THE PENTAGON

*A Mathematics Magazine for Students*

Volume 57 Number 2 Spring 1998

## Contents





Typeset in $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX Printed in the United States of America.

*The Pentagon* (ISSN 0031–4870) is published semiannually in December and May by Kappa Mu Epsilon. No responsibility is assumed for opinions expressed by individual authors. Papers written by undergraduate mathematics students for undergraduate mathematics students are solicited. Papers written by graduate students or faculty will be considered on a space-available basis. Submissions should be typewritten (double spaced with wide margins) on white paper, standard notation conventions should be respected and any special symbols not typed should be carefully inserted by hand in black ink. All illustrations must be submitted on separate sheets and drawn in black ink. Computer programs, although best represented by pseudocode in the main text, may be included as an appendix. Graphs, tables or other materials taken from copyrighted works MUST be accompanied by an appropriate release from the copyright holder permitting further reproduction. Student authors should include the names and addresses of their faculty advisors. Electronic copies of final versions in "text only" (ASCII) format are requested. Contributors to *The Problem Corner* or *Kappa Mu Epsilon News* are invited to correspond directly with the appropriate Associate Editor.

Individual domestic subscriptions: $5.00 for two issues (one year) or $10.00 for four issues (two years); individual foreign subscriptions: $7.00 (US) for two issues (one year). Library rate: $10.00 (US) for two issues (one year) or $20.00 (US) for four issues (two years). Correspondence regarding subscriptions, changes of address or back copies should be addressed to the Business Manager. Copies lost because of failure to notify the Business Manager of changes of address cannot be replaced. Microform copies are available from University Microfilms, Inc., 300 North Zeeb Road, Ann Arbor, Michigan 48106–1346 USA.

---

# Groups on Elliptic Curves

Kelli Polotaye, *student*

New York Lambda

C. W. Post Campus of Long Island University
Brookville, NY 11548

Presented at the 1997 National Convention and
awarded "top four" status by the Awards Committee.

In the seventeenth century, the French mathematician Pierre de Fermat proposed that there exists no positive integer solutions to the equation $X^n+Y^n = Z^n$ for $n > 2$. Over the past three and a half centuries numerous mathematicians have tried to prove this theorem and, in doing so, have created entire branches of mathematics. Among the ideas that have been explored are ellipic curves, whose general form is $y^2 = Ax^3+Bx^2+Cx+D$, and specifically Frey curves. Frey curves are a special kind of elliptic curve with the form $y^2 = x(x - A)(x + B)$, and they look like the example in figure 1.

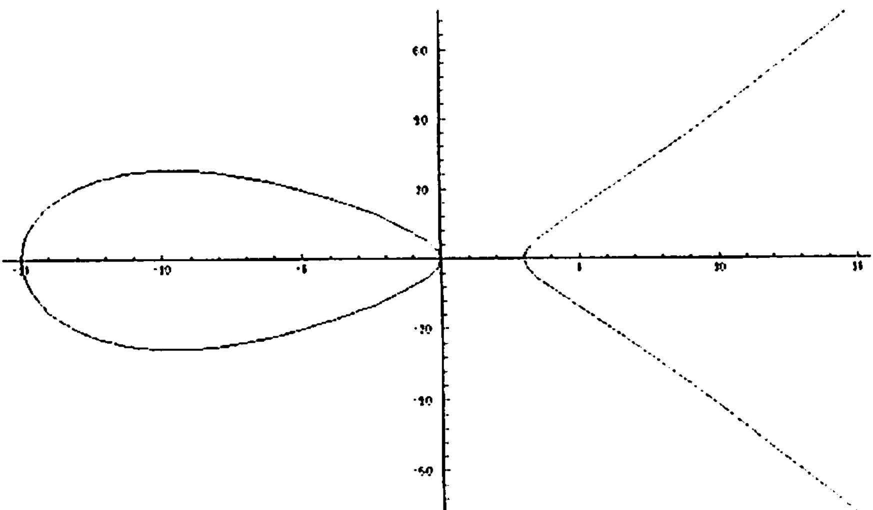


**Figure 1.** Example of a Frey curve.

It has been shown that the coefficients, $A$ and $B$, of a Frey curve could be determined by the properties of the solutions, $X^n$ and $Y^n$, of Fermat's equation, if they exist. Mathematicians conjectured that if a counterexample to Fermat's Last Theorem actually did exist, the $X^n$ and $Y^n$ of the counterexample would determine coefficients, $A$ and $B$, of a related Frey elliptic curve. Then, they proceeded to show that these Frey curves cannot actually exist because they would possess certain impossible properties. This, of course, is a conclusion that has been reached after three hundred and fifty years of work by some of history's greatest mathematicians using the most modern and powerful theory available. Along the way, mathematicians explored many preliminary properties of elliptic curves to better understand the nature of Frey curves. One such property is the group the rational points on an elliptic curve form under a special addition operation. A rational point of the elliptic curve has both coordinates, $x$ and $y$, as rational numbers. Mathematicians have shown that the rational points of an elliptic curve which does not contain singularities are closed and associative under a specific addition operation and contain identity and inverse elements, forming a group.

In order for the rational points of an elliptic curve to form a group, an addition operation which is closed and associative must be defined. To do so, form the chord between any two points, $P$ and $Q$, and then extend it to its third point of intersection, $R$, with the curve (see figure 2). Then, if $R$ has coordinates $(x, y)$, $R'$ will have coordinates $(x, -y)$, and the addition operation of the group is defined as $P + Q = R'$.

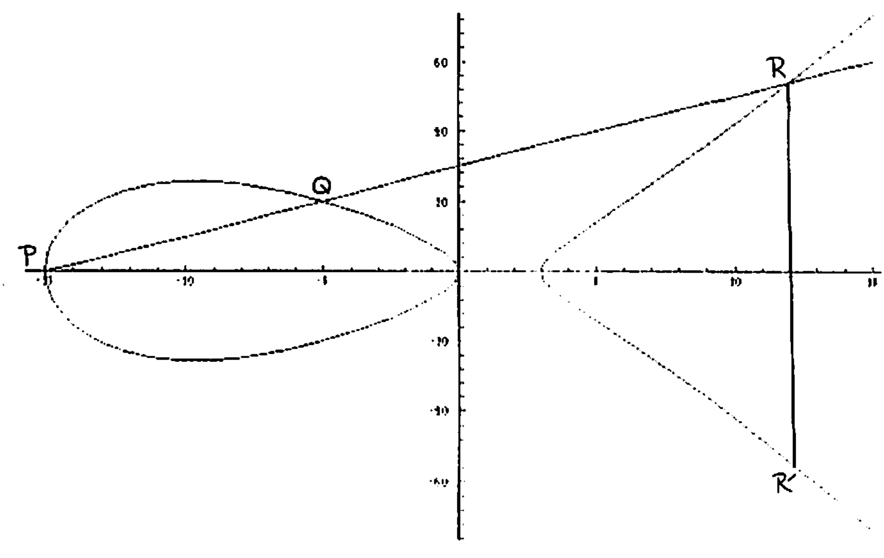



**Figure 2.** Frey curve addition; $P + Q = R'$.

For an example, consider the elliptic curve

$$y^2 = x(x-3)(x+15) \tag{1}$$

and the two points $P(-15,0)$ and $Q(-5,20)$, both of which are on the elliptic curve. The slope of the chord between these two points is 2, making the equation of the chord

$$y = 2(x+15) = 2x+30. \tag{2}$$

Both sides of this equation must be squared so it can be equated to (1) to determine the third point of intersection. Squaring both sides of (2) yields

$$y^2 = 4x^2 + 120x + 900. \tag{3}$$

Setting (1) equal to (3) and solving for $x$ yields

$$0 = x^3 + 8x^2 - 165x - 900.$$

Dividing this synthetically by the two $x$-coordinates, $-15$ and $-5$, the result is the $x$-coordinate of the third point of intersection. For this example, the $x$-coordinate of the third point of intersection is 12. To find the $y$-coordinate for this point, substitute $x = 12$ into (2), giving $y = 54$. This point, $R(12,54)$, is the third point of intersection of the line and curve, yielding $R'(12,-54)$. Therefore, $P(-15,0) + Q(-5,20) = R'(12,-54)$, as in figure 2.

In order to add a point $P$ to itself, use the equation of the line tangent to the curve at the point $P$ instead of the chord. Extend the tangent line to its intersection point, $R$, with the curve and again, the addition operation yields $P+P = R'$. It has been proven that each of the chords and tangents intersect the elliptic curve at three points which are not necessarily distinct. In the chord example the three intersection points were $P$, $Q$ and $R$. In the tangent example there only seems to be two points of intersection, $P$ and $R$. In order to understand where the third point of intersection lies in this case, think of the tangent "as the limiting case of a chord that gets shorter and shorter until finally its end points coincide," and then "the tangent is a chord that passes through the same point twice" (Ribet [1], p. 149). Therefore, the tangent case does indeed have three points of intersection, namely $P$, $P$, and $R$.

For example, we can add $P(9,36)$, which lies on the elliptic curve

$$y^2 = x(x-3)(x+15), \tag{4}$$

to itself. To determine the equation of the tangent line, first differentiate (4) using the chain rule, giving

$$y' = \frac{3x^2 + 24x - 45}{2\sqrt{x^3 + 12x^2 - 45x}},$$

and then evaluate (4) at $P(9,36)$. This gives a slope of 23/4, and the equation of the tangent line is

$$y = \frac{23}{4}x - \frac{63}{4}.$$

Squaring both sides of this equation and setting it equal to (4), we have

$$0 = x^3 - \frac{337}{16}x^2 - \frac{2178}{16}x - \frac{3969}{16}. \tag{5}$$

Since $P(9,36)$ represents two of the points of intersection of the line and the curve, divide (5) synthetically by 9 twice. This division gives the third point of intersection $R(49/16, 119/64)$, and therefore $P(9,36) + P(9,36) = R'(49/16, -119/64)$ (see figure 3).

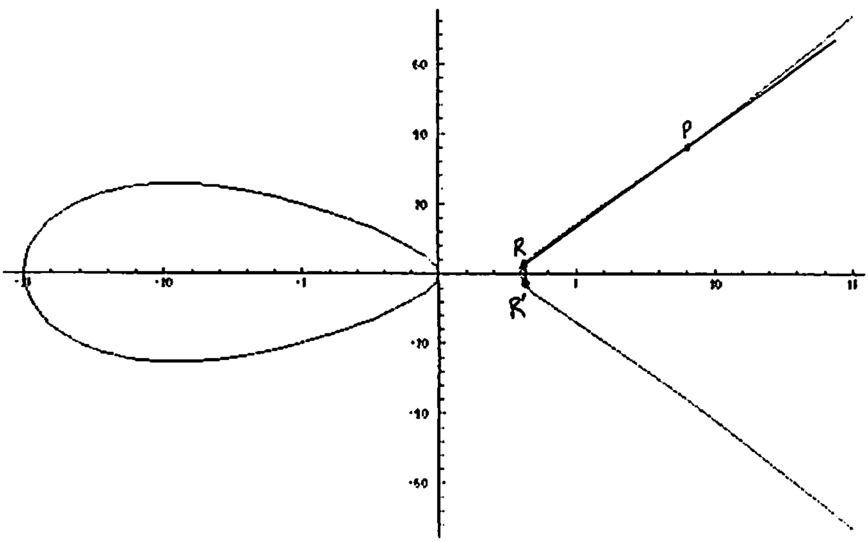



**Figure 3.** Frey curve addition; $P + P = R'$.

Another case that needs to be considered is the addition of a point to itself which happens to yield a vertical tangent line and seemingly no third point of intersection. To accommodate this case, an extra point at infinity, called the origin, is added to the curve. Now if the tangent line, which contains two of the intersection points, is extended up, it will strike this point at infinity, giving the third point of intersection. Therefore, this point at infinity must also be included in the group (see figure 4).

In order for the rational points of an elliptic curve to be a group, they must have identity and inverse elements. For this group, the identity element is the origin, $O$, and for any point $P$, $P + O = P$. This can be

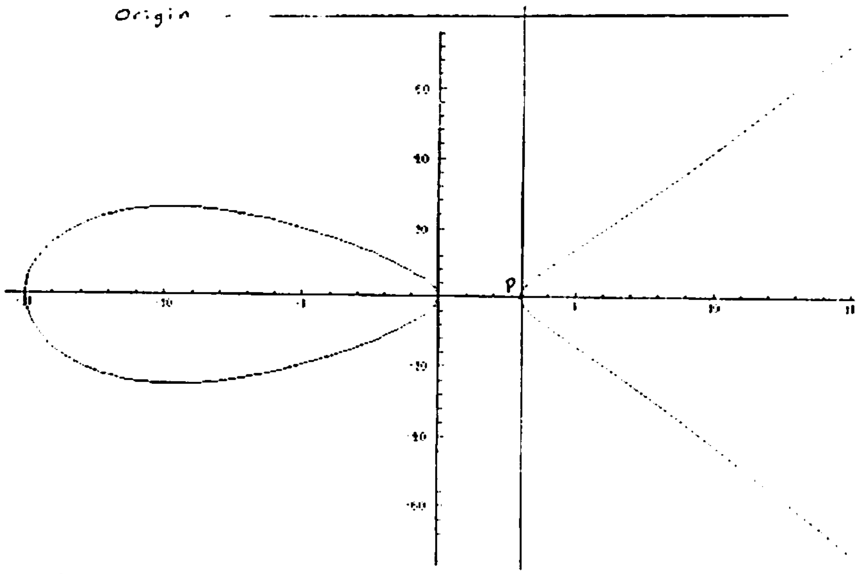



**Figure 4.** Frey curve addition; $P + P$ yields the point at infinity.

shown by drawing the vertical chord between $P(x, y)$ and $O$; $O$ can be represented by a horizontal line at the top of the graph. This chord can then be extended to its third point of intersection with the curve, which is insured to be $P'(x, -y)$ because of the curve's symmetry to the $x$-axis. Then, $P + O = (P')' = P$. Similarly, the inverse of any point $P$ is $P'$. This can be shown by constructing the (vertical) chord between $P(x, y)$ and $P'(x, -y)$. Extend this chord to its third point of intersection, $O$. Since $O' = O$, $P + P' = O' = O$.

