n} \cdot \frac{n^{2}}{\gamma n X_{n}-\delta \sum_{k=1}^{n} x_{k}} \\
& =\frac{(\alpha \delta+\beta \gamma) n^{2}}{\gamma n-\delta} \\
\Longleftrightarrow \delta U_{n} & \geq \frac{(\alpha \delta+\beta \gamma) n^{2}}{\gamma n-\delta}-n \beta=\frac{\alpha \delta n^{2}+\beta \delta n}{\gamma n-\delta} \\
\Longleftrightarrow U_{n} & \geq \frac{(\alpha n+\beta) n}{\gamma n-\delta},
\end{aligned}
$$

and we are done.
Proof 2.2. We have

$$
\begin{aligned}
&(\alpha n+\beta)^{2} X_{n}^{2}=\left(\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)\right)^{2} \\
&=\left(\sum_{k=1}^{n} \sqrt{\frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}} \cdot \sqrt{\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right)}\right)^{2} \\
& \leq\left(\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}\right)\left(\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right)\right) \\
& \Longleftrightarrow \sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}} \geq \frac{(\alpha n+\beta)^{2} X_{n}^{2}}{\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right)} \\
&= \frac{(\alpha n+\beta)^{2} X_{n}^{2}}{\alpha \lambda n X_{n}^{2}+(\beta \gamma-\alpha \delta) X_{n} \sum_{k=1}^{n} x_{k}-\beta \delta \sum_{k=1}^{n} x_{k}^{2}} \\
&= \frac{(\alpha n+\beta)^{2} X_{n}^{2}}{(\alpha \gamma n+\beta \gamma-\alpha \delta) X_{n}^{2}-\beta \delta \sum_{k=1}^{n} x_{k}^{2}}
\end{aligned}
$$

By AM-QM inequality we deduce that:

$$
\begin{equation*}
\sum_{k=1}^{n} x_{k}^{2} \geq \frac{1}{n}\left(\sum_{k=1}^{n} x_{k}\right)^{2}=\frac{X_{n}^{2}}{n} \tag{17}
\end{equation*}
$$

Then we obtain

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}} & \geq \frac{(\alpha n+\beta)^{2} X_{n}^{2}}{(\alpha \gamma n+\beta \gamma-\alpha \delta) X_{n}^{2}-\frac{\beta \delta}{n} X_{n}^{2}} \\
& =\frac{(\alpha n+\beta)^{2} n}{\alpha \gamma n^{2}+(\beta \gamma-\alpha \delta) n-\beta \delta} \\
& =\frac{(\alpha n+\beta) n}{\gamma n-\delta}
\end{aligned}
$$

Proof 2.3. We have

$$
U_{n}=\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}=\sum_{k=1}^{n} \frac{\left(\alpha X_{n}+\beta x_{k}\right)^{2}}{\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right)},
$$

and by Bergström's inequality we deduce that

$$
\begin{aligned}
U_{n} & \geq \frac{\left(\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)\right)^{2}}{\sum_{k=1}^{n}\left[\alpha \gamma X_{n}^{2}+(\beta \gamma-\alpha \delta) X_{n} x_{k}-\beta \delta x_{k}^{2}\right]} \\
& =\frac{(\alpha n+\beta)^{2} X_{n}^{2}}{\alpha \gamma n X_{n}^{2}+(\beta \gamma-\alpha \delta) X_{n}^{2}-\beta \delta \sum_{k=1}^{n} x_{k}^{2}}
\end{aligned}
$$

using (7), and we obtain the conclusion.

Proof 2.4. We consider the trinomial

$$
\begin{aligned}
T_{n}= & \left(\sqrt{\frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}} \cdot X-\sqrt{\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right)}\right)^{2} \\
= & \left(\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}\right) X^{2}-2\left(\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)\right) X \\
& +\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right) \\
= & \left(\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}\right) X^{2}-2\left(\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)\right) X \\
& +\sum_{k=1}^{n}\left[\alpha \gamma X_{n}^{2}+(\beta \gamma-\alpha \delta) X_{n} x_{k}-\beta \delta x_{k}^{2}\right]
\end{aligned}
$$

Because the values of this quadratic trinomial satisfy $T_{n}(x)>0$ for all $x>0$, we have

$$
\begin{aligned}
& \left(\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)\right)^{2} \\
\leq & \left(\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}\right)\left[\sum_{k=1}^{n} \alpha \gamma X_{n}^{2}+(\beta \gamma-\alpha \delta) X_{n} x_{k}-\beta \delta x_{k}^{2}\right] \\
= & \left(\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}\right)\left[\alpha \gamma n X_{n}^{2}+(\beta \gamma-\alpha \delta) X_{n}^{2}-\beta \delta \sum_{k=1}^{n} x_{k}^{2}\right] .
\end{aligned}
$$

Then, by (7), we obtain

$$
\begin{aligned}
& \sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)^{2} \\
& \quad \leq\left(\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}\right)\left(\alpha \gamma n+\beta \gamma-\alpha \delta-\frac{\beta \delta}{n}\right) X_{n}^{2} \\
& \Longleftrightarrow \sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}} \geq \frac{(\alpha n+\beta)^{2} X_{n}^{2}}{\left(\alpha \gamma n+\beta \gamma-\alpha \delta-\frac{\beta \gamma}{n}\right) X_{n}^{2}} \\
& \Longleftrightarrow \quad \sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}} \geq \frac{n(\alpha n+\beta)^{2}}{\alpha \gamma n^{2}+(\beta \gamma-\alpha \delta) n-\beta \delta}=\frac{n(\alpha n+\beta)}{\gamma n-\delta} .
\end{aligned}
$$

Observation 2.1. If $\alpha=0, \beta=\gamma=\delta=1$, then the relation (5) becomes:

$$
\sum_{k=1}^{n} \frac{x_{k}}{X_{n}-x_{k}} \geq \frac{n}{n-1}
$$

i.e. Nesbitt's inequality for $n$ variables, and for $n=3$, we obtain Nesbitt's inequality, i.e.:

$$
\frac{x_{1}}{x_{2}+x_{3}}+\frac{x_{2}}{x_{3}+x_{1}}+\frac{x_{3}}{x_{1}+x_{2}} \geq \frac{3}{2} .
$$

Theorem 6 Let $n \geq 2$ be a positive integer. Suppose $\alpha \geq 0 ; \beta, \gamma, \delta>0$; and $m \geq 2$. Suppose that for $1 \leq k \leq n, x_{k}>0$, and set $X_{n}=\sum_{k=1}^{n} x_{k}$. Assume $\gamma X_{n}>\delta \max _{1 \leq k \leq n} x_{k}$. Then
$\sum_{k=1}^{n} \frac{1}{\left(\alpha X_{n}+\beta x_{k}\right)^{m-2}\left(\gamma X_{n}-\delta x_{k}\right)^{m}} \geq \frac{(\alpha n+\beta)^{m} n^{m}}{(\gamma n-\delta)^{m}}\left[\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)^{2}\right]^{1-m}$.

Proof. We have

$$
\begin{aligned}
S_{n} & =\sum_{k=1}^{n} \frac{1}{\left(\alpha X_{n}+\beta x_{k}\right)^{m-2}\left(\gamma X_{n}-\delta x_{k}\right)^{m}} \\
& =\sum_{k=1}^{n} \frac{\left(\alpha X_{n}+\beta x_{k}\right)^{2}}{\left[\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right)\right]^{m}}
\end{aligned}
$$

and

$$
V_{n}=\sum_{k=1}^{n}\left(\alpha X_{n}+\beta x_{k}\right)^{2}=n \alpha^{2} X_{n}^{2}+2 \alpha \beta X_{n}^{2}+\beta^{2} \sum_{k=1}^{n} x_{k}^{2},
$$

so that

$$
\begin{equation*}
\frac{S_{n}}{V_{n}}=\sum_{k=1}^{n} \frac{\left(\alpha X_{n}+\beta x_{k}\right)^{2}}{V_{n}} \cdot \frac{1}{\left[\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right)\right]^{m}} \tag{19}
\end{equation*}
$$

The function $g:(0, \infty) \rightarrow(0, \infty)$ defined by

$$
g(x)=x^{m}
$$

satisfies $g^{\prime}(x)=m x^{m-1}$ and $g^{\prime \prime}(x)=m(m-1) x^{m-2}>0$ for all $x>$ 0 ; we deduce that $g$ is convex on $(0, \infty)$ and that by Jensen's inequality,

$$
\begin{equation*}
\sum_{k=1}^{n} \lambda_{k} g\left(x_{k}\right) \geq\left(\sum_{k=1}^{n} \lambda_{k}\right) \cdot g\left(\sum_{k=1}^{n} \lambda_{k} x_{k}\right) \tag{20}
\end{equation*}
$$

for all $\lambda_{k}, x_{k}>0$. If we take

$$
\lambda_{k}=\frac{\left(\alpha X_{n}+\beta x_{k}\right)^{2}}{V_{n}} \text { and } x_{k}=\frac{1}{\left[\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right)\right]^{m}},
$$

then by (10) we deduce that:

$$
\begin{aligned}
& \sum_{k=1}^{n} \frac{\left(\alpha X_{n}+\beta x_{k}\right)^{2}}{V_{n}} \cdot \frac{1}{\left[\left(\alpha X_{n}+\beta x_{k}\right)\left(\gamma X_{n}-\delta x_{k}\right)\right]^{m}} \\
\geq & \frac{1}{V_{n}^{2}}\left(\sum_{k=1}^{n} \frac{\alpha X_{n}+\beta x_{k}}{\gamma X_{n}-\delta x_{k}}\right)^{m},
\end{aligned}
$$

and taking account (5) we obtain that:

$$
\begin{aligned}
& \frac{S_{n}}{V_{n}} \geq \frac{1}{V_{n}^{m}} \cdot \frac{(\alpha n+\beta)^{m} n^{m}}{(\gamma n-\delta)^{m}} \\
\Longleftrightarrow \quad & S_{n} \geq \frac{(\alpha n+\beta)^{m} n^{m}}{(\gamma n-\delta)^{m}} V_{n}^{1-m},
\end{aligned}
$$

and the proof is complete.

Theorem 7 Let $n \geq 2$ be a positive integer. Suppose $a, b, c \geq 0$, with $a+b+c>0$. Let $m \geq 0$. Suppose that for $1 \leq k \leq n, x_{k}>0$. Then

$$
\begin{align*}
\sum_{k=1}^{n} \frac{x_{k}^{m+2}}{\left(a x_{k}+b x_{k+1}+c x_{k+2}\right)^{m}} & \geq \frac{1}{(a+b+c)^{m}} \sum_{k=1}^{n} x_{k}^{2}  \tag{21}\\
& \geq \frac{1}{n(a+b+c)^{m}}\left(\sum_{k=1}^{n} x_{k}\right)^{2}
\end{align*}
$$

where $x_{n+1}=x_{1}$ and $x_{n+2}=x_{2}$.

Proof. We have

$$
\begin{aligned}
W_{n} & =\sum_{k=1}^{n} \frac{x_{k}^{m+2}}{\left(a x_{k}+b x_{k+1}+c x_{k+2}\right)^{m}} \\
& =\sum_{k=1}^{n} \frac{\left(x_{k}^{2}\right)^{m+1}}{\left(a x_{k}^{2}+b x_{k} x_{k+1}+c x_{k} x_{k+2}\right)^{m}} \\
& \geq \sum_{k=1}^{n} \frac{\left(x_{k}^{2}\right)^{m+1}}{\left[a x_{k}^{2}+\frac{b}{2}\left(x_{k}^{2}+x_{k+1}^{2}\right)+\frac{c}{2}\left(x_{k}^{2}+x_{k+2}^{2}\right)\right]^{m}} \\
& =2^{m} \cdot \sum_{k=1}^{n} \frac{\left(x_{k}^{2}\right)^{m+1}}{\left[2 a x_{k}^{2}+b\left(x_{k}^{2}+x_{k+1}^{2}\right)+c\left(x_{k}^{2}+x_{k+2}^{2}\right)\right]^{m}},
\end{aligned}
$$

where we apply J. Radon's inequality and we deduce that:

$$
\begin{aligned}
W_{n} & \geq 2^{m} \cdot \frac{\left(\sum_{k=1}^{n} x_{k}^{2}\right)^{m+1}}{\left(\sum_{k=1}^{n}\left[2 a x_{k}^{2}+b\left(x_{k}^{2}+x_{k+1}^{2}\right)+c\left(x_{k}^{2}+x_{k+2}^{2}\right)\right]\right)^{m}} \\
& =2^{m} \cdot \frac{\left(\sum_{k=1}^{n} x_{k}^{2}\right)^{m+1}}{2^{m} \cdot(a+b+c)^{m} \cdot\left(\sum_{k=1}^{n} x_{k}^{2}\right)^{m}} \\
& =\frac{\sum_{k=1}^{n} x_{k}^{2}}{(a+b+c)^{m}},
\end{aligned}
$$

and by the arithmetic-quadratic mean inequaliy $\sum_{k=1}^{n} x_{k}^{2} \geq \frac{1}{n}\left(\sum_{k=1}^{n} x_{k}\right)^{2}$, we obtain (11), and we are done.

## 3. Applications

A3.1. If $t \in[2, \infty)$ and $x, y>0$, then in all triangles $A B C$ (with usual notations)

$$
\begin{equation*}
\sum_{\mathrm{cyc}} \frac{1}{m_{a}^{t-2}\left(x m_{b}+y m_{c}\right)^{t}} \geq \frac{3 \cdot 4^{t-1}}{(x+y)^{t}}\left(a^{2}+b^{2}+c^{2}\right)^{1-t} \tag{22}
\end{equation*}
$$

Proof. In equation (1), we take $\alpha=m_{a}, \beta=m_{b}, \gamma=m_{c}$. Using

$$
m_{a}^{2}+m_{b}^{2}+m_{c}^{2}=\frac{3}{4}\left(a^{2}+b^{2}+c^{2}\right),
$$

we obtain (12), and we are done.
A3.2. Let $A B C$ be a triangle (with the usual notations), and denote

$$
H(A B C)=h_{a}+h_{b}+h_{c}, M(A B C)=m_{a}+m_{b}+m_{c} .
$$

If $\alpha>0, \beta, \gamma, \delta>0$, and $t \in[2, \infty)$ such that

$$
\gamma H(A B C)>\delta \max \left\{h_{a}, h_{b}, h_{c}\right\},
$$

then

$$
\begin{align*}
& \sum_{\text {cyc }} \frac{1}{\left(\alpha H(A B C)+\beta h_{a}\right)^{t-2}\left(\gamma M(A B C)-\delta m_{a}\right)^{t}}  \tag{23}\\
\geq & \frac{3^{t}(3 \alpha+\beta)^{t}}{(3 \gamma-\delta)^{t}}\left(\sum_{\text {cyc }}\left(\alpha M(A B C)+\beta m_{a}\right)\right)^{1-t} .
\end{align*}
$$

Proof. Since $m_{a} \geq h_{a}, m_{b} \geq h_{b}$ and $m_{c} \geq h_{c}$ yields $M(A B C) \geq$ $H(A B C)$, we have

$$
\begin{aligned}
& \sum_{\text {cyc }} \frac{1}{\left(\alpha H(A B C)+\beta h_{a}\right)^{t-2}\left(\gamma M(A B C)-\delta m_{a}\right)^{t}} \\
\geq & \sum_{\text {cyc }} \frac{1}{\left(\alpha M(A B C)+\beta h_{a}\right)^{t-2}\left(\gamma M(A B C)-\delta m_{a}\right)^{t}},
\end{aligned}
$$

and by (8) with $n=3$, we obtain (13).

