THE PENTAGON

*A Mathematics Magazine for Students*

Volume 67 Number 2 Spring 2008

# Contents




Typeset in Scientific WorkPlace Printed in the United States of America.

The Pentagon (ISSN 0031-4870) is published semiannually in December and May by Kappa Mu Epsilon. No responsibility is assumed for opinions expressed by individual authors. Papers written by undergraduate mathematics students for undergraduate mathematics students are solicited. Papers written by graduate students or faculty will be considered on a space-available basis. Submissions must include both an electronic copy and a hard copy. The electronic copy should be included as an attachment to an e-mail sent to the editor. Either a TeX file or Word document is acceptable. It is the responsibility of the author to ensure that any images are in a camera-ready form that can be scaled to the page size of *The Pentagon*. The typewritten copy should be double-spaced with wide margins on white paper. Standard notational conventions should be respected. Graphs, tables, or other materials taken from copyrighted works MUST be accompanied by an appropriate release form the copyright holder permitting their further reproduction. Student authors should include the names and addresses of their faculty advisors. Contributors to The Problem Corner or Kappa Mu Epsilon News are invited to correspond directly with the appropriate Associate Editor.

Individual domestic subscriptions: $5.00 for two issues (one year) or $10.00 for four issues (two years); individual foreign subscriptions: $7.00 (US) for two issues (one year). Library rate: $10.00 (US) for two issues (one year) or $20.00 (US) for four issues (two years). Correspondence regarding subscriptions, changes of address or back copies should be addressed to the Business Manager. Copies lost because of failure to notify the Business Manager of changes of address cannot be replaced. Microform copies are available from University Microfilms, Inc., 300 North Zeeb Road, Ann Arbor, Michigan 48106-1346 USA.

---

## KME Website

The national KME website can be found at

http://www.kappamuepsilon.org/

Among the items on the site:

- Contact information for national officers
- Forms
- How to start a KME chapter
- Information on KME conventions

When you design a chapter homepage, please remember to make it clear that your page is for your chapter, and not for the national organization. Also, please include a link to the national homepage and submit your local chapter webpage's URL to the national webmaster, who can be contacted from the national website.

## *Subscription Renewals and Change of Address*

Your *Pentagon* subscription expires with the volume and number that appears in the upper right corner of your address label (see back cover). Since this issue is Volume 67 Number 2, if the code 67-2 appears on your label then this is your last issue!

**Early renewals save us the cost of mailing out renewal notices.** To renew, please send your check - just $10 for four more issues (domestic individuals only; see page 2 for rates for libraries and foreign subscriptions) - together with your name and address and a copy of your old address label to:

Don Tosh, *The Pentagon* Business manager
Department of Science and Technology
Evangel University
1111 N. Glenstone Ave.
Springfield, MO 65802-2191 USA

Please renew promptly to avoid gaps in your journal collection. **If you have a change of address,** please don't forget *The Pentagon*! Send your change of address to the business manager at the above address. Failure to do so may cost KME additional postage fees!

---

## *Back Issues*

Is your journal collection complete? Copies of most back issues of *The Pentagon* are still available for $5.00 per copy. Please send inquiries to:

Don Tosh, *The Pentagon* Business manager
Department of Science and Technology
Evangel University
1111 N. Glenstone Ave.
Springfield, MO 65802-2191 USA

# *Modeling Drug Dosages of Quinidine*

John Carr, *student*

MO Iota

Missouri Southern State University
Joplin, MO 64801

Presented at the 2007 National Convention and awarded "top three" status by the Awards Committee.

## 1. Introduction to Mathematical Modeling

Since the 1960's mathematical models have rapidly expanded in engineering, social, and natural sciences. Fields such as economics, sociology, politics, biology, chemistry, physics, engineering, and other fields have developed qualitative ways of describing phenomena. In this study, pharmacokinetics was chosen as the primary area of focus. In reality, most problems are messy and intertwined with other problems. Mathematical models simplify the problem and abstract out the mathematical relationships in the problem. By adding mathematical structures through in the format of equation, the model seeks to explain the problem. [4]

In order to construct a mathematical modeling, a series of logical steps must be followed. First, the problem must be identified. This includes a simplification and a complete understanding of the problem. After determining exactly what problem needs to be addressed, making assumptions and identifying the mathematical variables is crucial. Assumptions should follow some logic and be reasonable. The next step is solving the modeling using the mathematical variables. Models may explain data or explain the phenomena; the latter is often more beneficial. Testing the model will provide evidence to the accuracy of the model. If the model is not accurate, revaluate the assumptions and equations. [3]

## 2. Introduction to Quinidine

Discovered 350 years ago, Quinidine is a class I antiarrhythmic cardiac drug used to stabilize irregular heart beats. Quinidine works on heart tissue by equalizing fast sodium currents. If the heart rate is too high, Quinidine slows the heart; however, if the heart rate is too low, Quinidine raises the heart rate. Sodium channels opening too fast off set the electrical gradient causing rapid heart rate response, known as the rapid depolarization phase. Altering the sodium channels uptake alters the action potential and excitability of heart cells. Quinidine is also used to treat malaria; although the mechanism of treatment is still unknown. [1]

According to the same source, a stereoisomer of the more common quinine, which is found from Cinchona trees in South America. Exact discovery is unknown. Religious documents show the record of the drug during the mid 1700s. Romanticized stories exist about the Countess of Chinchon treating her husband around the same time. The first recorded research occurred in 1921 by a group of cardiologist who called themselves the Cardiac Club. A group of 265 patients suffered from irregular heart beats and were treated with Quinidine. From this group, 60 percent showed signs of treatment, 7 experienced embolism and 8 died for unexplained reasons. The conclusion from their study was "[Quinidine] is not with out risk." Even in the late 20th century, in depth studies surrounding Quinidine is limited. According to Douglas Zipes in 1985 in [1], "Pharmacological control of arrhythmia still remains predicted on individual, patient-by-patient, pharmacological experiment, determined by trial and error, and tempered by good clinical judgment."

## 3. Variables and Values for Quinidine

The goal of this project is to access dosages and time intervals of Quinidine and the effect of drug concentration in the body. Quinidine, as mentioned above, is a highly sensitive drug; therefore, these models attempt to solve the problem health care providers experience when administering or prescribing this drug. Unfortunately, limited data exists on Quinidine due to the time of its discovery. The drug is administered orally or intravenously. The half life is 6 to 8 hours; these models assume the half life will be 7 hours. From the 7 hours half life, a residual rate is 89 percent of drug remaining after each hour. The percent that makes it to the circulatory system after oral consumption is 70 to 80 percent. [2] Although different concentration levels are desirable for people depending on their age, size, etc., the effective blood concentration is assumed to be 200 to 700 mg for an average sized adult (150 lbs). This model will also assume instant and equal distribution of drug. The models will be broken into four phases.

## 4. Phase 1

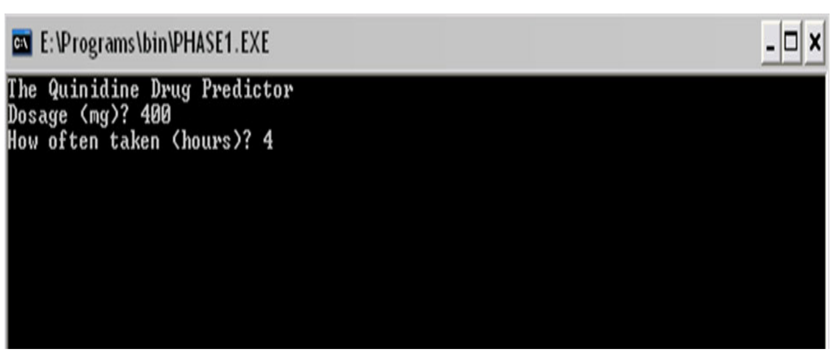
Using computer programming, the model developed through four phases. The first phase and most elementary is intravenous administration. The goal of the model is to test dosage and time interval combinations and determine if concentration would remain in the effective concentration range (200 – 700 mg) after one week (168 hours). The following algorithm was used:

$$U_{n+1} = k \cdot U_n + A_{DT}\left(t\left(n\right)\right),$$

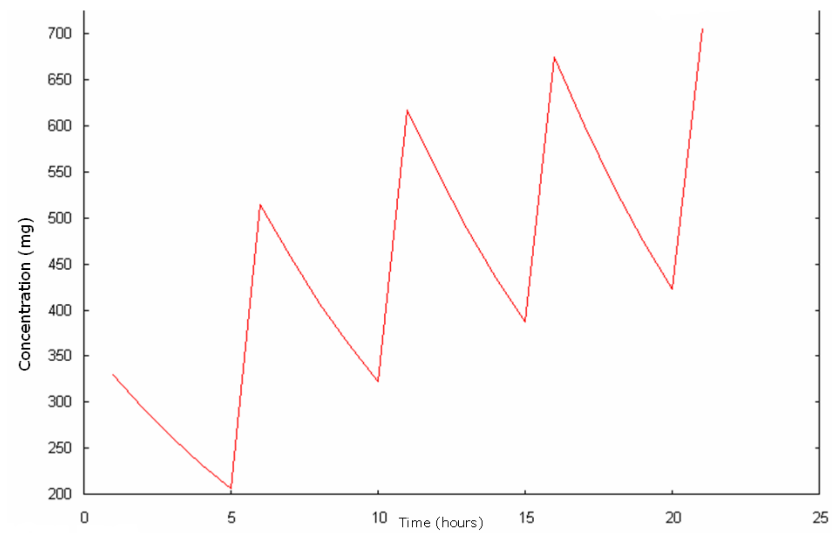
where

$$\begin{array}{lcl} U_n & = & \text{concentration of drug in blood} \\ k & = & \text{residual fraction (89\%)} \\ A & = & \begin{cases} D & \text{if} \quad n \text{ is a multiple of the dosage interval} \\ 0 & \text{if} \qquad\qquad \text{otherwise} \end{cases} \\ t\left(n\right) & = & \text{period (hours)} \\ D & = & \text{dosage (mg)} \end{array}$$

Using the above algorithm, a program was written to prompt user for a dosage amount and how often the drug was administered. The program outputs the concentration at each hour and a warning if the concentration exceeds the maximum or goes below the minimum concentration levels. Phase one example and graph of this algorithm:



```
The Quinidine Drug Predictor
Dosage Amount:  400 mg
Time Interval:  4 hours

Hours      Concentration
 1         400
 2         356
 3         316.84
 4         281.9876
 5         650.9689
 6         579.3624
 7         515.6325
 8         458.9129
 9         808.4325
Warning Concentration Of Quinidine is too high
```




The above graph shows the algorithm. A few practical disadvantages accompany this model. First, it only accounts for an instant amount of drug administration, such as a shot. An I.V. would administer a specific concentration of drug over a specific amount of time. Also, the input makes the user guess and check. Phase two solves the latter problem.

## 5. Phase 2

Phase two program loops though time intervals, then a nested loop of time intervals, then a nested loop of hours, recording the minimum and maximum concentration levels in a two-dimensional array. The program then only output drug dosages and time intervals that will fall within the minimum and maximum concentration range. This approach allows healthcare providers to view all the possible options of administration. Unfortunately, this approach still assumes instant distribution after the drug is administered. Phase four will address this problem using a differential equation approach.

## 6. Phase 3

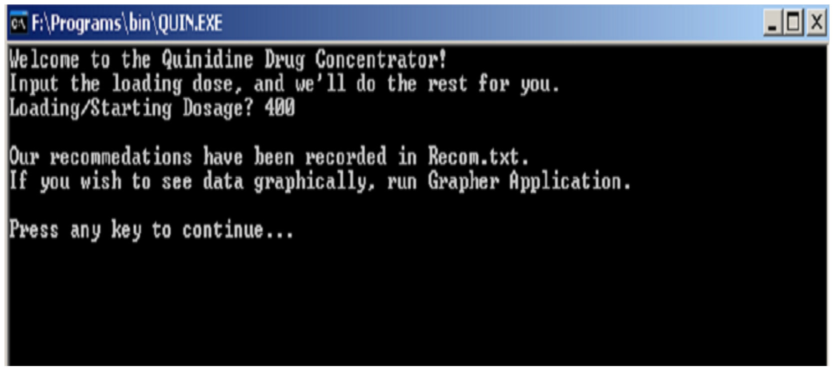
Phase three utilizes another model to solve for oral administration. Assuming a bioavailability (percent of drug that reaches the blood from the gastrointestinal tract each hour) of 70 percent, the following algorithm was developed:

$$U_{n+1} = kU_n + \left[(0.7) \cdot (0.3)^{n-1}\right] L,$$

where

$$\begin{aligned} U_n &= \text{concentration of drug in blood} \\ L &= \text{loading (initial) dose} \\ k &= \text{residual fraction (89\%)} \\ D &= \text{dosage (mg)} \end{aligned}$$

The loading dose is a larger dose to start the cycle. Loading doses are common with sensitive drugs. The phase three program loops through the following nested loops: time intervals, dosage, hours, and hours ago, then records minimum and maximum concentration. After recording these it compares to an ideal concentration (assumed to be 450 mg). The program then outputs all the time and dosage options and the minimum deviation. This allows the healthcare provider to see all the possible options and the one with the least deviation from an ideal concentration. By minimizing the deviation from the ideal concentration, it avoids a high then low effect of the drug. The following is an example output of this program:



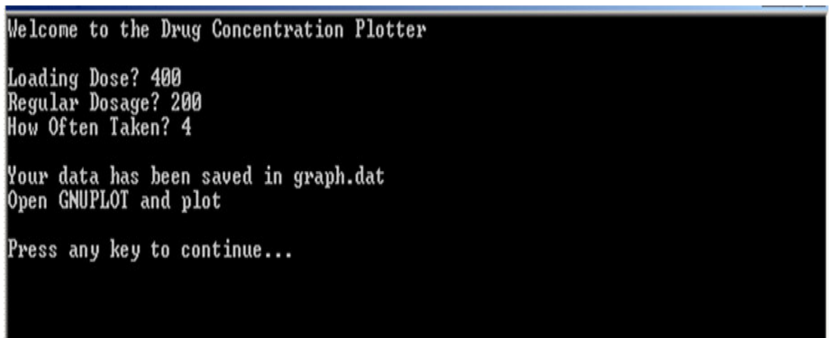
F:\Programs\bin\QUIN.EXE

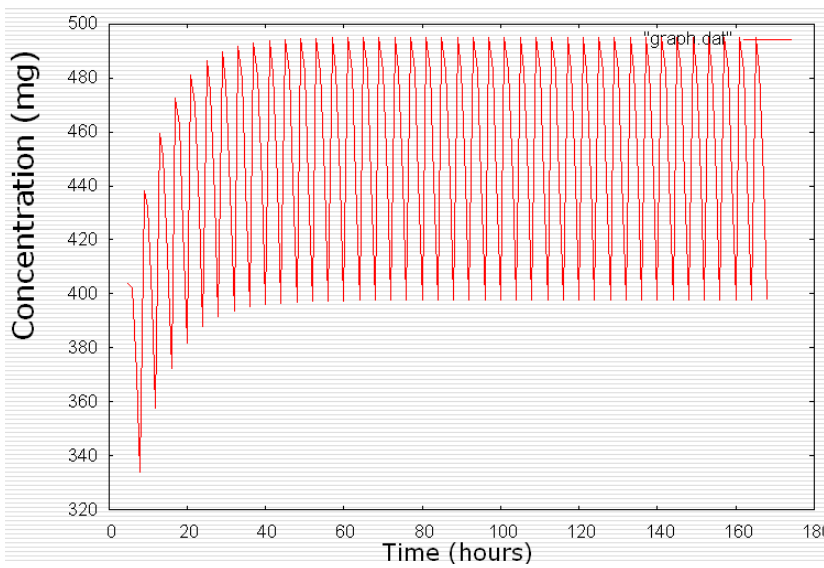
```
Welcome to the Quinidine Drug Concentrator!
Input the loading dose, and we'll do the rest for you.
Loading/Starting Dosage? 400

Our recommedations have been recorded in Recom.txt.
If you wish to see data graphically, run Grapher Application.

Press any key to continue...
```

| Take Every hours | Dose (mg) | Minimum Conc. | Maximum Conc. |
|---|---|---|---|
| 1 | 50 | 280 | 454.5456 |
| 2 | 100 | 333.2 | 468.7902 |
| 3 | 100 | 280.2045 | 363.9157 |
| 3 | 150 | 321.748 | 481.9868 |
| 4 | 150 | 293.9157 | 371.2931 |
| 4 | 200 | 293.9157 | 495.0574 |
| 5 | 200 | 263.853 | 406.8767 |
| 5 | 250 | 263.853 | 508.5959 |

Initial Dose = 400
Minimum Deviation = 68.79022

This program provides the user with all the possible options after inputting a loading dose. If desired, a looping through loading doses could easily be added to this program. In order to analyze this model graphically, another program was written. The graphing program looped similar loops to record concentrations at each hour. After choosing an option from the output above, the graphing program requests the loading dose, the regular dose, and the time interval. The following is output from the graphing program:

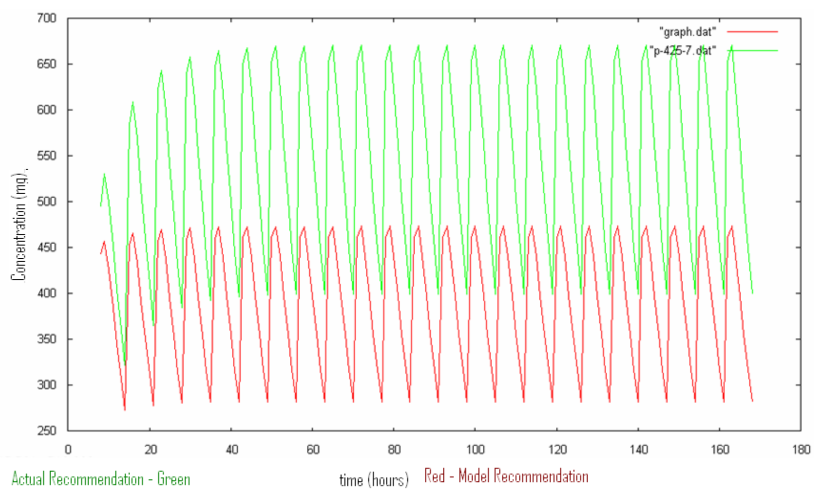



The above graph shows the function becomes essentially uniform after approximately 40 hours. Phase three is a practical model for oral administration. The recommend dosage for the average adult is 200 to 600 mg, three to four times daily. Assuming an average of these values of 425 mg every 7 hours, we can compare to phase three of this model. With a loading dose of 500 mg, the program recommends 300 mg every 7 hours, as shown below. The following graph is a comparison of the actual recommendation and the recommendation from the model.

| Take Every hours | Dose (mg) | Minimum Conc. | Maximum Conc. |
|---|---|---|---|
| 1 | 50 | 350 | 484.2631 |
| 2 | 100 | 416.5 | 480.2532 |
| 3 | 100 | 280.2045 | 437.3947 |
| 3 | 150 | 402.185 | 481.9868 |
| 4 | 150 | 298.2424 | 434.8162 |
| 4 | 200 | 367.3947 | 495.0574 |
| 5 | 200 | 299.4782 | 434.3869 |
| 5 | 250 | 329.8162 | 508.5959 |
| 6 | 250 | 292.8337 | 441.7375 |
| 6 | 300 | 294.3869 | 528.4265 |
| 7 | 300 | 262.2595 | 473.5008 |
| 7 | 350 | 262.2595 | 552.4175 |

Initial Dose = 500
Minimum Deviation = 80.2532




As evident from the above graph, the model is accurate. The model's recommendation (red) has less deviation than the actual recommendation (green).