The rational points of the elliptic curve must also be shown to be associative if they are to form a group, that is $P + (Q + R) = (P + Q) + R$ for all rational points $P$, $Q$, and $R$ on the curve. By choosing any three rational points on the elliptic curve and performing the additions in the prescribed order, it can be shown that the result of each side is the same. For example, choose $P(-5, 20)$, $Q(-15, 0)$ and $R(0, 0)$. Then $P + (Q + R) = P + (3, 0) = (-15/4, -135/8)$ and $(P + Q) + R = (12, -54) + R = (-15/4, -135/8)$, thus proving the associative nature of the curve for one example.

Thus far, the addition operation on the rational points of an elliptic curve has been defined and shown to be associative. The group also possesses an identity and inverse elements for each point on the curve. The only unfounded assumption made thus far is that a third rational point of intersection always exists for any chord or tangent construction. This is necessary to prove that the group is closed. To prove this assumption choose two general points, $P(h, n)$ and $Q(r, s)$, and a general Frey curve of the form

$$y^2 = x(x-a)(x+b). \tag{6}$$

As before, form the chord between the two points giving

$$y = mx - mh + n, \tag{7}$$

where $m = (s-n)/(r-h)$ is the slope. Squaring both sides of (7) gives

$$y^2 = m^2x^2 + m^2h^2 - 2m^2hx + 2mnx - 2mnh + n^2. \tag{8}$$

Equating (6) and (8) and rearranging yields

$$0 = x^3 - x^2(a+b) + abx - m^2x^2 - m^2h^2 + 2m^2hx - 2mnx + 2mnh - n^2.$$

Dividing this equation by the $x$-coordinates of our two general points, $h$ and $r$, yields a quotient of

$$x + h + r - a - b - m^2$$

and a remainder of

$$\begin{aligned} &x(ab + 2m^2h - 2mn - rh + (h+r)(h+r-a-b-m^2)) \\ &\quad - m^2h^2 + 2mnh - n^2 - rh((h+r) - a - b - m^2). \end{aligned}$$

Substituting $m = (s-n)/(r-h)$ and $n^2 = h^3 - h^2(a+b) + abh$, the remainder can be reduced to a sum of fractions which cancel one another out, proving the remainder equals zero and that the two $x$-coordinates divide evenly into the equation. The quotient is rational considering $h$ and $r$ are rational $x$-coordinates, $a$ and $b$ are the roots of the elliptic curve and $m$ is the slope of the chord. This number represents the third point of intersection of the general chord and the general curve, thus proving that it must exist and that it is rational. Therefore, the rational points on a Frey elliptic curve form a closed group under this addition operation.

## References

1. Ribet, Kenneth A. and Hayes, Brian, "Fermat's Last Theorem and Modern Arithmetic," *American Scientist* 82 (March-April 1994), 144–156.

---

"Mathematicians are like Frenchmen; whatever you say to them they translate into their own language and forthwith it is something entirely different." —Goethe

# A Simple Venn Diagram for Four Sets

Subhash C. Saxena, *faculty*
Paige Adams, *student*

Coastal Carolina University
Conway, SC 29526

It was in 1880 that John Venn (1834–1923) published his first paper [6] on "The Diagrammatic and Mechanical Representation of Propositions and Reasonings," which popularized diagrams named after him. In textbooks, they almost always show Venn diagrams for three or fewer sets, since for four or more sets *circles* cannot be used to display all possibilities. Moreover, any diagram illustrating four or more sets becomes complicated. In recent years, many have given ingenious schemes for representing four or more sets. We reference some of them: Margaret Baron [1] published a paper in *Mathematical Gazette*; Branko Grunbaum [4] discussed them in *Mathematics Magazine* in 1975; Shoeleh Mutameni [5] published an algorithm for generating Venn diagrams for an arbitrary number of sets in *Mathematics Teacher*. Kiran Chilakamarri, Peter Hamburger, and Raymond Pippert [2,3] have published a series of papers connecting Venn diagrams to planar graphs and Hamiltonian cycles.

The second author of this note, a student of the first author, came up with an idea of depicting four sets with a design which is easy to visualize and is simple in its construction. We use a $4 \times 4$ grid of regions represented by $g_{ij}$ $(i = 1,2,3,4;\ j = 1,2,3,4)$. Each cell represents one of the sixteen areas needed in the Venn diagram for four sets. They identify sixteen mutually exclusive regions. Let $R_i$ $(i = 1,2,3,4)$ and $C_j$ $(j = 1,2,3,4)$ represent rows and columns of four cells each, respectively. Let $A_1$, $A_2$, $A_3$, and $A_4$ be the four sets, and $A_1'$, $A_2'$, $A_3'$, and $A_4'$ be their complements. We let $A_1 = R_1 \cup R_2$, $A_2 = R_2 \cup R_3$, $A_3 = C_2 \cup C_3$, and $A_4 = C_3 \cup C_4$. See figure 1.

Since each row intersects with every column, we immediately arrive at the conclusion that $A_1 \cap A_2 \cap A_3 \cap A_4 = R_2 \cap C_3 = g_{23}$, which is the intersection of the second row (common to $A_1$ and $A_2$) and the third column (part of $A_3$ and $A_4$).

The beauty of this construction lies in its simplicity of visualization and comprehension. Starting with $g_{23}$, if we move left, we leave $A_4$; if we go right, we are out of $A_3$; going up takes us out of $A_2$; and shifting

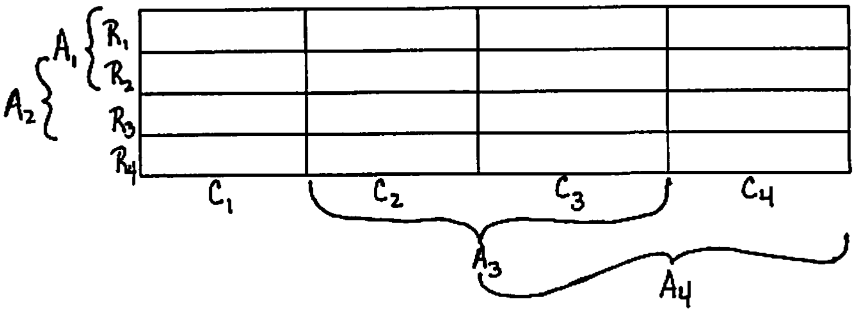



**Figure 1.** The diagram.

downwards gets us out of $A_1$. Figure 2 displays all sixteen possibilities. It is easy to identify any cell. For example, $g_{34}$ lies in the 3rd row and 4th column, belonging to $A_2$ and $A_4$ but not in $A_1$ nor in $A_3$. Hence, $g_{34} = A_1' \cap A_2 \cap A_3' \cap A_4$.

| | | | |
|---|---|---|---|
| $A_1 \cap A'_2 \cap A'_3 \cap A'_4$ | $A_1 \cap A'_2 \cap A_3 \cap A'_4$ | $A_1 \cap A'_2 \cap A_3 \cap A_4$ | $A_1 \cap A'_2 \cap A'_3 \cap A_4$ |
| $A_1 \cap A_2 \cap A'_3 \cap A'_4$ | $A_1 \cap A_2 \cap A_3 \cap A'_4$ | $A_1 \cap A_2 \cap A_3 \cap A_4$ | $A_1 \cap A_2 \cap A'_3 \cap A_4$ |
| $A'_1 \cap A_2 \cap A'_3 \cap A'_4$ | $A'_1 \cap A_2 \cap A_3 \cap A'_4$ | $A'_1 \cap A_2 \cap A_3 \cap A_4$ | $A'_1 \cap A_2 \cap A'_3 \cap A_4$ |
| $A'_1 \cap A'_2 \cap A'_3 \cap A'_4$ | $A'_1 \cap A'_2 \cap A_3 \cap A'_4$ | $A'_1 \cap A'_2 \cap A_3 \cap A_4$ | $A'_1 \cap A'_2 \cap A'_3 \cap A_4$ |

**Figure 2.** Representations of sets.

On the other hand, we can easily place any of the sets and/or their union or intersection in appropriate cells. The set $A_2 \cap A_3$ consists of four boxes: $g_{22}$, $g_{23}$, $g_{32}$, and $g_{33}$. Furthermore, it is easy to remember and reconstruct this. *Each of the original four sets has to occupy two adjacent rows or two adjacent columns.* The following observations make this scheme pedagogically sound: The *first* set gets the *first* row, and it contains the second row also, and no more. The *second* set has to contain *second* row, and gets the third row and no more. The last two sets take up columns. Naturally, the *third* set uses the *third* column, and it contains the second column also, and no more. The *fourth* set contains the *fourth* column, and includes the third column also, and no more.

The set $A_1 \cap A_2$ is the second row consisting of 4 cells. The set $A_3 \cap A_4$ is the the third column. The set $A_2 \cap A_3$ consists of four cells, $g_{22}$, $g_{23}$, $g_{32}$, and $g_{33}$. We illustrate intersections of two sets in figure 3 and intersections of four sets in figure 4.

The simplicity of these diagrams makes the solution of a complicated inclusion-exclusion problem involving four sets uncomplicated, as shown by the following example.

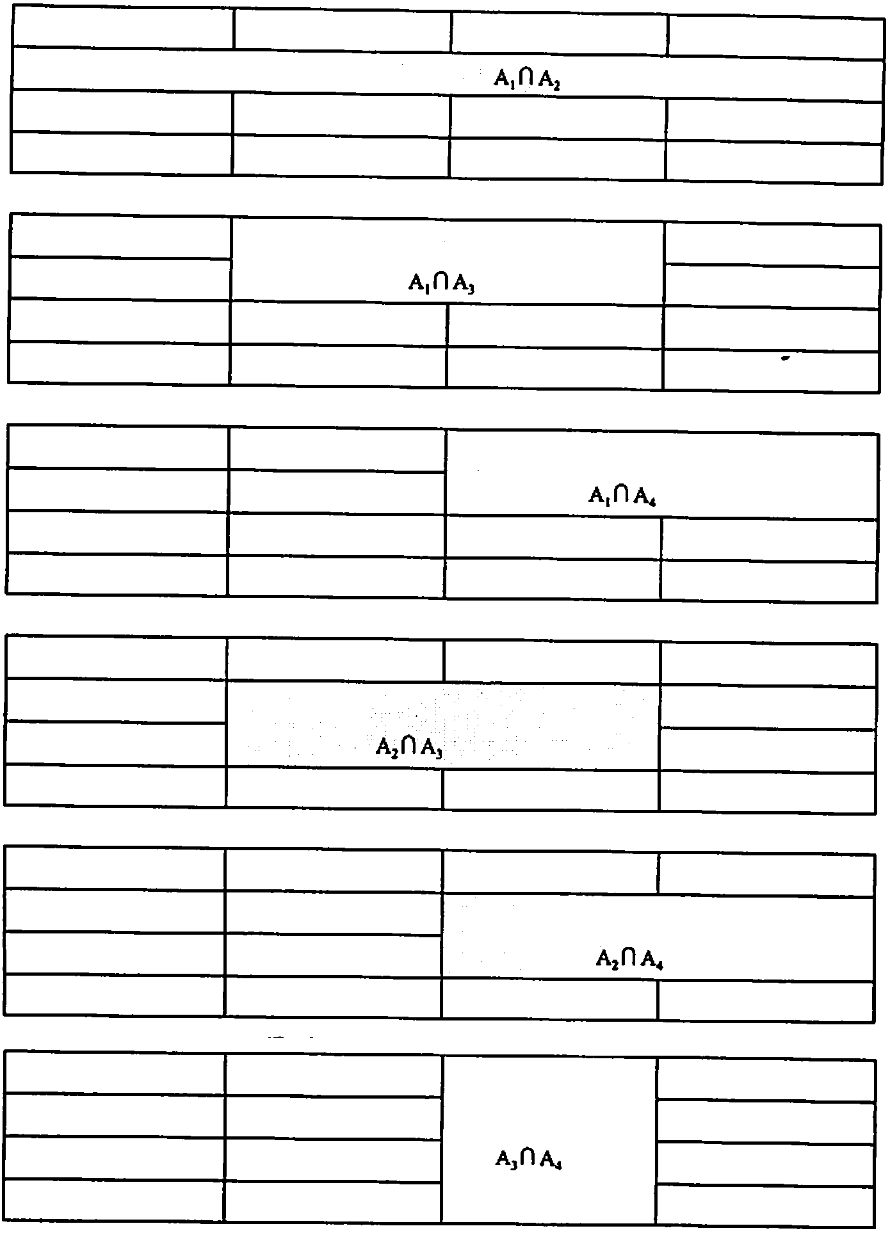



**Figure 3.** Intersections of two sets.

**Problem situation.** *In a state university along the east coast, there are 100 mathematics majors; 45 of them are taking Advanced Calculus ($A_1$); 44 are taking Abstract Algebra ($A_2$); 49 are registered in Modern Geome-*

