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Distribution Plots

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awarded "top three" status by the Awards Committee.

Introduction

Suppose you were handed the following table of data and were asked to pick out any unusual values corresponding to $\phi(k)$ ([4]; see figure 1).

k	ϕ(k)
0	4.04
1	5.31
2	6.64
3	8.06
4	9.46
5	10.80
6	12.19
7	13.46
8	14.41
9	16.10
10	17.41
11	18.89
13	22.55
14	25.19

Figure 1

Based solely on the above information, you might conclude that none of the values for k are unusual since $\phi(k)$ seems to be consistently increasing with k.

Suppose now that you were handed the following graph, and were asked

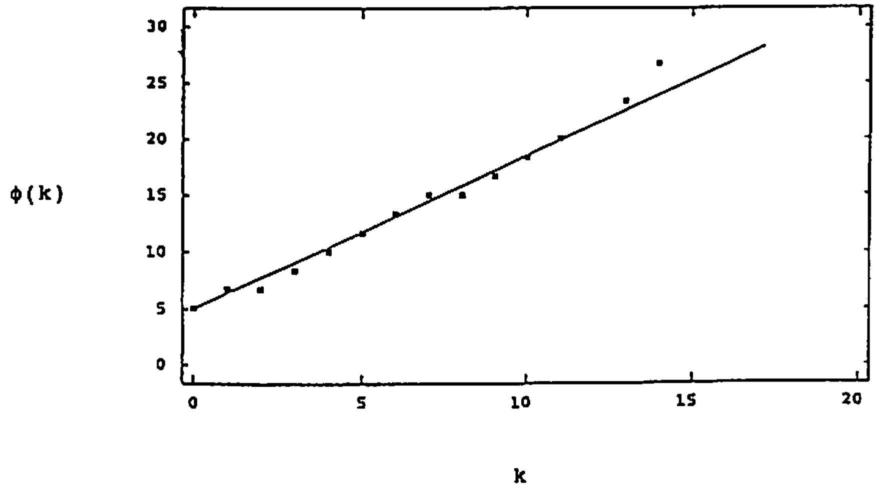


Figure 2

the same question as above (see figure 2).

What can you conclude now that perhaps you were not able to conclude from the table alone? The graph makes it clear that the value for $k = 14$ deviates from the straight line that the other data points follow, whereas such a deviation is not so obvious in the table. Not only did the graph allow a straightforward detection of the discrepant point, but it was almost instantaneous. It is much easier for our eye-brain system to conclude results about a data set from a graph of the data points than from a corresponding listing of the data points ([2]).

Because graphs give such a proficient visual representation of relationships within a particular data set, they often are used to construct or to verify statistical models. However, within a given data set, one cannot just plot any two quantities against each other and look for a straight line because the quantities might not be linearly related. Such an expectation would be an unproven assumption about the data. So, before beginning data analysis, one needs to understand basic characteristics of the data set. In other words, the data needs to be fit to a particular distribution.

There are several different distributions, all of which fall into one of three categories: discrete, continuous, or mixed. The type of distribution that will be focused on in this paper is a discrete distribution.

Discrete distributions arise from counting techniques, which means that there is a correspondence between the observed discrete data and the set $k = \{0, 1, 2, \ldots\}$. These correspondences are obtained by counting the number of occurrences, n_k, in the observed data that are equal to the outcome k. In this notation, n_0 of the observations pertain to the outcome 0, n_1 are associated with the outcome 1, and so on (as in [5]). Sometimes, however, the outcomes of an experiment are not initially in numerical terms. Such is the case with studies on hair color, gender, nationwide location, etc. Therefore, in order to perform statistical analysis, one needs to define a random variable, which is a function that assigns a real number to each outcome of a particular event (c. f. [1]).

Since random variables allow all the outcomes of an experiment to be considered for statistical analysis, one can define a probability density function (p.d.f.). Every distribution has an associated p.d.f., which assigns probabilities to each possible value of a random variable. P.d.f.s are usually denoted by $P(k)$, which is the probability of outcome k occurring.

Not only does every distribution have a p.d.f., but each depends on one or more associated parameters. A parameter is an unknown quantity that describes its distribution. If the parameter changes, so does the shape of the distribution. Since a researcher would like to have a completely defined distribution so that certain properties for a data set could be revealed, which in turn would allow the researcher to begin verifying assumptions about the data set, he/she needs to know the value for the parameter of the distribution. However, since parameters are unknown quantities, their true values can never be known. Thus, parameters need to be estimated.

Several methods for parameter estimation exist. In the beginning of this paper, graphs were praised as excellent statistical tools. One additional benefit of graphs that should now be introduced is that a carefully constructed graph can yield an estimate for an unknown parameter of a probability distribution. A graph of a linear equation whose slope and intercept are functions of the parameter of interest would yield a fitted line. The slope and intercept of this fitted line would in turn reveal something about the value of the unknown parameter. A researcher could therefore estimate a parameter of a distribution by plotting an appropriate graph. This estimated value would replace the unknown value of the parameter, thus describing its distribution. The researcher would then have all the necessary pieces to calculate any desired probability pertaining to the specified distribution.

Because each distribution has distinct characteristics, a researcher usually has a pretty good idea about which distribution a given data set follows. Therefore, he/she has an idea as to which associated parameter needs to be estimated. However, such assumptions must be verified. Several statisticians have proposed different graphical techniques that will show how closely a sample of data follows a particular distribution, but one such technique that is of particular interest is the "Poissonness plot."

A Poissonness Plot

As was stated before, there are many different kinds of discrete distributions. One that comes up quite often is the Poisson distribution, which counts the occurrence of rare events such as the number of misprints on a page of text, or the number of tornados per year in a particular area. The Poisson distribution has only one parameter, λ, which measures the "rate" of the occurrence. The p.d.f. for a Poisson distribution is

$$P_\lambda(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \lambda > 0 \text{ and } k = 0, 1, 2, \ldots.$$

This means that if a researcher thinks that the outcomes of a particular event are from the Poisson distribution, then he/she could find the probability that an outcome of, say, 3 would occur by calculating

$$P_\lambda(3) = \frac{e^{-\lambda}\lambda^3}{3!},$$

provided that the value of λ is known.

In order to be able to calculate a probability for any of the outcomes in the sample from the above equation, one needs to estimate the value for λ. Recall that an estimate of a desired parameter can be found by plotting a function of the parameter. So, one would want to determine the function of λ that would be useful.

Thus far, the only information that is known is that the outcomes of a particular event are assumed to be from the Poisson distribution. In order to estimate λ, one would first have to verify that the data being worked with is in fact Poisson. Now, either the outcomes are from a Poisson distribution, or they are not. Since the assumption is that the outcomes are from a Poisson distribution, one would expect a collection of observed frequencies of the event to be equal to the expected frequencies calculated from the Poisson p.d.f. Therefore, it would seem reasonable to set the two frequencies equal to each other in the following manner:

$$(1) \qquad n_k = NP_\lambda(k) = \frac{Ne^{-\lambda}\lambda^k}{k!},$$

where n_k is the observed frequency of outcome k, $P(k)$ is the probability of k occurring, and $N = n_0 + n_1 + n_2 + \cdots$ is the sample size of the data set ([3]).

A function has now been derived that contains the parameter of interest. However, it was initially stated that we could get information about a parameter from the slope of the fitted line, indicating that a linear function would be preferred over the one previously stated. To obtain such an equation, David Hoaglin, in his article "A Poissonness Plot" [4], proposes taking the natural logarithm of both sides of the above equation, which yields

$$(2) \qquad \ln n_k = \ln N - \lambda + k \ln \lambda - \ln k!.$$

Rearranging this equation gives

$$\ln\left(\frac{n_k k!}{N}\right) = -\lambda + k\ln\lambda. \tag{3}$$

The left-hand side of this equation is referred to as the count metameter, denoted $\phi(n_k)$, and its value is known from the sample information. So $\phi(n_k) = -\lambda + k\ln\lambda$ is an equation whose slope and intercept are functions of λ, the parameter of interest. Then, if the observed data are actually from a Poisson distribution, a plot of $\phi(n_k)$ against k should produce a straight line with a slope of $\ln\lambda$ and an intercept of $-\lambda$ ([3]).

However, in practice a researcher would never see a plot that precisely follows a straight line. So, for practical purposes, we call a line "straight" if it appears relatively so, with only a few exceptions. A researcher can then fit the observed line by whatever means he/she prefers (least squares estimate, fitting the line by eye, etc.). Calculating a slope for the fitted line is an easy task, and the value of the slope, call it m, is set equal to $\ln\lambda$. Then $e^m = e^{\ln\lambda} = \lambda$, so e^m is an estimate for λ, where m is the slope of the fitted line of the Poissonness plot.

If, on the other hand, the points on the derived Poissonness plot deviate significantly from a "straight" line, then a Poisson distribution is probably not an ideal fit to the data. Extreme outliers, definite curvature, or unexplained gaps in a plot are all strong indications that the observed data are from a distribution other than the Poisson ([2, p. 203]).

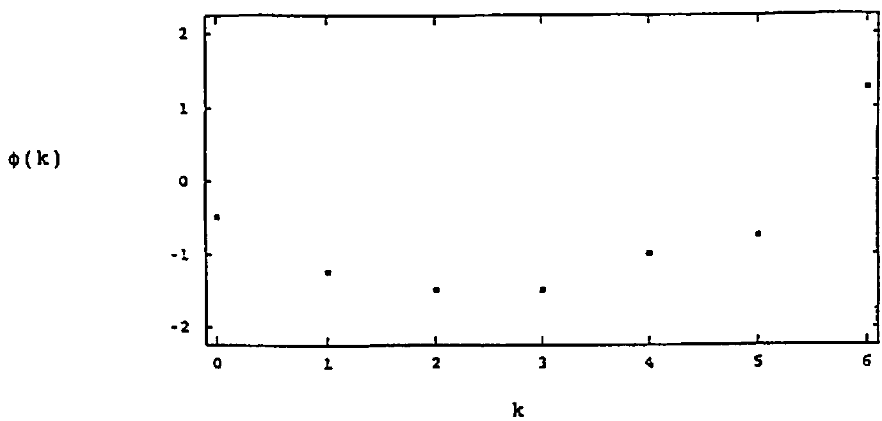


Figure 3

For an example, consider the Poissonness plot in figure 3, whose data pertain to the occurrences of the word "may" in the *Federalist Papers*. Each point on the plot corresponds to an observed data value that was inserted into the Poissonness plot equation to output the appropriate point. The curvature of the graph indicates that the Poisson distribution does not provide a good fit for the data. In fact, Moesteller and Wallace propose using the negative binomial distribution for this particular example (see [4]).

A Generalization of the Poissonness Plot for Other Distributions

The previous example brings to light a feature of the Poissonness plot: its concept is not restricted to only the Poisson distribution. A generalization of the Poissonness plot has been derived that includes certain other distributions, such as the binomial. Because of this, it would be more natural, not to mention less confusing, to simply call the plots "distribution plots." Then one could refer to the plotting process for the binomial distribution as a distribution plot for the binomial, and a Poissonness plot could be referred to as a distribution plot for the Poisson, which makes it clearer as to which distribution is being plotted.

To derive the generalization, refer back to equation (2) for the Poissonness plot and note that it is equivalent to

$$\ln n_k = \ln(N/k!) - \lambda + k \ln \lambda.$$

Now, the generalization proceeds by letting $a_k = N/k!$, a general term whose only constraint is that it depends on k; $b(\theta) = -\lambda$, a general function of the parameter of the distribution; and $c(\theta) = \ln \lambda$, another general function of the parameter. Note for future reference that $b(\theta) = -\lambda$ is equivalent to $-e^{b(\theta)} = e^{\lambda}$ and that $c(\theta) = \ln \lambda$ is the same as $e^{c(\theta)} = \lambda$. Now, the above substitutions result in the following generalized plotting equation:

$$\ln n_k - \ln a_k = b(\theta) + kc(\theta). \tag{4}$$

So fitting a theoretical distribution to a particular data set would entail plotting $\ln n_k - \ln a_k$, the count metameter, against k and looking for a straight line with slope $c(\theta)$ and intercept $b(\theta)$. Just like with the Poisson distribution, a plot which reveals a relatively straight line indicates that the distribution in question is a good fit to the observed data ([5, p. 376]).

But again, one runs into the problem of which functions of the parameter should be chosen. In other words, one would want to find values for $b(\theta)$ and $c(\theta)$. Recall that for the Poisson distribution we have $b(\theta) = -\lambda$ and $c(\theta) = \ln \lambda$, but these particular functions might not work for another distribution whose parameter(s) and p.d.f. differ from the Poisson. So, it would be in order to further generalize equation (4) by finding equations for $b(\theta)$ and $c(\theta)$.

To do this, one can regroup equation (1) and then replace λ and e^λ with the appropriate quantities mentioned above in the following manner:

$$n_k = \frac{N}{k!}\frac{1}{e^\lambda}\lambda^k = a_k \frac{1}{-e^{b(\theta)}} e^{c(\theta)}.$$

Note that the middle portion of this equation represents the p.d.f. of N trials of a Poisson experiment, while the right-hand side corresponds to the p.d.f. of a theoretical distribution. Since λ is the parameter of the Poisson distribution, it follows that $e^{c(\theta)}$ represents the parameter of the theoretical distribution; for simplicity's sake, call the quantity θ. In addition, because e^λ is a function of the parameter λ, $-e^{b(\theta)}$ is a function of the parameter of the theoretical distribution; call this quantity $f(\theta)$. Solving the equations

$$\theta = e^{c(\theta)}$$

and

$$f(\theta) = -e^{b(\theta)}$$

for $c(\theta)$ and $b(\theta)$, respectively, gives

$$c(\theta) = \ln \theta$$

and

$$b(\theta) = -\ln(f(\theta)),$$

so that generalized functions of the parameter of the theoretical distribution have been found.

It is important to note that the above substitutions give rise to the following:

$$n_k = a_k \frac{1}{f(\theta)} \theta^k. \tag{5}$$

Any distribution whose p.d.f. can be fit into the form of equation (5) can offer values for $b(\theta)$ and $c(\theta)$ which can be substituted into equation (4), thereby yielding a corresponding plot which will show how closely observed data points follow the proposed distribution.

To see that the generalized method works for distributions other than the Poisson, consider the binomial distribution. The binomial distribution measures such things as the number of heads that will come up in any given number of flips of a coin. Thus, it is often used in games of chance where only two outcomes (usually denoted success or failure) are possible. Its p.d.f. is

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots n,$$

where n is the number of trials performed and p is the probability of obtaining a "success" (for the coin-flipping problem, the probability of getting a head, or a success, is $\frac{1}{2}$).

The above p.d.f. is algebraically equivalent to

$$P(k) = \frac{\binom{n}{k}\left(\frac{p}{1-p}\right)^k}{(1-p)^{-n}},$$

which makes it easier to see that the binomial p.d.f. fits into equation (5), where

$$a_k = N\binom{n}{k};$$
$$\theta = p/(1-p); \text{ and}$$
$$f(\theta) = (1-p)^{-n}.$$

So,

$$c(\theta) = \ln\theta = \ln(p/(1-p))$$

and

$$b(\theta) = -\ln(f(\theta)) = -\ln((1-p)^{-n}) = n\ln(1-p).$$

Substituting these values into equation (4) gives

$$\ln n_k - \ln\left(N\binom{n}{k}\right) = n\ln(1-p) + k\ln(p/(1-p)),$$

so that a plot of $\phi(n_k) = \ln n_k - \ln\left(N\binom{n}{k}\right)$ against k would be expected to yield a straight line with a slope of $\ln(p/(1-p)) = c(\theta)$ and an intercept of $n\ln(1-p) = b(\theta)$. Thus, a distribution plot for the binomial distribution has been constructed, proving that the Poissonness plot method extends to distributions other than just the Poisson, provided that their p.d.f.s can fit into equation (5).

Now, recall that the Poisson distribution relies on only one parameter, λ. In general, for any discrete distribution with only one parameter, a plot of $\phi(n_k)$ against k produces a straight line where the slope identifies the parameter of the theoretical distribution ([5, p. 347]). However, some distributions, such as the binomial, rely on two parameters. In these cases, at least one of the parameters needs to be held constant, or assumed known. So, in order to do the above distribution plot for the binomial distribution whose two parameters are n and p, one of the parameters needs to be held constant. Since n is usually known, it is chosen as the constant parameter, so p is the parameter to be estimated ([5, p. 377]). Even though the binomial distribution relies on two parameters, we see that the distribution plot method works when n is known. Thus, a distribution doesn't have to have only one parameter for equation (4) to work; it could have several parameters, provided it has only one unknown parameter. Also, its p.d.f. must fit into the form of equation (5). If a particular distribution meets these conditions, then such a plot can be constructed for that distribution.

Applications of Distribution Plots

To help clarify the usefulness of distribution plots, consider the following situation: a researcher just received two data sets, shown in figure 4.

k	Φ(k)
0	93
1	65
2	34
3	7
4	1
5	0

DATA SET A

k	Φ(k)
0	43
1	69
2	50
3	31
4	6
5	1

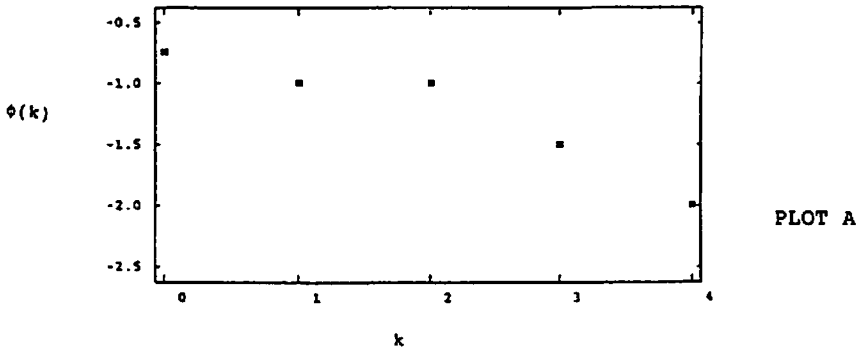
DATA SET B

Figure 4

Both data sets were constructed by a random number generator; one was generated according to the Poisson process with parameter λ, and the other corresponding to the binomial process with parameters n and p. The researcher wants to analyze the data, but is unsure as to which data set befits which distribution. In order to solve the problem, he/she decides to fit both data sets into the distribution plot for the Poisson distribution. Since one of the data sets is actually Poisson, the researcher knows that out of the two graphs that will be produced by the distribution plot procedure, one should reveal a relatively straight line (indicating the Poisson data set), and the other should stray significantly from a straight line (implying a data set from a distribution other than the Poisson).