A3.3. If $x, y>0$ and $t \in[2, \infty)$, then in all triangles $A B C$

$$
\begin{equation*}
\sum_{\mathrm{cyc}} \frac{1}{m_{a}^{t-2}\left(x h_{b}+y h_{c}\right)^{t}} \geq \frac{3 \cdot 4^{t-1}}{(x+y)^{t}}\left(a^{2}+b^{2}+c^{2}\right)^{1-t} \tag{24}
\end{equation*}
$$

Proof. Because, $m_{a} \geq h_{a}, m_{b} \geq h_{b}$ and $m_{c} \geq h_{c}$

$$
\begin{equation*}
\sum_{\mathrm{cyc}} \frac{1}{m_{a}^{t-2}\left(x h_{b}+y h_{c}\right)^{t}} \geq \sum_{\mathrm{cyc}} \frac{1}{m_{a}^{t-2}\left(x m_{b}+y m_{c}\right)^{t}} . \tag{25}
\end{equation*}
$$

Therefore, taking $\alpha=m_{a}, \beta=m_{b}$, and $\gamma=m_{c}$ in (1), by (15) we obtain

$$
\begin{aligned}
\sum_{\mathrm{cyc}} \frac{1}{m_{a}^{t-2}\left(x h_{b}+y h_{c}\right)^{t}} & \geq \sum_{\mathrm{cyc}} \frac{1}{m_{a}^{t-2}\left(x m_{b}+y m_{c}\right)^{t}} \\
& \geq \frac{3^{t}}{(x+y)^{t}\left(m_{a}^{2}+m_{b}^{2}+m_{c}^{2}\right)^{t-1}} \\
& =\frac{3 \cdot 4^{t-1}}{(x+y)^{t}\left(a^{2}+b^{2}+c^{2}\right)^{1-t}}
\end{aligned}
$$

and we are done.
A 3.4. If $a, b, c>0, m \geq 0$, and $a+b+c=1$, then

$$
\sum_{\mathrm{cyc}} \frac{a^{m+2}}{(a+b)^{m}} \geq \frac{1}{2^{m} \cdot 3}
$$

Proof. We have

$$
\begin{aligned}
U & =\sum_{\text {cyc }} \frac{a^{m+2}}{(a+b)^{m}} \\
& =\sum_{\text {cyc }} \frac{a^{2 m+2}}{\left(a^{2}+a b\right)^{m}} \\
& =\sum_{\text {cyc }} \frac{\left(a^{2}\right)^{m+1}}{\left(a^{2}+a b\right)^{m}} \\
& \geq \sum_{\text {cyc }} \frac{\left(a^{2}\right)^{m+1}}{\left(a^{2}+\frac{a^{2}+b^{2}}{2}\right)^{m}} \\
& =2^{m} \cdot \sum_{\text {cyc }} \frac{\left(a^{2}\right)^{m}}{\left(3 a^{2}+b^{2}\right)^{m}},
\end{aligned}
$$

where we apply J. Radon's inequality and we obtain

$$
U \geq 2^{m} \cdot \frac{\left(\sum_{\text {cyc }} a^{2}\right)^{m}}{\left[\sum_{\text {cyc }}\left(3 a^{2}+b^{2}\right)\right]^{m}}=2^{m} \cdot \frac{\left(\sum_{\text {cyc }} a^{2}\right)^{m+1}}{\left(4 \sum_{\text {cyc }} a^{2}\right)^{m}}=\frac{1}{2^{m}} \cdot \sum_{\text {cyc }} a^{2}
$$

Since $a^{2}+b^{2}+c^{2} \geq \frac{(a+b+c)^{2}}{3}$ by the AM-QM inequality, we deduce that

$$
U \geq \frac{1}{2^{m}} \cdot \frac{(a+b+c)^{2}}{3}
$$

and from the hypothesis $a+b+c=1$, we obtain

$$
U \geq \frac{1}{2^{m} \cdot 3},
$$

completing the proof.

## Remark 3.4.1.

- If $\sum_{k=1}^{n} x_{k}=1$, then by (11) we obtain

$$
\begin{equation*}
\sum_{k=1}^{n} \frac{x_{k}^{m+2}}{\left(a x_{k}+b x_{k+1}+c x_{k+2}\right)^{m}} \geq \frac{1}{n(a+b+c)^{m}} \text { for all } m \geq 0 \tag{26}
\end{equation*}
$$

- If $\sum_{k=1}^{n} x_{k}=1, a=b=1$, and $c=0$, then by (16) we obtain

$$
\begin{equation*}
\sum_{k=1}^{n} \frac{x_{k}^{m+2}}{\left(x_{k}+x_{k+1}\right)^{m}} \geq \frac{1}{2^{m} \cdot n}\left(\sum_{k=1}^{n} x_{k}\right)^{2}=\frac{1}{2^{m} \cdot n} \text { for all } m \geq 0 \tag{27}
\end{equation*}
$$

- If $n=3, x_{1}=a, x_{2}=b$, and $x_{3}=c$, then by (17) we obtain A 3.4., i.e.

$$
\begin{equation*}
\sum_{\text {cyc }} \frac{a^{m+2}}{(a+b)^{m}} \geq \frac{1}{2^{m} \cdot 3} \text { for all } m \geq 0 \tag{28}
\end{equation*}
$$

- If $m=1$, then by (18) we obtain the result of Problem 694. Therefore, A3.4. is a generalization of this problem.

A3.5. If $a, b, c>0, m \geq 0, x, y>0$, and $a+b+c=1$, then

$$
\sum_{\mathrm{cyc}} \frac{a^{m+2}}{(x a+y b)^{m}} \geq \frac{1}{3(x+y)^{m}}
$$

Proof. We have

$$
\begin{aligned}
V & =\sum_{\text {cyc }} \frac{a^{m+2}}{(x a+y b)^{m}} \\
& =\sum_{\text {cyc }} \frac{\left(a^{2}\right)^{m+1}}{\left(x a^{2}+y a b\right)^{m}} \\
& \geq \sum_{\text {cyc }} \frac{\left(a^{2}\right)^{m+1}}{\left(x a^{2}+\frac{y\left(a^{2}+b^{2}\right)}{2}\right)^{m}} \\
& =2^{m} \cdot \sum_{\text {cyc }} \frac{\left(a^{2}\right)^{m+1}}{\left[(2 x+y) a^{2}+y b^{2}\right]}
\end{aligned}
$$

where we apply J. Radon's inequality and we obtain

$$
\begin{aligned}
V & \geq 2^{m} \cdot \frac{\left(\sum_{\text {cyc }} a^{2}\right)^{m+1}}{\left[\sum_{\text {cyc }}\left((2 x+y) a^{2}+y b^{2}\right)\right]^{m}} \\
& =2^{m} \cdot \frac{\left(\sum_{\text {cyc }} a^{2}\right)^{m+1}}{2^{m}(x+y)^{m}\left(\sum_{\text {cyc }} a^{2}\right)^{m}} \\
& =\frac{1}{(x+y)^{m}} \cdot \sum_{\text {cyc }} a^{2} \\
& \geq \frac{(a+b+c)^{2}}{3(x+y)^{m}}
\end{aligned}
$$

so that $V \geq \frac{1}{3(x+y)^{m}}$.
Remark 3.5.1. For $x=y=1$, we reobtain the generalization of the Problem 694, i.e. A3.4.

## References

[1] Problem 682, The Pentagon, Spring 2011, p. 60.
[2] Problem 694, The Pentagon, Fall 2011, p. 54.

# On the Theory of Numbers: A Paper by Évariste Galois 

Emilie Huffman, student

TN Gamma
Union University
Jackson, TN 38305
Presented at the 2012 North Central Regional Convention

## 1. Introduction

The early 19th century French mathematician Évariste Galois is memorable for multiple reasons, both professional and personal. His short life (1811-1832) was dominated by both mathematical work and political activity. At the age of 20, he died in a duel-the causes leading up to this event are unknown. The state of Galois' life at the time was tumultuous at best. His father had committed suicide a few years earlier, Galois had been imprisoned for his political activities, and he had both failed to gain admission to the École Polytechnique, the most mathematically prestigious school in France, and failed to gain recognition for much of his work. Some popular speculation also includes a love interest, Stéphanie-Félicie Poterin du Motel, as a contributing factor to the duel.[1]

However, despite the brevity of his life, Galois is remembered for some major mathematical accomplishments. He laid the foundations for group theory in his most famous paper, "Memoir on the Conditions for Solvability of Equations by Radicals," where he found a criterion for determining whether or not a polynomial equation is solvable by radicals. As a result, he additionally introduced the concepts of normal subgroup and solvable group, which are still used today. The importance of this paper was recognized widely only after his death.

Galois did have work that was published during his lifetime, however, and this included the paper "On the Theory of Numbers,"2[2] published in 1830. This work is shorter than the "Memoir" and is not as well known, yet is the first to introduce the concept of a finite field. While prominent mathematicians of the time such as Abel had a clear notion of what is now called a "field," all fields that had been considered up until that time contained the rational number field, and thus were fields of characteristic zero.[3] Galois, however, found that he could construct finite fields by considering the incommensurable solutions for polynomials whose coefficients were elements of the set of integers modulo $p$, or $\mathbb{Z}_{p}$, where $p$ is a prime number. In his paper, Galois studies the structures of these fields and in doing so makes use of several ideas important in contemporary abstract algebra, such as congruence, cyclic groups, and cosets, however these ideas are somewhat obscured by the different manner of presentation.

The goal of this paper is to update the language, notation, and presentation of "On the Theory of Numbers" to that of contemporary writing at the undergraduate level. The Galois paper has the pedagogical benefit of providing insight through both revealing the manner in which these fields were constructed historically and relating these various concepts in a way different from the mode in which they are currently introduced. As we follow Galois' experimentations with these (then) new concepts and topics, the hope is that the motivations for the development of such concepts will become clearer, and that the added context will also spark interest in studies of related topics.

## 2. Mathematical Background

Galois begins by "regard[ing] as zero all quantities which, in algebraic calculations are found to be multiplied by $p$," thus he is interested in elements of the integers modulo $p$, where $p$ is a prime (which will now be denoted as $\mathbb{Z}_{p}$ ). He then considers equations involving polynomials in $\mathbb{Z}_{p}[x]$, a study initiated by Gauss. However, while Gauss had admitted only rational solutions, Galois decides to introduce other, "incommensurable" solutions, and it is here that in his own words, "I have attained some results which I consider new."

[^1]Consider the polynomial equation, $f(x)=0$, where $f(x)$ is degree $v$ and is an element of $\mathbb{Z}_{p}[x]$. Galois often refers to such equations as congruences in his paper. Suppose that $v>1$ and $f(x)$ is irreducible in $\mathbb{Z}_{p}[x]$, meaning there does not exist $\varphi(x), \psi(x) \in \mathbb{Z}_{p}[x]$, both of degree greater than zero, such that $\varphi(x) \psi(x)=f(x)$.

In that case, $f(x)$ cannot be factored in $\mathbb{Z}_{p}[x]$ and $f(x) \neq 0$ for any $x \in$ $\mathbb{Z}_{p}$, because if there existed an element $r \in \mathbb{Z}_{p}$ such that $f(r)=0$, then $x-r \in \mathbb{Z}_{p}[x]$ would be a factor of $f(x)$, contradicting the irreducibility of $f(x)$. Galois notes then that the polynomial, or "congruence," has no integer roots. He then denotes one root of this congruence with the symbol $i$, since it does not exist in the field of $\mathbb{Z}_{p}$. This reflects the common usage of $i$ to denote $\sqrt{-1}$, which does not exist in the field of real numbers.

Next Galois considers all expressions of the form

$$
\begin{equation*}
a+a_{1} i+a_{2} i^{2}+\ldots+a_{v-1} i^{v-1} \tag{29}
\end{equation*}
$$

where $a, a_{1}, a_{2}, \ldots, a_{v-1} \in \mathbb{Z}_{p}$. Since there are $p$ possible values that each coefficient could assume, as well as $v$ coefficients, thus there are $p^{v}$ expressions of this form. These elements form a Galois field $G F\left(p^{v}\right)$.

Galois' paper is concerned with the structure of these finite fields. We will see that the multiplicative group of Galois' constructed finite field is cyclic under multiplication, a property that in fact holds for all finite fields. Furthermore, we will concern ourselves with finding a generator for such a field.

As an example, consider the polynomial $g(x)=x^{3}+x+1 \in \mathbb{Z}_{2}[x]$ of degree $v=3$. It is straightforward to verify by method of exhaustion that $g(x)$ is irreducible in $\mathbb{Z}_{2}[x]$. Let $i$ denote one of the roots of $g(x)$, so then $i^{3}+i+1=0$. When we consider all expressions of the form

$$
\begin{equation*}
a+a_{1} i+a_{2} i^{2} \tag{30}
\end{equation*}
$$

where $a, a_{1}, a_{2} \in \mathbb{Z}_{2}$, we have $p=2$ and $v=3$, so we have $2^{3}=8$ elements in this form:

$$
\begin{equation*}
0,1, i, 1+i, 1+i+i^{2}, i+i^{2}, 1+i^{2}, i^{2} \tag{31}
\end{equation*}
$$

It is straightforward to verify by a Cayley table that these elements do, in fact, form a field (see Figures 1 and 2). The identity, $i^{3}+i+1=0$, is used extensively in achieving these results.

| + | 0 | 1 | $i$ | $1+i$ | $i^{2}$ | $1+i^{2}$ | $1+i+i^{2}$ | $i+i^{2}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |


| 0 | 0 | 1 | $i$ | $1+i$ | $i^{2}$ | $1+i^{2}$ | $1+i+i^{2}$ | $i+i^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 0 | $1+i$ | $i$ | $1+i^{2}$ | $i^{2}$ | $i+i^{2}$ | $1+i+i^{2}$ |
| $i$ | $i$ | $1+i$ | 0 | 1 | $i+i^{2}$ | $1+i+i^{2}$ | $1+i^{2}$ | $i^{2}$ |
| $1+i$ | $1+i$ | $i$ | 1 | 0 | $1+i+i^{2}$ | $i+i^{2}$ | $i^{2}$ | $1+i^{2}$ |
| $i^{2}$ | $i^{2}$ | $1+i^{2}$ | $i+i^{2}$ | $1+i+i^{2}$ | 0 | 1 | $1+i$ | $i$ |
| $1+i^{2}$ | $1+i^{2}$ | $i^{2}$ | $1+i+i^{2}$ | $i+i^{2}$ | 1 | 0 | $i$ | $1+i$ |
| $1+i+i^{2}$ | $1+i+i^{2}$ | $i+i^{2}$ | $1+i^{2}$ | $i^{2}$ | $1+i$ | $i$ | 0 | 1 |
| $i+i^{2}$ | $i+i^{2}$ | $1+i+i^{2}$ | $i^{2}$ | $1+i^{2}$ | $i$ | $1+i$ | 1 | 0 |

Figure 1: The Cayley table for the elements of $G F\left(2^{3}\right)$ under the operation of addition

|  | 0 | 1 | $i$ | $1+i$ | $i^{2}$ | $1+i^{2}$ | $1+i+i^{2}$ | $i+i^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | $i$ | $1+i$ | $i^{2}$ | $1+i^{2}$ | $1+i+i^{2}$ | $i+i^{2}$ |
| $i$ | 0 | $i$ | $i^{2}$ | $i+i^{2}$ | $1+i$ | 1 | $1+i^{2}$ | $1+i+i^{2}$ |
| $1+i$ | 0 | $1+i$ | $i+i^{2}$ | $1+i^{2}$ | $1+i+i^{2}$ | $i^{2}$ | $i$ | 1 |
| $i^{2}$ | 0 | $i^{2}$ | $1+i$ | $1+i+i^{2}$ | $i+i^{2}$ | $i$ | 1 | $1+i^{2}$ |
| $1+i^{2}$ | 0 | $1+i^{2}$ | 1 | $i^{2}$ | $i$ | $1+i+i^{2}$ | $i+i^{2}$ | $1+i$ |
| $1+i+i^{2}$ | 0 | $1+i+i^{2}$ | $1+i^{2}$ | $i$ | 1 | $i+i^{2}$ | $1+i$ | $i^{2}$ |
| $i+i^{2}$ | 0 | $i+i^{2}$ | $1+i+i^{2}$ | 1 | $1+i^{2}$ | $1+i$ | $i^{2}$ | $i$ |

Figure 2: The Cayley table for the elements of $G F\left(2^{3}\right)$ under the operation of multiplication

Galois then focuses on the operation of multiplication for this field. Because there are $p^{v}$ elements of the form (1), we conclude that there are $p^{v}-1$ non-zero elements. Choose one such element, $\alpha$. Because the field is finite, $\alpha^{n}=1$ for some $n \in \mathbb{N}$. Let $n$ be the smallest integer such that $\alpha^{n}=1$, denoted as $|\alpha|$, the order of $\alpha$; then $1, \alpha, \alpha^{2}, \ldots, \alpha^{n-1}$ are all distinct, and the set $\langle\alpha\rangle$ forms a cyclic subgroup of multiplicative group $G F\left(p^{v}\right) \backslash\{0\}$.

Multiplying each of these $n$ elements of $\langle\alpha\rangle$ by another element of $G F\left(p^{v}\right), \beta$, such that $\beta \notin\langle\alpha\rangle$, yields another set of $n$ quantities all distinct from those in $\langle\alpha\rangle$. Hence a coset, $\beta\langle\alpha\rangle$, of the multiplicative group $G F\left(p^{v}\right) \backslash\{0\}$ is formed. This process can be continued until all distinct cosets are found, yielding all of $G F\left(p^{v}\right) \backslash\{0\}$. Because each coset contains $n$ elements, the number $n$ must divide the total number of nonzero elements in $G F\left(p^{v}\right) \backslash\{0\}$. Thus we conclude that $n$ (which is $|\alpha|$ ) divides $p^{v}-1$. Note that Lagrange's theorem for finite groups also guarantees this result, as $G F\left(p^{v}\right) \backslash\{0\}$ is a multiplicative group.[4]

Hence

$$
\begin{equation*}
\alpha^{p^{v}-1}=1 \tag{32}
\end{equation*}
$$

or

$$
\begin{equation*}
\alpha^{p^{v}}=\alpha . \tag{33}
\end{equation*}
$$

As an aside, it is interesting to note that the integers modulo $p$, where $p$ is prime, form a finite field of order $p^{1}$. Thus we see that Fermat's Little theorem, which guarantees that $\alpha^{p} \equiv \alpha(\bmod p)$, where $a \in \mathbb{Z}$ and $p$ is prime, is in fact a special case of (5), which holds for all finite fields of order $p^{v}$.

