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- Contact information for national officers
- Initiation report form
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- Information on KME conventions

When you design a chapter homepage, please remember to make it clear that your page is for your chapter, and not for the national organization. Also, please include a link to the national homepage and submit your local chapter webpage's URL to the national webmaster. Currently, this is the National Secretary, Rhonda McKee. Her contact information is located in the list of National Officers on page 76 and under National Officers on the web site.

Analyzing a Spirograph Shape with an Outer Ellipse and Inner Rotating Circular Disk

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Presented at the 2003 National Convention and awarded "top four" status by the Awards Committee.

Abstract

Equations for the special class of curves, created using a Spirograph Design Toy, can be derived mathematically. The purpose of this research is to find the parametric equations of a curve described by a circular disk rolling inside an ellipse. The curve produced by the path of a point on a circular disk rolling inside a circle is revisited for comparison to the ellipse case. Each case is set up geometrically using a figure from which the parametric equations of the curves are derived. It is shown that the ellipse case proved to be more complicated than the circular case. Specifically the ellipse case required the use of elliptic integrals, resulting in computer software assistance.

1. Introduction

The Spirograph Design Toy has entertained children since the 1960's. The Spirograph is a classic toy that I enjoyed using when I was young. It stimulates not only the creative minds of children but also the analytical minds of mathematicians. How can an artistic children's toy be related to mathematics? The Spirograph Design Toy creates special curves known to mathematicians as epitrochoids and hypotrochoids.

My original mathematical interest in the Spirograph was to find the equations of the most common case, an outer fixed circular ring with an inner rotating circular wheel. Various mathematicians and educators have studied the Spirograph's relation to mathematics.

A trochoid is a curve produced by points carried by a circular disk rolling on a fixed circle. These curves are called epitrochoids or hypotrochoids according to whether the circular disk rolls on the outside or inside of the fixed circle, respectively. In the particular case when the point is on the circumference of the rolling circular disk, the curve is a cycloid [1, p. 46], or epicycloid or hypocycloid. Many mathematicians throughout history have studied the cycloid curve; for more details see [6, pp. 309-315].

More specifically on curve relations to the Spirograph Design Toy, are studies by Robert J. Whitaker, Dennis Ippolito, Alfinio Flores, and William Cavanaugh. Whitaker [5] derives the parametric equations for epicycloids, epitrochoids, hypocycloids, and hypotrochoids. Specific details on deriving the equations for hypocycloids and hypotrochoids can be found in the next section of this paper. Ippolito [3, p. 356] and the Standard Mathematical Tables and Formulae [7, p. 293] confirm Whitaker's equations.

My original question concerning the equations of Spirograph curves with a circular disk rotating inside a circle was answered through previous studies. However as part of my research, I purchased the newest version of the Spirograph Design Toy. It is very different from the Spirograph Design Toy I still had from my childhood. Besides having fewer gears and larger teeth, the new Spirograph has three outer ring shapes to choose from, a circle, an ellipse, and a kidney bean shape. My old Spirograph Design Toy did not have a kidney bean or an outer ellipse ring. This new ellipse ring inspired the topic of this paper, analyzing a Spirograph shape with an outer ellipse and inner rotating circular disk.

In all my research, I was unable to find any previous studies on any case besides the circular disk inside a circle case. One article [4, p. 1] did mention, as a challenge, to create a design using a combination of related parameterizations. The interest of this paper is to explore a specific case Spirograph design, an outer ellipse with a rotating inner circular disk.

2. The Circular Case Revisited

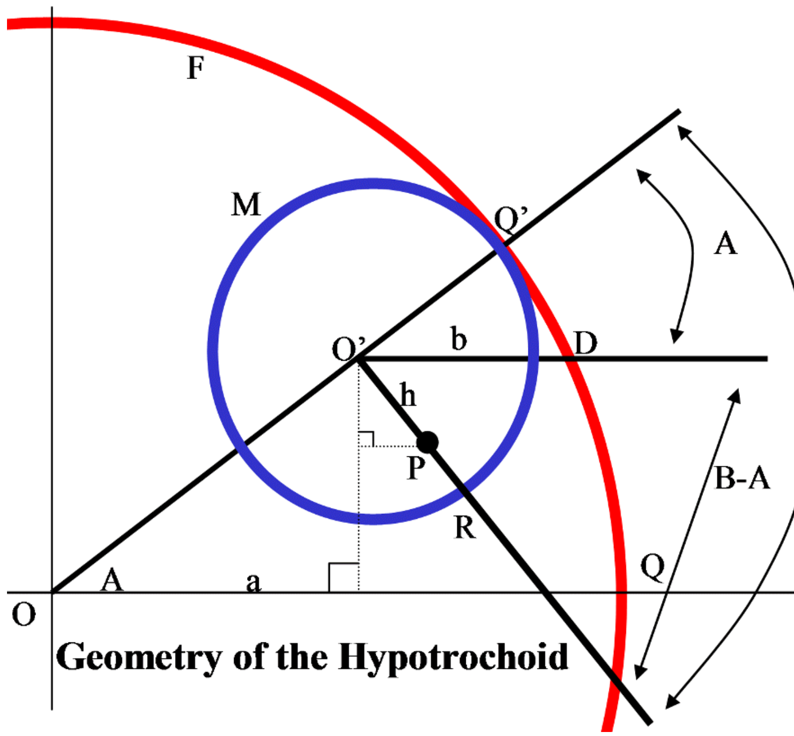


Figure 1

Note this entire section is based on the research of Robert Whitaker [5, pp. 555-557]. The ideas in this section are important to review, for they will be built upon in the elliptic case.

First refer to Figure 1 - Geometry of the Hypotrochoid [5, p. 557]. F is the fixed outer circle with radius a and center O. M is the inner rotating circular disk with radius b, and center O', where $b < a$. The inner circular disk will move counterclockwise around the inside of the outer circle causing the wheel to rotate clockwise, see figure 2.

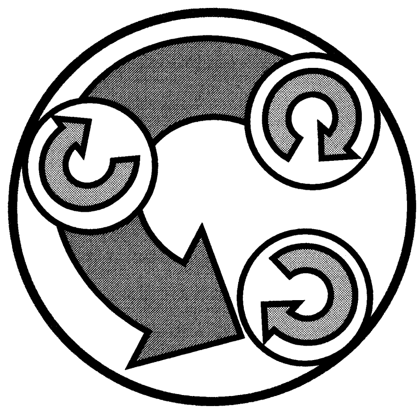


Figure 2

P is the point of the pencil placement, or the point that creates the curve. The distance $O'P$ is denoted by h, where $h < b$. Assume that initially the centers of both circles lie along the horizontal axis and the point R is concurrent with Q. Q' is the point of tangency of the two circles and angle $OO'Q'$ is a straight angle. The angle $Q'OQ$ is represented by A. A line parallel to the line OQ through O' will intersect F at D. By corresponding angles, angle $Q'O'D$ is congruent to angle A. Angle $Q'O'R$ will be angle B. Therefore angle $DO'R$ is angle $(B - A)$ and by alternate interior angles the angle at P, as seen in Figure 4, is also angle $(B - A)$.

Let O' be the point (x_1, y_1) and P be $(x_1 + x_2, y_1 - y_2)$. To follow the point P, the distances x_1, y_1, x_2, and x_2, as seen more closely in figures 3 and 4, need to be calculated.

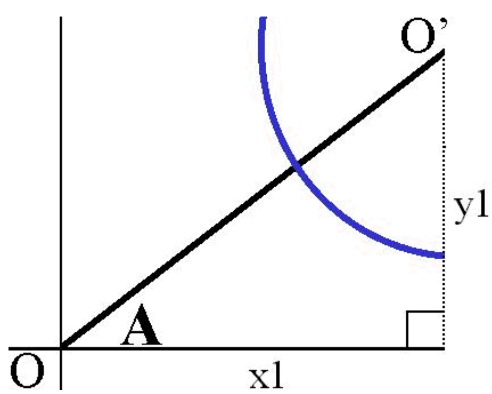


Figure 3

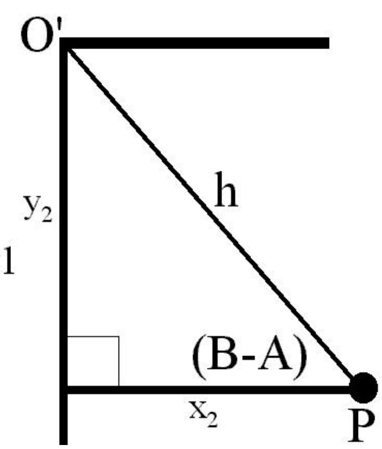


Figure 4

We have

$$\cos A = \frac{x_1}{a-b} \qquad \sin A = \frac{y_1}{a-b}$$
$$\cos(B-A) = \frac{x_2}{h} \qquad \sin(B-A) = \frac{y_2}{h}$$

Solving for x_1, x_2, y_1, and y_2 yields the equations

$$\begin{aligned} x_1 &= (a-b)\cos A \\ y_1 &= (a-b)\sin A \\ x_2 &= h\cos(B-A) \\ y_2 &= h\sin(B-A) \end{aligned}$$

Using these equations, the parametric equations of the path of P are:

$$\begin{cases} x = x_1 + x_2 \\ y = y_1 - y_2 \end{cases},$$

or, equivalently,

$$\begin{cases} x = (a-b)\cos A + h\cos(B-A) \\ y = (a-b)\sin A - h\sin(B-A) \end{cases}. \quad (1)$$

To express these equations in terms of angle A only, angle B can be expressed in terms of angle A. Since R was concurrent with Q before circle M rolled along F, arc QQ' = arc RQ'. Using the formula for arc length of a circle, arc $QQ' = aA$, and arc $RQ' = bB$. Substituting and solving these equations for B results in

$$B = \frac{aA}{b}.$$

Substituting this value of B into the parametric equations (1) results in

$$\begin{cases} x = (a-b)\cos A + h\cos\left(\dfrac{(a-b)A}{b}\right) \\ y = (a-b)\sin A - h\sin\left(\dfrac{(a-b)A}{b}\right) \end{cases}.$$

Any hypotrochoid or spirograph curve can be graphed using these equations for any $a > b > h$. This can be seen using a graphing calculator or graphing software. Figure 5(a) shows an example using Maple 9.5.

```
> restart;
> x := (a - b) * cos(A) + h * cos(((a - b)/b) * A);
  y := (a - b) * sin(A) - h * sin(((a - b)/b) * A);
```

$$
\begin{aligned}
x &= (a-b)\cos(A) + h\cos\left(\frac{(a-b)A}{b}\right) \\
y &= (a-b)\sin(A) - h\sin\left(\frac{(a-b)A}{b}\right)
\end{aligned}
$$

$> \mathbf{a := 48 : b := 30 : h := 17 :}$
$> \mathbf{plot}([\mathbf{x}, \mathbf{y}, \mathbf{A} = \mathbf{0..10} * \mathbf{Pi}]);$

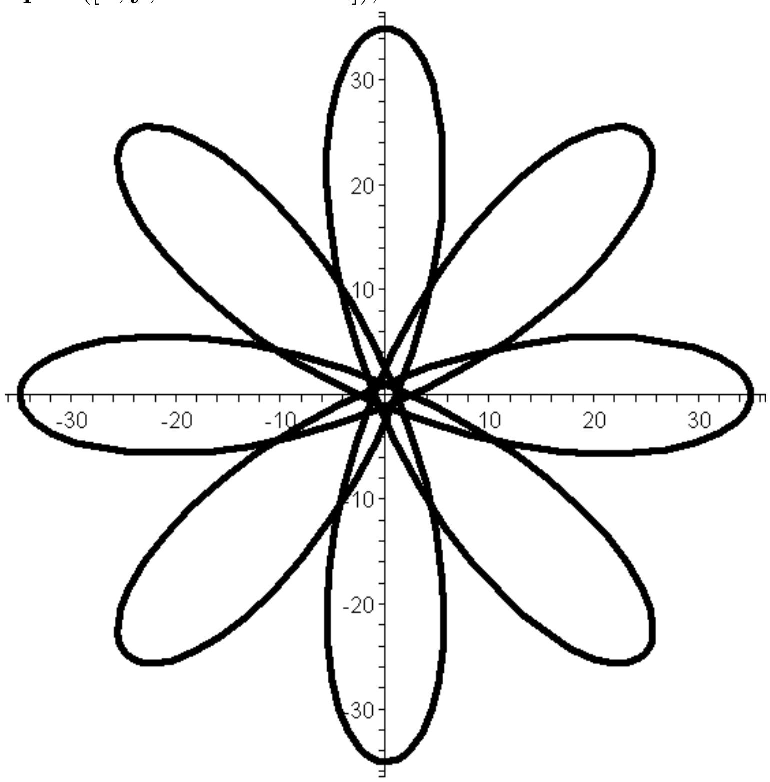


Figure 5(a)

A similar curve can be drawn using the Spirograph Design Toy with an outer ring with 48 teeth, inner rotating circle with 30 teeth, and hole number 5. The spirograph hand drawing is in figure 5(b).

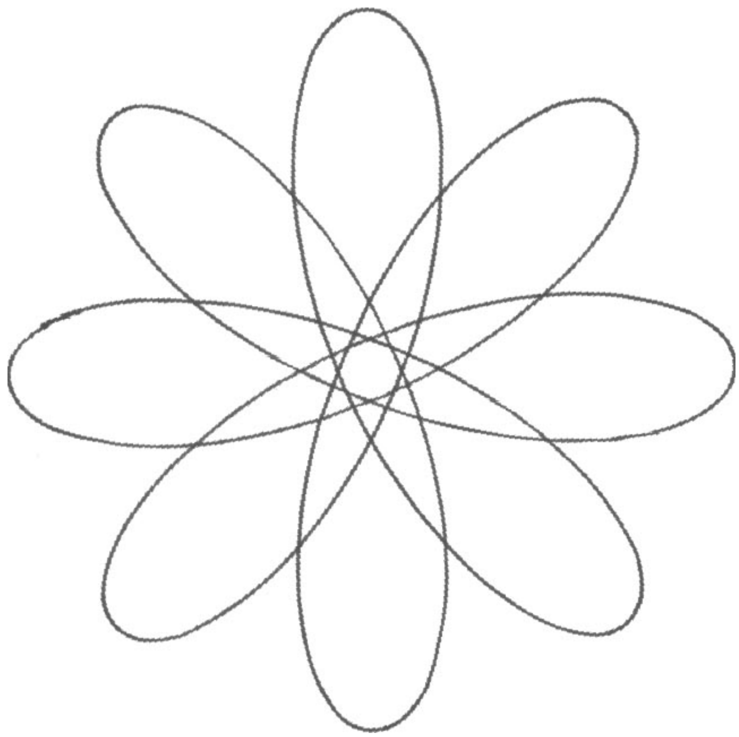


Figure 5(b)

Now that the set-up and process of deriving equations for the curves of the circle case has been reviewed, its logic can be applied and built upon in the following ellipse case.

3. The Ellipse Case

In the ellipse case, for more simplistic calculations along the way, a specific ellipse and circle will be assumed. Assume the ellipse:

$$\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases},$$

and a circle of radius 1. Note, that all calculations may be duplicated for other ellipse and circle pairs.

The first consideration, when choosing a circle to roll within a given ellipse, is the largest curvature of the ellipse. The formula for curvature using derivatives is

$$k = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}}.$$

For the given ellipse

$$\begin{aligned} k &= \frac{|-3\sin t \cdot (-2)\sin t - 2\cos t \cdot (-3)\cos t|}{\left[(-3\sin t)^2 + (2\cos t)^2\right]^{3/2}} \\ &= \frac{\left|6\sin^2 t + 6\cos^2 t\right|}{\left(9\sin^2 t + 4\cos^2 t\right)^{3/2}} \\ &= \frac{6}{\left(9\sin^2 t + 4\cos^2 t\right)^{3/2}}. \end{aligned}$$

One point of maximum curvature will occur at $t = 0$. When $t = 0$, $k = \frac{3}{4}$. To find the largest circle to roll inside the given ellipse, the radius of curvature needs to be calculated. The equation for radius of curvature is

$$\rho = \frac{1}{k}.$$

In this case, $\rho = \frac{4}{3}$. Therefore, the circle can have a radius r such that $r \leq \frac{4}{3}$. Hence using the unit circle, $r = 1$ for this specific case is appropriate.

The initial set up in the ellipse case is similar to that of the circular case; refer to figure 6 - Geometry of the Ellipse Case. The curve will be traced by the point $P\,(x_1 + x_2, y_1 - y_2)$. Thus x_1, y_1, x_2, and y_2 must all be found in terms of t. The angle $Q'OQ$ is represented by t. It is important to recognize that the points O, O', Q' are not collinear, as they were in the circular case, so that the coordinates of the point O' are not so easily found.

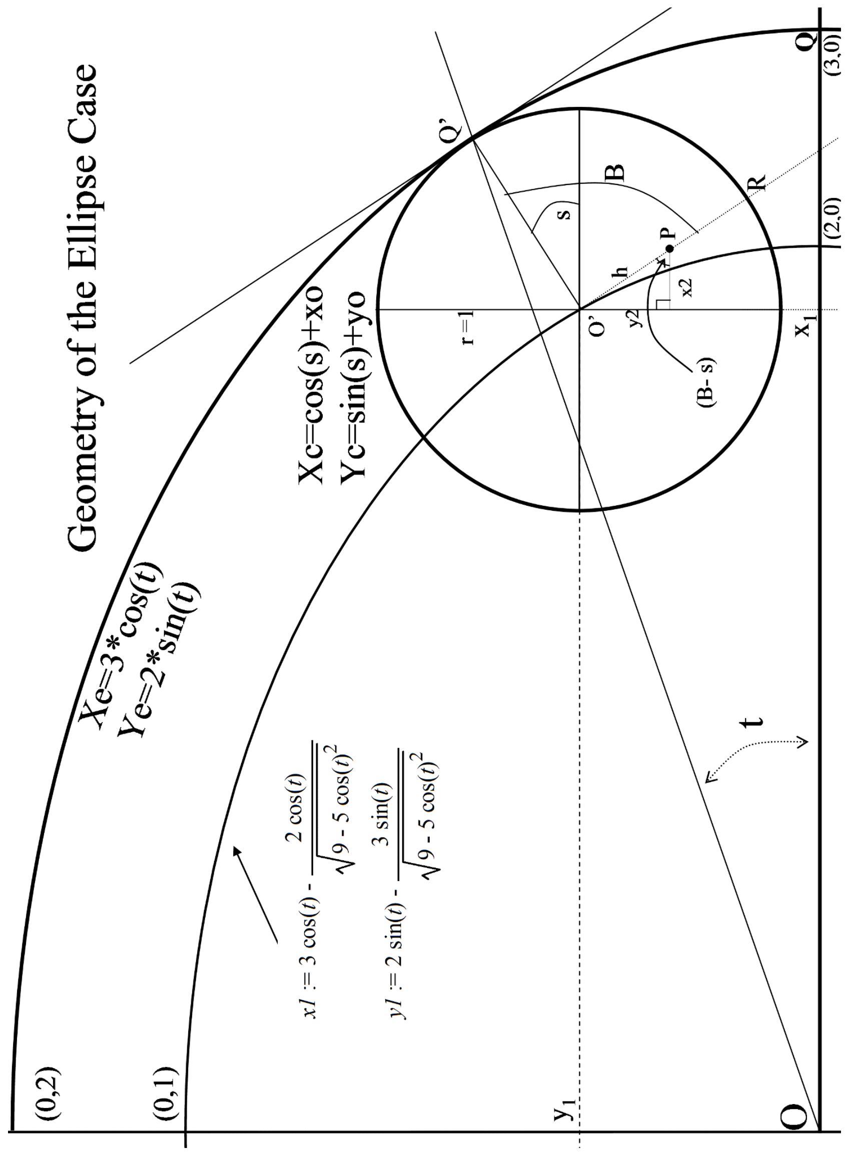

Geometry of the Ellipse Case
Xe=3*cos(t)
Ye=2*sin(t)
Xc=cos(s)+xo
Yc=sin(s)+yo
x1 := 3 cos(t) - 2 cos(t) / √(9 - 5 cos(t)^2)
y1 := 2 sin(t) - 3 sin(t) / √(9 - 5 cos(t)^2)
(0,2)
(0,1)
(2,0)
(3,0)
O
Q
Q'
O'
B
R
P
s
h
r =1
x2
y2
(B- s)
x1
y1
t

Figure 6

Using the fact that point Q' is on both the circle and the ellipse, and that the slope of the tangent line to the circle and the slope of the tangent line to the ellipse at Q' are equal, a system of two equations can be set up to find the path of point O'. The next section will be presented using Maple 9.5 input and output with supporting comments. [Ed. note: Some of the output has been reformatted for easier reading.]