Fitting each data set into equation (3), the researcher finds corresponding count metameters and produces the plots shown in figure 5. Notice that plot A reveals a relatively straight line, whereas plot B conveys a curve. This would strongly suggest to the researcher that data set A is from the Poisson distribution. Doing an analysis of variance on the above plots reveals the following slope and intercept for each of the fitted lines:

	data set A	data set B
slope	−0.315	0.227
intercept	−0.701	−1.269



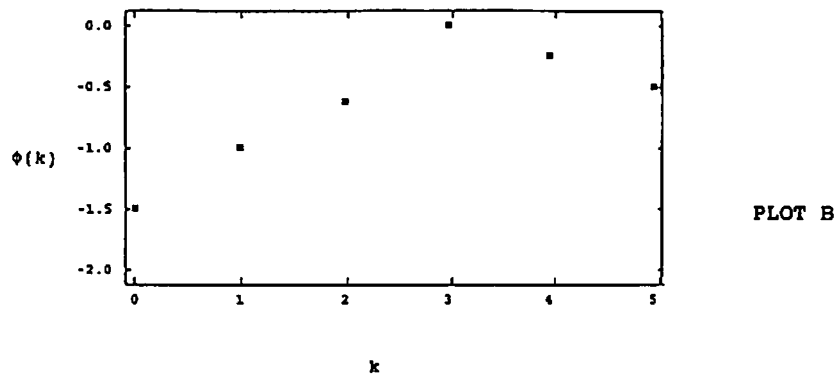


Figure 5

Recall that the distribution plot for the Poisson distribution should reveal a straight line with a slope of $\ln\lambda$ and an intercept of $-\lambda$. Since plot A conveyed a straight line, and since values of the slope and intercept of the fitted line were found, the researcher can estimate the value of the unknown parameter, λ, as follows:

$$\text{slope} = -0.315 = \ln\lambda \Rightarrow \lambda = e^{-0.315} = 0.730.$$
$$\text{intercept} = -0.701 = -\lambda \Rightarrow \lambda = 0.701.$$

The researcher can now conclude that data set A is from the Poisson distribution with $\lambda \approx 0.715$. (Data set A was actually generated from the Poisson process with $\lambda = 0.8$, indicating that the distribution plot should produce a straight line with a slope of $\ln 0.8 = -0.223$ and an intercept of -0.8. Note that these values are very close to what was observed from the data of plot A. In contrast, the data from plot B reveals an estimate of approximately 1.26 for λ, which is not nearly as accurate.)

The researcher could conclude, then, that the other data set is the binomial set. But, just to make sure, he/she should plug the values of data set B into the distribution plot for the binomial distribution, which would yield the plot in figure 6.

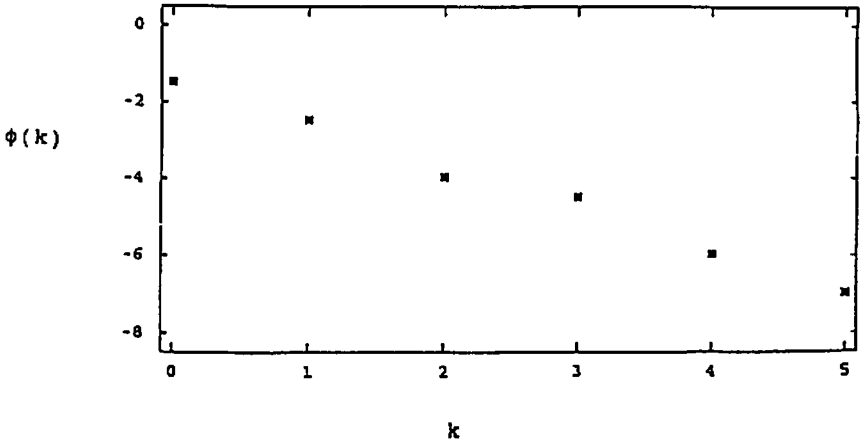


Figure 6

The straight line in the plot of figure 6 confirms the researcher's belief that the second data set is, in fact, binomial. Computing an analysis of variance for this plot reveals a slope of -1.103 and an intercept of -1.684, which gives way to an estimate of approximately 0.245 for the binomial parameter p. The value for the parameter n is fixed at 6, because there are six values for k in data set B. (Data set B was actually generated by a binomial process with $n = 6$ and $p = 0.25$, implying a value of -1.0986 for the slope, and -1.726 for the intercept.)

By employing distribution plots, the researcher was not only able to figure out which data set was from which distribution, but was also able to estimate the values for their corresponding parameters. Analysis of the data can now be performed.

Advantages of Distribution Plots

When working with graphs, it would be useful to be able to "incorporate resistance." That is, it would be advantageous to work with a graph that would enable a person to easily detect unusual data points. This would be beneficial because sometimes there is a reasonable explanation for stray data points. For example, consider the Poissonness plot in figure 7, whose original observed data points correspond to the number of incidents of international terrorism over a period of about six years.

Note that the first five points seem to be following a relatively straight line, but that the point at $k = 12$ is discrepant. However, examining this

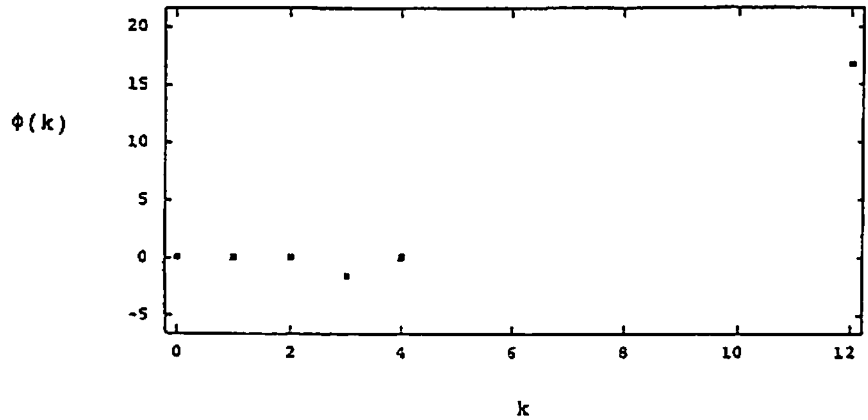


Figure 7

particular data point more closely reveals that the point corresponds to the month of July, 1968, and that 11 of the 12 terrorist incidents were tied to a certain anti-Castro group ([5, p. 351]). Since this unusual data point can be explained, it could be omitted from the other data points, thereby preventing misleading results. One wants to be sure, however, not to complicate matters by choosing a graph that draws unnecessary attention to good data points. Just as a researcher doesn't want a "bad" point to go unnoticed, he/she doesn't want to waste time checking a "good" point that only appears discrepant. Such a possibility might occur if a particular plot were to incorporate two different observed values, say n_k and n_{k-1}, into its count metameter. Then a discrepant n_k would affect two points on the plot. Even if n_{k-1} were not a discrepant point, it would appear so on the plot due to its dependence on the discrepant n_k. This would make the data seem to be more discrepant from a straight line than it actually is, possibly causing a researcher to disregard a plausible distribution.

The advantage of distribution plots is that they satisfy the above requirements. A distribution plot incorporates only one observed value, n_k, into its count metameter, which means that each point on the plot is independent of all the others. Therefore, if n_k is a discrepant point, it will affect only one point on the plot. Moreover, distribution plots are constructed so that discrepant points often will be obvious enough to be easily detected.

Drawbacks of Distribution Plots

Refer back to the distribution plot of the *Federalist Paper* data (figure 3). The data points were plotted under the assumption that the Poisson distribution was a good fit to the data, but the plot indicated otherwise. Now, if a researcher has an idea as to an alternative distribution that might provide a better fit (such as the negative binomial), then if its p.d.f. fits into equation (1), the assumption can be tested using Equation (2). But what if the person doing the plotting has absolutely no idea as to which distribution might fit the data? Theoretically, the person could try one distribution plot after another until one is found which yields a straight line, but this method would be both monotonous and extremely time-consuming. So, the disadvantage of distribution plots is that if the calculated plot does not reveal a straight line for the distribution of interest, it gives no indication as to which distribution might give a better fit.

Conclusion

The advantages of distribution plots are that they tell how closely a given set of data follow a particular distribution, they give an estimate of the unknown parameter of a distribution, and they incorporate resistance. However, the main disadvantage of a distribution plot is that if the plot does not reveal a straight line, it gives no indication as to a distribution that possibly would provide a fit to the data. One possible solution to this drawback is the Ord plot, which can suggest a fitting distribution to a set of data. Such a task is accomplished by examining the shape of the derived line on the plot; interested readers are encouraged to explore this idea in Ord's article [6].

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The Formation of the Rainbow

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Presented at the 1998 Region III Convention and awarded "best undergraduate paper" by the Awards Committee.

The rainbow is one of nature's most breathtaking sites. After a storm, the rainbow provides a calm peace to the sky. However, it is not only the rainbow's beauty that appeals to man. There is a great mystery behind the rainbow. What could cause a band of colors to suddenly appear in the sky? Many theories, ranging from Biblical to mythological, have been given in attempt to explain the rainbow. Interestingly, the formation of the rainbow can be explained mathematically.

Early Fascination with the Rainbow

Man's fascination with the rainbow can be traced back for centuries. The most familiar explanation of the rainbow can be found in the Bible. In Genesis 9, God promises Noah that He will never destroy the earth by flood again. He creates the rainbow to remind Noah and the people of His promise. Other early explanations of the colorful bow can be found in Greek mythology. The Greeks believed the goddess Iris "used the rainbow as a sign both of warning and of hope" [3, p. 42]. What was a symbol of beauty to the Greeks, was seen in African mythology as a deadly "snake coming out to graze after the storm" [3, p. 42]. American Indians viewed the rainbow as a bridge leading from one world to the next. Since the rainbow seems to touch the earth, many say there is a pot of gold at the end of the rainbow. There are many theories explaining the rainbow. However, we will explore the mathematical reasoning behind this colorful arc.

Early Mathematicians and the Rainbow

To many, the early explanations of the rainbow were not satisfying. It was not long before great minds began to observe relationships between the sun, rain, and rainbow. In fact, in 578 BC, a Greek scholar named

Anaximenes theorized that clouds bent the sun's light to form the rainbow. Aristotle theorized that geometric reasoning lay behind the circular arc. Although his reflection laws were incorrect, his idea was valid [3, p. 42]. However, it was not long until scholars realized that the reflection and refraction of light played an important role in the formation of the rainbow. Then in the fourteenth century, Theodoric of Freiburg and the Persian scholar Kamal al-Din al Farisis independently decided that drops of rain were the key [3, p. 42].

However, it was the mathematicians of the seventeenth century that put all of these pieces together. Fermat theorized about the path of light. Descartes traced light rays to demonstrate how light is reflected and refracted in water, which led to explanation of the circular arc. Then, Isaac Newton finalized the explanation in his book, *Optics*, which discussed the distinct colors of the rainbow [5, p. 68]. We will use many of these mathematical ideas as we explore the formation of the rainbow.

The Path of Light in Water

Descartes knew the path of light was essential to the formation of the rainbow. To understand the path light rays follow when they encounter water, one must first understand the principles of reflection and refraction. When light encounters a water droplet, a portion of the light rays bounce off the drop's surface, while the remaining rays enter the drop.

To understand the idea of reflection, examine the rays which bounce off the water's surface. Consider the following diagram where P is the source of light rays such that at least one light ray passes through point Q after reflecting off the surface at some point R (figure 1).

Fermat's principle states, "Light follows a path which minimizes the total travel time." Thus, the ray will travel from P, bounce off the surface at R, and pass through Q in the least amount of time possible. In order to do so, R should be positioned so the path PRQ has minimum length, assuming the speed of light is constant [3, p. 43].

Now, the length of path PRQ is equivalent to the length of path PR plus the length of path RQ. These lengths can be calculated using the Pythagorean theorem for the triangles in figure 1. We can represent the entire path length as a function of x. Then,

$$L(x) = \sqrt{p^2 + x^2} + \sqrt{q^2 + (d - x)^2}.$$

The minimum length is found by setting the derivative, $L'(x)$, equal to zero. Now,

$$L'(x) = \frac{1}{2}(p^2 + x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}(q^2 + (d - x)^2)^{-\frac{1}{2}}(2)(d - x)(-1).$$

Thus, we want

$$\frac{1}{2}(p^2 + x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}(q^2 + (d - x)^2)^{-\frac{1}{2}}(2)(d - x)(-1) = 0.$$

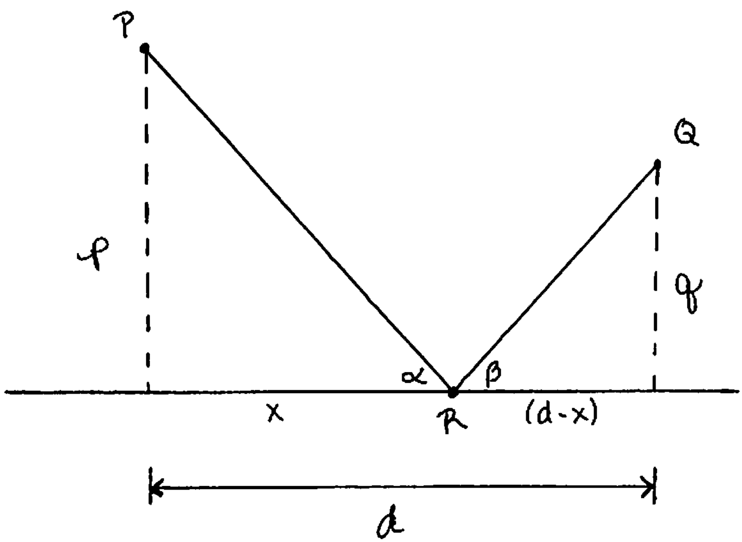

Figure 1

Simplifying,

$$\frac{x}{\sqrt{p^2+x^2}} - \frac{d-x}{\sqrt{q^2+(d-x)^2}} = 0,$$

or

$$\frac{x}{\sqrt{p^2+x^2}} = \frac{d-x}{\sqrt{q^2+(d-x)^2}}.$$

Looking at figure 1,

$$\sin\alpha = \frac{x}{\sqrt{p^2+x^2}}$$

and

$$\sin\beta = \frac{d-x}{\sqrt{q^2+(d-x)^2}}.$$

Thus, $\sin\alpha = \sin\beta$ for $L'(x) = 0$. Since these two angles are between 0 and $\frac{\pi}{2}$, the angles must also be equal. So, $\alpha = \beta$.

We must verify that this is a minimum, for it could be a maximum or an inflection point. To do so, we will solve the first derivative for x and plug the root into the second derivative. Since the computations can become rather intense, *Mathematica* is an excellent tool to aid in this process.

For $L'(x) = 0$, we previously derived

$$\frac{x}{\sqrt{p^2+x^2}} - \frac{d-x}{\sqrt{q^2+(d-x)^2}} = 0;$$

solving yields

$$x = \frac{d^2 - p^2 + q^2}{2d}.$$

Now,

$$\begin{aligned}
L''(x) &= x\left(\tfrac{-1}{2}\right)(p^2+x^2)^{-\frac{3}{2}}(2x) + \frac{1}{\sqrt{p^2+x^2}} \\
&\quad - (d-x)\left(\tfrac{-1}{2}\right)(q^2+(d-x)^2)^{-\frac{3}{2}}(2)(d-x)(-1) + \frac{1}{\sqrt{q^2+(d-x)^2}} \\
&= -\frac{x^2}{(p^2+x^2)^{3/2}} + \frac{1}{\sqrt{p^2+x^2}} - \frac{(d-x)^2}{(q^2+(d-x)^2)^{3/2})} \\
&\quad + \frac{1}{\sqrt{q^2+(d-x)^2}}.
\end{aligned}$$

Plugging $x = \frac{d^2-p^2+q^2}{2d}$ into this equation, we have

$$L\left(\frac{d^2-p^2+q^2}{2d}\right) = 8(p^2+q^2)\left(\frac{d^4+(p^2-q^2)^2+2d^2(p^2+q^2)}{d^2}\right)^{-3/2}$$

All of the values in this equation are positive, because each of the quantities is squared. Thus, the entire expression is positive. This means that the graph is concave up. Hence, this is a minimum.

Since this is a minimum, the light ray travels the least distance when $\alpha = \beta$. Fermat's principle guarantees the light ray will travel this path when coming from a source, P, reflecting off the surface at some point R, and continuing to pass through Q. Using Fermat's principle, we have derived the *law of reflection*: for reflection, the angle of incidence is equal to the angle of reflection (c.f. [3, p. 44]).

However, not all light rays are reflected from the water's surface. Some rays enter the water drop. These light rays are said to be refracted. We can also determine the path of these refracted rays. Consider figure 2, where P is the source of light. Notice that the light ray passes through some point Q, which is in the water, so that the ray crosses into the water at some point R. In the diagram, α will represent the angle of incidence, the angle the path PR makes with the line perpendicular to the water's surface. Also, β will represent the angle of refraction, the angle between the path RQ and the perpendicular (c.f. [3, p. 45]).