We note that (5) in fact, holds for zero in addition to the non-zero elements of $G F\left(p^{v}\right)$, and thus we now come to what Galois terms the "remarkable result," that the elements in the Galois field are exactly the distinct roots of the equation

$$
\begin{equation*}
x^{p^{v}}=x . \tag{34}
\end{equation*}
$$

From here, Galois gives an additional result in which he returns to his original irreducible polynomial $f(x)$. We know that $i$ is one of the roots to this polynomial, and due to Galois' remarkable result, (6), we can find the other roots as well. Galois relates the roots of $f(x)$ to each other by first noting that for $n \geq 1$,

$$
\begin{equation*}
(f(x))^{p^{n}}=f\left(x^{p^{n}}\right) \tag{35}
\end{equation*}
$$

in $\mathbb{Z}_{p}[x]$. This is a well-known result.[5]
We know that if $x$ is a root of $f(x)$ then the left hand side of equation (7) must be zero. Thus the right hand side must be zero as well, meaning $x^{p^{n}}$ is a root as well. Since we know $i$ is a root of $f(x)$, the following
powers of $i$ must also be roots:

$$
\begin{equation*}
i^{p}, i^{p^{2}}, \ldots, i^{p^{v-1}} \tag{36}
\end{equation*}
$$

Since all the roots of $f(x)$ are also roots of $x^{p^{v}}-x$, then Galois concludes that one can write

$$
\begin{equation*}
\psi(x) f(x)=x^{p^{v}}-x \tag{37}
\end{equation*}
$$

where $\psi(x)$ is a polynomial in $\mathbb{Z}_{p}[x]$.
Returning to our example in which $p=2, v=3$ and $g(x)=x^{3}+x+1$, we know that there are $2^{3}-1=7$ nonzero elements in the Galois field. All the elements in this Galois field are roots of the equation

$$
\begin{equation*}
x^{8}=x, \tag{38}
\end{equation*}
$$

as can be verified by the multiplicative Cayley table. Additionally, as one of the roots of $g(x)$ is $i$, so the remaining two roots of this polynomial should be $i^{2}$ and $i^{2^{2}}$. These roots reduce to $i^{2}$ and $1+i+i^{2}$, and are verified to be roots of $g(x)$ by the Cayley tables. Finally, if we let $\psi(x)$ be $x^{5}+x^{3}+x^{2}+x$, then we do indeed have

$$
\begin{equation*}
\psi(x) f(x)=\left(x^{5}+x^{3}+x^{2}+x\right)\left(x^{3}+x+1\right)=x^{8}-x \tag{39}
\end{equation*}
$$

in $\mathbb{Z}_{2}[x]$, just as the theory has shown.
Galois asserts that there are certain values of $\alpha \in G F\left(p^{v}\right)$ for which $|\alpha|=p^{v}-1$, and then shows how these elements can be constructed. He calls such values primitive elements (or primitive roots), and they can also be defined as elements $\alpha \in G F\left(p^{v}\right)$ such that all the nonzero elements of $G F\left(p^{v}\right)$ can be written as $\alpha^{k}$ for some $k \in \mathbb{Z}$. The nonzero elements of the field can thus be generated solely from the powers of any one of these primitive elements. This implies, of course, that the group $G F\left(p^{v}\right) \backslash\{0\}$ is cyclic under the operation of multiplication.

In the previous example using $G F\left(2^{3}\right)$ then, the multiplicative group in this field is cyclic, so there must exist a primitive root of order 7 whose powers generate all the other nonzero elements of the field. In fact it is straightforward to search by exhaustion through the seven nonzero elements for the primitive root using the multiplicative Cayley table. The primitive root for $G F\left(2^{3}\right)$ is hence found to be $1+i^{2}$.

In providing a method for obtaining primitive elements, Galois himself uses an example in which $p=7$ and $v=3$. Thus the resulting Galois field, with $7^{3}=343$ elements total, is much larger than that of our example. His irreducible polynomial in $\mathbb{Z}_{7}[x]$ is $x^{3}-2$. Following Galois' notation, $i$ will represent an incommensurable root of the irreducible polynomial $x^{3}-2$. Hence $i^{3}-2=0$.

As he begins his search for the primitive root, Galois notes that by definition, he must find "a form of expression which, when raised to all possible powers, gives all the roots of the congruence"

$$
\begin{equation*}
x^{7^{3}-1}=1 \tag{40}
\end{equation*}
$$

Because the roots of equation (12) are exactly the elements of $G F\left(7^{3}\right) \backslash$ $\{0\}$, Galois seeks the root of (12) which, when raised to all powers, yields all the other roots of this equation. Hence, in modern terminology, Galois seeks a member of $G F\left(7^{3}\right)$ whose powers yield all the elements of $G F\left(7^{3}\right) \backslash\{0\}$.

Galois simplifies the problem by observing that $7^{3}-1=2^{1} 3^{2} 19$, so to find the primitive root for $G F\left(7^{3}\right) \backslash\{0\}$, he first seeks a primitive root for each of the following equations in $\mathbb{Z}_{7}[x]$.

$$
\begin{equation*}
x^{2}=1, x^{3^{2}}=1, x^{19}=1 \tag{41}
\end{equation*}
$$

Now, if $a_{1}, a_{2}$, and $a_{3}$ are elements defined such that $\left|a_{1}\right|=2,\left|a_{2}\right|=3^{2}$, and $\left|a_{3}\right|=19$, then $\left(a_{1} a_{2} a_{3}\right)^{2^{1} 3^{2} 19}=1$. Thus $\left|a_{1} a_{2} a_{3}\right|$ must divide the number $2^{1} 3^{2} 19=7^{3}-1$. By finding primitive roots for the equations in (13), Galois finds elements of such orders. But $2,3^{2}$ and 19 are also pairwise relatively prime-thus we conclude that $\left|a_{1} a_{2} a_{3}\right|$ is exactly $2^{1} 3^{2} 19=7^{3}-1[6]$.

When the three primitive roots of (13) are multiplied together then, the primitive root for the multiplicative group $G F\left(7^{3}\right) \backslash\{0\}$ will be obtained. For the first equation in (13), Galois notes that the primitive root for $x^{2}=1$ is $6(\bmod 7)$. For $x^{3^{2}}=1$, he recognizes that $x^{3^{2}}=\left(x^{3}\right)^{3}$. Because $2^{3} \equiv 1(\bmod 7)$, he finds that the root is given by the equation $x^{3}=2$, or $x^{3}-2=0$. Thus he denotes this root by $i$. Finally, in order to find the root for $x^{19}=1$, Galois first checks to see if one exists in the form $a_{1}+a_{2} i$. It turns out that one does indeed exist, namely, $-1+i$. Now that Galois has these three roots, $-1, i$, and $-1+i$, he multiplies them together:

$$
\begin{equation*}
(-1)(i)(-1+i)=i-i^{2} \tag{42}
\end{equation*}
$$

and concludes that $i-i^{2}$ is the primitive root for the multiplicative group $G F\left(7^{3}\right) \backslash\{0\}$. All $7^{3}-1=342$ elements can be generated from powers of this root.

## 3. Applications to Pedagogy

Galois' approach to the subject of finite fields differs in many ways from the current approach to introducing these fields to undergraduate students. A few of these differences yield some pedagogical advantage for the introduction of finite fields. While contemporary pedagogy introduces finite fields first with $\mathbb{Z}_{p}$ and then with quotient rings $\mathbb{Z}_{p}[x] /(p(x))$, Galois bypasses these more advanced concepts altogether by starting with the irreducible polynomial, $p(x)$, and then searching for its roots. The roots are then shown to be the elements of the finite field of order $p^{v}$, where $v$ is the degree of $p(x)$. This process of constructing finite fields can be grasped before the idea of quotient rings has been introduced.

Not only does the Galois paper offer an alternate way for finite fields to be introduced, but it also gives additional context to the ideas of "cyclic" and "coset" in group theory. While Galois never uses the term "coset" in his paper, he certainly uses this concept in order to show his remarkable result that every element of $G F\left(p^{v}\right)$ is a root of the polynomial equation $x^{p^{v}}=x$. In this way, the concept of a coset can be introduced to undergraduates with clear purpose, as it plays a necessary part in obtaining the field. Likewise, while Galois never uses the term "cyclic," he focuses a large part of his paper on finding "primitive elements." The existence of these elements implies that the multiplicative group of the nonzero elements of a finite field must be cyclic. The motivation for this concept is now clear, since it aids in describing a finite field-a construction motivated by the search for the roots of an irreducible polynomial. Furthermore, the original Galois paper offers a method for finding a generator for the cyclic multiplicative group of nonzero elements in the finite field. While current texts give the proof that a finite field is cyclic, often there lacks the process of actually finding the generator. Galois' text, on the other hand, offers a strategy for finding this generator, or what he calls the "primitive element."

Furthermore, in addition to offering motivation for certain concepts, Galois' paper offers new perspective and use for some well-known results in group theory. There is a strong connection to Lagrange's theorem for finite groups in the development of Galois' argument for his remarkable result, and an alternative line of reasoning from which to prove Fermat's Little theorem can also be found in this paper. Additionally, this paper motivates further study of other results and topics such as the formula $(f(x))^{p^{n}}=f\left(x^{p^{n}}\right)$ for $\mathbb{Z}_{p}[x]$, and orders of group elements and their products.

Finally, in following Galois through his discovery of these "results which [he] consider[s] new," the student is offered a view of the creative, constructive nature of mathematics. By following the historical progression of ideas, the methods by which and motivations for which the concepts were developed become transparent. When finite fields are developed later using more sophisticated ideas, the student already has an understanding that these clever and well-organized concepts were not always in existence, but developed over time from simpler ones.

## 4. Conclusion

Now that the language, notation, and presentation of "On the Theory of Numbers" have been updated to that of contemporary writing, at a level understandable to undergraduate students, it is the hope of this author that the paper is now more easily understandable, and that some pedagogically advantageous elements in the Galois approach, including the relatively simple construction of the fields, the motivated usage of group theory concepts, and the strategy for finding "primitive elements" have been demonstrated.

Acknowledgements. The author thanks Dr. Matt Lunsford, who motivated the project and gave helpful comments, references and direction throughout the development of this paper.

## References

[1] L. Rigatelli, Évariste Galois, Birkhauser Verlag, 1999.
[2] S. Stahl, Introductory Modern Algebra: A Historical Approach, John Wiley and Sons, Inc., 1997.
[3] B. L. van der Waerden, A History of Algebra. Out of print.
[4] J. R. Durbin, Modern Algebra An Introduction, 6th ed., p. 88, John Wiley and Sons, Inc., 2009.
[5] S. Lang, Undergraduate Algebra, 3rd ed., p. 311, Springer-Verlag, New York, NY. 2005.
[6] J. Rotman, A First Course in Abstract Algebra, 2nd ed., p. 175. Prentice Hall, Upper Saddle River, NJ. 2000.

## The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before October 1, 2013. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2013 issue of The Pentagon. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859)-622-3051)

## NEW PROBLEMS 711-721

Problem 711. Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.

Prove that there are infinitely many squares that are each the sum of two cubes and infinitely many cubes that are each the sum of two squares.

Problem 712. Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.

Find, with proof, all positive integer powers of 2 that can be written as the sum of two squares (of positive integers). Do the same for powers of 5.

Problem 713. Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.

Show that every positive integer power of 2 greater than 1 is either the sum of two triangular numbers or the sum of a triangular number and a pentagonal number.

Problem 714. Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.

The triangular numbers are $1,3,6,10, \ldots$ and satisfy the formula $T_{n}=$ $\frac{n(n+1)}{2}$.
a) Prove that the squares of infinitely many triangular numbers can be written as the sum of a square and a cube.
b) Prove that an infinite sequence of squares of nontrivial multiples of triangular numbers can be written as the sum of a square and a cube.

Problem 715. Proposed by D. M. Batinetu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania.

For $t>0$, define

$$
x_{n}(t)=n^{1-t}\left(\frac{(\sqrt[n+1]{(n+1)!})^{2 t}}{(n+1)^{t}}-\frac{(\sqrt[n]{n!})^{2 t}}{n^{t}}\right)
$$

Calculate $\lim _{n \rightarrow \infty} x_{n}(t)$.
Problem 716. Proposed by D. M. Batinetu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania.

Prove that if $m \geq 0$, then in any triangle $A B C$, the following holds:

$$
\cot ^{m+1} \frac{A}{2}+\cot ^{m+1} \frac{B}{2}+\cot ^{m+1} \frac{C}{2} \geq 3^{(m+3) / 2}
$$

Problem 717. Proposed by Pedro H. O. Pantoja (student), Natal-RN, Brazil.

Let $A$ and $B$ be $4 \times 4$ matrices with integer entries such that $\operatorname{det} A+$ $\operatorname{det} B=0$. Prove that

$$
|\operatorname{det}(A+2 B)+15 \operatorname{det} A-3 \operatorname{det}(A+B)-\operatorname{det}(A-B)|
$$

is a multiple of 6 .

Problem 718. Proposed by Jose Luis Diaz-Barrero, BARCELONA TECH, Barcelona, Spain.

Let $x, y, z$ be positive real numbers and $n$ a positive integer. Show that

$$
\frac{1}{2}\left(\sum_{\text {cyclic }} \frac{x^{n+2}+y^{n+2}}{x^{n}+y^{n}}\right) \geq \sum_{\text {cyclic }} \frac{x^{2} y^{2}}{x^{2}+y^{2}} .
$$

Problem 719. Proposed by Jose Luis Diaz-Barrero, BARCELONA TECH, Barcelona, Spain.

Find, with proof, all real solutions to the equation

$$
3 \cdot 1331^{x}+4 \cdot 363^{x}=34 \cdot 99^{x}+77 \cdot 27^{x} .
$$

Problem 720. Proposed by Jeremy Wade, Pittsburg State University, Pittsburg, KS.

Let $\left\{n_{k}\right\}$ be an increasing sequence of positive integers. Prove that

$$
\sum_{k=1}^{\infty} \frac{n_{k+1}-n_{k}}{n_{k+1}}=\infty
$$

Problem 721. Proposed by the editor.
The Northeastern Section meeting of the MAA in November 2012 was devoted to mathematical problems and problem solving. It was such a good meeting that the following alphametic was created. Find the two solutions base 8 to the alphametic. (Each different letter is replaced by a different digit so that the summation is true.)

$$
\begin{array}{rccc} 
& \mathrm{N} & \mathrm{E} & \mathrm{~S} \\
+ & \mathrm{M} & \mathrm{~A} & \mathrm{~A} \\
\hline \mathrm{G} & \mathrm{O} & \mathrm{O} & \mathrm{D}
\end{array}
$$

## SOLUTIONS TO PROBLEMS 689-698

Problem 689. Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.

Prove that there are infinitely many triangular numbers which are the sum of three pentagonal numbers.

Solution by Michelle Zeng, Fort Hays State University, Hays, KS.
Let $T_{n}$ represent the $n^{\text {th }}$ triangular number and $P_{n}$ the $n^{\text {th }}$ pentagonal number. For any integer $k \geq 2$, if $n=k^{2}$, then the sum $P_{n-k}+P_{n}+P_{n+k}$ equals $T_{3 n}$. To see this, we use the formulas for pentagonal and triangular numbers.

$$
\begin{aligned}
& P_{n-k}+P_{n}+P_{n+k} \\
= & \frac{3(n-k)^{2}-(n-k)}{2}+\frac{3 n^{2}-n}{2}+\frac{3(n+k)^{2}-(n+k)}{2} \\
= & \frac{3 n^{2}-6 n k+3 k^{2}-n+k+3 n^{2}-n+3 n^{2}+6 n k+3 k^{2}-n-k}{2} \\
= & \frac{9 n^{2}+6 k^{2}-3 n}{2} \\
= & \frac{9 n^{2}+6 n-3 n}{2} \text { since } n=k^{2} \\
= & \frac{3 n(3 n+1)}{2} \\
= & T_{3 n} .
\end{aligned}
$$

The proposer proves that $T_{12 k+5}=3 P_{4 k+2}$ and $T_{6 k+2}=3 P_{2 k+1}$.

Problem 690. Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.

Show that the sequence $91,8911,889111,88891111, \ldots$ (i.e., put one more 8 on the front and one more 1 on the end) consists solely of triangular numbers.

Solution Anna Faina and Keeli Garroutte (students), Northeastern State University, Tahlequah, OK.

Define sequences $\left\{a_{n}\right\}$ and $\left\{T_{n}\right\}$ by

$$
a_{1}=13, a_{n}=10 \cdot a_{n-1}+3 \text { for } n \geq 2
$$

and

$$
T_{n}=\frac{a_{n}\left(a_{n}+1\right)}{2} .
$$

Then $T_{1}=91, T_{2}=8911$, and $\left\{T_{n}\right\}$ is the automatically-generated sequence, and it is written in the form of a triangular number. In particular, it is $T_{n}$, with $n=133 \ldots 3$.

Also solved by Wade Combs (student), Eastern Kentucky University, Richmond, KY; Missouri State University Problem Solving Group, Missouri State University, Springfield, MO; Michelle Zeng, Fort Hays State University, Hays, KS; Ioan Viorel Codreanu, Satulung, Maramures, Romania; and the proposer.

Problem 691. Proposed by Tom Moore, Bridgewater State University, Bridgewater, MA.