$>$ **restart;**

Parametric equations of the given ellipse are

$>$ **xe:=3*cos(t);ye:=2*sin(t);**

$$\begin{aligned} x_e &= 3\cos t \\ y_e &= 2\sin t \end{aligned}$$

Let $DyDx$ stand for $\dfrac{d(ye)}{d(xe)}$, the slope of the line tangent to the given ellipse.

$>$ **DyDx:=diff(ye,t)/diff(xe,t);**

$$DyDx := -\frac{2\cos t}{3\sin t}$$

Let $EQ1$ be the general equation of a unit circle with center (x_1, y_1).

$>$ **EQ1:=(x-x1)^2+(y-y1)^2=1;**

$$EQ1 : (x - x_1)^2 + (y - y_1)^2 = 1$$

Using implicit differentiation, $\dfrac{dy}{dx}$, the slope of the line tangent to the circle at any (x, y), is given by

$$2\,(x - x_1) + 2\,(y - y_1)\,\frac{dy}{dx} = 0.$$

Denoting $\dfrac{dy}{dx}$ for the circle by $dydx$ and solving this equation for $dydx$ results in

$>$ **dydx:=implicitdiff(EQ1,y,x);**

$$dydx = -\frac{x - x_1}{y - y_1}$$

At the point Q', the slope of the line tangent to the circle and the slope of the line tangent to the ellipse are equal. Equation $EQ2$ expresses this relationship.

$>$ **EQ2:=dydx=DyDx;**

$$EQ2 : -\frac{x - x_1}{y - y_1} = -\frac{2\cos t}{3\sin t}$$

In terms of t, the point Q' which lies on both the ellipse and the circle is

$>$ **x:=xe; y:=ye;**

$$\begin{aligned} x &= 3\cos t \\ y &= 2\sin t \end{aligned}$$

Restatements of $EQ1$ and $EQ2$ with the given point are

$>$ **EQ1;EQ2;**

$$\begin{aligned} (3\cos t - x_1)^2 + (2\sin t - y_1)^2 &= 1 \\ -\frac{3\cos t - x_1}{2\sin t - y_1} &= -\frac{2\cos t}{3\sin t} \end{aligned}$$

Solving the system of equations for (x_1, y_1) results in two solution pairs.

$>$ **allvalues(solve({EQ1,EQ2}, {x1,y1}));**

$$\begin{aligned} y_1 &= \frac{\left(2\sqrt{5\sin^2 t + 4} + 3\right)\sin t}{\sqrt{5\sin^2 t + 4}}, \\ x_1 &= \frac{1}{3}\cos t\left(5 + \frac{2\left(2\sqrt{5\sin^2 t + 4} + 3\right)}{\sqrt{5\sin^2 t + 4}}\right); \\ y_1 &= \frac{\left(2\sqrt{5\sin^2 t + 4} - 3\right)\sin t}{\sqrt{5\sin^2 t + 4}}, \\ x_1 &= \frac{1}{3}\cos t\left(5 + \frac{2\left(2\sqrt{5\sin^2 t + 4} - 3\right)}{\sqrt{5\sin^2 t + 4}}\right). \end{aligned}$$

Simplifying and expanding the results yields more interpretable expressions.

$>$ **outside_x1:=**
expand(simplify(1/3*cos(t)*(5+2*(2*(5*sin(t)^2+4)^(1/2)+3)
/(5*sin(t)^2+4)^(1/2))));

$$outside\_x_1 = 3\cos t + \frac{2\cos t}{\sqrt{9 - 5\cos^2 t}}$$

$>$ **outside_y1:=**
expand(simplify((2*(5*sin(t)^2+4)^(1/2)+3)*sin(t)
/(5*sin(t)^2+4)^(1/2)));

$$outside\_y_1 = 2\sin t + \frac{3\sin t}{\sqrt{9 - 5\cos^2 t}}$$

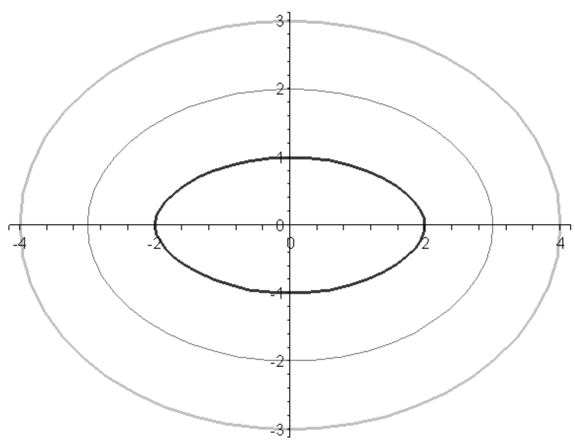
> **x1 :=**
expand(simplify(1/3*cos(t)*(5+2*(2*(5*sin(t)^2+4)^(1/2)-3)
/(5*sin(t)^2+4)^(1/2))));

$$x_1 = 3\cos t - \frac{2\cos t}{\sqrt{9-5\cos^2 t}}$$

> **y1 :=**
expand(simplify((2*(5*sin(t)^2+4)^(1/2)-3)*sin(t)
/(5*sin(t)^2+4)^(1/2)));

$$y_1 = 2\sin t - \frac{3\sin t}{\sqrt{9-5\cos^2 t}}$$

It is now clear from the above equations that the first result creates a path outside the ellipse and the second result is the appropriate path of point O' inside the ellipse. This can be seen in the plot below. The middle curve is the ellipse, the outer path is the result outside the ellipse, and the inner path is the result (x_1, y_1) for the path of O' needed.
> **plot([[x1,y1,t=0..2*Pi], [outside_x1,outside_y1,t=0..2*Pi],**
[xe,ye,t=0..2*Pi]],thickness=[2,2,1], color=[red, gray, sienna]);



It can also be noted that

$$\sqrt{9-5\cos^2 t} = \sqrt{(2\cos t)^2 + (3\sin t)^2} = \sqrt{x_e^2 + y_e^2}.$$

The expression in the denominator of x_1 and y_1 relates directly to the given ellipse.

Now, $x_2 = h\cos(B-s)$ and $y_2 = h\sin(B-s)$. Thus B and s must be found in terms of t. Using s to parameterize the circle, the equations of the circle are:

> **xc:=x1+cos(s);yc:=y1+sin(s);**

$$\begin{cases} x_c = 3\cos t - \dfrac{2\cos t}{\sqrt{9-5\cos^2 t}} + \cos s \\ y_c = 2\sin t - \dfrac{3\sin t}{\sqrt{9-5\cos^2 t}} + \sin s \end{cases}$$

When solving for s in terms of t, x_e can be set equal to x_c, and y_e can be set equal to y_c, because both sets of equations describe point Q'.

> **sx:=solve(xe=xc,s);sy:=solve(ye=yc,s);**

$$\begin{cases} s_x = \arccos\left(\dfrac{2\cos t}{\sqrt{9-5\cos^2 t}}\right) \\ s_y = \arcsin\left(\dfrac{3\sin t}{\sqrt{9-5\cos^2 t}}\right) \end{cases}.$$

It is expected that the two solutions above should be identical, that is that $s_x = s_y$ for all t. However, that is not the case. The problem lies in the fact that the range of the arccosine function is $[0, \pi]$, whereas the range of the arcsine function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Backing up a step, since x_e can be set equal to x_c and y_e can be set equal to y_c,

$$\begin{cases} x_c = 3\cos t - \dfrac{2\cos t}{\sqrt{9-5\cos^2 t}} + \cos s = x_e = 3\cos t \\ y_c = 2\sin t - \dfrac{3\sin t}{\sqrt{9-5\cos^2 t}} + \sin s = y_e = 2\sin t \end{cases}.$$

Hence $\cos s = \dfrac{2\cos t}{\sqrt{9-5\cos^2 t}}$, and $\sin s = \dfrac{3\sin t}{\sqrt{9-5\cos^2 t}}$. These equations for $\cos s$ and $\sin s$ show that $\cos s$ and $\cos t$ will always have the same sign as will $\sin s$ and $\sin t$. Therefore, s and t are always in the same quadrant. So, for $0 \le t \le \pi$, s_x, above, gives the correct value of s, but for $\pi < t \le 2\pi$, s must be defined as $2\pi - s_x$. Looking ahead, s will only be used as an input for cosine and sine functions; thus, using $-s_x$ instead of $2\pi - s_x$ is sufficient. This pattern of s_x and $-s_x$ needs to be continued for all values of t in the domain of the final ellipse spirograph curve equation. The following function will accomplish this alternating pattern for every interval of length π. Let $s = (-1)^{\left\lfloor \frac{t}{\pi} \right\rfloor} s_x$.

> **s:=(-1)^floor(t/Pi)*sx;**

$$s = (-1)^{\left\lfloor \frac{t}{\pi} \right\rfloor} \arccos\left(\frac{2\cos t}{\sqrt{9-5\cos^2 t}}\right)$$

Now, angle B needs to be expressed in terms of t. Using logic from the circular case, the arc length of QQ' equals the arc length of RQ'. The arc length of RQ' is B, because $r = 1$. Therefore, B equals the arc length of

QQ'. If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$, where f' and g' are continuous and not simultaneosly zero on $[\alpha, \beta]$, and if C is traversed exactly once as t increases from α to β, then the length of C [2, p. 419] is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Applying this formula for the given ellipse gives

> **B:=int(sqrt((3*sin(tau))^2+(2*cos(tau))^2), tau=0..t);**

$$B = \int_0^t \sqrt{9\sin^2\tau + 4\cos^2\tau}\, d\tau.$$

This type of integral is called an *elliptic integral of the second kind*. Elliptic integrals cannot be evaluated exactly, only estimated, using tables, calculators, or computers. Here Maple 9.5 will calculate the approximate values needed.

The values x_2 and y_2 are found in the same manner as in the circular case.

> **x2:=h*cos(B-s);y2:=h*sin(B-s)**;

$$\begin{aligned} x_2 &= h\cos[-\int_0^t \sqrt{9\sin^2\tau + 4\cos^2\tau}\, d\tau \\ &\quad + (-1)^{\lfloor \frac{t}{\pi} \rfloor} \arccos\left(\frac{2\cos t}{\sqrt{9 - 5\cos^2 t}}\right)] \\ y_2 &= -h\sin[-\int_0^t \sqrt{9\sin^2\tau + 4\cos^2\tau}\, d\tau \\ &\quad + (-1)^{\lfloor \frac{t}{\pi} \rfloor} \arccos\left(\frac{2\cos t}{\sqrt{9 - 5\cos^2 t}}\right)] \end{aligned}$$

Note that, since B and s have both been found in terms of t, x_2 and y_2 are functions of t. As in the circular case, the equations of the path of P are expressed as

$$\begin{cases} x = x_1 + x_2 \\ y = y_1 - y_2 \end{cases}.$$

The final equations of the curve produced from a given point on a circular disk rolling inside the given ellipse are

> **spiro_x:=x1+x2;spiro_y:=y1-y2;**

$$\begin{aligned} spiro\_x \quad : \quad &= 3\cos t - \frac{2\cos t}{\sqrt{9-5\cos^2 t}} \\ &+h\cos[-\int_0^t \sqrt{9\sin^2\tau + 4\cos^2\tau}\,d\tau \\ &+(-1)^{\lfloor \frac{t}{\pi} \rfloor} \arccos\left(\frac{2\cos t}{\sqrt{9-5\cos^2 t}}\right)] \\ spiro\_y \quad : \quad &= 2\sin t - \frac{3\sin t}{\sqrt{9-5\cos^2 t}} \\ &+h\sin[-\int_0^t \sqrt{9\sin^2\tau + 4\cos^2\tau}\,d\tau \\ &+(-1)^{\lfloor \frac{t}{\pi} \rfloor} \arccos\left(\frac{2\cos t}{\sqrt{9-5\cos^2 t}}\right)] \end{aligned}$$

The equations for $spiro\_x$ and $spiro\_y$ are expressed solely in terms of t. The desired spirograph curve can now be drawn for any given interval of t values. First a specific value of h must be chosen.

> **h:=.5:**

A plot of the curve can be seen in Figure 7(a).

> **plot([[spiro_x,spiro_y,t=0..80*Pi], [3*cos(t),2*sin(t),t=0..2*Pi]]);**

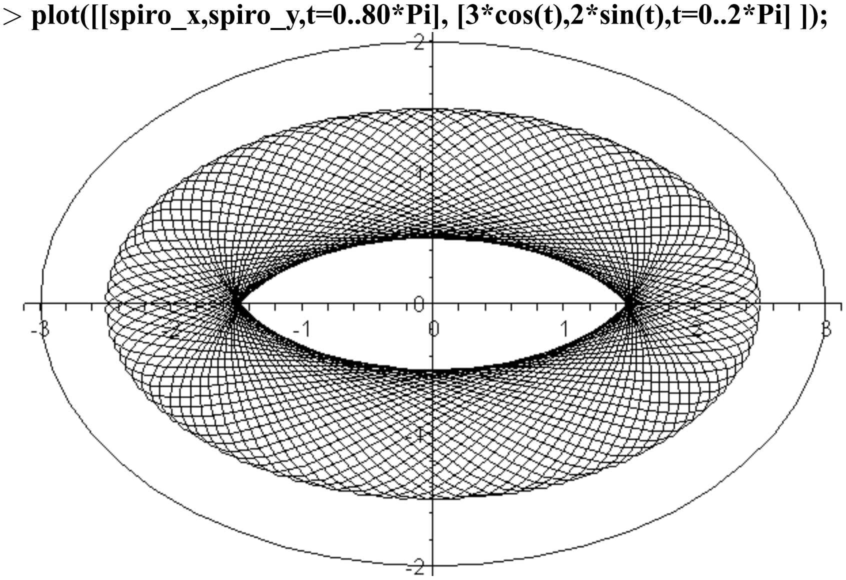


Figure 7(a)

Figure 7(b) is a curve drawn using the Spirograph Design Toy with an outer ellipse with 64 teeth, inner rotating circular disk with 30 teeth, and hole position number 8.

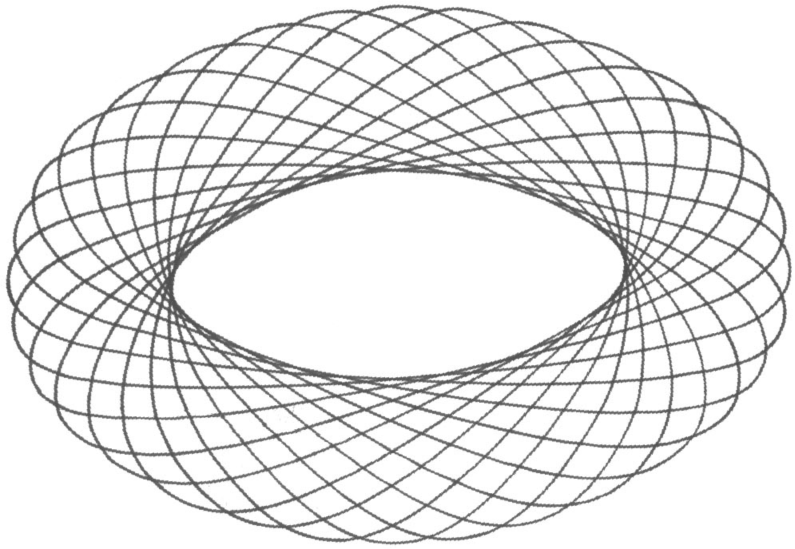


Figure 7(b)

4. Conclusion

In conclusion, the ellipse case proved to have more lengthy and complicated calculations than the circular case. This can be partially attributed to the nice properties of circles that do not hold for an ellipse. The analysis of the Spirograph curve with an outer ellipse and inner rotating circular wheel was a challenging project. This study has shown that Spirograph curves involve many levels of mathematics ranging from basic knowledge of parametric equations, trigonometry, and geometry to more advanced special functions and geometric situations.

Further research can be done on other Spirograph curves using any combination of outer and inner shapes. The Spirograph Design Toy inspires many curve possibilities to study and explore.

Acknowledgements: I would like to acknowledge my advisor, Dr. Rhonda McKee, Professor, Central Missouri State University, whose guidance is always helpful and appreciated. I would also like to acknowledge Andrew Ray, a graduate student at Central Missouri State University, for his help in redefining s.

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Pythagorean Rectangles

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Given a line segment of length x, it is simple to construct (using only a compass and straightedge) a square of area x^2 with sides congruent to the original segment. The "Pythagorean theorem" is the remarkable fact that *three* line segments forming the sides of a *right* triangle have corresponding squares such that the area of the largest is the sum of the areas of the other two. More precisely, this theorem states that $x^2 + y^2 = z^2$ where x and y are the lengths of the "legs" (the shorter two sides) and z is the length of the "hypotenuse" (the longest side) of *any* right triangle. If the lengths of these line segments are integers, $a = x, b = y, c = z$, the triple (a, b, c) is called a "Pythagorean triple".

Given an angle θ as well as a line segment of length x, we can extend the line segment to double its length (using a compass) and use it as the diagonal of a rectangle in which the angle determines the placement of the other diagonal (see Figure 1).