Again, consider Fermat's principle, which claims that the point R is positioned so as to make the total time of travel a minimum. To find this minimum, we will not represent this path as a function of length as we did in the derivation of the law of reflection. To do so would be to assume that the speed of light in air is equivalent to the speed of light in water, since

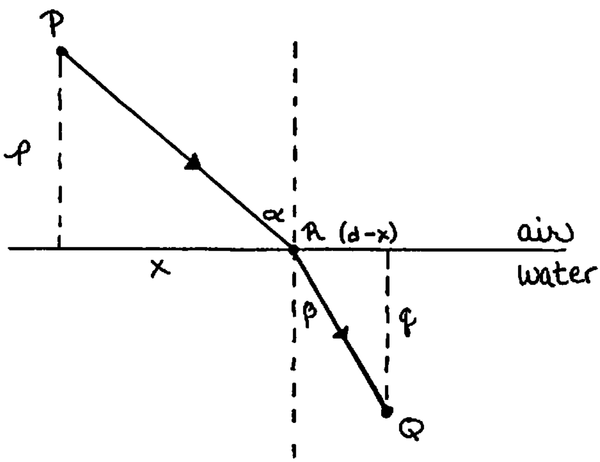

Figure 2

distance = rate × speed. Since the speed of the ray changes when it crosses into water, we should consider both speeds. Let a be the speed of light in air and w be the speed of light in water. Because time is distance divided by speed,

$$\frac{\sqrt{p^2+x^2}}{a}$$

represents the time it takes to travel path PR. Likewise,

$$\frac{\sqrt{q^2+(d-x)^2}}{w}$$

represents the time it takes to travel path RQ. Thus, the total time can be represented as

$$T(x)=\frac{\sqrt{p^2+x^2}}{a}+\frac{\sqrt{q^2+(d-x)^2}}{w}.$$

To find the minimum, set the derivative of $T(x)$ equal to 0 and solve, i.e.,

$$\frac{1}{a}\cdot\frac{x}{\sqrt{p^2+x^2}}-\frac{1}{w}\cdot\frac{d-x}{\sqrt{q^2+(d-x)^2}}=0$$

or

$$\frac{\sin\alpha}{a}-\frac{\sin\beta}{w}=0,$$

yielding

$$\frac{\sin\alpha}{\sin\beta}=\frac{a}{w}.$$

To verify that this is a minimum, let's examine the second derivative:

$$\begin{aligned}
T''(x) &= \frac{1}{w\sqrt{(q^2+(d-x)^2)}} - \frac{(d-x)^2}{w(q^2+(d-x)^2)^{3/2}} \\
&\quad - \frac{x^2}{a(p^2+x^2)^{3/2}} + \frac{1}{a\sqrt{p^2+x^2}} \\
&= \frac{1}{w}\cdot\frac{q^2+(d-x)^2-(d-x)^2}{(q^2+(d-x)^2)^{3/2}} - \frac{1}{a}\cdot\frac{x^2-(p^2+x^2)}{(p^2+x^2)^{3/2}} \\
&= \frac{1}{w}\cdot\frac{q^2}{(q^2+(d-x)^2)^{3/2}} + \frac{1}{a}\cdot\frac{p^2}{(p^2+x^2)^{3/2}}.
\end{aligned}$$

Because the variables are squared and a and w have positive values, $T''(x)$ will always be positive. Thus, no matter what value we examine for $x > 0$, $T(x)$ will be concave up. In other words, the graph will be concave up no matter what the distance the ray must travel. Thus, this equation reflects the minimum time it takes to travel from P to Q. Therefore, path PRQ is the shortest path.

Since this is the minimum time it takes to travel path PRQ, Fermat's principle guarantees that the light ray travels this path so that the ratio of the incidence angle to the angle of refraction equals some constant. This constant, $\frac{a}{w}$, is the ratio of the speed of light in air to the speed of light in water. Furthermore, this constant can be calculated. Data has been gathered that gives the ratio of the speed of light in an vacuum to the speed of light in various media. Now, the ratio of the speed of light in a vacuum, v, to the speed of light in water is approximately 1.33. So, $\frac{v}{w} = 1.33$. This is called the index of refraction for water. Similarly, the ratio of the speed of light in a vacuum to the speed of light in air is close to 1. Thus, $\frac{v}{a} = 1$, where 1 is the index of refraction for air. It follows that $\frac{a}{w} = 1.33$.

Therefore, we have derived the *law of refraction*: the ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant. Although we used Fermat's principle to obtain the law of refraction, Willebrord Snell experimentally discovered this law in 1621. Understandably, the law of refraction is often referred to as Snell's law [3, p. 46]. However, Descartes also discovered this result when he traced rays of light through a water drop. He published his work before Snell, who died before his work was published. Many were angered when Descartes claimed this discovery and refused to give recognition to his accomplishment. However, the law of refraction is referred to as Descartes' law in his home country of France [2, p. 2].

Notice that the derivation of this law is independent of direction. So, it does not matter if the source of light comes from P or from Q. With this in mind, once again consider what we have proven, $\frac{\sin\alpha}{\sin\beta} = \frac{a}{w}$. If the light travels from one medium to another, where the second medium has a higher index of reflection, the light is bent toward the line perpendicular to the surface between the two mediums. This perpendicular is often referred to as the normal. Remember, the index of refraction of a substance is the ratio of the speed of light in a vacuum to the speed of light in that particular substance. So, if the second medium has a higher refraction index, the numerator will be greater than the denominator. Thus, $\sin\alpha$ must be greater than $\sin\beta$. Since both angles are between 0 and $\frac{\pi}{2}$, it follows that α must be greater than β. Thus, it appears that the light ray bends toward the normal. Similarly, if light travels from one medium to another with a lower index of reflection, the light ray bends away from the normal. Knowing the index of refraction for both mediums will enable us to predict the path of the light rays.

Formation of the Primary Bow

In discussing the formation of the rainbow, we will utilize the principles of reflection and refraction. For a rainbow to be visible, the sun must be positioned behind you, such that its light shines on droplets of water in the atmosphere in front of you. This could be compared to a movie theater, where the sun acts as the film projector and the water droplets act as the movie screen [1, p. 47]. Rainbows form when the light interacts with these water drops in the atmosphere. Each time the light rays strike the surface of the drop, a portion of the rays are reflected, while the remaining rays are refracted. For clarification, examine figure 3.

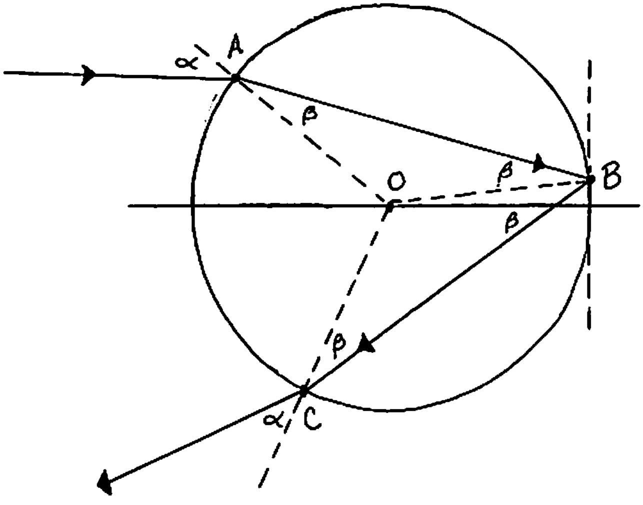

Figure 3

Now, the shape of a raindrop depends on several factors, but for a good approximation, it is fairly safe to assume that it is spherical [3, p. 46]. Thus, the circle centered at O will represent a cross-section of the drop of water. Suppose the sun is on the left side of this water drop. Then, light enters the drop at some point A. Some of the light rays are reflected off of the drop and do not interact with the water. Since those rays play no part in the formation of the rainbow, we will not consider them. We want to focus on the remaining rays that are refracted into the water. Since the refraction index of water, 1.33, is larger than that of air, approximately 1, these light rays will be bent toward the normal. We know from geometry that a circle's radius through a point on the circle is perpendicular to the circle's tangent at that particular point. Thus, radius OA represents the normal. We will again refer to α as the angle of incidence and β as the angle of refraction. After these rays are bent toward OA, they continue to travel through the circle until they encounter the other side of the drop at some point B. Again, a portion of the rays are reflected back into the drop, while the remaining rays are refracted into the air. We will not see the light that is refracted back into the atmosphere, because we are on the left-hand side of the drop. So, we will follow the reflected rays. From the law of reflection, the angle of incidence equals the angle of reflection. Thus, $\angle ABO$ is equal to $\angle OBC$. The light continues until it strikes the drop again at point C. Once again, part of the light is reflected back into the drop and part is refracted into the air. The rays that are refracted into the atmosphere are bent away from the normal, OC, since air has a lower refractive index than water.

This is only one of the many paths light rays travel inside the water drop. The rays could continue to bounce around inside the sphere. However, each time the rays strike a surface, part of the rays are reflected and part are refracted. After each interaction, the light is less intense than the original ray. Thus, we want to consider the rays that strike the interface the least amount of times to examine the brightest light. Therefore, we have traced the simplest path that contributes to the rainbow.

If any light ray that was reflected once inside the water drop produced a rainbow, we would see an infinite number of rainbows in the sky. In fact, we will see that a light ray which enters the drop at a special angle will contribute to the formation of the rainbow. To explore this issue, we should examine the deflection of the light ray.

We have said that the light enters the drop at some point A. Now, since we picked the sun to be on the left side of the drop, point A could be anywhere on the left half of the circle. If the ray enters on the upper half of the circle, it will exit on the lower half. Likewise if it enters on the lower half, it will exit on the upper half. Because the upper and lower halves of the circle are symmetrical, we can focus on only the upper half. For rays that enter on the upper-left quadrant of the circle, α will range from 0 to 90 degrees.

We are interested in how much the ray is deflected once it exits the drop. The idea of deflection is illustrated in figure 4. The ray enters the drop at S and emerges at some point E. Angle D represents the angle of deflection, or the measure of deviation of the emergent ray from its original direction [4].

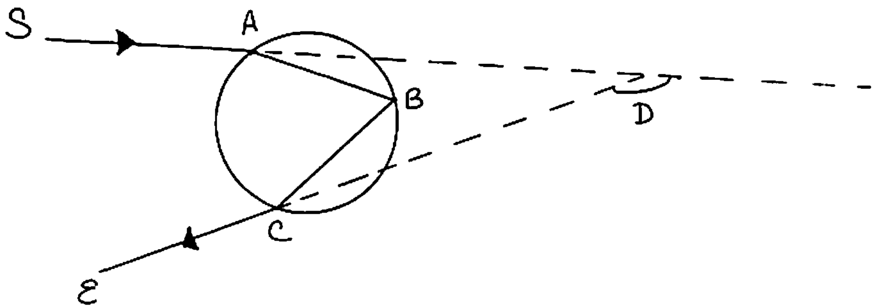

Figure 4

To demonstrate the idea of deflection, consider the drop that enters along the diameter of the circle. Its angle of incidence is zero. By the law of refraction, its angle of refraction is also zero. Now, this ray continues to be reflected directly off the back of the drop. Thus, the ray exits along the same diameter that it entered on [3, p. 48]. The total deflection would be 180 degrees.

As point A moves around the circle, the angle of incidence changes. As the angle of incidence changes, the deflection angle also changes. Thus, the angle of deflection can be represented as a function of the incidence angle. Once again, consider the path this ray follows.

We see from figure 3 that once the ray enters the drop, it does not continue in a straight line. We know from the law of refraction that the ray is bent. If we consider the angle vertical to be α and subtract angle β, we can see the ray has rotated clockwise $\alpha - \beta$ degrees. The ray continues until it is reflected off the back of the drop. If we imagine the ray continuing in a straight line and subtract 2β, the angle of incidence and angle of refraction, we are left with θ. In short, the ray is rotated θ, or $180 - 2\beta$ degrees. At point C, the ray is again refracted. Following the law of refraction, the ray is bent away from the normal. If we consider the angle α and subtract the angle vertical to β, we see the ray has been rotated clockwise $\alpha - \beta$ degrees. Now, consider the deflection angle as a function of the angle of incidence where $D(\alpha) = \alpha - \beta + 180 - 2\beta + \alpha - \beta$, or $D(\alpha) = 180 + 2\alpha - 4\beta$. As $D(\alpha)$ is written here, it is a function of both α and β. However, β can be expressed as a function of α. Recall the law of refraction, where $\frac{\sin\alpha}{\sin\beta} = \frac{a}{w}$, or $\frac{\sin\alpha}{\sin\beta} = 1.33$.

We have already said when $\alpha = 0$, $D(\alpha) = 180$. As α increases, the angle of deflection at first decreases. But, we will see that $D(\alpha)$ has a minimum. After it reaches that minimum, the angle of deflection increases. We can determine this minimum by taking the derivative of $D(\alpha)$ with respect to α:

$$D'(\alpha) = 2 - 4\frac{d\beta}{d\alpha}.$$

Recall that

$$\frac{\sin\alpha}{\sin\beta} = \frac{a}{w}.$$

Let's represent $\frac{a}{w}$ by k. So,

$$\sin\alpha = k\sin\beta.$$

Again, differentiate with respect to α to get

$$\cos\alpha = k\cos\beta\frac{d\beta}{d\alpha}.$$

Thus,

$$\frac{d\beta}{d\alpha} = \frac{\cos\alpha}{k\cos\beta}.$$

Substituting into $D'(\alpha)$, we have

$$D'(\alpha) = 2 - 4\frac{\cos\alpha}{k\cos\beta}.$$

To find the critical value, we will set the first derivative equal to zero. Thus,

$$2 - 4\frac{\cos\alpha}{k\cos\beta} = 0,$$

or

$$\frac{k}{2} = \frac{\cos\alpha}{\cos\beta}.$$

We are solving for α and want to eliminate β, which can be expressed in terms of α. Squaring both sides of the equation yields

$$\frac{k^2}{4} = \frac{\cos^2\alpha}{\cos^2\beta}.$$

From trigonometry, $\cos^2\beta = 1 - \sin^2\beta$. Substituting this into the equation gives us

$$\frac{k^2}{4} = \frac{\cos^2\alpha}{1 - \sin^2\beta}.$$

But, remember $\sin\beta = \frac{1}{k}\sin\alpha$. Substituting,

$$\frac{k^2}{4} = \frac{\cos^2\alpha}{1 - \frac{\sin^2\alpha}{k^2}}.$$

Now, multiplying this equation by $\frac{1}{k^2}$ gives us

$$\frac{1}{4} = \frac{\cos^2\alpha}{k^2 - \sin^2\alpha}.$$

Again, substitute the trigonometric identity $\sin^2\alpha = 1 - \cos^2\alpha$ to get

$$\frac{1}{4} = \frac{\cos^2\alpha}{k^2 - (1 - \cos^2\alpha)}.$$

Now, cross-multiply to get

$$k^2 - 1 + \cos^2\alpha = 4\cos^2\alpha.$$

To solve for α,

$$3\cos^2\alpha = k^2 - 1.$$

Thus,

$$\cos\alpha = \sqrt{\frac{k^2 - 1}{3}}.$$

This gives us the cosine of a critical incidence angle. In the formation of the primary bow, we have said $k = 1.33$. So, $\cos\alpha \approx .5063$. Thus,

$$\alpha \approx 59.58°.$$

Now, if $\alpha \approx 59.58°$, then

$$\frac{\sin 59.58°}{\sin\beta} = 1.33.$$

So,

$$\beta \approx 40.42°.$$

Now, we can substitute these values into $D(\alpha)$:

$$D(59.58) = 180 + 2(59.58) - 4(40.42) = 137.48.$$

So, when a light ray enters a raindrop at an angle of 59.58°, the ray is deflected 137.48° when it leaves the drop. To prove this is the minimum

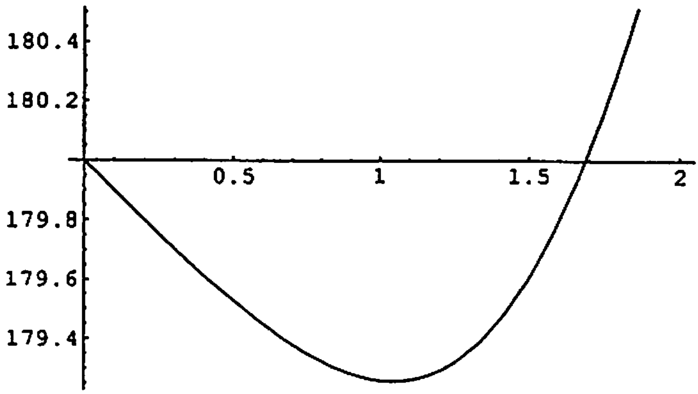

Figure 5

deflection, and not a maximum or inflection point, we should plug the critical values into the second derivative of $D(\alpha)$.

Using *Mathematica 3.0*, we can determine

$$D''(\alpha) = -\frac{4\cos^2\alpha\sin\alpha}{k^3\cos^3\beta} + \frac{4\sin\alpha}{k\cos\beta}.$$

When we substitute $\alpha = 59.58°$, $\beta = 40.42°$ and $k = 1.33$, then $D''(\alpha) = 2.55513$. Notice that this value is positive, thus the graph is concave up. At $\alpha = 59.58°$ the first derivative of $D(\alpha)$ is zero while the second derivative is positive. Thus, this represents the minimum of $D(\alpha)$.

Examine the graph of $D(\alpha)$ in figure 5. The graph is in radians. By looking at the graph we can tell that the minimum occurs at approximately 1.04 radians or $\alpha = 59.58°$.

Thus, any ray that enters the droplet at 59.58° is deflected from the droplet the least, and $D'(\alpha) = 0$ at $\alpha = 59.58°$. Thus, the difference equation $\frac{\Delta D(\alpha)}{\Delta\alpha}$ is small in magnitude for all α near $\alpha_0 = 59.58°$. This means that there is not much change in the deflection angle of the rays whose incidence angles are near 59.58°. In other words, the light rays that enter the raindrop near 59.58° get deflected by about the same amount [3, p. 49]. Rays that are not near this special angle are spread out more when they exit the drop. It makes sense that these rays with α near 59.58° are the brightest and most visible, since these are the highest concentration of rays deflected from the drop. Now, these rays together are called the *rainbow ray*. When the observer looks at the rainbow, he actually sees these special rays that enter the drop near 59.58° and are deflected 137.48°. Together, these rays form the rainbow's band of light.

The rainbow ray is often referred to as the Descartes' ray for Rene' Descartes, who discovered it in 1637 [2, p. 3]. In his laboratory experiments, Descartes followed each of these rays. He was aware that rainbows could be made artificially in sprays of water and that small water drops are spherical [6, p. 42]. Figure 6 illustrates Descartes' findings. The lines represent the paths the rays of light follow when they enter a drop of water. Notice the concentration of deflected rays. These rays form the rainbow ray.