Which pentagonal numbers are the positive integer power of a single prime?
Solution by the proposer.
Pentagonal numbers have the form $P_{n}=\frac{n(3 n-1)}{2}$. Suppose that $P_{n}=$ $p^{a}$, where $p$ is a prime and $a$ is a positive integer. Then we have $n(3 n-1)=$ $2 p^{a}$. Since $n$ and $3 n-1$ are relatively prime, it follows that 2 divides one of them but not both.

- If 2 divides $n$, then $n=2 q$, and we get $2 q(6 q-1)=2 p^{a}$, so that $q(6 q-1)=p^{a}$. Since $q$ and $6 q-1$ are relatively prime, $p$ divides one of them but not both. The other one must be one.
- If $q=1$, then $6 q-1=5$, and $p=5$, with $a=1$.
- If $6 q-1=1, q$ is not an integer.

Thus, in the case of $n$ even, the only solution is 5 .

- If 2 divides $3 n-1$, then $n$ is odd, say $n=2 q+1$. Then $n(3 n-1)=$ $(2 q+1)(3 q+1)=p^{a}$. By the Euclidean Algorithm, the factors $(2 q+1)$ and $(3 q+1)$ are relatively prime, which forces two cases.
- If $2 q+1=1$ and $3 q+1=p^{a}$, then $q=0$ and $1=p^{a}$, causing $a=0$, but we assumed $a>0$.
- If $2 q+1=p^{a}$ and $3 q+1=1$, then $q=0$ and $1=p^{a}$, which causes $a=0$, but we assumed $a>0$.

Thus, the only pentagonal prime power is 5 .
Partial solution by Paige Lewis (student), Northeastern State University, Tahlequah, OK. Full Solutions by Wade Combs (student), Eastern Kentucky University, Richmond, KY; Michelle Zeng, Fort Hays State University, Hays, KS; and Ioan Viorel Codreanu, Satulung, Maramures, Romania.

Problem 692. Proposed by D. M. Batinetu-Giurgiu, Matei Basarab
National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania.

Let

Calculate

$$
\begin{gathered}
x_{n}=\sqrt[n]{\sqrt{2!} \cdot \sqrt[3]{3!} \cdots \sqrt[n]{n!}} \\
\lim _{n \rightarrow \infty}\left[\frac{(n+1)^{2}}{x_{n+1}}-\frac{n^{2}}{x_{n}}\right]
\end{gathered}
$$

Solution by Anastasios Kotronis, Athens, Greece.
Set $z_{n}=\frac{n^{n}}{x_{n}^{n}}$. Then

$$
\begin{equation*}
\frac{z_{n+1}}{z_{n}}=\left(1+\frac{1}{n}\right)^{n}(n+1) \cdot \frac{x_{n}^{n}}{x_{n+1}^{n+1}}=\left(1+\frac{1}{n}\right)^{n} \cdot \frac{n+1}{(n+1)!\frac{1}{n+1}} \rightarrow e^{2} \tag{43}
\end{equation*}
$$

by Stirling's formula, and

$$
\begin{align*}
\lim _{n \rightarrow \infty} z_{n}^{1 / n} & =\exp \left(\lim _{n \rightarrow \infty} \frac{\ln z_{n}}{n}\right)=\exp \left(\lim _{n \rightarrow \infty} \ln \frac{z_{n+1}}{z_{n}}\right)  \tag{44}\\
& =\exp \left(\ln \lim _{n \rightarrow \infty} \frac{z_{n+1}}{z_{n}}\right)=e^{2} \tag{45}
\end{align*}
$$

by Cezaro Stolz. Furthermore,

$$
\begin{equation*}
\left(\frac{(n+1) z_{n+1}^{\frac{1}{n+1}}}{n z_{n}^{\frac{1}{n}}}\right)=\left(1+\frac{1}{n}\right)^{n} \cdot \frac{z_{n+1}}{z_{n} z_{n+1}^{\frac{1}{n+1}}} \rightarrow e \tag{46}
\end{equation*}
$$

by (1) and (2), so

$$
\begin{aligned}
\lim _{n \rightarrow \infty}\left[\frac{(n+1)^{2}}{x_{n+1}}-\frac{n^{2}}{x_{n}}\right]= & \lim _{n \rightarrow \infty} z_{n}^{\frac{1}{n}} \cdot \frac{\frac{(n+1) z_{n+1}^{\frac{1}{n+1}}}{n z_{n}^{n}}-1}{\ln \frac{(n+1) z_{n+1}^{\frac{1}{n+1}}}{n z_{n}^{n}}} \\
& \cdot \ln \left(\frac{(n+1) z_{n+1}^{\frac{1}{n+1}}}{n z_{n}^{\frac{1}{n}}}\right)^{n} \\
= & e^{2} \cdot\left(\lim _{x \rightarrow 0} \frac{e^{x}-1}{x}\right) \cdot 1 \\
= & e^{2}
\end{aligned}
$$

where the next-to-last equality follows from (2) and (3).
Also solved by the proposer.

Problem 693. Proposed by D. M. Batinetu-Giurgiu, Matei Basarab
National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzau, Romania.

Prove that if $x, y, m>0$, then in any triangle $A B C$, the following holds

$$
\begin{aligned}
& \frac{\tan ^{2 m+1} \frac{A}{2}}{\left(x \cot \frac{B}{2}+y \cot \frac{C}{2}\right)^{m}}+\frac{\tan ^{2 m+1} \frac{B}{2}}{\left(x \cot \frac{C}{2}+y \cot \frac{A}{2}\right)^{m}} \\
+\quad & \frac{\tan ^{2 m+1} \frac{C}{2}}{\left(x \cot \frac{A}{2}+y \cot \frac{B}{2}\right)^{m}} \geq \frac{(4 R+r) r^{m}}{(x+y)^{m} s^{m+1}},
\end{aligned}
$$

where $R$ is the circumradius, $r$ is the inradius, and $s$ is the semiperimeter.
Solution by the proposers.
We have
$U=\sum \frac{\tan ^{2 m+1} \frac{A}{2}}{\left(x \cot \frac{B}{2}+y \cot \frac{C}{2}\right)^{m}}=\sum \frac{\tan ^{m+1} \frac{A}{2}}{\left(x \cot \frac{A}{2} \cot \frac{B}{2}+y \cot \frac{A}{2} \cot \frac{C}{2}\right)^{m}}$,
and by Radon's inequality, we deduce that

$$
U \geq \frac{\left(\sum \tan \frac{A}{2}\right)^{m+1}}{(x+y)^{m}\left(\sum \cot \frac{A}{2} \cot \frac{B}{2}\right)^{m}}
$$

From the identities

$$
\sum \tan \frac{A}{2}=\frac{4 R+r}{s} \text { and } \sum \cot \frac{A}{2} \cot \frac{B}{2}=\frac{4 R+r}{r}
$$

we obtain the conclusion.
Also solved by Ioan Viorel Codreanu, Satulung, Maramures, Romania.

Problem 694. Proposed by Jose Luis Diaz-Barrero, BARCELONA TECH, Barcelona, Spain.

Let $a, b, c$ be positive real numbers such that $a+b+c=1$. Prove that

$$
\frac{a^{3}}{a+b}+\frac{b^{3}}{b+c}+\frac{c^{3}}{c+a} \geq \frac{1}{6} .
$$

Solution by the proposer.
From the inequality $(a-b)^{2}(3 a+b) \geq 0$, it follows that $3 a^{3} \geq 5 a^{2} b-$ $a b^{2}-b^{3}$. Adding $5 a^{3}$ to both sides, we obtain $8 a^{3} \geq 5 a^{3}+5 a^{2} b-a b^{2}-b^{3}$, which is equivalent to

$$
\frac{a^{3}}{a+b} \geq \frac{5 a^{2}-b^{2}}{8}
$$

Likewise, we get

$$
\frac{b^{3}}{b+c} \geq \frac{5 b^{2}-c^{2}}{8} \text { and } \frac{c^{3}}{c+a} \geq \frac{5 c^{2}-a^{2}}{8} .
$$

Adding these three inequalities yields

$$
\frac{a^{3}}{a+b}+\frac{b^{3}}{b+c}+\frac{c^{3}}{c+a} \geq \frac{a^{2}+b^{2}+c^{2}}{2}
$$

Applying Cauchy's inequality to the vectors $\mathbf{u}=\langle a, b, c\rangle$ and $\mathbf{v}=\frac{1}{\sqrt{6}}\langle 1,1,1\rangle$, we get

$$
\frac{a^{2}+b^{2}+c^{2}}{2} \geq\left(\frac{a}{\sqrt{6}}+\frac{b}{\sqrt{6}}+\frac{c}{\sqrt{6}}\right)^{2}=\frac{1}{6}
$$

with the last equality because $a+b+c=1$.
Also solved by Pedro H. O. Pantoja (student), Natal-RN, Brazil; and Ioan Viorel Codreanu, Satulung, Maramures, Romania.

Problem 695. Proposed by Jose Luis Diaz-Barrero, BARCELONA TECH, Barcelona, Spain.

Let $a>1$ be a real number such that $\{a\}+\{1 / a\}=1$, where $\{x\}$ represents the fractional part of $x$. [Note that the golden ratio $(1+\sqrt{5}) / 2$ has this property.] Compute

$$
\lim _{n \rightarrow \infty} \sum_{k=1}^{n}\left(\left\lfloor a^{k}\right\rfloor+\left\lfloor\frac{1}{a^{k}}\right\rfloor+\lfloor a\rfloor+\left\lfloor\frac{1}{a}\right\rfloor+2\right)^{-1}
$$

where $\lfloor x\rfloor$ denotes the floor of $x$.
Solution by the proposer.
When $m$ is an integer, any solution $a$ of the equation $a^{2}-m a+1=0$ satisfies

$$
\{a\}+\{1 / a\}=1
$$

We have

$$
\begin{aligned}
& \sum_{k=1}^{n}\left(\left\lfloor a^{k}\right\rfloor+\left\lfloor\frac{1}{a^{k}}\right\rfloor+\lfloor a\rfloor+\left\lfloor\frac{1}{a}\right\rfloor+2\right)^{-1} \\
= & \sum_{k=1}^{n}\left(\left\lfloor a^{k}\right\rfloor+\left\lfloor\frac{1}{a^{k}}\right\rfloor+\left\{a^{k}\right\}+\left\{\frac{1}{a^{k}}\right\}+\lfloor a\rfloor+\left\lfloor\frac{1}{a}\right\rfloor+\{a\}+\left\{\frac{1}{a}\right\}\right)^{-1} \\
= & \sum_{k=1}^{n}\left(a^{k}+\frac{1}{a^{k}}+a+\frac{1}{a}\right)^{-1} \\
= & \sum_{k=1}^{n} \frac{a^{k}}{\left(a^{k-1}+1\right)\left(a^{k+1}+1\right)} .
\end{aligned}
$$

Splitting into partial fractions, we have

$$
\frac{a^{k}}{\left(a^{k-1}+1\right)\left(a^{k+1}+1\right)}=\frac{A}{a^{k-1}+1}+\frac{B}{a^{k+1}+1},
$$

from which it follows that

$$
A=-B=\frac{a}{a^{2}-1} .
$$

Thus,

$$
\begin{aligned}
\sum_{k=1}^{n} \frac{a^{k}}{\left(a^{k-1}+1\right)\left(a^{k+1}+1\right)} & =\frac{a}{a^{2}-1} \cdot \sum_{k=1}^{n}\left(\frac{1}{a^{k-1}+1}-\frac{1}{a^{k+1}+1}\right) \\
& =\frac{a}{a^{2}-1}\left(\frac{1}{2}+\frac{1}{a+1}-\frac{1}{a^{n+1}+1}\right)
\end{aligned}
$$

Finally,
$\lim _{n \rightarrow \infty} \sum_{k=1}^{n}\left(\left\lfloor a^{k}\right\rfloor+\left\lfloor\frac{1}{a^{k}}\right\rfloor+\lfloor a\rfloor+\left\lfloor\frac{1}{a}\right\rfloor+2\right)^{-1}=\frac{a}{a^{2}-1}\left(\frac{1}{2}+\frac{1}{a+1}\right)$.
Problem 696. Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.
Let $\alpha>4$ be a real number. Find in closed form the value of the sum

$$
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\lfloor\sqrt{n+m}\rfloor^{\alpha}}
$$

where $\lfloor x\rfloor$ denotes the floor of $x$.
Solution by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.

Note that $\lfloor\sqrt{n+m}\rfloor=k$ is equivalent to $k^{2} \leq n+m<(k+1)^{2}$. The number of pairs of positive integers $(n, m)$ such that $n+m=j$ is $j-1$. Therefore the number of pairs such that $\lfloor\sqrt{n+m}\rfloor=k$ is

$$
\sum_{j=k^{2}}^{k^{2}+2 k}(j-1)=2 k^{3}+3 k^{2}-k-1
$$

Hence,

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(\lfloor\sqrt{n+m}\rfloor)^{\alpha}} \\
= & \sum_{k=1}^{\infty} \frac{2 k^{3}+3 k^{2}-k-1}{k^{\alpha}} \\
= & 2 \sum_{k=1}^{\infty} \frac{1}{k^{\alpha-3}}+3 \sum_{k=1}^{\infty} \frac{1}{k^{\alpha-2}}-\sum_{k=1}^{\infty} \frac{1}{k^{\alpha-1}}-\sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} \\
= & 2 \zeta(\alpha-3)+3 \zeta(\alpha-2)-\zeta(\alpha-1)-\zeta(\alpha),
\end{aligned}
$$

where $\zeta(x)$ is the Riemann zeta function.
Also solved by Anastasios Kotronis, Athens, Greece; and the proposer.

Problem 697. Proposed by Pedro H. O. Pantoja (student), Natal-RN, Brazil.

Let $A$ and $B$ be $2 \times 2$ matrices with real entries such that $\operatorname{Tr}(B A)=0$ and
$\operatorname{Det}\left(A \cdot A^{t}+B^{t} \cdot B\right)+\operatorname{Det}\left(A \cdot A^{t}-B^{t} \cdot B\right)+\operatorname{Det}\left(2(A B)^{2}\right)=-1$.
Compute $(A B)^{2012}$.
Solution by the proposer.
Let $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$ and $B=\left[\begin{array}{cc}x & y \\ z & w\end{array}\right]$ with real entries. We have $\operatorname{det}(A+B)+\operatorname{det}(A-B)=2 \operatorname{det} A+2 \operatorname{det} B$.
Thus

$$
\begin{aligned}
\operatorname{det}\left(A A^{t}+B^{t} B\right)+\operatorname{det}\left(A A^{t}-B^{t} B\right) & =2 \operatorname{det}\left(A A^{t}\right)+2 \operatorname{det}\left(B^{t} B\right) \\
& =2 \operatorname{det} A \operatorname{det} A^{t}+2 \operatorname{det} B \operatorname{det} B^{t} \\
& =2\left[(\operatorname{det} A)^{2}+(\operatorname{det} B)^{2}\right] \\
& \geq 4 \operatorname{det} A \operatorname{det} B \\
& =4 \operatorname{det}(A B) .
\end{aligned}
$$

By hypothesis,
$-1-4 \operatorname{det}(A B)^{2}=\operatorname{det}\left(A A^{t}+B^{t} B\right)+\operatorname{det}\left(A A^{t}-B^{t} B\right) \geq 4 \operatorname{det}(A B)$.
Hence,

$$
\begin{aligned}
1+4(\operatorname{det} A B)^{2} & \leq-4 \operatorname{det} A B \\
(2 \operatorname{det} A B+1)^{2} & \leq 0 \\
\operatorname{det} A B & =-\frac{1}{2}
\end{aligned}
$$

By the Cayley-Hamilton Theorem,

$$
\begin{aligned}
(A B)^{2}-\operatorname{tr}(A B) \cdot(A B)+(\operatorname{det} A B) \cdot I & =0 \\
(A B)^{2}-\operatorname{tr}(B A) \cdot A B+(\operatorname{det} A B) \cdot I & =0
\end{aligned}
$$

Since $\operatorname{tr}(B A)=0,(A B)^{2}+(\operatorname{det} A B) \cdot I=0$, or $(A B)^{2}=\left[\begin{array}{cc}1 / 2 & 0 \\ 0 & 1 / 2\end{array}\right]$.
Finally, $(A B)^{2012}=\left[\begin{array}{cc}\frac{1}{2^{1006}} & 0 \\ 0 & \frac{1}{2^{1006}}\end{array}\right]$.

Problem 698. Proposed by Mathew Cropper, Eastern Kentucky University, Richmond, KY.

Start with an $8 \times 8$ array of squares. Remove the central $4 \times 4$ square. Note that 48 squares remain and 48 is divisible by 3 . Determine the number of ways in which the 48 squares can be tiled with tiles made up of three squares in the shape of an L as indicated below.


Solution by Wade Combs (student), Eastern Kentucky Univ., Richmond, $K Y$.

We refer to three squares in the shape of an L as an L -shaped tromino. First, we determine whether the 48 squares can be completely tiled with 16 L -shaped trominos. Observe that a $3 \times 2$ block of squares can be tiled with two L -shaped trominos put together in the following way:


By symmetry, the two L-shaped trominos can actually be put into the $3 \times 2$ block in two different ways. Now, the 48 squares in our figure can be broken into eight $3 \times 2$ blocks in two different ways as illustrated below.