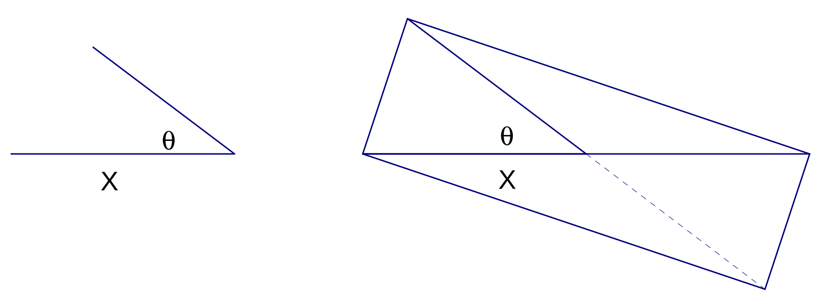

Figure 1: The rectangle from a line segment x and an angle θ

For a right triangle, we may carry out this rectangle construction twice, using the legs (x and y) and the angles (θ and ϕ) they form with the hypotenuse (see Figure 2).

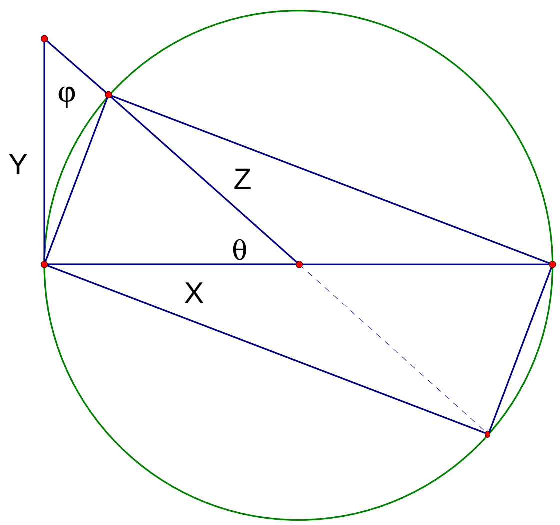

Figure 2: The rectangle from a leg of a right triangle

It is my purpose to investigate the relative proportions of these "Pythagorean rectangles" for arbitrary right triangles in terms of the lengths x, y, z of the legs and hypotenuse. In particular, what can be said about these proportions when the lengths form a Pythagorean triple (a, b, c)?

If we bisect the angle θ (see Figure 3) we create a triangle with sides of half the height and half the width of the rectangle.

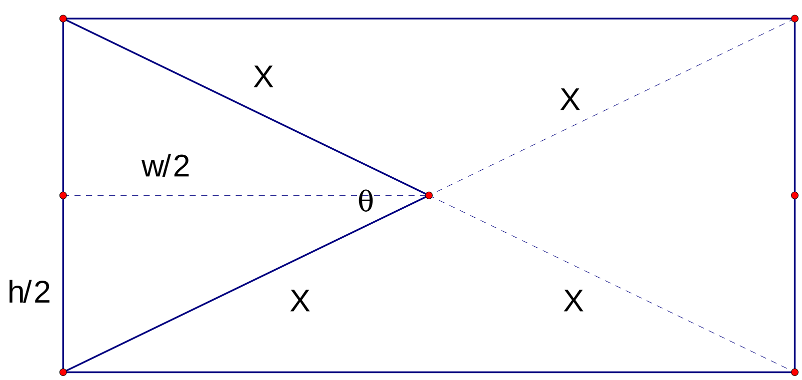

Figure 3: The height-to-width ratio using the half angle

Since the ratio of these halves is the same as the relative proportion of the sides of the rectangle, we can use the "tangent half-angle formula" to find the ratio of the height to the width of the rectangle. For the rectangle drawn using side x,

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-x/z}{1+x/z}} = \sqrt{\frac{z-x}{z+x}}$$

and for the rectangle drawn using side y,

$$\tan\left(\frac{\phi}{2}\right) = \sqrt{\frac{1-\cos\phi}{1+\cos\phi}} = \sqrt{\frac{1-y/z}{1+y/z}} = \sqrt{\frac{z-y}{z+y}}$$

For arbitrary x and y values, these square roots may not be rational numbers. But if they are, the fractions under the radicals must be ratios of perfect squares after all common factors have been removed. For rectangles with rational proportions, we thus need that the lengths x, y, z form a *primitive* Pythagorean triple (a, b, c) for integers a, b, c with no common factor and that the ratios $\frac{z-x}{z+x}$ and $\frac{z-y}{z+y}$ are ratios of perfect squares. For the well-known $(5, 12, 13)$ primitive Pythagorean triple, $\frac{z-x}{z+x} = \frac{8}{18} = \frac{4}{9}$ and $\frac{z-y}{z+y} = \frac{1}{25}$ are indeed such ratios. Will this hold true in general?

Before we can consider this question, we must first characterize all primitive Pythagorean triples; that is, triples (a, b, c) of positive integers with no common factors such that $a^2 + b^2 = c^2$.

Which of these numbers can be odd and which even? a and b cannot both be even since if they were, the sum of their squares would then be even as well and a, b, c would have a common factor of at least 2. a and b cannot both be odd since if they were, the sum of their squares would be divisible by 2 but not by 4 while the square of c would be even and hence divisible by 4. Therefore, a and b must be of opposite parities and c must then be odd. Renaming if necessary, we may choose a even and b odd.

Writing $a^2 + b^2 = c^2$ as $a^2 = c^2 - b^2 = (c+b)(c-b)$, we have factored a^2 as the produce of two even numbers, $c + b$ and $c - b$. Since each is divisible by 2, we may divide through by 4 to obtain

$$\left(\frac{a}{2}\right)^2 = \left(\frac{c+b}{2}\right)\left(\frac{c-b}{2}\right)$$

The left side is a square, so the right side expressions must either have something in common or be squares themselves. But any divisor of two integers also divides their sum and difference. Thus any common factor of $(c+b)/2$ and $(c-b)/2$ is also a common factor of $\frac{c+b}{2} + \frac{c-b}{2} = c$ and $\frac{c+b}{2} - \frac{c-b}{2} = b$. But c and b have no common factors and therefore $(c+b)/2$ and $(c-b)/2$ are squares. Letting $(c+b)/2 = m^2$ and $(c-b)/2 = n^2$, we have $c = m^2 + n^2$ and $b = m^2 - n^2$, and then $a^2 = c^2 - b^2 = (m^2+n^2)^2 - (m^2-n^2)^2 = 4m^2n^2$. Thus

$$a = 2mn, b = m^2 - n^2, c = m^2 + n^2$$

where, of course, $m > n > 0$ have no common factor (see Hardy and Wright [2]).

Returning to our question about Pythagorean rectangles, we may take these expressions for the Pythagorean triple (a, b, c) and substitute them into our two height-to-width ratios found earlier using the tangent half-angle formula:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{c-a}{c+a}} = \sqrt{\frac{(m^2+n^2)-2mn}{(m^2+n^2)+2mn}} = \sqrt{\frac{(m-n)^2}{(m+n)^2}} = \frac{m-n}{m+n}$$

and

$$\tan\left(\frac{\phi}{2}\right) = \sqrt{\frac{c-b}{c+b}} = \sqrt{\frac{(m^2+n^2)-(m^2-n^2)}{(m^2+n^2)+(m^2-n^2)}} = \sqrt{\frac{2n^2}{2m^2}} = \frac{n}{m}$$

These equations show that for any triangle similar to a right triangle corresponding to a Pythagorean triple, the height-to-width ratios of the resulting Pythagorean rectangles will be ratios of integers.

This brings us to something very interesting about the $3-4-5$ triangle. For this triangle, $m = 2$ and $n = 1$. As you can see, these values give us height-to-width ratios of $\frac{1}{3}$ and $\frac{1}{2}$, and this is the *only* instance where the ratios are *both* of the form "1 to an integer".

Acknowledgements. This paper is my explanation of the above final result which is from the Grattan-Guiness book [1] *The Rainbow of Mathematics*, and is based on the Mathematics Seminar ("MTH 90") presentation I prepared under the direction of Dr. Andrew M. Rockett and gave at the C.W. Post Campus of Long Island University in November 2001.

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Mathematical Background of the Simplex Tableau

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1. Introduction

Production managers in the manufacturing field are constantly faced with the pressure of maximizing profits with the limited resources available. In the short run, a firm is forced to work within fixed constraints such as square footage, labor hours, supplies, and product demand. Each product line requires a set of inputs, and when compared to its profit-contribution, a manager can decide how best to use these limited resources. The simplex tableau allows business managers to maximize profits just by changing which products to produce and how many of each.

I encountered the simplex tableau in the Analytical Methods class required for accounting and business majors in the Breech School of Business at Drury University. When first exposed to the simplicity of the simplex tableau, it seemed like a magic trick, and left me with a desire to research the mathematical basis of this chaotic model. Having concluded my research, it has become very apparent that the simplex tableau offers a mathematically simple way for managers to find profit-maximizing levels of output. The main purpose of this paper is to understand the mathematical background for the business model simplex tableau. First I will recreate what was presented in the business class, and then I will show how this parallels to the calculations in the mathematical model. Finally I will test the mathematical model for problems with constraints beyond two dimensions.

2. The Business Model

To understand the computations of the business model simplex tableau, follow the problem facing Electrocomp Corporation and its production facility. The explanation is one very similar to that in the business class, in which students were only instructed on how to do the computations, and not why to do the computations.

The Electrocomp Corporation manufactures two electrical products: air conditioners and large fans. The assembly process for each is similar in that both require a certain amount of wiring and drilling. Each air conditioner takes 3 hours of wiring and 2 hours of drilling. Each fan must go through 2 hours of wiring and 1 hour of drilling. During the next production period, 240 hours of wiring time are available and up to 140 hours of drilling time may be used. Each air conditioner sold yields a profit of \$25. Each fan assembled may be sold for a \$15 profit. [1]

The short term limitation for Electrocomp in this problem is labor hours; they must decide if it is profitable to produce both air conditioners and fans, and then decide the most that they can produce of each. Since each product requires different amounts of labor, and each contributes a different amount of profit per unit, the simplex tableau is needed to simultaneously consider all variables. To solve this problem using the business model simplex tableau, first identify the appropriate equations, with the number of air conditioners to be produced denoted as x_1, the number of large fans as x_2, and all variables greater than or equal to zero:

$$P = 25x_1 + 15x_2 \text{ (Profit Equation)}$$

$$\left.\begin{array}{r} 240 \geq 3x_1 + 2x_2 \\ 140 \geq 2x_1 + x_2 \end{array}\right\} \text{(Constraint Equations)}$$

$$x_1 \geq 0, x_2 \geq 0$$

For the simplex tableau, these relationships need to be set to equalities rather than inequalities. There exists some value greater than or equal to zero that can be added to the right hand side of the relationships to equalize the given amounts. This extra value is known as the slack variable because it represents the slack in the inequality (for example, the amount of available wiring hours not employed). The new equalities with slack variables s_1 and s_2 for the two equations appear as follows:

$$\begin{array}{lllllllll} 240 & = & 3x_1 & + & 2x_2 & + & s_1 & & \\ 140 & = & 2x_1 & + & x_2 & & & + & s_2 \end{array}$$

The next step is to insert these formulas into the simplex tableau:

	1	2	3	4	5	6	7
A	$C_j \longrightarrow$			25	15	0	0
B	$\downarrow$	Sol. Mix	Quantity	x_1	x_2	s_1	s_2
C	0	s_1	240	3	2	1	0
D	0	s_2	140	2	1	0	1
E		Z_j					
F		$C_j - Z_j$					

This is just a skeleton of the business model simplex tableau, filling in only the parts that are derived directly from the constraint and profit equations. C_j shows the profit-contributing values for variables x_1, x_2, s_1, and s_2. These values are the coefficients listed in the profit equation,

$$P = 25x_1 + 15x_2 + 0s_1 + 0s_2.$$

The sol. mix is the solution mix, or the variables which are being considered to maximize profits. There are two types of variables to be considered in the solution mix: basic variables, which have values greater than zero, and non-basic variables, which have values equal to zero. There will always be both basic and non-basic variables in the solution mix, but only the basic variables are listed in the tableau. It was not explained in the business class, but the procedure is to begin with the two slack variables as the basic variables, and x_1 and x_2 as the non-basic variables. So if x_1 and x_2 are equal to zero, the slack variables s_1 and s_2 will have a quantity, or value, equal to the left hand side of the equalities, 240 and 140 respectively. The remaining cells in rows C and D are the coefficients from the two constraint equations,

$$240 = 3x_1 + 2x_2 + 1s_1 + 0s_2$$

and

$$140 = 2x_1 + 1x_2 + 0s_1 + 1s_2,$$

underneath their respective variables in row B.

To calculate the rest of the table, use the following procedure:

1. Cells $E3 - E7$ in the Z_j row are calculated by multiplying each number in the same column, from two rows above it, times the C_j within the same row, and then adding the two numbers together.
$$(E3 = C3 \cdot C1 + D3 \cdot D1, E4 = C4 \cdot C1 + D4 \cdot D1)$$
2. Cells $F4 - F7$ are calculated by subtracting Z_j from C_j, or the values in cells $E4 - E7$ from the values in the respective cells $A4 - A7$.
$$(F4 = A4 - E4, F5 = A5 - E5)\,.$$

The completed table following the procedure now reads as follows:

	1	2	3	4	5	6	7
A	C$_j$ $\longrightarrow$			25	15	0	0
B	$\downarrow$	Sol. Mix	Quantity	x_1	x_2	s_1	s_2
C	0	s_1	240	3	2	1	0
D	0	s_2	140	2	1	0	1
E		Z_j	0	0	0	0	0
F		$C_j - Z_j$		25	15	0	0

If there are any positive numbers in cells $F4 - F7$, another table must be computed. Below is the procedure that was followed in the business class.

Steps to achieve an optimal solution [1]

Repeat the following 5 steps until an optimum solution is reached:

1. Select the nonbasic variable with the largest C-Z value as the entering variable—the column of this variable is called the pivot column. (25 is the greatest value, so column 4 is the pivot column)
2. Select the row to be replaced (the leaving variable) as the row with the smallest positive quantity-to-pivot column ratio—call this the pivot row and identify the intersection of this row and the pivot column the pivot element. (row C ratio = C3/C4 = 80, row D = D3/D4 = 70; since 70 is smallest, 2 becomes the pivot element)
3. Calculate the new values for the pivot row by dividing each number in the row by the pivot element. (Row D is the pivot row, and each new cell equals the current number divided by 2)
4. Calculate the new values for all other rows as follows:

   New value $=$ Old value

   – (pivot-column value) · (new row value of entering variable)

(Cells in row Cnew = [current values – (3 * values of row Dnew)]; C5new = C5 – (C4 * D5new) = $\frac{1}{2}$)

5. Calculate the Z and C-Z row values for the revised tableau—if there is a positive value remaining in the C-Z row, then this solution is not optimal—Repeat steps a-e until an optimal solution is reached.

The second table following the above procedure appears below:

	1	2	3	4	5	6	7
A	$C_j \longrightarrow$			25	15	0	0
B	$\downarrow$	Sol. Mix	Quantity	x_1	x_2	s_1	s_2
C	0	s_1	30	0	$\frac{1}{2}$	1	$-\frac{3}{2}$
D	25	x_1	70	1	$\frac{1}{2}$	0	$\frac{1}{2}$
E		Z_j	1750	25	$\frac{25}{2}$	0	$\frac{25}{2}$
F		$C_j - Z_j$		0	$\frac{5}{2}$	0	$-\frac{25}{2}$

Since there is a positive number remaining in the bottom row, there is yet another table to be computed. The new solution mix will be $x1$ and $x2$, with $\frac{1}{2}$ as the pivot element. Table 3 will also be computed following the procedure listed above:

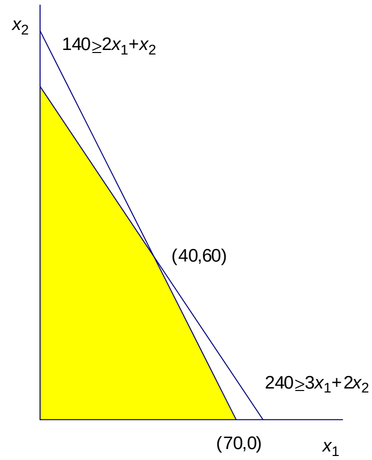
	1	2	3	4	5	6	7
A	$C_j \longrightarrow$			25	15	0	0
B	$\downarrow$	Sol. Mix	Quantity	x_1	x_2	s_1	s_2
C	15	x_2	60	0	1	2	-3
D	25	x_1	40	1	0	-1	2
E		Z_j	1900	25	15	5	5
F		$C_j - Z_j$		0	0	-5	-5

The solution mix now includes x_1 and x_2. Since there are no positive numbers in cells F4 – F7, the simplex tableau is complete. The solution to this profit-maximizing problem is 40 units of air conditioners and 60 units of large fans for a total profit of $1900.

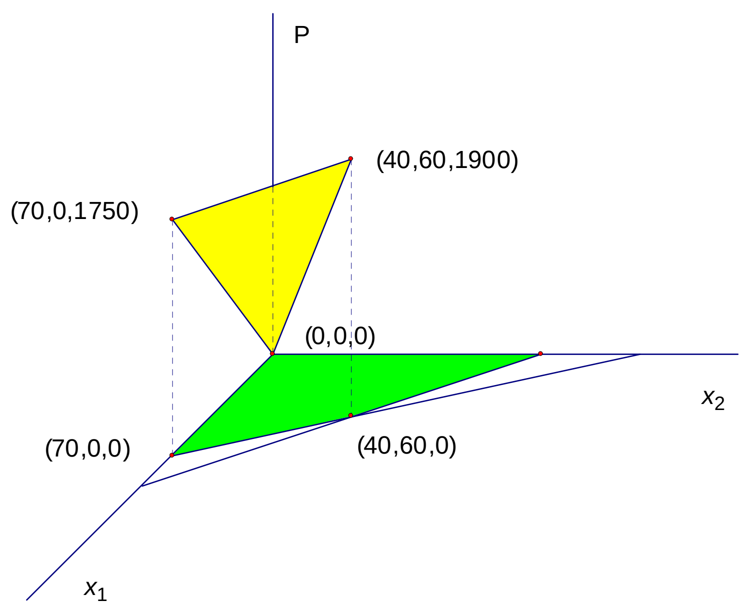
The business model simplex tableau is very confusing because it appears chaotic at best. The simple calculations involving ratios and algebra seem most confusing because it does not seem possible that these calculations can simultaneously consider so many variables. Let's refer now to the mathematical model to make sense of these previous calculations.

3. The Mathematical Model

It might be simpler to understand how the mathematical model relates to the simplex tableau with the graphical solution near at hand:



Two-Dimensional Model of Constraints



Three-Dimensional Model of Constraints and Profit Plane

Referring to the two-dimensional graph, the feasible region is the convex area created between the axis and the intersection of the constraint inequalities. Any point in the shaded region would satisfy all the constraints; however, by choosing a solution set that lies on the boundary lines of the shaded region, the solution lies on the line of equality, which diminishes the non-profit-contributing slack variable completely. The profit equation is a plane that rests above the graph of constraint equations, a portion of which can be seen in the three-dimensional graph. Theory states that the solution to the maximization problem in two-dimensions will lie on the boundary points of the convex region created by lines; in a three-dimensional problem will lie on the vertices or lines of the boundary of the convex region created by planes. This paper will not pursue proof of this theory, but will use it to explain the computations found in the business and mathematical models of the simplex tableau.