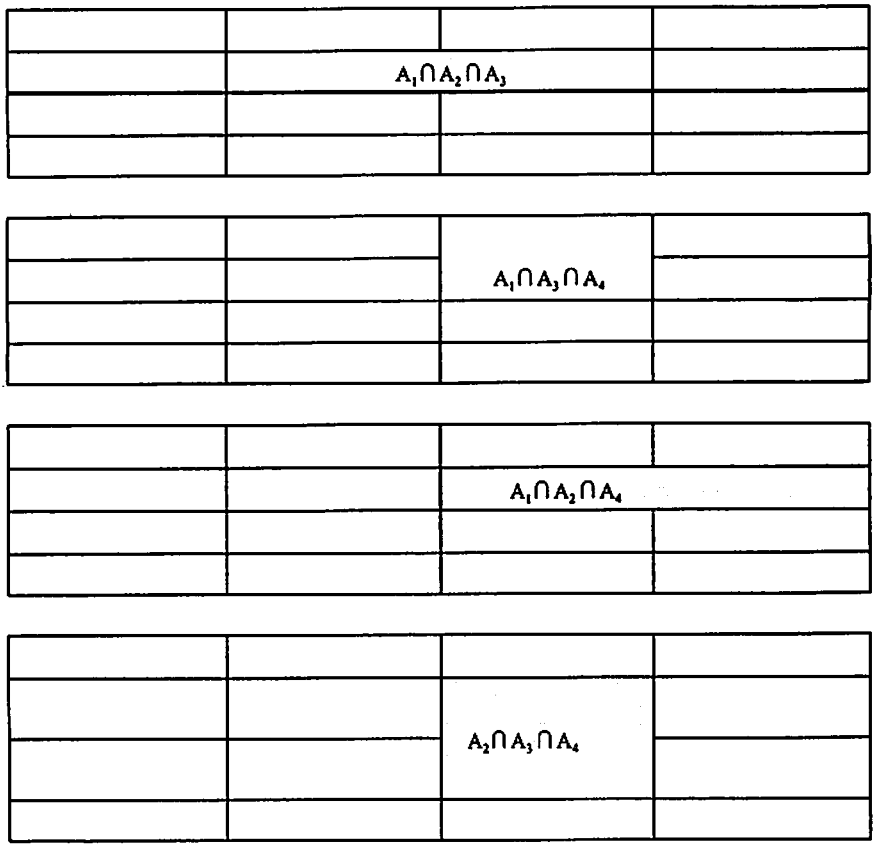



**Figure 4.** Intersections of three sets.

*try ($A_3$); 48 are enrolled in Linear Algebra ($A_4$); 22 are taking Advanced Calculus and Abstract Algebra ($A_1 \cap A_2$); 25 are registered for Advanced Calculus and Modern Geometry ($A_1 \cap A_3$); 23 are enrolled in Advanced Calculus and Linear Algebra ($A_1 \cap A_4$); 26 are taking Abstract Algebra and Modern Geometry ($A_2 \cap A_3$); 23 are enrolled in Abstract Algebra and Linear Algebra ($A_2 \cap A_4$); 24 are registered for Modern Geometry and Linear Algebra ($A_3 \cap A_4$); 15 are registered for Advanced Calculus, Abstract Algebra, and Modern Geometry ($A_1 \cap A_2 \cap A_3$); 14 are taking Advanced Calculus, Abstract Algebra and Linear Algebra ($A_1 \cap A_2 \cap A_4$); 12 are enrolled in Advanced Calculus, Modern Geometry, and Linear Algebra ($A_1 \cap A_3 \cap A_4$); 13 are registered in Abstract Algebra, Modern Geometry, and Linear Algebra ($A_2 \cap A_3 \cap A_4$); and 10 are enrolled in all four ($A_1 \cap A_2 \cap A_3 \cap A_4$).*

With this information, we can answer any quantitative question about the cardinality of any set involving these four sets. We pose a few of them.

(a) How many are taking Advanced Calculus, Modern Geometry, and Linear Algebra, but not Abstract Algebra? (b) How many are taking Abstract Algebra and Modern Geometry, but not the other two? (c) How many are enrolled in Modern Geometry only out of these four? (d) How many are not registered for any one the four courses?

To answer these questions, we draw the diagram, and start with the last piece of information for $A_1 \cap A_2 \cap A_3 \cap A_4$, and go sequentially backward, using the numbers for shaded diagrams shown earlier. The complete diagram is shown in figure 5.

| | | | |
|---|---|---|---|
| 45 - (8+2+7+4+10+5+3)= 6 | 25-(5+10+2) = 8 | 12 - 10 = 2 | 23 - (4 + 10 + 2) = 7 |
| 22-(5+10+4) = 3 | 15 - 10 = 5 | 10 | 14 - 10 = 4 |
| 44 - (8+3+6+4+10+5+3) = 5 | 26 - (3+10+5) = 8 | 13 - 10 = 3 | 23 - (4+10+3) = 6 |
| 100 - (6+8+2+7+3+5+10+4+5 +8+3+6+4+9+7) = 13 | 49 - (8+5+8+2+10+3+9)= 4 | 24 - (3+10+2) = 9 | 8 - (6+4+7+2+10+3+9)= 7 |

**Figure 5.** Diagram for the problem.

To answer (a), we use the number for $A_1 \cap A_2' \cap A_3 \cap A_4$, which is given in $g_{13}$, and is **2**. For (b), we need the cardinal number for $A_1' \cap A_2 \cap A_3 \cap A_4'$, and $g_{32}$, which is **8**. For (c), we have to look for the cell for $A_1' \cap A_2' \cap A_3 \cap A_4'$, and the solution is **4**. For (d), we use $A_1' \cap A_2' \cap A_3' \cap A_4'$, which is in $g_{41}$, and the answer is **13**.

This example has illustrated the relative simplicity of the procedure in the solution of a tedious problem dealing with four sets.

## References


1. Baron, Margaret, "A Note on Historical Development of Logic Diagrams: Leibniz, Euler and Venn," *The Mathematical Gazette* LIII No. 383 (May 1969), 113–125.
2. Chilakamarri, Kiran; Hamburger, Peter and Pippert, Raymond, "Hamilton Cycles in Planar Graphs and Venn Diagrams," *Journal of Combinatorial Theory, Series B* 67 No. 2 (1996), 296–303.
3. Chilakamarri, Kiran; Hamburger, Peter and Pippert, Raymond, "Venn Diagrams and Planar Graphs," *Geometriae Dedicata* 62 No. 1 (1996), 73–91.
4. Grunbaum, Branko, "Venn Diagrams and Independent Families of Sets," *Mathematics Magazine* 48 No. 1 (1975), 12–23.
5. Mutameni, Shoeleh, "Readers' Reflections," *Mathematics Teacher* 87 No. 9 (1994), 744–746.

6. Venn, John, "Diagrammatic and Mechanical Representation of Propositions and Reasonings," *The London, Edinburgh, and Dublin Philos. Magazine and Journal of Science* 9 (1880), 1–18.

---

## Selections from the Cumulative Subject Index

The cumulative subject index of *The Pentagon* is available online at:

www.cst.cmich.edu/org/kme/indpent.htm

Below are a few selections from the index. There are literally hundreds of additional interesting articles. Most libraries of institutions with active KME chapters have a good collection of past issues. Back issues can also be ordered (see page 40).

Items in the index contain the volume number in which it was printed, the issue number in the volume (1 is fall, 2 is spring; 1–2 is for years during WWII in which the issues were combined), the page numbers, the title of the article, and the author(s) of the article. Volume 1, Number 1 was Fall 1941. Therefore, the year of publication for fall issues is volume+1940, and the year for spring issues is volume+1941. For example, an item listed 15, 2, 34–38 would be in Volume 15 Number 2 (Spring 1956) on pages 34–38.



**See page 26 for more selections!**

# Pinochle Probability

Crystal Vacura, *student*

Kansas Epsilon

Fort Hays State University
Hays, KS 67601

Presented at the 1996 Region IV Convention

When I was growing up, my grandparents taught me how to play pinochle. At that time, I just kept hoping the dealer would pass a queen of spades and a jack of diamonds my way. I never stopped to consider the probability that a hand contained a pinochle. In the contents of this paper, the focus will be directed at the above question — the probability that a hand contains a pinochle. To answer this question, permutations will be used.

Before examining the method of permutations, let me explain a little about the card game. The game of pinochle starts with a deck that consists of 48 cards. We divide these 48 cards into four suites: hearts, clubs, diamonds and spades. Within these suites, we have six different pairs of cards: 9, 10, jack, queen, king and ace. The dealer deals each of the four players 12 cards from the aforementioned deck. These 12 cards are referred to as a hand, which we will refer to later.

The purpose of the game is to get the most points. There are many different ways of accomplishing this. The one we will focus on today is the pinochle. A pinochle is what we call the combination of the queen of spades and the jack of diamonds.

Now, let us move back to our discussion of probability. To compute the probabilities of certain hands, simply dividing the number of distinct hands is not correct. Some hands occur more often than others. Consider, for example, the hand containing all 12 of the clubs. There is only one way to obtain this hand — each and every club must be included. Modify this hand by switching one of the 9 of clubs with a 9 of diamonds, while leaving the rest of the hand intact. The second hand will occur more frequently (four times as often) than the first, because the second hand can contain either of the two 9's of clubs and either of the two 9's of diamonds.

The probability of obtaining all clubs is

$$\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{48\cdot 47\cdot 46\cdot 45\cdot 44\cdot 43\cdot 42\cdot 41\cdot 40\cdot 39\cdot 38\cdot 37}$$
$$=\frac{4.7900\cdot 10^8}{3.3371\cdot 10^{19}}=1.4354\cdot 10^{-11}.$$

With the first card a player is dealt, they have 12 ways to place that club in the hand. With the second card, there is only 11 ways and so forth down to one way. The denominator denotes the total number of ways. The first card can be any of the 48 ways. The second is 47 ways, etc. to the 12th card which can be placed 37 different ways.

The number of ways that a hand contains only clubs, represented by the numerator, is $4.7900\cdot 10^8$. From the denominator, we see that there are $3.3371\cdot 10^{19}$ ways to get any hand. Both of these are correct assuming the deck contains all the cards and the deck was well shuffled. Therefore, the probability that a hand contains all clubs is $1.4354\cdot 10^{-11}$.

Modifying the first hand to include one 9 of diamonds gives us an occurrence of

$$\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 2\cdot 2}{48\cdot 47\cdot 46\cdot 45\cdot 44\cdot 43\cdot 42\cdot 41\cdot 40\cdot 39\cdot 38\cdot 37}$$
$$=\frac{1.9169\cdot 10^9}{3.3371\cdot 10^{19}}=5.7415\cdot 10^{-11}.$$

We use the same process as before in determining the probability of the modified hand. The numerator again shows that the first card has 12 positions it can fill or can be placed in 12 ways. This continues till the last position is filled. The extra 2's at the end of the expression represent the two 9's of clubs and diamonds. When choosing the cards, we can pick either one of these 9's. The numerator, when computed, shows $1.9160\cdot 10^9$ ways this hand can occur. The denominator still represents the total possible ways. Therefore, the probability of the modified hand is $5.7415\cdot 10^{-11}$.

When looking at the ratio, we do see that the second hand does occur four times more frequently than the first:

$$\frac{5.7415\cdot 10^{-11}}{1.4354\cdot 10^{-11}}=4.0000 \text{ times}.$$

Keeping this idea in mind, we can determine the probability that a hand contains a pinochle. The method examined in this process is permutations.

A permutation is defined as any (linear) arrangement of a given collection of $n$ distinct objects which cannot be repeated. In general, if there are $n$ distinct objects and $r$ is an integer with $1\le r\le n$, then by the rule of product, the number of permutations of size $r$ for the $n$ objects is

$$n\cdot(n-1)\cdot(n-2)\cdots(n-r+1),$$

where there are $n$ choices for the object to be put in the 1st position, $n-1$ choices for the object to be put in the 2nd position, etc., ending with $n-r+1$ choices for the object to be put in the $r$th position.

The basic idea of a permutation can be thought of as filling $r$ slots in a line with one object in each slot by drawing these objects one at a time from a pool of $n$ distinct objects. The first slot can be filled in $n$ ways. Extending this reasoning to $r$ slots, we have that the number of ways of filling all $r$ slots is

$$\begin{aligned} n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) &\cdot \frac{(n-r)\cdot(n-r-1)\cdots 3\cdot 2\cdot 1}{(n-r)\cdot(n-r-1)\cdots 3\cdot 2\cdot 1} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

The expression above is commonly referred to as $P(n,r)$ and called the number of permutations of $n$ objects taken $r$ at a time.

We can derive the probability of a pinochle by using permutations. In this case, we assume that the order in which the cards are received matters, so that the two hands {jack of diamonds, queen of spades, ... } and {queen of spades, jack of diamonds, ... } are regarded as different. We must select, in order, 12 cards to complete a hand.

Let $s_1$ denote the index (from 1 to 11) of the first selected card that is a jack of diamonds or a queen of spades. Let $s_2$ denote the index (from $s+1$ to 12) of the first selection that completes the pinochle. Thus, if the index $s_1$ corresponds to choosing a queen of spades, then $s_2$ corresponds to the index of the first jack of diamonds that is selected.

With this information, we can now determine the probability of $s_1$. Because cards 1 through $s_1 - 1$ are not jacks of diamonds or queens of spades, their probability of selection are

$$\frac{44}{48}, \frac{43}{47}, \frac{42}{46}, \ldots, \frac{44-s_1+2}{48-s_1+2},$$

with product

$$\frac{P(44, s_1-1)}{P(48, s_1-1)}.$$

The $s_1$st card can be any of four (two jacks of diamonds and two queens of spades) out of the remaining $48 - s_1 + 1$ cards. Once this card is chosen, its twin must be included in the allowed choices for selections $s_1 + 1$ to $s_2 - 1$. Similarly, we figure the probability of $s_2$ given that the first of the pair was drawn at $s_1$ as

$$\frac{P(46-s_1, s_2-s_1-1)}{P(48-s_1, s_2-s_1-1)}.$$

```
PROGRAM PINOCHLE (INPUT, OUTPUT);

{PRE: NOTHING.                                                  }
{POST: THE PROGRAM FIGURES THE SUM OF PINOCHLE PROBABILITY.     }
{USES: NOTHING.                                                 }

VAR  S1,                      {FOR THE SUMMATION OF OUTSIDE LOOP}
     S2,                      {FOR THE SUMMATION OF INSIDE LOOP}
     A, B, D, E,              {FOR THE PERMUTATION LOOPS}
     NUM1, NUM2,              {FOR THE NUMERATORS OF MAJOR LOOPS}
     DENOM1, DENOM2,          {FOR THE DENOMINATORS OF MAJOR LOOPS}
     SUM,                     {FOR THE SUM OF INSIDE LOOP}
     TOTSUM:   INTEGER;       {FOR THE FINAL PROBABILITY}

BEGIN          {MAIN PROGRAM}

     NUM1 := 1;
     DENOM1 := 1;
     TOTSUM := 0;

     FOR S1 := 1 TO 11 DO
     BEGIN

          FOR A := 44 DOWNTO (44-(S1-1)) DO
               NUM1 := NUM1 * A;
          FOR B := 48 DOWNTO (48-(S1-1)) DO
               DENOM1 := DENOM1 * B;

          SUM := 0;
          FOR S2 := (S1+1) TO 12 DO
          BEGIN

               FOR D := (46-S1) DOWNTO ((46-S1)-(S2-S1-1)) DO
                    NUM2 := NUM2 * D;
               FOR E := (48-S1) DOWNTO ((48-S1)-(S2-S1-1)) DO
                    DENOM2 := DENOM2 * E;
               SUM := SUM + ((NUM2/DENOM2) * (2/(49-S2)));

          END;

          TOTSUM := TOTSUM + ((NUM1/DENOM1) * (4/(49-S1)) * SUM);

     END;

     WRITELN ('THE PROBABILITY IS: ', TOTSUM:10:6);

END.           {MAIN PROGRAM}
```

**Figure 1. PASCAL program.**

The $s_2$nd card must be one of the two possibilities to complete the pinochle, and the rest of the hand is unrestricted.