Regard each of these $3 \times 2$ blocks as an independent event. Then, by the Principle of Multiplication, each of the ways to break up the 48 squares into $3 \times 2$ blocks can be tiled by L-shaped trominos in $2^{8}=256$ ways. Thus there are $2 \cdot 256=512$ tilings that involve breaking the 48 squares into $3 \times 2$ blocks.


Now we must determine the number of ways the 48 squares can be tiled without relying on $3 \times 2$ blocks. Clearly, placing an L-shaped tromino on one of the outer corners of the removed $4 \times 4$ center square will prevent any tiling using $3 \times 2$ blocks. It now follows that there is a unique tiling of the 48 squares starting from this tromino. In fact, all four corners must be covered by an L-shaped tromino with one behind it and back-to-back trominos in the middle of the sides as shown below.


Hence there are 513 ways to tile the 48 squares with L-shaped trominos.
Also solved by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO; Ethan Peck and Mitchell Ernsten (students), Northeastern State University, Tahlequah, OK; Alexander Bogomolny, Cut the Knot Software; and the proposer.

# Kappa Mu Epsilon News 

Edited by Peter Skoner, Historian

## Updated information as of November 2012

Send news of chapter activities and other noteworthy KME events to
Peter Skoner, KME Historian
Saint Francis University
117 Evergreen Drive, 313 Scotus Hall
Loretto, PA 15940
or to
pskoner@francis.edu

## Installation Report

Tennessee Zeta Chapter
Lee University
The installation of the Tennessee Zeta Chapter of Kappa Mu Epsilon was held in the Lecture Hall of the Science and Mathematics Complex on the campus of Lee University in Cleveland, TN on Monday, November 5, 2012, at 6:00 P.M. The meeting was conducted by Dr. Blayne Carroll, and the installing officer was National President Ron Wasserstein. Dr. Carroll is already a member of KME. In addition, Caroline Maher-Boulis was initiated as a faculty member. The following students were initiated as the charter members. (Those who were also installed as officers are noted below).

Julie A. Hardesty (Chapter President)
Erica Swindle (Vice President)
Sarah Alexandra Dawe (Secretary)
Latishua Overton (Treasurer)
Anna M. Clay
Hollie Marie German
Brittany Kanerva
Nicholas B. Ramsey
Sharise Riether
Tyler Smith
Michael Odell Yokosuk

Dr. Carroll was installed as the corresponding secretary and Dr. MaherBoulis as the faculty sponsor. Faculty members Jerry Adams and Bob Griffith also attended the proceedings, as did numerous family members and guests. Prior to the installation and initiation, Dr. Wasserstein presented a talk entitled "What Probability and Forrest Gump Teach Us About the Tennessee Lottery." After the ceremony, a wonderful spread of food, including desserts from Egypt (care of Dr. Maher-Boulis), was enjoyed by all the participants.

Installation of the Tennessee Zeta Chapter at Lee University on November 5, 2012.


Pictured above from left to right are: Dr. Wasserstein, Dr. MaherBoulis, Nicholas Ramsey, Brittany Kanerva, Julie Hardesty, Erica Swindle, Sarah Dawe, Latishua Overton and Dr. Carroll.

## Chapter News

## AL Alpha - Athens State University

Chapter President - Kathryn Henderson; 13 Current Members, 15 New Members
Other Spring 2012 Officers: Julie Hill, Vice President; Paige Riggs, Secretary; and Patricia Glaze, Corresponding Secretary and Faculty Sponsor.
New Initiates - Mallory Alberti, Ashley Leann Avans, Meagan Danielle Box, Robin LaShay Brown, Kalie Michelle Godwin, Jessica Kathryn Henderson, Julie Marie Hill, Matthew Keith McHargue, Ariel Marie Murray, Brittany Leighann Peak, James Michael Alexander Ponder, Paige Spiller Riggs, Lauren Walker, Tiffany Diane Whitten, and David J. Williams.

## AL Gamma - University of Montevallo

Corresponding Secretary - John David Herron; 4 New Members
New Initiates - Katherine Burks, Cecilla Johnson, John Petters, and Vincent Rodriguez.
AL Zeta - Birmingham-Southern College
Corresponding Secretary - Bernadette Mullins; 21 New Members
New Initiates - Linda Amaya, Allison Boyd, Andrew Conner, Allyson Cox, Kaitlyn Flagg, Jennifer Frego, Marissa Gacek, George Paterson Graham, Nathan Haywood, Alexandra Hunsucker, Nelly Kaneza, Raeann Lamere, Ruo Li, Courtney Mauck, Erica Ogle, Jacob Porch, Stephen Russell, Aaron Sherrill, Whitney Smith, Lindsay Erin Wainwright, and Junzhu Wang.

## AL Theta - Jacksonville State University

Chapter President - Nicholas Charles; 50 Current Members, 24 New Members
Other Spring 2012 Officers: Noel Overton, Vice President; Allison Clark, Secretary; Brittney Kingery, Treasurer; and Dr. David Dempsey, Corresponding Secretary and Faculty Sponsor
On March 5, 2012, the Alabama Theta chapter initiated 24 new student members who received their certificates, pins, and honor cords. The afternoon ceremony was held on the 11th floor of Houston Cole Library. During the April 12, 2012 chapter meeting, new officers were elected.
New Initiates - Kyle Mason Bradshaw, Christopher Burdette, Nicholas Joseph Charles, Brittney G. Cole, Karen Davidson, Jennifer Gurley, Alexandra N. Hall, Jennifer A. Holthof, Joi Jacobs, Mary Kathryn Killion, Brittney Lauren Kingery, Tiffany Nicole Mcllwain, Matthew Lee Miller, Laura Francis Morris, Meagan Ingram Morrow, Gabriel Phillips, LaShandra Rivers, Zachary Mark Searels, Emily Alyss Shelton, Katie LeAnna Smith, Alecia Brooke Stovall, Ashley Elizabeth Trotter, Ralph Rogers Wheeler III, and Ashley Danielle Wiggins.

## CO Beta - Colorado School of Mines

Corresponding Secretary -Terry Bridgman; 21 New Members
New Initiates - Shad Allen, Abigail Branch, Evan Breikss, Grant Emery, Jennifer Fantich, Julianne Fantich, Rebecca Ferguson, Michelle Griffith, Kaycie Lane, Brian Lindstadt, Kate Lyssy, Jacqueline Mays, Maxwell Mazzocchi, Scott McClary, Kathryn Monopoli, Lindsay Parr, Samantha Powell, Aaron Troyer, Caroline Woody, Kautsar Zamanuri, and Mengnan Zhou.
CT Beta - Eastern Connecticut State University
Corresponding Secretary - Christian L. Yankov; 10 New Members Other Spring 2012 Officer: Mizan R. Khan, Treasurer.
New Initiates - Paula Bailey, Sarah Chouinard, Kyle Courtemanche, Daniel Delany, Sarah Joy, Christopher Laurich, Lisa Lawrence, Michael McCarthy, Jessalyn Salisbury, and Nicholas Vitterito.

## FL Beta - Florida Southern College

Chapter President - Calaam Berry; 15 Current Members, 8 New Members Other Spring 2012 Officers: Brian Covello, Vice President; Kate Stromberg, Secretary and Treasurer; and Shawn Hedman, Corresponding Secretary and Faculty Sponsor.
New Initiates - Calaam J. Berry, Chelsea Michelle Bradshaw, Kenneth W. Curry, Sarah E. Dittmer, Beth N. Lawson, Nathaniel P. Stambaugh, Katherine A. Stromberg, and Zhuo Yang.
GA Beta - Georgia College \& State University
Corresponding Secretary - Rodica Cazacu; 1 New Member
New Initiate - Katherine E. Austin.

## HI Alpha - Hawaii Pacific University

Chapter President - Janet Calkins; 9 Current Members, 4 New Members Other Spring 2012 Officers: Isaac Kim, Vice President; Laura Mitchell, Secretary; Randolf Uclaray, Treasurer; Tara Davis, Faculty Sponsor; and Tara Davis and Chez Neilson, Corresponding Secretaries.
An initiation ceremony took place on May 2, 2012. On February 17, 2012 there was a meeting to elect officers, play Dominion, and work on the Pentagon problems.
New Initiates - Cami Maria Atwood, Kevin Goo, Lauren Smoot, and Matthew David Kristofer Troglia.

## IA Alpha - University of Northern Iowa

Chapter President - Adam Feller, 35 Current Members, 9 New Members Other Spring 2012 Officers: Renee Greiman, Vice President; Hannah Andrews, Secretary; Lucas Thomas, Treasurer; and Mark D. Ecker, Corresponding Secretary and Faculty Sponsor.

Our first spring KME meeting was held on February 22, 2012 at Professor Mark Ecker's residence where student member David Ta talked about his KME paper entitled "Blackjack Winning Strategies." Our second meeting was held on March 28, 2012 at Professor Doug Mupasiri's residence where student member Renee Grieman presented her paper on "Finding the Center of a Circle." Student member Kassaundra Young addressed the spring initiation banquet with "Determinants of Property Crime." Our banquet was held at Godfather's Pizza in Cedar Falls on April 25, 2012 where nine new members were initiated.
New Initiates - Kaitlin Bruden, Stacy Carnahan, Jeremy Meyer, Madelyn Mosiman, Emily Stumpff, Byron Tasseff, Katie Wilford, Erik Wolter, and Kassaundra Young.

## IA Delta - Wartburg College

Chapter President - Daniel Mysnyk; 38 Current Members, 20 New Members
Other Spring 2012 Officers: Adam Kucera, Vice President; Nicole Boesenberg, Secretary; Alyssa Hanson, Treasurer; Brian Birgen, Corresponding Secretary; and Dr. Joy Becker, Faculty Sponsor.
In March, twenty new initiates were welcomed at our annual banquet and initiation ceremony, which was combined with the initiation ceremony for Sigma Pi Sigma, the Physics Honor Society. Our speakers were Jade Holst Groen and Saran Andreesen Olsem, Wartburg Alums and math majors who work at Rockwell Collins in Cedar Rapids. In May, together with the Physics and Computer Science clubs, we hosted the departmental end of the year picnic.
New Initiates - Nicole Boesenberg, Marcela Correa, Oliver de Quadros, Courtney Egts, Ashley Freese, Alyssa Hanson, Allison Huedepohl, Loni Kringle, Matthew Kristensen, Adam Kucera, Joshua Lehman, Jennifer Lynes, Paul Masterson, Kristen Nielsen, Derek Norton, Nevena Ostojic, Megan Puls, Samuel Read, Alexander Schaefer, and Ellen Schwarz.

## IL Delta - University of St. Francis

Corresponding Secretary - Richard J. Kloser; 11 New Members
New Initiates - Kimberly A. Askew, Nicholas A. Blatti, Michael C. Cisarik, William J. Garcia, Sean M. Hansen, Rebecca M. Max, Scott A. Ratzburg, Kyle P. Schomer, Ashley N. Thompson, Feng Wang, and Allison E. Zagrzebski.

## IL Zeta - Dominican University <br> Chapter President - Daniel Dziarkowski; 40 Current Members, 12 New Members <br> Other Spring 2012 Officers: Lisa Gullo, Vice President; Claudia Ramirez, Secretary; Magdalena Kolek, Treasurer; and Aliza Steurer, Corresponding Secretary and Faculty Sponsor <br> The Illinois Zeta Chapter of KME operates together with Dominican University's Math Club. Thus, the above officers are members of KME or Math Club (or both) and the activities were prepared by both groups. Spring 2012 was an exciting semester. We organized two faculty presentations, open to the entire university community, about how mathematics is used everyday. We held a "Smarties for Smarties" competition in which students around the university could win Smarties candy by completing problems from basic algebra through calculus. As we do every spring, we passed out free slices of pie on Pi Day. To conclude a great semester, we held a party in April so that students could take a break from end-of-theyear activities. <br> New Initiates - Courtney Adams, Drew Adduci, Lisa Gullo, Mark Hodges Ph.D., Magdalena Kolek, Ivonne Machuca, Joanna Sasara, Sara Seweryn, Willa Skeehan, Demirhan Tunc M.S., Michael Wesolowski, and Matthew Zitkus.

## IL Theta - Benedictine University

Chapter President - Natalia Poniatowska; 15 Current Members, 7 New Members
Other Spring 2012 Officers: Betsy Williams, Vice President; Sandra TovalinSchmidt, Secretary; Christina Rubik, Treasurer; Dr. Anthony DeLegge, Faculty Sponsor, and Dr. Thomas Wangler, Corresponding Secretary Some of our selected activities for the year included: A math competition, where students from college algebra, trigonometry, and calculus tested themselves against others for gift cards and bragging rights. Over 50 people attended the competition this year! We had two club-sponsored guest speakers, an actuarial scientist who discussed his profession, and a professor sharing some of his research on "The Price is Right." Pi Day was held March 14, 2012 and included a pie decorating contest, pie eating contest, and "Pi a Teacher in the Face," where students could donate money to the professor they would most like to see a whipped cream pie thrown in his/her face. The event raised significant money for the club and was a lot of fun... except for the one who got pied, of course.
New Initiates - Amreen Barde, Jennifer Cagney, Briana Grenke, Margaret Muller, Kiran Munir, Jessica Rovner, and Mehak Sandhu.

## IN Alpha - Manchester University

Corresponding Secretary - Jim Brumbaugh-Smith; 7 New Members
New Initiates - Sarah Carman, Matthew Harter, David Herrmann, Sarah Leininger, Stephanie Miller, Eva Sagan, and Marcus Wyatt.

## IN Delta - University of Evansville

Corresponding Secretary - Adam Salminen; 27 New Members
New Initiates - Mason T. Blankenship, Haylee M. Bryant, Brandon D. Causey, Efoise Eigbobo, Craig M. Gore, Matthew W. Harris, Christopher D. Harrison, Logan A. Herwehe, Christine L. Hopp, Brianna F. Kelley, Lisa M. Letterman, Marianne Kay Marcotte, Christopher J. Matlak, Chase V. Miller, Samuel J. Mires, Heather Passey, Elizabeth N. Reis, Drew A. Reisinger, Jake S. Schwartz, Alex J. Schwinghamer, Justin T. Simerly, Dalton T. Snyder, Ezequiel F. Suar, Nicholas J. Takebayashi, Melissa Jo Thompson, Alan Vandagriff, and Raymond Charles Watkin.

## KS Alpha - Pittsburg State University

Corresponding Secretary - Tim Flood; 46 New Members
New Initiates - Calin Archer, Megan Bertone, Dale Brauer, Silas Bloesser, Kylie Breitenbach, Aaron Burns, Jennifer Butler, Tanner Cheney, Andrew Clark, Jean Coltharp, Jacob Crazybear, Aaron Flood, Aisha Ford, Danielle Frey, Andrew Frink, Samuel Hardy, Blake Hartman, Amanda Hay, Hannah Hays, Nicholas Hicks, Chelsea Johnson, Michael Kephart, Song-Yi Kim, Aaron Kolich, Jae Min Lee, Reine Loflin, Daniel Macclymont, Matthew McDonald, Michael McGill, Jonathan McPherson, Kira Moravec, Kirsten Nemecek, Christopher Nusbaum, Rachael Osborn, Julie Oswald, Savanna O’Toole, Leslie Root, Dennis Rosser, Graciela Saldivar, Jordan Soucie, Edwin Stremel, Timothy Walker, Brian Welch, Chelsea Welch, William White, and Christopher Yarnell.

## LA Delta - University of Louisiana at Monroe

Corresponding Secretary - Brent Strunk; 27 New Members
New Initiates - Alexandra Babin, Brandie Book, Quintin Brown, Brittany Carson, Daniel Cordell, Daniel Graves, Sara Gwerder, Dwayne Hammer, Ashlea Hart, Hannah Holbrook, Tracie Johnson, Brittany Kuethe, Emily Lara, Morgan Ledbetter, Jenna Lee, Ali Lucas, Tenerica Madison, Jared Marquis, William Morrison, Demi Morvant, Garrett Owens, Lauren Pardue, Rachel Robinson, Jessica Sims, Allison Stevens, Anthony Viramontez, and Shelby Wells.

## MA Beta - Stonehill College

Chapter Pres. - Cortney Logan; 11 Current Members, 12 New Members
Other Spring 2012 Officers: Laura Bercume, Vice President; Kathleen
Zarnitz, Secretary; and Timothy Woodcock, Corresponding Secretary and Faculty Sponsor

The Massachusetts Beta Chapter of Kappa Mu Epsilon at Stonehill College initiated 12 new members on March 30, 2012, including one faculty member. Prior to the ceremony, new initiates, active members, and family shared a buffet dinner in the College's Cleary Dining Room. Over the first week of May, our Chapter's student members held drop-by help sessions for calculus students, preparing for their final exam.
New Initiates - Corey Adams, Olivia Almeida, Daniel Bouchard, Patrick Clark, Joshua Cunningham, Lauren Hinchey, William Pellegrino, Elizabeth Perkins, Kelsey Roberts, Danielle Sutherby, Laura Sweeney, and Heiko Todt.