## 7. Phase 4

Phase four uses differential equations to model intravenous administration. These equations were used from materials proved in a mathematical modeling course under Charles Curtis in the Fall of 2006. The following model was used:

$$\frac{dA}{dt} = D(t) - P(A),$$

where

$$\begin{array}{lcl} A(t) & = & \text{amount of drug in the bloodstream at time } t \\ A(0) & = & 0 \\ D(t) & = & \left\{ \begin{array}{lll} a & \text{if} & t \in S \\ 0 & \text{if} & t \notin S \end{array} \right. \\ t & = & \text{time elapsed (hours)} \\ a & = & \text{amount administered (mg)} \\ \tau & = & \text{length of time the drug is turned on (assumed 1 hour)} \\ T & = & \text{how often the drug is administered (the duration of each period)} \\ S & = & \text{the set of intervals during which the drug is turned on} \\ P(A) & = & r \cdot A(t) \\ r & = & \text{rate of drug leaving blood stream (11\% per hour)} \end{array}.$$

Note that

$$S = [0, \tau] \cup [T + \tau] \cup [2T + \tau] \cup \cdots.$$

These models create a need for two submodels: a model for $t \in S$, and a model for $t \notin S$.

- If $t \in S$, then
$$\frac{dA}{dt} = a - r \cdot A(t),$$
and
$$A(t_j) = A_j,$$
where
$$t_j = j \cdot T,$$
so that
$$a - r \cdot A = Ce^{rt}.$$
Solving gives
$$A(t) = \frac{a}{r}\left[1 - e^{-r(t-t_j)}\right] + A_j e^{-r(t-t_j)}.$$
- $t \notin S$, then
$$\frac{dA}{dt} = -r \cdot A(t),$$

and

$$A\left(s_j\right) = B_j,$$

where

$$s_j = \tau + j \cdot T,$$

so that

$$A = C \cdot e^{-rt}.$$

Solving gives

$$A\left(t\right) = B_j e^{-r\left(t - s_j\right)}.$$

Solutions of these sub models produce additional variables $A(j)$ and $B(j)$, $j = 0, 1, 2, ...$, representing the low and high values of $A\left(t\right)$ in the $j^{th}$ period $\left[jT, \left(j+1\right)T\right]$. We can solve for $A(j)$ and $B(j)$:

$$\begin{aligned} A\left(j\right) &= B_{j-1} e^{-r\left(t_j - s_j - 1\right)} \\ &= B_{j-1} e^{-r\left(T - \tau\right)}, \end{aligned}$$

and

$$\begin{aligned} B\left(j\right) &= \frac{a}{r}\left[1 - e^{-r\left(s_j - t_j\right)}\right] + A_j e^{-r\left(s_j - t_j\right)} \\ &= \frac{a}{r}\left[1 - e^{-r\left(T - \tau\right)}\right] + A_j e^{-r\tau}. \end{aligned}$$

The above models simplify the $A(j)$ and $B(j)$. However, substitution can make these equations independent of each other.

$$\begin{aligned} A\left(j\right) &= \left[\frac{a}{r}\left(1 - e^{-r\tau}\right) + A_{j-1} e^{-r\tau}\right] e^{-r\left(T - \tau\right)} \\ &= \frac{a}{r}\left(1 - e^{-r\tau}\right) e^{-r\left(T - \tau\right)} + A_{j-1} e^{-r\tau}, \end{aligned}$$

and

$$\begin{aligned} B\left(j\right) &= \frac{a}{r}\left(1 - e^{-r\tau}\right) + B_{j-1} e^{-r\left(T - \tau\right)} e^{-r\tau} \\ &= \frac{a}{r}\left(1 - e^{-r\tau}\right) + B_{j-1} e^{-r\tau}, \end{aligned}$$

where

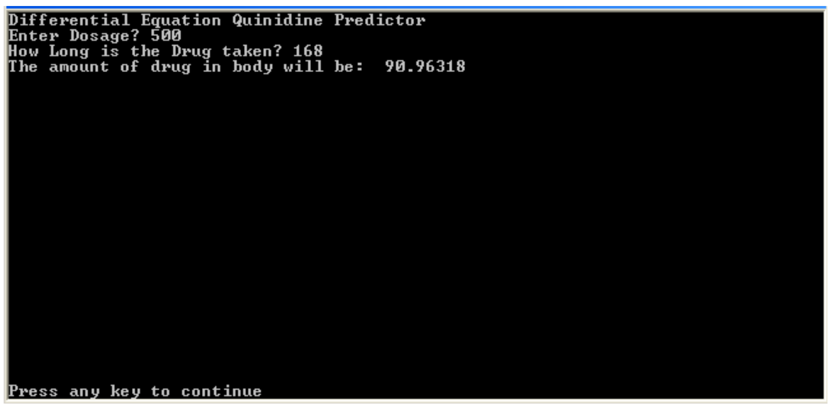
$$A\left(0\right) = 0,$$

and

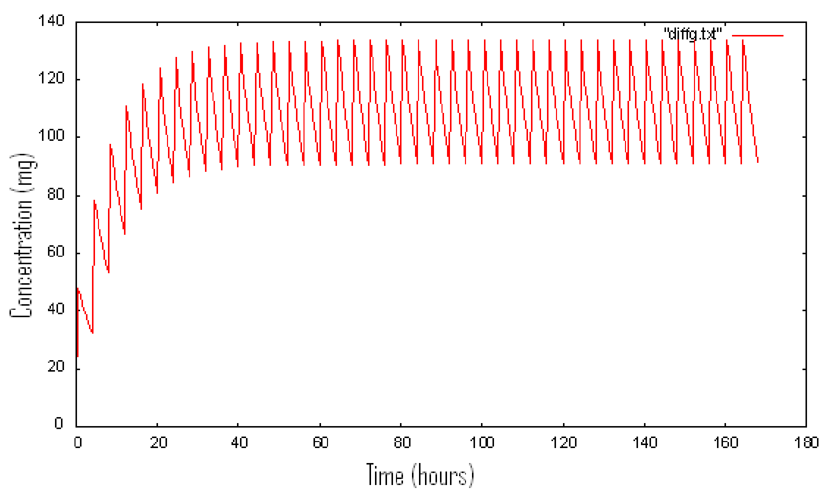
$$B\left(0\right) = \frac{a}{r}\left(1 - e^{-r\tau}\right).$$

The program for phase four prompts the user to input the amount of drug, $a$, to be administered each time the drug is turned on and the elapsed time $t$. The program then loops through $A(j)$ and $B(j)$ values. The program assumes $\tau = 1$ hour and $T = 4$ hours (these values are easily changeable). The program determines if the $t$ valued entered is occurring when drug is on or off. Then the appropriate model is used.

Here is an example.




An alternative program was written recording the concentration of Quinidine in the blood as a function of time:




The above graph may appear piece-wise linear; however, the graph is slightly concave up on the intervals during which the drug is leaving the body.

## 8. Conclusion

Each phase of the mathematical model had it advantages and disadvantages as discussed in each section. All the models showed the concentration of drug becomes uniform as time approaches infinity. The recommendations from the models paralleled the recommendations for actual dosing. Quinidine is a difficult drug to prescribe; modeling programs can help find the appropriate prescription. Unfortunately, the actual data for concentration tests for Quindine was not easily accessible. Comparing to actual data would be very beneficial to these types of models. Choosing a newer drug would make finding this information easier. The differential model could be developed further and optimized as the other three discrete models were. Also broadening to a range of drugs would be interesting.

## *References*

[1] M. Duke, *The Development of Medical Techniques and Treatments*, International Universities Press Inc., 1991.

[2] A.G. Gilman, L.S. Goodman, T.W. Rall, and F. Murad, *The Pharmacological Basis of Therapeutics*, MacMillian Publishing Company, 1985.

[3] F.R. Giordano, M.D. Weir, and W.P. Fox, *A First Course in Mathematical Modelling*, Brooks/Cole – Thomson Learning, 2003.

[4] T.L. Saaty and J.M. Alexander, *Thinking With Models*, Permagon Press Inc., 1981.

# Maximizing Functionals: A Generalization of Problem B5 from the 67th Putnam Competition

Nathan Bloomfield, *student*

MO Kappa

Drury University
Springfield, MO 65802

Presented at the 2007 National Convention.

## 1. Introduction

One of the problems on the 2006 Putnam exam read as follows:

**1** *For each continuous function $f : [0,1] \to \mathbb{R}$, let*

$$I(f) = \int_0^1 x^2 f(x)\,dx \text{ and } J(f) = \int_0^1 x\,(f(x))^2\,dx.$$

*Find the maximum value of $I(f) - J(f)$ over all such functions.*

Here $I - J$ is an example of a *functional*, a function whose domain is a set of functions and whose codomain is $\mathbb{R}$. One possible solution is as follows:

$$\begin{aligned}\int_0^1 x^2 f(x) - x\,(f(x))^2\;dx &= \int_0^1 \frac{x^3}{4} - x\left(f - \frac{x}{2}\right)^2\,dx \\ &\leq \int_0^1 \frac{x^3}{4}\,dx \\ &\leq \frac{1}{16},\end{aligned}$$

with equality when $f(x) = \frac{x}{2}$. See [1].

Presently we wish to know if this problem can be made more general. In particular, the choice of 2 in the exponent seems arbitrary, as do the limits of integration and the $x$ inside the integral. Let's define a more general functional, $P_{g,k}$, as follows. Set

$$P_{g,k}\left(f(x)\right) = \int_a^b g^k f - g f^k \, dx, \tag{1}$$

where $g$ is nonnegative on $[a,b]$, both $g$ and $f$ are continuous on $[a,b]$, and $k$ is an integer. We consider $g$ and $k$ to be parameters of $P$. The original problem is a special case with $g(x) = x$, $k = 2$, $a = 0$, and $b = 1$. We can now propose a more general question:

**2** *Let $g\left(x\right)$ be a nonnegative, continuous function on $[a,b]$, and let $k$ be an integer. Define the functional $P_{g,k}$ on all functions $f$ continuous on $[a,b]$ as follows: $P_{g,k}\left(f(x)\right) = \int_a^b g^k f - g f^k \, dx$. Find the extrema of $P$.*

In this paper, we will solve this problem.

## 2. Restrictions on $k$

As stated, our problem allows parameter $k$ to be any integer. We suspect that some $k$ will render $P$ unmaximizable. For instance, consider $P_{g,k}\left(cg\right)$, where $c$ is a real constant. Then

$$\begin{aligned} P_{g,k}\left(cg\right) &= \int_a^b g^k cg - g\left(cg\right)^k \, dx \\ &= \int_a^b c g^{k+1} - c^k g^{k+1} \, dx \\ &= \left(c - c^k\right) \int_a^b g^{k+1} \, dx. \end{aligned}$$

Note that $g$ and $k$ are fixed, and hence $\int_a^b g^{k+1} \, dx$ is some fixed number. We will now consider the behavior of $c - c^k$ under various restrictions on $k$.

- If $k = 0$, then $\lim\limits_{c \to \infty} c - c^k = \infty$ and $\lim\limits_{c \to -\infty} c - c^k = -\infty$. Hence, $P_{g,k}$ has no global extrema.
- If $k = 1$, then $P_{g,k}\left(cg\right) = 0$. This does not give us any information about the extrema of $P_{g,k}$, so we will examine this case in detail later.
- If $k$ is negative, then $\lim\limits_{c \to \infty} c - c^k = \infty$ and $\lim\limits_{c \to -\infty} c - c^k = -\infty$. Hence,

$P_{g,k}$ has no global extrema.

- If $k$ is positive, odd, and not 1, then $c$ and $c^k$ always have the same sign and $|c^k| > |c|$. In that case, $\lim\limits_{c\to\infty} c - c^k = -\infty$ and $\lim\limits_{c\to-\infty} c - c^k = \infty$. Hence, $P_{g,k}$ has no global extrema.
- If $k$ is even and positive, then $c^k$ is positive and $c < c^k$. Thus, $\lim\limits_{c\to\infty} c - c^k = -\infty$. Hence, $P_{g,k}$ has no minimum.

So, if $k$ is not a positive, even integer and $k \neq 1$, then $P_{g,k}$ has no global extrema, and if $k$ is a positive, even integer, $P_{g,k}$ has no minimum. We will now examine the two indeterminate cases in detail; $k = 1$ and $k \equiv 0 \pmod 2$ where $k > 0$.

## 3. The Trivial Case $k = 1$

If $k = 1$, then

$$\begin{aligned} P_{g,k}(f) &= \int_a^b gf - gf\,dx \\ &= \int_a^b 0\,dx \\ &= 0. \end{aligned}$$

Thus, if $k = 1$, $P_{g,k}(cg)$ has the trivial maximum and minimum 0, which is achieved by all $f$.

## 4. Maximizing $P_{g,k}$ for even $k$

To find the maximizer of $P$ in the general even case, we will first examine the cases $k = 2$ and $k = 4$. We will then attempt to generalize these solutions for all even positive $k$.

### The case $k = 2$

First, we'll try to find a function $f_2$ that might maximize $P_{g,2}$. Then we will show that $f_2$ does in fact maximize $P_{g,2}$. (We already suspect what form $f_2$ takes from the above derivation, but for the current exposition a more detailed analysis will be enlightening.) Note that if $f_2$ maximizes $P_{g,2}$, then $P_{g,2}(f_2) - P_{g,2}(f_2 + v) \geq 0$ for all $v$ continuous on $[a, b]$. Moreover,

$$
\begin{aligned}
& P_{g,2}\left(f_2\right) - P_2\left(f_2 + v\right) \\
= & \int_a^b g^2\left(f_2\right) - g(f_2)^2 dx - \int_a^b g^2\left(f_2 + v\right) - g(f_2 + v)^2 dx \\
= & \int_a^b g^2 f_2 - g{f_2}^2\, dx - \int_a^b g^2 f_2 + g^2 v - g\left({f_2}^2 + 2vf_2 + v^2\right)\, dx \\
= & \int_a^b g^2 f_2 - g{f_2}^2\, dx - \int_a^b g^2 f_2 + g^2 v - g{f_2}^2 - 2gvf2 - gv^2\, dx \\
= & \int_a^b g^2 f_2 - g{f_2}^2 - g^2 f_2 - g^2 v + g{f_2}^2 + 2gvf_2 + gv^2\, dx \\
= & \int_a^b 2gvf_2 - g^2 v + gv^2\, dx \\
= & \int_a^b gv\left(2f_2 - g\right) + gv^2\, dx \\
= & \int_a^b gv\left(2f_2 - g\right)\, dx + \int_a^b gv^2\, dx.
\end{aligned}
$$

Since $g$ is positive on $[a, b]$, we know that $\int_a^b gv^2\, dx$ is nonnegative for all $v$. Thus this expression can be negative only if $\int_a^b gv\left(2f_2 - g\right)\, dx$ is negative. To avoid this, we will simply require that this integral evaluate to $0$; specifically, that $2f_2 - g = 0$. Simple algebra reveals that $f_2 = \frac{g}{2}$ is a candidate for our maximizer of $P_2$.

Now to prove that $f_2$ maximizes $P_2$, we must show that $P_2\left(f_2\right) - P_2\left(h\right) \geq 0$ for all functions $h$ continuous on $[a, b]$. To that end, if $h$ is such a function, then

$$
\begin{aligned}
P_2\left(f_2\right) - P_2\left(h\right) & = \int_a^b g^2\left(f_2\right) - g(f_2)^2 dx - \int_a^b g^2\left(h\right) - g(h)^2 dx \\
& = \int_a^b g^2\left(\frac{g}{2}\right) - g\left(\frac{g}{2}\right)^2\, dx - \int_a^b g^2\left(h\right) - g(h)^2 dx.
\end{aligned}
$$

Simplifying gives

$$\begin{aligned}
P_2(f_2) - P_2(h) &= \int_a^b \frac{g^3}{2} - \frac{g^3}{4} - g^2 h + g h^2 \, dx \\
&= \int_a^b \frac{g^3}{4} - g^2 h + g h^2 \, dx \\
&= \int_a^b \frac{1}{4} g \left(g^2 - 4gh + 4h^2\right) \, dx \\
&= \int_a^b \frac{1}{4} g (g - 2h)^2 \, dx.
\end{aligned}$$

Clearly, the last integrand is always nonnegative on $[a, b]$, and thus the integral is always nonnegative. Hence, $P_2(f_2) - P_2(h) \geq 0$ for all $h$ continuous on $[a, b]$.

To summarize, $P_2(f)$ achieves its maximum value when $f(x) = \frac{g}{2}$, which (for future reference) can also be expressed as $f(x) = 2^{-\frac{1}{1}} g$.

The case $k = 4$

Now we'll try to find a function $f_4$ that might maximize $P_{g,4}$. Note that if $f_4$ maximizes $P_{g,4}$, then $P_{g,4}(f_4) - P_{g,4}(f_4 + v) \geq 0$ for all $v$ continuous on $[a, b]$. Moreover,

$$\begin{aligned}
& P_{g,4}(f_4) - P_{g,4}(f_4 + v) \\
= & \int_a^b g^4 (f_4) - g(f_4)^4 dx - \int_a^b g^4 (f_4 + v) - g(f_4 + v)^4 dx \\
= & \int_a^b g^4 f_4 - g f_4^4 \, dx \\
& - \int_a^b g^4 f_4 + g^4 v - g\left(f_4^4 + 4 f_4^3 v + 6 f_4^2 v^2 + 4 f_4 v^3 + v^4\right) \, dx \\
= & \int_a^b g^4 f_4 - g f_4^4 \, dx \\
& - \int_a^b g^4 f_4 + g^4 v - g f_4^4 - 4 g f_4^3 v - 6 g f_4^2 v^2 - 4 g f_4 v^3 - g v^4 \, dx \\
= & \int_a^b g^4 f_4 - g f_4^4 - g^4 f_4 - g^4 v + g f_4^4 + 4 g f_4^3 v + 6 g f_4^2 v^2 \\
& + 4 g f_4 v^3 + g v^4 \, dx.
\end{aligned}$$

Combining like terms gives

$$
\begin{aligned}
& P_{g,4}\left(f_4\right)-P_{g,4}\left(f_4+v\right) \\
= & \int_a^b 4 g f_4^3 v-g^4 v+g v^4+4 g f_4 v^3+6 g f_4^2 v^2 \, d x \\
= & \int_a^b g v\left(4 f_4^3-g^3\right)+g v^2\left(v^2+4 f_4 v+6 f_4^2\right) \, d x \\
= & \int_a^b g v\left(4 f_4^3-g^3\right) \, d x+\int_a^b g v^2\left(v^2+4 f_4 v+4 f_4^2+2 f_4^2\right) \, d x \\
= & \int_a^b g v\left(4 f_4^3-g^3\right) \, d x+\int_a^b g v^2\left(\left(v+2 f_4\right)^2+2 f_4^2\right) \, d x .
\end{aligned}
$$

Clearly, the right integrand of this line is nonnegative for all $v$. Thus this last expression can be negative only if $\int_a^b gv\left(4f_4^3 - g^3\right)\, dx$ is negative. As before, we will simply require that this integral evaluate to 0; specifically, that $4f_4^3 - g^3 = 0$. Simple algebra reveals that $f_4 = 4^{-\frac{1}{3}}g$ is a candidate for our maximizer of $P_{g,4}$.