Thus, we obtain the following expression for the probability of a pinochle by combining the probabilities of $s_1$ and $s_2$:

$$\sum_{s_1=1}^{11}\left(\frac{P(44,s_1-1)}{P(48,s_1-1)}\cdot\frac{4}{49-s_1}\right.$$
$$\left.\cdot\left(\sum_{s_2=s_1+1}^{12}\left(\frac{P(46-s_1,s_2-s_1-1)}{P(48-s_1,s_2-s_1-1)}\cdot\frac{2}{49-s_2}\right)\right)\right).$$

Each part of the expression is disjoint from the others. Therefore, the probability of the whole is the sum of the probability of each part.

In order to solve this problem, I wrote a PASCAL program utilizing the "for loop" process. The program is listed in figure 1.

From running this program, we discover that the probability that we will actually be dealt a pinochle is approximately .185708 or 18.57% of the time.

In the span of about 15 minutes we have solved the problem of probability of a pinochle. In order to do this, we used the method of permutation. There are other combinatorial methods that could be utilized for this problem such as combinations, complements, and generating functions. I choose permutations because it is the basis for the other three.

The next time you play pinochle, I hope you don't depend on that pinochle because the odds are against you.

---

## *Subscription Renewals and Change of Address*

Your *Pentagon* subscription expires with the volume and number that appears in the upper right corner of your address label (see back cover). Since this issue is Volume 57 Number 2, if the code 57-2 appears on your label then this is your last issue!

**Early renewals save us the cost of mailing out renewal notices.** To renew, please send your check — just $10 for four more issues (domestic individuals only; see page 2 for rates for libraries and foreign subsciptions) — together with your name and address and a copy of your old address label to:

*The Pentagon* Business Manager
Division of Mathematics and Computer Science
Emporia State University
Emporia, KS 66801 USA

Please renew promptly to avoid gaps in your journal collection.

**If you have a change of address,** please don't forget *The Pentagon*! Send your change of address to the business manager at the above address. Failure to do so may cost KME additional postal fees!

# Progression of Chaos Theory

Mary Kay Vaske, *student*

South Dakota Alpha

Northern State University
Aberdeen, SD 57401

Presented at the 1996 Region IV Convention

Chaos is a branch of science, mostly mathematics and physics, that suggests an odd order and pattern exist in what was formerly seen as random and unpredictable behavior. Because this behavior was often erratic the study became known as chaos. At times the study of chaos itself seems wild and unpredictable, but it is just as fascinating as it is puzzling. And it is precisely this combination that has granted chaos such a broad audience.

The exploration and development of chaos theory is apparent in the behavior of the weather, of an airplane in flight, of cars grouping on a freeway, and of water flowing in underground pipes. Because of its universal benefit, distinguished scientists say that 20th century science will be remembered for just three things: relativity, quantum mechanics, and chaos ([2], pp. 5–6).

Ancient philosophers in Greece used the term *chaos* to describe the "mysterious and unorganized material which was used by the gods to fabricate the visible universe." In modern English, chaos came to mean "a completely unorganized, turbulently unpredictable state" ([3], p. 1). During the middle ages, matters of complexity were debated by "natural philosophers." These people were scientists and philosophers simultaneously. With the entrance of Sir Isaac Newton (1642–1727), complexity theory would come to rely more on science. Newton contributed differential calculus and the laws of classical mechanics. With these tools it was possible to calculate and thus determine the dynamics of an object by straightforward equations. More and more, complexity was being explored and seemed to form a closed universal system, able to produce the answer to any question asked. When a question could not be answered by classical dynamics it was considered a "psuedoproblem ... that was devoid of fundamental significance" ([5], p. 3). In 1986, Sir James Lighthill, speaking to the Royal Society, said this ([1], pp. xii–xiii):

> "The enthusiasm of our forebearers for the marvelous achievements of Newtonian mechanics led them to make generalizations in this area of predictability which indeed, we may have generally tended to believe before 1960, but which we now recognize were false."

For hundreds of years the predictability of classical mechanics was unchallenged. After all, mechanics was the essential element of the scientific world. In the last three hundred years there has been a slow movement toward both deterministic concepts and chaotic concepts playing the essential role. During the climax of Newtonian physics, scientists believed future events were determined by past states of an object/situation. It was thought that Newton's three laws of motion were enough to predict the planets' paths as well as future events. Eighteenth-century thinking could be summed up with the following: with the correct equation anything could be determined. Thus determinism became the focus of scientific philosophy.

Determinism can be traced back to the time of Socrates (470–399 B.C.), and has always been attractive to many sectors of society, not just scientists ([1], p. 7). Determinism provided one with the power to give lengthy dictation without challenge because one could simply invoke the authority of the laws of science! Pierre Simon Laplace (1749–1827) developed his own theory of determinism. "Laplacian determinism" asserted future events as predictable when complete knowledge of the present state was acquired ([3], p. 4). Laplace's book *Celestial Mechanics* continued to spread his deterministic philosophy. Laplace was not a religious writer, but oddly, his book also reinforced the religious belief of a god that never interferes with the progress of a predetermined plan. Although Newton related his work to a creator, Laplace did not. His intention was to show a universe functioning rationally according to the laws of mechanics. Napoleon chided Laplace for not mentioning a god in connection with his "rationally functioning universe." Laplace replied that he had no need for such a hypothesis ([4], p. 210). In Laplace's deterministic world there would be no chance, no choice, no free will, no uncertainty; everything would be predetermined.

Mathematicians and physicists of the nineteenth century took aim at identifying the tiniest level to which application of dynamics and its deterministic qualities could apply. This would in turn serve as the foundation for explaining all observable phenomena. Scientists began to reconsider the concepts of determinism when in 1927 Karl Heisenberg put forward his thoughts of uncertainty. The Heisenberg Uncertainty Principle stated that both position and velocity of an object could never be known simultaneously with absolute certainty ([3], p. 4). Several members of the scientific community took this in terms of the physical universe being "... a manifestation of swirling changes in an underlying soup of uncertainty" ([3], p. 4). Incidentally, many other scientists, Albert Einstein among them, took issue with this impression. Over the years, many of those scientists who scoffed at Heisenberg's theory (firm believers in Laplacian determinism) continued to claim impending success in capturing knowledge of future events. The newest sensation, computers, were used to gather the so-called "knowledge of the present state" which would be used in solving the mysteries of the universe. Surprisingly it was these computations that began to disprove Laplace's determinism and give credence to Heisenberg. The computers produced large numbers of physical systems that appeared deterministic but were so sensitive to initial conditions that any hope of predictability, although theoretically possible, was fundamentally impossible ([3], pp. 4–5). This surprising discovery was dubbed deterministic chaos because of the complex behavior exhibited by such sensitive deterministic systems.

Where chaos begins classical science stops. The number of deterministic chaotic systems produced by computers soon became infinite. With every gain in science there is always loss. In this situation it was the retirement of deterministic predictability: life's sequence of events is part of an equation and that once that equation is discovered we can see the events of the future. Chaos theory inserts the existence of "sensitive dependence on initial conditions." This means that one or more constants in the equation will suddenly change, causing the outcome to be different than the original equation would predict. Chaos brings the predictability back into perspective with the suggestion that the sudden change(s) is determined by yet another equation that can be solved and so on and so on. One can now realize the depth of this science is in the infinite number of dependent equations produced.

Today most scientists still hold on to deterministic theory in varying degrees. These individuals define deterministic chaos as a retreat from full determinism rather than a defeat of all determinism. It is true that while objects can no longer be described in a predictable manner, their behavior can be anticipated in degrees of certainty. As years of studying chaos went on, it became clear that statistical probability theory and deterministic chaos theory are closely related subjects.

In everyday speech, "chaos" can be used when referring to noise and commotion. Its technical meaning, however, is quite different. In this sense it refers to randomness that arises in a deterministic system. As mentioned previously, chaos possesses a "sensitive dependence on initial conditions." This means that although in principle it should be possible to predict the future dynamics of a system as a function of time, in reality it is impossible because the error in describing an initial condition (and there is always an error in varying degree) leads to an erroneous predicted outcome.

Edward Lorenz, a meteorologist at MIT, was interested in mathematical models for the behavior of the earth's atmosphere. In the early 1960's he studied the use of three ordinary differential equations to define the state of the atmosphere in terms of $x$, $y$, and $z$:

$$\begin{aligned} x &= -\sigma x + \sigma y \\ y &= -xz + rx - y \\ z &= xy - bz \end{aligned}$$

When the parameters $\sigma$, $r$, and $b$ are changed, the characteristics of the system are changed. In Lorenz's system this would represent physical properties of the atmosphere's air. The new computers of his day carried out the integrations for a typical solution to the system. Lorenz had this to say regarding his findings: "It implies that two states differing by imperceptible amounts may eventually evolve into two considerably different states" ([2], p. 12).

Phase space is the center court of dynamic phenomena. Systems occupying this space will behave in strange patterns, move wildly, and sometimes stop altogether. An example of this is a wound-up yo-yo. When allowed to unwind from the hand it will wind itself back up for successively shorter distances. Ultimately, it will come to rest at the lowest point allowed by the length of the string. This stable equilibrium point is called a fixed-point attractor ([1], p. 59). The word *attractor* is used because of the behavior of a system in phase space. Specifically, when a system in phase space is in the vicinity of an attractor, it tends to assimilate the state exhibited by that attractor.

Strange attractors are a second type of attractors operating in phase space. Unlike fixed-point attractors, these involve aperiodic dynamical systems and appear extreme in fluctuation. All of the different types of attractors are related to patterns, even the strange attractors associated with chaos. This relation to pattern contributes greatly to the understanding of phenomena that appears random but has aspects of organization within ([1], pp. 72–73). An example of this would be the weather. In 1963, Professor Lorenz wrote a paper about his three-dimensional equations, now called the Lorenz equations. In his paper he asserts long-term weather prediction to be impossible due to the sensitivity to initial conditions. This sensitive dependence on initial conditions is representative of what Lorenz called "the butterfly effect." If the earth's weather is a chaotic system, then "the flap of a butterfly's wings in Brazil today may make the difference between calm weather and a tornado in Texas next month" ([1], p. 12).

T. Y. Li and J. A. Yorke published a paper in 1975 entitled "Period Three Implies Chaos." It was the first time the term *deterministic chaos*, as we know it today, appeared in the scientific literature. Their article described certain deterministic flows as chaotic. Then in 1976 Robert May, a prominent mathematical biologist, referred to the Li-Yorke paper in his own publication. May's paper discussed seemingly simple equations that can produce complicated dynamics ([1], p. 16). May's paper was widely read and contributed to the acceptance of the term deterministic chaos.

Technical terms used by the scientific community tend to become less precise when the general public becomes active participants. As chaos became accessible to the public through magazines and television it became known as "chaos" rather than its original "deterministic chaos." And once the media began to catch on, they were finding chaos everywhere. Publications and television series utilized the following examples: a running faucet with a flow that breaks into pattern and then returns to a united stream, a flag snapping in the wind, and a column of smoke rising into swirls ([2], p. 5).

The study of deterministic chaos exploded. Articles ranged from technical wording to vibrant visuals. One example of a visual picture of chaos is the Mandelbrot set. In 1975 Benoit B. Mandelbrot (born in 1924) created a new subject called fractal geometry. *Fractal* comes from the Latin *fractus* meaning "broken." This new branch of mathematics studied "rough and fractured systems in materials science" ([3], p. 5). Fractal dimension, the measure of ruggedness and space-filling ability, has become one of the most important aspects in chaos studies today.

In order to explain the benefits of the Mandelbrot set, it is necessary to go back to some earlier challenges in mathematics. One such challenge was a puzzling value that continued to reappear; the solutions of $x^2 + 1 = 0$ included a quantity $(-1)^{1/2}$, or the square root of negative one. In 1645, Italian mathematician Girolamo Cardano described numbers involving $(-1)^{1/2}$ as "imaginary numbers." The symbol $i$ was used to denote this previously unseen value. By the early 1800's, German mathematician Karl Friedrich Gauss showed that a physical meaning could be assigned to $i$ if one looked at how numbers in a two-dimensional plane could be represented in two-dimensional space using: $P_1 = (a + ib)$, where $a$ is the $x$-coordinate, $b$ is the $y$-coordinate, $i$ is the directional measurement of $b$ at right angles to $a$, and $P_1$ is the object's site (pixel location) ([3], p. 87). $P_1$ is known today as $c$ for a "complex number" that includes both real and imaginary parts. Repeating these equations for successive values is called iteration. Creating computer images from several iterations leads to astonishing movements of the pixel location and results in the magnificent Mandelbrot set. The very first picture of a Mandelbrot set was generated at Harvard in 1980. As with many scientific discoveries, it was not intentional. The creation was the result of experiments focusing on pixel size ([3], p. 100). Today, commercial software generates images of hundreds to thousands of iterations. One important point to be made is that photos give only two dimensions to these sets. Many books, too, neglect to mention that Mandelbrot sets can exist in three-dimensional space. In this dimension the images viewed should be considered cross-sections of a three-dimensional Mandelbrot set.

In the 1970's, a few scientists in the United States and Europe began to look at the erratic and unpredictable sides of nature. Individuals from disciplines that were previously separated were brought together by the study of chaos theory. Mathematicians, physicists, biologists, chemists and others searched to find links in the many forms of discontinuity. The cooperation among different fields has developed into a cycle. Scientists continue to discover new initial conditions. Engineers then design mechanisms to use the information. Sociologists contemplate how society can benefit. And finally, artists express what has or may occur as a result.