## MD Delta - Frostburg State University

Chapter President - Kevin Loftus; 31 Current Members, 12 New Members Other Spring 2012 Officers: Justin Good, Vice President; Jesse Otto, Secretary; Marcus Carter, Treasurer; Mark Hughes, Corresponding Secretary and Faculty Sponsor; and Frank Barnet, Faculty Sponsor

The Maryland Delta Chapter welcomed twelve new members during its Initiation Ceremony on February 26, 2012. The ceremony featured a presentation by faculty sponsor Dr. Mark Hughes on Legendre's proof of the Euler Characteristic formula. In mid-March we had a very successful Pi Day Bake Sale where nearly $\$ 100$ was raised. Our March meeting featured a joint presentation by student member Justin Zimmermann and Dr. Mark Hughes. The topic was Descartes' original approach on his Total Angular Defect Theorem. Towards the end of April we had our last meeting of the semester. Featured was a presentation by student Justin Good on the Poincaré-Hopf Theorem. This meeting also saw the election of new officers for the coming year. Kevin Loftus will continue as President with new officers: DeVonte' McGee (Vice President), Debbie Wiles (Secretary) and Meghan Voelkel (Treasurer). We finished the term with an end of semester cookout where a great time was had by all. Also worthy of note is the participation of KME member Justin Good in the First Annual Undergraduate Research Symposium here at Frostburg State University. Justin displayed a poster on his study of the curvature of Lissajous Figures. Finally, we offer best wishes to graduating seniors Justin Good, John Cupp, Marcus Carter, Richard Wall and Jesse Otto.
New Initiates - Devota Aabel, Raymond Azenadaga, Joshua Green, Joshua McDonald, DeVonte' McGee, Steven Moon, Jacob Reed, Andrew Siemann, Anna Struhar, Nicholas Torgerson, Deborah K. Wiles, and Justin C. Zimmermann.

## MD Epsilon - Stevenson University

Chapter President - Rebecca Hollins; 37 Current Members
Other Spring 2012 Officers: Kristina Pugh, Vice President; Charles Schuster, Secretary; Rachel Buchanan, Treasurer; and Dr. Christopher E. Barat, Corresponding Secretary and Faculty Sponsor
KME members actively assisted in hosting the Spring Meeting of the MD-DC-VA Section of the Mathematical Association of America. The Meeting was held in April on Stevenson's Owings Mills Campus. In May, 22 students were identified as candidates for the Fall 2012 initiation ceremony, and invitations will be sent to these students at the end of the Summer.

## MI Delta - Hillsdale College

Chapter President - Gladys Anyenya; 59 Current Members, 6 New Members
Other Spring 2012 Officers: David Montgomery, Vice President; Aubrey Annis, Secretary; Casey Gresenz, Treasurer; and Dr. David Murphy, Corresponding Secretary and Faculty Sponsor
During the Spring 2012 semester, we celebrated the 15th anniversary of our installation (April 30, 1997) and initiated 6 new members. In addition, we participated in Hillsdale College's third annual Geek Week, again hosting our Paper Airplane Challenge and taking part in the Honorama bowling tournament to raise money for Circle K , the college branch of $\mathrm{Ki}-$ wanis International that each year provides dictionaries and thesauruses to students throughout our county. Some of our members also volunteered to supervise or assist in events for the Michigan Region 9 Science Olympiad tournament hosted at Hillsdale College on March 31. On March 3, some of our faculty and student members attended the annual Michigan Undergraduate Math Conference held at Sienna Heights University in Adrian, MI. Our outreach events for the semester included co-sponsoring (with the department) several math talks, including "Can pursuing self-interest lead to benevolence? A game-theoretic look at Adam Smith's 'Invisible Hand"" by Dr. Tom Treloar (Hillsdale College), "Are the Real Numbers Really Real?" by John Derbyshire (author of Prime Obsession), and "Symmetry Groups: the mathematical connection between patterns in Moorish architecture and the artwork of M.C. Escher" by Dr. Dave Reimann (Albion College), as well as a game night following our viewing of the movie 21.

New Initiates - Abigail M. Loxton, Joshua Mirth, Megan Moss, Brett R Pasche, Alisha Pehlert, and Paulina Volosov.

## MI Epsilon - Kettering University

Chapter President - Jessi Harden (A Section); 186 Current Members
Other Spring 2012 Officers: Ryan McGuire, President elect; Bryan Coburn (A Section) and Starla Walters (B Section), Vice Presidents; Derek Hazard and Michael Steinert (A Section) and Shahnoor Amin (B Section), Secretaries; Kasey Simons (A Section), Treasurer; Boyan N. Dimitrov, Corresponding Secretary; and Ruben Hayrapetyan (Section A) and Ada Cheng (Section B), Faculty Sponsors

Kettering University enjoys an active KME Society life. In the Winter and Spring 2012, Dr. Hayrapetyan and Ada Cheng offered the traditional Pizza/Movie noon meetings. The Winter term movie, shown on March 13, was "Lost at Sea: The Search for Longitude. Ada's Spring choice was the movie Galileo's Battle for the Heavens; the story was full of history of Italy, the Catholic church and Vatican's fight with the scientists of the Renaissance. It was surprising how many discoveries in Mathematics, Physics and Astronomy are due to Galileo's work.

A new scientific mathematics Seminar was started in the Fall 2011, and continued in the Winter and Spring, despite the fact that Professor Hayrapetyan has been on vacation and sabbatical; he is the initiator and main presenter/lecturer of this Seminar on Control Theory. It is amazing how Professor Hayrapetyan makes this seemingly difficult and complicated matter look easy, understandable, and touchable. It is pleasure and joy to see how these things allow multiple applications in various hot areas as Biomathematics, Statistics, Optimization, etc.

Another new initiative this year was the Provost Series called Distinguished Faculty Speaker Series. The First lecturer, of course, was our KME Chapter founder, Professor Brian McCartin; you also can be mystified by his lecture "Mysteries of Equilateral Triangle," and be witness of the virtuosity of his speech and selection of this historically rich stuff. It can be watched either on the Kettering Mathematics Department site www.kettering.edu/math, or on You Tube www. youtube.com/ watch?v=teqyejsnryw\&feature=player_embedded\#!

In May, the KME corresponding secretary Professor Boyan Dimitrov revealed the mystery of your body internal clock in his lecture "Longer Life Through Math." His lecture also can be seen and heard on the Kettering Mathematics Department site, or on You Tube www. youtube. com/ watch?v=2GouA8Y1oJA\&feature=player_embedded. We are proud of the achievements of our senior KME faculty officers.

In the coming Fall term, Kettering University will host the 12th High School Mathematics Olympiad (by the beginning of December). Every high school student working towards a high school degree who is currently
enrolled in a public school, private school, or a home-school program can sit for the examination. The competition consists of six challenging problems and has a time limit of four hours. The problems range from "mindbenders" that require little mathematical skills to problems that require the knowledge of geometry, trigonometry and beginning calculus. No calculators are permitted for this competition. For additional information contact Dr. Ruben Hayrapetyan, Dr. Joe Salacuse, Ada Cheng, or Daniela Szatmari-Voicu, or visit www.kettering.edu/math /ketterings-math-olympiad.

## MO Alpha - Missouri State University

Chapter President - Rebecca Wood; 28 Current Members, 6 New Members
Other Spring 2012 Officers: Rachel Siemen, Vice President; Sarah Kramer, Secretary; Marissa Mullen, Treasurer; and Jorge Rebaza, Corresponding Secretary and Faculty Sponsor.

KME Seminars were held during the spring semester: February 14, 2012-Ardeshir Dalal (Economics, MSU) on the use of mathematics in economics; March 12, 2012-Sarah Wingfield (Wolfram Inc.), on mathematica in education and research; and April 24, 2012-Denny Bosch, Jacob Swett, Blake Wallace (MSU), senior seminar projects.
New Initiates - Kelsey Guemmer, Rachel Siemen, Alex Thomson, Manda Tiwari, Chelsea West, and Rebecca Wood.

## MO Zeta - Missouri University of Science and Technology

Corresponding Secretary - Dr. Vy K. Le; 39 New Members
New Initiates - Amanda Baker, Jeffrey Becker, Sean Brady, Erica Budler, Zach Ciurczak, Molly Clement, Brittany Davis, Nicholas D'Errico, Ryan Gibbs, Benjamin Glover, Lindsay Golem, Jace'Karmon Heard, Kathryn Hendricks, Katie Isbell, Courtney Johnson, Autumn Kolkmeier, Neal Mahoney, Chris Marchman, Ian McGhee, Luke Meyer, Lindsay Miller, David Muller, Madison Nahrup, John Nance, Stewart Nelson, Alicia Pajda, Brendan Proske, Christine Raczka, Matthew Robinson, Jeffrey Rodgers, Juston Sanger, Meagan Schneier, Kristina Sevy, Kyle Thicke, Neal Thomas, Ann Torack, Jacy Waldrop, Matt Whitwell, and Brandon Wolk.
MO Eta - Truman State University
Corresponding Secretary - Jason Miller; 7 New Members
New Initiates - Margaret T. Avery, Thomas M. Cassily, Courtney Cisler, Abigail M. Faron, Anne E. Fernandez, Jancee L. Jarman, and Alexandra E. Potter.

## MO Theta - Evangel University

Chapter Pres. - Joshua Forsman; 16 Current Members, 7 New Members
Other Spring 2012 Officers: Elizabeth Baumeister, Vice President; and Don Tosh, Corresponding Secretary and Faculty Sponsor

Meetings were held monthly. In January we initiated 7 new members. In March we hosted the North Central Regional convention, and one of our members, Katie Strand, received an award for one of the top four papers. In April we had our end-of-year ice cream social at the home of Don Tosh. New Initiates - Laura E. Balch, Joshua Forsman, Ryan C. Geppert, Emily Johnson, Kevin Mackey, Hope Moorhead, and Jared Strader.

## MO Iota - Missouri Southern State University

Corresponding Secretary - Chip Curtis; 11 New Members
Other Spring 2012 Off.: Rich Laird and Grant Lathrom, Faculty Sponsors New Initiates - Gayle Alverson, Bethany Burney, Benjamin Davis, Chasity Henson, Kayla N. Janson, Bidusha Poudyal, Ashley Robbins, Jacob Seaton, Timothy Stillings, Bart Starkey, and Corey Wattelet.
MO Kappa - Drury University
Chapter President - Rosalia Alcoser
Other Spring 2012 Officers: Sayan Patra, Vice President; Cynthia Lombardo, Secretary; Mat Duncan, Treasurer; and Carol Browning, Corresponding Secretary.

Last year we had two parties at Dr. Browning's house. Pizza was served, games were played, and an awesome time was had by all. Other events during the year included a bonfire at Dr. Sigman's where we played Apples to Apples and made s'mores; a Christmas study party with cookies and coffee; a barbecue of hamburgers in the computer lab; six movie nights; and an end-of-the-year party with shacks. In addition contests happened, conferences happened, senior talks happened, as did the KME initiation and banquet.
MO Mu - Harris-Stowe State University
Corresponding Secretary - Ann Podleski; 5 New Members
New Initiates - Jasmin Ceric, Christina Graves, Kiearea Henderson, David Marango, and Tyra Scott.

## MO Nu - Columbia College

Chapter Pres. - Tomas Horvath; 15 Current Members, 5 New Members
Other Spring 2012 Officers: Kyle Christian, Vice President; Carolyn Summers, Secretary; Ran Kim, Treasurer; and Dr. Kenny Felts, Corresponding Secretary and Faculty Sponsor
New Initiates - Rachel Garrett, Carroline Kirtley, Michael Little, Brandy Poag-Dorado, and Colleen Whalen.

## MS Alpha - Mississippi University for Women

Chapter President - Leigh Ellen Barfield; 10 Current Members, 2 New Members
Other Spring 2012 Officers: Tshering Sherpa, Vice President; Menuka Ban, Secretary; and Joshua Hanes, Treasurer, Corresponding Secretary and Faculty Sponsor
New Initiates - Leigh Ellen Barefield and Tshering Lama Sherpa.
MS Gamma - University of Southern Mississippi
Chapter President - Abbie Deselle; 6 Current Members, 6 New Members Other Spring 2012 Officers: Chasmine Flax, Vice President; Kevin Tran, Secretary; and Samuel Lyle, Corresponding Secretary and Faculty Sponsor New Initiates - Nicole Cotten, Abbie Desselle, Melissa Dyess, Chasmine Flax, Amber Sumner, and Kevin Tran.

## NC Epsilon - North Carolina Wesleyan College

Chapter President - John Williamson; 21 Current Members, 5 New Members
Other Spring 2012 Officers: Melita Lewis, Vice President; Matt Dougherty, Secretary; Samantha House, Treasurer; and Bill Yankosky, Corresponding Secretary and Faculty Sponsor.

The North Carolina Epsilon Chapter Kappa Mu Epsilon mathematics honor society initiation ceremony was held on Monday, April 2 in the Trustees Room at North Carolyn Wesleyan College. This year marked the fifth KME ceremony at NCWC and brought the total number of members up to 42 . After the ceremony two of our recent math major graduates, Daniel Moore and Andrew Webb, spoke to those in attendance about their experiences as far as part of East Carolina University's Master's program in mathematics. Daniel and Andrew are both enrolled in a special program specifically designed for those wishing to teach mathematics at the community college level. The evening was a great success.
New Initiates - Jeremy Davis, Steven Franklin, Fred Lemongo, Temple Annette Mills, and Zach Seitter.
NC Zeta - Catawba College
Corresponding Secretary - Doug Brown; 5 New Members
New Initiates - Cameron Beard, Jordan Hunsaker, Luke Kooyman, Andrew McCollister, and Suzanne Williams.

## NC Eta - Johnson C. Smith University

Chapter President - Marcia Higgins; 12 Current Members
Other Spring 2012 Officers: Merischia Griffin, Secretary; Ashley Moore, Treasurer; Dr. Lakeshia Legette, Corresponding Secretary; and Dr. Brian Hunt, Faculty Sponsor.

Kappa Mu Epsilon co-sponsored with the Math Club a workshop called "How To Study Math" presented by Dr. Brian Hunt, KME Advisor, to help students learn study strategies when preparing for their upcoming math midterm exams. This workshop was held on March 6th and approximately 5 students attended. Kappa Mu Epsilon co-sponsored with the Math Club a Game Night for KME and Math Club members so that faculty and students could have an opportunity to interact with each other in an informal setting outside of the classroom. This activity was held on April 23rd and approximately 7 people attended.

## NE Beta - University of Nebraska Kearney

Chapter Pres. - April Christman; 20 Current Members, 9 New Members Other Spring 2012 Officers: Brent Wheaton, Vice President; Ali Titus, Secretary; Brittany Spiehs, Treasurer; and Dr. Katherine Kime, Corresponding Secretary and Faculty Sponsor.

In April, KME sponsored the visit of Dr. Arthur Benjamin, the "Mathemagician," of Harvey Mudd College. Dr. Benjamin gave his fascinating performance to a group of several hundred, many from the community. We received financial support from student government based on the proposal of KME secretary Ali Titus. KME member Claire Aylward corresponded with Dr. Benjamin, and others helped with the planning.
New Initiates - Malaika Albin, Nathan Brady, Shaina Bryan, Michael Christen, Jordan Engle, Daniel Farlin, Natalie Hanisch, Kelan Schumacher, and Danielle Thornton.

## NE Delta - Nebraska Wesleyan University

Chapter Pres. - Alex Whigham; 13 Current Members, 4 New Members Other Spring 2012 Officers: Laura Booton, Vice President; Jayme Prenosil, Secretary and Treasurer; and Rebecca Swanson, Faculty Sponsor (Spring 2012) and Melissa Erdmann, Corresponding Secretary and Faculty Sponsor (Fall 2012).

We were pleased to have an initiation ceremony followed by a picnic combined with the Physics Club. Brats, burgers, dogs, and assorted sides were enjoyed by all. In March, we had our second annual Pi Run of 3.14 miles and about 75 people participated.
New Initiates - Paul Dorenbach, Katie Fagot, Jayme Prenosil, and Alex Whigham.

## NJ Delta - Centenary College of New Jersey

Chapter Treasurer - Linda Ritchie; 15 Current Members, 9 New Members Other Spring 2012 Officer: Corresponding Secretary and Faculty Sponsor Kathy Turrisi.

In order for a student to become aware of what they can do with a mathematics degree, Kappa Mu Epsilon (KME) along with the math team offered free tutoring to students seeking to pass the Praxis exam. This tutoring serves to assist students to choose if this is the career path they may want to investigate for their future. Professor Kathy Turrisi's advisee was the first student to present at a student math conference at the 26th Annual Moravian College Student Mathematics Conference on Saturday, February 18,2012 . The student started working on a topic early September and presented in February with Paul Turrisi-Chung, another student from Stevens Institute of Technology, working under the guidance of the math team, Prof. Ritchie, Prof. Search, and Prof. Turrisi. The math team investigated becoming involved with a business internship. Kathy was the faculty advisor for the math students (KME members) participating in this business internship, VITA (Volunteer Income Tax Assistance) program. This was the first year that students from our department participated with the business students in this community outreach, which counted for two internship credits. Each student, as a pre-requisite for participation in VITA, had to pass both an ethics test and a tax preparation test at the basic level. Kathy led a student study group for these exams in her home. Kathy, along with these students, achieved certification at the basic level. She also coordinated site visits for this internship with Professor Jim Ford. This proved to be very successful and the feedback from the math students was positive. On Sunday, April 29, 2012 the Mathematics and Natural Sciences Department held the Department Award Ceremony and the KME Initiation. Two KME students, Peter DeMary and David Garriques, received the Mathematics Merit Award, given to students who exemplify outstanding achievement in mathematics. Centenary College initiated nine new members into the New Jersey Delta Chapter. Lastly, three members of KME, Peter DeMary, David Garriques, and Sara McCollum, graduated from Centenary College at the May 2012 graduation. We are very proud of their accomplishments and wish them well.
New Initiates - Johanne Barthelemy, Jennifer Catherine Ciecwisz, David R. Garriques, Brittany Howell, Kimberly Kopesky, Allison Nowicki, Michelle Rogers, Brooke Alicia Smith, and Carissa Utter.