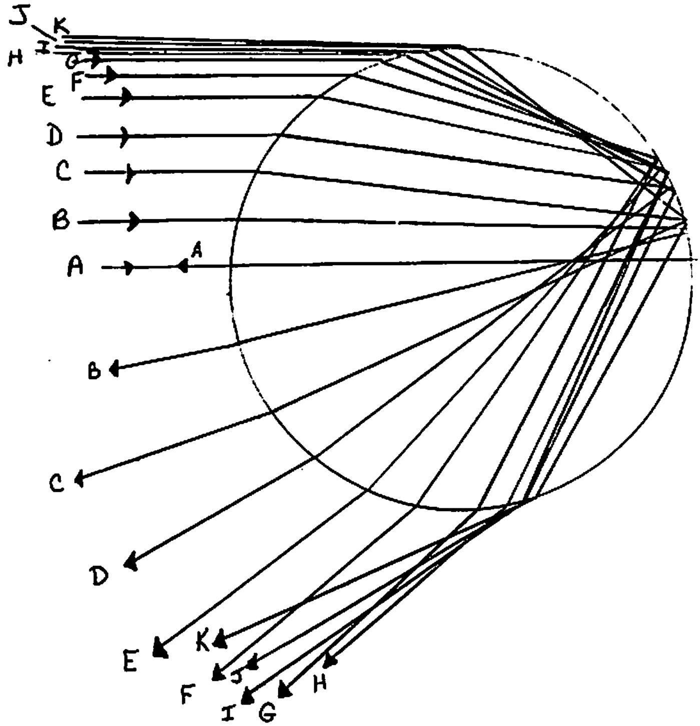

Figure 6

If the observer looks up at an angle of 42.52°, his eyes should meet the rainbow ray. Examine figure 7. Now, a few of these rays will enter the drop near 59.58°. These rays will be deflected 137.48°. Consider the sun's rays to shine horizontally. Now, if the sun could shine directly through the diameter of the drop, the rays would form a straight line. However, we are examining the rays that are deflected 137.48°. Thus, the supplementary angle of 42.52° is formed. This angle is called the *rainbow angle*. The rainbow angle is the angle between the rainbow ray and the rays entering the drop directly from the sun. Since the light rays entering the drop are parallel to the ground, alternate interior angles are formed. Thus, the rainbow angle is congruent to the angle formed by the rainbow ray and the ground. Therefore, the observer should look up at an angle of 42.52° to see

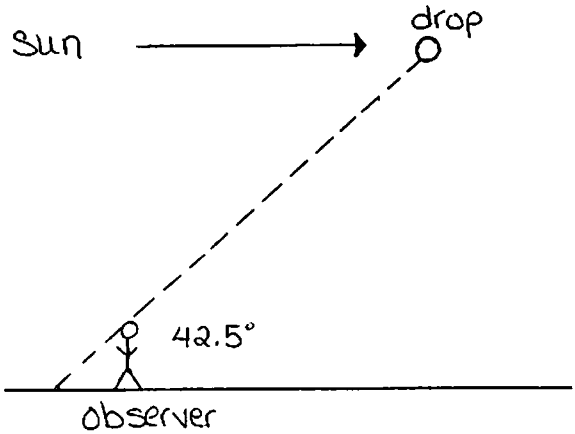

Figure 7

the rainbow. For clarification, examine figure 7.

It may be surprising that the drops inclined 42.5° from the observer are the only drops that contribute to the rainbow. However, as long as the rays are horizontal, the drops which are inclined 42.5° will appear brighter than drops viewed from a lower angle. For the observer to view the light deflected from drops inclined at a lower angle, the angle of deflection must be greater than 137.5°. As we have already discussed, rays that are deflected more than 137.5° are not as bright as those deflected near 137.5°. Now, drops higher in the sky must have an angle of deflection less than 137.5°. However, 137.5° is the minimum deflection for rays deflected once internally. Therefore, no such rays exists unless they were not reflected or reflected more than once inside the drop.

Experience tells us that not every rainbow is viewed 42.5° from the ground. This is because the sun's rays are not always horizontal. Thus, they are not always parallel to the ground. Suppose we view a rainbow at an angle of 25° degrees from the horizontal. This scenario could easily take place if the sun is inclined. The rays we see are still deflected from the drops at an angle of 137.5°; the rainbow angle is still 42.5°. The question remains — what is the sun's inclination? We can still consider the horizontal line parallel to the ground to form alternate interior angles. Since we see the rainbow at an angle of 25°, the angle between the horizontal and rainbow ray is 25°. We are left with an angle of 17.5° between the sun's rays entering the drop and the horizontal. Thus, the sun is inclined 17.5°.

We know that if the sun's rays are parallel to the ground, we should look up at an angle of 42.5° to see the rainbow. However, even if the sun is inclined, we can still easily find a rainbow if it exists. We must simply look at an angle of 42.5° away from the *antisolar point.* The antisolar point lies on the line from the sun passing through the eye of the observer. Since the sun is behind us, it will cast our shadow onto the ground. The shadow of the observer's head marks the antisolar point. If we look at this point, our line of vision will be almost parallel to the sun's rays. We must simply look 42.5° away from the antisolar point for our eyes to catch the rainbow ray and see the rainbow. For clarification, examine figure 8.

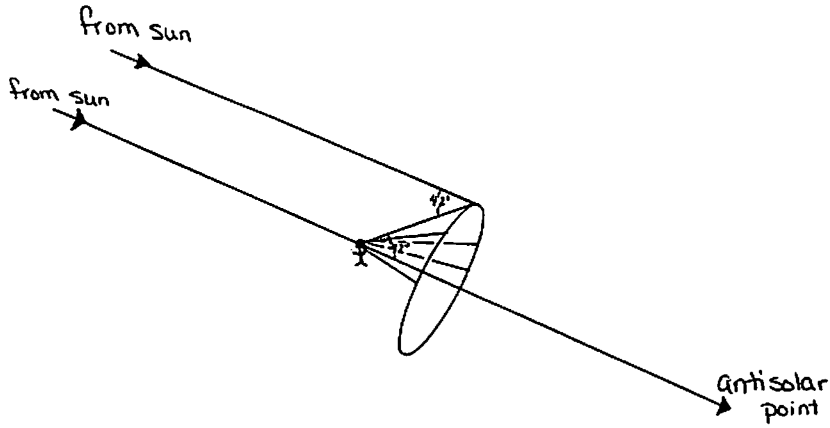

Figure 8

Perhaps there is still confusion why the rainbow is in a circular arc. To illustrate, imagine a cone with vertex angle equal to twice the rainbow angle. Now, if you stand at the vertex of the cone and a plane is cut perpendicular to its axis, you will see a circular cross-section. It is the drops on this cross-section that form the rainbow [3, p. 50]. You could also consider yourself a painter. Now, suppose your hand had to stay in a fixed position and the brush had to remain at a 42.5° angle. The only figure you could draw would be a circle. Note, in this illustration, your hand represents the antisolar point [1, p. 48]. In actuality, the rainbow is a circle. However, the horizon usually intercepts the bow so that all the observer sees is the arc.

Colors of the Rainbow

We have discussed the rays that form the bow and the shape of the bow. We have yet to explain the origin of the colors of the bow. Light is actually an "electromagnetic wave" [3, p. 50]. It is made up of many wavelengths. However, our eyes are only sensitive to wavelengths ranging from 7000 to 4240 angstroms. Red light has a wavelength of about 6470 to 7000 angstroms; light with 4000 to 4240 is perceived as violet. All other colors fall between these two. Since the wavelengths of these light rays are different, the refractive index of water varies depending on which color of light passes through it [3, p. 50].

The refractive index is around 1.3318 when red light with wavelength 6563 angstroms travels from air to water. We can calculate the angle of deflection and rainbow angle for this ray of light. First, we must calculate α. Recall that

$$\cos\alpha = \sqrt{\frac{k^2-1}{3}}.$$

We have said $k = 1.3318$. Thus $\cos\alpha = .5078$, and $\alpha = 59.48°$. We also need the value for β. Since $\frac{\sin\alpha}{\sin\beta} = k$, we have $\frac{\sin 59.48°}{\sin\beta} = 1.3318$, $\sin\beta = .6468$, and $\beta = 40.3°$. We can use these two values to find the angle of deflection:

$$D(\alpha) = 180 + 2\alpha - 4\beta = 180 + 2(59.48) - 4(40.3) = 137.75.$$

Thus, the rainbow angle is $(180 - 137.75)°$ or $42.25°$.

Now the refraction index for violet light with wavelength 4047 angstroms is about 1.3435. We can also determine the angle of deflection and rainbow angle for these rays. We have $\cos\alpha = .518$, and $\alpha = 58.8°$. Also, $\sin\beta = .6367$, and $\beta = 39.54°$. We substitute these values to compute the angle of deflection:

$$D(\alpha) = 180 + 2\alpha - 4\beta = 180 + 2(58.8) - 4(39.54) = 139.42.$$

So, the angle of deflection is $139.42°$. The rainbow angle is $(180 - 139.42)°$ or $40.58°$.

In short, the observer must look at an angle of $42.25°$ from the horizontal or antisolar point to see the red ray of the rainbow. However, he must look at an angle of $40.58°$ to see the violet ray of the rainbow. Obviously, the red ray is above the violet ray. The other colors of the rainbow have refractive indices that cause them to fall between these two rays. Thus, the colors of the primary rainbow are always ordered red, orange, yellow, green, blue, indigo, and violet. It is interesting to note that Isaac Newton was the first to make these calculations to explain the systematic order of the colors of the rainbow [3, p. 51].

The Secondary Bow

Occasionally, a second rainbow, called the *secondary bow*, is visible just outside the primary bow. Its colors are much fainter than the primary bow and appear in reverse order. This bow is formed when light rays are reflected twice inside the water drop. When we traced the path of the light ray in the primary bow, we only allowed one reflection off the back of the drop. We said that this ray would produce the brightest light ray to form the rainbow. However, light rays that are reflected twice within the water

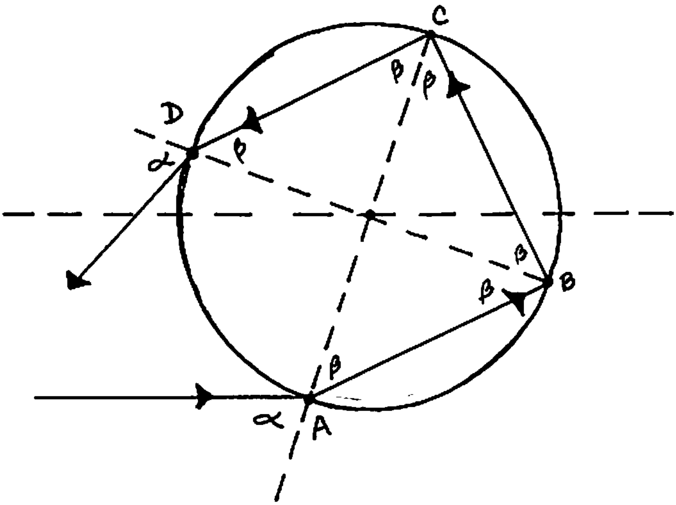

Figure 9

drop are occasionally visible. These rays form the secondary bow. Let's track this ray; see figure 9.

Notice the light ray enters the bottom of the drop. This is the ray that will reach the observer. We can again express the total deflection as a function of α. The path this ray travels is very similar to the primary rainbow's ray. However, there are a few differences. This ray is reflected twice within the drop. So, when we derive the formula for $D(\alpha)$, we must add another $180 - 2\beta$ to compensate for the second reflection. Thus, the total deflection is represented by

$$(\alpha - \beta) + (180 - 2\beta) + (180 - 2\beta) + (\alpha - \beta)$$

or

$$360 - 2\alpha - 6\beta.$$

Since a deflection of 360 degrees sends the ray in the same direction it started, we can disregard this term and represent the total deflection as

$$2\alpha - 6\beta.$$

The second difference between the paths of light in the formation of the primary and secondary rays is that the light travels clockwise in the water drop of the primary bow, while it travels counter-clockwise in the drop of the secondary bow. In order to compare this ray with the original ray we examined, we can multiply the total deflection by -1 which will cause the deflections to occur clockwise. Thus, the deflection function can be represented by

$$D_2(\alpha) = 6\beta - 2\alpha.$$

Again, when $\alpha = 0$, the ray enters the drop and is reflected. The law of reflection says that in this case $\alpha = \beta$. So, $\beta = 0$. Thus, $D_2(0) = 0$. However, as α increases $D_2(\alpha)$ also increases. To determine if D_2 increases infinitely with α, we should find the critical points by setting the derivative equal to zero:

$$D_2'(\alpha) = 6\frac{d\beta}{d\alpha} - 2 = 0.$$

Recall that

$$\frac{d\beta}{d\alpha} = \frac{\cos\alpha}{k\cos\beta}.$$

Substituting, we have

$$D_2'(\alpha) = \frac{6\cos\alpha}{k\cos\beta} - 2 = 0.$$

To solve for α, we obtain

$$\frac{6\cos\alpha}{\cos\beta} = 2k,$$

or

$$\frac{\cos\alpha}{\cos\beta} = \frac{k}{3}.$$

Squaring both sides,

$$\frac{\cos^2\alpha}{\cos^2\beta} = \frac{k^2}{9}.$$

Substituting the trigonometric identity $\cos^2\beta = 1 - \sin^2\beta$ implies

$$\frac{\cos^2\alpha}{1 - \sin^2\beta} = \frac{k^2}{9}.$$

However, $\sin\beta = \frac{1}{k}\sin\alpha$. Substituting,

$$\frac{\cos^2\alpha}{1 - \frac{\sin^2\alpha}{k^2}} = \frac{k^2}{9},$$

or

$$\frac{\cos^2\alpha}{\frac{k^2 - \sin^2\alpha}{k^2}} = \frac{k^2}{9}.$$

If we multiply both sides of this equation by $\frac{1}{k^2}$, we get

$$\frac{\cos^2\alpha}{k^2 - \sin^2\alpha} = \frac{1}{9}.$$

Now, substitute $\sin^2\alpha = 1 - \cos^2\alpha$ to get

$$\frac{\cos^2\alpha}{k^2 - 1 + \cos^2\alpha} = \frac{1}{9}.$$

We can cross-multiply to get

$$9\cos^2\alpha = k^2 - 1 + \cos^2\alpha.$$

When we simplify we find

$$\cos\alpha = \sqrt{\frac{k^2-1}{8}}.$$

We should determine if this critical value is a minimum, maximum or inflection point. Since $k = 1.33$, $\cos\alpha = .3100$. Thus, $\alpha = 71.94°$. It follows that $\sin\beta = \frac{\sin 71.94°}{1.33}$. Thus, $\beta = 45.63°$. Now, we can substitute these values into the second derivative of $D(\alpha)$. Using *Mathematica 3.0*, we find

$$D_2''(\alpha) = \frac{6\cos^2\alpha\sin\alpha}{k^3\cos^3\beta} - \frac{6\sin\alpha}{k\cos\beta}.$$

When we substitute $\alpha = 71.94°$, $\beta = 45.63°$, and $k = 1.33$, we find that $D_2''(\alpha) = -5.4519$. This tells us that the graph of $D_2''(\alpha)$ is concave down at the critical value $\alpha = 71.94°$. Thus this point is a maximum. We can graph $D_2''(\alpha)$ (see figure 10). Again the graph is in radians. Notice the maximum appears to occur at 1.26 radians or 71.94°.

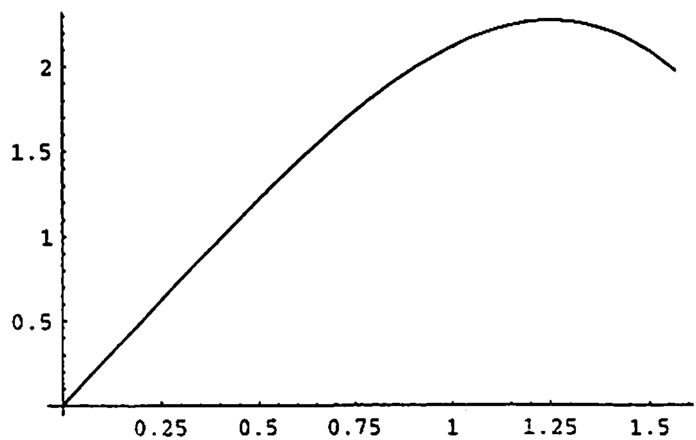

Figure 10

When we substitute these values into D_2, we find

$$D_2(71.94) = 6(45.63) - 2(71.94) = 129.9.$$

This gives us the maximum angle of deflection.

Thus, the rainbow ray of the secondary rainbow is deflected about 129.9°. Notice in figure 10 that the maximum deflection is about 2.3 radians or 130 degrees. Thus, the rainbow angle is $(180 - 129.9)°$ or 50.1°. To see the secondary rainbow, the observer should look up at an angle of 50.1° from the antisolar point.

Colors of the Secondary Bow

We have proven that D_2 is concave down. Thus we are discussing maximums instead of minimums. We will find that the red light rays of the secondary bow are deflected more than the violet light rays instead of less.

Recall that the refraction index for the red light ray is 1.3318. Thus,

$$\cos\alpha = \sqrt{\frac{1.3318^2 - 1}{8}} = .3109.$$

Thus, $\alpha = 71.88°$. Also,

$$\sin\beta = \frac{\sin 71.88°}{1.3318},$$

and $\beta = 45.53°$. We should substitute this into $D_2(\alpha)$ to find the angle of deflection:

$$D_2(\alpha) = 6(45.53) - 2(71.88) = 129.43.$$

The angle of deflection for red light is 129.43°. Thus, it can be seen at an angle of $(180 - 129.43)°$ or 50.57°.

We can do a similar computation to determine the reflection angle of the violet ray. If $k = 1.3435$, we will find $\alpha = 71.5055°$ and $\beta = 44.9°$. Substituting these values into $D_2(\alpha)$, we find that the angle of deflection is approximately 126.4°. It follows that the angle the observer should look to see the violet ray of the secondary bow is 53.62°. Since the observer must look at a greater angle to see the violet ray in the secondary bow, it is obvious that the violet ray appears higher than the red ray in the secondary bow. Again, the other colors fall between red and violet. Thus, the colors of the secondary bow appear in reverse order than those of the primary bow.

Rays with n Internal Reflections

As stated earlier in the paper, light rays can be reflected any number of times inside the water drop. Theoretically, each of these classes of rays forms another rainbow [3, p. 53]. Unfortunately, these bows are usually too dim to be seen unless in a special laboratory set-up. However, we can still derive the deflection function and critical points for these rays.

First, let's trace the path of these rays. Experience, along with common sense, tells us any ray will be deflected by an angle of $\alpha - \beta$ when it enters and exits the drop. Also, each time the ray is reflected off the drop, the ray is deflected by $180° - 2\beta$. This gives us the deflection function,

$$D_n = 2(\alpha - \beta) + n(180 - 2\beta).$$

Obviously, this function holds true for the two cases with which we have worked, one internal reflection and two internal reflections. We can also find the critical points of this function:

$$\begin{aligned} D_n'(\alpha) &= 2 - 2(n+1)\frac{d\beta}{d\alpha} \\ &= 2 - 2(n+1)\frac{\cos\alpha}{k\cos\beta}. \end{aligned}$$

Again, we should set this derivative equal to zero:

$$2 - 2(n+1)\frac{\cos\alpha}{k\cos\beta} = 0.$$

When we simplify, we see that

$$\frac{k}{n+1} = \frac{\cos\alpha}{\cos\beta}.$$

Squaring both sides,

$$\frac{k^2}{(n+1)^2} = \frac{\cos^2\alpha}{\cos^2\beta}.$$

Recall that $\cos^2\beta = 1 - \sin^2\beta$; thus

$$\frac{k^2}{(n+1)^2} = \frac{\cos^2\alpha}{1 - \sin^2\beta}.$$

However, $\sin\beta = \frac{1}{k}\sin\alpha$. Substituting,

$$\frac{k^2}{(n+1)^2} = \frac{\cos^2\alpha}{1 - \frac{\sin^2\alpha}{k^2}}$$

or

$$\frac{k^2}{(n+1)^2} = \frac{\cos^2\alpha}{\frac{k^2 - \sin^2\alpha}{k^2}}.$$

Multiply through by $\frac{1}{k^2}$, and we get

$$\frac{1}{(n+1)^2} = \frac{\cos^2\alpha}{k^2 - \sin^2\alpha}.$$

Now, substitute $\sin^2\alpha = 1 - \cos^2\alpha$ to get

$$\frac{1}{(n+1)^2} = \frac{\cos^2\alpha}{k^2 - 1 + \cos^2\alpha}.$$

We can cross multiply to get

$$(n+1)^2\cos^2\alpha = k^2 - 1 + \cos^2\alpha.$$

When we simplify we find

$$\cos\alpha = \sqrt{\frac{k^2-1}{(n+1)^2-1}}.$$

Again, we can see that this holds for the examples we have worked where $n = 1$ and $n = 2$.