For the remainder of this analysis, the table found in [2] will be used because it is easier to accommodate for multi-dimensional problems. The equations for the mathematical model are derived in the same manner as the business model, except for the profit maximizing formula. It is shifted so the profit solution lies on the same side as the inputs, and is referred to as the **objective row**.

$$\begin{aligned} 1P - 25x_1 - 15x_2 + 0s_1 + 0s_2 &= 0 \\ 3x_1 + 2x_2 + 1s_1 + 0s_2 &= 240 \\ 2x_1 + 1x_2 + 0s_1 + 1s_2 &= 140 \\ x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 &\geq 0 \end{aligned}$$

The initial table of the mathematical model for Electrocomp Corporation appears below, with all values taken directly from the above equations:

BV	P	x_1	x_2	s_1	s_2	RHS
P	1	-25	-15	0	0	0
s_1	0	3	2	1	0	240
s_2	0	2	1	0	1	140

The mathematical model appears more like a traditional linear algebra matrix. BV stands for basic variables, P for profit, and RHS for the right hand side of the equation.

Just like in the business model, the initial table begins at the point where all slack variables are basic and real variables are non-basic. When the model begins with s_1 and s_2 equal to all of the slack in production, and x_1 and x_2 equal to zero, there is no production. In the business model, calculation is complete when there were no remaining positive variables in the last row; in the mathematical model, calculation is complete when there are no negative values in the top row. These are equal statements because of the sign change. The first step to solving the mathematical model is to identify the pivot element (number). The selection of the pivot element can be located following three rules for the maximization problem. [2]

- Rule 1: The pivot column is selected by locating the most negative entry in the objective row. This is the negative entry whose absolute value is largest.
- Rule 2: Divide each entry in the last column (the RHS) by the corresponding positive entry (from the same row) in the pivot column. (Ignore any rows in which the pivot column entry is less than or equal to 0.) The row in which the smallest nonnegative ratio is obtained is the pivot row.
- Rule 3: The pivot element is the entry at the intersection of the pivot row and pivot column. Note that the pivot element is never in the objective row.

Following these rules, x_1 will be the new basic variable in the pivot column, and s_2 will be become a non-basic variable in the pivot row. Notice that this is the same ratio taken in the business model (using cells C3 – E3 instead of RHS), and the same selection of the pivot column and pivot row.

So why do both the mathematical and business models have an entering and a leaving variable? Graphically, the entering of one variable and the passing of another creates a movement in the direction of greatest profit potential, in this case x_1. So how much can x_1 be increased and still maintain the constraints of the problem? Consider the following inequalities, under the assumption that x_2 is still equal to zero. The ratio taken really represents a solution to how much the entering variable can be increased according to the system of equations. For example:

$$\begin{aligned} 240 &= 3x_1 + 2x_2 + s_1 \text{ or } 240 = 3x_1 + s_1 \rightarrow x_1 \leq 80 \\ 140 &= 2x_1 + x_2 + s_2 \text{ or } 140 = 2x_1 + s_2 \rightarrow x_1 \leq 70 \end{aligned}$$

If x_1 is to be maximized, it will force the other variable in its equation to zero. If it is to be maximized under both equations, it will be forced to stop at 70. If x_1 becomes 70,

$$\begin{aligned} 240 &= 3x_1 + 2x_2 + s_1 \text{ or } 240 = 3(70) + (30 \text{ slack}) \\ 140 &= 2x_1 + x_2 + s_2 \text{ or } 140 = 2(70) + (0 \text{ slack}) \end{aligned}$$

Here, s_2 has been forced to zero in the system of equations. This assessment of the system of equations explains why it can be simplified to a ratio: with at least one variable in the system still equal to zero, and another variable forced to move to zero, the incoming variable can be calculated by dividing the RHS by the incoming variable's coefficient. Thus the ratio is used in both models to determine the leaving variable, or the variable that was forced to zero with the maximization of the entering variable. Graphically, the solution has moved from $(0, 0)$ in the direction of the x_1 axis to the point of $(70, 0)$, which is a corner point of the convex feasible region.

Once the pivot element is found, the next step is to divide the entire pivot row by whatever number necessary to make the pivot element equal to 1. All other values in that column must then be forced to zero through matrices computations. In the table below x_1 has replaced s_2 in the bottom row, and the other two rows have been calculated so that the values in the pivot column, other than the pivot element, are equal to 0.

BV	P	x_1	x_2	s_1	s_2	RHS
P	1	0	$-\frac{5}{2}$	0	$\frac{25}{2}$	$1,750$
s_1	0	0	$\frac{1}{2}$	1	$-\frac{3}{2}$	30
x_1	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	70

Row x_1 was divided by 2 to make the pivot element cell equal to 1. Row s_1 is comprised of its original entries plus (-3) times row x_1; row P is the original plus 25 times row x_1. Notice that these are the same values as in the second table of the business model, and though the visual manipulation is different, the calculations are the same.

	1	2	3	4	5	6	7
A	C$_j$ $\longrightarrow$			25	15	0	0
B	$\downarrow$	Sol. Mix	Quantity	x_1	x_2	s_1	s_2
C	0	s_1	30	0	$\frac{1}{2}$	1	$-\frac{3}{2}$
D	25	x_1	70	1	$\frac{1}{2}$	0	$\frac{1}{2}$
E		Z_j	1750	25	$\frac{25}{2}$	0	$\frac{25}{2}$
F		$C_j - Z_j$		0	$\frac{5}{2}$	0	$-\frac{25}{2}$

The Second Table from the Business Model

The RHS values parallel to values in C3 – E3 (though not in order), and the values in the bottom two rows of the math model (not including the P and RHS columns) are the same as in cells C4 – C7 and D4 – D7. The values in cells F4 – F7 are the opposites of those in the objective row of the mathematical model because the profit equation coefficients are opposites in the two models. The apparently chaotic manipulation of numbers can now be explained by the matrix algebra needed the make all entries in the pivot column, other than the pivot row, equal to 0.

Referring to the second table of the math model, there is a negative value in the objective row, which will indicate that the company can increase its profit by increasing x_2 from its current value of zero. So the question is once more, how much can x_2 be increased, with the added condition that s_2 equals zero:

$$\begin{aligned} 30 &= \frac{1}{2}x_2 + s_1 - \frac{3}{2}s_2 \text{ or } 30 = \frac{1}{2}x_2 + s_1 \rightarrow x_2 \leq 60 \\ 70 &= x_1 + \frac{1}{2}x_2 + \frac{1}{2}s_2 \text{ or } 70 = x_1 + \frac{1}{2}x_2 \rightarrow x_2 \leq 140 \end{aligned}$$

For these two equations to be satisfied, x_2 can be maximized at 60 units. Referring back to the geometrical analysis, this is a shift of the solution along the boundary line of constraint $240 \geq 3x_1 + 2x_2$ to the corner point $(40, 60)$.

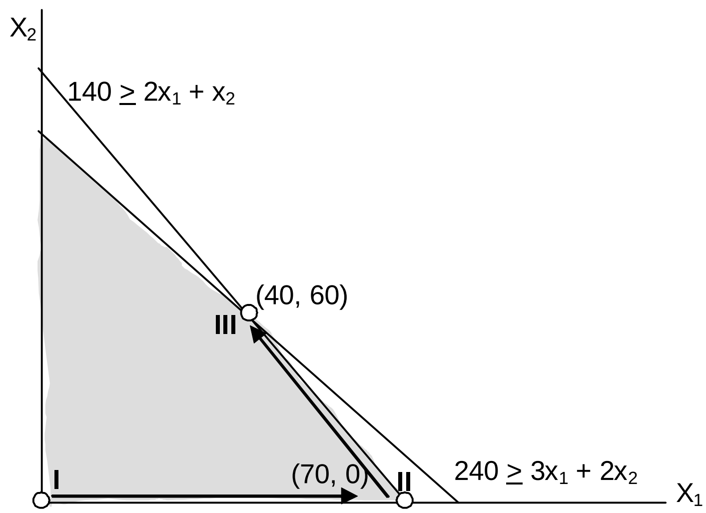
With the new pivot element equal to $\frac{1}{2}$, the third table will be calculated by multiplying the pivot row times 2 so that the pivot element will become 1, and the other two rows will be added to a multiple of the pivot row so that the other values in the pivot column are equal to 0 (following the same procedure from the first and second tables).

BV	P	x_1	x_2	s_1	s_2	RHS
P	1	0	0	5	5	$1,900$
x_2	0	0	1	2	-3	60
x_1	0	1	0	-1	2	40

Notice that this table also parallels to the third table of the business model, following the same explanation from the previous tables. The objective row has only positive numbers in it, and following the rules for pivot selection, there is no greatest negative value. This means that there is not a variable that could be entered into the solution mix so that profit could be increased from its current value.

To review the tabular calculations, follow the graphical interpretation on the following page. The Roman numerals refer to the table number and the arrows depict the direction of movement to the next point. The solution set began at (0,0). In the first table it was determined that the x_1 variable had greater per-unit-profit-contribution, so the matrix calculations to create table two forced the solution set to x_1's maximum value, 70. The second table then determined that greater profit could still be achieved if

x_2 was introduced into the solution set. The introduction of x_2, with s_2 being forced to zero, shadowed a movement along the constraint $140 > 2x_1 + x_2$ in the graphical solution. The greatest that x_1 and x_2 could both reach while satisfying the two constraint equations was at (40,60). The third table revealed that it would not be profitable to introduce any other variables into the solution mix, and thus (40,60) was the optimum solution, which can also be seen in the three-dimensional graphical solution at the beginning of this section as the point with the greatest vertical distance from the profit plane.



Depiction of the Graphical Movements of the Electrocomp Solution

The business tableau is just a reproduction of the movements of matrix algebra in a condensed and methodical fashion. It can safely be concluded that these two tables are calculated by the same methods. The seemingly chaotic additions and multiplications found in the business model can be mathematically derived from a mathematical model of the simplex tableau that is formatted for a matrix algebra problem, which provides for the movement along the graphical solution.

The comparison of these two models helped me to understand the computations behind the business model, but it then opened a new question for me: why waste time with the simplex tableau when the graphical solution is much quicker? Under three- and higher dimensions it becomes much more difficult to quickly analyze a graphical solution. Consider the following example of how the mathematical model can still help to shadow the movements of the solution along a feasible region in three-space.

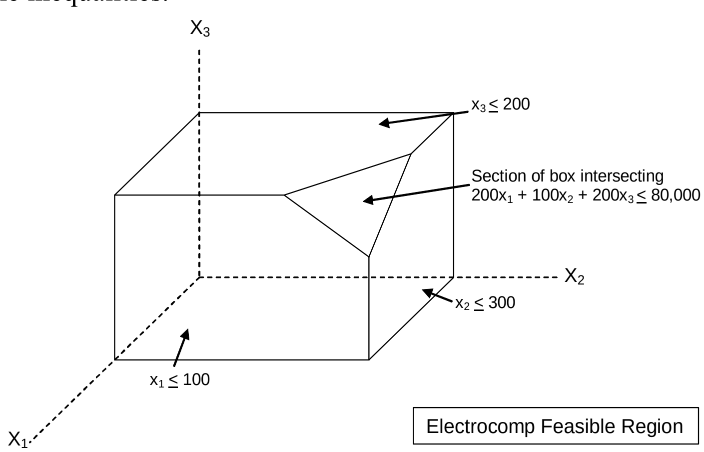
4. Three-Dimensional Model

Assume that the manager at Electrocomp Corporation has upgraded the technology for the company so that production is much more efficient, and time is no longer a constraint for the company. With this new level of efficiency, he is contemplating whether it would be profitable to introduce a new product line, electric space heaters. In the short-run, the only constraints the company faces are square-footage of the production facility and consumer demand. Air conditioners contribute \$25 of profit per unit, large fans \$15, and space heaters \$20. The total square footage of the company is 80,000 square feet. Each unit of air conditioners' production requires 200 square feet, each unit of large fans requires 100 square feet, and each unit of space heaters requires 200 square feet. The total anticipated consumer demand for air conditioners is 100 units, large fans 300 units, and space heaters 200 units.

The number of air conditioners to be produced shall be denoted as x_1, the number of large fans as x_2, and the number of electric space heaters as x_3. The constraint relationships appear below:

$$\begin{aligned} 200x_1 + 100x_2 + 200x_3 &\leq 80,000 \\ x_1 &\leq 100 \\ x_2 &\leq 300 \\ x_3 &\leq 200 \\ x_1 &\geq 0, x_2 \geq 0, x_3 \geq 0 \end{aligned}$$

To visualize these relationships, refer to the graphical interpretation on the following page. This object represents only the convex region that satisfies all of the inequalities.



After entering the slack variables into the inequalities, the profit and constraint equations appear as follows:

$$\begin{aligned}
1P - 25x_1 - 15x_2 - 20x_3 &= 0 \\
200x_1 + 100x_2 + 200x_3 + 1s_1 &= 80,000 \\
1x_1 + 1s_2 &= 100 \\
1x_2 + 1s_3 &= 300 \\
1x_3 + 1s_4 &= 200 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \quad & \quad s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, s_4 \geq 0
\end{aligned}$$

These constraints can then be entered into the first table of the mathematical model:

BV	P	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
P	1	-25	-15	-20	0	0	0	0	0
s_1	0	200	100	200	1	0	0	0	$80,000$
s_2	0	1	0	0	0	1	0	0	100
s_3	0	0	1	0	0	0	1	0	300
s_4	0	0	0	1	0	0	0	1	200

Since there is a negative value in the objective row, another table must be calculated. Here, x_1 will be the entering variable. If x_2 and x_3 are equal to zero, x_1 can be maximized by solving the following system:

$$\begin{aligned}
200x_1 + 1s_1 &= 80,000 \rightarrow x_1 \leq 400 \\
1x_1 + 1s_2 &= 100 \rightarrow x_1 \leq 100 \\
1s_3 &= 300 \rightarrow \text{no constraint on } x_1 \\
1s_4 &= 200 \rightarrow \text{no constraint on } x_1
\end{aligned}$$

The greatest that the solution set can move in the x_1 direction is 100. If x_1 becomes 100, s_2 will be forced to zero and becomes the leaving variable; the pivot element is 1. Following the same matrix calculations as used in the two-dimensional problem, the next table will be calculated so that the pivot element is the only non-zero value in its column. Since it is already equal to 1, the pivot row will stay the same. The second table with a solution of $(100, 0, 0)$ appears below:

BV	P	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
P	1	0	-15	-20	0	25	0	0	$2,500$
s_1	0	0	100	200	1	-200	0	0	$60,000$
x_1	0	1	0	0	0	1	0	0	100
s_3	0	0	1	0	0	0	1	0	300
s_4	0	0	0	1	0	0	0	1	200

The greatest negative value in the objective row is x_3, which becomes the pivot column. If x_2 and s_2 are equal to zero, s_4 becomes the pivot row, and the pivot element is 1:

$$\begin{aligned} 200x_3 + 1s_1 &= 60,000 \rightarrow x_3 \leq 300 \\ 1x_3 + 1s_4 &= 200 \rightarrow x_3 \leq 200 \end{aligned}$$

With s_4 as the leaving variable, the solution set to the third table below is at $(100, 0, 200)$, created by a movement along the edge of the constraint $x_1 \leq 100$ up to the intersection with constraint $x_3 \leq 200$.

BV	P	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
P	1	0	-15	0	0	25	0	20	$6,500$
s_1	0	0	100	0	1	-200	0	-200	$20,000$
x_1	0	1	0	0	0	1	0	0	100
s_3	0	0	1	0	0	0	1	0	300
x_3	0	0	0	1	0	0	0	1	200

The greatest negative value in the objective row, and thus the pivot column, is x_2. If s_2 and s_4 are equal to zero, s_1 will become the pivot row, making 100 the pivot element:

$$\begin{aligned} 100x_2 + 1s_1 = 20,000 &\quad \rightarrow \quad x_2 \leq 200 \\ 1x_2 + 1s_3 = 300 &\quad \rightarrow \quad x_2 \leq 300. \end{aligned}$$

If x_2 is maximized at 200, then s_1 will be forced to zero. The new solution set for the fourth table below is $(100, 200, 200)$, which is produced by a movement along the graphical intersection of constraints $x_1 \leq 100$ and $x_3 \leq 200$. The movement stops at the intersection of the third constraint $200x_1 + 100x_2 + 200x_3 \leq 80,000$.

BV	P	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
P	1	0	0	0	$\frac{15}{100}$	-5	0	-10	$9,500$
x_2	0	0	1	0	$\frac{1}{100}$	-2	0	-2	200
x_1	0	1	0	0	0	1	0	0	100
s_3	0	0	0	0	$-\frac{1}{100}$	2	1	2	100
x_3	0	0	0	1	0	0	0	1	200

The greatest negative value in the objective row is now s_4. Assuming that s_2 and s_1 are equal to zero, s_3 will become the pivot row, making 2 the pivot element:

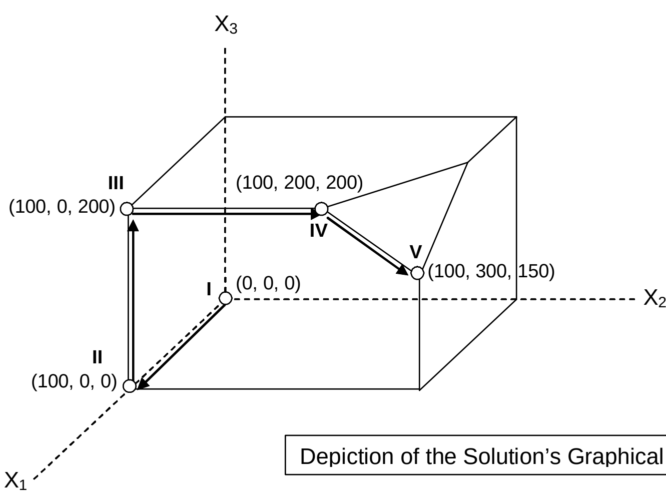
$$\begin{aligned} 1s_3 + 2s_4 &= 100 \rightarrow s_4 \leq 50 \\ 1x_3 + 1s_4 &= 200 \rightarrow s_4 \leq 200 \end{aligned}$$

With s_3 now equal to zero, the solution set moves along the edges of the constraints $200x_1 + 100x_2 + 200x_3 \leq 80,000$ and $x_1 \leq 100$ until it reaches the intersection with constraint $x_2 \leq 300$ at $(100, 300, 150)$. The fifth table below displays this solution:

BV	P	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
P	1	0	0	0	$\frac{10}{100}$	5	5	0	$10,000$
x_2	0	0	1	0	0	0	1	0	300
x_1	0	1	0	0	0	1	0	0	100
s_4	0	0	0	0	$-\frac{1}{200}$	1	$\frac{1}{2}$	1	50
x_3	0	0	0	1	$\frac{1}{200}$	-1	$-\frac{1}{2}$	0	150

There are no longer any negative values in the objective row, so a profit of \$10,000 with 300 units of large fans, 100 units of air conditioners, and 150 units of space heaters is the optimum solution. There are 50 slack units in the demand for space heaters.