We could prove that $f_4$ maximizes $P_{g,4}$ (it does), but instead we will use the functional forms of $f_2$ and $f_4$ to conjecture what function $f_k$ maximizes $P_{g,k}$ in the more general case where $k$ is even.

## The case $k = 2n$

From the previous analysis, it appears that, for even integers $k$, $f_k = gk^{-\frac{1}{k-1}}$ may maximize $P_{g,k}$. If and only if that is the case, we will have $P_{g,k}\left(f_k\right) - P_{g,k}(h) \geq 0$ for all $h$ continuous on $[a,b]$. So, if $h$ is such a function, we have

$$
\begin{aligned}
P_{g,k}\left(f_k\right)-P_k(h) & =\int_a^b g^k\left(f_k\right)-g\left(f_k\right)^k d x-\int_a^b g^k(h)-g(h)^k d x \\
& =\int_a^b g^k g k^{-\frac{1}{k-1}}-g\left(g k^{-\frac{1}{k-1}}\right)^k-g^k h+g h^k \, d x \\
& =\int_a^b g^{k+1} k^{-\frac{1}{k-1}}-g^{k+1} k^{-\frac{k}{k-1}}-g^k h+g h^k \, d x \\
& =\int_a^b g^{k+1} k^{\frac{k-1}{k-1}} k^{-\frac{k-1}{k-1}} k^{-\frac{1}{k-1}}-g^{k+1} k^{-\frac{k}{k-1}} \\
& \quad -g^k h+g h^k \, d x .
\end{aligned}
$$

Simplifying gives

$$\begin{aligned} P_{g,k}\left(f_k\right) - P_k\left(h\right) &= \int_a^b g^{k+1} k k^{-\frac{k}{k-1}} - g^{k+1} k^{-\frac{k}{k-1}} - g^k h + g h^k \, dx \\ &= \int_a^b \left(k-1\right) g^{k+1} k^{-\frac{k}{k-1}} - g^k h + g h^k \, dx \\ &= \int_a^b g \left[ \frac{k-1}{k^{\frac{k}{k-1}}} g^k - g^{k-1} h + h^k \right] dx. \end{aligned}$$

Clearly, because $g$ is positive on $[a, b]$, this integral is nonnegative when the bracketed expression is nonnegative; we will now attempt to prove that this is the case. To that end, let

$$z(x, y) = a x^k - x^{k-1} y + y^k,$$

where

$$a = \frac{k-1}{k^{k/(k-1)}}.$$

Because $h$ is a real valued function, proving that $z(x, y) \geq 0$ is sufficient to conclude that the last integrand is nonnegative.

Since

$$\frac{\partial z}{\partial x} = k a x^{k-1} - (k-1) x^{k-2} y$$

and

$$\frac{\partial z}{\partial y} = k y^{k-1} - x^{k-1},$$

$\nabla z = \left( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right)$ vanishes precisely where the lines $y = \left( \frac{k}{k-1} \right) a x$ and $y = k^{-1/(k-1)} x$ intersect; since $a = \frac{k-1}{k^{k/(k-1)}}$, these lines coincide. Hence $\nabla z(x, y) = \vec{0}$ when $y = k^{-\frac{1}{k-1}} x$; this line contains all the critical points of $z$.

There are three possible cases concerning $(x, y)$: either $y = k^{-1/(k-1)} x$, $y > k^{-1/(k-1)} x$, or $y < k^{-1/(k-1)} x$. Of course if $(x, y)$ is on the critical line then $z(x, y) = 0$. We will now show that in the other two cases, $z(x, y) > 0$.

1. If $y > k^{-1/(k-1)}x$, then we have
$$\begin{aligned} z_y(x,y) &= ky^{k-1} - x^{k-1} \\ &> k\left[k^{-\frac{1}{k-1}}x\right]^{k-1} - x^{k-1} \\ &= kk^{-1}x^{k-1} - x^{k-1} \\ &= x^{k-1} - x^{k-1} \\ &= 0 \end{aligned}$$
Thus if $y > k^{-1/(k-1)}x$, then $\frac{\partial z}{\partial y} > 0$.

2. If $y < k^{-1/(k-1)}x$, then we have
$$\begin{aligned} z_y(x,y) &= ky^{k-1} - x^{k-1} \\ &< k\left[k^{-\frac{1}{k-1}}x\right]^{k-1} - x^{k-1} \\ &= kk^{-1}x^{k-1} - x^{k-1} \\ &= x^{k-1} - x^{k-1} \\ &= 0 \end{aligned}$$
Thus if $y > k^{-1/(k-1)}x$, then $\frac{\partial z}{\partial y} < 0$.

Hence we have the following situation: in the positive $y$ direction, $z$ decreases until it reaches zero at the critical line. It then increases without bound. Therefore $z(x,y) > 0$ for all $(x,y)$. We can now revisit $P_{g,k}(f_{2n}) - P_{g,k}(h)$:

$$P_{g,k}(f_k) - P_{g,k}(h) = \int_a^b g\left[\frac{k-1}{k^{\frac{k}{k-1}}}g^k - g^{k-1}h + h^k\right]dx.$$

We now know that the bracketed expression is nonnegative for all $g$ and $h$, and thus the integral is nonnegative. That is,
$$P_{g,k}(f_k) - P_{g,k}(h) \geq 0$$
for all $h$ continuous on $[a,b]$. Therefore, if $k$ is an even positive integer, $P_k(f)$ reaches its maximum when $f_k = gk^{-1/(k-1)}$.

Moreover,

$$\begin{aligned}
P_{g,k}(f_k) &= \int_a^b g^k \left(gk^{\frac{-1}{k-1}}\right) - g\left(gk^{\frac{-1}{k-1}}\right)^k dx \\
&= \int_a^b g^{k+1}k^{\frac{-1}{k-1}} - g^{k+1}k^{\frac{-k}{k-1}} dx \\
&= \left(k^{\frac{-1}{k-1}} - k^{\frac{-k}{k-1}}\right)\int_a^b g^{k+1}\, dx \\
&= \left(k^{\frac{k-1}{k-1}}k^{-\frac{k-1}{k-1}}k^{\frac{-1}{k-1}} - k^{\frac{-k}{k-1}}\right)\int_a^b g^{k+1}\, dx \\
&= \left(kk^{\frac{-k}{k-1}} - k^{\frac{-k}{k-1}}\right)\int_a^b g^{k+1}\, dx \\
P_{g,k}(f_k) &= k^{-\frac{k}{k-1}}(k-1)\int_a^b g^{k+1}\, dx.
\end{aligned}$$

Hence, the maximum value of $P_{g,k}$ is $k^{-\frac{k}{k-1}}(k-1)\int_a^b g^{k+1}\, dx$.

## 5. Uniqueness of $f_k$

We wish to know whether $f_k$, the maximizer of $P_{g,k}$ where $k$ is even and positive, is unique. First, we assume that there exists a function $h$, not equal to $f_k = gk^{-1/(k-1)}$, such that $h$ maximizes $P_{g,k}$. Then it is true that $P_{g,k}(f_k) - P_{g,k}(h) = 0$; moreover,

$$\begin{aligned}
& P_{g,k}(f_k) - P_{g,k}(h) \\
= & \int_a^b g^k gk^{-1/(k-1)} - g\left(gk^{-1/(k-1)}\right)^k - g^k h + gh^k\, dx \\
= & \int_a^b \left(k^{\frac{-1}{k-1}} - k^{\frac{-k}{k-1}}\right)g^{k+1} - g^k h + gh^k\, dx \\
= & \int_a^b g\left[\frac{k-1}{k^{\frac{k}{k-1}}}g^k - g^{k-1}h + h^k\right] dx.
\end{aligned}$$

Note that we have already proven that the bracketed expression is non-negative for all $g$ and $h$. Hence this integral must evaluate to 0 only when the bracketed expression is 0. Previously we showed that this is the case precisely when $h = gk^{-1/(k-1)}$; this is a contradiction. Thus, if $k$ is an even positive integer, then the function $f_k$ which maximizes $P_{g,k}$ is unique.

## 6. Conclusion

**Theorem 1** *Let $g\left(x\right)$ be a nonnegative, continuous function on $\left[a,b\right]$, and let $k$ be an integer. Define the functional $P_{g,k}$ on all functions $f$ continuous on $\left[a,b\right]$ as follows: $P_{g,k}\left(f(x)\right)=\int_a^b g^k f - g f^k\,dx$.*

1. *If $k=1$, the maximum and minimum of $P_{g,k}$ is $0$, and is achieved by all $f$.*
2. *If $k\equiv 0\left(\mathrm{mod}\,2\right)$ and $k>0$, then the maximum of $P_{g,k}$ is*

$$k^{-\frac{k}{k-1}}\left(k-1\right)\int_a^b g^{k+1}dx,$$

*and is achieved only by $f_k\left(x\right)=gk^{-1/(k-1)}$. Moreover, $P_{g,k}$ does not have a minimum.*

3. *Otherwise, $P_{g,k}$ has no local extrema.*

As an example, consider the original Putnam problem. Now we can solve it by calculating $\max\left(P_{x,2}\right)$, with $a=0$ and $b=1$:

$$\begin{aligned}\max\left(P_{x,2}\right) &= 2^{-\frac{2}{2-1}}\left(2-1\right)\int_0^1 x^{2+1}\,dx\\ &= \frac{1}{4}\left[\frac{x^4}{4}\right]_0^1\\ &= \frac{1}{16}.\end{aligned}$$

Also, taking note of the role of $g$ in this exposition, we can propose the following conjecture:

**Conjecture 1** *Let $g\left(x\right)$ be a nonpositive, continuous function on $\left[a,b\right]$, and let $k$ be an integer. Define the functional $P_{g,k}$ on all functions $f$ continuous on $\left[a,b\right]$ as follows: $P_{g,k}\left(f(x)\right)=\int_a^b g^k f - g f^k\,dx$. Then $P_{g,k}$ has a global minimum when and only when $k$ is an even, positive integer.*

*Acknowledgements*: I'd like to acknowledge my advisor on this paper, Dr. Robert Robertson.

### *Reference*

[1] http://www.unl.edu/amc/a-activities/a7-problems/putnam/-pdf/2006s.pdf.

# *Finding an Optimal Solution to Bachet's Scale Problem*

Walter O. Krawec

New York Pi

Mount Saint Mary College
Newburgh, NY 12550

## 1. Introduction

In 1612 AD, Claude-Gaspar Bachet introduced a series of mathematical problems in his book *Problèmes plaisans et delectables qui se font par les nombres*. Contained within this text were several arithmetical puzzles including what is now called the Bachet Scale Problem.[1] Finding an answer to this problem will be the main focus of this paper; specifically we will apply a genetic algorithm in order to find an 'optimal' solution set.

The Bachet Scale Problem is relatively simple in its design. Imagine you're given a balance containing two 'pans' and using this balance you wish to find the integral weight of any object up to $T$ units. The problem asks what is the smallest number of integer weights you would need to 'purchase' in order to accomplish such a task. As an example, if you wanted to weigh an object up to and including 10 units ($T = 10$), you could purchase ten weights of value 1 - 10 each however the same effect can be created with weights of 1, 3, and 9 units. Using these three weights you could for example weigh a three unit object easily by placing your object on the right pan and the three unit weight on the left pan causing the balance to equal out. Furthermore, if you wanted to weigh say a two-unit object, place the object and the one-unit weight on the right pan and the three-unit weight on the left pan - the balance would equal out with three units of weights on each side (see Table 1). Bachet's Scale Problem poses the following question. What is the smallest number of weights capable of weighing any integral object up to T units [1].

| **Object Weight** | **Left Pan Weights** | **Right Pan Weights** |
| --- | --- | --- |
| 1 | 1 | 0 |
| 2 | 3 | 1 |
| 3 | 3 | 0 |
| 4 | $3+1$ | 0 |
| 5 | 9 | $3+1$ |
| 6 | 9 | 3 |
| 7 | $9+1$ | 3 |
| 8 | 9 | 1 |
| 9 | 9 | 0 |
| 10 | $9+1$ | 0 |

Table 1. Weights of 1, 3, and 9 units

## 2. Finding a Solution

In order to find a solution to this problem, let's start by first defining some important terms.

To construct some integer C, it needs to be of the form $C = |L - R|$ with $L, R \in \mathbb{N}$. $L$ and $R$ represent the total weight placed on each pan of the scale and C represents the object being weighed. Without loss of generality, we will assume throughout this paper that the object being weighed ($C$) is placed on the right pan therefore the left pan ($L$) must equal the sum of this object ($C$) and the weights placed on the right pan ($R$) in order for the scale to balance out. This implies that $L = C + R$ as you'll see in the following tables.

**Definition 1** *A weight set $W(n) = \{\omega_1, \omega_2, ..., \omega_n\}$ shall be the set of $n \geq 1$ positive integers such that the elements within $W$ can be used to construct any positive integer number (such that L and R are the sums of unique elements in $W(n)$) up to and including $S_n = \omega_1 + \omega_2 + \cdots \omega_n$, the total sum of the weights in $W(n)$.*

We shall begin by trying to find a weight set for $T = 1$. The only solution to this is the obvious $W(1) = \{1\}$. There can be no smaller set for this value of $T$ (as a weight set must contain at least one element). Also, the first value must be a 1 for any other value would require a weight in $R$ to construct $C = 1$.

Now let's say we wish to construct an integer larger than 1. We can build onto our set by adding a new element, but what should we choose for our second weight? If $\omega_2 = 2$, we would have the set $W(2) = \{1, 2\}$, which would allow us to construct the values shown in Table 2. This is not the best choice, however, if we wish to achieve a maximum effect. Choosing a 3 for our new element would allow us to construct the values shown in Table 3 which includes all of Table 2's values with the addition of $C = 4$. If, however, we choose a 4 for our new element (giving us the set $W(2) = \{1, 4\}$) we would be unable to construct $C = 2$.

If we think about this algorithmically, the best choice for our next element should be a value (say $a$) such that the sum of the elements in $W(1)$ subtracted from $a$ should equal the next integer value we need to construct (2 in this case or in more general terms, the next integer we need to construct would be the sum of the elements in $W(1)$ plus 1). In other words: $a - 1 = 1 + 1 \Rightarrow a = 3$ which would imply that $W(2) = \{1, 3\}$. This new set of weights can construct any integer up to and including $S_2 = 1 + 3 = 4$ (as shown in Table 3).

| **C** | **L** | **R** |
|---|---|---|
| 1 | 1 | 0 |
| 2 | 2 | 0 |
| 3 | $1+2$ | 0 |

Table 2. Possible weights using $W(2) = \{1, 2\}$

| **C** | **L** | **R** |
|---|---|---|
| 1 | 1 | 0 |
| 2 | 3 | 1 |
| 3 | 3 | 0 |
| 4 | $3+1$ | 0 |

Table 3. Possible weights using $W(2) = \{1, 3\}$

Taking this a step further, let's say we wish to construct an integer greater than 4. We'll obviously need to add a third element to our weight set so let's use the same logic as before to choose the optimal weight to achieve a maximum effect. The new element that's to be added to $W(3)$ should be a number (say $b$) such that the sum of the elements in $W(2)$ subtracted from this new element should equal the sum of the elements in $W(2)$ plus 1. In other words: $b - 4 = 4 + 1 \Rightarrow b = 9$ meaning that $W(3) = \{1, 3, 9\}$ and this can be used to construct integers as shown in Table 4. We can apply this logic in more general terms to say that the most efficient choice for the next element in $W(n+1)$ should be an integer (say $\omega_{n+1}$) such that: $\omega_{n+1} - S_n = S_n + 1 \Longrightarrow \omega_{n+1} = 2S_n + 1$.

At this point we can begin to see a pattern: all the elements in $W(3)$ are powers of 3. That is: $W(3) = \{3^0, 3^1, 3^2\}$. But does this pattern hold

| C | L | R | C | L | R |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 8 | 9 | 1 |
| 2 | 3 | 1 | 9 | 9 | 0 |
| 3 | 3 | 0 | 10 | 9 + 1 | 0 |
| 4 | 3 + 1 | 0 | 11 | 9 + 3 | 1 |
| 5 | 9 | 3 + 1 | 12 | 9 + 3 | 0 |
| 6 | 9 | 3 | 13 | 9 + 3 + 1 | 0 |
| 7 | 9 + 1 | 3 | | | |

Table 4. Possible weights using $W(3) = \{1, 3, 9\}$

for any $n$? Let's suppose that for any $n$, $W(n) = \{\omega_1, \omega_2, \ldots, \omega_n\}$ with $\omega_n = 3^{n-1}$. The sum of the elements of $W(n)$ would be:

$$S_n = \sum_{i=1}^{n} \omega_i = \sum_{i=1}^{n} 3^{i-1} = 3^0 + 3^1 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$$

A weight set capable of constructing any positive integer up to and including $T$, is $W\left(\left\lceil \frac{\log(2T+1)}{\log 3} \right\rceil\right) = \left\{\omega_1, \omega_2, ..., \omega_{\left\lceil \frac{\log(2T+1)}{\log 3} \right\rceil}\right\}$, where $\omega_n = 3^{n-1}$.