Within ten years of its start, an abundance of conferences and journals were highlighting the progress of chaos. In the 1980's, government program managers in charge of military research, the CIA, and the Department of Energy soon set up bureaucracies to handle the financing for their chaos research. Today, nearly every major university researches chaos, as does nearly every major corporation ([2], p. 4). Upon recognizing the benefits of chaos, changes were seen in the way business executives make decisions about insurance, the way astronomers look at the solar system, and also the way sociologists discuss stresses that lead to armed conflict ([2], p. 5).

Chaos and fractals have become buzzwords. They are the subjects of motion pictures, wall calendars, and political strategies. The overwhelming popularity of the term "chaos" does cause confusion between its scientific meaning and its everyday use. Nevertheless, the publicity serves to educate the public about a scientific concept that benefits everyone.

Science is about studying cause-and-effect relationships and the predictions that can be drawn from them. Chaos seems to undermine this effort. This does not, however, mean advance planning is in vain. It simply reinforces what is already known; a degree of uncertainty is normal. Once this is accepted, improvements in forecasting can be studied. This is what chaos theory is all about — improving the accuracy of predictions.

*Acknowledgements.* I would like to acknowledge my research advisor, Dr. Elton Fors, for his assistance in this project. Dr. Fors received his Ph.D. in mathematics from the University of Oklahoma and is professor emeritus of mathematics at Northern State University in Aberdeen, South Dakota. Dr. Fors provided the essential guidance and direction I needed to explore the boundless world of chaos theory.

## References

1. Cambel, A. B., *Applied Chaos Theory: a paradigm for complexity*, Academic Press, San Diego, 1993.
2. Gleick, James, *Chaos: making of a new science*, Viking Penguin, New York, 1987.
3. Kaye, Brian, *Chaos and Complexity*, VCH Publishers, New York, 1993.

4. Kline, M., *Mathematics in Western Culture*, Oxford University Press, New York, 1953.
5. Prigogine, I., *From Being to Becoming*, W. H. Freeman, San Francisco, 1980.

## *Selections From the Cumulative Subject Index*

For an explanation of the key, see page 14.



The subject headings available in the cumulative subject index are Abstract Algebra and Algebraic Systems; Applications; Complex Variables; Computers and Computer Science; Differential Equations and Difference Equations; Discrete Mathematics and Set Theory; Fractals, Chaos, Dynamical Systems; Games and Game Theory; Geometry; History of KME and *The Pentagon*; History of Mathematics; Issues of the Profession; Logic, Foundations, Structures; Mathematical Modeling; Mathematics Education; Matrices, Linear Algebra, and Vector Spaces; Multi-Variable Calculus and Analysis; Number Theory and Arithmetic; Numerical Methods; Philosophy, Commentary, etc.; Precalculus Mathematics; Probability, Combinatorics, and Statistics; Recreational Mathematics; Single Variable Calculus and Analysis; and Topology.

# Tinkering with the Quaternion

Dawn M. Weston, *student*

Kansas Gamma

Benedictine College
Atchison, KS 66002

Presented at the 1996 Region IV Convention

When the parents of Sir William Rowan Hamilton conceived this genius, they did not have the slightest notion that the product of their union would become a legend in mathematics. Nowhere is it noted that they decided one day to produce a master of science. The creation of Hamilton can be expressed in mathematical terms:

> *Let all the human bodies that ever existed or will ever exist be domain $B$, and all the souls that exist be domain $S$. Define a function $f$ from $B \times S$ to $H$, the set of all humans ever born or ever to be born. Take a unique element $b$ from $B$ and a unique element $s$ from $S$ and map the pair to a unique element $h$ in $H$. Creation of a unique human can be represented mathematically as $f(b, s) = h$.*

Sometimes such a union is the making of a genius. Just such a union occurred at the creation of William Rowan Hamilton. Hamilton was born into an affluent Dublin, Ireland family in 1805. At a young age he went to live with an uncle. He was a child prodigy and was home schooled. At 18 years old he entered Trinity College. At the tender age of 21 and as an undergraduate he became a professor of astronomy at the college. When he was 27, he received the Cunningham Medal of the Royal Society and was elected into the society. He served as its president between 1837 and 1845 ([1], p. 693). Hamilton was knighted at age 30.

In 1843, Hamilton made what has been called his greatest discovery, quaternions. Hamilton struggled for many years with the problems whose answers were "the algebra of quaternions" ([4], p. 468). This was the first ring to be discovered in which the commutative property does not hold ([4], p. 468). Hamilton turned his attention to a search for a generalization of complex numbers, that is, numbers of the form $a + ib$ (where $i = \sqrt{-1}$). He hoped that such generalized complex numbers would serve to represent rotations in three-dimensional space in much the same way as ordinary complex numbers serve to represent rotations in the plane ([5], p. 137).

For Hamilton, the vector product in quaternion four-space is a rotation in a plane. Depending on the order of the vectors being operated on, the rotation is either counterclockwise (positive rotation) or clockwise (negative rotation). It took many years for Hamilton to find a solution to this problem of multiplying triples. It took a walk to Dublin for him to find the answer to his puzzle.

We can look at our own lives to see how a revelation can occur when least expected. Have you ever struggled with a mathematical problem? No matter how hard you sought the answer as you struggled, the solution never came out of hiding. Then you left your studying environment and found other distractions only to have a vision. This phenomenon has also occurred among those we label as geniuses. William Rowan Hamilton is one of these geniuses that experienced a mathematical revelation while occupying himself in other than academic pursuits. Hamilton's disclosure in mathematical terms:

> *If we define $G$ as the set of all geniuses that have lived and $K$ as the set of all ideas, we can define a function $h$ from $G \times K$ to $D$, the set of discoveries. Take a unique element $g$ (Hamilton) from $G$ and a unique element $k$ (quaternions) from $K$ and map to a unique element $d$ (the algebra of quaternions) from $D$. This new algebra can be represented mathematically as $h(g, k) = d$.*

Eves ([2], p. 102) describes the event this way:

> Hamilton has told the story that the idea of abandoning the commutative law of multiplication came to him in a flash, after fifteen years of fruitless meditation, while walking along the Royal Canal near Dublin with his wife just before dusk. He was so struck by the unorthodoxy of the idea that he took out his penknife and scratched the gist of this multiplication table into one of the stones of Brougham Bridge. A cement tablet embedded in the stone of the bridge tells the story:
>
> > Here as he walked by
> > on the 16th of October 1843
> > Sir William Rowan Hamilton
> > in a flash of genius discovered
> > the fundamental formula for
> > quaternion multiplication
> > $i^2 = j^2 = k^2 = ijk = -1$
> > & cut it in a stone of this bridge.

The Brougham Bridge was affectionately named by his children the Quaternion Bridge.

The thought of a system that was not commutative was unheard of. The possibility that given two elements $a$ and $b$, $ab$ not equal to $ba$ was a violation of a fundamental law of algebra.

There are many daily activities that illustrate that actions are not always commutative. What about moving left and right? If I first turn left, walk one mile then turn right and walk another mile, I will be in a different location than if I first turn right, walk one mile, turn left and walk another mile. This operation in mathematical terms:

> *If $L$ is a one unit move to the left and $R$ is a one unit move to the right, a left followed by a right does not equal a right followed by a left. This can be expressed $RL \neq LR$. This is a noncommutative operation. See figure 1.*

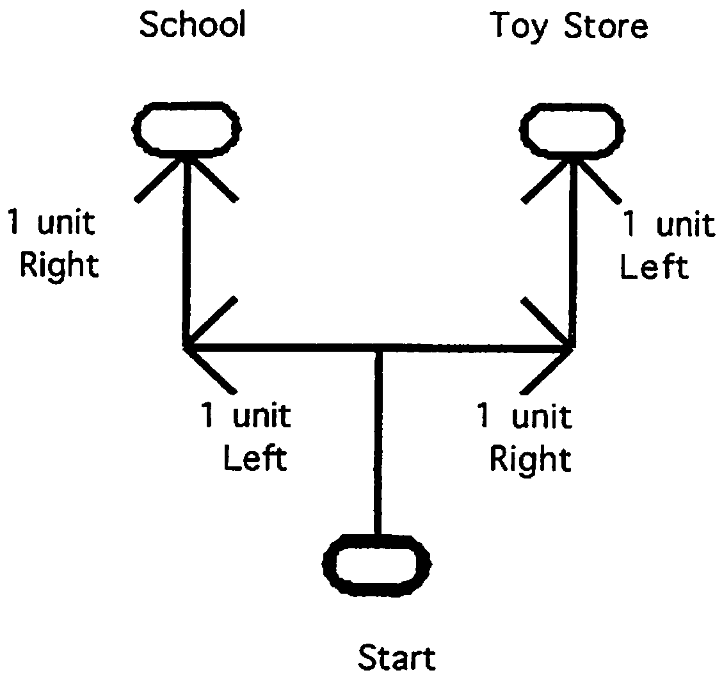



**Figure 1. A noncommutative operation.**

Look around in the natural world. There are many such examples in life. So if mathematics describes the natural world, why was it impossible to conceive of a system in mathematics being noncommutative? The discovery of just such a system was possible for Sir William.

Hamilton's view of algebra helped free him from its constraints. He wrote "The study of Algebra may be pursued in three very different schools, the Practical, the Philological or the Theoretical, according as Algebra itself is accounted an Instrument or a Language, or a Contemplation" ([3], p.

257). Hamilton studied his quaternions without being tied down to the current mindset governing algebra. Hamilton modeled "complex numbers by pairs of ordered numbers, which can represent rotation in a two-dimensional space, and from here he went on trying to represent rotations in a three-dimensional space by triplets" ([1], p. 693). He realized that he needed a four-tuple to master his new discovery. Quaternions became "ordered real number quadruples $(a, b, c, d)$ having both the real and complex numbers embedded within them" ([2], p. 102).

Hamilton's view of complex numbers as ordered pairs where the complex number $a + bi$ ($a$ and $b$ are real numbers) could be seen as the ordered pair $(a, b)$ led to his discovery. The $a$ would be mapped onto the horizontal axis and the imaginary component $b$ mapped onto the vertical axis. Hamilton extended this into ordered quadruples, $(a, b, c, d)$, where $a$ is a real component and $b$, $c$ and $d$ are coefficients of the imaginary components $i$, $j$ and $k$. The vectors $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ are unit vectors with magnitude 1 and directions represented by $i$, $j$ and $k$ in this system.

We now picture the quaternion components $i$, $j$ and $k$ as vectors in three-dimensional space. They are unit vectors with direction. Using the right hand-hand rule, the vector product of $i$ with $j$ is $k$. That is, $i \times j = k$. This is a 90 degree rotation in the counterclockwise direction. The vector product of $j$ with $i$ is the opposite of $k$, namely $j \times i = -k$. This is a 90 degree rotation in the clockwise direction. A geometrical representation of $i$, $j$ and $k$ is in figure 2.

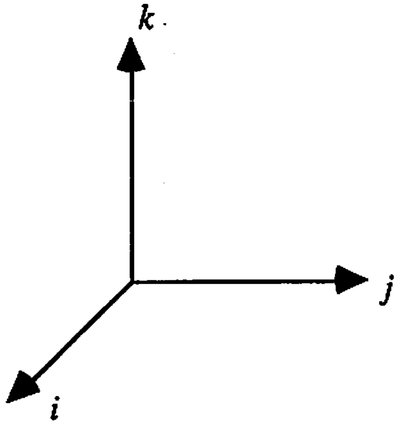



**Figure 2. The vectors $i$, $j$, and $k$.**

All the quaternion relations can be formed by using figure 3. The clockwise rotation of the group taking the first element cross product with the next clockwise element produces the third element. The counterclockwise rotation, one element cross product with the next counterclockwise element, produces the negative of the third element. For example, clockwise rotation produces $ij = k$, and counterclockwise rotation produces $ji = -k$. There are various ways to visualize the quaternion relations. Tinkertoys can be utilized to build a three-dimensional representation similar to that shown in figure 2.

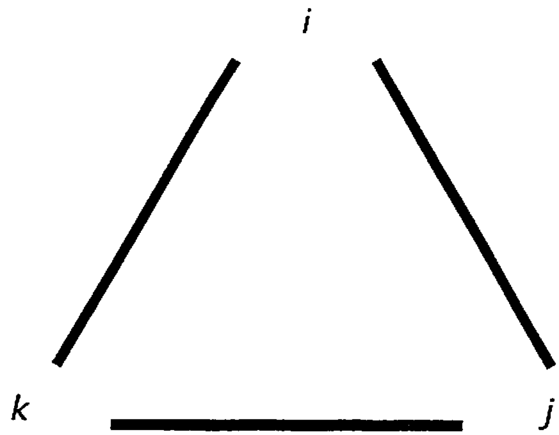



**Figure 3. Triangular relationship of $i$, $j$ and $k$.**

Using Hamilton's bridge notation, a Cayley graph can be created. We will use the following relations in the quaternions for the graph:

$$i^2 = j^2 = k^2 = -1$$
$$ij = k, jk = i, ki = j$$
$$ji = -k, kj = -i, ik = -j.$$

The elements of this system can be generated with $i$, $j$, and $k$, by $\langle i \rangle = \{1, i, -1, -i\}$, $\langle j \rangle = \{1, j, -1, -j\}$, and $\langle k \rangle = \{1, k, -1, -k\}$.

These relationships can be used to fill in a Cayley table. Using the definition of quaternions as an expression $a + bi + cj + dk$, where $a$, $b$, $c$ and $d$ are real numbers and $i$, $j$ and $k$ are unit vectors, and the operation on the elements as cross product of vectors, we can use the right-hand rule to create a Cayley table. The operations on 1 and $-1$ are the same as multiplying vectors by scalar multiples. See figure 4 for the Cayley table.

Hamilton's quaternion group can also be represented by a Cayley digraph. Using the quaternion relationships $i^2 = j^2 = k^2 = (ij)^2 = -1$, $i^4 = j^4 = k^4 = 1$, $ij = k$ and $ji = -k$, $i$ and $j$ can generate all elements of the quaternion group. See figure 5.