To review the movements detailed above, with respect to the graphical depiction of this problem, refer to the following diagram.



Depiction of the Solution's Graphical Movements

The Roman numerals refer to the table number, and the arrows depict the direction of movement to the next solution set. Analyzing the first table with solution $(0, 0, 0)$ showed that it would be profitable to increase the x_1 variable. The solution for the second table $(100, 0, 0)$ was a movement in the x_1 direction stopping at the corner point of the constraint $x_1 \leq 100$. In the third table, the solution $(100, 200, 0)$ was found by moving the solution in the direction of x_3, until the constraint $x_3 \leq 200$. The fourth table's solution stopped at the intersection of three constraints at the point $(100, 200, 150)$. The optimum solution was then found in the fifth table when it moved along the intersection of constraints $x_1 \leq 100$ and $200x_1 + 100x_2 + 200x_3 \leq 80,000$, before stopping at the corner point on the feasible region $(100, 300, 150)$.

5. Conclusion

Upon researching the business model of the simplex tableau, as identified in the Analytical Methods course at Drury University, I found that the apparently chaotic calculations were based on a similar table used in most mathematical textbooks. These tables use matrix algebra calculations to shadow movements along the edges of the convex region created in the graphical solution. As a new variable enters the solution mix, it must be maximized by forcing another variable to zero. This puts the solution along the edge of the inequality, and the movement of a particular solution stops once it reaches a corner point (the next solution). Upon analyzing the table to see if this is the optimum solution, this process may be repeated until it is no longer profitable to change the solution mix, and thus profit is maximized and the problem is solved. The advantage to using the simplex tableau or mathematical model, when compared to analyzing the graphical solution, is that it can simultaneously consider up to thousands and millions of constraints. To do this by hand would of course be ludicrous, but that is why the simplex tableau has been converted into linear programming.

Acknowledgements: Special thanks to Dr. Charles Allen for help in exploring the topic of this paper.

References

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[2] A. Mizrahi and M. Sullivan., *Mathematics for Business, Life Sciences, and Social Sciences*, 5th ed., John Wiley & Sons, New York, 1993.

When Good Data Goes Bad

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Information is all around us, and digital information has become especially pervasive. We depend on an endless flow of data far more than we realize. Computers are the tools we use to operate our modern society, and they assign numbers to just about everything. And if our lives are so inextricably linked to those numbers, then their accurate transmission is vital.

Unfortunately, disaster is constantly smiting our precious data. Clumsy fingers can ruin codes being typed by hand. Radio interference can wreck transmissions. Sunspots can topple whole power grids. Unblockable gamma radiation can zip through and fry anything in its path. It's a dangerous world for information.

Of course, errors don't occur all that often. But when they do, they can be showstopping. For instance, when downloading a program from the internet, it is imperative that every single bit is perfectly transmitted; otherwise, the program may not run. So how does our modern world keep on turning, sunspots and gamma radiation notwithstanding?

1. Error Detection

The simplest way to avert digital catastrophe is to merely detect errors when they happen, then prompt for retransmission. Perhaps the most widely-known error-detection scheme is the Universal Product Code (UPC) barcode system. Barcodes are notoriously stubborn, and often refuse scan. This happens when the computer reads in a barcode and finds that the check digit (the right-most digit) doesn't equal what it is supposed to. The check digit is obtained by performing a tricky calculation on the barcode data. Were it not for this failsafe, we would constantly be charged for the wrong items.

8 21935 11112 4

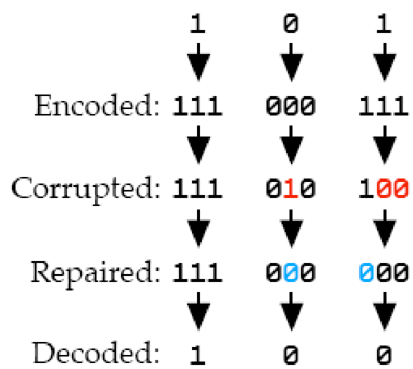
Another simple error-detection scheme is known as parity checking. Computers store all characters as binary numbers consisting only of zeroes and ones. The English language, complete with a respectable set of punctuation, requires about 80 characters, which can be represented using 7 binary digits, which are called bits. A parity check scheme sets an eighth bit so that the number of 1s in the character is even. For instance, the letter a is represented as $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. This character has three 1s in it, so the parity bit is set to 1, making a total of 4 ones.

$$a = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Of course, it would be nice if rather than just throwing up its hands in disgust, the computer could actually *repair* any errors that occur. For doing that, however, we need something a little juicier than a single parity bit.

2. Error-Correction

A very basic example is one in which every bit is repeated three times. Upon receiving the data, we simply select the word which is most likely to have been the original. Thus if the codeword 000 is corrupted into 010, the computer will assume the original was most likely 000. Unfortunately, this code system breaks down if two errors have occurred, and fixes the corrupted word incorrectly. The biggest problem with this code, however, is that it is simply too big: it expands to a whopping three times the size of the original data. However, it demonstrates the basic idea of adding redundant checks onto the data to be encoded.



Triple-Repetition Code

The error-correcting codes studied here are known as *linear codes*, which operate with linear algebra. A code is linear if it is a subspace of n-tuples in the field on which it is defined. Since coding theory is usually applied with a computer, that field is almost always the binary numbers. Linear codes create check bits that are combinations of data bits; those combinations are represented in a generating matrix. There must also exist a check matrix, which verifies that these equations still hold on a received word. Thus, to test whether the word was transmitted accurately, one multiplies by the check matrix.

We also have three important terms we used to describe words:

1. *weight*: the number of nonzero components in **u**, denoted $w(\mathbf{u})$.

   Example: The weight of the word $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$ is 2. Note that the the weight is the dot product of the word vector with itself.

2. *Hamming Distance*: the number of coordinates in which two binary words **u** and **v** differ, denoted $d(\mathbf{u}, \mathbf{v})$.

Example:

$$\begin{cases} u = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \\ v = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \end{cases};$$

$d(\mathbf{u}, \mathbf{v}) = 1$, since **u** and **v** differ only in position 2.

3. *sum*: the vector sum modulo 2, denoted $\mathbf{u} + \mathbf{v}$.

   Example:

$$\begin{cases} \mathbf{u} = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \\ \mathbf{v} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \\ \mathbf{u} + \mathbf{v} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \end{cases}$$

The sum is equivalent to an exclusive-or (XOR) operation in boolean logic.

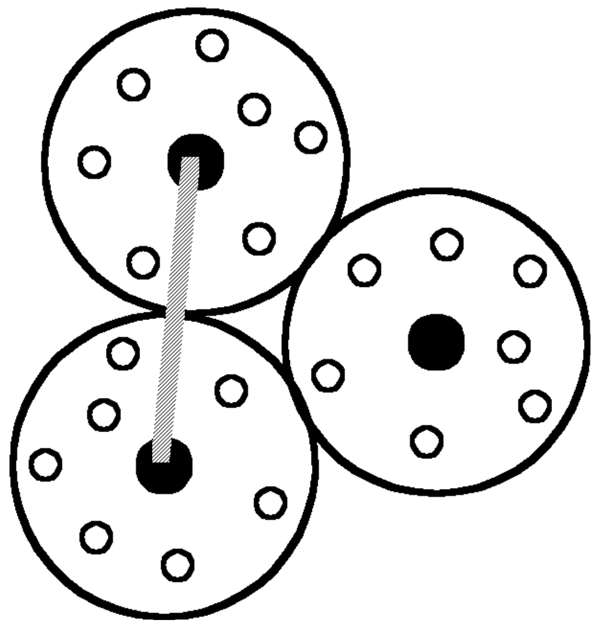
It turns out that the Hamming Distance really is a good distance function, satisfying all of the necessary properties to be a metric of binary numbers: [2, p. 15]

1. $d(\mathbf{u}, \mathbf{v})$ is always a nonnegative real number.
2. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
3. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$
4. $d(\mathbf{u}, \mathbf{w}) \leq d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{u})$

In practical terms, we use the Hamming Distance to measure how far codewords are from each other. We want each codeword to be a *minimum distance* away from each other codeword. As long as our codewords (indicated in the following graphic with the large black dots) are far enough apart, smaller errors (represented by the open circles) aren't too far away from the codeword. Corrupted words are corrected by associating them with the closest legitimate codeword. If each of these circles is disjoint, then each corruption (white circle) will always be associated only with the codeword (black dot) in the same large circle. Thus, the radius of each circle must be less than half the minimum distance. If d is the minimum distance, then a code can correct a maximum of

$$\left\lfloor \frac{d-1}{2} \right\rfloor$$

errors.



Minimum Distance: Each codeword is a certain distance from every other codeword.

With these concepts in place, we are now ready to discuss two actual codes.

3. Hamming Codes

The most famous and perhaps the most elegant error-correcting code is the Hamming Code. Its namesake, coding-pioneer Richard Hamming, developed these codes in 1950. Words are encoded and checked using matrix multiplication.

The Hamming (7,4) code encodes 4-bit words as 7-bit codewords. The generator matrix is given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The sum of any two rows of H has a weight of at least 3, so the minimum distance between any two codewords is 3. Thus, the Hamming code can correct $\left\lfloor \frac{3-1}{2} \right\rfloor = 1$ error.

The generator matrix H appends the additional check bits onto the original data:

$$\begin{pmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 & \mathrm{a}_4 \end{pmatrix} H =$$

$$\begin{pmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \mathrm{a}_3 & \mathrm{a}_4 & (a_2{+}a_3{+}a_4) & (a_1{+}a_3{+}a_4) & (a_1{+}a_2{+}a_4) \end{pmatrix}$$

When the word is transmitted, we expect the last three bits in the received word should still satisfy the equations by which they were defined. So, if the received word is denoted $\begin{pmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \end{pmatrix}$, then we expect that

$$\begin{aligned} b_2 + b_3 + b_4 + b_5 \bmod 2 &= 0 \\ b1 + b_3 + b_4 + b_6 \bmod 2 &= 0 \\ b_1 + b_2 + b_4 + b_7 \bmod 2 &= 0 \end{aligned}$$

To check any Hamming word, we need only convert these equations into a check matrix.

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Received words multiplied by this check matrix produce what is called a *syndrome*, which diagnoses the kind of error that has occurred: a zero syndrome means that no errors were detected; a nonzero syndrome indicates that data has been corrupted. In this simple Hamming Code, all we need to do is match up our syndrome with a row in the check matrix.

For example, consider the following received words and syndromes, in which only the first bit differs between codewords:

$$\begin{aligned} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} P &= \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} P &= \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \end{aligned}$$

The first syndrome indicates that no errors have occurred. The second syndrome, however, is nonzero. It matches the first row in the check matrix, indicating that the error occurred in the first position.

4. Implementing the Hamming Code

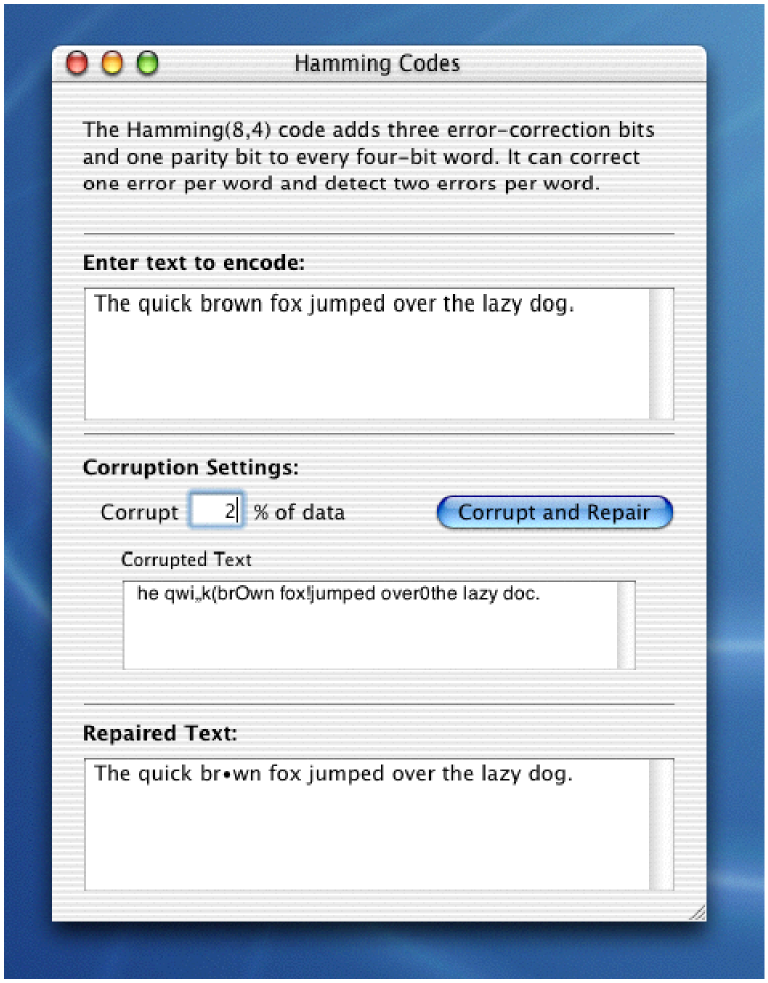
Several simple modifications make automated decoding much easier. First, note that each row of the check matrix is the binary representation of a number between 1 and 7. If the matrix P is sorted in numeric order, the syndrome automatically reads off the binary equivalent of the position where the error occurred. (A corresponding rearrangement must also be performed to the generator matrix H.) The revised matrices now read as follows:

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

To implement these codes on the computer, I set each bit according to the equation specified in the matrix. The resulting program accepts a user-specified string of text, encodes it with the Hamming check bits, and corrupts a specified percentage of the bits in the string. The program then decodes the string and displays the output.

The program also is extended to use a parity checking scheme to detect double errors. When the parity bit is wrong, it is assumed that one error has occurred. However, when the parity bit is correct, it is possible either that no errors have occurred, or that the word has two corruptions. Thus, whenever a correct parity bit is received, the syndrome is calculated anyway. A zero syndrome confirms that the word is intact; a nonzero syndrome indicates that an uncorrectable double error has occurred. The parity information was especially easy to incorporate, since computers use eight bits to represent each character, and Hamming codewords are only seven bits long.

Ironically, the biggest problem implementing this code came not from the error correction itself, but from the data representation. In the C programming language (the most widespread language in use today), a series of eight zeroes is set aside as the string termination signal. Unfortunately, the Hamming Code with an even parity check bit often generates the string termination signal. I solved the problem by switching from an even to an odd parity scheme. Thus, there is always at least one 1 in each byte. Unfortunately, there is no guarantee that random errors wont change that 1 into a 0. I simply rely on such an error being rare.



The Hamming Code, with parity checking. Unrecoverable errors are indicated with a ·.

The computer program lends itself to identifying trends. Even at 1% corruption, the Hamming Code occasionally fails. At higher corruption rates, it becomes readily apparent that the Hamming code loses too much information. While in theory, Hamming codes can correct data corrupted up to 12 1/2% (1 error per 8 bits), the Hamming code comes nowhere close to that number in actual practice. The problem lies in the tendency of errors to bunch up together. Hamming Codes can only correct errors that are nicely-spaced in 8-bit partitions.

5. Golay Codes

In the search for bigger and better codes, we naturally arrive at the Golay Code. Closely related to the Hamming Code, it can correct a stunning three errors in every 24 bits. The Golay Code was sufficiently powerful for NASA to utilize it to protect images sent from the Voyager spaceprobes.

The Golay code G_{24} encodes 12-bit words as 24-bit codewords. It can be generated by the matrix $G_1 = [I_{12}\,|A]$, where

$$A = \begin{bmatrix} 0&1&1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&0&1&1&1&0&0&0&1&0 \\ 1&1&0&1&1&1&0&0&0&1&0&1 \\ 1&0&1&1&1&0&0&0&1&0&1&1 \\ 1&1&1&1&0&0&0&1&0&1&1&0 \\ 1&1&1&0&0&0&1&0&1&1&0&1 \\ 1&1&0&0&0&1&0&1&1&0&1&1 \\ 1&0&0&0&1&0&1&1&0&1&1&1 \\ 1&0&0&1&0&1&1&0&1&1&1&0 \\ 1&0&1&0&1&1&0&1&1&1&0&0 \\ 1&1&0&1&1&0&1&1&1&0&0&0 \\ 1&0&1&1&0&1&1&1&0&0&0&1 \end{bmatrix}.$$

This generator matrix can be created by piecing together two Hamming codes, where the second is obtained by reversing the order of bits in the first. [2, pp. 72-74] However, when Marcel Golay introduced his code to the world in a one-page paper in 1949, he neglected to give the slightest mention of how he obtained his generator matrices. Here we shall be content to enjoy the codes marvelous properties apart from its development.

The first of these important properties is that G_{24} is *self-dual*; that is, every codeword is orthogonal to another codeword. Because of this property, not only is $G_1 = [I_{12}\,|A]$ a generator matrix, but so is $G_2 = [A\,|I_{12}]$.

Additionally, the Golay code G_{24} has a minimum weight and minimum distance of 8. Thus G_{24} can correct $\left\lfloor \dfrac{8-1}{2} \right\rfloor = 3$ or fewer errors per 24-bit codeword. At first glance, this doesn't seem like much of an improvement over the Hamming (7,4) code, which can mend 1 error per 8 bits. The important difference is that the Golay code can correct those three errors anywhere inside a long codeword. Thus, Golay codewords are less likely to become irreparably corrupted.

Decoding the Golay code is challenging, and a number of different methods have been proposed. Unfortunately, because the Golay code can correct more than one error, the simple method of comparing the syndrome to rows in the check matrix (as seen in the Hamming code) is not available. A brute-force approach of comparing syndromes would require the construction of a onerous table having $\frac{2^{24}}{2^{12}} = 2^{12} = 4096$ different syndromes, which is not especially practical. Thankfully, a means of decoding the Golay code based on its structure does exist. [3, pp. 226-227]

Recall first that since G_{24} is self-dual, $G_1 = [I_{12}\,|A]$ and $G_2 = [A\,|I_{12}]$ are both parity and check matrices. We shall suppose that 3 or fewer errors have occurred in some received word $\mathbf{x}$, and call $\mathbf{e}$ the 24-bit error vector. If we can find that error vector, we can easily obtain the original codeword $\mathbf{c}$, since $\mathbf{c} + \mathbf{e} = \mathbf{x}$. For example, let

$$\begin{aligned} \mathbf{x} &= \mathbf{c}+\mathbf{e} = [\begin{matrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{matrix} \\ &\qquad \begin{matrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{matrix}] \end{aligned}$$

and

$$\begin{aligned} \mathbf{e} &= [\begin{matrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \\ &\qquad \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}]. \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{c} &= [\begin{matrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{matrix} \\ &\qquad \begin{matrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{matrix}]. \end{aligned}$$

Also, we shall say $\mathbf{e} = [\mathbf{f}\,|\mathbf{g}]$, where $\mathbf{f}$ and $\mathbf{g}$ have length 12. Now we may compute the syndromes using both parity check matrices as follows:

$$\begin{aligned} S_1 &= \mathbf{e}G_1^T = [\mathbf{f}\,|\mathbf{g}]\left[\frac{I_{12}}{A}\right] = \mathbf{f} + \mathbf{g}A \\ S_2 &= \mathbf{e}G_2^T = [\mathbf{f}\,|\mathbf{g}]\left[\frac{A}{I_{12}}\right] = \mathbf{f}A + \mathbf{g} \end{aligned}$$

Also, we note the important fact that the syndrome of a received word is identical to the syndrome of its corresponding error word. [2, p. 57]

Proof: Let c be a transmitted codeword and $x = c + e$ be received. Let H be the check matrix. Then

$$Hx^T = Hc^T + He^T = 0 + He^T.$$

So $Hx^T = He^T$. QED.