Conclusion

We have seen that rainbows are the amazing result of an interaction between sunlight and water. In fact, the path light follows when it encounters a water droplet is essential to the formation of the rainbow. Each time light interacts with water, a portion of the light is reflected, while the remaining rays are refracted back into the water drop. We developed and examined the laws of reflection and refraction. Then, we used these laws to determine the path of light in a water drop. We found the first and second derivatives of this path. We were then able to find the ray with the minimum angle of deflection. These rays contribute to the rainbow ray, which forms the rainbow's band of light. Knowing the rainbow ray, we found the rainbow angle, which gives rise to the bow's circular arc. We then finalized our exploration of the primary bow by examining the separation of light into distinct colors. Finally, we followed these same procedures to explore the formation of the secondary bow. It is fascinating that a phenomenon that has intrigued mankind for years can be explained with mathematics such as calculus, geometry, and trigonometry.

Acknowledgements. I would like to thank Dr. Matt Lunsford for his advice on this project.

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In Memoriam: Sister Helen Sullivan

Sister Helen Sullivan, former KME national historian and editor of Kappa Mu Epsilon News from 1943 to 1947, died December 22, 1998 at the age of 91. She earned many honors in her life, including being named a "Distinguished Member" at the fiftieth anniversary celebration of Kappa Mu Epsilon in 1981. Sister Helen was founding faculty sponsor of the Kansas Gamma chapter of Kappa Mu Epsilon at what was then called Mount St. Scholastica College (now Benedictine College) in 1940, and remained active with the chapter for several decades.

Sister Helen Sullivan was born Monica Elizabeth Sullivan on April 10, 1907, in Effingham, Kansas. She obtained an AB degree from St. Benedict's College, and a master's degree in physics and doctorate in mathematics from the Catholic University of America. Though she served most of her career at Mount St. Scholastica, she was very well-traveled and spent leaves in several different positions around the country and the world, including a sabbatical at the University of Minnesota as part of a geometry writing team. She was active in many professional organizations, and was honored by the Smithsonian Institution for being one of the first women to earn a doctorate in mathematics. She also spent three of her later years working in the Jesuit School of Theology in Berkeley, California.

Sister Helen Sullivan's name can be found in the very first issue of *The Pentagon*, Fall 1941, as faculty sponsor of Kansas Gamma. Besides serving as editor of KME News 1943–1947, she also served as associate editor for installation of new chapters, 1961–1970. A great number of her students had papers published in this journal in the 1950's and 1960's as prize-winners at KME national conventions, including a sweep of the top three prizes in 1965.

A picture of Sister Helen can be found on page 22 of the Fall 1990 issue of *The Pentagon*.

Boundary Value Problems in Hollow Rectangular Beams

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Presented at the 1998 Region IV Convention and awarded "top three" status by the Awards Committee.

In this paper, we examine the mathematics in the vibration problems developed in Samuel French's paper entitled "Beam Vibration Problems in Hollow Rectangular Towers" [1]. There are three cases which have been investigated.

The first case is the simplest case in which only the effects of flexure are considered, neglecting the effects of the physically present rotary inertia and shear. The partial differential equation for this case as presented in [2] and slightly altered for consistency is:

$$w_{xxxx} + \frac{m}{EI} w_{tt} = 0. \tag{1}$$

(The subscript notation is used to denote the partial derivative with respect to the subscript.) This simple case is only an approximation of the actual behavior. To improve this approximation, one would include the effects of rotary inertia in addition to flexure. This complicates the partial differential equation by adding an additional term, namely,

$$v_{xxxx} - \frac{mr^2}{EI} v_{xxtt} + \frac{m}{EI} v_{tt} = 0. \tag{2}$$

This, too, is just an approximation. The most accurate equation is also the most complex. The inclusion of shear along with flexure and rotary inertia gives the most accurate results. The equation again has an additional term and becomes

$$u_{xxxx} - \frac{mr^2}{EI}\left[1 + \frac{2(1+\mu)}{\kappa}\right] u_{xxtt} + \left[\frac{mr^2}{EI}\right]^2 \frac{2(1+\mu)}{\kappa} u_{tttt} + \frac{m}{EI} u_{tt} = 0. \tag{3}$$

The simplifications required for the first case (equation (1)), that included only the effects of flexure, are limiting. The case involving both rotary inertia and flexure (equation (2)) is a more accurate approximation. Also, the solution of equation (3), which includes the effects of flexure, rotary inertia, and shear, is contingent upon the results of solving equation (2). Therefore, the focus of our examination is the second case considering the effects of both flexure and rotary inertia.

The positive constants in the equations are as follows:

$$
\begin{aligned}
m &= \text{mass per unit length;}\\
\mu &= \text{Poisson's Ratio;}\\
E &= \text{Young's modulus;}\\
\kappa &= \text{shear conversion factor (5/6 for rectangles);}\\
I &= \text{inertia;}\\
r &= \text{radius of gyration.}
\end{aligned}
$$

The variable x is used to describe the spatial displacement of the tower while t is used for time.

The solution of the partial differential equation (2) is obtained by utilizing the technique of separation of variables. In order to do so, we assume that the function $v(x,t)$ can be written as the product of two single-variable functions $Y(x)$ and $T(t)$; that is, $v(x,t) = Y(x)T(t)$. This assumption allows us to take the partial derivatives more freely. Equation (2) requires three distinct partial derivatives:

$$
\begin{aligned}
v_{xxxx} &= Y^{(4)}(x)T(t)\\
v_{xxtt} &= Y''(x)T''(t)\\
v_{tt} &= Y(x)T''(t).
\end{aligned}
$$

The primes are used to denote the derivative with respect to the variable in parentheses. The $Y^{(4)}(x)$ denotes the fourth derivative of Y with respect to x. Inserting these forms of the partial derivatives into equation (2) gives us

$$Y^{(4)}(x)T(t) - \frac{mr^2}{EI}Y''(x)T''(t) + \frac{m}{EI}Y(x)T''(t) = 0.$$

Moving two terms to the right side of the equality and dividing through by the time function gives

$$Y^{(4)}(x) = \left(\frac{mr^2}{EI}Y''(x) - \frac{m}{EI}Y(x)\right)\frac{T''(t)}{T(t)}.$$

In order to separate the variables, we isolate the time function by dividing through by the spatial portion within the large parenthesis. The separation of variables techniques requires that the two isolated portions of the separated equation be equal to a separation constant. Here we define this separation constant as $-\omega^2$ for convenience. This results in the equation

$$\frac{Y^{(4)}(x)}{\frac{mr^2}{EI}Y''(x) - \frac{m}{EI}Y(x)} = \frac{T''(t)}{T(t)} = -\omega^2.$$

This provides us with two distinct, ordinary differential equations. First we set the time functions ratio equal to the separation constant. Then we multiply both sides by the denominator and subtract the right side of the equation from both sides creating the familiar ordinary differential equation

$$T''(t) + \omega^2 T(t) = 0.$$

Thus we have

$$T(t) = G\cos(\omega t) + H\sin(\omega t)$$

as the general solution. However, our interest lies in the spatial differential equation.

Using the same procedure, we arrive at the following fourth-order, ordinary differential equation governing the spatial aspects of the vibration problem:

$$Y^{(4)}(x) + \frac{m\omega^2}{EI}r^2 Y''(x) - \frac{m\omega^2}{EI}Y(x) = 0. \tag{4}$$

In order to simplify the mathematics, we let

$$\lambda = \frac{m\omega^2}{EI}r^2.$$

This leads to the simplified form of the ordinary differential equation

$$Y^{(4)}(x) + \lambda Y''(x) - \frac{\lambda}{r^2}Y(x) = 0. \tag{5}$$

The next issue is the boundary conditions that complete our problem. Using French's analysis based on Timoshenko's text [3], we have both the deflection and slope equal to zero at the end where x is equal to zero. At the top of the tower, or in our case when x is equal to one, the moment is also zero. The final condition (when the shearing force is equal to zero) involves a relationship between the third and first derivatives of Y. Hence, we consider the following boundary conditions:

$$\begin{aligned} Y(0) = Y'(0) &= 0 \\ Y''(1) &= 0 \\ Y'''(1) + \lambda Y'(1) &= 0. \end{aligned} \tag{6}$$

Equation (5) and the boundary conditions (6) are in the special class of boundary value problems known as eigenvalue problems. Consequently, the eigenvalue problem describing the spatial portion of the vibration problem for a hollow rectangular tower when neglecting the effects of shear and considering only flexure and rotary inertia is:

$$(7) \qquad Y^{(4)}(x) + \lambda Y''(x) - \frac{\lambda}{r^2}Y(x) = 0, \qquad \begin{aligned} Y(0) = Y'(0) &= 0 \\ Y''(1) &= 0 \\ Y'''(1) + \lambda Y'(1) &= 0. \end{aligned}$$

We shall refer to λ as the eigenvalue. It is interesting to note that the eigenvalue appears in two of the terms in the differential equation as well as in one of the boundary conditions.

For the eigenvalue problem (7), we first prove that the eigenvalues are both positive and real. To prove that the eigenvalues must be positive and real we assume that the function is complex and multiply equation (5) through by the conjugate of the function ($\overline{Y}(x)$) and integrate both sides over the interval $[0,1]$:

$$\int_0^1 \overline{Y}(x)Y^{(4)}(x) + \lambda\overline{Y}(x)Y''(x) - \frac{\lambda}{r^2}\overline{Y}(x)Y(x)\,dx = 0.$$

Integrating by parts twice, applying the boundary conditions, and solving for λ, we see that λ must be both positive and real since

$$\lambda = \frac{\int_0^1 |Y''(x)|^2\,dx}{\int_0^1 |Y'(x)|^2\,dx + \frac{1}{r^2}\int_0^1 |Y(x)|^2\,dx}, \qquad |Y(x)|^2 = Y(x)\overline{Y}(x).$$

By taking λ equal to zero in (7), we obtain only the trivial solution and so λ must be positive and real.

We let the characteristic equation for (5) be denoted by

$$s^4 + \lambda s^2 - \frac{\lambda}{r^2} = 0.$$

Using the quadratic formula to solve for s^2, we get

$$s^2 = \frac{1}{2}\left(-\lambda \pm \sqrt{\lambda^2 + 4\frac{\lambda}{r^2}}\right)$$

and

$$s = \pm\sqrt{\frac{1}{2}\left(-\lambda \pm \sqrt{\lambda^2 + 4\frac{\lambda}{r^2}}\right)}.$$

Using the fact that λ is positive and real, we define k and l in terms of λ to be

$$k = \sqrt{\frac{1}{2}\left(\lambda + \sqrt{\lambda^2 + 4\frac{\lambda}{r^2}}\right)}$$

and

$$l = \sqrt{\frac{1}{2}\left(-\lambda + \sqrt{\lambda^2 + 4\frac{\lambda}{r^2}}\right)}.$$

The general solution for the differential equation (5) is then

$$Y(x) = A\cos(kx) + B\sin(kx) + C\cosh(lx) + D\sinh(lx),$$

where k and l are as defined above and A, B, C and D are arbitrary constants. By taking the first three derivatives of $Y(x)$, we obtain

$$\begin{aligned}
Y(x) &= A\cos(kx) + B\sin(kx) + C\cosh(lx) + D\sinh(lx);\\
Y'(x) &= -Ak\sin(kx) + Bk\cos(kx) + Cl\sinh(lx) + Dl\cosh(lx);\\
Y''(x) &= -Ak^2\cos(kx) - Bk^2\sin(kx) + Cl^2\cosh(lx) + Dl^2\sinh(lx);\\
Y'''(x) &= Ak^3\sin(kx) - Bk^3\cos(kx) + Cl^3\sinh(lx) + Dl^3\cosh(lx).
\end{aligned}$$

Applying the boundary conditions results in the following algebraic system of equations for the constants A, B, C, and D:

$$\begin{aligned}
A + C &= 0\\
Bk + Dl &= 0\\
-Ak^2\cos k - Bk^2\sin k + Cl^2\cosh l + Dl^2\sinh l &= 0\\
A(k^3 - \lambda k)\sin k - B(k^3 - \lambda k)\cos k + C(l^3 + \lambda l)\sinh l &\\
+D(l^3 + \lambda l)\cosh l &= 0.
\end{aligned}$$

We use a matrix format to write this system as

$$\begin{bmatrix} 1 & 0 & 1 & 0\\ 0 & k & 0 & l\\ -k^2\cos k & -k^2\sin k & l^2\cosh l & l^2\sinh l\\ (k^3 - \lambda k)\sin k & -(k^3 - \lambda k)\cos k & (l^3 + \lambda l)\sinh l & (l^3 + \lambda l)\cosh l \end{bmatrix}$$
$$\cdot \begin{bmatrix} A\\ B\\ C\\ D \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\\ 0 \end{bmatrix}.$$

Using cofactor expansion and setting the determinant of the coefficient matrix equal to zero, we obtain the eigenvalue equation, namely

$$2 - r\sqrt{\lambda}\sin k\sinh l + (2 + r^2\lambda)\cos k\cosh l = 0 \tag{8}$$

or

$$\begin{aligned}
2 - r\sqrt{\lambda}\sin\sqrt{\frac{1}{2}\left(\lambda+\sqrt{\lambda^2+4\frac{\lambda}{r^2}}\right)}\sinh\sqrt{\frac{1}{2}\left(-\lambda+\sqrt{\lambda^2+4\frac{\lambda}{r^2}}\right)} \\
+ (2+r^2\lambda)\cos\sqrt{\frac{1}{2}\left(\lambda+\sqrt{\lambda^2+4\frac{\lambda}{r^2}}\right)}\cosh\sqrt{\frac{1}{2}\left(-\lambda+\sqrt{\lambda^2+4\frac{\lambda}{r^2}}\right)} \\
= 0.
\end{aligned}$$

Equation (8) allows us to find the values of λ for which the coefficient matrix is not invertible. If the coefficient matrix is not invertible, then there exists solutions of this matrix equation that are non-trivial. At this point we have found the eigenvalue equation in terms of λ and r. From Dr. French, we have learned that r realistically falls in the interval of real numbers from 8 to 16, inclusive.

Using *Mathematica*, we defined the eigenvalue equation by means of the function `f[p,r]`, where $p = \lambda$. By doing so, we can use the `Plot` function to generate a graph of the eigenvalue equation for any value of r. These graphs can be utilized to obtain initial estimates of the roots of the eigenvalue equation. *Mathematica*'s built-in function `FindRoot`, which utilizes Newton's method, was used to numerically approximate the value of λ. We can then solve the matrix equation for the constants A, B, C and D. The method implemented in this situation was that of substitution in the general case which was transformed into a *Mathematica* program. Therefore, once a value of r is chosen and a value for λ found, they can be applied to this program that will return the eigenfunction. This function, `EigenFunction[p,r]`, returns the solution in its simplest form. It is simple and quick to find the eigenfunction using *Mathematica* and the programs we have written. The functions can then be easily plotted using *Mathematica*.

The eigensolutions of the eigenvalue problem (7), when flexure and rotary inertia are present, are solved in this manner. We note that when λ is known, the value for ω can be readily found as

$$\omega^2 = \frac{\lambda EI}{mr^2}$$

and can be used in the time-dependent equation as well. The *Mathematica* programs developed through the work done for this paper have reduced the solution of this problem to a single question: "What is the value of r?"

The methods used in the other two cases are similar. As mentioned previously, the approximation considering only the effects of flexure is intuitively inaccurate and therefore is ignored. The final case including flexure, rotary inertia, and shear is contingent upon the results of the solution of the approximation we have discussed. The boundary conditions require the results of solving problem (5) and thus becomes a boundary value problem. With the substitution of

$$\alpha = \frac{2(1+\mu)}{\kappa}$$

and allowing λ to be defined as above, we solve this boundary value problem in an analogous procedure. Using similar *Mathematica* functions, it is easy to compare the two sets of solutions.

The concern of the accuracy of the approximation of the second case for the third is definitely important. If it is unreasonable to neglect shear, then one can only use the situation which includes flexure, rotary inertia and shear as defined by equation (3). If we are able to graphically view the two situations simultaneously, we will be able to visualize the differences between each of the cases. However, this does not seem to correspond with the findings of Dr. French. This may be due to a normalization of the vertical scale that was not done in our analysis. It appears in our graphs that the case which only includes flexure and rotary inertia is not a good approximation for the case also including shear. A more thorough investigation into the proper values for r could lead to more definitive results. It is unclear what values of r were used by Dr. French in his appendix. The results of this paper rely heavily on the value or values of r. Thus the concern of the accuracy of the approximations requires a deeper investigation.

Acknowledgements. The author was directed by Dr. Philip Schaefer of the University of Tennessee, Knoxville throughout this project. The work was a project for the Research Experience for Undergraduates sponsored by the National Science Foundation and the Tennessee Science Alliance. The program was directed by Dr. Suzanne Lenhart of the University of Tennessee, Knoxville. The author thanks Dr. Schaefer for his direction and professionalism and Dr. Lenhart, the National Science Foundation, and the Tennessee Science Alliance for the opportunity.

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A Ratio Proof of the Pythagorean Theorem

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This is a research project in the course Modern Geometry. A new proof of the Pythagorean theorem has been found. This proof is not included in the collection of 370 proofs in [1].

Put two congruent right triangles ABC and $A'BC'$ in position as in figure 1 such that B, C, and A' are collinear and A, C', and B are collinear. Let P be the intersection point of AC and $A'C'$. Let $BC = a = BC'$, $AC = b = A'C'$, $AB = c = A'B$ and $AP = x$. Then $AC' = A'C = c - a$ and $PC = b - x$.