**Theorem 2** *A weight set capable of constructing any positive integer up to and including $T$, is $W\left(\left\lceil \frac{\log(2T+1)}{\log 3} \right\rceil\right) = \left\{\omega_1, \omega_2, ..., \omega_{\left\lceil \frac{\log(2T+1)}{\log 3} \right\rceil}\right\}$, where $\omega_n = 3^{n-1}$.*

We can determine whether or not the powers of three can be used to create a minimum set of weights. As mentioned previously, if given a minimum set of weights $W(n)$, the next element in $W(n+1)$ should be $\omega_{n+1} = 2S_n + 1$. We can easily show that this identity holds true:

$$\begin{aligned} & \omega_{n+1} = 2S_n + 1 \\ \iff\quad & 3^{(n+1)-1} = 2\left(\frac{3^n - 1}{2}\right) + 1 \\ \iff\quad & 3^n = 3^n - 1 + 1 \\ \iff\quad & 3^n = 3^n. \end{aligned}$$

We can take this one step further. Suppose we were asked to find a minimum set of weights that can construct any positive integer up to and including $T$. By definition, $W(n)$ can construct any positive integer up to and including $S_n$. Thus,

$$\begin{array}{rcl} T & \leq & S_n \\ & \iff & T \leq \dfrac{3^n - 1}{2} \\ & \iff & 2T \leq 3^n - 1 \\ 2T + 1 & \leq & 3^n \\ n & \geq & \log_3 (2T + 1) \,. \end{array}$$

Since we want $n$ to be an integer greater than or equal to $\log_3 (2T + 1)$, we set $N = \lceil \log_3 (2T + 1) \rceil = \left\lceil \frac{\log(2T+1)}{\log 3} \right\rceil$, where $\lceil r \rceil$ is the ceiling function.

## 3. An Optimal Solution

In Theorem 2 we define a general approach to creating a weight set for some target value of $T$ however there are many more solutions to this problem some of which are perhaps 'better' than others. Theorem 2 will provide us with a set of weights that can construct integers up to $S_n$ which can be greater than $T$ our target value. Obviously this provides a set of the most efficient weights (meaning we can weigh more objects than our target) however if we know exactly the highest value we needed to weigh, wouldn't it be best to purchase smaller weights? Perhaps smaller weights are cheaper than heavier ones; or perhaps we don't want to have to carry around these heavier weights if we really don't need them.

For example, if we wanted to weigh objects up to and including 10 units, using Theorem 2 we would have the set $W(3) = \{1, 3, 9\}$ which is capable of constructing integers up to and including $1 + 3 + 9 = 13$. However another possible solution to this problem would be the set of weights: $W = \{1, 2, 7\}$ which is capable of constructing integers up to and including $1 + 2 + 7 = 10$ exactly. Using the second set, there is no 'waste'. Alternatively the set $W = \{1, 3, 6\}$ could be used with the same effect. So perhaps we should expand the problem by asking not only what is the smallest set of weights capable of constructing any integer value up to and including $T$, but instead what is an *optimal* solution? We make the following definitions.

**Definition 2** *Let's define a solution as a set of positive non-zero integer weights $W$ such that $W = \{\omega_1, \omega_2, ...,\}$, where $\omega_1$, $\omega_2$, ... can be used to construct any positive integer up to and including some target value $T$. Also, we'll define the following:*

$$\varepsilon = \textit{number of integers not constructible} \tag{2}$$

$$\sigma(W) = \sum_{i=1}^{|W|} \omega_i. \tag{3}$$

**Definition 3** *A set $W$ of weights is optimal if*

- The elements of $W$ can be used to construct any positive integer up to and including $T$.
- $|W|$ is a minimum. (We still want the fewest weights possible.)
- $\sigma(W)$ is a minimum.

We are now presented with a problem: how do we go about searching for an optimal solution?

## 4. Genetic Algorithms

When searching for an optimal solution to a problem there are many strategies we could apply. In this paper however we shall make use of a genetic algorithm (GA) - a particular kind of algorithm that turns out to be well suited to this particular problem. In general the algorithm works through a form of "natural selection" where a collection of possible solutions are found and only those that are considered 'better' than the others are kept while the rest are discarded - basically "survival of the fittest" [2].

The algorithm works by first creating an initial 'population' or collection of random sets of weights (or possible solutions). Since these are all random, the probability that even a single solution is correct (i.e. has $\epsilon = 0$) is very small yet for each of these possible solutions we assign a fitness value - a single value $[0, \infty)$ that determines how optimal a possible solution is (the closer to $0$, the better). After each set of weights has been assigned a fitness value, the algorithm chooses some of the very best solutions (along with a few random ones) and, using information from these sets, forms a new collection of possible solutions. These new possible solutions are created using the crossover operation which takes two weight sets and forms a new third (see Definition 4). The algorithm then performs a 'mutation' operation on a small portion of randomly selected solutions from the population (this operation randomly modifies a small element of a solution - see Definition 5). Each new possible solution is assigned a fitness value and the algorithm repeats [3].

**Definition 4** *If given two possible solutions $A = \{\alpha_1, \alpha_2, ...\}$ and $B = \{\beta_1, \beta_2, ...\}$, then we'll define the crossover function as*

$$A \otimes B = \left\{\alpha_1, \alpha_2, ..., \alpha_{\frac{|A|}{2}}, \beta_{\frac{|B|}{2}}, \beta_{\frac{|B|}{2}+1}, ..., \beta_{|B|}\right\}.$$

**Definition 5** *If given a possible solution $A = \{\alpha_1, \alpha_2, ...\}$, then the mutation function will do one of three things:*

- Change one random element in $A$ (say $\alpha_r$) such that $1 \leq \alpha_r \leq T$.
- Add $n$ random integers of value between 1 and $T$ to $A$.
- Remove $n$ random elements from $A$ such that $1 \leq n < |A|$.

We now have enough information to formally define our algorithm.

Genetic Algorithm for Bachet's Scale Problem

- Create initial random population $P$
- Repeat
  - Transfer 25% of the strongest solutions (those with the lowest fitness value) from $P$ into a new set $S$.
  - Transfer 25% of random solutions from $P$ into set $S$.
  - Repeat.
    * Choose two random solutions from $S$; call them $A$ and $B$.
    * Add two new solutions to $P$ : $A \otimes B$ and $B \otimes A$ (see 4)
    * Remove $A$ and $B$ from $S$ and return them to $P$.
  - until $|S| \leq 1$
  - Perform mutation operation on $5\%$ of randomly selected solutions in $P$.
  - Destroy 10% of the weakest solutions (those with the highest fitness value) in $P$.
- until done

With the definitions of $\epsilon$ and $\sigma$ (Equations 2 and 3 respectively) in mind, we can begin to formulate our fitness function (the function responsible for determining how optimal a solution is) however this must be done with care for even a slight change to the fitness function could cause the genetic algorithm to yield completely different results (and we'll see this later). From Definition 3, we know that there are three factors to take into account when evaluating a possible solution $W$'s fitness. These factors are: the error of $W$ (we obviously want $\epsilon = 0$); the number of elements in $W$; and the sum of the elements in $W$ (or $\sigma(W)$). These three factors however are not all of equal importance (for example having $\epsilon = 0$ is far more important than having a small $\sigma(W)$). Our first attempt at a fitness function is $\gamma_1(W)$.

$$\gamma_1(W) = \epsilon^3 + |W|^2 + \sigma(W) \tag{4}$$

Notice that smaller values of $\gamma$ represent more optimal solutions. Also notice that greater importance is placed upon the error of the solution ($\epsilon$ which is cubed) than the sum of the solution ($\sigma(W)$).

Using the above definitions, a computer program can be created to attempt to find an optimal solution. The PC program was written in C++ and is available for download at `http://www.geocities.com/walter.krawec/bachet`. Running this program proved to generate some interesting results as you can see in Table 5. You'll notice that there were times where the GA would generate solutions that contained more elements than Theorem 2 would suggest is needed yet their total sum was much smaller. The reason our GA turned out such solutions is due in part to our fitness function ($\gamma_1(W)$). If we change it to the equation shown below ($\gamma_2(W)$) the algorithm generates a completely different set of solutions.

$$\gamma_2(W) = 2T(|W| - N)^5 + \epsilon^3 + \sigma(W), \tag{5}$$

with $N$ defined in the proof of Theorem 2.

Using $\gamma_2(W)$ as our fitness function, the solutions shown in Table 6 are generated. You can see that in this new function we place a lot more importance on the size of the generated weight set ($|W|$) in comparison to the size of the weight set that would be generated by Theorem 2 ($N$). Because of this modification the algorithm generates even more optimal solutions (that is they contain fewer elements than $\gamma_1(W)$ produced and their sum is still small).

| T | $W(n)$ | $S_n$ | GA weight set | $\sigma$ |
|---|---|---|---|---|
| 10 | $\{1,3,9\}$ | 13 | $\{1,2,7\}$ | 10 |
| 42 | $\{1,3,9,27,81\}$ | 121 | $\{1,3,7,7,24\}$ | 42 |
| 125 | $\{1,3,9,27,81,243\}$ | 364 | $\{1,2,5,16,18,83\}$ | 125 |
| 528 | $\{1,3,9,27,81,243,729\}$ | 1093 | $\{10,25,32,34,42,53,71,77,80,194\}$ | 618 |
| 1287 | $\{1,3,9,27,81,243,729,2187\}$ | 3280 | $\{5,37,44,58,59,70,122,274,832\}$ | 1501 |

Table 5. Weight Sets using Fitness Function Defined by $\gamma_1(W)$

| $T$ | $W(n)$ | $S_n$ | GA weight set | $\sigma$ |
|---|---|---|---|---|
| 10 | $\{1,3,9\}$ | 13 | $\{1,3,6\}$ | 10 |
| 42 | $\{1,3,9,27,81\}$ | 121 | $\{1,3,5,14,19\}$ | 42 |
| 125 | $\{1,3,9,27,81,243\}$ | 364 | $\{2,6,7,25,29,62\}$ | 131 |
| 528 | $\{1,3,9,27,81,243,729\}$ | 1093 | $\{1,3,9,22,43,124,329\}$ | 531 |
| 1287 | $\{1,3,9,27,81,243,729,2187\}$ | 3280 | $\{8,15,24,36,46,153,297,831\}$ | 1410 |

Table 6. Weight Sets using Fitness Function Defined by $\gamma_2(W)$

## 5. Conclusion

Finally, there are a number of improvements that could be made to the genetic algorithm program. The fitness function ($\gamma_2(W)$) could most likely be even further improved upon. The software does seem to have difficulty converging on a solution for $T > 2000$. This problem could be the result of the fitness function or a timing issue. This of course leads to the other possible improvement which would be to optimize the C++ code allowing it to find a possible solution faster. Currently the program runs inefficiently (on a Intel Core 2 CPU 6400 PC it takes roughly 90 minutes to converge on a solution for $T = 1287$). This slow pace could also be why the program has difficulty converging on a solution for large values of $T$ - the program might simply need more time to run in order to find a solution. I encourage the reader to download the software and attempt to improve upon it.

*Acknowledgements*: The author would like to thank Dr. Mike Daven and Dr. Lee Fothergill for their invaluable assistance in the writing of this paper.

## *References*

[1] School of Mathematics and Statistics University of St Andrews, Scotland: `http://www-groups.dcs.st-and.ac.uk/history/Biographies/Bachet.html`.

[2] D.E. Goldberg, *Genetic Algorithms in Search, Optimization, and Machine Learning*, Addison-Wesley, 1989.

[3] R.L. Haupt and S.E. Haupt, *Practical Genetic Algorithms*, Wiley, 1998.

# PascGalois Triangles: Visualizing Abstract Algebra Concepts Using Pascal's Triangle and Group Theory

Casey Kuhn, *student*

KS Alpha

Pittsburg State University
Pittsburg, KS 66762
Presented at the 2007 National Convention.

Note: Since the diagrams in this article make extensive use of color, we invite you to view the article on-line at kappamuepsilon.org.

## 1. Introduction

Algebra has been around for a really long time. Some estimate it has been around over 4,000 years. It is, essentially, "finding the unknown number." This was the only type of algebra until 200 years ago [8]. This new and modern type of algebra is more theoretical. Instead of looking at individual problems, it looked more into structures. This branch of mathematics is now known as Abstract Algebra [7, p. ix].

A big part of abstract algebra is the theory of groups. From 1770 until 1876, there were advances in four different fields of mathematics. Lagrange was studying polynomial equations, while Gauss was involved in number theory; Klein was working on transformations and also conferring with Lie and Poincare on the subject of analysis. Certain similarities were discovered between these branches of mathematics (Kleiner). All of their research seemed to produce results that had the same structure. It included a non-empty set together with a binary operation that is closed, associative, has an identity element and every element has an inverse. This would evolve to be the definition of a group [19].

As the name implies, PascGalois Triangles have their roots based with two mathematicians: Pascal and Galois. Blaise Pascal was a child prodigy. He showed great skill in mathematics from an early age. His interests were mainly in projective geometry and probability theory. He also worked in other areas as well. One of the topics he explored was binomial coefficients. Pascal noticed a pattern and developed a rule for the combinatorial identity about binomial coefficients. It states that for any natural number $n$ we have

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

where $1 \leq k \leq n$, and $\binom{n}{k}$ is a binomial coefficient. This would come to be known as Pascal's Rule [11]. The proof from [10] for this rule follows.

We need to show

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.$$

Begin by writing the left-hand side as

$$\frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-(k-1))!}.$$

Getting a common denominator and simplifying, we have

$$\begin{aligned} & \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-(k-1))!} \\ = \quad & \frac{(n-k+1)\,n!}{(n-k+1)\,k!\,(n-k)!} + \frac{kn!}{k\,(k-1)!\,(n-k+1)!} \\ = \quad & \frac{(n-k+1)\,n! + kn!}{k!\,(n-k+1)!} \\ = \quad & \frac{(n+1)\,n!}{k!\,((n+1)-k)!} \\ = \quad & \frac{(n+1)!}{k!\,((n+1)-k)!} \\ = \quad & \binom{n+1}{k}. \end{aligned}$$

## 2. Pascal's Triangle

Upon further review of the binomial coefficients, Pascal noticed that arranging them in a certain way yielded interesting results. At the age of 30, he published Traité du triangle arithmétique ("*Treatment of the Arithmetical Triangle*"). This paper talked about "a convenient tabular presentation for binomial coefficients" [2]. This would eventually come to be known as Pascal's Triangle [12].

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

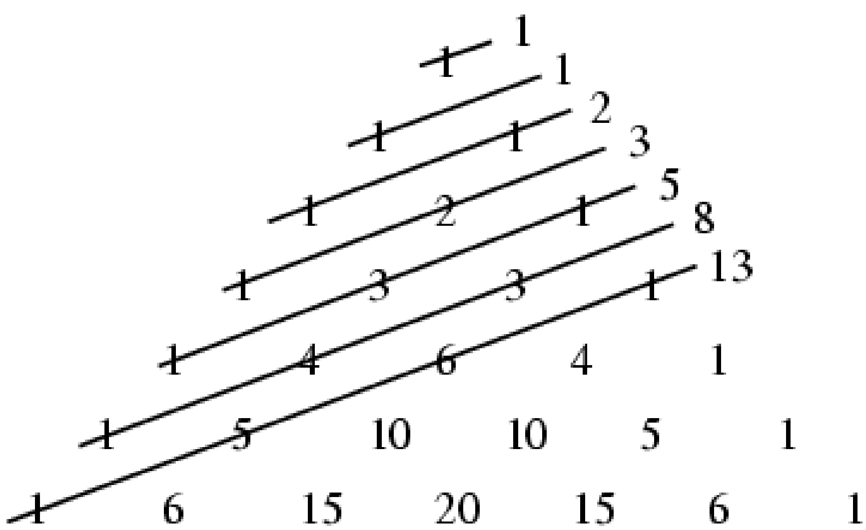
1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1

1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1

There are many interesting properties in Pascal's Triangle. Every number on the triangle (except the top 1 and the second row) is the sum of the two numbers above it. This property is shown in Pascal's Rule. Both the left and right diagonals contain only 1's. Moving inward, the next diagonals contain the natural numbers in order. Continuing inward, the diagonals are the triangular numbers in order. Another diagonal in contains the tetrahedral numbers, followed by the pentatope numbers. This pattern continues [12]. The Fibonacci numbers also appear when more shallow diagonal lines are summed. [18]




As with many mathematical terms, "Pascal's Triangle" was not named after the first mathematician to address subject. Omar Khayyam and Yang Hui both looked at the concept long before Pascal was even around [12]. It is therefore known as the Yang Hui Triangle in China [18].

**3. Examples of Groups**

As the name implies, Galois' contributions to mathematics are also a big part of the PascGalois Project. In abstract algebra, there is a branch known as Galois Theory. Abstract algebra looks at algebraic structures, while Galois Theory focuses in on two specific structures: fields and groups ([6]. Galois originated the idea of a group. These groups will become the elements of the PascGalois Triangles.

There are many common examples of groups. For instance, the Real Numbers with addition form a group. Adding any real numbers will produce a real number. The real numbers are associative with addition. The identity element is 0; that is, adding zero to any real number will produce that real number. Each real number also has an inverse that, when added together, produce zero, the identity element [13].

Another example of a group is $Z_n$ (where $n$ is any positive integer) with addition. For instance $Z_{12}$ is similar to a clock. There are 12 elements that when added together, continue on a cycle instead of infinitely growing like the real numbers do. Here is the addition table for $Z_{12}$ [3].

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 |
| 7 | 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 8 | 8 | 9 | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 9 | 9 | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 10 | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 11 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

You can combine $Z_n$ with another $Z_n$ to get a group knows as $Z_n \times Z_n$. The most known example of this is $Z_2 \times Z_2 = \{(0,0), (0,1), (1,0), (1,1)\}$. Each element contains two parts, one coming for the first $Z_2$ and the second part coming from the last $Z_2$. Here is the addition table for $Z_2 \times Z_2$.

| $*$ | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
|---|---|---|---|---|
| $(0,0)$ | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
| $(0,1)$ | $(0,1)$ | $(0,0)$ | $(1,1)$ | $(1,0)$ |
| $(1,0)$ | $(1,0)$ | $(1,1)$ | $(0,0)$ | $(0,1)$ |
| $(1,1)$ | $(1,1)$ | $(1,0)$ | $(0,1)$ | $(0,0)$ |

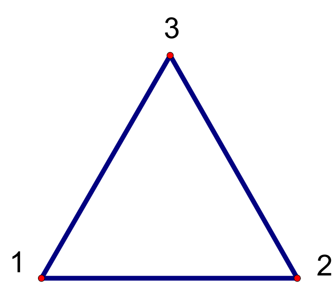
### 4. Dihedral Groups

All the above groups have all dealt with integers, but there are other types of groups. For instance, there are Dihedral Groups. "The dihedral group $D_n$ is the symmetry group of an $n$-sided regular polygon" [17]. In other words, it is "the rotations and reflections [that are preformed on] a regular $n$-gon [4]. Using $R$ for rotations and $F$ for flips or reflections, you can define the dihedral group [5] as:

$$D_n = \left\{R^i F^j : R^n = e, F^2 = e, FRF^{-1} = R^{-1}\right\}.$$

There are $n$ rotations for an $n$-sided polygon. You can rotate the polygon $2\pi/n$ radians (or $360/n$ degrees) about the center to get a symmetric polygon. You can also flip or reflect the polygon about a line through a vertex and the center. When you combine these two facts, the order of the dihedral group is $2n$. This means that there are $2n$ different symmetries that can happen on an n-sided polygon. It should also be noted that performing no rotations or reflections produces the identity element [5]. Each element (rotation, reflection, or combination) also had an inverse. This means that there is a way to get from each element to the identity element by combining it with an element.