Combining this fact with what we observed about the syndrome of $\mathbf{e}$, we see that

$$\begin{aligned} xG_1^T &= S_1 = \mathbf{f} + \mathbf{g}A \\ xG_2^T &= S_2 = \mathbf{f}A + \mathbf{g} \end{aligned}$$

Although $\mathbf{f}$ and $\mathbf{g}$ are currently unknown, we can certainly compute S_1 and S_2, because we do know what the received word is. Once we compute the syndromes, we can derive their relationship to **f** and **g**. Now let us consider the different cases that may arise when three or fewer errors have corrupted a received word; that is, when $0 < w(\mathbf{e}) \leq 3$.

1. If $w(\mathbf{f}) = 0$ and $w(\mathbf{g}) \leq 3$, then $S_2 = \mathbf{0}A + \mathbf{g} = \mathbf{g}$ and so
$$\mathbf{e} = [\mathbf{f}\,|\mathbf{g}] = [\mathbf{0}\,|\mathbf{g}] = [\mathbf{0}\,|S_2]\,.$$
Also,
$$w(S_2) = w(\mathbf{g}) \leq 3 \text{ and } w(S_1) = w(\mathbf{g}A) \geq 5.$$
The second inequality requires closer scrutiny. Note that
$$\mathbf{x}G_1^T = S_1 = 0 + \mathbf{g}A.$$
Also,
$$\mathbf{x}G_1^T = \mathbf{e}G_1^T = [\mathbf{0}\,|\mathbf{g}]\begin{bmatrix} I_{12} \\ \hline A \end{bmatrix} = \mathbf{g}A.$$
Since $w(\mathbf{g}) \leq 3$, the expression $[\mathbf{0}\,|\mathbf{g}]\begin{bmatrix} I_{12} \\ \hline A \end{bmatrix}$ is selecting 3 or fewer rows out of $[I_{12}\,|A]$. Since G_{24} has a minimum distance of 8, the sum of these rows must have a weight of at least 8. However, the first half of that sum consists only of entries from the identity matrix, so the first half of the sum can have a weight of at most 3. But since the sum must have weight 8, we know that the second half of the sum (contributed by A) must have a weight of 5 or greater. Thus, $\mathbf{g}A$ must also have a weight of 5 or greater. This is an important tool in decoding the Golay Code. If we encounter syndrome weights greater than 5, we know that the error vector cannot be **0** on that half.

2. If $w(\mathbf{f}) \leq 3$ and $w(\mathbf{g}) = 0$, then $S_1 = \mathbf{f}$ and so
$$\mathbf{e} = [\mathbf{f}\,|\mathbf{g}] = [\mathbf{f}\,|\mathbf{0}] = [S_1\,|\mathbf{0}]\,.$$
Also,
$$w(S_1) = w(\mathbf{f}) \leq 3 \text{ and } w(S_2) = w(\mathbf{f}A) \geq 5.$$
This inequality follows from reasoning similar to that in Case 1.

3. If $w(\mathbf{f}) \leq 1$ and $w(\mathbf{g}) \geq 1$, then $w(S_1) \geq 5$ and $w(S_2) \geq 5$. So, as long as neither S_1 nor S_1 has a weight greater than 3, it is very easy to determine **e**. The error simply corresponds to the syndrome appended with **0**. However, all is not lost if an error has occurred in both halves of the received word. We must instead perform a more complicated computation to retrieve it. There are two cases to consider about the form of this new error vector:
   a. $w(\mathbf{f}) = 1$ and $w(\mathbf{g}) = 1$ or 2. To compute the error, we make use of the vector $\mathbf{e}_i$, which has length 12, with a 1 in the ith position and

zeroes elsewhere. For example,

$$\mathbf{e}_2 = \left[\begin{array}{cccccccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Now under this case, $\mathbf{f} = \mathbf{e}_i$, for some i, since $w(\mathbf{f}) = 1$. Thus, $\mathbf{e} = \mathbf{e}_i\mathbf{g}$. We now define $\mathbf{y}_u$ such that

$$\mathbf{y}_u = (\mathbf{x} + \mathbf{e}_u\mathbf{0})\, G_2^T = (\mathbf{e}_i\mathbf{g} + \mathbf{e}_u\mathbf{0})\, G_2^T = \mathbf{e}_i A + \mathbf{g} + \mathbf{e}_u A.$$

We want $\mathbf{y}_u$ to give us the value of $\mathbf{g}$. In order for that to happen, $\mathbf{e}_i A$ and $\mathbf{e}_u A$ must annihilate each other. Since we are working in a modulo 2 subspace, $\mathbf{e}_i A$ and $\mathbf{e}_u A$ will cancel each other only when $i = u$. Thus, under case (a), we can calculate $\mathbf{e}$ by computing y_u for $u = 1, 2, ..., 12$ and looking for $w(\mathbf{y}_u) \leq 2$.

b. $w(\mathbf{f}) = 2$ and $w(\mathbf{g}) = 1$ Case (b) can be solved by making a similar computation as in case (a), only with G_1 instead of G_2. Since $w(\mathbf{g}) = 1$, $g = \mathbf{e}_i$. This time, we define $\mathbf{z}_u$ such that

$$\begin{aligned} \mathbf{z}_u &= (\mathbf{x} + \mathbf{0}\mathbf{e}_u)\, G_1^T = (\mathbf{e} + \mathbf{e}_u\mathbf{0})G_1^T = (\mathbf{f}\mathbf{e}_j + \mathbf{0}\mathbf{e}_u)G_1^T \\ &= \mathbf{f} + \mathbf{e}_j A + \mathbf{e}_u A. \end{aligned}$$

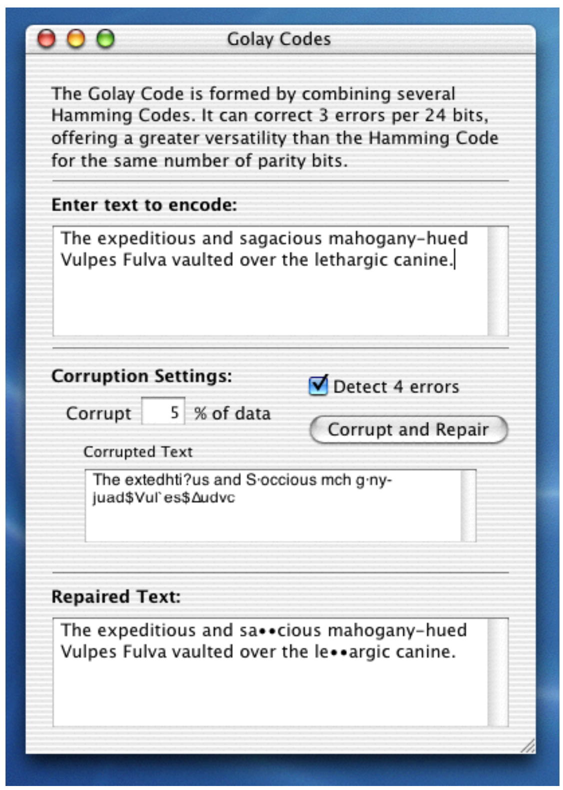
We want $\mathbf{z}_u$ to give us the value of $\mathbf{f}$, so $\mathbf{e}_j A$ and $\mathbf{e}_u A$ should erase each other. As before, $\mathbf{e}_j A$ and $\mathbf{e}_u A$ will cancel out only when $j = u$. Thus, under case (b), we can calculate $\mathbf{e}$ by computing $\mathbf{z}_u$ for $u = 1, 2, ..., 12$ and looking for $w(\mathbf{z}_u) \leq 2$. In conclusion, corrupted Golay codewords can be repaired by calculating at most 26 different syndromes. Of course, that is not exactly easy to do by hand. However, the task is excellently suited to the computer.

6. Implementing the Golay Code

Once one has figured out how to decode the Golay code, implementing it is not exceedingly difficult. Much of the logic transfers directly from the Hamming program already written. The biggest difference is that while the Hamming code worked very nicely in bytes, the Golay code works in chunks of 1 1/2-byte words and three-byte codewords. This did not adapt quite as naturally to predefined data types (like characters), but was not a significant obstacle.

Actually, the Golay program was bedeviled by the same problem as the Hamming program: the appearance of the eight-zero string termination symbol in the process of regular encoding. Rather than relying on an odd parity scheme, the Golay program solves the problem by saving the original string length and working with that throughout the program. This raises the legitimate question of how one is to transmit that string length; after all, the entire decoding process is dependent on that information. Should such a system be used for a critical application, one would assume the string length would be transmitted with additional error-checking, or that a much more complicated termination symbol would be used!

Additionally, my Golay program is extended to detect (but of course not correct) four errors. As in the Hamming program, this is accomplished using a parity-checking scheme. However, the parity information is already built in to the Golay code: every Golay codeword has an even weight. In fact, by removing any bit from the Golay-24 code, we retain the entire code structure. My aim, however, was to retain and use this extra information.



The Golay Code, with quadruple-error detection enabled

Because of its impressive triple-error correction, the Golay code is much more reliable than the Hamming code. This is especially evident when running both programs side-by-side. The Golay code repairs almost every error when using less than 5% corruption, while the Hamming code is already becoming useless at that point.

7. Conclusion

As it turns out, both these codes, while important, are fairly obsolete. However, error correction definitely influences everyone's life. For instance, CDs are written with codes that protect them against minor flaws. Another striking example is the Maxicode barcode used by UPS on its packages. Thanks to the error-correction abilities in this code, up to 25% of the barcode can be destroyed, and the code will still scan perfectly - far more corruption than either the Hamming or Golay code can tolerate! However, the more complicated codes used today owe their existence to the pioneering concepts of these linear codes. Learning these basics has been an enjoyable and fruitful experience.

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The Problem Corner

Edited by Catherine Kong and Kenneth M. Wilke

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before January 1, 2006. Solutions received after the publication deadline will be considered also until the time when copy is prepared for publication. The solutions will be published in the Spring, 2006 issue of *The Pentagon*, with credit being given to the student solutions. Affirmation of student status and school should be included with solutions. Solutions to problems in this issue should be addressed to Kenneth M. Wilke, Department of Mathematics, 275 Morgan Hall, Washburn University, Topeka, Kansas 66621 (e-mail: ken.wilke@washburn.edu). All new problem proposals and all solutions for problems appearing in columns after this one should be addressed to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859)622-3051)

PROBLEMS 585-589

Problem 575. *Proposed by Pat Costello, Eastern Kentucky University, Richmond, Kentucky. (Corrected)*

Given a positive integer m, take the sum of its digits to obtain a different number, take the sum of the digits of this new number to obtain yet another number, and so on, until the remaining number has only one digit. we call the one digit number the digital root of m. Suppose that we have a recursive sequence defined by $x_1 = 3$, $x_2 = 6$, $x_3 = 9$, and $x_n = x_{n-1} + x_{n-2} + x_{n-3}$ for all integers $n \geq 4$. Show that the digital root of x_n is always 3, 6, or 9.

Problem 579. *Proposed by M. Khoshnevisan, Griffith University, Gold Coast, Queensland, Australia. (Corrected)*

A Generalized Smarandache Palindrome (GSP) is a concatenated number of the form $a_1a_2\cdots a_na_n\cdots a_2a_1$ or $a_1a_2\cdots a_{n-1}a_na_{n-1}\cdots a_2a_1$, where $a_1, a_2, ..., a_n$ are positive integers of various numbers of digits. Find the number GSP of four digits which are not palindromic numbers.

Problem 585. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Suppose that the roots $z_1, z_2, ..., z_n$ of

$$z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1z^1 + a_0 = 0$$

are in arithmetic progression with difference d. Prove that

$$d^2 = \frac{12(n-1)a_{n-1}^2 - 2na_{n-2}}{n^2(n-2)}.$$

Problem 586. *Proposed by Pat Costello, Eastern Kentucky University, Richmond, Kentucky.*

Let f_n denote the n^{th} Fibonacci number: i.e.,

$$f_1 = 1, f_2 = 1,$$

and for all integers $n > 2$,

$$f_n = f_{n-1} + f_{n-2}.$$

Find the exact value of the infinite series

$$\sum_{n=1}^{\infty} \frac{f_n (\bmod 3)}{3^n}.$$

Problem 587. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Show that if A, B, C are the angles of a triangle, and a, b, c its sides, then

$$\prod_{\text{cyclic}} \sin^{1/3}(A-B) \leq \sum_{\text{cyclic}} \frac{(a^2+b^2)\sin(A-B)}{3ab}.$$

Problem 588. *Proposed by the editor.*

Prove that the first 2005 digits after the decimal point in the decimal expansion of

$$\left(7+\sqrt{48}\right)^{2005}$$

are nines.

Problem 589. *Proposed by the editor.*

Find a, b, c, d, and e so that the number $a8b2cd7e3$ is divisible by both 73 and 137, where a, b, c, d, and e are distinct integers chosen from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and $a \neq 0$

Please help your editor by submitting problem proposals.

SOLUTIONS 570 – 574

Problem 570. *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan.*

Let ABC be a triangle and H its orthocenter. Prove that

$$(AH + BH + CH)\sqrt{3} = AB + BC + CA$$

if and only if at least one of the angles of triangle ABC is 60°.

Solution *by the proposer.*

We shall prove both directions simultaneously. It is well known that in any triangle, the following relations hold:

$$AH = 2R\cos A, BH = 2R\cos B, CH = 2R\cos C$$

and

$$a = 2R\sin A, b = 2R\sin B, c = 2R\sin C,$$

where R denotes the radius of the circumscribed circle. The equality given above is equivalent to

$$\begin{aligned}
& \sqrt{3}\sum_{\text{cyclic}}\cos A = \sum_{\text{cyclic}}\sin A \\
\Longleftrightarrow\quad & \sum_{\text{cyclic}}\left(\sqrt{3}\cos A - \sin A\right) = 0 \\
\Longleftrightarrow\quad & \sum_{\text{cyclic}}\left(\frac{\sqrt{3}}{2}\cos A - \frac{1}{2}\sin A\right) = 0 \\
\Longleftrightarrow\quad & \sum_{\text{cyclic}}\sin\left(60^\circ - A\right) = 0 \\
\Longleftrightarrow\quad & \sin\left(60^\circ - A\right) + \sin\left(60^\circ - B\right) + \sin\left(60^\circ - C\right) = 0.
\end{aligned}$$

Using the standard angle-sum and double-angle formulae, we obtain

$$\begin{aligned}
& \sin\left(60^\circ - A\right) + \sin\left(60^\circ - B\right) + \sin\left(60^\circ - C\right) \\
= \quad & 2\sin\left(\frac{120^\circ - A - B}{2}\right)\cos\left(\frac{B-A}{2}\right) \\
& +2\sin\left(\frac{60^\circ - C}{2}\right)\cos\left(\frac{60^\circ - C}{2}\right) \\
= \quad & 0.
\end{aligned}$$

Since $\dfrac{120^\circ - A - B}{2} = \dfrac{C - 60^\circ}{2}$, we have

$$2\sin\left(\frac{C-60^\circ}{2}\right)\left[\cos\left(\frac{B-A}{2}\right) - \cos\left(\frac{60^\circ - C}{2}\right)\right] = 0.$$

But

$$\begin{aligned}
& \cos\left(\frac{B-A}{2}\right) - \cos\left(\frac{60^\circ - C}{2}\right) \\
= \quad & 2\sin\left(\frac{60^\circ - C - B + A}{4}\right)\sin\left(\frac{B - A + 60^\circ - C}{4}\right) \\
= \quad & 2\sin\left(\frac{A-60^\circ}{2}\right)\sin\left(\frac{B-60^\circ}{2}\right).
\end{aligned}$$

Thus we have

$$2\sin\left(\frac{A-60^\circ}{2}\right)\sin\left(\frac{B-60^\circ}{2}\right)\sin\left(\frac{C-60^\circ}{2}\right) = 0.$$

Then, since $A, B, C \in (0^\circ, 180^\circ)$, we must have one of $A, B, C = 60^\circ$. Note that if any one of the angles, say A, is 60°, then $\dfrac{B+C}{2}$ is also 60°, so that the equality $(AH + BH + CH)\sqrt{3} = AB + BC + CA$ is easy to prove. Observe that

$$\begin{aligned}
& \sqrt{3}\,(\cos A + \cos B + \cos C) \\
= \;& \sqrt{3}\left(\frac{1}{2} + \cos B + \cos C\right) \\
= \;& \frac{\sqrt{3}}{2} + \sqrt{3}\left[2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\right] \\
= \;& \frac{\sqrt{3}}{2} + \sqrt{3}\cos\left(\frac{B-C}{2}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{\text{cyclic}} \sin A &= \sin 60^\circ + \sin B + \sin C \\
&= \frac{\sqrt{3}}{2} + 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) \\
&= \frac{\sqrt{3}}{2} + \sqrt{3}\cos\left(\frac{B-C}{2}\right).
\end{aligned}$$

Problem 573. *Proposed by Pat Costello, Eastern Kentucky University, Richmond, Kentucky.*

A pair of numbers is called *amicable* if the sum of the proper divisors of the first number is equal to the second number and vice-versa. One way of constructing amicable pairs is to develop a "Thabit-Rule" which is just a set of values determined by a single input that need to be checked for primality. If all values turn out to be primes, they can be put together in some way to form an amicable pair.

Bill discovered the following "Thabit-Rule": if $p = 11 \cdot 2^n - 1$, $q = 2p - 1$, $r = 4p - 1$, and $s = (q+1)(r+1) - 1$ are all primes, then the numbers $2^{n+2}pqr$ and $2^{n+2}ps$ form an amicable pair. He was so excited because he quickly calculated that when $n = 2$, $p = 43$, which is prime. His excitement waned when he computed $q = 85$, which is not prime. He was very disappointed when it turned out that not all four of the values p, q, r, s are prime. Find an amicable pair using Bill's "Thabit-Rule", or show that none exists.

Solution *by Joe Flowers, Texas Lutheran University, Seguin, Texas.*)

Unfortunately, Bill's "Thabit Rule" never produces an amicable pair. We prove this by establishing the following.