What can we learn from the life of Hamilton? We can learn more than just how to manipulate a four-tuple. When faced with a difficult problem for what seems like too long of a time, get away from your work area.

| | 1 | i | j | k | - 1 | - i | - j | - k |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | i | j | k | - 1 | - i | - j | - k |
| i | i | - 1 | k | - j | - i | 1 | - k | j |
| j | j | - k | - 1 | i | - j | k | 1 | - i |
| k | k | j | - i | - 1 | - k | - j | i | 1 |
| - 1 | - 1 | - i | - j | - k | 1 | i | j | k |
| - i | - i | 1 | - k | j | i | - 1 | k | - j |
| - j | - j | k | 1 | - i | j | - k | - 1 | i |
| - k | - k | - j | i | 1 | k | j | - i | - 1 |

**Figure 4.** Cayley table.

Find an enjoyable diversion. Go for a walk. Play with tinkertoys. Spend time with a loved one. You may make that once-in-a-lifetime discovery. Mathematics can answer many questions about our natural world and our natural world can help us discover a lot about mathematics.

*Acknowledgements.* I wish express a special thank you to Sister Jo Ann Fellin, O.S.B., Ph.D. for her patience and faith in this student. She kept me focused when I was ready to give up.

## References

1. Balaban, Alexandru T., "W. R. Hamilton: His Genius, His Circuits, and the IUPAC Nomenclature for Fulleranes," *Journal of Chemical Education* **72** No. 8 (August 1995), 693–698.
2. Eves, Howard, *Great Moments in Mathematics (After 1650),* The Mathematical Association of America, Washington, DC, 1981.
3. Fisch, Menachem, "'The emergency which has arrived': the problematic history of nineteenth-century British algebra — a programmatic outline," *British Journal for the History of Science* **27** No. 9 (September 1994), 247–

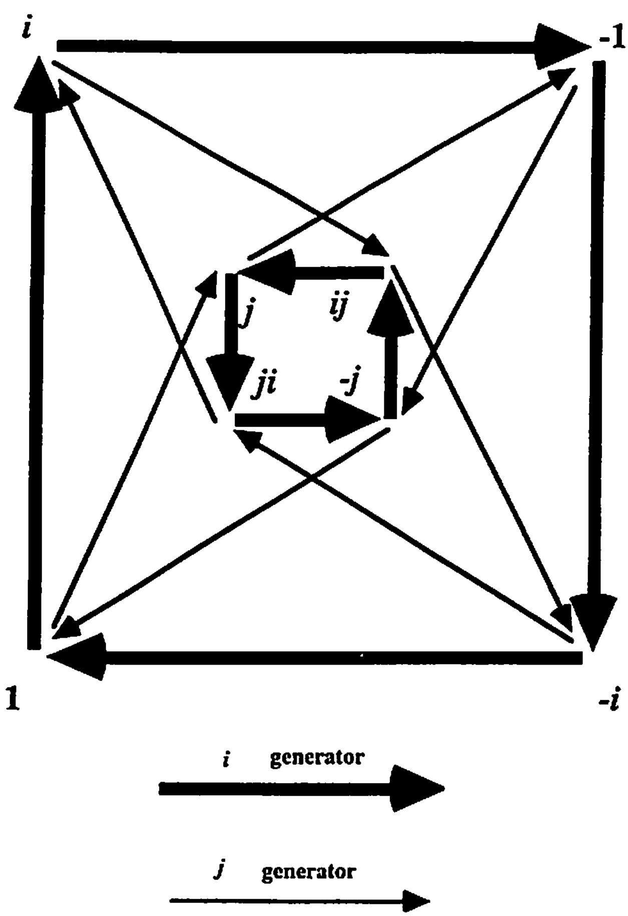



Figure 5


276.

4. Gallian, Joseph A., *Contemporary Abstract Algebra,* D. C. Heath and Company, Lexington, Massachusetts, 1994.
5. Grossman, Israel and Magnus, Wilhelm, *Groups and Their Graphs,* Random House, New York, 1964.


## *Twisted Tic-Tac-Toe*

Have you ever wanted to play tic-tac-toe on a torus or on a Klein bottle? Try this web site by Jeff Weeks:

www.northnet.org/weeks/TorusGames/TorusGames.html

# Choices, Choices, Choices

Jeffrey D. Blanchard, *student*

Kansas Gamma

Benedictine College
Atchison, KS 66002

Presented at the 1997 National Convention

If a child found a small sphere of gold weighing 12 ounces, his parents may choose to have it sold and the proceeds of this discovery placed in a savings account to earn interest until the child attends college. This could lead to a rather fine amount of money to be put towards education. On the other hand, using mathematics in their favor, the parents could choose to take this small amount of gold to a man in South America who claims to have performed the first real-world application of the Banach-Tarski paradox. He claims that, using a pair of tweezers, a small jeweler's saw, and a computer program that can perform the Banach-Tarski paradox, he has taken a twelve-ounce sphere of gold and created a sphere of twice the diameter, weighing 49.58 ounces. It took almost seven months according to his claims ([1], p. 119). If the child's twelve ounces of gold can be increased fourfold every seven months, then his parents have certainly made a better financial choice.

The Banach-Tarski paradox of 1924 is the most famous of arguments surrounding the acceptance of the axiom of choice. The paradox is a mathematical truth when we accept the axiom of choice. However, controversies such as the success of this anonymous South American mathematician/alchemist, who now claims to create five pounds of gold a week out of nothing ([1], p. 119), have created a heated debate surrounding the axiom of choice.

First we must understand exactly what the axiom of choice means in order to discuss the controversy surrounding it. We will use the following definition of the axiom of choice in this discussion:

**Axiom of choice.** *For any set $A$, there is a function, $f$, such that for any non-empty subset $B$ of $A$, $f(B) \in B$.*

The function $f$ is referred to as the choice function for $A$. The domain of the function is the family of non-empty subsets of $A$, and the range is $A$.

In order to clarify what a choice function actually does, Suppes offers a wonderfully simple example in [8] (p. 239). In the following example the axiom of choice is not needed. Let there be a set $A$ and all of the non-empty subsets of $A$ defined as

$$A = \{1,2\}, \quad B_1 = \{1\}, \quad B_2 = \{2\}, \quad B_3 = \{1,2\}.$$

Let $f_1$ and $f_2$ be distinct choice functions for $A$. Then

$$f_1(B_1) = 1$$

and

$$f_2(B_1) = 1.$$

These two are equal because the choice functions $f_1$ and $f_2$ must return exactly one element of the subset. Since there is only one element, they must return the same element. Therefore, it follows that

$$f_1(B_2) = f_2(B_2) = 2.$$

We now see where the two functions become distinct. In order for them to be distinct they must not create an identical set of outputs. Therefore, it follows that

$$f_1(B_3) = 1$$

and

$$f_2(B_3) = 2,$$

or vice versa. From this demonstration we obtain a formal definition for the choice function as postulated by the axiom of choice ([8], p. 239):

**Definition.** *We say $f$ is a choice function for $A$ if and only if $f$ is a function whose domain is the family of non-empty subsets of $A$ and for every $B \subseteq A$ with $B \neq \emptyset$, $f(B) \in B$.*

This seems to be exceptionally similar to the axiom of choice. This does not occur accidentally. While we can simply define a choice function for a finite set, the axiom of choice postulated the existence of the choice function for infinite sets.

Let us look at a simple theorem that does not require the axiom of choice, but is more like an argument for the existence of choice functions.

**Theorem.** *Every finite set has a choice function.*

This, too, is obvious in that if a set has $n$ elements then it has $\prod_{k=2}^{n} k^{C(n,k)}$ distinct choice functions. These distinct choice functions must only follow the behavior of the choice functions in the previous example. Although these can be easily defined for the finite, the choice functions for the infinite, said to exist by the axiom of choice, have created the controversy mathematicians discuss today.

We now must look at the applications of the axiom of choice in order to further the knowledge base necessary to make accurate arguments about the validity of the axiom. We first examine a proof that requires the axiom of choice. We shall use the simple proof of the following theorem:

**Theorem.** *If a set is infinite then it has a countably infinite subset.*

*Proof.* Let $A$ be an infinite set and let $f$ be a choice function for $A$ as postulated by the axiom of choice. First we define a function $g$ on the non-negative integers such that

$$g(0) = f(A)$$

and

$$g(n) = f(A - \{g(k) \mid k < n\}).$$

From this definition we obtain

$$\begin{aligned} g(1) &= f(A - \{g(0)\}) \\ g(2) &= f(A - \{g(0), g(1)\}) \\ g(i) &= f(A - \{g(0), g(1), \ldots, g(i-1)\}). \end{aligned}$$

We can see from the behavior of the function $g$ that $g$ assigns $x_0 = f(A)$ to 0, $x_1 = f(A - \{x_0\})$ to 1, $x_2 = f(A - \{x_0, x_1\})$ to 2, etc. Since $f(B) \in B$ is always true, then $g$ is a one-to-one function.

Now suppose there is an $n$ such that

$$A - \{g(k) \mid k < n\} = \emptyset;$$

then $A$ has $n$ elements. This is contrary to the hypothesis that $A$ is an infinite set, so

$$A - \{g(k) \mid k < n\} \neq \emptyset.$$

Therefore, the range of $g$ is a subset of $A$ with a one-to-one correspondence with the non-negative integers. Therefore $g$ creates a countably infinite subset ([8], p. 241).

As we can see from this proof, the axiom of choice provides the choice function which is used to show the existence of the subset that we now see to be countably infinite. However, it gives no method by which the subset is constructed, creating one of the problems with the axiom of choice.

Now let us turn our attention to the arguments surrounding the acceptance of the axiom of choice. First of all, can the axiom of choice be derived from the other axioms of Zermelo-Fraenkel set theory? There is absolutely no controversy surrounding any of the other axioms of Zermelo-Fraenkel set theory. For several sets, even infinite sets, these axioms can be used to show the existence of a choice function. Yet in 1963, Paul Cohen proved that there exist sets that require the axiom of choice to show the existence of a choice function ([7]). Therefore, the axiom of choice can not be derived from the other six axioms of Zermelo-Fraenkel set theory. This answers the first of three questions, all of which must be true for the mathematical world to accept the axiom of choice as an axiom.

The second question was answered in 1938 by Kurt Gödel. The question can take two equivalent forms: "Is the axiom of choice consistent with the other six axioms of Zermelo-Fraenkel set theory?" or "Does the axiom of choice lead to a contradiction?" Gödel proved in 1938 "that the axiom of choice is relatively consistent, that is, if the other axioms of [Zermelo-Fraenkel] set theory are consistent, the addition of the axiom of choice will not lead to a contradiction" ([8], p. 250). This proof is the second of three necessary steps to accepting the axiom of choice.

The final question is the one still debated today. Should we accept the axiom of choice as an axiom? *Mathematics Dictionary* defines the term axiom as "a self-evident and generally accepted principle" ([4], p. 25). Let us first look at the second part of the definition: "generally accepted." Since the axiom of choice is not accepted by everyone, the question becomes an interpretation of the word "generally" and the argument continues. Does "generally" have an exact quantity that can be used as a measuring tool? Do ninety-nine percent of the mathematicians have to accept it, or does it just need to be over fifty percent? There is no numerical quantity assigned to the word "generally." However, another dictionary defines an axiom as "a proposition, principle, role, or maxim that has found general acceptance or is thought worthy thereof on the basis of an appeal to self-evidence" ([2], p. 153). This leads back to the first half of the original definition, an argument of the self-evidence of the axiom of choice.

The arguments for the acceptance of the axiom of choice do not attempt to show self-evidence of the axiom; they argue that the axiom simply provides the existence of something self-evident, which is a choice function. They contend a choice function is self-evident because it merely selects an element from each set ([5], p. 59). The remaining arguments are more along the lines that we must accept it because it is useful and necessary to prove a large portion of set theory. If we do not accept the axiom of choice, then we cannot accept any of its equivalents. A secondary part to this argument is that without the axiom of choice, we cannot accept any mathematical statement which requires the axiom for its proof. These arguments for accepting the axiom are weak. Simply because it is useful does not allow us to accept it as a mathematical truth. It does, however, pose new questions: what mathematics do we get with the axiom of choice, and what mathematics do we get without it?

There are four main arguments against the acceptance of the axiom of choice. First, it is obvious that the axiom of choice is non-constructive; that is, it creates a set of elements which it has chosen using a choice function, without providing any sort of procedure in doing so. Many mathematicians believe that if we cannot show how the set is formed or how we obtained the choice function, then the axiom of choice is invalid on the grounds that it is useless. The second argument is that the axiom of choice is confusing and not as aesthetically pleasing as the other axioms of Zermelo-Fraenkel set theory ([7]). This is a matter of opinion and not an acceptable argument in a mathematical debate. These first two arguments are no better than the arguments for the acceptance of the axiom of choice. These could be easily dismissed and the debate would be over. It is the final two arguments that keep the controversy alive today.

The third argument is that the axiom of choice is equivalent to many statements that are not self-evident, the first argument to address the definition of an axiom. However, it is the fourth argument that receives the most attention. The final argument is that the axiom of choice can be utilized to derive many results that are not self-evident. The most famous derivation is the Banach-Tarski paradox. In 1924, Banach and Tarski, two Polish mathematicians, proved that any two bounded sets in Euclidean space $\mathbf{R}^n$ are equidecomposable if they contain interior points and if $n > 2$ ([6], p. 973). This allows us to state that a "pea may be cut up into infinitely many pieces which can be rearranged to yield the sun (in volume if not in substance)" ([6], p. 973). They go on to tell us that these pieces cannot be obtained "using scissors or other cutting devices. They are obtained using the axiom of choice" ([6], p. 973). To further this argument I will now prove the following:

**Theorem.** *Any object in Euclidean space* $\mathbf{R}^3$ *can be composed into a volume greater than any given volume.*

*Proof.* Let $n = 3$ in the Banach-Tarski theorem of 1924 that any two bounded sets in Euclidean space $\mathbf{R}^n$ are equidecomposable if they contain interior points and if $n > 2$. Let the given volume be $10^n$ meters$^3$. Let the first object be any object in three-space, say a normal casino die. Let the second object be a cube with each side measuring $10^{(n+1)/3}$ meters. The two objects do not necessarily need to be the same shape. This is done for simplicity. The two objects are equidecomposable according to the Banach-Tarski theorem of 1924. Now the die has been decomposed and rearranged into a cube with a volume of $10^{n+1}$ meters$^3$. Therefore, if the given volume is $10^n$ meters$^3$, the original object can be composed into an object with the volume $10^{n+1}$ meters$^3$.