To visualize the dihedral groups, think about a triangle.




| $*$ | H | $R_1$ | $R_2$ | $F_1$ | $F_2$ | $F_3$ |
|---|---|---|---|---|---|---|
| H | H | $R_1$ | $R_2$ | $F_1$ | $F_2$ | $F_3$ |
| $R_1$ | $R_1$ | $R_2$ | H | $F_2$ | $F_3$ | $F_1$ |
| $R_2$ | $R_2$ | H | $R_1$ | $F_3$ | $F_1$ | $F_2$ |
| $F_1$ | $F_1$ | $F_3$ | $F_2$ | H | $R_2$ | $R_1$ |
| $F_2$ | $F_2$ | $F_1$ | $F_3$ | $R_1$ | H | $R_2$ |
| $F_3$ | $F_3$ | $F_2$ | $F_1$ | $R_2$ | $R_1$ | H |

This table uses $H$ for the identity element symbolizing no rotations or reflections. $R_1$ is the rotation of $120°$ and $R_2$ is the rotation of $240°$. The element $F_1$ is the flip or reflection about the line through the vertex 1 and the center. Similarly, $F_2$ is the reflection about the line though the vertex 2 and the center. Also, $F_3$ is the reflection about the line through the vertex 3 and the center [19].

## 5. Basics of Group Theory

There are certain properties or concepts general to all groups. For instance, there is the term isomorphic. Two groups are isomorphic if there exists a function from one group to the other that is one-to-one, onto, and it preserves the group operation [7, p. 118]. The term isomorphic is basically calling the groups the same, but with different names for the elements [19].

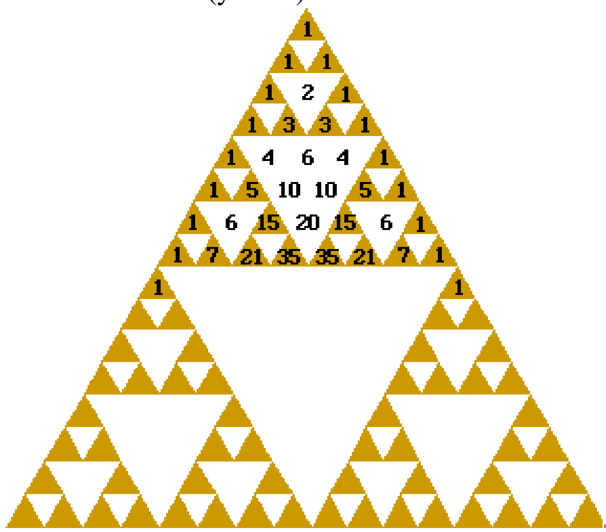
Another term common to all groups is the "subgroup." A subgroup is a subset of the elements of a group that form a group themselves. There are two subgroups of every group: the identity element and the entire group. There are referred to as the trivial subgroups [7, p. 131].

You can take one of these subgroups and create cosets. If you multiply ever element in a subgroup by one of the elements in the group, you will get a coset. You can either multiply on the left or right, creating left or right cosets. Basically, all the elements in the subgroup are grouped together because when you multiply the subgroup by the identity, you will get the subgroup [7, p. 186]. Similar to cosets are quotient groups. "If $H$ is a subgroup of $G$, the group $G/H$ that consists of the cosets of $H$ in $G$ is called the quotient group of $G$ by $H$". Cosets evenly divide the group. They create a quotient group if you consider each coset to be an element [7, p. 197].

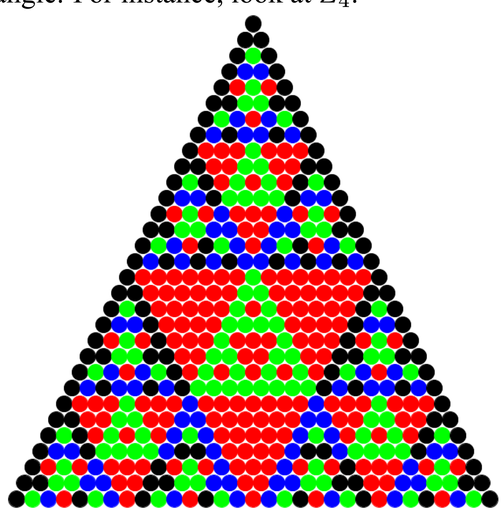
## 6. The PascGalois Triangle

These concepts are somewhat "abstract." Many times people studying the topic of abstract algebra (mainly undergraduate students) have difficulties visualizing the concepts and ideas. For this reason, Drs. Michael Bardzell and Kathleen Shannon came up with the idea of the PascGalois Triangle [1]. It takes the idea of generating a triangle by adding the two elements above it, similar to Pascal's Triangle. However, instead of just using the integers, it uses elements from different groups in abstract algebra. Then, it assigns each element a different color. The simplest of these PascGalois Triangles occurs when you make all the even numbers in Pascal's Triangle white and the odd numbers yellow. The picture of this is also known as the Sierpinski Gasket [15].

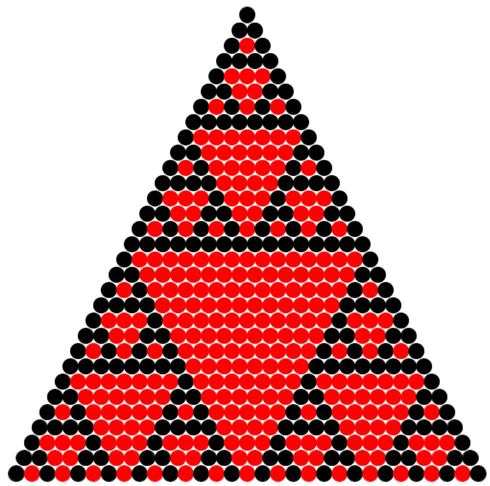
This is the same as the PascGalois Triangle $Z_2$ because it takes the integers separates them into two groups (odds and evens) which act like the elements of $Z_2$, $\{0, 1\}$, because when you add two odd numbers together, you get an even number and when you add an even to an odd number, you get an odd. The 0 is representative of the even numbers (white), and the 1 is representative of the odds (yellow).



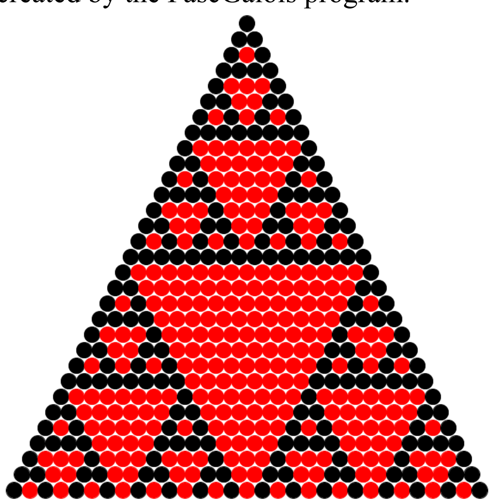

Following this pattern, we can use other $Z_n$ as the group to generate a PascGalois Triangle. For instance, look at $Z_4$.



From this you can see that there are four different elements, $\{0, 1, 2, 3\}$, denoted by four different colors. In this image, 0 is red, 1 is black, 2 is green and 3 is blue. You can create a quotient group if you take the cosets generated by the element $\{2\}$. You will end up with two sets in your quotient group: $\{0, 2\}$, $\{1, 3\}$. If you color the elements in each set the same, you'll end up with the following PascGalois Triangle.

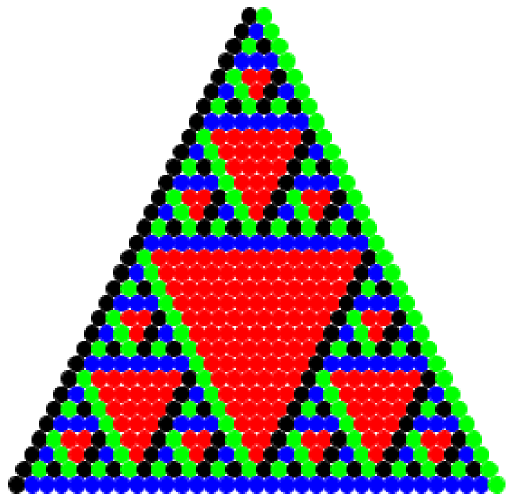


The pattern in this triangle looks similar to the Sierpenski Gasket. Compare it with $Z_2$ created by the PascGalois program.



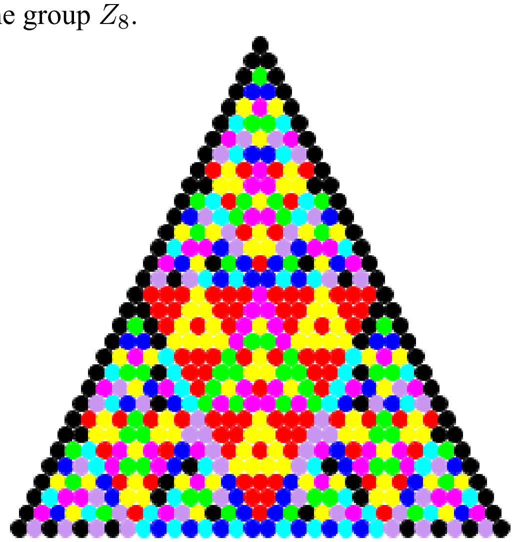
These two triangles have the same PascGalois pattern, which makes them isomorphic. Keep in mind that color doesn't matter (although it happens to be in this example that the colors are the same). The important aspect is the pattern the colors make.

Now look at $Z_2 \times Z_2$, which has 4 elements $\{(0,0),(0,1),(1,0),(1,1)\}$. In order for all 4 elements to appear, you need to start with two elements, one for each outer diagonal $(0,1)$ and $(1,0)$.

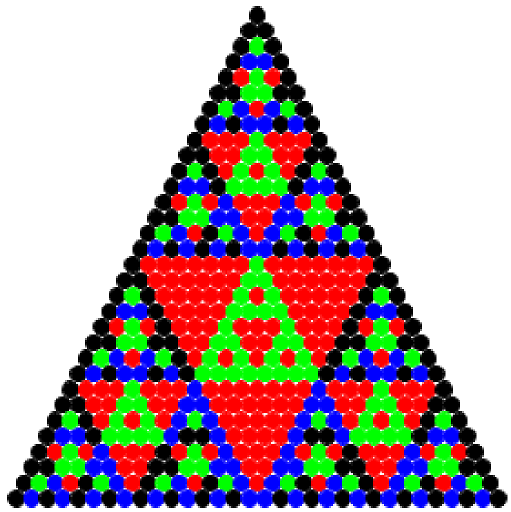


This group has the potential to be isomorphic to $Z_4$ because they have the same number of elements. However, when you compare the color patterns, you notice that $Z_2 \times Z_2$ has a line going down the inside triangle that $Z_4$ doesn't have. This means that $Z_4$ and $Z_2 \times Z_2$ are not isomorphic to each other.

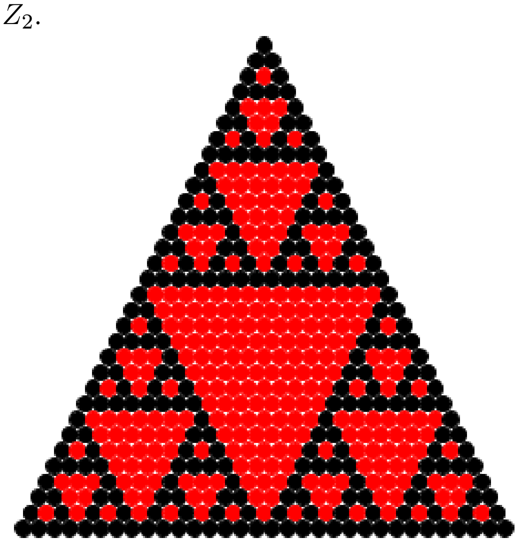
Consider the group $Z_8$.



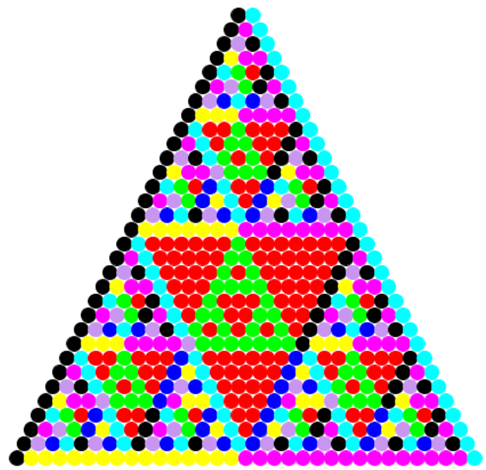
If you group the elements by left cosets of the element $\{4\}$, you will end up with 4 different cosets: $\{0,4\},\{1,5\},\{2,6\},\{3,7\}$. When each coset is colored similarly, it will produce an image that looks similar in pattern to $Z_4$.



If the elements are grouped by the left cosets of the element $\{2\}$, you will end up with 2 cosets: $\{0,2,4,6\}$ and $\{1,3,5,7\}$. Coloring these cosets the same will produce an image that has the same pattern as $Z_2$, the Sierpinski gasket. This means that the quotient group $Z_8/<2>$ is isomorphic to $Z_2$.

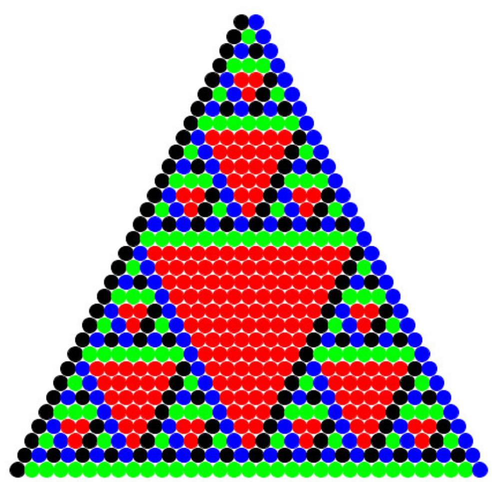


In addition to $Z_n$, PascGalois Triangles allow the visualization of other groups, such as the Dihedral Groups.



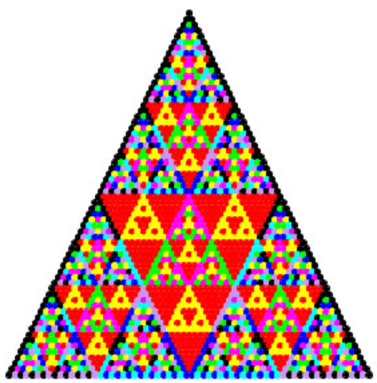
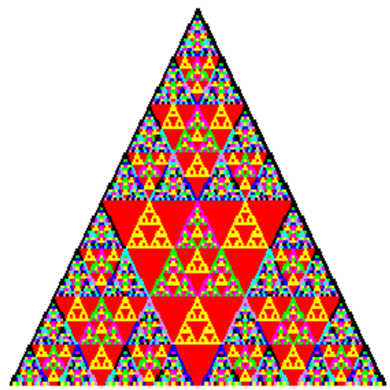
As an example, $D_4$ is a group with order 8. Therefore, there is a possibility that it is isomorphic to $Z_8$. However, upon examination, you notice that there are red down-pointing triangles in the $D_4$ that are not present in the $Z_8$ PascGalois triangle. This means that these two groups are not isomorphic.

Consider the dihedral group $D_2$, which has 4 elements.



When you look at this image, the pattern is identical to the pattern of $Z_2 \times Z_2$. The only difference is that the green and blue dots have switched. This means that $D_2$ and $Z_2 \times Z_2$ are isomorphic.

There are some elements of the PascGalois Triangles that need to be addressed. For instance, comparing triangles for isomorphisms can only happen if you are displaying the same number of rows in the triangles. If you have two triangles with varying row numbers, the images won't look similar. Unless, you happen to look get a row number that shows the self-similar property of PascGalois triangles. For instance, $Z_8$ with 64 rows will pretty much the same as $Z_8$ with 125 rows because of the self-similar property [14].

You can notice the big red down-pointing triangle with three little yellow up-pointing triangles in both images. The look similar, but the one on the left looks like it is a clearer image.

Another important aspect of PascGalois Triangles is the color schemes. Sometimes it is hard to see the similar patterns in two triangles when the colors are different. By changing around the colors, you can see the isomorphisms (or lack there of) much easier.

PascGalois Triangles present a way for abstract algebra to be visualized. Students can literally "see" if a group is isomorphic to another group. They can "see" how two elements of a group are added together. Visual learners are sometimes left out in mathematics, especially algebra, because of the logically nature of the subject. This is one way to include the visual learners.

The possibilities of the applications of the PascGalois Triangles are infinite. Fifteen pages aren't enough to explain all the ins and outs of these figures. There are many types of groups not addressed here. Numerous concepts and theorems (including the few acknowledged here) in abstract algebra can be explained through PascGalois Triangles. Mathematics is an ever-growing field needing new ways of presenting both old and new information.

### ***References***


[1] M.J. Bardzell and K.M. Shannon,.The PascGalois Project: Visualizing Abstract Algebra, *Focus*, March 2002, pp. 4 – 5.

[2] http://en.wikipedia.org/wiki/Blaise_Pascal

[3] http://britton.disted.camosun.bc.ca/modart/jbmodart.htm

[4] http://www.mathreference.com/grp,dih.html.

[5] B. Ettinger and G.M. Guy, "The Dihedral Group," Grand Valley State University.

[6] http://en.wikipedia.org/wiki/Galois_theory

[7] J. Gilbert and L. Gilbert, *Elements of Modern Algebra*, 6th ed., Belmont: Thompson Brooks/Cole, 2005.

[8] <http://www.ucs.louisiana.edu/~sxw8045/history.htm>.

[9] I. Kleiner, The Evolution of Group Theory: A Brief Survey, *Mathematics Magazine,* October 1986.

[10] http://planetmath.org/?op=getobj&from=objects&id=259.

[11] http://en.wikipedia.org/wiki/Pascal_rule.

[12] http://en.wikipedia.org/wiki/Pascal_triangle.

[13] R. Sandler and L.S. Foster, *Modern Algebra*, Harper & Row Publishers, 1978.

[14] K.M. Shannon and M.J. Bardzell, Patterns in Pascal's Triangle – with a Twist, *Journal of Online Mathematics and its Applications*.

[15] "Sierpinski Triangle." Pascal's Triangle. Math Forum: Drexel School of Education, 2 May 1998:

[16] http://mathforum.org/workshops/usi/pascal/pascal_sierpinski.html.