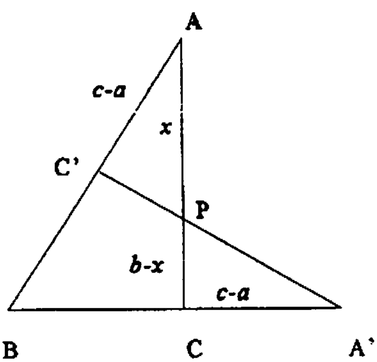

Figure 1

Since triangle ABC is similar to triangle APC', we have

$$\frac{c-a}{b} = \frac{x}{c}$$

and

$$x = \frac{c^2 - ac}{b}.$$

Since triangle $A'BC'$ is similar to triangle $A'PC$, we have

$$\frac{b-x}{a} = \frac{c-a}{b}$$

and

$$b - x = \frac{ac - a^2}{b}.$$

Thus,

$$b = \frac{ac - a^2}{b} + \frac{c^2 - ac}{b}$$

and

$$b^2 + a^2 = c^2.$$

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1. Loomis, Elisha, *The Pythagorean Proposition,* National Council of Teachers of Mathematics, 1968.

Subscription Renewals and Change of Address

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Two Proofs of the Pythagorean Theorem using Area

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In doing research projects for Modern Geometry last year, Tu and Tuan Tran [3] and Søren Poulsen [2] found three proofs of the Pythagorean theorem using ratio and area combined together. Here I give two proofs using area only. These proofs are not included in Elisha Loomis' book [1].

Proof 1. Let ABC and ECD be two congruent right triangles in a position as in figure 1 such that D is on AC and DE is perpendicular to AC. The intersection points of AB with ED and EC are F and G, respectively. Through point E, draw EH perpendicular to DE and meeting the extension of CB at H. Draw AI parallel to DE and meeting HE at I.

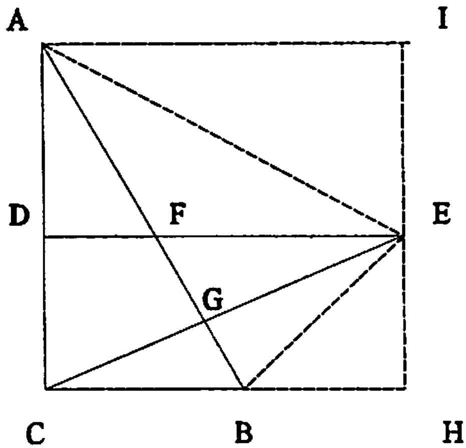

Figure 1

It is easily seen that AG is perpendicular to EC because the sum of angles ECD and DAG is a right angle. Let $BC = CD = a$, $AC = ED = b$ and $AB = EC = c$. Then $AI = CH = b$, $HI = b$, $EH = a$, $EI = b - a$ and $BH = b - a$.

The following equality involving areas is trivial:

$$\text{Area}(ACHI) = \text{Area}(AEI) + \text{Area}(BHE) + \text{Area}(ACE) + \text{Area}(EBC).$$

Therefore,

$$\begin{aligned} b^2 &= \frac{b(b-a)}{2} + \frac{a(b-a)}{2} + \frac{AG \cdot EC}{2} + \frac{BG \cdot EC}{2} \\ &= \frac{b^2 - a^2}{2} + \frac{EC(AG + BG)}{2} \\ &= \frac{b^2 - a^2}{2} + \frac{c^2}{2}. \end{aligned}$$

Thus,

$$a^2 + b^2 = c^2.$$

Proof 2. We discuss two cases, one for scalene right triangles and one for isosceles right triangles.

Case 1. Let ABC and $A'B'A$ be two scalene congruent right triangles as in figure 2 such that B' is on AC and AA' is perpendicular to AB' and P is the intersection point of AB and $A'B'$. Connect $A'B$ and BB'.

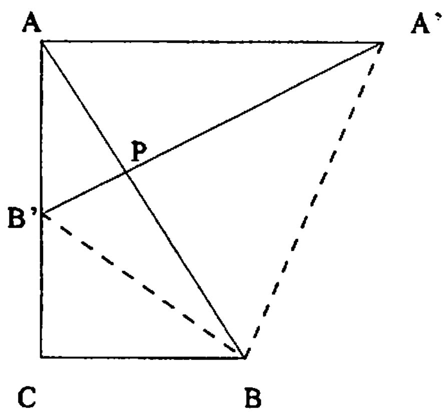

Figure 2

It is easily seen that AB is perpendicular to $A'B'$. Let $BC = a$, $AC = b$ and $AB = c$. Then $AB' = a$, $A'A = b$, $A'B' = c$ and $B'C = b - a$.

The following equality involving areas is trivial:

$$\text{Area}(ACBA') = \text{Area}(A'B'A) + \text{Area}(A'B'B) + \text{Area}(B'BC).$$

Therefore,

$$\frac{(a+b)b}{2} = \frac{AP \cdot A'B'}{2} + \frac{PB \cdot A'B'}{2} + \frac{(b-a)a}{2}$$

and

$$b^2 + a^2 = A'B'(AP + PB);$$

thus

$$a^2 + b^2 = c^2.$$

Case 2. Let ABC and $A'CA$ be two congruent isosceles right triangles as in figure 3, where P is the intersection point of AB and $A'C$. Let $BC = AC = a$ and $AB = c$. Then $AA' = a$ and $A'C = c$. It is easy to see that $ACBA'$ is a square and $A'C$ is perpendicular to AB.

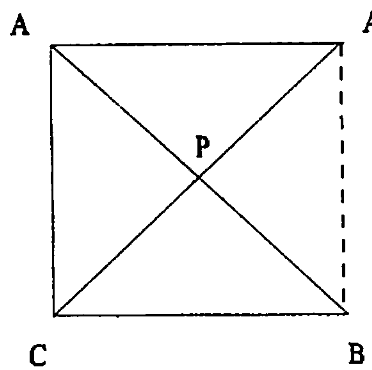

Figure 3

The following equality is obvious:

$$\text{Area}(ACBA') = \text{Area}(A'CA) + \text{Area}(A'CB) = 2\,\text{Area}(A'CA).$$

Hence,

$$a^2 = 2((c^2/2)/2),$$

and

$$c^2 = 2a^2.$$

By the discussion of these two cases, we know that the Pythagorean theorem is true.

Acknowledgements. The author thanks her advisor Dr. Jingcheng Tong sincerely.

References

1. Loomis, Elisha, *The Pythagorean Proposition,* National Council of Teachers of Mathematics, 1968.
2. Poulsen, Søren, "Another Proof of the Pythagorean Theorem," *The Pentagon* **58** No. 1 (1998), 19–20.
3. Tran, Tu and Tran, Tuan, "Two New Proofs of the Pythagorean Theorem," *The Pentagon* **58** No. 1 (1998), 16–18.

A Rose by Any Other Name ...

In the middle ages, the Pythagorean theorem went by the name of "Magister Matheseos."

An Algorithmic Method for the Construction of a 4×4 Magic Square Consisting Only of Prime Numbers

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Presented at the 1997 National Convention

Introduction

Last spring I had the pleasure to take number theory, and the class was exciting, interesting and challenging. One topic that we discussed was magic squares. However, the magic squares that we will be discussing here are slightly different than just ordinary magic squares. We shall discuss an algorithmic method for constructing 4×4 magic squares in which all of the entries are unique prime numbers.

The Chinese were the first to record magic squares; the story goes "a Man was brought a magic square by a turtle from the River Lo in the days of the Emperor Yii." This was one of the leading factors in the development of a method to solve simultaneous linear equations [1, p. 197]. Albrecht Durer, in 1514, was regarded as the first to use magic squares in art in his engraving *Melancholia* [1, pp. 296–297]. In *Melancholia*, the magic square is in the upper-right-hand corner.

What is a magic square? A magic square is a square array of numbers with the property that the rows, columns, main diagonal and anti-diagonal all sum to the same number. The following theorem is also helpful (c.f. [2, p. 51]).

Theorem. *If M and N are addition magic squares, so are (1) $M + N$; (2) kM for any k; and (3) M^{T}.*

The Algorithm

Step 1. Arrange the integers 1, 3, 7 and 9 on the main diagonal in any order (Why 1, 3, 7 and 9? Because all primes longer than one digit end in either 1, 3, 7 or 9). See figure 1.

1			
	3		
		7	
			9

Figure 1

Step 2. In either of the interior two positions on the anti-diagonal, put the two numbers that appear in the corner positions (figure 2).

1			
	3	9	
	1	7	
			9

Figure 2

Step 3. Continue entering the numbers 1, 3, 7 and 9 so that each number appears once and only once in each row, column or diagonal. When finished this is our *original square* (figure 3).

1	9	3	7
7	3	9	1
9	1	7	3
3	7	1	9

Figure 3. Original square.

Step 4. Transpose the original square (figure 4). This is the *transposed square.*

1	7	9	3
9	3	1	7
3	9	7	1
7	1	3	9

Figure 4. Transposed square.

Step 5. Examine a list of primes and find sets of twin primes such that each set shares the same digits in all positions except for the ones place. Now that we have found the sets of twin primes, change the last digit in the numbers to zero. We need four of these numbers (i.e., repeat three times). Example:

11	13	17	19	$\Rightarrow$ 10
101	103	107	109	$\Rightarrow$ 100
191	193	197	199	$\Rightarrow$ 190
821	823	827	829	$\Rightarrow$ 820

Step 6. Substitute the four numbers that we found in step 5 for the numbers 1, 3, 7, 9 in our transposed square (figure 5). This is our *second square.*

10 ➡ 1　　100 ➡ 3

190 ➡ 7　　820 ➡ 9

Transposed Square

1	7	9	3
9	3	1	7
3	9	7	1
7	1	3	9

➡

Second Square

10	190	820	100
820	100	10	190
100	820	190	10
190	10	100	820

Figure 5

Step 7. Add the original square and second square, as pictured in figure 6.

1	9	3	7
7	3	9	1
9	1	7	3
3	7	1	9

Original Square

+

10	190	820	100
820	100	10	190
100	820	190	10
190	10	100	820

Second Square

Figure 6

Result (figure 7): a 4×4 magic square consisting of unique prime numbers!

11	199	823	107
827	103	19	191
109	821	197	13
193	17	101	829

Figure 7

Additional examples of the result are given in figures 8 and 9. The *magic number* of each square, i.e. the sum of the entries of each row, column, and diagonal, is listed for each square.

Conclusion

How I came about this method was truly by mistake. My instructor, Dr. LeVan, asked as a homework problem to make either a 3×3 or 5×5 magic square consisting of unique primes. I tried and tried and thought that I could compromise by doing a 4×4 and get credit, but it didn't work. I still got no credit.

3463	5659	9431	13007
13001	9437	5653	3469
5657	3461	13009	9433
9439	13003	3467	5651

"Magic" Number is 31560

1489	1871	2083	3257
3253	2087	1879	1481
1877	1483	3251	2089
2081	3259	1487	1873

"Magic" Number is 8700

5657	1489	3251	9433
9431	3253	1487	5659
1483	5651	9439	3257
3259	9437	5653	1481

"Magic" Number is 19830

Figure 8

21017	19429	18911	18043
18041	18913	19427	21019
19423	21011	18049	18917
18919	18047	21013	19421

"Magic" Number is 77400

15649	15737	16061	22273
16063	22271	15647	15739
22277	16069	15733	15641
15731	15643	22279	16067

"Magic" Number is 69720

21019	16061	15733	19427
19423	15737	16069	21011
16067	21013	19421	15739
15731	19429	21017	16063

"Magic" Number is 72240

Figure 9

References

1. Boyer, Carl B., *A History of Mathematics*, John Wiley and Sons, New York, 1991.
2. LeVan, Marijo, *Number Theory with Applications in Secondary Mathematics*, Eastern Kentucky University, Richmond, KY, 1995.

All Digits of Largest Known Prime

The current record for the largest known prime number is $2^{3021377} - 1$. This number has 909,526 digits in base 10. Unfortunately, that is too large to print in this journal, even if the entire volume was dedicated to the project. Hence, if we wish to print all the digits, we need a different base. Instead of moving to octal or hexadecimal, neither of which give a great improvement on printability, the natural choice is to print it in base 2^{503563} (of course!). Then, since the digit for $2^{503562} - 1$ in that base is X and the digit for $2^{503563} - 1$ is Y (you mean you didn't already know that?), all digits of the largest known prime can be printed as

$$XYYYYY.$$

Using All Nine Digits

There is an interesting property of the following two numbers, their sum, and their difference. Can you name the property? The numbers are 371294568 and 216397845. Their sum and difference are:

$$371294568 + 216397845 = 587692413$$
$$371294568 + 216397845 = 154896723.$$

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The Problem Corner

Edited by Kenneth M. Wilke

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before January 1, 2000. Solutions received after the publication deadline will be considered also until the time when copy is prepared for publication. The solutions will be published in the Spring 2000 issue of *The Pentagon*. Address all communications to Kenneth M. Wilke, Department of Mathematics, 275 Morgan Hall, Washburn University, Topeka, Kansas 66621 (e-mail: xxwilke@acc.wuacc.edu).

PROBLEMS 525–529

Problem 525. Proposed by Pat Costello, Eastern Kentucky University, Richmond, Kentucky.

Determine the last two digits of the number

$$N = 19^{19^9} + 29^{29^9} + 39^{39^9} + 49^{49^9} + 59^{59^9} + 69^{69^9} + 79^{79^9} + 89^{89^9} + 99^{99^9}.$$

Problem 526. Proposed by Bryan Dawson, Union University, Jackson, Tennessee.

Given $\triangle ABC$ and its image $\triangle A'B'C'$ under an unknown glide reflection, give a compass-and-straightedge construction that determines both the line of reflection and the vector of translation parallel to that line that constitute the unknown glide reflection.

Problem 527. Proposed by Carol Collins, Drury College, Springfield, Missouri.

Prove that in the expansion of $(x^2 + x + 1)^n$, the coefficient of the x term is n and the coefficient of the x^2 term is $n(n+1)/2$ for all integers $n \geq 1$.

Problem 528. Proposed by the editor.

Consider a paired number $p(n)$ to be formed by concatenating the same number twice; e.g. $p(1234) = 12341234$. What is the smallest integer n for which $p(n)$ is a perfect square? What is the next smallest integer nn for which $p(nn)$ is a perfect square and nn has more digits than n does?

Problem 529. Proposed by Bryan Dawson, Union University, Jackson, Tennessee.

Let $\overline{BC}$ be a fixed line segment, ℓ a line parallel to $\overline{BC}$, and A an arbitrary point on ℓ. Describe (with proof) the path followed by the orthocenter of $\triangle ABC$ as A moves along ℓ.

Please help your editor by submitting problem proposals.

SOLUTIONS 515–519

Problem 515. Proposed by Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin.

Consider the following equation:

$$(*) \qquad 4(x^2 - x + 1)^3 - 27(x-1)^2x^2 = (x-2)^2(2x-1)^2(x+1)^2.$$

Either (a) prove that the equation (*) holds for each real number x using elementary algebra or (b) find a real number x such that the left side of (*) does not equal the right side of (*).

Solution by Russell Euler and Jawad Sadek (jointly), Northwest Missouri State University, Maryville, Missouri.

Notice

$$\begin{aligned} 4(x^2 - x + 1)^3 - 27(x-1)^2x^2 &= 4(x^2 - x + 1)^2 - 27\left((x^2 - x + 1)^2 - 1\right) \\ &= 4y^3 - 27y^2 + 54y - 27. \end{aligned}$$

Using the rational root theorem, it is easy to show that the roots of the polynomial $f(y) = 4y^3 - 27y^2 + 54y - 27$ are $\frac{3}{4}$, 3 and 3 so that

$$\begin{aligned} 4y^3 - 27y^2 + 54y - 27 &= (4y-3)(y-3)^2 = (4x^2 - 4x + 1)(x^2 - x - 2)^2 \\ &= (2x-1)^2(x-2)^2(x+1)^2. \end{aligned}$$

Also solved by: Ben Ault, Eastern Illinois University, Charleston, Illinois; Karl Bittinger, student, Austin Peay State University, Clarksville, Tennessee; Phil Carden, student, Florida Southern College, Lakeland, Florida; Carol Collins, Drury College, Springfield, Missouri; Kristi Karber, Missouri Southern State College, Joplin, Missouri; Matthew Zhou, student, California State University, Fresno, California and the proposer.

Editor's comment. Most solutions involved expanding both sides of the given equation and showing that the results were equal. The featured solution (and the proposer's) avoided that by artfully using one side of the given equation to derive the other.

Problem 516. Proposed jointly by Underwood Dudley, DePauw University, Greencastle, Indiana and Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri.

For every positive integer n, prove that there exists a prime p with n digits.

Solution by M. Ysabel Cervantes, student, California State University, Fresno, California.

Bertrand's Postulate states that for any natural number $k \geq 2$, there is always a prime number between k and $2k$. Taking $k = 2$ we have 3 is the prime between 2 and $2 \cdot 2 = 4$. Hence we have a one-digit prime which satisfies the condition of the problem. Now for $n \geq 2$ we take $k = 10^{n-1}$. Then $2k = 2 \cdot 10^{n-1}$ and both k and $2k$ have exactly n digits. But by Bertrand's Postulate there is always (at least) one prime number p such that $k < p < 2k$, and by our choice of k, p has exactly n digits. This completes the proof.

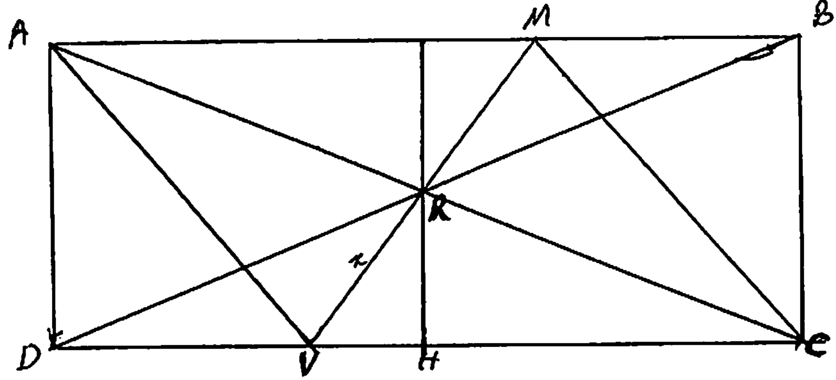
Also solved by: Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin and the proposers.

Problem 517. Proposed by the editor.

Consider a rectangular piece of paper $ABCD$ where $AB = CD = 24$ inches and $BC = DA = 10$ inches. Next bring point A into coincidence with point C and fold the sheet, creating a crease from AB to CD. How long is the crease?

Solution by Rosa V. Huerta, student, California State University, Fresno, California. (Revised by the editor.)