## NJ Epsilon - New Jersey City University

Corresponding Secretary - Beimnet Teclezghi; 25 New Members
New Initiates - Allison Arbitblit, Mariam Aziz, Joseph Bulatowicz, Jaclene Diaz, Melvyn
Drag, Chinwendu Emelumba, Andrew Getz, Marcil Gobraiel, Jeimy Hirujo, Rusty Laracuenti, Eric Ledesma, Kayla Lennon, Bernard Lipat, Guido Molina, Maryan Morgan, Duke Nyangweso, Maham Saeed, Julissa Saenz, William Schaffer, Dariusz Serwach, Stephen Szulc, Mariam Tawfike, Elisa Tejada, Brenda Sarai Valladares, and Kyle Woolley.

## NM Alpha - University of New Mexico

Corresponding Secretary - Pedro F. Embid; 5 New Members
New Initiates - Barbara Auvaa, Katherine Belvin, Yuridia Leiva, Amber Weinstein, and Reed Young.
NY Eta - Niagara University
Corresponding Secretary - Maritza M. Branker; 14 New Members
New Initiates - Kelsey Atwater, Andrea Lynn DeRosa, Emily French, Victoria M. Gilliland, Paige Houston, Emily James, Danielle M. Kennedy, Christina Knapp, Jolene L. Lambert, Karen Miranda, Alyssa Porrino, Kristen Angela Prabucki, Caitlin Riegel, and Morgan Elizabeth Ryan.
NY Kappa - Pace University
Corresponding Secretary - Lisa Fastenberg; 6 New Members
New Initiates - Liana Brancati, Catherine Flores, Raymond Hunce, Leslie Piedra, Mary Tracy, and Alexander Vassilopoulos.

## NY Lambda - C.W. Post Campus of Long Island University

Chapter President - Daniel Barone; 50 Current Members, 8 New Members Other Spring 2012 Officers: Elyse Capozza, Vice President; Jennifer Hanly, Secretary; and Dr. James B. Peters, Corresponding Secretary and Faculty Sponsor.
The annual banquet was held on March 25 and the new members were initiated.
NY Nu - Hartwick College
Chapter President - Ashley Hunt; 13 Current Members, 15 New Members Other Spring 2012 Officers: Jessica Bentley, Vice President; Rhianna Morgan, Secretary; Alyssa Failey, Treasurer, and Ron Brzenk, Corresponding Secretary and Faculty Sponsor.
New Initiates - Mercy Alila, Jessica Bentley, Alyssa Failey, Matthew Feeman, Desiree Fuller, Steven Grzeskowiak, Brain Heller, Leanne Keeley, Jordan Liz, Rhianna Morgan, Nathan Nichols, James Orlando, Aaron Parisi, Joseph Seney, and Jaime Toboada.

## NY Omicron - St. Joseph's College

Chapter President - Salvatore J. Alfredson; 35 Current Members, 17 New Members
Other Spring 2012 Officers: Alexander DeRidder, Vice President; Mercedees E. Jordan, Secretary; Stephen A. Bates, Treasurer; Elana Reiser, Corresponding Secretary; and Dr. Donna Marie Pirich, Faculty Sponsor.

The NY Omicron chapter held an initiation ceremony and welcomed 16 new members. Our members have each volunteered at least two Saturday mornings to tutor local high school students at our math clinic. In addition to several talks, socials, and fundraisers, our chapter served over 500 local high school students during the academic year by providing free tutoring services at the Math Clinic (see http://www.sjeny.edu/ Academics/Math-Clinic/573). Pictures and videos from recent events can be found here: http://www.sjcny.edu/Academics/ Math-Clinic/573
New Initiates - Rachelle L. Amendola, Kimberly L. Avelin, Stephen A. Bates, Cassandra Benedict, Daniel G. Ferguson, Janéce S. Guerra, Mercedees E. Jordan, Jaclyn R. Lazio, Corine A. Maglione, Stephanie Mann, Kasey C. Melzer, Sarah E. Redding, Emma K. Schrader, Nicolas J. Simonetti, Douglas M. Smith, Christopher P. Vandenberge II, and Geeta Vir.

## OH Epsilon - Marietta College

Chapter Pres. - Misty Hussing; 25 Current Members, 11 New Members
Other Spring 2012 Officers: Evan Winklmann, Vice President; and John Tynan, Corresponding Secretary and Faculty Sponsor.
New Initiates - Laura Carpenter, Darci Combs, Caleb Drennon, William Ervin, Misty Hussing, Sarah Gallahan, Matthew Mastracci, Michael Phillips, Tyler Thomas, Evan Winklmann, and Magdalena Zook.

## OK Alpha - Northeastern State University

 Chapter President - Zac Kindle; 53 Current Members, 6 New Members Other Spring 2012 Officers: James Sherrell, Vice President; Gregory Palma, Secretary; Kalin Lee, Treasurer; and Dr. Joan E. Bell, Corresponding Secretary and Faculty Sponsor.Our spring initiation brought six new members into our chapter. At our January meeting, Dr. Elwyn Davis, Pittsburg State University, spoke on "Adventures on the Sphere." This hands-on activity used the "Lenart Sphere" to differentiate between spherical geometry and plane Euclidean geometry. In February, members worked together on a problem in The College Mathematics Journal, and submitted two different solutions to the journal.

OK Alpha members Ryan Berkley, Abraham (Rho) Middleton, Joshua Gregory, Miranda Sawyer, Gregory Palma, and future member, Joshua Killer, attended the 74th annual Oklahoma-Arkansas section meeting of the Mathematics Association of America. Winning second place in the jeopardy portion of the competition were: Ryan Berkley, Rho Middleton and Joshua Killer. Also attending the meeting were NSU mathematics faculty Dr. Darryl Linde, Dr. Joan E. Bell, and Dr. John Diamantopoulos. The speaker at the April meeting was Tracy Slate, Gore High School mathematics teacher. The talk explored ways to use the Texas Instrument CBR Motion Sensor device. She demonstrated how it is used to present distance/time graphs. We also enjoyed ice cream sundaes at this last meeting of the year.
New Initiates - Ryan A. Berkley, Jerry J. Capps, Haley R. Crane, Anna E. Faina, Chad T. Hollifield, and Falicia R. Mansfield.

## OK Epsilon - Oklahoma Christian University

Chapter President - Jon McCallun; 22 Current Members
Other Spring 2012 Officers: Bart Niyibizi, Vice President; Shaylee Davis, Secretary; Talon Unruh, Treasurer; Ray Hamlett, Corresponding Secretary; and Craig Johnson, Faculty Sponsor.

In the fall the Oklahoma Epsilon Chapter hosts a "math day" for Christian Heritage High School. Competitive exams are given in Algebra I, II, and Calculus. Awards are given at a luncheon ceremony.

## PA Alpha - Westminster College

Corresponding Secretary - Natacha Fontes-Merz; 11 New Members
New Initiates - Julie Bearer, Jeffrey Boerner, Gregory Clark, Kellie Amber Hill, Jenna Huston, Connor Mackenzie, Brandon Mosley, Allison Rice, Julie Rice, Laura Smallhoover, and Emily Walther.

## PA Beta - La Salle University

Chapter President - Stephen Kernytsky; 19 New Members
Other Spring 2012 Officers: Rose Venuto, Secretary; Ryan Cunningham, Treasurer; and Stephen Andrilli, Corresponding Secretary and Faculty Sponsor.

Our KME Chapter holds initiations every two years. This Spring, we initiated 19 new members (qualifying sophomores, juniors, seniors, and graduate students). Prior to the initiation, Dr. Robert Styer of Villanova University, PA gave an invited talk entitled "Bouncing Balls and Geometric Series".

Our KME Chapter is closely affiliated with our Math Club and MAA Student Chapter. Together, these organizations sponsored a problem-solving competition, watched a humorous MAA math video (The United States of Mathematics Presidential Debate with Colin Adams and Thomas Garrity) on Pi Day (after baking pies the previous night to eat during the video), held several poker nights (at which probability was undoubtedly studied!), and sponsored an evening of "Bowling for Primes" - a type of bowling where the rules are altered mathematically in an appropriate manner. A student representative was also sent to the Careers in Mathematics conference for students in West Chester, PA last Fall.
New Initiates - D. Joseph Barron, Daniel Bowers, Kellen Burke, Salvatore Calvo, RoseMary Carberry, Anthony Carbone, John Celley, David Comberiate, Alex Confer, Meghan Dondero, Georgia Hansen, Joseph Jung, Dominick Macaluso, John Miller, Anna Nguyen, Steven Rose, Amanda Russo, Olivia Shoemaker, and Marina Solomos.

## PA Gamma - Waynesburg University

Corresponding Secretary - James R. Bush; 10 New Members
New Initiates - Juan Carlos Angel, Leah N. Bujaky, David D. Cobb, Jordan R. Harvey, Megan C. Knowlson, Benjamin R. Lohr, Emily M. Miller, Brianna K. Wenger, Adam J. Roberge, and Timothy Q. VanRiper.
PA Delta - Marywood University
Corresponding Secretary - Thomas Kent; 3 New Members
New Initiates - Alexandra Burge, Kaitlyn Jones, and Erin Macduff.

## PA Kappa - Holy Family University

Chapter President - Gidget Mantelibano; 7 New Members
Other Spring 2012 Officers: Emily Anick, Vice President; and Sister Marcella Wallowicz, CSFN, Corresponding Secretary and Faculty Sponsor.

Seven students were initiated during Spring 2012. Prior to initiation, candidates typically engage in a service project. This year's group established a "drop-in" math tutoring center for students enrolled in a full range of math courses from developmental math through Calculus I.
New Initiates - Ahmed Benjaani, Kaitlin Bonner, Carly Cofer, Michelle Green, Anthony Ivanoski, William Kane, and Alexander Sowa.

## PA Lambda - Bloomsburg University of Pennsylvania

Corresponding Secretary - Eric Kahn; 9 New Members
New Initiates - Conor Flynn, Dagaen Golomb, Korilyn M. Grady, Paul Gregorowicz, Courtney Heiser, Kyle J. Higgins, Kristen Klock, Christine Seward, and James M. Van Horn.

## PA Mu - Saint Francis University

Chapter President - Laura Stibich; 61 Current Members, 17 New Members Other Spring 2012 Officers: Marissa Basile, Vice President; Katie Dacanay, Secretary; Matthew Skoner, Treasurer; Peter Skoner, Corresponding Secretary; and Katherine Remillard, Faculty Sponsor.

Pi Day (March 14) gave the mathematics department an opportunity to share sweet treats, socializing, and mathematics education with the campus community. A large number of round pies were consumed by the many who visited during the day at this annual event. Students from Dr. Katherine Remillard's courses demonstrated various ways to estimate pi, including Infinite Secrets, The Circle's Measure, and Pi Line, while other students presented posters illustrating present-day conditions and challenges related to: ending poverty and hunger, achieving universal primary education, promoting gender equality and empowering powering women, reducing child mortality, improving maternal health, combating HIV/AIDS/Malaria, ensuring environmental sustainability and developing a global partnership for development. Addison Fox's mathematics research poster "A Course Invariant for Sigma Stable Spaces" was also presented. Those sharing in the day also contributed to CAFOD's (Catholic Agency for Overseas Development) Lenten Appeal focusing on projects that provide clean water sources in the world's poorest countries.

KME members Quy Cao and Michael DeLyser accompanied Dr. Brendon LaBuz to the Spring 2012 Meeting of the Allegheny Mountain Section of the Mathematical Association of America. The mathematics competition was held at West Virginia University on Friday, April 13, 2012. Our students enjoyed talks by other students and participated in the student problem solving competition where they solved seven out of nine problems; both students won prizes for their problem solving abilities. The initiation ceremony was held on Wednesday, April 25, 2012 in the Christian Hall Conference Room. The evening began with a prayer from chapter Assistant Chaplain, Chapter Photographer, and KME member Br. Paul Johns, continued with dinner, followed by the initiation ceremony for the seventeen new members, followed by a talk "Newton's Method," by Assistant Professor of Mathematics and KME member Dr. Brendon LaBuz, and concluded with a closing prayer from KME member and Chapter Chaplain Fr. Joe Chancler. The Pennsylvania Mu chapter now has 343 members initiated through the years.
New Initiates - Abenezer Alemu, Jeffrey Chastel, Ann Colligan, Christopher DeLaney, Michael DeLyser, Collier Devlin, Phuong Minh N. Do, Julia Havko, Rob Hodgson, Bemnet Kebede, Elise Löfgren, Michelle Maher, Jacob McCloskey, Jessie Minor, James Shiring, Sean Veights, and Selam Woldemeskel.

## PA Nu - Ursinus College

Corresponding Secretary - Jeff Neslen; 11 New Members
New Initiates - Gina Brienza, Brandon Fitts, Shannon Hansell, William Molden, Alexandra Signoriello, Sam Snodgrass, Jennifer Sroba, Michelle Tanco, Alina Tinjala, Jayant Velagala, and Tricia Wiegert.
PA Pi - Slippery Rock University
Corresponding Secretary - Elise Grabner, 1 New Member
New Initiate - Chelsea Benzie.
PA Sigma - Lycoming College
Corresponding Secretary - Santu de Silva; 10 New Members
New Initiates - David Brown, Robert Brown, Julie Martinez, Jason Mifsud, Jenna Morgan, Trey Nenow, Alexandria Parizek, Cortney Schoenberger, David Surmick, and Kristin Tate.
PA Tau - DeSales University
Chapter President - Sarah A. Capano; 17 New Members
Other Spring 2012 Officers: Robert A. Zanneo, Vice President; Kelsey R. Foster, Secretary; Michael P. Russo, Treasurer; and Br. Daniel P. Wisniewski, O.S.F.S., Corresponding Secretary and Faculty Sponsor.
New Initiates - Austin M. Benner, Sarah A. Capano, Carrie A. Caswell, Jennifer L. Duncan, Kelsey R. Foster, Annmarie Houck, Joseph A. Marlin, Lauren N. Metz, Tripty Modi, Daniel J. Papson, Michael P. Russo, Colleen M. Shelley, Mary E. Simone, Andrew J. Tevington, Robert A. Zanneo, Rebecca A. Zysk, and Daniel P. Wisniewski, O.S.F.S.

## PA Xi - Cedar Crest College

Corresponding Secretary - Patrick Ratchford; 4 New Members
New Initiates - Megan Brainard, Megan Sando, Deborah Shartle, and Glorivee Suarez.

## SC Gamma - Winthrop University

Corresponding Secretary - Matthew R. Clark; 5 New Members
New Initiates - Robert Wayne Anderson, Michael Capps, Zachary Curry, Rachel Ivey, and Heather Martin.
TN Beta - East Tennessee State University
Chapter President - Lindsey Fox
Other Spring 2012 Officers: Chelsea Herald, Vice President; Jessica Lunsford, Secretary; Terrance McDermott, Treasurer, Robert Gardner, Corresponding Secretary and Faculty Sponsor, and Robert Beeler, Faculty Sponsor.

Tennessee Beta chapter member Cade Herron had his solution to The Pentagon Problem 669 published in the Fall 2011 issue and had his name mentioned as solver of Problem 677 in the same issue. The chapter held a problem session on January 20, 2012 to attempt to solve other problems posted in The Pentagon. The session was held at local watering hole Poor Richard's.

On April 12, 2012, the Tennessee Beta chapter held the initiation ceremony for new members. The ceremony was held at the ETSU Department of Mathematics and Statistics' annual Honors Banquet and included a presentation by guest speaker Dr. Sherry Towers, research professor in the Mathematics Department of Arizona State University. Dr. Towers' presentation involved her travels through academia, to her current position at ASU. The chapter's cumulative membership reached 788 with the initiation. Details of the banquet, along with some photographs, can be found online at: http://faculty.etsu.edu/gardnerr/ KME/Spring-2012/Spring2012.html.
New Initiates - Jack Hartsell, Chelsea Herald, Chelsea Holtzsclaw, Erin McMullen, Tabitha McCoy, Olivia Miller, Eric Seguin, Clayton Walvoort, and Andrew Young.