[17] http://mathworld.wolfram.com/DihedralGroup.html.

[18] http://mathworld.wolfram.com/PascalsTriangle.html.

[19] C. Woodburn, Pittsburg State University. Class Presentation and Notes, Abstract Algebra, MATH 613 – 01. Fall 2005.


All of the PascGalois Triangle images were generated from the PascGaloisJE, platform-independent PascGalois software created by Donald Spickler.

# *The Problem Corner*

Edited by Pat Costello

*The Problem Corner* invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before January 1, 2009. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring, 2009 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859)-622-3051)

## NEW PROBLEMS 624-631

**Problem 624.** *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, IL.*

Given a tetrahedron, prove that two triangles can be formed such that the lengths of the six triangle sides equal the lengths of the six edges of the tetrahedron. Prove that the converse is not true.

**Problem 625.** *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, IL.*

All of the integers from 1 through 999999 are written in a row. All of the zeros are erased. Each of the remaining digits is separately inverted and the sum, $S$, is computed. Let $T$ be the sum of the reciprocals of the digits 1 through 9. Show that $S/T$ is an integer and find it.

**Problem 626.** *Proposed by David Rose, Florida Southern College, Lakeland, FL.*

Two values are randomly selected from the uniform distribution on the interval $(0, L)$. They create three subintervals of the interval $[0, L]$. What is the probability that the lengths of the three subintervals are the lengths of the sides of some triangle?

**Problem 627.** *Proposed by Ken Dutch, Eastern Kentucky University, Richmond, KY.*

Suppose that the artist Krypto wants to form several rows of blocks 10 feet wide. He only wants to use two types of blocks. One type is one foot wide, and the other is two feet wide. He wants to form a row for every possible pattern of blocks (order matters). How many rows will he have to make? How many of each type of block will he have to use?

**Problem 628.** *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let $F_n$ be the $n^{th}$ Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that

$$\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} F_k \tanh F_k\right)^2 + \left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} F_k \operatorname{sech} F_k\right)^2 \leq F_n F_{n+1}.$$

**Problem 629.** *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let $a$, $b$, $c$ be real numbers, with $a, b, c \geq 1$. Prove that

$$\frac{a^{1/a}}{b^{1/b} + c^{1/c}} + \frac{b^{1/b}}{a^{1/a} + c^{1/c}} + \frac{c^{1/c}}{b^{1/b} + a^{1/a}} < 2.$$

**Problem 630.** *Proposed by the editor.*

Suppose that $\log_x y + \log_y x$ is a positive integer. Prove that $(\log_x y)^n + \left(\log_y x\right)^n$ is an integer for all positive integers $n$.

**Problem 631.** *Proposed by the editor.*

The Columbus State University Problem of the Week for March 10, 2008 asked for the three smallest positive integers that could not be written as the difference of two positive prime numbers. These turn out to be primes. Prove that there are infinitely many positive primes that cannot be written as the difference of two positive prime integers. Also prove that there are infinitely many pairs of positive integers $(n, n+2)$ that cannot be written as the difference of two positive primes.

## SOLUTIONS 611-615

**Problem 611.** *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Find all triplets $(x, y, z)$ of positive numbers that satisfy the system of equations:

$$\begin{cases} x^3 - 3x + \ln\left(x^2 - x - 1\right) = y \\ y^3 - 3y + \ln\left(y^2 - y - 1\right) = z \\ z^3 - 3z + \ln\left(z^2 - z - 1\right) = x \end{cases}.$$

**Solution** *by the proposer.*

Consider the function $f : \left(\dfrac{1+\sqrt{5}}{2}, \infty\right) \to \mathbb{R}$ defined by

$$f(x) = x^3 - 3x + \ln\left(x^2 - x - 1\right).$$

[The logarithm would be undefined for $x \leq \dfrac{1+\sqrt{5}}{2}$.] Since

$$f'(x) = 3\left(x^2 - 1\right) + \frac{2x - 1}{x^2 - x - 1} > 0$$

for $x > \dfrac{1+\sqrt{5}}{2}$, $f$ is increasing. We may assume that $x \geq y \geq z$. On the other hand, from $f(x) = y$, $f(y) = z$, $f(z) = x$, we get $z = f(y) \geq f(z) = x$. It follows that $x = y = z$. We observe that the only positive number that satisfies the equation $x^3 - 3x + \ln\left(x^2 - x - 1\right) = x$ [since $x^3 - 4x + \ln\left(x^2 - x - 1\right)$ is also increasing] is $x = 2$. Thus the only triplet is $(2, 2, 2)$.

*Also solved by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

**Problem 612.** *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let $n$ be a nonnegative integer. Prove that

$$\sqrt{\frac{F_n}{F_n + 2F_{n+1}}} + \sqrt{\frac{F_{n+1}}{F_{n+1} + 2F_n}} \geq 1,$$

where $F_n$ represents the $n^{th}$ Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$.

**Solution** *by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

Since

$$F_n + 2F_{n+1} = F_{n+2} + F_{n+1} = F_{n+3}$$

and

$$F_{n+1} + 2F_n = F_{n+2} + F_n \leq F_{n+2} + F_{n+1} = F_{n+3},$$

$$\sqrt{\frac{F_{n+1}}{F_{n+1} + 2F_n}} \geq \sqrt{\frac{F_{n+1}}{F_{n+3}}}$$

and so

$$\sqrt{\frac{F_n}{F_n + 2F_{n+1}}} + \sqrt{\frac{F_{n+1}}{F_{n+1} + 2F_n}} \geq \sqrt{\frac{F_n}{F_{n+3}}} + \sqrt{\frac{F_{n+1}}{F_{n+3}}}.$$

To show the desired inequality, it suffices to show that the last sum is greater than or equal to 1. The square of the last sum is

$$\begin{aligned} \frac{F_n}{F_{n+3}} + \frac{F_{n+1}}{F_{n+3}} + \frac{2\sqrt{F_n}\sqrt{F_{n+1}}}{F_{n+3}} &= \frac{F_{n+2} + 2\sqrt{F_n}\sqrt{F_{n+1}}}{F_{n+3}} \\ &\geq \frac{F_{n+2} + 2\sqrt{F_{n+1}/4}\sqrt{F_{n+1}}}{F_{n+3}} \end{aligned}$$

since $\dfrac{F_{n+1}}{F_n} < 4$. The last fraction simplifies to

$$\frac{F_{n+2} + F_{n+1}}{F_{n+3}} = \frac{F_{n+3}}{F_{n+3}} = 1.$$

Taking square roots of the resulting inequality gives the desired result.

*Also solved by Cher Chang and Cher Yang (students), California State University at Fresno and the proposer.*

**Problem 613.** *Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

A point $P$ is moving on a quarter circle of center $O$ which is bounded by two points $A$ and $B$. Let $PQ$ be the perpendicular from $P$ to the radius $OA$. The point $M$ is chosen on the ray $OP$ such that the length of $OM$ = length of $OQ$ + length of $QP$. Let $N$ be a point on the radius $OP$ such that $ON = OQ$. Show that the center of the locus of points $M$ as $P$ moves along the quarter circle is located on the locus of the points $N$.

**Solution** *by the proposers.*

Since the triangles ONA and OPQ are congruent (see solution to 607), the angle ONA is 90 degrees. Hence the locus of N is the (half) circle of diameter OA. Since the angle OMA is 45 degrees (see problem 607), the locus of M is an arc of the circle circumscribed to the triangle OMA. This arc is unique as P moves along the quarter circle. In fact, it is the arc from which the viewing angle of the line segment OA is 45 degrees. (The construction of this kind of arc is known in elementary geometry. I will sketch a method for completeness. Draw a ray OX making a 45 degree angle with OA outside the quarter circle. Draw another ray OY from O that that is perpendicular to the ray OX. OY crosses the perpendicular bisector of OA at a point H. H is the center of the circle of which the locus of M is a part) . Clearly, the angle AHO is 90 degrees. Thus H is on the circle of diameter OA, which is the locus of N as mentioned above.

**Problem 614** *Proposed by the editor.*

Let $\tau(n)$ represent the number of divisors of $n$. For example $\tau(10) = 4$ because $1, 2, 5, 10$ are the divisors of 10. Let $\sigma(n)$ represent the sum of the divisors of $n$. For example, $\sigma(10) = 1 + 2 + 5 + 10 = 18$. Prove that the infinite sum $\sum_{n=1}^{\infty} \frac{4^{\tau(n)}}{5^{\sigma(n)}}$ is bounded above by the fraction $\frac{364}{375}$.

**Solution** *by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

If $n \geq 2$, then $\tau(n) \leq \frac{n}{2}$ and $\sigma(n) \geq n + 1$. It follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{4^{\tau(n)}}{5^{\sigma(n)}} &= \frac{4^{\tau(1)}}{5^{\sigma(1)}} + \sum_{n=2}^{\infty} \frac{4^{\tau(n)}}{5^{\sigma(n)}} \\
&\leq \frac{4}{5} + \sum_{n=2}^{\infty} \frac{4^{n/2}}{5^{n+1}} \\
&\leq \frac{4}{5} + \frac{1}{5} \sum_{n=2}^{\infty} \left(\frac{2}{5}\right)^n \\
&= \frac{4}{5} + \frac{4}{75} \\
&= \frac{64}{75} \\
&\leq \frac{364}{375}.
\end{aligned}$$

*Also solved by Dane Maness and Chris Taylor (students), California State University at Fresno and the proposer.*

**Problem 615.** *Proposed by the editor.*

The sequence $a_1, a_2, a_3, ...$ is a monotone increasing sequence of natural numbers. It is known for any $k$ that $a_{a_k} = 3k$. Find a formula for $a_k$ and find the particular value $a_{2007}$.

**Solution** *by the proposer.*

Because the sequence is monotone increasing, $a_k \geq k$ for all $k$. Let $a_1 = n$. Then the condition $a_{a_k} = 3k$ means that $n \leq a_n = a_{a_1} = 3 \cdot 1 = 3$. Consider two cases:

- Suppose that $a_1 = 1$. Then $a_1 = a_{a_1} = 3 \cdot 1 = 3$, which contradicts the assumption that $a_1 = 1$.
- Suppose that $a_1 = 3$. Then $a_3 = a_{a_1} = 3 \cdot 1 = 3$, which contradicts the assumption that the sequence is monotone increasing.

Hence $a_1 = 2$. Then $a_2 = a_{a_1} = 3 \cdot 1 = 3$ and $a_3 = a_{a_2} = 3 \cdot 2 = 6$. If $a_k = m$, then $a_m = a_{a_k} = 3k$. Thus $a_{3k} = a_{a_m} = 3m$. In particular, $a_{3^1} = 2 \cdot 3^1$ from the above. Now assume that $a_{3^k} = 2 \cdot 3^k$. Then $a_{3^{k+1}} = a_{3 \cdot 3^k} = 3 \cdot a_{3^k} = 3 \cdot \left(2 \cdot 3^k\right) = 2 \cdot 3^{k+1}$. Then by induction, $a_{3^k} = 2 \cdot 3^k$ for all $k$. We also have $a_2 = 3 = 3^1$ and in general $a_{2 \cdot 3^k} = a_{a_{3^k}} = 3 \cdot 3^k = 3^{k+1}$. Now consider subscripts of the form $3^k + m$, where $1 \leq m < 3^k$:

$$a_{3^k} = 2 \cdot 3^k, a_{3^k+1}, a_{3^k+2}, ..., a_{3^k+3^k} = 3^{k+1} = 2 \cdot 3^k + 3^k.$$

There are exactly $3^k - 1$ values for $m$ and exactly $3^k - 1$ integers between $2 \cdot 3^k$ and $3^{k+1}$. Since the sequence is monotone inecreasing, we must have $a_{3^k+m} = 2 \cdot 3^k + m$. Finally, consider subscripts of the form $2 \cdot 3^k + m$, where $1 \leq m < 3^k$. We have

$$a_{2 \cdot 3^k + m} = a_{a_{3^k+m}} = 3\left(3^k + m\right) = 3^{k+1} + 3m.$$

We have found formulas for all possible subscripts. In particular,

$$a_{2007} = a_{2 \cdot 3^6 + 549} = 3^7 + 3\,(549) = 3834.$$

*Also solved by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

# *Announcement of the Thirty-Seventh Biennial Convention of Kappa Mu Epsilon*

The Thirty-Seventh Biennial Convention of Kappa Mu Epsilon will be held April 2-4, 2009, at the Doubletree Hotel in historic Philadelphia, PA. KME President-Elect Ron Wasserstein (ron@amstat.org) will send additional details to chapters soon. Each attending chapter will receive the usual travel expense ($.35/mile) reimbursement from the national office as described in Article VI, Section 2, of the Kappa Mu Epsilon Constitution.

A significant feature of our national convention is presentations by student members of Kappa Mu Epsilon. The mathematical topic selected by each student should be of interest to the author and of such scope that it can be given adequate treatment in a timed oral presentation. Senior projects and seminar presentations have been a popular way for faculty to get students to investigate suitable topics. Student talks to be presented at the convention will be chosen prior to the convention by the Selection Committee on the basis of the materials submitted. At the convention, the Awards Committee will judge the selected talks on both content and presentation, and the top four papers will receive an award.

**Who may submit a paper?**

Any undergraduate or graduate student member of Kappa Mu Epsilon may submit an abstract for consideration as a talk at the national convention. Presentations may be coauthored.

**Presentation topics**

Presentations at the convention should discuss material understandable by undergraduates who have completed only calculus courses. The Selection Committee will favor abstracts that satisfy this criterion and which can be presented with reasonable completeness within the time allotted. Presentations may be original research by the student(s) or exposition of interesting but not widely known results. Presenters should always cite authors if presenting exposition of known results.

**Presentation time limits**

Presentations at the convention should take between 15 minutes and 25 minutes.

**How to submit an abstract**

Students who wish to make a presentation at the national convention should submit an abstract of up to 500 words explaining the nature of the presentation and indicating the results. The abstract should be accompanied by a letter from the student's project advisor, certifying that the student is doing the work specified in the abstract and the advisor's belief the student will have a fully prepared presentation by the time of the convention.

Please send the abstract and advisor letter by electronic mail not later than February 2, 2009 to:

Dr. Ron Wasserstein, KME President-Elect
Executive Director, The American Statistical Association
ron@amstat.org

**Selection of papers for presentation**

A Selection Committee will review the abstracts submitted by undergraduate students and will choose approximately fifteen papers to be judged for awards at the convention. Graduate students and undergraduate students whose papers are not selected for judging may be offered the opportunity to present their papers at a parallel session of talks during the convention. The President-Elect will notify all authors of the status of their submissions after the Selection Committee has completed its deliberations.

**Judging criteria**

Judging criteria include

- Choice and originality of topic
- Literature sources and references
- Depth, significance, and correctness of content
- Clarity and organization of materials
- Adherence to the time constraints
- Effective use of graphs and/or visual aids
- Overall effect

**Prizes**

All presenters at the convention will be given two-year extensions of the subscription to The Pentagon. Authors of the four best presentations by undergraduates, as decided by the Awards Committees, will each receive a cash prize.

**Publication**

A presenter who has not prepared a formal written paper by the time of the convention is encouraged to do so soon after the convention, so that the paper can be submitted for possible publication in *The Pentagon*. Unless published elsewhere, papers prepared from the prize-winning presentations will be published in *The Pentagon* after any necessary revisions have been completed. All other papers will be considered for publication. The editor of *The Pentagon* will schedule a brief meeting with each author during the convention to review his or her presentation and discuss requirements for publication.

To have a paper considered for publication, prepare it as a Microsoft Word document or .tex file, and include it as an attachment to an e-mail to the editor at curtis-c@mssu.edu. The electronic copy of the paper will be sent to a referee who will prepare an anonymous report. If the referee recommends publication and space is available, the paper will be published in one of the next several issues.

***Thank You Referees!***

The editor wishes to thank the following individuals who refereed papers submitted to The Pentagon during the last two years.

Dale Bachman
University of Central Missouri
Warrensburg, Missouri

V. S. Bakhshi
Virginia State University
Petersburg, Virginia

Sarah Cook
Washburn University
Topeka, Kansas

Yuanjin Liu
Missouri Southern State University
Joplin, Missouri

William Livingston
Missouri Southern State University
Joplin, Missouri

Elizabeth Mauch
Bloomsburg University
Bloomsburg, Pennsylvania

Evangelos Skoumbourdis
Liberty University
Lynchburg, Virginia

Lisa Townsley
Benedictine University
Lisle, Illinois

Marion Weederman
Dominican University
River Forest, Illinois

Also thanks to the many other individuals who volunteered to serve as referees but were not used during the past two years. Referee interest forms will be sent by electronic mail in the near future so that interested faculty may volunteer. If you wish to volunteer as a referee, feel free to contact the editor (see page 2) to receive a referee interest form.

# *Kappa Mu Epsilon News*

Edited by Connie Schrock, Historian

**Updated information as of March 2008**

Send news of chapter activities and other noteworthy KME events to

Connie Schrock, KME Historian
Department of Mathematics, Computer Science, and Economics
Emporia State University
1200 Commercial Street
Campus Box 4027
Emporia, KS 66801
or to
schrockc@emporia.edu

## Installation Report

HI Alpha
Hawaii Pacific University, Honolulu, HI

The installation of the Hawaii Alpha Chapter of Kappa Mu Epsilon was held in the Buca di Beppo restaurant in Honolulu on Monday, October 22, 2007, at six o'clock in the evening. A delicious meal was enjoyed by all and the installation ceremony followed.

KME National President Don Tosh was the installing officer. Dr. Tosh began the ceremony with a brief description of the history of KME. Nina Ribbat was introduced as the Vice President and instructed her fellow initiates on the purposes of KME. Dr. Tosh then asked the initiates to affirm their wish to join in the pursuit of these ideals, and to participate actively in KME and abide by the society's constitution. All agreed, and each initiate signed the Hawaii Alpha Chapter Roll. Each initiate was presented with a membership certificate and a KME pin. Zena Rengulbai, the chapter president, described the history and meaning of the various insignias of KME. The organization was then declared to be the Hawaii Alpha Chapter of Kappa Mu Epsilon and the chapter's charter was presented to the president.

Next, the officers of the Hawaii Alpha chapter were installed. Each officer was charged with the responsibilities of his or her new office, and each chose to accept those responsibilities. Zena Rengulbai was installed as President. Nina Ribbat was installed as Vice President. Katharine Schnare and Ronnie Crane were both installed as Corresponding Secretaries and Faculty Sponsors. Professor Schnare was instrumental in getting Hawaii Alpha formed, but her reserve unit has been activated and she will be taking a two year leave of absence beginning in November of 2007. Ronnie Crane agreed to assume the responsibilities of Corresponding Secretary and Faculty Sponsor upon her departure.