Let MV denote the crease as shown in the figure on page 62. Draw AC and BD and let R denote the intersection of AC and MV. Since C folds onto A, triangle AMC is isosceles. Also since angles MRA and MRC fold onto each other, they are right angles. Hence MR bisects AC and R is the midpoint of AC. Furthermore, the crease MRV is perpendicular to diagonal AC. Draw BD. Since the diagonals of a rectangle bisect each other, BD passes through the point R. From R draw a line which is perpendicular to CD which intersects CD at H. Right triangles RVH and CVR are similar; so are CVR and CRH. Let x denote the length RV. Then $x/RH = RC/CH$. Clearly $AD = BC = 10$, $AB = CD = 24$, $AC = DB = 26$, $RH = 5$, $CH = 12$ and $RC = 13$. Then $x = (13 \cdot 5/12) = 65/12$. Finally $MV = 2x = 65/6$.



Also solved by: Scott H. Brown, Auburn University, Montgomery, Alabama; Aaron Peters, Liberty University, Lynchburg, Virginia; and Russell Euler and Jawad Sadek (separately), Northwest Missouri State University, Maryville, Missouri.

Editor's comment. One can easily verify that $MV = 2x$ by drawing a line through M perpendicular to CD. Let J be the point of intersection with line CD. Then triangles MVJ and RVH are similar, with $MJ = 2RH$. Then $MV = 2x$.

Problem 518. Proposed by Russell Euler and Jawad Sadek, jointly, Northwest Missouri State University, Maryville, Missouri.

Let x be a positive integer greater than 1. Prove that $x^{20k+4}+x^{10k+2}+1$ is composite for all nonnegative integers k.

Solution by Bryan Chaffe and Jeanette Pires (jointly), students, California State University, Fresno, California.

Note that

$$x^{20k+4} + x^{10k+2} + 1 = (x^{10k+2} + x^{5k+1} + 1)(x^{10k+2} - x^{5k+1} + 1).$$

It remains to be shown that $x^{10k+2} - x^{5k+1} + 1 > 1$. Since x is an integer greater than 1 and k is a nonnegative integer, then x^{5k+1} is an integer greater than 1 and

$$1 < x^{5k+1} < (x^{5k+1})^2 = x^{10k+2},$$

so that

$$x^{10k+2} - x^{5k+1} + 1 > 1.$$

Also solved by: Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin and the proposers.

Problem 519. Proposed by the editor.

Define a sequence of integers $a_1, a_2, a_3, \ldots$ where a_1 is an arbitrarily chosen positive integer and for $k > 1$, $a_k = (3a_{k-1}/2) + 1$. Can one find a value for a_1 such that a_{1001} is odd and a_k is even for all integers $k < 1001$?

Solution by Carol Collins, Drury College, Springfield, Missouri.

We shall prove the following claim: If $a_1 = 3^q \cdot 2^n - 2$ for any positive integer q, then a_m is even for all integers $m < n + 1$ and a_{n+1} is odd. As a result of this claim, if it is true, the value $a_1 = 3 \cdot 2^{1000} - 2$ satisfies the conditions of the problem.

For $n = 1$, we have $a_1 = 3^q \cdot 2^1 - 2$ which is even since $q > 0$ is an integer. Then

$$a_2 = 3(a_1/2) - 2 = 3(3^q \cdot 2 - 2)/2 + 1 = (3^{q+1} \cdot 2 - 2 \cdot 3)/2 + 1 = 3^{q+1} - 2,$$

which is an odd integer.

Now assume that the claim holds for an arbitrary integer k and proceed by mathematical induction; i.e: for any sequence b_i defined for any positive integer p by $b_1 = 3^p \cdot 2^k - 2$ and $b_i = 3b_{i-1}/2 + 1$ for $i > 1$ where i is a positive integer, then b_m is even for all integers less than $k + 1$ and b_{k+1} is odd. We take $a_1 = 3^q \cdot 2^{k+1} - 2$ and we will show that a_m is even for all integers $m < k + 2$ and that a_{k+2} is odd. Since $a_1 = 3^q \cdot 2^{k+1} - 2$, then

$$\begin{aligned} a_2 &= [3(3^q \cdot 2^{k+1} - 2)/2] + 1 = [(3^{q+1} \cdot 2^{k+1} - 3 \cdot 2)/2] + 1 \\ &= 3^{q+1} \cdot 2^k - 3 + 1 = 3^{q+1} \cdot 2^k - 2. \end{aligned}$$

Now let $b_1 = a_2$ and $p = q + 1$. Then $b_j = a_{j+1}$ for all $j \geq 1$. By the induction hypothesis, b_m is even for all integers $1 \leq m < k + 1$ and b_{k+1} is odd; i.e. a_m is even for all integers $2 \leq m < k + 2$ and a_{k+2} is odd. Since a_1 is even, the proof of our claim is complete and taking $a_1 = 3 \cdot 2^{1000} - 2$ satisfies the conditions of the problem.

Also solved by: Rodrigo Miguel, student, California State University, Fresno, California and Russell Euler and Jawad Sadek (separately), Northwest Missouri State University, Maryville, Missouri.

Kappa Mu Epsilon News

Edited by Don Tosh, Historian

News of chapter activities and other noteworthy KME events should be sent to Don Tosh, Historian, Kappa Mu Epsilon, Mathematics Department, Evangel University, 1111 N. Glenstone, Springfield, MO 65802, or to toshd@evangel.edu.

CHAPTER NEWS

AL Gamma Chapter President — Dorthy Gearhart
University of Montevallo, Montevallo 18 actives

Other Fall 1998 officers: John Woodruff, vice president; Ginger Hand, secretary; Pauline Kennard, treasurer; Larry Kurtz, corresponding secretary; Michael Sterner, faculty sponsor.

AL Zeta Chapter President — Melanie Styers
Birmingham Southern College, Birmingham 26 actives, 13 associates

The fall program included the installation of new members. Other Fall 1998 officers: Kelly O'Donnell, vice president; Elizabeth White, secretary/treasurer; Mary Jane Turner, corresonding secretary; Shirley Branan, faculty sponsor.

AL Eta Chapter President — Justin Smith
University of West Alabamba, Livingston 11 actives

Other Fall 1998 officers: James Zimlich, vice president; Jaime Shutt, secretary; Jason Overstreet, treasurer; Michael Reekie, corresponding secretary; Julia Massey, faculty sponsor.

CA Gamma Chapter President — Jeff Mintz
California Polytechnic State University, San Luis Obispo 22 actives, 2 associates

We have meetings every other week. We held a book sale of mathematics, statistics, and computer science textbooks. We had a session for graduate school advising for mathematics and related fields. Other Fall 1998 officers: Carrie Mortensen, vice president; Judy Fetcho, secretary/treasurer; Kent Morrison, corresponding secretary/faculty sponsor.

CO Delta

Chapter President — John Bright

Mesa State College, Grand Junction

23 actives, 9 associates

We started this year with a bagel party/meeting on September 9. Pins and certificates were presented to those initiated last April, and members discussed possible activities for the year. On November 18, 1998, we held our first Fall initiation. Thirty-five initiates, members, and guests attended the initiation reception. At the business meeting which followed, students voted to order pink/silver honor cords for graduation and decided to work with the Math Club in hosting area high school math students for a day of problem solving and competition. This "Math Extravaganza" will be held in January. Other Fall 1998 officers: Amanda Widel, vice president; Sarah Kennedy, secretary; David Wing, treasurer; Donna Hafner, corresonding secretary; Kenneth Davis, faculty sponsor.

GA Alpha

Chapter President — Nancy Bryson

State University of West Georgia, Carrollton

25 actives

Once again, Georgia Alpha sponsored a food and clothing drive for the needy with the proceeds being delivered to the Salvation Army in December. We also had our Fall Social at a local Mexican restaurant in early December. The social was well-attended and a fine time was had by all. Other Fall 1998 officers: Tonya McElwaney, vice president; Nancy Boyette, secretary; Roger Huffstetler, treasurer; Joe Sharp, corresponding secretary/faculty sponsor; Mark Faucette faculty sponsor.

IL Zeta

Chapter President — Karen Jarosz

Rosary College, River Forest

15 actives

Fall activities included math tutoring and two meetings which were held to discuss activities for the school year. Other Fall 1998 officers: Christa Lee, vice president; Heather Wasielewski, secretary; Anna Cantal, treasurer; Paul Coe, corresponding secretary/faculty sponsor.

IL Theta

Chapter President — Julie Deroche

Benedictine University, Lisle

20 actives

The main event for the fall was a Calculus Competition. Three different levels were featured, with one-hour, multiple choice exams for each. Prizes included bookstore certificates for winners and pencils (engraved with "Reach Your Limits with Calculus") for honorable mention. Over 70 students from current calculus courses participated. Other Fall 1998 officers: Dennis Wozniak, vice president; Lisa Townsley Kulich, corresponding secretary/faculty sponsor.

IA Alpha

Chapter President — Suzanne Shontz

University of Northern Iowa, Cedar Falls

38 actives

The KME Homecoming Coffee hosted by emeritus professors Carl and

Wanda Wehner in October was quite successful in spite of rather inclement weather conditions. Students presenting papers at local KME meetings included Gary Spieler on "Strassen's Algorithm for Matrix Multiplication," Douglas Stockel on "Ideal Membership and Gröbner Bases," Brooke Brill on "Hypatia: The Woman Behind the Philosopher, the Astronomer, the Mathematician" and Manuel Chapa on "Population Growth Models." Tanya Sperry addressed the Fall initiation banquet on "The Origins of Graph Theory: Euler's Representation of the Königsberg Bridge Problem." Dr. Mark Ecker is providing additional faculty leadership for Iowa Alpha chapter. Other Fall 1998 officers: Gary Spieler, vice president; Beth Koch, secretary; Mary Noga, treasurer; John Cross, corresponding secretary/ faculty sponsor.

IA Delta

Chapter President — Emily Bailey

Wartburg College, Waverly

40 actives

The September meeting consisted of establishing three committees for our club's participation in Homecoming 1998. We also established a committee to develop ideas for a new KME t-shirt. In October, Katie Straub talked about her summer internship at Rockwell Collins Engineering Forcasting. The November program was jointly presented by Dr. Olson, Dr. Birgen and Will Smith, director of the Career Development Center. Graduate school and job application processes were discussed. Our December meeting was a Christmas pizza party. Other Fall 1998 officers: Christine Morrissey, vice president; Joel Nelson, secretary; Keith Cummer, treasurer; August Waltmann, corresponding secretary; Mariah Birgen, faculty sponsor.

KS Alpha

Chapter President — Mandy Fritz

Pittsburg State University, Pittsburg

actives 52, associates 10

The first meeting of the fall semester was held on September 2. Fundraising activities were discussed and President Mandy Fritz gave an interesting presentation on "The Lo Shu Magic Square." During the month of September, the chapter ran a "Kiss the Cow" contest with collection jars for people to deposit money for their choice of which faculty member or teaching assistant should get to kiss one of professor Tim Flood's calves. The second meeting was held on October 8. Pictures were taken for the university yearbook, an initiation ceremony was conducted, and the winner of the "Kiss the Cow" contest, graduate assistant Amy Ferguson, collected her "prize." A pizza party followed. The third meeting was held on November 4. Graduate assistant Jennifer Laswell enlightened those in attendance about "Probabilities and Wild Card Poker." The final meeting of the semester took place on December 2. KME member Erin Reavley gave an excellent PowerPoint presentation on "Three Wise Women (Grace

Chisholm Young, Sophie Germain, and Sonya Kovalevsky)." The annual KME Christmas party was held that evening at the home of Mathematics Department chair Elwyn Davis. Other Fall 1998 officers: Catherine Ellis, vice president; Lisa Collier, secretary; Jeremy Dill, treasurer; Cynthia Woodburn, corresponding secretary; Yaping Liu, faculty sponsor.

KS Beta Chapter President — Melanie Kurtz

Emporia State University, Emporia 28 actives, 6 associates

We had a very good semester. One of the new activities we decided to be involved in this semester was making a float for the homecoming parade. The theme was "Hat's off to ESU" so we decided to built a pyramid of hats. Computers were placed at the bottom. All went well until the day of the parade. It started raining and before it was over it was pouring. KME members had fun but we got wet through and through. Our shield and sign may never be the same. Other Fall 1998 officers: Phillip Jost, vice president; Tracy Kitson(fall) and Jason Robben(spring), secretary/treasurer; Brian Albright, historian; Connie Schrock, corresponding secretary; Larry Scott, faculty sponsor.

KS Gamma Chapter President — Kevin Slattery

Benedictine College, Atchison 5 actives, 15 associates

On September 10, Kansas Gamma members gathered for their fall picnic at Schroll Center on campus. At the end of September two seniors gave presentations on their summer employment experiences. Donnie Eason told about creating database software at NASA's space station. President Kevin Slattery described electronic note taking, a project in his Research Experience for Undergraduates at DePauw University in Greencastle, IN. In mid-October many KS Gamma members gathered for the afternoon reception honoring Dr. Mary Gray of American University. That evening Dr. Gray gave the second lecture in the Mary L. Fellin Lecture Series. She spoke on "Justice by the Numbers: Pensions, Prisoners, and Ice Hockey." The Christmas Wassail party was again hosted by faculty moderator Sister Jo Ann Fellin at Marywood. Alum and former KS Gamma president Matt McIntosh completed the Ph. D. degree in statistics at the University of Missouri where commencement was held on December 18. Sister Helen Sullivan, OSB, former professor and chair of the mathematics department at the college and foundress of KS Gamma, died on December 22 at the age of 91. Recipient of the fourth George R. Mach Distinguished Service Award and active professionally, she is fondly remembered by many. Other Fall 1998 officers: Curtis Sander, vice president; Jo Ann Fellin, corresponding secretary/faculty sponsor.

KS Delta Chapter President — Laurie Payeur

Washburn University, Topeka 30 actives

The Kansas Delta chapter met in tandom with the Washburn mathematics club, Mathematica, for two afternoon meetings throughout the semester. Lunch was enjoyed and mathematical games were played. Other Fall 1998 officers: Stephanie Lambert, vice president; Justin Freeby, secretary/treasurer; Allan Riveland, corresponding secretary; Ron Wasserstein, Donna LaLonde, faculty sponsors.

KS Epsilon Chapter President — Mariam Riazi

Fort Hays State University, Hays 24 actives

Monthly meetings were held and featured various speakers. Some members attended the KATM Conference in Hutchinson, KS. Other Fall 1998 officers: Adam North, vice president; Drew Heiman, secretary/treasurer; Chenglie Hu, corresonding secretary; Linda Kallam, faculty sponsor.

KS Zeta Chapter President — Thyrza Mucambe

Southwestern College, Winfield 11 actives

Other Fall 1998 officers: Jeff Rahm, vice president; Tary Helmer, secretary; Tory Helmer, treasurer; Mehri Arfaei, corresponding secretary; Reza Sarhangi, faculty sponsor.

KY Alpha Chapter President — Brandy Smith

Eastern Kentucky University, Richmond 24 actives

The semester began with floppy disk sales (together with the ACM chapter) to students in the computer literacy class and the Mathematica class. At the September meeting, we made paper versions of the Instant Insanity Cube and tried to solve the puzzle. The second event was a joint KME/ACM picnic with faculty. The picnic was held at Lake Reba Park on a nice Sunday in October. At the late October meeting Brandy Smith gave a talk on "Hyperbolic Tesselations." In December we had our white elephant gift exchange at the Christmas party. During the fun, we were visited by three faculty members who shared a rousing rendition of the song "Math Exams" to the tune of "Santa Claus is Coming to Town." Other Fall 1998 officers: Charles Woolum, vice president; Amy Brewer, secretary; Shannon Purvis, treasurer; Pat Costello, corresponding secretary.

MD Beta Chapter President — Jenny Addeo

Western Maryland College, Westminster 26 actives

In September we held a picnic for mathematics majors and had a business meeting at which we elected new officers. In the October meeting we inducted three new members and heard a presentation about career opportunities by a recent alumnus, Jason Barr. At the November meeting we showed the movie "Good Will Hunting." Ongoing weekly activities include tutoring sessions for lower-level mathematics classes. Our vice president, Tom Lapato, was named the Burger King Division III Scholar Athlete of the Year and also received an NCAA Graduate Scholarship. Other Fall 1998 officers: Tom Lapato, vice president; Christina Addeo, secretary; David Meckley, treasurer; Linda Eshleman, corresponding secretary; Harry Rosenzweig, faculty sponsor.

MD Delta — Chapter President — Sean Carley

Frostburg State University, Frostburg — 27 actives

Our chapter held several "get-acquainted" meetings open to all students, and explored the possibility of forming a math club on campus. In October, the group enjoyed a picnic at Rocky Gap State Park. Other Fall 1998 officers: Julie Robison, vice president; Katherine Taylor, secretary; Andrew Adam, treasurer; Edward White, corresponding secretary; John Jones, faculty sponsor.

MI Epsilon — Chapter President — Martin Przyjazny

Kettering University, Flint — 27 actives

MI Epsilon's first full academic year began with the election of Martin Przyjazny as president, replacing recent graduate Michael Fisackerly. Sheri Houston volunteered to coordinate the design and creation of a KME display case to highlight our activities and members. Term activities included a showing of the movie "Mathematical Mystery Tour" and a pizza party. Retiring professor Duane McKeachie spoke at the pizza party on "50 Years of Mathematics at 1700 West Third Avenue," giving us some highlights of his many years on our faculty, from the school's days as General Motors Institute, then as GMI Engineering & Management Institute, and finally as Kettering University. Other Fall 1998 officers: Joel Pfauth, vice president; Derek Fisackerly, secretary; Jeremy Plenzler, treasurer; Jo Smith, corresponding secretary; Brian McCartin, faculty sponsor.

MS Alpha — Chapter President — Gordona Bauhan

Mississippi University for Women, Columbus — 12 actives, 5 associates

Monthly meetings were held during the fall. Additionally, an initiation was held in September and a bake sale was held in December. Also in December, Ms. Cecily McNair, a teacher recruiter with the Mississippi Department of Education, gave a talk about scholarship opportunities for education majors in Mississippi. Other Fall 1998 officers: Julie Torrent, vice president; Jaime Rickert, secretary; Jacqueline Tharp, treasurer; Shaochen Yang, corresponding secretary; Beate Zimmer, faculty sponsor.

MS Gamma Chapter President — Jason Haight

University of Southern Mississippi, Hattiesburg 20 actives, 4 associates

Other Fall 1998 officers: Paula Thigpen, vice president; Adrienne Davis, secretary; Alice Essary, treasurer/corresponding secretary; Bill Horner and Jose Contreras, faculty sponsors.