## TN Gamma - Union University

Chapter President - Emilie Huffman; 24 Current Members, 6 New Members
Other Spring 2012 Officers: Kimberly Lukens, Vice President; Rachel Carbonell, Secretary; Seth Kincaid, Treasurer; Michelle Nielsen, Corresponding Secretary; and Matt Lunsford, Faculty Sponsor

The annual initiation banquet for the Tennessee Gamma Chapter of KME was held at the Old Country Store on April 16, 2012. Guest speaker Kristin Kirk, a Union University alumnus ('09), spoke about her experience teaching high school mathematics at South Gibson County High School. Members of the Tennessee Gamma Chapter of KME attended the North Central Regional KME Convention March 30-31 in Springfield, Missouri. Evangel University hosted the event. The Tennessee Gamma Chapter was represented by faculty members Drs. Bryan Dawson, Matt Lunsford, and Michelle Nielsen and student members Ms. Kimberly Lukens and Ms. Emilie Huffman. Students from KME chapters in Missouri, Kansas, and Tennessee presented scholarly talks at the convention. Emilie Huffman presented her mathematics honors research paper entitled "On the Theory of Numbers - A Paper by Evariste Galois" and Kimberly Lukens presented "Oddities in C x C", a research paper taken from her summer REU (Research Experiences for Undergraduates) at the University of Wyoming. Both papers were well received and Kimberly was awarded one of three "Best Paper" awards at the convention. The award carries both a certificate and a cash stipend.
New Initiates - Alexandria Archer, David Clark, Patrick Joseph, Caroline McConnell, Grace Morriss, and Corey Wilson.

TN Delta - Carson-Newman College
Corresponding Secretary - Kenneth Massey; 4 New Members
New Initiates - Kristin Eubanks, Stephen Henderson, Andrew Manning, and Andrew Sweeney.
TX Gamma - Texas Woman's University
Chapter President - Jennifer Kirk; 30 Current Members, 8 New Members
Other Spring 2012 officers: Rachel Berry, Vice President; Danesh Chowritmootoo, Secretary; Kendra Murphy, Treasurer; and Dr. Mark S. Hamner, Corresponding Secretary and Faculty Sponsor.
TX Lambda - Trinity University
Corresponding Secretary - Diane Saphire; 12 New Members
New Initiates - Jordan Bush, Garner Cochran, Shelby Guenthardt, Spencer Hayes, Chandler Hicks, Rahaman Navaz Gangji, Michael Rodriguez, Ethan Rudd, Jerel Xaver San Gabriel, Benjamin Scheiner, Leah Wesselman, and Molly Zumbro.

## VA Delta - Marymount University

Chapter President - Matthew Villemarette; 31 Current Members, 1 New Member
Other Spring 2012 Officers: Myriam Joga, Vice President; Matthew Villemarette and Myriam Joga, Secretaries/Treasurers; William Heuett, Corresponding Secretary; and Elsa Schaefer, Faculty Sponsor.

## WI Gamma - University of Wisconsin-Eau Claire

Chapter President - Josh Frinak; 41 Current Members, 21 New members
Other Spring 2012 Officers: Kaisey Garrigan, Vice President; Kristina Bleess, Secretary; Bret Meier, Treasurer; and Dr. Carolyn Otto, Corresponding Secretary and Faculty Sponsor.

This semester, the UW-Eau Claire chapter of KME national honor society inducted 21 new members. The initiation ceremony was held on February 9. New officers were selected for next year and these officers will lead the Math Club. The new officers will be: President: Kaisey Garrigan, Vice President: Meghan Christenson, Secretary: Cassandra Dale, Treasurer: Lindsey Alger.
New Initiates - Roxanne Accola, Ivan Alias, Sarah Baker, Sophia Bolle, Meghan Christenson, Cassandra Dale, Cory Davis, Kayla S. Deutscher, Laura Elder, Robert Erickson, Ryan Horstman, Abagail Huebner, Jessica Joniaux, Zachary Kelliher, Jenniffer Kulesa, Hannah Leary, Bethanna Petersen, Stephanie Prahl, Blake R. Smith, Sudharishini Subramani, and Kayla VandenLangenberg.

## WV Beta - Wheeling Jesuit University

Corresponding Secretary - Theodore S. Erickson; 3 New Members
New Initiates - Nathaniel Maya, Kevin Meigh, and Jessica Steve.

# Kappa Mu Epsilon National Officers 



KME National Website:
http://www.kappamuepsilon.org/

# Active Chapters of Kappa Mu Epsilon 

## Listed by date of installation

| Chapter | Installation Date |  |
| :---: | :---: | :---: |
| OK Alpha | Northeastern State University, Tahlequah | 18 Apr 1931 |
| IA Alpha | University of Northern Iowa, Cedar Falls | 27 May 1931 |
| KS Alpha | Pittsburg State University, Pittsburg | 30 Jan 1932 |
| MO Alpha | Missouri State University, Springfield | 20 May 1932 |
| MS Alpha | Mississippi University for Women, Columbus | 30 May 1932 |
| MS Beta | Mississippi State University, Mississippi State | 14 Dec 1932 |
| NE Alpha | Wayne State College, Wayne | 17 Jan 1933 |
| KS Beta | Emporia State University, Emporia | 12 May 1934 |
| AL Alpha | Athens State University, Athens | 5 Mar 1935 |
| NM Alpha | University of New Mexico, Albuquerque | 28 Mar 1935 |
| IL Beta | Eastern Illinois University, Charleston | 11 Apr 1935 |
| AL Beta | University of North Alabama, Florence | 20 May 1935 |
| AL Gamma | University of Montevallo, Montevallo | 24 Apr 1937 |
| OH Alpha | Bowling Green State University, Bowling Green | 24 Apr 1937 |
| MI Alpha | Albion College, Albion | 29 May 1937 |
| MO Beta | University of Central Missouri, Warrensburg | 10 Jun 1938 |
| TX Alpha | Texas Tech University, Lubbock | 10 May 1940 |
| KS Gamma | Benedictine College, Atchison | 26 May 1940 |
| IA Beta | Drake University, Des Moines | 27 May 1940 |
| TN Alpha | Tennessee Technological University, Cookeville | 5 Jun 1941 |
| MI Beta | Central Michigan University, Mount Pleasant | 25 Apr 1942 |
| NJ Beta | Montclair State University, Upper Montclair | 21 Apr 1944 |
| IL Delta | University of St. Francis, Joliet | 21 May 1945 |
| KS Delta | Washburn University, Topeka | 29 Mar 1947 |
| MO Gamma | William Jewell College, Liberty | 7 May 1947 |
| TX Gamma | Texas Woman's University, Denton | 7 May 1947 |
| WI Alpha | Mount Mary College, Milwaukee | 11 May 1947 |
| OH Gamma | Baldwin-Wallace College, Berea | 6 Jun 1947 |
| CO Alpha | Colorado State University, Fort Collins | 16 May 1948 |
| MO Epsilon | Central Methodist College, Fayette | 18 May 1949 |
| MS Gamma | University of Southern Mississippi, Hattiesburg | 21 May 1949 |
| IN Alpha | Manchester College, North Manchester | 16 May 1950 |
| PA Alpha | Westminster College, New Wilmington | 17 May 1950 |
| IN Beta | Butler University, Indianapolis | 16 May 1952 |
| KS Epsilon | Fort Hays State University, Hays | 6 Dec 1952 |
| PA Beta | LaSalle University, Philadelphia | 19 May 1953 |
| VA Alpha | Virginia State University, Petersburg | 29 Jan 1955 |
| IN Gamma | Anderson University, Anderson | 5 Apr 1957 |
| CA Gamma | California Polytechnic State University, San Luis Obispo | 23 May 1958 |
| TN Beta | East Tennessee State University, Johnson City | 22 May 1959 |
| PA Gamma | Waynesburg College, Waynesburg | 23 May 1959 |
| VA Beta | Radford University, Radford | 12 Nov 1959 |
| NE Beta | University of Nebraska-Kearney, Kearney | 11 Dec 1959 |
| IN Delta | University of Evansville, Evansville | 27 May 1960 |

OH Epsilon
MO Zeta
NE Gamma
MD Alpha
CA Delta
PA Delta
PA Epsilon
AL Epsilon
PA Zeta
AR Alpha
TN Gamma
WI Beta
IA Gamma
MD Beta
IL Zeta
SC Beta
PA Eta
NY Eta
MA Alpha
MO Eta
IL Eta
OH Zeta
PA Theta
PA Iota
MS Delta
MO Theta
PA Kappa
CO Beta
KY Alpha
TN Delta
NY Iota
SC Gamma
IA Delta
PA Lambda
OK Gamma
NY Kappa
TX Eta
MO Iota
GA Alpha
WV Alpha
FL Beta
WI Gamma
MD Delta
IL Theta
PA Mu
AL Zeta
CT Beta
NY Lambda
MO Kappa
CO Gamma
Ma

| Marietta College, Marietta | 29 Oct 1960 |
| :---: | :---: |
| University of Missouri-Rolla, Rolla | 19 May 1961 |
| Chadron State College, Chadron | 19 May 1962 |
| College of Notre Dame of Maryland, Baltimore | 22 May 1963 |
| California State Polytechnic University, Pomona | 5 Nov 1964 |
| Marywood University, Scranton | 8 Nov 1964 |
| Kutztown University of Pennsylvania, Kutztown | 3 Apr 1965 |
| Huntingdon College, Montgomery | 15 Apr 1965 |
| Indiana University of Pennsylvania, Indiana | 6 May 1965 |
| Arkansas State University, Jonesboro | 21 May 1965 |
| Union University, Jackson | 24 May 1965 |
| University of Wisconsin-River Falls, River Falls | 25 May 1965 |
| Morningside College, Sioux City | 25 May 1965 |
| McDaniel College, Westminster | 30 May 1965 |
| Dominican University, River Forest | 26 Feb 1967 |
| South Carolina State College, Orangeburg | 6 May 1967 |
| Grove City College, Grove City | 13 May 1967 |
| Niagara University, Niagara University | 18 May 1968 |
| Assumption College, Worcester | 19 Nov 1968 |
| Truman State University, Kirksville | 7 Dec 1968 |
| Western Illinois University, Macomb | 9 May 1969 |
| Muskingum College, New Concord | 17 May 1969 |
| Susquehanna University, Selinsgrove | 26 May 1969 |
| Shippensburg University of Pennsylvania, Shippensburg | 1 Nov 1969 |
| William Carey College, Hattiesburg | 17 Dec 1970 |
| Evangel University, Springfield | 12 Jan 1971 |
| Holy Family College, Philadelphia | 23 Jan 1971 |
| Colorado School of Mines, Golden | 4 Mar 1971 |
| Eastern Kentucky University, Richmond | 27 Mar 1971 |
| Carson-Newman College, Jefferson City | 15 May 1971 |
| Wagner College, Staten Island | 19 May 1971 |
| Winthrop University, Rock Hill | 3 Nov 1972 |
| Wartburg College, Waverly | 6 Apr 1973 |
| Bloomsburg University of Pennsylvania, Bloomsburg | 17 Oct 1973 |
| Southwestern Oklahoma State University, Weatherford | 1 May 1973 |
| Pace University, New York | 24 Apr 1974 |
| Hardin-Simmons University, Abilene | 3 May 1975 |
| Missouri Southern State University, Joplin | 8 May 1975 |
| State University of West Georgia, Carrollton | 21 May 1975 |
| Bethany College, Bethany | 21 May 1975 |
| Florida Southern College, Lakeland | 31 Oct 1976 |
| University of Wisconsin-Eau Claire, Eau Claire | 4 Feb 1978 |
| Frostburg State University, Frostburg | 17 Sep 1978 |
| Benedictine University, Lisle | 18 May 1979 |
| St. Francis University, Loretto | 14 Sep 1979 |
| Birmingham-Southern College, Birmingham | 18 Feb 1981 |
| Eastern Connecticut State University, Willimantic | 2 May 1981 |
| C.W. Post Campus of Long Island University, Brookville | 2 May 1983 |
| Drury University, Springfield | 30 Nov 1984 |
| Fort Lewis College, Durango | 29 Mar 1985 |

NE Delta
TX Iota
PA Nu
VA Gamma
NY Mu
OH Eta
OK Delta
CO Delta
PA Xi
MO Lambda
TX Kappa
SC Delta
SD Alpha
NY Nu
NH Alpha
LA Gamma
KY Beta
MS Epsilon
PA Omicron
MI Delta
MI Epsilon
KS Zeta
TN Epsilon
MO Mu
GA Beta
AL Eta
NY Xi
NC Delta
PA Pi
TX Lambda
GA Gamma
LA Delta
GA Delta
TX Mu
NJ Gamma
CA Epsilon
PA Rho
VA Delta
NY Omicron
IL Iota
WV Beta
SC Epsilon
PA Sigma
MO Nu
MD Epsilon
NJ Delta
NY Pi
OK Epsilon
HA Alpha
NC Epsilon
MA

| Nebraska Wesleyan University, Lincoln | 18 Apr 1986 |
| :---: | :---: |
| McMurry University, Abilene | 25 Apr 1987 |
| Ursinus College, Collegeville | 28 Apr 1987 |
| Liberty University, Lynchburg | 30 Apr 1987 |
| St. Thomas Aquinas College, Sparkill | 14 May 1987 |
| Ohio Northern University, Ada | 15 Dec 1987 |
| Oral Roberts University, Tulsa | 10 Apr 1990 |
| Mesa State College, Grand Junction | 27 Apr 1990 |
| Cedar Crest College, Allentown | 30 Oct 1990 |
| Missouri Western State College, St. Joseph | 10 Feb 1991 |
| University of Mary Hardin-Baylor, Belton | 21 Feb 1991 |
| Erskine College, Due West | 28 Apr 1991 |
| Northern State University, Aberdeen | 3 May 1992 |
| Hartwick College, Oneonta | 14 May 1992 |
| Keene State College, Keene | 16 Feb 1993 |
| Northwestern State University, Natchitoches | 24 Mar 1993 |
| Cumberland College, Williamsburg | 3 May 1993 |
| Delta State University, Cleveland | 19 Nov 1994 |
| University of Pittsburgh at Johnstown, Johnstown | 10 Apr 1997 |
| Hillsdale College, Hillsdale | 30 Apr 1997 |
| Kettering University, Flint | 28 Mar 1998 |
| Southwestern College, Winfield | 14 Apr 1998 |
| Bethel College, McKenzie | 16 Apr 1998 |
| Harris-Stowe College, St. Louis | 25 Apr 1998 |
| Georgia College and State University, Milledgeville | 25 Apr 1998 |
| University of West Alabama, Livingston | 4 May 1998 |
| Buffalo State College, Buffalo | 12 May 1998 |
| High Point University, High Point | 24 Mar 1999 |
| Slippery Rock University, Slippery Rock | 19 Apr 1999 |
| Trinity University, San Antonio | 22 Nov 1999 |
| Piedmont College, Demorest | 7 Apr 2000 |
| University of Louisiana, Monroe | 11 Feb 2001 |
| Berry College, Mount Berry | 21 Apr 2001 |
| Schreiner University, Kerrville | 28 Apr 2001 |
| Monmouth University, West Long Branch | 21 Apr 2002 |
| California Baptist University, Riverside | 21 Apr 2003 |
| Thiel College, Greenville | 13 Feb 2004 |
| Marymount University, Arlington | 26 Mar 2004 |
| St. Joseph's College, Patchogue | 1 May 2004 |
| Lewis University, Romeoville | 26 Feb 2005 |
| Wheeling Jesuit University, Wheeling | 11 Mar 2005 |
| Francis Marion University, Florence | 18 Mar 2005 |
| Lycoming College, Williamsport | 1 Apr 2005 |
| Columbia College, Columbia | 29 Apr 2005 |
| Stevenson University, Stevenson | 3 Dec 2005 |
| Centenary College, Hackettstown | 1 Dec 2006 |
| Mount Saint Mary College, Newburgh | 20 Mar 2007 |
| Oklahoma Christian University, Oklahoma City | 20 Apr 2007 |
| Hawaii Pacific University, Waipahu | 22 Oct 2007 |
| North Carolina Wesleyan College, Rocky Mount | 24 Mar 2008 |

CA Zeta
NY Rho
NC Zeta
RI Alpha
NJ Epsilon
NC Eta
AL Theta
GA Epsilon
FL Gamma
MA Beta
AR Beta
PA Tau
TN Zeta

| Simpson University, Redding | 4 Apr 2009 |
| :---: | ---: |
| Molloy College, Rockville Center | $21 \mathrm{Apr}, 2009$ |
| Catawba College, Salisbury | $17 \mathrm{Sep}, 2009$ |
| Roger Williams University, Bristol | $13 \mathrm{Nov}, 2009$ |
| New Jersey City University, Jersey City | $22 \mathrm{Feb}, 2010$ |
| Johnson C. Smith University, Charlotte | $18 \mathrm{Mar}, 2010$ |
| Jacksonville State University, Jacksonville | $29 \mathrm{Mar}, 2010$ |
| Wesleyan College, Macon | $30 \mathrm{Mar}, 2010$ |
| Southeastern University, Lakeland | $31 \mathrm{Mar}, 2010$ |
| Stonehill College, Easton | $8 \mathrm{Apr}, 2011$ |
| Henderson State University, Arkadelphia | $10 \mathrm{Oct}, 2011$ |
| DeSales University, Center Valley | $29 \mathrm{Apr}, 2012$ |
| Lee University, Cleveland | $5 \mathrm{Nov}, 2012$ |


[^0]:    1 The author has been unable to find a reference for a previous proof of this theorem.

[^1]:    ${ }^{2}$ Bulletin des Sciences mathematiques de M. Ferussac, Vol. 13, June 1830; with the folowing note: "This memoire forms part of the research of Mr. Galois on the theory of permutations and algebraic equations."