1. If n is odd, p is not prime, because $p \equiv 0 (\bmod 3)$.
2. If $n = 4k$ for some integer k, then p is not prime because $p \equiv 0 (\bmod 5)$.
3. If $n = 4k + 2$ for some integer k, then p is not prime because $p \equiv 0 (\bmod 5)$.

Proof of 1: If $n = 2k + 1$, for some integer $k \geq 0$, then

$$\begin{aligned} p &= 11 \cdot 2^{2k+1} - 1 \\ &\equiv 2 \cdot 2^{2k+1} - 1 \\ &\equiv 2^{2k+2} \\ &\equiv 4^{k+1} - 1 \\ &\equiv 0 (\bmod 3). \end{aligned}$$

Proof of 2: If $n = 4k$ for some integer $k \geq 0$, then

$$\begin{aligned} p &= 11 \cdot 4^{k} - 1 \\ &\equiv 16^{k} - 1 \\ &\equiv 0 (\bmod 5). \end{aligned}$$

Proof of 3: If $n = 4k + 2$ for some integer $k > 0$, then

$$\begin{aligned} p &= 11 \cdot 2^{4k+2} - 1 \\ &\equiv 44 \cdot 16^{k} - 1 \\ &\equiv -2 \\ &\equiv 3 (\bmod 5), \end{aligned}$$

so that

$$\begin{aligned} q &= 2p - 1 \\ &\equiv 5 \\ &\equiv 0 (\bmod 5). \end{aligned}$$

Note that when $k = 2$ we get Bill's original values for p, q, and r. This completes the proof.

Also solved by the proposer.

Problem 576. *Proposed by Albert White, St. Bonaventure University, St. Bonaventure, New York.*

Find the maximum of the function $y = f(x)$ on the interval $[e, \pi]$, given that

$$\ln f(x) = \frac{xf'(x)}{f(x)} \text{ and } f(\pi) = e.$$

Solution *by Tim Wertz, Alma College, Alma, Michigan.*

Let $y = f(x)$ be a function on the interval $[e, \pi]$ such that $\ln f(x) = \dfrac{xf'(x)}{f(x)}$ and $f(\pi) = e$. Note that this is a variable-separable differential equation. Thus $\dfrac{1}{x}dx = \dfrac{1}{y \ln y}dy$. Integrating both sides, we obtain $\ln |x| = \ln |\ln y| + K$, for some constant K, or $y = e^{Cx}$, for some constant C. Since $f(\pi) = e$, solving for C yields $C = \dfrac{1}{\pi}$. Thus

$$f(x) = e^{x/\pi}.$$

Differentiating, we obtain

$$f'(x) = \frac{e^{x/\pi}}{\pi},$$

which is always positive on the interval $[0, \pi]$; thus f is a strictly increasing function. Hence the maximum value for $f(x)$ on the given interval occurs at π.

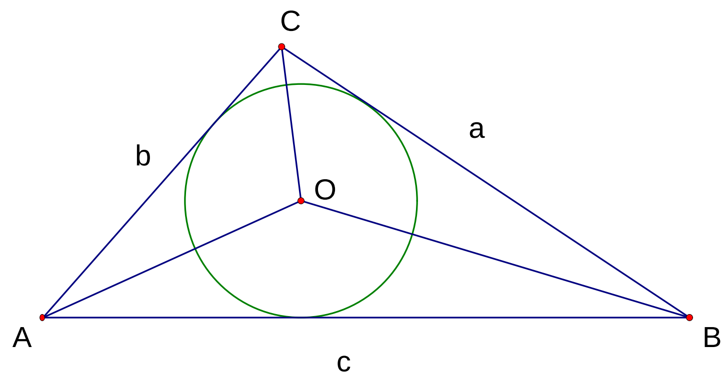
Also solved by Russell Euler and Jawad Sadek (jointly), Northwest Missouri State University, Maryville, Missouri, Ovidiu Furdui, student, Western Michigan University, Kalamazoo, Michigan, and the proposer.

Problem 577. *Proposed by Thomas Chu, Austin Texas*

In triangle ABC, let O denote the center of the inscribed circle. Prove that

$$OA^2 + OB^2 + OC^2 = \frac{a^2(b + c) + b^2(c + a) + c^2(a + b) - 3abc}{a + b + c},$$

where a, b, c denote the sides of triangle ABC, and OA, OB, OC denote the distances from the incenter O to the respective vertices of the triangle.



Solution *by Scott H. Brown, Auburn University, Montgomery, Alabama.*

In problem 574, we found that

$$\begin{cases} OA^2 = \dfrac{-abc + b^2c + bc^2}{a+b+c} \\ OB^2 = \dfrac{-abc + a^2c + ac^2}{a+b+c} \\ OC^2 = \dfrac{-abc + a^2b + ab^2}{a+b+c} \end{cases}.$$

Adding, we obtain

$$\begin{aligned} & OA^2 + OB^2 + OC^2 \\ = \quad & \frac{-abc + b^2c + bc^2 - abc + a^2c + ac^2 - abc + a^2b + ab^2}{a+b+c}. \end{aligned}$$

The right-hand side of this last equation can easily be simplified to complete the proof.

Also solved by the proposer. One incorrect solution was received.

Problem 578. *Proposed by Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let triangle ABC be a triangle such that $\sin A$, $\sin B$, $\sin C$ are in arithmetic progression. Prove that

$$\tan\frac{A}{2}\tan\frac{C}{2} = \frac{1}{3}.$$

Since no solutions have been received, this problem will remain open for another issue.

Kappa Mu Epsilon News

Send news of chapter activities and other noteworthy KME events to
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1200 Commercial Street
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or to
schrockc@emporia.edu

Installation Report

New York Omicron
St. Joseph's College, Patchogue

The installation of the New York Omicron chapter of Kappa Mu Epsilon was held on Saturday afternoon, 1 May 2004, at the Reverend Gennaro D'Ecclesiis Auditorium in O'Connor Hall on the Patchogue Campus of St. Joseph's College. Dr. Andrew M. Rockett, a former editor of The Pentagon, was the installing officer. The fifty-eight members installed during the ceremony were presented by Dr. Bogumila Lai, who is a member of New York Lambda and will be joining the faculty of St. Joseph's College this coming fall. The new members received their membership certificates and KME pins during the ceremony. Dr. Sharon Kunoff, who served as The Pentagon business manager during Dr. Rockett's tenure as editor, joined him in making a general address on the history of KME and included well-received reminiscences of various KME conventions since her 1951 initiation into New York Alpha.

Officers installed were: Eileen Bracciodieta, president; Meredith Miller, vice-president; Nicole Acosta, recording secretary; Kristin L. Cinquemani, treasurer; Professor Barbara A. Thorpe, corresponding secretary; and Dr. Donna Marie Pirich, faculty sponsor.

Sister Elizabeth Hill, President of St. Joseph's College, delivered the welcome address, and closing remarks were made by: Sister Loretta McGrann, Vice-President of Academic Affairs; Dr. Randall Krieg, Academic Dean of the School of Arts and Sciences; Dr. David Seppala-Holtzman, Chairman of the Department of Mathematics and Computer Science; and Eileen Bracciodieta, President of New York Omicron.

The afternoon concluded with a dinner reception held in the McGann Conference Center.

Chapter News

AL Zeta – Birmingham Southern College

Chapter President– Matt Woods., 5 Current Members, 5 New Members
Other fall 2004 officers: Wiley Truss , Vice–President; Meredith Kirkpatrick, Secretary; Meredith Kirkpatrick, Treasurer; Mary Jane Turner, Corresponding Secretary.

The formal initiation of new members was held Tuesday, December 14, 2004. Five new members were initiated. All mathematics faculty and current members were present at the initiation. Refreshments and a social time were held afterward. The names and addresses of the new members are being sent on the proper form.

CA Delta – California State Polytechnic University

Chapter President– Brandon Duffey.
Other fall 2004 officers: Brian Chen, Vice–President; Aily Xie, Secretary; Chris Fragoso,
Treasurer; Patricia Hale, Corresponding Secretary.

CA Epsilon – California Baptist University

Chapter President– Ryan Tucker. 19 Current Members
Other fall 2004 officers: Ryan Williams, Vice–President; Kao Saechao, Secretary; Kao Saechao, Treasurer; Catherine Kong, Corresponding Secretary.

On September 18, 2004, the California Epsilon chapter had a beach party kick–off. Then on Oct. 21, seventeen students participated in their math fun day at Jefferson Elementary School, Riverside, California. CBU students created their own math activities for more then 100 K–6 children according to themes provided by Brandon Davis, HEARTS (after school program) director. The chapter also held a tutoring–relief night on Dec. 2. Then the chapter ended the year with a Christmas/ end of semester party on Dec. 9.

CO Gamma – Fort Lewis College

New Initiates - Jamie Aweida, Amanda Brodbeck, Cameron Cooper, Nick Cummings, Leslie Goldstein, Justin Hanel, Jenna Jewell, Matthew Kraft, Greg Lazarev, Jeremiah Marsh, Jeremy May, Diane Neff, Terra Plank, Krystal Roskowinski, Jason Stafford, Shanna Vaughan.

CT Beta – Eastern Connecticut State University

New initiates – Robert K. Bowers, II, Keira M. Mazy, Shannielle L. Danner, Marlena L. Scialla, Olinda A. LaBeef, Chrystal A. Urbani, Laura Waz, Brian Whitehead, Amy L Bedner.

FL Beta – Florida Southern College

Chapter President– Louis Rufo. 22 Current Members

Other fall 2004 officers: Wade Williams, Vice–President; Lindsay Anne Stambaugh, Secretary; Lindsay Anne Stambaugh, Treasurer; Allen Wuertz, Corresponding Secretary.

GA Alpha – University of West Georgia

Chapter President– Heather Morse. 16 Current Members

Other fall 2004 officers: Sarah Blair, Vice–President; David Yarbrough, Secretary; Jill Copeland, Treasurer; Joe Sharp, Corresponding Secretary.

From November 10 through December 10, 2004, the Georgia Alpha Chapter of KME conducted its annual Food and Clothing Drive for the Needy with the proceeds going to the Salvation Army. On November 6, 2004, we held our Fall Social at a local restaurant with about a dozen members and guests attending. A fine time was had by all.

GA Gamma – Piedmont College

New Initiate – Anthony Baldridge.

IA Alpha – University of Northern Iowa

Chapter President– Cindee Calton.

Other fall 2004 officers: Lindsey Aronson, Vice–President; Melissa Potter, Secretary; Michelle Boelman, Treasurer; Mark D. Ecker, Corresponding Secretary.

Student member Ryan Dunkel presented his paper "A Brief History of Probabilistic Thinking during the Classical Period: 1650–1800" at our first spring KME meeting on September 16, 2004 at Professor Mark Ecker's residence. The University of Northern Iowa Homecoming Coffee was held at Professor (emeritus) Carl Wehner's residence on October 9, 2004. Our second meeting was held on October 21, 2004 at Professor Jerry Ridenhour's residence where student member Martha Aragon presented her paper on "Mayan Mathematics". Our third meeting was held on November 18, 2004 at Professor Ben Schafer's residence where student member Donald Daws presented his paper on "Aronszajn Trees". Student member Emily Borcherding addressed the fall initiation banquet with "Probability: A Focus on Law through Court Cases, DNA and Bayes Theorem". Our banquet was held at The Brown Bottle restaurant in Cedar Falls on December 9, 2004 where six new members were initiated.

IA Delta – Wartburg College

Chapter President– Angela Kohlhaas. 35 Current Members

Other fall 2004 officers: Nicholas Wuertz, Vice–President; Kristin Granchalek, Secretary; Ben Brady, Treasurer; Dr. Brian Birgen, Corresponding Secretary.

At the Wartburg Homecoming Renaissance Fair, our club successfully ran our traditional annual fundraiser by selling egg–cheeses. During October we ran a series of successful math talks. Two of our students who had completed summer REU's presented on their experience; we also invited a finishing graduate student from the University of Iowa to present on her research. There is some discussion of selling temporary tattoos with mathematical designs as a fundraiser.

IL Zeta – Dominican University

New Initiates – Melisa Biskup, Michelle Cali, Nicole Gentile, Conan Jurkowski, Lisa Leonard, Marsel Oxhaku, Isidro Padilla, Lisette A. Rodriguez, Marion Weedermann.

IN Alpha – Manchester College

New Initiates – Kyle Helfrich, Jerred Jagger.

IN Beta – Butler University

Chapter President– Eric Nelson. 10 Current Members, 10 New Members
Other fall 2004 officers: Brittany Brown, Vice–President; Scott Guttman, Secretary; Dirk Conner, Treasurer; Amos Carpenter, Corr. Sec.

The IN Beta chapter hosted two presenters during the semester.

New Initiates – Douglas S. Adams, Nicholas R. Berkeley, Adam D. Conner, Dan J. Gibas, Scott M. Guttman, Jeremiah A. Kline, Ezekiel J. Maier, Rebecca C. Mitchell, Eric M. Nelson, Diego Quevedo

KS Alpha – Pittsburg State University

Chapter President– Keith Smeltz. 30 Current Members, 8 New Members
Other fall 2004 officers: Angela Steele, Vice–President; Kristie Julian, Secretary; Josh Gier, Treasurer; Tim Flood, Corresponding Secretary.

During the Spring 2005 semester, Kansas Alpha held monthly meeting with free pizza and pop. Speakers included Dr. Elwyn Davis (Adventures on the Sphere), Keith Smeltz (Solving Cubic Equations) and Josh Gier (Probabiltiy and Fibonacci Numbers).

New Initiates – Christine Baker, Jeremy Burnison, John Cauthon, Lyndsey Crosswhite, Kyle Heater, Bradley Smith, Jamie VanLeeuwen, Siyi Wang.

KS Beta – Emporia State University

Chapter President– Chris Dobbs. 24 Current Members, 5 New Members
Other fall 2004 officers: Shannon Wooton, Vice–President; Meaghan Jackson, Secretary; Jason Miller, Treasurer; Connie Schrock, Corresponding Secretary.

Our chapter again hosted a calculator workshop as one of our service projects for ESU. We work to help ESU student learn to use their calculator so they will be successful in their mathematics classes. The members also help as proctors for the ESU math day where over 600 high school students come to campus to compete in mathematics.

New Initiates –Karen Bitler, Grant Dreiling, Mike Moore, Renee Sull, Dustin Wilhelmi.

KS Delta –Washburn University

Chapter President– Fred Hollingshead. 25 Current Members

Other fall 2004 officers: Jo Marie Rozzelle, Vice–President; Jan Misak, Secretary; Jan Misak, Treasurer; Kevin Charlwood, Corr. Sec.

Fall 2004 new from Kansas Delta KME chapter: The Kansas Delta chapter of KME met for three luncheons with the Washburn Math Club during the fall semester. Speakers and/or mathematics presentations were typically part of the meetings. Two KME students, Fred Hollingshead and Jo Marie Rozzelle, presented their summer research work to the faculty of the Natural Sciences Division at their first fall meeting. This work is part of that to be presented at the KME national meeting at Schreiner University in April 2005.

KY Alpha – Eastern Kentucky University

Chapter President– Tracie Prater. 29 Current Members

Other fall 2004 officers: Robert Bassett, Vice–President; Jenna Dovyak, Secretary; Erica Cepietz, Treasurer; Pat Costello, Corr. Sec.

At the September meeting, we had the election of officers and discussed plans for the year. At the October meeting, Dr. Vineet Gupta (new to the department) gave a talk entitled "DNA Computing." The annual picnic was held in conjunction with the Stat Club at Million Park. In December we had our White Elephant Gift Exchange at the Christmas party. Everyone helped draw and decorate a Christmas tree on the chalkboard with colored chalk.

KY Beta – Cumberland College

Chapter President– Joshua White. 15 Current Members, 13 New Members

Other fall 2004 officers: Stephanie Isaacs, Vice–President; Gretchen Phelps, Treasurer; Jonathan Ramey, Corresponding Secretary.

On September 7, the Kentucky Beta chapter officers helped to host an ice cream party for the freshmen math and physics majors. Along with the Mathematics and Physics Club and Sigma Pi Sigma, the chapter had a picnic at Briar Creek Park on October 5. On December 9, the entire department, including the Math and Physics Club, the Kentucky Beta chapter, and Sigma Pi Sigma had a Christmas party with about 36 people in attendance.

New Initiates – Alejandro Beiderman, Roger Carpenter, Heather Hall, David Morris, Lauren Papenfuss, Gretchen Phelps, Hannah Rogers, Andrea Gray Sawyers, Thomas J. Scheithauer, Ben Smith, Dan Sommers, Joshua White, Robert Wilder, Noel Zvonar.

MD Alpha – College of Notre Dame of Maryland

New Initiates – Lanette Andrews, Alexandra Chaillou, Jamel Epps, Susan C. James, Heather Karbonski, Allison Kingsland, Karen Reed, Amanda Reiner, Penny Robinson, Erin Routzahn, Tiffany E. Russell, Tiffany M. Russell, Erin Snyder, Sarah Wassink.

MD Beta – McDaniel College

New Initiate – Randall May.

MI Delta – Hillsdale College

Chapter President– Michael Nikkila. 14 Current Members
Other fall 2004 officers: Joel Clark, Vice–President; Erin Bartee, Treasurer; Dr. John H. Reinoehl, Corresponding Secretary.

Free tutoring for students of mathematics classes. MI Delta sponsored speech on different voting techniques by Hillsdale College faculty member.

MI Epsilon (Section A) – Kettering University

Chapter President– Lynette Fulk. 104 Current Members
Other fall 2004 officers: Gayle Ridenour, Vice–President; Kathleen Moufore, Secretary; George Hamilton, Treasurer; Boyan N. Dimitrov, Corresponding Secretary.

There was a Pizza Party/Movie on 5th Tuesday (August 10) at noon. The movie was "Galileo's Battle for the Heavens" – Part 1. The second Pizza Party/Movie was on 8th Tuesday (August 31) at noon. It was curiosity, fun, enjoyment, and lots of news from the life of this giant of the physics, mathematics and astronomy was revealed for our students, which vast majority is in engineering field. Pins and membership cards were distributed.

MI Epsilon (Section B) – Kettering University

Chapter President– Rebecca Barthlow .121 Current Members
Other fall 2004 officers: Julie Xiong, Vice–President; Justin Via , Secretary; Jamie Taylor, Treasurer; Boyan N. Dimitrov, Corr. Sec.

Rather than the traditional Pizza Party/Movie over the lunch hour it was tried something new this term. There was a Pizza Party/Movie on Thursday (October 27) from 7 in the evening called "Fall 2004 Movie Night and the movie which the KME members and many other students watched was The Beautiful Mind. We at Kettering decide that it is a good idea to start collecting CDs, and videotapes with movies for mathematics discoveries, and for great mathematicians and other interesting fields in the sciences. We look forward for getting similar information from other KME chapters, and start exchanging it for presentations like ours.

MO Alpha – Southwest Missouri State University

Chapter President– Michael Sallee. 37 Current Members, 5 New Members
Other fall 2004 officers: April Williams, Vice–President; John Hammond, Secretary; Samantha Cash, Treasurer; John Kubicek, Corresponding Secretary.