The student charter members of Hawaii Alpha are (in alphabetical order) Amber P. Barbin, Abigail E. Barbour, Erika E. Baldwin, Zena K. Rengulbai (President), Nina Ribbat (Vice President), Derrick Tanalgo Solidum, and Marie Wourms. The faculty charter members of Hawaii Alpha are Katharine G. Schnare and Ronnie Crane (Corresponding Secretaries and Faculty Sponsors), Barbara Burke, Daniel Gefroh, Randolph Goldman, and Evelyn Puaa.

## Chapter News

**AL Alpha – Athens State University**

*Chapter President– Jenna O'Neal, 15 Current Members, 0 New Members*
*Other fall 2007 officers: Curt Merchant, Vice–President; Susan Webb, Secretary; Dottie Gasbarro, Corresponding Secretary.*

Along with members of the Math and Computers Science Club (MACS), KME officers and members worked in our food booth during the 2007 Old Time Fiddler's Convention on the Athens State Campus the first weekend in October. Thousands of attendees from all over North America attend the convention each year. This year, the approximately 7000 in attendance enjoyed the music, competitions, arts & crafts vendors, and the 30+ refreshment booths offered by ASU campus organizations in the 2-day event. KME and MACS members raised about $1500 for student scholarships and awards during the two day "cook and serve-a-thon" as we cooked and sold hundreds of hot dogs, hamburgers, and served chips and soft drinks as well. Along with lots of fun, we enjoyed serving the community with our hard work. Also in December, we joined with other campus organizations in raising money and gifts for Operation Santa Claus in Limestone County.

**AL Epsilon – Huntingdon College**

*Sally Clark, Corresponding Secretary.*

New initiates – Jeremy Dwain Driver, Kyle Jordan Eller, Emily Diane Hand, Caleb Allen Hartin, Tiffany Nicole Jordan.

**CO Delta – Mesa State College**

*Chapter President – Joshua Garland, 198 Current Members, 9 New Members*

*Other fall 2007 officers: Steven Hartman, Vice-President; Christopher Aquinto, Secretary; Heather Boette, Treasurer; Erik Packard, Corresponding Secretary.*

We met in May and initiated 9 new members.

**HI Alpha – Hawaii Pacific University**

New Initiates – Erika Baldwin, Amber Barbin, Abigail Barbour, Barbara Burke, Ronnie Crane, Dan Gefroh, Randy Goldman, Evelyn Puaa, Zena Rengulbai, Nina Ribbat, Katharine Schnare, Derrick Solidum, Marrie Wourms.

**IA Alpha – University of Northern Iowa**

*Chapter President – Erin Conrad, 17 Current Members, 5 New Members*

*Other fall 2007 officers: Adam Schneberger, Vice–President;Kellen Miller, Secretary; Keth Kolsrud, Treasurer; Mark D. Ecker, Corresponding Secretary.*

Our first Fall KME meeting was held on September 17, 2007 at Professor Jerry Ridenhour's residence where student member Ben Zaugg presented his paper entitled "Winning in the NFL". The University of Northern Iowa Homecoming Coffee was held at Professor Suzanne Riehl's residence on October 13, 2007 and our second meeting was held on November 12, 2007 at Professor Mark Ecker's. Student member Kellen Miller addressed the fall initiation banquet with "Special Teams in the NFL: Are they Really Important". Our Fall banquet was held at Diamond Dave's restaurant in Cedar Falls on December 3, 2007 where five new members were initiated.

New Initiates – Brian Curlott, Robert Clough, Xiang Ling Chay, Briana Ritter, Darcy Thomas.

**IA Delta – Wartburg College**

*Chapter President – Tim Schwickerath, 25 Current Members*

*Other fall 2007 officers: Jill Wiebke, Vice–President; Sarah Danner, Secretary; Man-Ling Fan, Treasurer; Dr. Brian Birgen, Corresponding Secretary.*

At the Wartburg Homecoming Renaissance Fair, warm weather contributed to an exceptionally successful fundraiser (our traditional egg-cheese booth).

**IL Theta – Benedictine University**

*Chapter President – Brad Callard, 15 Current Members, 0 New Members*

*Other fall 2007 officers: Debra Witczak, Vice-President; Jeff Herning, Secretary; Dr. Lisa Townsley, Corresponding Secretary.*

The Math Club in conjunction with KME chapter held a chess competition this fall. They also held a math competition for which teachers awarded extra credit for participation. The group designed and sold a T-shirt with math slogans as a fund-raiser for charity. Spring plans include a dvd (Colin Adams et al debating Pi vs E), a library display for math awareness month, and a Pi Day pie eating competition as well as our annual induction ceremony.

**IL Zeta – Dominican University**

*Chapter President – Cassie Hileman and Melissa Wegener, 20 Current Members, 0 New Members*

*Other fall 2007 officer: Aliza Steurer, Corresponding Secretary.*

Here is a brief summary of the events that KME was involved with in the fall 2007 semester. Dr. Paul Coe (chair of the math department) gave a talk on how to make platonic solids out of origami. We held four monthly meetings and two officers' meetings.

**IN Delta – University of Evansville**

*Chapter President – Luanne Benson-Lender, 45 Current Members, 18 New Members*

*Other fall 2007 officers: Sarah Schonaman, Vice-President; Amanda Watkins, Secretary; Dr. Adam Salminen, Correspond Secretary.*

The main activities for the Indiana Delta Chapter of KME were the following.

- Provided free tutoring for students before final exams.
- Sponsored the University of Evansville Mathematics Competition.
- Supported students travel expenses for the University of Evansville's team at the Indiana College Mathematics Competition.

**KS Beta – Emporia State University**

*Chapter President– Mike Moore, 33 Current Members, 10 New Members*

*Other fall 2007 officers: Emily Wassenberg, Vice–President; Mirel Howard, Secretary; Jared Leads, Treasurer; Connie Schrock, Corresponding Secretary.*

KS Beta chapter held a calculator workshop for algebra students. We also hosted a Math Jeopardy and participated in Math Day. Several presentations were held throughout the semester a few of them included "The Use of Mathematics in Pharmaceutical Research" by Lane Senne and "The Straight and Narrow" by Clyde Martin. The movie Flatland was also shown followed by pizza.

New Initiates – Michael Brown, Ashley Brown, Bonnie Young, Amber Roan, Phyllis Conner, Qiquan Lu, Zheng Yang, Mary Daque, Qiang Shi, Le Zhang.

**KS Delta –Washburn University**
*Chapter President– Tamela Bolen, 30 Current Members*
*Other fall 2007 officers: Brandy Mann, Vice–President; Richard Nelson, Secretary; Richard Nelson, Treasurer; Kevin Charlwood, Corresponding Secretary.*

During the Fall semester, our KME chapter had three luncheon meetings with our math club, Club Mathematica. We hosted an actuary with Blue Cross/Blue Shield of Kansas, Randy Edwards, FSA, and he gave a presentation about the actuarial profession. We have at least two students preparing KME projects for presentation at the KME regional meeting coming up in April 2008 at Pittsburg State University.

**KS Gamma – Benedictine College**
*Chapter President – Chris G'Sell , 16 Current Members*
*Other fall 2007 officers: Erica Goedken , Vice-President; Danny Noonan, Secretary; Dr. Linda Herndom, Corresponding Secretary.*

We sponsored a back-to-school picnic at a local park on September 13. Since several students are interested in actuarial science, we had an actuarial Fellow speak to us about what an actuary is and does.

**KY Beta – University of the Cumberlands**
*Chapter President- Kathryn DeLozier, 32 Current Members, 0 New Members*
*Other fall 2007 officers: Charlotte Abel, Vice–President; Dustin Ursrey, Secretary; Deidre Higgins, Treasurer; Dr. Jonathan Ramey, Corresponding Secretary.*

On September 4, the Kentucky Beta chapter helped to host an ice cream party for the freshmen math and physics majors. Along with the Mathematics and Physics Club and Sigma Pi Sigma, the chapter had a picnic at Briar Creek Park on October 11. On December 7, the entire department, including the Math and Physics Club, the Kentucky Beta chapter, and Sigma Pi Sigma had a Christmas party with about 27 people in attendance.

**MD Alpha – College of Notre Dame of Maryland**
*Chapter President – Nicole Kotulak, 13 Current Members, 5 New Members*
*Other fall 2007 officers: Irene McNulty, Vice-President; Amanda Hubbel, Secretary; Krista Bujanowski, Treasurer; Dr. Margaret Sullivan, Corresponding Secretary.*

At our October Initiation, Tiffany Russell, a 2006 alumna and KME member, talked about her work at NASA as well as the projects she is involved in as an Aerospace-Engineering student at University of Maryland, College Park.

**MD Beta – McDaniel College**

*Dr. Harry Rosenzweig, Corresponding Secretary.*

New Initiates – Stephen Hardy, Christopher Alan, Dan Thornton.

**MD Delta – Frostburg State University**

*Chapter President – Shay Mallory, 29 Current Members, 0 New Members. Other fall 2007 officers: Matthew Bucchino, Vice-President; Nicole Garber, Secretary; Courtney Kamauf, Treasurer; Dr. Mark Hughes, Corresponding Secretary.*

The Maryland Delta Chapter started the semester with a meeting in early September where we planned our participation in a “majors fair” held in the student center. This event was held in order to introduce new students to the various majors and student organizations present on campus. Our members represented the Department of Mathematics and KME. Displays and multimedia presentations were prepared during our meeting and the fair went very nicely. During our October and November meetings, we were treated to a two part lecture by Dr. Frank Barnet on shrinking a curve by the heat equation where some very nice graphics were presented. We express best wishes to our treasurer Courtney Kamauf who graduated this semester and mention with pride that she served as a commencement speaker in December.

**MD Epsilon – Villa Julie College**

*Chapter President – Courtney Naff, 17 Current Members, 7 New Members Other fall 2007 officers: Catherine Gerber, Vice-President; Krystal Burns, Secretary; Amy Walsh, Treasurer; Dr. Christopher E. Barat, Corresponding Secretary.*

The Chapter’s 3rd Annual Initiation Ceremony was held on Wednesday, September 19, 2007. Seven new members were initiated. The guest speaker was Dr. J. Stephen Dumler of the Department of Pathology at The Johns Hopkins University School of Medicine. Planned activities for the spring semester include a fund-raising "pi(e)" sale and a joint meeting with other local chapters of KME to celebrate Mathematics Awareness Month, as well as VJC’s 60th anniversary.

New Initiates - Krystal Burns, Brandon Cooper, Catherine Gerber, Ashley Hoffman, Lindsay Koepper, Jennifer O’Bier, Elizabeth Schlimm.

**MO Alpha – Missouri State University**

*Chapter President– Thomas Buck, 30 Current Members, 12 New Members Other fall 2007 officers: Chris Inabnit, Vice–President; Bobby Gregory, Secretary; Michael McDonald, Treasurer; Jorge Rebaza, Corresponding Secretary.*

Our Fall 2007 activities included

- 09/20/07 KME Picnic
- 10/16/07 KME Seminar. Speaker: Robert Roe, University of Missouri, Rolla.
- 11/15/07 KME Seminar. Speakers: Tom Buck and Chris Inabnit, MSU.

New Initiates – Craig Carmack, Joseph Dwyer, Evan Kloeppel, Myles Loffler, Kyle McKee, Immanuel McLaughlin, Nancy McMillion, Michelle Noe, Courtney Smith, Brandon Turner, Nicole Typaldos, Carrie Whittle.

**MO Beta – Central Missouri State University**

*Dr. Rhonda McKee, Corresponding Secretary.*

New Initiates – Andrew Dotson, Michael Howard, Benjamin Larkin, Cale Magnuson, Kimberly Tritsch.

**MO Nu – Columbia College**

*Chapter President – Laura Weaver, 15 Current Members, 0 New Members*
*Other fall 2007 officers: Catryna Palmer, Vice-President; Ashley Wesche, Secretary; Sara Demma, Treasurer; Dr. Ann Bledsoe, Corresponding Secretary.*

The Missouri Nu Chapter of Kappa Mu Epsilon hosted a math alumni social, participated in Columbia College homecoming, and held four monthly meetings.

**MO Theta – Evangel University**

*Chapter President – Lindsay Hull, 8 Current Members, 0New Members.*
*Other fall 2007 officers: Monica Pedersen, Vice-President; Don Tosh, Corresponding Secretary.*

Meetings were held every month. In November several members attended a local math conference held at Missouri State University and cohosted by Evangel. The final meeting of the semester was a Christmas party held at the home of Dr. Tosh.

**MS Alpha – Mississippi University for Women**

*Chapter President – Dana Derrick, 12 Current Members, 1 New Member*
*Other fall 2007 officers: Michelle L. Hitt, Vice-President; Dana Derrick, Secretary; Michelle L. Hitt , Treasurer; Dr. Shaochen Yang, Corresponding Secretary.*

As a service project, we prepared four shoe boxes of Christmas presents for "Operation Christmas Child". On Nov. 14, our speaker, Dr. Ratnasingham Shivaji from Department of Mathematics and Statistics at Mississippi State University.

New Initiate – Stacy Elmore.

**NE Alpha – Wayne State College**

*Chapter President – Spencer Alewel, 10 Current Members, 0 New Members*

*Other fall 2007 officers: Jed Martens, Vice-President; Spencer Alewel, Secretary; Jed Martens, Treasurer; Tami Worner, Corresponding Secretary.*

KME held a welcome lunch for new math majors. Current majors and faculty also attended.

**NE Beta – University of Nebraska at Kearney**

*President – Adam Haussler, 13 Current Members, 2 New Members*

*Other fall 2007 officers: Amber Nabity, Vice-President; Amber Norman, Secretary; Sasha Anderson, Treasurer; Dr. Katherine Kime, Corresponding Secretary.*

In December, Dr. John Schneider of Hastings College gave a talk entitled “My Life as Middle School Sub”. Dr. Schneider teaches the mathematical methods course at Hastings, and had the opportunity to serve as a substitute teacher during his sabbatical. Prior to the talk, we initiated two new members.

KME member John Auwerda graduated in December and was honored by the College of Natural and Social Sciences. John student-taught at Norris High School, south of Lincoln, this past semester . He will be looking for a teaching position in the area of Ames Iowa, where his new wife, Kandra Johnson, also a KME member, attends graduate school in Chemistry.

New Initiates – Justin Arellano, Haewoo Jeong.

**NE Delta – Nebraska Wesleyan University**

*Chapter President– Marcus Hatfield, 14 Current Members, 0 New Members*

*Other fall 2006 officers: Rebecca Brown, Vice President; Brian Grummert, Secretary; Brian Grummert, Treasurer; Melissa Erdmann, Corresponding Secretary.*

We enjoy picnics, game nights, and doing Adopt a Highway. Tonight we are having a holiday party where those who wish can bring an inexpensive math-related gift for an exchange. An apple pie with a stick (natural log) taped on top is an example of such a gift. Mathematical puzzles and games appear frequently. In short, KME is alive and well at Nebraska Wesleyan University.

**NJ Delta – Centenary College of New Jersey**

*Chapter President – Brittany Garcia, 23 Current Members, 7 New Members*

*Other fall 2007 officers: Nita Connell, Vice-President; Michael Pignoloso III, Secretary; Linda Ritchie, Treasurer; Kathy Turrisi, Corresponding Secretary.*

On Sunday, December 9th, 2007 eight members were inducted into the KME Delta Chapter of NJ. The Opening Remarks were given by Prof Robert Search who spoke of what great promise all the inductees had to give to the math community. The President, Brittany Garcia, thanked the math faculty, Linda Ritchie, Robert Search, Joseph Repice, Kathy Turrisi and the graduate mentor, Kathryn Curley for their hard work, dedication, and guidance to the Kappa Mu Epsilon students throughout the semesters and the ceremony. The Closing Remarks were given by Dr. Don Tosh, National KME President who presented the Charter and Crest to Kathy Turrisi, Faculty Sponsor and Corresponding Secretary and Nita Connell, Student Vice President for KME Delta. At the close of the ceremony, all KME members, new and former, posed for pictures with faculty, family, and friends and enjoyed light refreshments.

Two members of Centenary College's Delta Chapter, Seth Wisner and Jane Guglielmello, graduated on January 12, 2008.

New Initiates - Xandria Katelynne Matlock, Stephanie Osinski-Rea, Amy Sprofera, Constance Karima Edouard, Caitlin Burghoffer, Stephanie M. Kowalak, Candice Zappile.

**NY Iota – Wagner College**

*Dr. Zohreh Shahvar, Corresponding Secretary.*

New Initiates – Rebecca Giannattassio, Taylor Wheaton, Nicole Fitzgerald, Charles DiNicolais, Frank Costanza, Jaquelyn Laurie, Danielle Tatusko, Joseph LaPreta.

**NY Nu – Hartwick College**

*Chapter President – Joseph Fayton, 22 Current Members*

*Other fall 2007 officers: Caitlin Gilman, Vice-President; Anees Gharzita, Secretary; Dustin Jones, Treasurer; Ron Brzenk, Corresponding Secretary.*

New Initiates – Carol Ann Fischer, Marjorie McMillan, Gregory Mease, Nancy Perrotti, Mark Powers, Esther Rushing, William Silvanic, Joanne Pedersen Squilla, Nancy Vogelsang.

**OH Epsilon – Marietta College**

*Chapter President – Phil DeOrsey, 20 Current Members, 0 New Members*

*Other fall 2007 officers: Kelsie McCartney, Vice-President; Dr. John C. Tynan, Corresponding Secretary.*

**OK Alpha – Northeastern State University**

*Chapter President – Caleb Knowlton, 55 Current Members, 10 New Members*

*Other fall 2007 officers: Ramona Medlin, Vice–President; Felicia Lotchleas, Secretary; Amanda Barker, Treasurer; Dr. Joan E. Bell, Corresponding Secretary.*

Our fall initiation brought ten new members into our chapter. To celebrate the Oklahoma Centennial (Nov. 16, 2007), Dr. Wendell Wyatt, NSU, showed us how to make Oklahoma-stellated-octahedrons. We also ate Oklahoma themed refreshments. We participated in the annual NSU Halloween carnival with our "KME Pumpkin Patch" activity. The children fished for pumpkins with meter stick fishing poles. During our October meeting, we worked on a problem from a math journal, and submitted the solution for publication! We ended the semester with a Christmas party for KME members, math majors, and faculty. The pizza, made by our department chair, Dr Darryl Linde, was incredible! We also played the game "Catch Phrase."

New Initiates: Walter Colston, Joshua Johnston, Derek Lane, Volha Leeper, Traci Meadows, Kimberly Nazarman, Bryan L. Parker, Richard Peters, Derrel Walters, Callie Wilson.

**OK Gamma – Southwest Oklahoma State University**

*Bill Sticka, Corresponding Secretary.*

New Initiates – Kevin Freeman, Danielle McNair, Abigail Ntreh, Stuart Payne, Justin Silkwood, Ashleigh Streit.