It doesn't actually stop there. There is no volume that is unachievable according to this theorem. If the volume is continually increased for the second object, then the volume created by the original object can always be greater, using the Banach-Tarski paradox.

This is obviously not self-evident, that a die, or any object for that matter, can be decomposed and rearranged into a volume greater than any given volume. Yet if we accept the axiom of choice, we must accept the Banach-Tarski proof of 1924 and therefore we must accept this theorem. This is only one example of the many apparent contradictions that can be derived using the axiom of choice. This argument along with the third argument against the axiom of choice are the two best arguments surrounding the acceptance of the axiom of choice.

Unfortunately, the debate does not stop there. This proof I have just shown cannot be proven to be invalid, however, it can be dismissed as a philosophical interpretation of mathematical truths. That is to say, it can be used as the basis to dismiss the axiom of choice according to constructivism, the leading school of thought among applied mathematicians. Constructivists believe that the only valid proofs are constructive proofs and that anything that cannot be constructed is invalid, for it is useless. Mathematicians have not yet been able to apply the Banach-Tarski paradox, nor construct a method used by the choice functions postulated by the axiom of choice. There are, however, two other main schools of thought in mathematics today: Platonism and formalism. A Platonist believes that all mathematics is either true or false and will accept the axiom of choice if and only if they believe it to be true. The formalist will accept any mathematical principle that is true in any given mathematical setting ([7]).

Until a definite proof or disproof of the axiom of choice is published and accepted, the axiom of choice should be used in a formalistic manner. If the axiom of choice is kept strictly within the bound of Zermelo-Fraenkel set theory, it is harmless. As previously mentioned, it is independent of the other axioms and does not cause a contradiction. However, when taken outside of strict, abstract set manipulation, as in the case of the Banach-Tarski theorem of 1924, it can become an apparent contradiction as proven above.

The axiom of choice, developed to rid set theory of the antinomies, has possibly become the single most debated topic in mathematics this century. Norstrand proclaims "that it is a mathematical truism ... that the more generally a theorem applies, the less deep it is" ([3], p. vi). This is not the case in regards to the axiom of choice. The more freedom we give to the boundaries in which the axiom of choice is applied, the more controversial it becomes. A formalistic approach to the use of the axiom of choice and a definite restriction to set manipulation within Zermelo-Fraenkel set theory allows us to maintain its existence independent of human intuition. Maybe, the anonymous South American entrepreneur actually exists, and will finally have made enough gold to come forward with his techniques. Until then, the choices surrounding the axiom of choice shall remain controversial.

*Acknowledgements.* A sincere thank you is in order to Mr. Richard Farrell for all of his help in constructing this project, and to all the members of Benedictine College who helped me during the production of this paper.

References

1. Dewdney, A. K., "Computer Recreations: A Matter Fabricator Provides Matter for Thought," *Scientific American* 260 No. 4 (April 1989), 116–119.
2. Gove, Philip Babcocke, ed., *Webster's Third New International Dictionary of the English Language Unabridged,* 1981.
3. Halmos, Paul R., *Naive Set Theory,* D. Van Norstrand Company, Princeton, NJ, 1960.
4. James, Glenn and James, Robert C., ed., *Mathematics Dictionary,* D. Van Norstrand Company, Princeton, NJ, 1949.
5. Kaplansky, Irving, *Set Theory and Metric Spaces,* Allyn and Bacon, Boston, 1972.
6. Kleiner, I. and Movshovitz-Hadar, N., "The Role of Paradoxes in the Evolution of Mathematics," *The American Mathematical Monthly* 101 No. 10 (December 1994), 963–974.
7. McGough, Nancy, "The axiom of choice," http://daisy.uwaterloo.ca/~alopez-o/math-faq/node36.html (1994). Verified September, 1996.
8. Suppes, Patrick, *Axiomatic Set Theory,* D. Van Norstrand Company, Princeton, NJ, 1960.

## *Back Issues*

Is your journal collection complete? Copies of most back issues of *The Pentagon* are still available for $5.00 per copy for individuals, $10.00 per copy for libraries. Please send inquiries to:

*The Pentagon* Business Manager
Division of Mathematics and Computer Science
Emporia State University
Emporia, KS 66801 USA

## *Pseudoquote*

Will Rogers the mathematician: "I never met an element of the empty set I didn't like."

# Newton to Chaos: An Unexpected Turn in Numerical Analysis

Kari A. Hamm, *student*

Kansas Alpha

Pittsburg State University
Pittsburg, KS 66762

Presented at the 1997 National Convention

During my History of Mathematics course two semesters ago, I did a research paper on Sir Isaac Newton. Since then, I have been fascinated with how much he has contributed to mathematics. In my Numerical Analysis I course, my instructor introduced Newton's method as a fixed-point iteration: given $x$, the next $x$ is given by $Nx = g(x) = x - f(x)/f'(x)$. For a few weeks, we studied how Newton's method worked and ways that it could fail. It was at this time that I decided to pursue the topic of chaos. I started noticing different aspects of chaos. I became aware of the logistic map and the period-doubling road to chaos. The logistic map $g(x) = \lambda x(1-x)$ maps the unit interval $[0,1]$ into itself when $0 \leq \lambda \leq 4$. It can be shown that there is a sequence $0 < \lambda_1 < \cdots < \lambda_n < \cdots < 4$ of values of $\lambda$ with the property that for each $\lambda_n$ there is a point $x$ in $[\lambda_n, \lambda_{n+1})$ whose orbit is periodic with period $2^n$. I knew that Newton's method could not converge for $f(x) = x^2 + 1$ since the initial guess and all iterates were real numbers while the roots of $f(x)$ were complex. Moreover, the Newton iterates appeared to be "chaotic." It is this interesting concept that I have decided to focus on. I will show in this paper the basic properties that are needed for chaos and that the Newton iterates for $f(x) = x^2 + 1$ are in fact chaotic.

To begin, I need to introduce some basic terms and concepts. First, let $g : S \to S$ be a function where $S$ is an infinite set of real numbers. In my examples, $S$ will be the real line or an interval of real numbers. The main concept needed is the idea of a chaotic function. I will now lay before you the definition of a chaotic function, but there will be several terms that will need to be defined, which I will do thereafter in a stepwise fashion.

## Definitions

The ultimate goal of this first section is to understand what a *chaotic* *function* is:

**Definition 1.** *We say $g : S \to S$ is a chaotic function if*
*1) the periodic points of $g$ are dense in $S$,*
*2) $g$ is topologically transitive, and*
*3) $g$ exhibits sensitive dependence on initial conditions.*

Now that the immediate goal is in sight, I will define the necessary concepts for chaos. In the following definitions, $g$ is a function from $S$ to $S$.

**Definition 2.** *Let $x \in S$. The* orbit *of $x$ is the sequence of $g$-iterates of $x$; that is, the orbit of $x$ is the sequence $x_0 = x$, $x_1 = g(x_0)$, $x_2 = g(x_1)$, ..., $x_{n+1} = g(x_n)$, ....*

**Definition 3.** *Let $x \in S$. Then $x$ is a* fixed point *of $g$ if $g(x) = x$; that is, the orbit of $x$ is the constant sequence $x, x, x, \ldots$.*

*Example 3.1.* The fixed points of $g(x) = x^{1/2}$ on the interval $[0, \infty)$ are $\lambda = 0$ and $\lambda = 1$.

**Definition 4.** *The* doubling map *is the map $D : [0,1) \to [0,1)$ defined as follows:*

$$D(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2x - 1 & \text{if } \frac{1}{2} \le x < 1. \end{cases}$$

The doubling map will be very important for the remainder of this paper. Points with periodic orbits are easy to compute for the doubling map. The examples for definitions 5 and 6 are periodic points over the doubling map.

**Definition 5.** *Let $x \in S$. Then $x$ is* periodic *with period $n \ge 1$ if the first $g$-iterates $x_0 = x, \ldots, x_{n-1}$ are distinct but $x_n = x_0$.*

*Example 5.1.* Notice $x = 1/3$ is a periodic point with orbit $1/3 \to 2/3 \to 1/3 \to \cdots$.

*Example 5.2.* Note that $x = 1/5$ is a periodic point with orbit $1/5 \to 2/5 \to 4/5 \to 3/5 \to 1/5 \to \cdots$.

**Definition 6.** *Let $x \in S$. Let $x_0 = x, \ldots, x_{n-1}, x_n$ be the orbit or $g$-iterates of $x$. Then $x$ is* eventually periodic *if there exists $N \ge 1$ so that*
*1) the $N$th iterate $x_N$ is periodic, and*
*2) if $N > 1$, $x_{N-1}$ is not periodic.*

*Example 6.1.* Notice $x = 1/14$ is eventually periodic with orbit $1/14 \to 1/7 \to 2/7 \to 4/7 \to 1/7 \to \cdots$.

**Definition 7.** *Let $D$ be the real line or an interval of real numbers. A set $Z \subset X$ is* dense *in $D$ if for every $x \in D$ and for every open interval $I_x^{\delta_x} = (x - \delta_x, x + \delta_x)$ with center $x$ and radius $\delta_x > 0$ there is a point of $Z$ in the interval.*

*Examples 7.1.* The rational numbers are dense on the real line. The dyadic rationals (rational numbers whose denominators are a power of 2) are dense on the real line.

**Definition 8.** *We say $g : X \to X$ is* topologically transitive *if for any pair of points $x$ and $y$ and any $\epsilon > 0$ there is a third point $z$ within $\epsilon$ of $x$ whose orbit comes within $\epsilon$ of $y$; that is, there is a point in the orbit of $z$ whose distance from $x$ is less than $\epsilon$ and a point in the orbit of $z$ whose distance from $y$ is less than $\epsilon$.*

**Definition 9.** *We say $g : X \to X$ is* sensitively dependent on initial conditions *if there is a $\delta > 0$ such that for any $x$ and $\epsilon > 0$, there is a $y$ within $\epsilon$ of $x$ and a $k \geq 1$ such that the distance between $x_k = g^k(x)$ and $y_k = g^k(y)$ is at least $\delta$.*

Now that I have the basic concepts I need, I can get to the heart of the matter. I will show that for $f(x) = x^2+1$, the map $g(x) = x - f(x)/f'(x)$ is chaotic through the following steps: 1) I will prove that the doubling map $D$ on $[0,1)$ is chaotic; 2) I will consider how the doubling map $D$ and the Newton iterate map $g(x) = x - f(x)/f'(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$ for $f(x) = x^2+1$ are related. To do this, I will need to introduce the concept of two topologically conjugate maps and discuss the importance of topological conjugacy for showing that $g(x)$ is indeed chaotic.

**Step 1: The doubling map is chaotic.**

We have seen examples of points in $[0,1)$ where orbits under the doubling map $D$ are periodic and eventually periodic. After repeated calculations, I noticed the following patterns:

**Proposition 1.** *Let $D : [0,1) \to [0,1)$ be the doubling map and let $x = a/b > 0$ where $b \neq 0$ and $a$ and $b$ have no common factors. Then*

*1) if $b$ is a power of two (that is, a dyadic rational), then except for a finite number of terms the orbit of $x$ consists of zeros;*

*2) if $b$ is odd, then $x$ is periodic; and*

*3) if $b$ is even but not a power of two, then $x$ is eventually periodic.*

*Proof.* All three assertions are apparent if we consider $D$ from the proper perspective. For $x \in [0,1)$, let $x_1/2 + x_2/2^2 + x_3/2^3 + \cdots = 0.x_1x_2x_3\ldots$ be a binary expansion of $x$. Then, $D(x) = x_2/2 + x_3/2^2 + x_4/2^3 + \cdots = 0.x_2x_3x_4\ldots$; that is, $D$ shifts the binary expansion of $x$ by one to the left, dropping the first binary digit $x_1$.

To clarify this, consider three examples.

Example of assertion 1: let $x = 5/8 = 1/2 + 1/8 = 0.101$. Then $D(x) = 1/4 = 0.01$, $D^2(x) = 1/2 = 0.1$, and $D^3(x) = 0 = 0.0$.

Example of assertion 2: let $x = 11/15$. Then $11/15 \to 7/15 \to 14/15 \to 13/15 \to 11/15 \to \ldots$. This orbit in terms of the binary expansion of $x$ is

$$\begin{aligned}
11/15 &= 0.1011\ 1011\ 1011 \cdots \to \\
7/15 &= 0.011\ 1011\ 1011 \cdots \to \\
14/15 &= 0.11\ 1011\ 1011 \cdots \to \\
13/15 &= 0.1\ 1011\ 1011 \cdots \to \\
11/15 &= 0.1011\ 1011 \cdots \to \cdots .
\end{aligned}$$

Example of assertion 3: let $x = 9/22$. Then $9/22 \to 9/11 \to 7/11 \to 3/11 \to 6/11 \to 1/11 \to 2/11 \to 4/11 \to 8/11 \to 5/11 \to 10/11 \to 9/11 \ldots$. This orbit in terms of the binary expansion of $x$ is

$$\begin{aligned}
9/22 &= 0.0\ 1101000101\ 1101000101 \cdots \to \\
9/11 &= 0.1101000101\ 1101000101 \cdots \to \\
7/11 &= 0.101000101\ 1101000101 \cdots \to \\
3/11 &= 0.01000101\ 1101000101 \cdots \to \\
6/11 &= 0.1000101\ 1101000101 \cdots \to \\
1/11 &= 0.000101\ 1101000101 \cdots \to \\
2/11 &= 0.00101\ 1101000101 \cdots \to \\
4/11 &= 0.0101\ 1101000101 \cdots \to \\
8/11 &= 0.101\ 1101000101 \cdots \to \\
5/11 &= 0.01\ 1101000101 \cdots \to \\
10/11 &= 0.1\ 1101000101 \cdots \to \\
9/11 &= 0.1101000101 \cdots \to \cdots .
\end{aligned}$$

**Proposition 2.** *The set of periodic points of the doubling map is dense in* $[0, 1)$.

*Proof.* Since $Z = \{a/b \mid b > 0,\ a \text{ and } b \text{ have no common factors, and } b \text{ is odd}\}$ is dense in $[0, 1)$ and each point of $Z$ is a periodic point of $D$, then the first set of periodic points of the doubling map is dense in $[0, 1)$.