MS Epsilon Chapter President — Ken Byars

Delta State University, Cleveland 12 actives

Other Fall 1998 officers: Chad Huff, vice president; Amanda Seward, secretary/treasurer; Paula Norris, corresponding secretary; Rose Strahan, faculty sponsor.

MO Alpha Chapter President — Angie Horton

Southwest Missouri State University, Springfield 20 actives, 5 associates

In the fall of 1998, the Missouri Alpha chapter of KME hosted the mathematics department picnic and held monthly meetings. Other Fall 1998 officers: Michael Byrd, vice president; Samual Blisard, secretary; Jessica McDonnell, treasurer; John Kubicek, corresonding secretary/faculty sponsor.

MO Beta Chapter President — Darin Tessier

Central Missouri State University, Warrensburg 20 actives, 10 associates

The Missouri Beta chapter of KME held monthly meetings during the fall semester. In September, the group watched the first "Life By the Numbers" video. Initiation was held in October. Seven full and six associate members were initiated, after which Scotty Orr gave a demonstration of the new electronic classroom. Jachin Misko was presented the 1997–98 Claude H. Brown Mathematics Acheivement Award for Freshmen. At the November meeting, Dr. David Ewing presented "Bubble, Bubble, Toil and Trouble ... or Determining Minimum Surface Area of 3–D Frames." In December, KME hosted a Christmas party and invited all of the other student organizations in the Department of Mathematics and Computer Science. Other Fall 1998 officers: Aaron Shaefer, vice president; Andrew Feist, secretary; Warren Christensen, treasurer; Tammy Surfus, historian; Rhonda McKee, corresponding secretary; Larry Dilley, Phoebe Ho, Scotty Orr, faculty sponsors.

MO Epsilon Chapter President — David Bates

Central Methodist College, Fayette 8 actives

Other Fall 1998 officers: Christina Miller, vice president; Sheryll Rector, secretary/treasurer; William McIntosh, corresponding secretary; Linda Lembke and William McIntosh, faculty sponsors.

MO Eta

Chapter President — Shawn Logan

Truman State University, Kirksville — 8 actives, 6 associates

We held meetings every other week. The main business discussed was our math expo to be held in February for area high school students. We held initiation in October. Other Fall 1998 officers: Bryan Bischel, vice president; Angela Kell, secretary; Chad Muse, treasurer; Angela Kell, corresponding secretary; Mary Beersman, faculty sponsor.

MO Theta

Chapter President — Jeremy Osborne

Evangel University, Springfield — 8 actives, 1 associate

Monthly meetings were held. The fall social was held at the home of Don Tosh. Other Fall 1998 officers: Mandy Wilson, vice president; Don Tosh, corresponding secretary/faculty sponsor.

MO Kappa

Chapter President — Nathan Ratchford

Drury College, Springfield — 13 actives, 5 associates

The chapter started the semester off by watching a video on Fermat's Last Theorem. The winners of the annual Math Contest this year were Kristen Hannah for the Calculus II and above division and Steven Gradney for the Calculus I and below division. Prize money was awarded to the winners at a pizza party held for all contestants. The chapter participated in the Annual Exploration in Mathematics and the Physical Sciences, which is a recruitment workshop designed for high school students. Sub sandwiches were served to the chapter at the Senior Talk given by Dennis Powell. The Math Club has also been running a tutoring service for both the day school and the continuing education division (Drury Evening College) as a money-making project. The semester ended with a Christmas party at Dr. Reich's house. Other Fall 1998 officers: Kristen Hannah, vice president; Dena Wisner, secretary; Billy Kimmons, treasurer; Charles Allen, corresponding secretary; Pamela Reich, faculty sponsor.

MO Lambda

Chapter President — Robert Horton

Missouri Western State College, St. Joseph — 38 actives

Ten new members were initiated on September 20. Elaine Hauschel, a faculty member who was one of the initiates, was the speaker for the program. Mrs. Hauschel described her work as a statistician in estimating groundwater contamination. Other fall activities of the Missouri Lambda chapter included participation in Family Day activities, hosting a Thanksgiving carry-in dinner for mathematics and computer science students and faculty, and participation in departmental colloquia. Steve Saffell, senior mathematics major and KME member, gave a colloquium talk entitled "Hypocycloids and Other Fascinating Curves." Other Fall 1998 officers: Stephanie Tingler, vice president; David McCay, secretary; Shaun Piatt, treasurer, John Atkinson, corresponding secretary; Jerry Wilkerson, faculty sponsor.

NE Alpha Chapter President — Stacie Olmer
Wayne State College, Wayne 33 actives, 7 associates

The Nebraska Alpha chapter initiated seven new members this semester and competed in many of the homecoming activities. They won second place in the Organization Olympics. It was the first year they had competed. Other Fall 1998 officers: Emily Negus, vice president; Gina Vaselaar, secretary; Dave May, treasurer; Matt Jansen, historian; John Fuelberth, corresponding secretary; Jim Paige, faculty sponsor.

NE Beta Chapter President — Kala Devi Ramalingam
University of Nebraska at Kearney, Kearney 12 actives, 3 associates

Other Fall 1998 officers: Michael Sullivan, vice president; Tisha Maas, secretary; Peter Okumah, treasurer; Stephen Bean, corresponding secretary; Richard Barlow, faculty sponsor.

NE Gamma Chapter President — Andy Boell
Chadron State College, Chadron 14 actives, 6 associates

Other Fall 1998 officers: Shaun Daugherty, vice president; Craig Bruner, Jr., secretary; Kendra Pedersen, treasurer; James Kaus, corresponding secretary; Robert Stack, faculty sponsor.

NH Alpha Chapter President — Allen Barriere
Keene State College, Keene 19 actives

Other Fall 1998 officers: Melissa Shepard, vice president; Laura Devoid, secretary; Travis Wakefield, treasurer; Vincent Ferlini, corresponding secretary; Ockle Johnson, faculty sponsor.

NM Alpha Chapter Co-Pres. — Dolores Gabaldon and Jennifer Gill
University of New Mexico, Albuquerque 90 actives, 15 associates

News about our meetings and the Fall 1998 banquet is available on the NM Alpha WWW pages at math.unm.edu/~kme. Other Fall 1998 officers: Holly Dison, vice president; Merlin Decker, webmaster; Archie Gibson, corresponding secretary/faculty sponsor.

NY Alpha Chapter President — William D'Angelo
Hofstra University, Hempstead

Fall activities included a volleyball game and a social and dinner for faculty and students. Other Fall 1998 officers: Michael Dallal, vice president; Drew Batkin, secretary; Andrea Genzale, treasurer; Aileen Michaels, corresponding secretary/faculty sponsor.

NY Eta

Chapter President — Brett Richner

Niagara University, Niagara

20 actives

Our chapter has combined meetings with the campus math club. Future plans include a career day with alumni participating, fundraising activities, presentations by faculty members, and a possible trip to the national convention in Florida. Other Fall 1998 officers: Mike Simons, vice president; Kristen Grimm, secretary; Alan Hunt, treasurer; Robert Bailey, corresponding secretary; Wendy Duignan, faculty sponsor.

NY Lambda

Chapter President — Jill Kahan

C. W. Post Campus of Long Island University, Brookville

16 actives

Other Fall 1998 officers: Nicole Garofalo, vice president; Tanya Palacio, secretary; David Joseph, treasurer; Andrew Rockett, corresponding secretary; John Stevenson, faculty sponsor.

NY Mu

St. Thomas Aquinas College, Sparkill

66 actives, 9 associates

Other Fall 1998 officers: Joseph Keane, corresponding secretary/faculty sponsor.

OH Gamma

Chapter President — Anila Xhunga

Baldwin-Wallace College, Berea

12 actives

The chapter, in conjunction with the campus computer club, sponsored a fall picnic for students and faculty. Other Fall 1998 officers: Duke Hutchings, vice president; Mary Guinn, secretary; Corina Moise, treasurer; David Calvis, corresponding secretary/faculty sponsor; Chungsim Han, faculty sponsor.

OK Alpha

Chapter President — Melina Weigle

Northeastern State University, Tahlequah

32 actives, 6 associates

Our fall initiation ceremonies for seven students were held in the banquet room of Roni's Pizza. It was well attended by many faculty, their families, and other students. We sponsored a booth at the Halloween Carnival sponsored by the Northeastern State Government Association. The children of Tahlequah are invited to attend this annual free "trick-or-treat" event. KME officers and math professors Joan Bell and Darryl Linde contributed their assistance at the 87th annual technical meeting of the Oklahoma Academy of Science which was held at NSU in November. The annual book sale was held in November. We extend our thanks to the faculty who donated old texts to us for this sale. In December, we sponsored a visit from Jeff Lazalier, chief meteorologist of the 2-NEWS/NBC Storm Team in Tulsa, Oklahoma. He showed a weather video, which included his 5-second performance in the movie "Twister." We provided pop and pizza to the more than sixty students attending. It was great! We continue to have joint activities with NSU's student chapter of the MAA and participate in The Problem Solving Competition, sponsored by the MAA. Our Christmas party in December ended our fall activities. Other Fall 1998 officers: Tina Wolfe, vice president; Tera McGrew, secretary; Gregg Eddings, treasurer; Joan Bell, corresponding secretary/faculty sponsor.

OK Delta — Chapter President — Daniel Gregory

Oral Roberts University, Tulsa — 15 actives

Other Fall 1998 officers: Vidar Ligard, vice president; Vince Dimiceli, treasurer; Dorothy Radin, corresponding secretary; Vince Dimiceli, faculty sponsor.

PA Alpha — Chapter President — Shannon Mack

Westminster College, New Wilmington — 13 actives

The chapter sponsored an ice cream social for all first-year students. Other Fall 1998 officers: Stephanie Tangora, vice president; Dena Streit, secretary; Mike Leiper, treasurer; Warren Hickman, corresponding secretary; Carolyn Cuff, faculty sponsor.

PA Iota — Chapter President — Donald Miller

Shippensburg University, Shippensburg — 22 actives

We held a number of meetings and co-sponsored activities with the SU Math Club. In the fall we inducted nine new members, bringing our current total to 22 (not including faculty members). Other Fall 1998 officers: Thomas Ruffner, vice president; Jaymie Kenny, secretary; Michael Seyfried, treasurer/corresponding secretary; Cheryl Olsen, faculty sponsor.

PA Kappa — Chapter Co-Pres. — Linda Bruce and Lindsay Janka

Holy Family College, Philadelphia — 5 actives, 3 associates

The PA Kappa chapter met on the first Wednesday of each month during the fall to work on problems for submission to *The Pentagon*, *Math Horizons* and *The College Journal of Mathmatics*. On October 20, the chapter sponsored and hosted a math competition for local high school students. On December 3, co-president Lindsay Janka and secretary/treasurer Brian Minster presented their seminar papers to the math faculty and upper-division math majors. Lindsay's topic was map coloring and graph theory. Brian spoke on non-orientable two-dimensional manifolds. Other Fall 1998 officers: Brian Minster, secretary/treasurer; Marcella Wallowicz, corresponding secretary/faculty sponsor.

PA Mu — Chapter President — Troy Mohney

Saint Francis College, Loretto — 16 actives

We participated in the Adopt-a-Highway program and picked litter in

October along our two miles of adopted highway. Several KME members and faculty took part in the college's annual Science Day, which brought more than 300 high school students to campus for interesting presentations, a science bowl, poster and video competitions, a public speaking competition, and a library scavenger hunt. Among the presentations were "The Year 2000 Problem" by KME member and alumnus John Miko and "The Tacoma Narrows Bridge Collapse" by KME member and faculty member John Harris. KME student members served as session moderators, registration helpers, Science Bowl moderators, scorekeepers, timers, and judges. Other Fall 1998 officers: Tracy Paxon, vice president; Rebecca Espenlamb, secretary; Kourosh Barati-Sedeh, treasurer; Pete Skoner, corresponding secretary; Amy Miko, faculty sponsor.

SC Gamma Chapter President — Leslie Hogan

Winthrop University, Rock Hill 8 actives

Other Fall 1998 officers: Kelly Ann Clardy, vice president; Kortnee Barnett, secretary; Stephanie Boswell, treasurer; Donald Aplin, corresponding secretary; James Bentley, faculty sponsor.

TN Beta Chapter President — Shannon Gosnell

East Tennessee State University, Johnson City 21 actives

During our fall meeting, an election of officers was conducted. A social at the Olive Garden Restaurant was held. Plans are being made for attendance at the regional meeting of the Southeastern Section of the Mathematical Association of America in Memphis in March. Other Fall 1998 officers: Ken Proffitt, vice president; Susan Hosler, secretary; Justin Hyder, treasurer; Lyndell Kerley, corresponding secretary/faculty sponsor.

TN Gamma Chapter President — Lori Davis

Union University, Jackson 15 actives

The chapter sponsored two social events during the semester. A joint back-to-school pizza party with ACM was held on September 29 and the annual KME Christmas pot-luck party was held on December 4. Both events occurred in the McAfee Commons building on campus. The chapter also sponsored a needy child from the community for Christmas through the Carl Perkins Center. Other Fall 1998 officers: Mandy Davidson, vice president; Lindsey Crain, secretary; Cathie Scarbrough, treasurer; Don Richard, corresponding secretary; Matt Lunsford, faculty sponsor.

TN Delta Chapter President — Robert Johnson

Carson-Newman College, Jefferson City 13 actives, 6 associates

Other Fall 1998 officers: Melissa Holland, vice president; Sarah Montgomery, secretary; Brian Renninger, treasurer; Catherine Kong, corresponding secretary/faculty sponsor.

TN Epsilon Chapter President — Jennifer Dowdy
Bethel College, McKenzie 10 actives, 2 associates

Other Fall 1998 officers: Jonathan Lankford, vice president; Christina Hill, secretary; James Wiggleton, treasurer; Russell Holder, corresponding secretary; David Lankford, faculty sponsor.

TX Alpha Chapter President — Jeffrey Braidon Hood
Texas Tech University, Lubbock 5 actives, 5 associates

Other Fall 1998 officers: Deanna Burns McLendon, vice president; Carrie Lee Bates, secretary; Charles "Lance" Cowey, treasurer; Victor Shubov, corresponding secretary.

TX Kappa Chapter President — Mary Bruton
University of Mary Hardin-Baylor, Belton 12 actives

Other Fall 1998 officers: Alicia Kuehl, vice president; Belinda Smith, secretary; Peter Chen, corresponding secretary; Maxwell Hart, faculty sponsor.

New Editor

Steven D. Nimmo of Morningside College (Iowa Gamma) has been named the next editor of *The Pentagon*. Any correspondence for the editor arriving after June 1, 1999 should be sent to the following address:

Steven D. Nimmo
Morningside College
P. O. Box 6400
Sioux City, IA 51106
email: sdn001@alpha.morningside.edu

Dr. Nimmo will be the tenth editor of this journal. Previous editors were C. V. Newsom (NM Alpha) 1941–1943; Harold D. Larsen (NM Alpha and MI Alpha) 1943–1953; Carl V. Fronabarger (MO Alpha) 1953–1959; Fred W. Lott, Jr. (IA Alpha) 1959–1965; Helen Kriegsman (KS Alpha) 1965–1971; James K. Bidwell (MI Beta) 1971–1979; Kent Harris (IL Eta) 1979–1989; Andrew M. Rockett (NY Lambda) 1989–1995; and C. Bryan Dawson (KS Beta and TN Gamma) 1995–1999.

More information about Steve Nimmo can be found at http://www.morningside.edu/acad/math/nimmo.htm. As of the date of press, the next business manager had not been named.

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Kappa Mu Epsilon, Mathematics Honor Society, was founded in 1931. The object of the Society is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics due to its demands for logical and rigorous modes of thought; to provide a Society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; and to disseminate the knowledge of mathematics and familiarize the members with the advances being made in mathematics. The official journal of the Society, *The Pentagon*, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the Chapters.

Active Chapters of Kappa Mu Epsilon

Listed by date of installation.

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 April 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Southwest Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
MS Beta	Mississippi State University, Mississippi State	14 Dec 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
NM Alpha	University of New Mexico, Albuquerque	28 March 1935
IL Beta	Eastern Illinois University, Charleston	11 April 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 April 1937
OH Alpha	Bowling Green State University, Bowling Green	24 April 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	Central Missouri State University, Warrensburg	10 June 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
TX Beta	Southern Methodist University, Dallas	15 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 June 1941
NY Alpha	Hofstra University, Hempstead	4 April 1942
MI Beta	Central Michigan University, Mount Pleasant	25 April 1942
NJ Beta	Montclair State University, Upper Montclair	21 April 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 March 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 June 1947
CO Alpha	Colorado State University, Fort Collins	16 May 1948
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 April 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959

IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
IL Epsilon	North Park College, Chicago	22 May 1963
OK Beta	University of Tulsa, Tulsa	3 May 1964
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 April 1965
AL Epsilon	Huntingdon College, Montgomery	15 April 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
AR Alpha	Arkansas State University, State University	21 May 1965
TN Gamma	Union University, Jackson	24 May 1965
WI Beta	University of Wisconsin—River Falls, River Falls	25 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	Western Maryland College, Westminster	30 May 1965
IL Zeta	Rosary College, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 March 1971
KY Alpha	Eastern Kentucky University, Richmond	27 March 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 April 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 April 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State College, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sept 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis College, Loretto	14 Sept 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury College, Springfield	30 Nov 1984

CO Gamma	Fort Lewis College, Durango	29 March 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 April 1986
TX Iota	McMurry University, Abilene	25 April 1987
PA Nu	Ursinus College, Collegeville	28 April 1987
VA Gamma	Liberty University, Lynchburg	30 April 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 April 1990
CO Delta	Mesa State College, Grand Junction	27 April 1990
NC Gamma	Elon College, Elon College	3 May 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 April 1991
SD Alpha	Northern State University, Aberdeen	3 May 1992
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 March 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 April 1997
MI Delta	Hillsdale College, Hillsdale	30 April 1997
MI Epsilon	Kettering University, Flint	28 March 1998
KS Zeta	Southwestern College, Winfield	14 April 1998
TN Epsilon	Bethel College, McKenzie	16 April 1998
MO Mu	Harris-Stowe College, St. Louis	25 April 1998
GA Beta	Georgia College and State University, Milledgeville	25 April 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
NY Xi	Buffalo State College	12 May 1998

KME Website

The national KME website can be found at:

www.cst.cmich.edu/org/kme_nat/

Below are just a few of the things that can be found on the site:

- How to start a KME chapter
- Information on KME conventions
- The cumulative subject index of *The Pentagon*
- Lists of KME chapters
- How to contact national officers
- KME History

Please remember to submit local chapter URLs to the national webmaster!