**PA Iota– Shippensburg University**

*Chapter President – Michelle Baker, 12 Current Members, 2 New Members*

*Other fall 2007 officers: David Miller, Vice – President; Valerie Koontz, Secretary; Rebecca Krumrine , Treasure; Dr. Paul Taylor, Corr. Sec.*

At our fall banquet we inducted two new members, Valerie Koontz and Kate Ibbetson. Both presented excellent essays on Cryptography and Valerie was awarded the prize for best essay.

**PA Kappa – Holy Family University**

*Chapter President – Danielle Fortuna, 8 Current Members, 1 New Members*

*Other fall 2007 officers: Shawn Kane, Vice-President; Nusrat Jahan, Secretary; Kevin Anderson, Treasurer; Sister Marcella Louise Wallowicz, Corresponding Secretary.*

Our Fall 2007 activities included

- October 24, 2007 – The Chapter hosted an Evening of Mathematical Suspense (Math Murder Mystery). University students solved math problems in order to obtain clue. There was a "Halloween" ambience. Pizza, snacks and soda were available throughout the evening. Those who correctly determined “whodunit” received prizes with the university logo.

- December 3, 2007 – The chapter sponsored a Meet-and-Greet prior to the Fall NSM Colloquium. Dr. Mark Tuckerman, Graduate Chair from NYU, spoke on the mathematics of protein folding: “Warping Rubber Sheets.”
- December 4, 2007 – The senior math and math-secondary education majors gave mini-presentations on their senior research projects. The talks were open to the university community. A pizza supper followed.

In addition, chapter members serve as student ambassadors to high school juniors and seniors who are interested in applying to the university as math or math-ed majors.

**PA Lambda – Bloomsburg University of Pennsylvina**

New Initiates – Jason Elsinger, Jacqueline Leslie, David Lusby, Michael Polites, Stephen Rhein, Quintin Todd, Kyle Vandermay.

**PA Mu – Saint Francis University**

*Chapter President – Jonathan Miller, 261 Current Members*

*Other fall 2007 officers: Tim Gaborek, Vice – President; David Kirby, Secretary; Joe Rosmus, Treasurer; Dr. Peter Skoner, Corresponding Secretary.*

Several KME members participated in the Adopt-A-Highway road litter cleaning program on October 2, 2007 as part of the University’s Day of Reflection. Twenty people picked litter off Manor Drive near the University.

Several KME faculty and student members participated in the Fourteenth Annual Science Day held on campus on November 20, 2007. Some KME members served as session moderators for faculty making presentations. Others served as moderators, judges, scorekeepers, and timers for the Science Bowl. A total of 386 high school students from 27 high schools attended.

**PA Pi – Slippery Rock University**

*Chapter President – Emily Hendrickson*

*Other fall 2007 officers: Michelle Komo, Vice-President; Tyler Druschel, Secretary; Dr. Elise Grabner, Corresponding Secretary.*

Undergraduate mathematics major, Emily Hendrickson, Conway, PA, presented her research, *Stabilizing the vibrations of a thermoelastic beam*, in a poster session at The Joint Meetings of the Mathematical Association of America and The American Mathematical Society in San Diego, CA, January 5-9. Her research advisor, Dr. Richard Marchand, also presented a paper at the meetings entitled, *Comparing mechanical and thermal damping in elastic beams*.

**PA Sigma– Lycoming College**

*Chapter President – Seth Burns, 11 Current Members, 0 New Members*
*Other fall 2007 officers: Justin Hughes, Vice-President; Stephanie Hartman, Secretary; Adam Hughes, Treasurer; Dr Santu de Silva, Corresponding Secretary.*

**PA Theta – Susquehanna University**

*Lisa Orloff Clark, Corresponding Secreetary.*

**SC Gamma – Winthrop University**

*Chapter President – Josh Jones, 10 Current Members*
*Other fall 2007 officers: Lauren Cairco , Vice – President; Kristen Huete, Secretary; Kyle Neeley, Treasurer; Dr. Trent Kull, Corresponding Secretary.*

**SC Epsilon – Francis Marion University**

*Damon Scott, Corresponding Secretary.*

New Initiates – Courtney Lynne Phipps, Lindsey Springs, Veronica Lynnette Thomas.

**TN Epsilon – Bethel College**

*Chapter President – Jessica Smith, 11 Current Members, 3 New Members*
*Other fall 2007 officers: Chastity Fortenberry, Vice-President; William Robertson, Secretary; Justin Dubruiel, Treasurer; Russell Holder, Corresponding Secretary.*

New Initiates – Randon S. Prather, William A. Robertson, Irene D. Smith.

**TN Gamma – Union University**

*Chapter President – Matthew Dawson. 14 Current Members, 0 New Members*
*Other fall 2007 officers: Will Trautman, Vice–President; Robbyn Reynolds, Secretary; Robbyn Reynolds, Treasurer; Joshua Brooks, Webmaster; Bryan Dawson, Corresponding Secretary.*

The first KME event of the semester was our annual cookout, this time held at the home of Dr. Dawson on September 24. The Great Dawsoni made another appearance with a mathematical trick, and Drs. Riggs and Hail also performed an impromptu trick. Some students had the chance to meet briefly with the great geometer Dr. Thomas Banchoff in October when he made a visit to campus, and later that month former chapter president Brian Taylor gave a talk about his doctoral research in a medical physics program at M.D. Anderson in Houston. Three students gave presentations of their Senior Seminar projects on December 4. The last KME event of the semester was the Christmas party, which included a “Dirty Santa” gift exchange and a screening of the movie Flatland.

### TX Gamma – Texas Women's University

*Dr. Mark Hammer, Corresponding Secretary.*

New Initiates – Katie Vandermeer, Cathy Szpet, Kendi Sanchez, Julie Wardell, Kandis Schroeder, Carolina Gutierrez, Iris Royal, Colleen Dow, Karen Stevens, Elizabeth Zepeda, Cristal Retana, Cheryl Erickson, Bogdan Obarse, Araceli Perez, Ann Ahlborn, Alex Montminy, Julianne Hockaday, Rachel Musgrove.

### TX Mu – Schreiner University

*Chapter President – Meagan Goodson 9 Current Members, 0 New Members*

*Other fall 2007 officers: Lynn Stow, Vice-President; Ashley Moore, Secretary; Amy Vickers, Treasurer; William Sliva, Corresponding Secretary.*

### WI Gamma – University of Wisconsin-Eau Claire

*Dr. Simei Tong, Corresponding Secretary.*

New Initiates – Erin Andre, Christian Cortner, Sarah Fisher, Katie Fitzpatrick, Matthew Fjerstad, Katrina Geske, Brenna Hall, Elizabeth Harter, Haley Haus, Jason Jossie, Katherine Knutson, Alison Lau, Adam Malone, Courtney Moy, Michael Nerbovig, Stephanie Pederson, April Timm, Michael Vandertie, Adam Wolfe, Yunyun Yang.

# Kappa Mu Epsilon National Officers

Don Tosh *President*

Department of Science and Technology
Evangel University
1111 N. Glenstone Avenue
Springfield, MO 65802
toshd@evangel.edu

Ron Wasserstein *President-Elect*

American Statistical Association
732 N Washington Street
Alexandria, VA 22314-1943
ron@amstat.org

Rhonda McKee *Secretary*

Department of Mathematics
University of Central Missouri
Warrensburg, MO 64093-5045
mckee@ucmo.edu

Cynthia Woodburn *Treasurer*

Department of Mathematics
Pittsburg State University
Pittsburg, KS 66762-7502
cwoodbur@pittstate.edu

Connie Schrock *Historian*

Department of Mathematics
Emporia State University
Emporia, KS 66801-5087
schrockc@emporia.edu

Kevin Reed *Webmaster*

Department of Science and Technology
Evangel University
1111 N. Glenstone Avenue
Springfield, MO 65802

KME National Website:
http://www.kappamuepsilon.org/

# *Active Chapters of Kappa Mu Epsilon*

*Listed by date of installation*

| Chapter | Location | Installation Date |
|---|---|---|
| OK Alpha | Northeastern State University, Tahlequah | 18 April 1931 |
| IA Alpha | University of Northern Iowa, Cedar Falls | 27 May 1931 |
| KS Alpha | Pittsburg State University, Pittsburg | 30 Jan 1932 |
| MO Alpha | Missouri State University, Springfield | 20 May 1932 |
| MS Alpha | Mississippi University for Women, Columbus | 30 May 1932 |
| MS Beta | Mississippi State University, Mississippi State | 14 Dec 1932 |
| NE Alpha | Wayne State College, Wayne | 17 Jan 1933 |
| KS Beta | Emporia State University, Emporia | 12 May 1934 |
| AL Alpha | Athens State University, Athens | 5 March 1935 |
| NM Alpha | University of New Mexico, Albuquerque | 28 March 1935 |
| IL Beta | Eastern Illinois University, Charleston | 11 April 1935 |
| AL Beta | University of North Alabama, Florence | 20 May 1935 |
| AL Gamma | University of Montevallo, Montevallo | 24 April 1937 |
| OH Alpha | Bowling Green State University, Bowling Green | 24 April 1937 |
| MI Alpha | Albion College, Albion | 29 May 1937 |
| MO Beta | University of Central Missouri, Warrensburg | 10 June 1938 |
| TX Alpha | Texas Tech University, Lubbock | 10 May 1940 |
| KS Gamma | Benedictine College, Atchison | 26 May 1940 |
| IA Beta | Drake University, Des Moines | 27 May 1940 |
| TN Alpha | Tennessee Technological University, Cookeville | 5 June 1941 |
| MI Beta | Central Michigan University, Mount Pleasant | 25 April 1942 |
| NJ Beta | Montclair State University, Upper Montclair | 21 April 1944 |
| IL Delta | University of St. Francis, Joliet | 21 May 1945 |
| KS Delta | Washburn University, Topeka | 29 March 1947 |
| MO Gamma | William Jewell College, Liberty | 7 May 1947 |
| TX Gamma | Texas Woman's University, Denton | 7 May 1947 |
| WI Alpha | Mount Mary College, Milwaukee | 11 May 1947 |
| OH Gamma | Baldwin-Wallace College, Berea | 6 June 1947 |
| CO Alpha | Colorado State University, Fort Collins | 16 May 1948 |
| MO Epsilon | Central Methodist College, Fayette | 18 May 1949 |
| MS Gamma | University of Southern Mississippi, Hattiesburg | 21 May 1949 |
| IN Alpha | Manchester College, North Manchester | 16 May 1950 |
| PA Alpha | Westminster College, New Wilmington | 17 May 1950 |
| IN Beta | Butler University, Indianapolis | 16 May 1952 |
| KS Epsilon | Fort Hays State University, Hays | 6 Dec 1952 |
| PA Beta | LaSalle University, Philadelphia | 19 May 1953 |
| VA Alpha | Virginia State University, Petersburg | 29 Jan 1955 |
| IN Gamma | Anderson University, Anderson | 5 April 1957 |
| CA Gamma | California Polytechnic State University, San Luis Obispo | 23 May 1958 |
| TN Beta | East Tennessee State University, Johnson City | 22 May 1959 |
| PA Gamma | Waynesburg College, Waynesburg | 23 May 1959 |
| VA Beta | Radford University, Radford | 12 Nov 1959 |
| NE Beta | University of Nebraska—Kearney, Kearney | 11 Dec 1959 |
| IN Delta | University of Evansville, Evansville | 27 May 1960 |
| OH Epsilon | Marietta College, Marietta | 29 Oct 1960 |

| | | |
|---|---|---|
| MO Zeta | University of Missouri—Rolla, Rolla | 19 May 1961 |
| NE Gamma | Chadron State College, Chadron | 19 May 1962 |
| MD Alpha | College of Notre Dame of Maryland, Baltimore | 22 May 1963 |
| CA Delta | California State Polytechnic University, Pomona | 5 Nov 1964 |
| PA Delta | Marywood University, Scranton | 8 Nov 1964 |
| PA Epsilon | Kutztown University of Pennsylvania, Kutztown | 3 April 1965 |
| AL Epsilon | Huntingdon College, Montgomery | 15 April 1965 |
| PA Zeta | Indiana University of Pennsylvania, Indiana | 6 May 1965 |
| AR Alpha | Arkansas State University, State University | 21 May 1965 |
| TN Gamma | Union University, Jackson | 24 May 1965 |
| WI Beta | University of Wisconsin—River Falls, River Falls | 25 May 1965 |
| IA Gamma | Morningside College, Sioux City | 25 May 1965 |
| MD Beta | McDaniel College, Westminster | 30 May 1965 |
| IL Zeta | Domincan University, River Forest | 26 Feb 1967 |
| SC Beta | South Carolina State College, Orangeburg | 6 May 1967 |
| PA Eta | Grove City College, Grove City | 13 May 1967 |
| NY Eta | Niagara University, Niagara University | 18 May 1968 |
| MA Alpha | Assumption College, Worcester | 19 Nov 1968 |
| MO Eta | Truman State University, Kirksville | 7 Dec 1968 |
| IL Eta | Western Illinois University, Macomb | 9 May 1969 |
| OH Zeta | Muskingum College, New Concord | 17 May 1969 |
| PA Theta | Susquehanna University, Selinsgrove | 26 May 1969 |
| PA Iota | Shippensburg University of Pennsylvania, Shippensburg | 1 Nov 1969 |
| MS Delta | William Carey College, Hattiesburg | 17 Dec 1970 |
| MO Theta | Evangel University, Springfield | 12 Jan 1971 |
| PA Kappa | Holy Family College, Philadelphia | 23 Jan 1971 |
| CO Beta | Colorado School of Mines, Golden | 4 March 1971 |
| KY Alpha | Eastern Kentucky University, Richmond | 27 March 1971 |
| TN Delta | Carson-Newman College, Jefferson City | 15 May 1971 |
| NY Iota | Wagner College, Staten Island | 19 May 1971 |
| SC Gamma | Winthrop University, Rock Hill | 3 Nov 1972 |
| IA Delta | Wartburg College, Waverly | 6 April 1973 |
| PA Lambda | Bloomsburg University of Pennsylvania, Bloomsburg | 17 Oct 1973 |
| OK Gamma | Southwestern Oklahoma State University, Weatherford | 1 May 1973 |
| NY Kappa | Pace University, New York | 24 April 1974 |
| TX Eta | Hardin-Simmons University, Abilene | 3 May 1975 |
| MO Iota | Missouri Southern State University, Joplin | 8 May 1975 |
| GA Alpha | State University of West Georgia, Carrollton | 21 May 1975 |
| WV Alpha | Bethany College, Bethany | 21 May 1975 |
| FL Beta | Florida Southern College, Lakeland | 31 Oct 1976 |
| WI Gamma | University of Wisconsin—Eau Claire, Eau Claire | 4 Feb 1978 |
| MD Delta | Frostburg State University, Frostburg | 17 Sept 1978 |
| IL Theta | Benedictine University, Lisle | 18 May 1979 |
| PA Mu | St. Francis University, Loretto | 14 Sept 1979 |
| AL Zeta | Birmingham-Southern College, Birmingham | 18 Feb 1981 |
| CT Beta | Eastern Connecticut State University, Willimantic | 2 May 1981 |
| NY Lambda | C.W. Post Campus of Long Island University, Brookville | 2 May 1983 |
| MO Kappa | Drury University, Springfield | 30 Nov 1984 |
| CO Gamma | Fort Lewis College, Durango | 29 March 1985 |
| NE Delta | Nebraska Wesleyan University, Lincoln | 18 April 1986 |

| | | |
|---|---|---|
| TX Iota | McMurry University, Abilene | 25 April 1987 |
| PA Nu | Ursinus College, Collegeville | 28 April 1987 |
| VA Gamma | Liberty University, Lynchburg | 30 April 1987 |
| NY Mu | St. Thomas Aquinas College, Sparkill | 14 May 1987 |
| OH Eta | Ohio Northern University, Ada | 15 Dec 1987 |
| OK Delta | Oral Roberts University, Tulsa | 10 April 1990 |
| CO Delta | Mesa State College, Grand Junction | 27 April 1990 |
| PA Xi | Cedar Crest College, Allentown | 30 Oct 1990 |
| MO Lambda | Missouri Western State College, St. Joseph | 10 Feb 1991 |
| TX Kappa | University of Mary Hardin-Baylor, Belton | 21 Feb 1991 |
| SC Delta | Erskine College, Due West | 28 April 1991 |
| SD Alpha | Northern State University, Aberdeen | 3 May 1992 |
| NY Nu | Hartwick College, Oneonta | 14 May 1992 |
| NH Alpha | Keene State College, Keene | 16 Feb 1993 |
| LA Gamma | Northwestern State University, Natchitoches | 24 March 1993 |
| KY Beta | Cumberland College, Williamsburg | 3 May 1993 |
| MS Epsilon | Delta State University, Cleveland | 19 Nov 1994 |
| PA Omicron | University of Pittsburgh at Johnstown, Johnstown | 10 April 1997 |
| MI Delta | Hillsdale College, Hillsdale | 30 April 1997 |
| MI Epsilon | Kettering University, Flint | 28 March 1998 |
| KS Zeta | Southwestern College, Winfield | 14 April 1998 |
| TN Epsilon | Bethel College, McKenzie | 16 April 1998 |
| MO Mu | Harris-Stowe College, St. Louis | 25 April 1998 |
| GA Beta | Georgia College and State University, Milledgeville | 25 April 1998 |
| AL Eta | University of West Alabama, Livingston | 4 May 1998 |
| NY Xi | Buffalo State College, Buffalo | 12 May 1998 |
| NC Delta | High Point University, High Point | 24 March 1999 |
| PA Pi | Slippery Rock University, Slippery Rock | 19 April 1999 |
| TX Lambda | Trinity University, San Antonio | 22 November 1999 |
| GA Gamma | Piedmont College, Demorest | 7 April 2000 |
| LA Delta | University of Louisiana, Monroe | 11 February 2001 |
| GA Delta | Berry College, Mount Berry | 21 April 2001 |
| TX Mu | Schreiner University, Kerrville | 28 April 2001 |
| NJ Gamma | Monmouth University | 21 April 2002 |
| CA Epsilon | California Baptist University, Riverside | 21 April 2003 |
| PA Rho | Thiel College, Greenville | 13 February 2004 |
| VA Delta | Marymount University, Arlington | 26 March 2004 |
| NY Omicron | St. Joseph's College, Patchogue | 1 May 2004 |
| IL Iota | Lewis University, Romeoville | 26 February 2005 |
| WV Beta | Wheeling Jesuit University, Wheeling | 11 March 2005 |
| SC Epsilon | Francis Marion University, Florence | 18 March 2005 |
| PA Sigma | Lycoming College, Williamsport | 1 April 2005 |
| MO Nu | Columbia College, Columbia | 29 April 2005 |
| MD Epsilon | Villa Julie College, Stevenson | 3 December 2005 |
| NJ Delta | Centenary College, Hackettstown | 1 December 2006 |
| NY Pi | Mount Saint Mary College, Newburgh | 20 March 2007 |