For the fall of 04 the Missouri Alpha Chapter hosted the annual Mathematics Department picnic, and held three monthly meetings. The meetings included two faculty presentations, one from the mathematics department and one from the Chemistry Department, and a DVD about Archimedes.

New Initiates – Tina Akers–Porter, Elizabeth Farrand, Grant Gelven, Theresa Sorey, Steven Straatmann.

MO Beta – Central Missouri State University

Chapter President– Melissa Libbert. 30 Current Members, 9 New Members
Other fall 2004 officers: Norman Elliott, Vice–President; Spencer Louden, Secretary; Bryce Holthouse, Treasurer; Rhonda McKee, Corresponding Secretary.

Missouri Beta Chapter held three meetings in the fall semester. On Sept. 16, Dr. Decker, director of undergraduate research, spoke on opportunities and funding for undergraduate research. Initiation was held in Oct. Nine new members were initiated and Dr. Bachman spoke on public key cryptography. Dr. Wang spoke on control theory at the November meeting. The semi–annual book sale was held in November and the holiday party consisted of bowling at the union and a chili supper at Dr. McKee's house.

New Initiates – Jennifer Delana, Charissa Eichman, Sarah Frezelle, Megan Gerhart, David Haarmann, Robert Lashlee, Jon Schulte, Andrew Thompson, Jill Watkins.

MO Gamma – William Jewell College

Chapter President– Sabrina Denny. 18 Current Members
Other fall 2004 officers: Michelle Richards, Vice–President; Lindsay Lewick, Secretary; Dr. Mayumi Sakata, Treasurer; Dr. Mayumi Sakata, Corresponding Secretary.

MO Theta – Evangel University

Chapter President– Kevin Reed.12 Current Members
Other fall 2004 officers: Nicolas Thompson, Vice–President; Don Tosh, Corresponding Secretary.

Meetings were held monthly and were generally well attended. The pizza helped. Plans are already being made to attend the national convention in April. Elections and initiations will be held in January.

MO Iota – Missouri Southern

10 Current Members, 9 New Members

Chip Curtis, Corresponding Secretary.

We worked concession stands at Missouri Southern's home football game as a fundraiser in preparation for our spring activities. As an end-of –semester celebration, we had a party jointly with the Chemistry Club and Environmental Health Club.

New Initiates - Nicole Barnes, Amy Barnicle, William Beene, Yvette Bowden, Micah Griffin, Rahila Khan, Jamin Perry, Kevin Talbert, Sarah Wallace.

MO Kappa – Drury University

Chapter President– Nicki Kennedy.20 Current Members

Other fall 2004 officers: Adam Scott, Vice–President; Tim Ischnor, Secretary; Tim Ischnor, Treasurer; Charles Allen, Corr. Sec.

The first activity of the semester was a pizza party held at Dr. Allen's house. The winner of the annual Math Contest was Vikram Somal for the Calculus I and below Division and Vikram also won the Calculus II and above Division. Prize money was awarded to the winners at a pizza party held for all contestants. A sub–sandwich luncheon was held for the reports of undergraduate research projects (potential KME papers) by Patrick Muehlam, Layna Boyd, and Chad Forshee. The Math Club has also been running a tutoring service for both the day school and the continuing education division as a money–making project.

MO Lambda – Missouri Western State College

Chapter President– Amy Pankau. 40 Current Members

Other fall 2004 officers: Robert Smith, Vice–President; Greg Hughes, Secretary; Jeff Puckett Treasurer; Don Vestal, Corresponding Secretary.

MS Alpha – Mississippi University for Women

New Initiates – Ken Aboge, Vasile Bratan, Josh Huwe

MS Beta – Mississippi State University

Chapter President– Megan Woodard. 7 Current Members, 1 New Member

Other fall 2005 officers: Amy Marot, Vice–President; Karolina Sarnowska, Secretary; Michael Muffelleto, Treasurer; Tuncay Aktosun, Corresponding Secretary.

NE Alpha – Wayne State College

Chapter President– Rochelle Swanson. 18 Current Members

Other fall 2004 officers: Gabe Fejfar, Vice–President; Christina Gathje, Secretary; Kristy Kounovsky , Treasurer; John Fuelberth, Corresponding Secretary.

KME had three meetings in the fall semester of 2004. At one meeting, there was a barbeque at Dr. Fuelberth's house. KME also had pizza for all math students on one day during study week.

NE Delta – Nebraska Wesleyan University

Chapter President– Brad Randazzo. 13 Current Members

Other fall 2004 officers: Nicole Heath, Vice President; Jennifer Choutka, Secretary; Melissa Erdmann, Corresponding Secretary.

NH Alpha – Keene State College

New Initiates – Premin Arnold, Rudolf Amofa–Baah, Daniel Connell, Amy Greenspan, William Hagen, Tara Jones–Hudson, Jennifer Letourneau, Eric Momnie, Irina Solovieva, Nathan Swartz, Jillian Trask.

NJ Gamma – Monmouth University

Chapter President– Paul Zoccali. 72 Current Members

Other fall 2004 officers: Jessica Gregory, Vice–President; Catharine Russamano, Secretary; Toni Festa, Treasurer; Lauren Kovacs, Historian; Lauren Grobelny and Lisa Marchalonis, Junior Liasons; Judy Toubin, Corresponding Secretary.

The chapter is continuing with the soda tab collection for the Ronald McDonald House Charities. So far two 55 oz containers have been filled. The fund raising project is ongoing so that enough money can be raised for the spring induction ceremony.

NY Alpha – Hofstra University

New Initiates – Christian Hilaire, Laura Fanelli, Jeffrey Haruthunian, Melissa Oddo, Christina Pawlowski, Craig A. Riha, Jyoti, Satkalmi, Mary Beth Simmons, Pamela Smith, Jennifer Anne Weinstein.

NY Eta – Niagara University

Chapter President– Matt Nethercott. 20 Current Members

Other fall 2004 officers: Megan Zdrojewski, Vice–President; Megan Zdrojewski, Secretary; Adam Meyer, Treasurer; Robert Bailey, Corresponding Secretary.

NY Iota – Wagner College

New Initiates – Erin Frawley, Maura Melvin, Tiffany Angelico, April Holcomb, Kerry Jaeger, Joy Galagher, Kaitlin Buffington, Phil De Paul, Diana D'Onorio–DeMeo, Brittany Corn, John Cichon, Jared Jax, Allison Murray, Kristen Wisniewski, Yuliana Toderika, Gregory Soja.

NY Kappa – Pace University

New Initiates – Matthew Buco, Robert Donley, Katie Finn, Jeremy Fornaro, Md. S. Mahmud, Jonathan Tom.

NY Mu – St. Thomas Aquinas College

104 Current Members, 0 New Members

J. Keane, Corresponding Secretary.

NY Onicron – St. Joseph's College

Chapter President– Eileen Bracciodieta. 34 Current Members

Other fall 2004 officers: Meredith Miller, Vice–President; Nicole

Acosta, Secretary; Krishn C. Cinqulmani, Treasurer; Barbara Thorpe, Corresponding Secretary.

OH Epsilon – Marietta College

Chapter President– Casey Trail. 20 Current Members

Other fall 2004 officers: John Tynan, Corresponding Secretary.

OK Alpha – Northeastern State University

Chapter President– Mary Kelly–Harper. 58 Current Members, 12 New Members

Other fall 2004 officers: Leticia Stone, Vice–President; Teri O'Neal, Secretary; Andy Hathcoat, Treasurer; Dr. Joan E. Bell, Corr. Sec.

The Fall initiation brought 12 new students into the Oklahoma Alpha chapter. We participated in the annual NSU Halloween carnival with our "KME Pumpkin Patch" activity. The children fished for pumpkins with meter stick fishing poles. Our president, Mary, then quizzed each child on a math problem before giving them candy. Our annual fall books sale was again a hit, bringing in $171 to fund our activities. We had several speakers this fall. Dr. William Jaco, Professor of Mathematics at Oklahoma State University, spoke on "Research in Mathematics: What might a research mathematician do?" We were very pleased to have Maurice Turney speak about the early days of NSU and KME. Mr. Turney became a member of our Oklahoma Alpha chapter in 1945! The theme for our 6th annual KME T–shirt was "Top ten reasons why e is better than pi." We sold 73 shirts! Our Christmas party was great! After pizza and Christmas goodies, we played the games SET and TABOO.

New Initiates – Kimberly D. Adams, Jerry D. Austin, Lindsey N. Box, Christopher P. Ducker, Josh D. Engle, Lee Harrell, Timothy C. King, Stephanie K. Layman, Casey A. Miller, Ashley L. Sisco, Gila K. Sooter, Jennifer S. Womeldorff.

OK Delta – Oral Roberts University

Chapter President –Matthew Sterns. 10 Current Members, 12 New Members

Other fall 2004 officers: Gabriel Cap, Vice–President, Willie Bustinza, Secretary

During the fall we were able to initiate 12 new members but the report will be received during the spring semester.

OK Gamma – Southwest Oklahoma State University

New Initiates – Heather Baker, Bhaskar Basnet, Shyla Fast, Kenneth Ground, Vishnu Pokhrel, Ashis Shrestha, Joy Oxford.

PA Eta – Grove City College

New initiates – Michelle Borza, Justin Brown, Donald Burkhart, Anthony DiPietro, Rebecca Donnell, Brad Dutton, Joel Gilliland, Ryan Gleason, Joshua Kunzmann, Russell Lodge, Roberta Long, Nicole Macy, Gretchen Reid, Braden Robinson, Jonathan Sanders, Michael Scarpitti, Lynn Schwartz, Megan Wilson.

PA Lambda – Bloomsburg University of Pennsylvina

New Initiates – Tennille Allman, Dustin Frey, Cynthia Kuhnle, Charisa McGowan, Kasey Pruzinsky.

PA Mu – Saint Francis University

Chapter President– Bridget Campbell. 38 Current Members

Other fall 2004 officers: Lesley Wenzel, Vice–President; Shannon Decker, Secretary; Rebecca Dombrowski, Treasurer; Dr. Peter Skoner, Corresponding Secretary.

The mathematics education students who are members of the Pennsylvania Mu chapter attended the annual meeting of the Pennsylvania Council of Teachers of Mathematics held on October 28–29, 2004 in Erie. Two KME members, Bridget Campbell and Tsega Workalemahu, assisted faculty member Paul Deskevich in his presentation on the Powerball Lottery.

The Pennsylvania Mu chapter helped host the Eleventh Annual Science Day on Friday, November 19, 2004. A record total of 426 high school students from 33 area high schools attended. The day included a 16–team science bowl run by Saint Francis University students, presentations by faculty, alumni, and industry professionals, and a variety of games and challenges. An induction ceremony was held on Tuesday, February 8, 2005, in the JFK student center lounge. Inductions featured the largest cohort ever, 25 students, to bring the total membership of Pennsylvania Mu to 237. The induction ceremony followed the dinner.

PA Omicron – Univ. of Pittsburgh at Johnstown

15 Current Members

Dr. Nina Girard, Corresponding Secretary.

Our chapter has been inactive on the campus for one academic year. No new students have been inducted. Plans to hold an initiation ceremony are slated for this spring semester.

TN Beta – East Tennessee State University

New Initiates – James Gardner, Bethany Jablonski, Daniel Lamb, Bryan Lilly, Courtney Sanders, Amber Simpson, Olivia Thomas, Beverly Tomlinson, Andrew Young.

TN Gamma – Union University

Chapter President– Brian Taylor. 16 Current Members

Other fall 2004 officers: Jennifer Ellis, Vice–President; Denise Baughman, Secretary; Willie George, Treasurer; Bryan Dawson, Corresponding Secretary.

The TN Gamma event calendar for Fall 2004 began with a dessert social on September 30; continuing members were welcomed back and new associate members were recruited. The next event consisted of Brian

Taylor sharing the mathematics he used in his second summer of work at the St. Jude Children's Hospital. Throughout the semester, some members attended the monthly department colloquium, each featuring a talk given by a faculty member. The final events of the semester occurred on December 6, when two KME members gave their senior seminar presentations, followed by a Christmas party held at the home of Dr. Lunsford. The chapter also continued its tradition of sponsoring a needy child for Christmas and participating in a service project on Union University's Day of Remembrance.

TX Alpha – Texas Tech University

Chapter President– Brandon Lewis. 25 Current Members

Other fall 2004 officers: Courtney Smartt, Vice–President; Christopher Taylor, Secretary; Sean Hannon, Treasurer; Dr. Anatoly B. Korchagin, Corresponding Secretary.

TX Eta – Hardin–Simmons University

New initiates – Megan Campbell, Mica Hill, Stephanie Irwin, Lynsey Mankins, Melissa McClanahan, Aaron McLaughlin, Tava Peralta, Jessica Rieger, Stephanie Rollins, Randall Volcko.

TX Iota – McMurry University

New initiates – Amy L. Bell, Christopher D. Dean, Jonathan M. Farrer, Li Liu, Kelly L. Perkins, Bonnie K. Schneider, Russel T. Sliter, Clipper W. Strickland, Michael R. Vives, Richard A. Williams, Dr. Cynthia Martin.

TX Kappa – University of Mary Hardin–Baylor

Chapter President– Andrew Ellis. 10 Current Members

Other fall 2004 officers: Andrea Weldy, Vice–President; James Parten, Secretary; Peter H. Chen, Corresponding Secretary.

TX Mu – Schreiner University

Chapter President- Rebekha Collins, 11 Current Members

Ofter fall 2004 officers: Charnelyn Fortune, Vice–President; Matthew Casey, Secretary-Treasurer; William M. Silva, Corresponding Secretary

The chapter has been very busy preparing for the national convention in the spring of 2005.

VA Alpha – Virgina State University

New Initiates – DeQuincy E. Faulcon, Rachael E. Gatling, Kathy A. Goodson, LaToya M. Lee, Victoria A. Morrison, Maisha T. Pollard, Jamel M. Robinson, Wanda M. Williams, Shevis D. Wimbush, Kiera A. Branch, Nadia J. Calhoun, Latres T. Davis, Ronald W. Davis II.

VA Delta – Marymount University

18 New Members

Dr. Elsa Schaefer, Corresponding Secretary.

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New Problem Corner Editor

This issue marks the start of a transition to a new editor of the Problem Corner of The Pentagon. Patrick Costello, of the Kentucky Alpha chapter, Eastern Kentucky University, will gradually assume these duties. Send new problem proposals and solutions to problems from future issues to:

Pat Costello
Department of Mathematics and Statistics
Eastern Kentucky University
521 Lancaster Avenue
Richmond, KY 40475-3102
e-mail: pat.costello@eku.edu, fax: (859)622-3051

Send solutions to problems in the current issue to:

Kenneth M. Wilke
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275 Morgan Hall
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Dr. Costello will be the ninth editor of the Problem Corner. The Problem Corner was inaugurated with the Fall, 1947 issue. Previous Problem Corner Editors were Judson W. Foust, Central Michigan College of Education (1947-1952), Frank C. Gentry, University of New Mexico (1953-1957), J.D. Haggard, Kansas State Teachers College (1957-1963), F. Max Stein, Colorado State University (1964-1965), H. Howard Frisinger, Colorado State University (1966-1968), Robert L. Poe, Berry College (1968-1973), Kenneth M. Wilke, Washburn University (1974-2005), Catherine Kong, California Baptist University (2004-2005). Catherine and Ken served as co-editors during this past year. Kenneth M. Wilke began his duties as editor of the Problem Corner with the Fall, 1974 issue of The Pentagon, thirty years ago. He has thus served longer than all of the previous editors of the Problem Corner put together. His dedication to Kappa Mu Epsilon has been truly remarkable. Thank you, Ken and Catherine.

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 April 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Southwest Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
MS Beta	Mississippi State University, Mississippi State	14 Dec 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
AL Alpha	Athens State University, Athens	5 March 1935
NM Alpha	University of New Mexico, Albuquerque	28 March 1935
IL Beta	Eastern Illinois University, Charleston	11 April 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 April 1937
OH Alpha	Bowling Green State University, Bowling Green	24 April 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	Central Missouri State University, Warrensburg	10 June 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 June 1941
NY Alpha	Hofstra University, Hempstead	4 April 1942
MI Beta	Central Michigan University, Mount Pleasant	25 April 1942
NJ Beta	Montclair State University, Upper Montclair	21 April 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 March 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 June 1947
CO Alpha	Colorado State University, Fort Collins	16 May 1948
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 April 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960

OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 April 1965
AL Epsilon	Huntingdon College, Montgomery	15 April 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
AR Alpha	Arkansas State University, State University	21 May 1965
TN Gamma	Union University, Jackson	24 May 1965
WI Beta	University of Wisconsin—River Falls, River Falls	25 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	McDaniel College, Westminster	30 May 1965
IL Zeta	Domincan University, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 March 1971
KY Alpha	Eastern Kentucky University, Richmond	27 March 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 April 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 April 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State University, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sept 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis University, Loretto	14 Sept 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury University, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 March 1985

NE Delta	Nebraska Wesleyan University, Lincoln	18 April 1986
TX Iota	McMurry University, Abilene	25 April 1987
PA Nu	Ursinus College, Collegeville	28 April 1987
VA Gamma	Liberty University, Lynchburg	30 April 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 April 1990
CO Delta	Mesa State College, Grand Junction	27 April 1990
NC Gamma	Elon College, Elon College	3 May 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 April 1991
SD Alpha	Northern State University, Aberdeen	3 May 1992
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 March 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 April 1997
MI Delta	Hillsdale College, Hillsdale	30 April 1997
MI Epsilon	Kettering University, Flint	28 March 1998
KS Zeta	Southwestern College, Winfield	14 April 1998
TN Epsilon	Bethel College, McKenzie	16 April 1998
MO Mu	Harris-Stowe College, St. Louis	25 April 1998
GA Beta	Georgia College and State University, Milledgeville	25 April 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
NY Xi	Buffalo State College, Buffalo	12 May 1998
NC Delta	High Point University, High Point	24 March 1999
PA Pi	Slippery Rock University, Slippery Rock	19 April 1999
TX Lambda	Trinity University, San Antonio	22 November 1999
GA Gamma	Piedmont College, Demorest	7 April 2000
LA Delta	University of Louisiana, Monroe	11 February 2001
GA Delta	Berry College, Mount Berry	21 April 2001
TX Mu	Schreiner University, Kerrville	28 April 2001
NJ Gamma	Monmouth University	21 April 2002
CA Epsilon	California Baptist University, Riverside	21 April 2003
PA Rho	Thiel College, Greenville	13 February 2004
VA Delta	Marymount University, Arlington	26 March 2004
NY Omicron	St. Joseph's College, Patchogue	1 May 2004
TX Nu	Texas A&M University - Corpus Christi, Corpus Christi	8 May 2004
IL Iota	Lewis University, Romeoville	26 February 2005
WV Beta	Wheeling Jesuit University, Wheeling	11 March 2005
SC Epsilon	Francis Marion University, Florence	18 March 2005
PA Sigma	Lycoming College, Williamsport	1 April 2005
MO Nu	Columbia College, Columbia	29 April 2005