**Proposition 3.** *The doubling map is transitive.*

In definition 8, I defined topologically transitive. From that definition and the following lemma, I will prove that $D$ is topologically transitive.

**Lemma.** *If two numbers $x, y$ in $[0, 1)$ are close, that is, if the binary expansions of $x$ and $y$ have the first block of $k$ digits in common, then* $d(x, y) \leq 1/2^{k-1}$.

In order to prove that $D$ is topologically transitive, it suffices to show that there is a $z$ in $[0,1)$ so that the orbit of $z$ comes arbitrarily close to both $x$ and $y$.

*Proof* (of proposition 3). Let $x = 0.x_1x_2\ldots$ and $y = 0.y_1y_2\ldots$ be two distinct points of $[0,1)$. By a block of length $k$, I will mean the set of binary expansions of $0,\ldots,2^{k-1}$. Thus, the blocks of length $k = 1$, 2, and 3 are as follows:

$$\begin{aligned} &\text{for } k=1: && 0,1;\\ &\text{for } k=2: && 00,01,10,11;\\ &\text{for } k=3: && 000,001,010,011,100,101,110,111. \end{aligned}$$

Let $z$ be the number in $[0,1)$ whose binary digits are obtained by concatenating the blocks of length $k \geq 1$; that is,

$$z = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111\ldots.$$

I claim that the orbit of $z$ (which consists of the sequence obtained by shifting the binary expansion of $z$ to the left one digit at a time) is close to both $x$ and $y$. We also need the following notation: for $n \geq 1$, we let $\sigma^n z$ be the number obtained by shifting the binary expansion of $z$ $n$ times to the left.

Let $\epsilon > 0$ be given. Pick $n \geq 1$ so that $1/2^n < \epsilon$. Let $b_x$ ($b_y$) be the block consisting of the first $n+1$ digits of the binary expansion of $x$ ($y$). Both the blocks $b_x$ and $b_y$ occur in $z$. Thus, we can shift $z$ to the left until the block $b_x$ occurs in the first digits of $z$; that is, there is $k_x \geq 0$ so that the binary expansions of $x$ and $\sigma^{k_x} z$ match. Therefore, $d(x, \sigma^{k_x} z) \leq 1/2^n < \epsilon$; that is, $\sigma^{k_x} z$ is close to $x$. A similar argument shows that some element in the orbit of $z$ is close to $y$.

In definition 9, I defined sensitive dependence. From that definition, I will prove proposition 4.

**Proposition 4.** *The doubling map is sensitively dependent on initial conditions.*

*Proof.* It suffices to show that there is $\delta > 0$ so that for any $x$ in $[0,1)$ and for any $\epsilon > 0$, there is a $y$ so that $d(x,y) < \epsilon$ and there is a $k \geq 0$ so that the $k$th points in the orbits of $x$ and $y$ differ by at least $\delta$; that is, $d(\sigma^k x, \sigma^k y) \geq \delta$. Let $\delta = 1/2$. Let $x = 0.x_1x_2\ldots$ be in $[0,1)$. Let $\epsilon > 0$ be given. Pick $k \geq 1$ so that $1/2^k < \epsilon$. Let $y = 0.x_1x_2\ldots$ be the binary number with $y_i = x_i$ for all $i \geq 1$ except for $i = k+2$. Let $y_{k+2} = 1 - x_{k+2}$. Then $d(x,y) \leq 1/2^k < \epsilon$ but $d(\sigma^{k+1}x, \sigma^{k+1}y) = 1/2 = \delta$.

By the last three propositions, we now know that the doubling map $D$ is chaotic.

**Step 2: The Newton iterate map is chaotic.**

We will show that the Newton iterate map $g(x) = x - f(x)/f'(x) = \frac{1}{2}(x - \frac{1}{x})$ is chaotic because it is topologically conjugate to the doubling map $D$.

**Definition 10.** *If a function $f$ is one-to-one, onto, and continuous and its inverse is continuous, then the function $f$ is* a homeomorphism.

**Definition 11.** *Let $f : X \to X$ and $g : Y \to Y$ be functions. Then $f$ is* topologically conjugate *to $g$ if there is a homeomorphism $\tau : X \to Y$ such that $\tau \circ f = g \circ \tau$; then, $\tau$ is called* a topological conjugacy.

This relationship is represented by the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\tau"'] & X \arrow[d, "\tau"] \\ Y \arrow[r, "f"'] & Y \end{tikzcd}$$

Topological conjugacy is important because it sometimes allows for the relation of a known chaotic map, such as the doubling map, to a map we are interested in studying. To be more precise, the following lemma is a working tool for those who thrive on chaos. For a proof of this proposition, please refer to [2] (pp. 89–91) or any standard text on chaos.

**Proposition 5.** *If $f$ and $g$ are topologically conjugate, then $f$ is chaotic if and only if $g$ is chaotic.*

In [1], Robert L. Devaney argues that the doubling map $D$ and the Newton iterate map $g(x) = \frac{1}{2}(x - \frac{1}{x})$ for $f(x) = x^2 + 1$ are topologically conjugate using the conjugacy map $\tau(x) = \cot(\pi x)$, and thus the following commutative diagram holds:

$$\begin{tikzcd} {[0,1)} \arrow[r, "D"] \arrow[d, "\tau"'] & {[0,1)} \arrow[d, "\tau"] \\ \mathbf{R} \arrow[r, "g"'] & \mathbf{R} \end{tikzcd}$$

It would follow by proposition 5 that since $D$ is chaotic, $g$ is chaotic.

There are, I have noticed, several problems with this theory. The trouble is that the diagram is not quite accurate. There are two obvious difficulties:

1) $\tau(0)$ is not defined. It is true, however that $\tau : (0,1) \to (0,1)$ is a homeomorphism.

2) $g$ is not defined on $\mathbf{R}$ since $g(0)$ is undefined. Things are even worse: there are many points $x \in \mathbf{R} \setminus \{0\}$ for which the $g$-orbit of $x$ is not defined.

I will consider the second difficulty first. Once we have identified an appropriate domain for $g$, the first difficulty will disappear!

I now list some points $x$ in $\mathbf{R}$ for which the $g$-orbit of $x$ is not defined.

$$\begin{aligned}
0 &\to \text{does not exist (d.n.e.)} \\
1 &\to 0 \to \text{d.n.e.} \\
-1 &\to 0 \to \text{d.n.e.} \\
1+\sqrt{2} &\to 1 \to 0 \to \text{d.n.e.} \\
1-\sqrt{2} &\to 1 \to 0 \to \text{d.n.e.} \\
-1+\sqrt{2} &\to -1 \to 0 \to \text{d.n.e.} \\
-1-\sqrt{2} &\to -1 \to 0 \to \text{d.n.e.} \\
&\text{etc.}
\end{aligned}$$

To properly find a domain on which $g$ is defined and for which all points have $g$-orbits, we need to remove from $\mathbf{R}$ the points whose $g$-orbits eventually contain zero. Since $\tau(1/2) = 0$, we can find the points of $\mathbf{R}$ whose orbits eventually become zero by finding the points in $(0,1)$ whose $D$-orbits eventually become $1/2$ and then take their image under the map $\tau$. A brief calculation shows the following:

$$\begin{aligned}
1/4 &\to 1/2 \\
3/4 &\to 1/2 \\
1/8 &\to 1/4 \to 1/2 \\
3/8 &\to 3/4 \to 1/2 \\
5/8 &\to 1/4 \to 1/2 \\
7/8 &\to 3/4 \to 1/2 \\
1/16 &\to 1/8 \to 1/4 \to 1/2 \\
3/16 &\to 3/8 \to 3/4 \to 1/2
\end{aligned}$$

The set of points in $[0,1)$ whose orbits eventually become $1/2$ can now be defined as follows:

$$\begin{aligned}
DQ &= \{a/b \mid 0 < a < b, \text{ where } b \text{ is a power of two and } a \text{ is odd}\} \\
&= \text{ the set of dyadic rationals in the interval } (0,1).
\end{aligned}$$

We are now able to fix those few problems and make $\tau$ a topological conjugacy. First, let $DQ$ be the set of dyadic rationals in $(0,1)$ and let

$\Delta = [0,1) \setminus DQ = (0,1) \setminus DQ$. Then, let $D^*(x) = D(x)$ for $x \in \Delta$. Then $D^*(x)$ is chaotic on $\Delta$. Let $\mathbf{R}^* = \mathbf{R} \setminus \tau(DQ)$ and let $\tau^*(x) = \tau(x)$ for $x \in \Delta$. Then $\tau^*$ is a homeomorphism from $\Delta$ to $\mathbf{R}^*$. Finally, let $g^*(x) = g(x)$ for $x \in \mathbf{R}^*$. Using these definitions, the maps $D^* : \Delta \to \Delta$, $g^* : \mathbf{R}^* \to \mathbf{R}^*$, and $\tau^* : \Delta \to \mathbf{R}^*$ are properly defined and $\tau^*$ is a homeomorphism.

**Proposition 6.** *For the maps as defined above, $t^* \circ D^* = g^* \circ \tau^*$; that is, $\tau^*(D^*(x)) = g^*(\tau^*(x))$ for all $x \in \Delta$ or, equivalently, the following commutative diagram holds:*

$$\begin{array}{ccc} \Delta & \xrightarrow{D^*} & \Delta \\ {\scriptstyle \tau^*}\downarrow & & \downarrow{\scriptstyle \tau^*} \\ \mathbf{R}^* & \xrightarrow[g^*]{} & \mathbf{R}^* \end{array}$$

*Proof.* Consider $g*(x)$. We have

$$\begin{aligned} \tau^*(D^*(x)) &= \cot(\pi D(x)) = \cot(2\pi x) \\ &= \frac{\cos^2(\pi x) - \sin^2(\pi x)}{2\sin(\pi x)\cos(\pi x)} \\ &= \frac{1}{4}\cot(\pi x) - \frac{1}{\cot(\pi x)} = g^*(\tau^*(x)). \end{aligned}$$

Therefore, $g^*(x)$ is conjugate to the doubling function $D^*(x)$.

Now that we have established topological conjugacy between the two maps, we can focus on showing that $g(x)$ is chaotic.

**Proposition 7.** *Both $D^*$ and $g^*$ are chaotic.*

*Proof.* The details of this proof are not difficult. I will outline the details for both $D^*$ and $g^*$.

Let $PP = \{a/b \mid b \text{ is an odd number and } 0 < a < b\}$. Then $PP$ consists of periodic points of $D^*$ in $\Delta$ and $PP$ is dense in $\Delta$. Also, $\tau^*(PP)$ consists of periodic points of $g^*$ in $\mathbf{R}^*$ and $\tau^*(PP)$ is dense in $\mathbf{R}^*$. An example goes as follows: (1) $x = 1/3 \to 2/3 \to 1/3$ is a periodic orbit of $D^*$ in $\Delta$; (2) $\tau^*(1/3) = \cot(\pi/3) = \cos(\pi/3)/\sin(\pi/3) = 1/\sqrt{3}$; and (3) $\tau^*(x) = 1/\sqrt{3} \to -1/\sqrt{3} \to 1/\sqrt{3}$ is a periodic orbit of $g^*$ in $\mathbf{R}^*$.

The orbit of $z = 0\ 1\ 00\ 01\ 10\ 11 \cdots \in \Delta$ is dense in $\Delta$ and thus $D^*$ is topologically transitive. The orbit of $\tau^*(z) \in \mathbf{R}^*$ is dense in $\mathbf{R}^*$ and thus $g^*$ is topologically transitive.

Similarly, since $z \in \Delta$, the sensitive dependence of $D$ on $[0,1)$ implies the sensitive dependence of $D^*$ on $\Delta$. Finally, the sensitive dependence of $g^*$ follows from that of $D^*$ and the fact that $\tau^*$ is a homeomorphism.

Now all problems have been fixed!!! Therefore $D^*$ is chaotic on $\Delta$ which indicates that $g^*$ is chaotic on $\mathbf{R}^* = \mathbf{R} \setminus \tau(DQ)$.

With that, I conclude my presentation of a Newton's method trip to chaos. I know that the trip has been quick and short, but I hope quite interesting. Taking such a known concept as Newton's method and getting an unexpected response was quite enjoyable. The one thing that I am sure of: The trip from Newton to Chaos is well worth the time!

*Acknowledgements.* I would like to give a special "thank you" to Dr. Gary McGrath for all of his time helping me prepare this paper.

References

1. Devaney, Robert L., *A First Course in Chaotic Dynamical Systems: Theory and Experiment*, Addison-Wesley, New York, 1992.
2. Holmgren, Richard A., *A First Course in Discrete Dynamical Systems*, second edition, Springer-Verlag, New York, 1996.

## *From the Pages of ...*

"The following is an excerpt from a fascinating document contributed to the editor's [J. M. Sachs'] collection by a friendly statistician. The document is meant to apply to statistical design and quality control but the mathematical representation seems worth repeating here. There is no source indicated on the material so credit cannot be given.

"Years ago when the Universe was relatively easy to explain the famous Finagle Constant $K_f$ was introduced so that $x' = x + K_f$, where $x$ represents a measured variable, $x'$ its theoretical counterpart and $K_f$ is arbitrary. Later as difficulties compounded $F_f$, the fudge factor appeared and $x' = F_f x + K_f$ was used as an aid and comfort to those in distress. In World War II the multiplicity of experiments made a stronger influence imperative and some unsung hero rose to the occasion with the diddle factor $F_d$ so that it was now possible to use $x' = F_d x^2 + F_f x + K_f$. This helped a lot. It is felt that for the present reality can be brought into reasonable agreement with theory by the use of these three constants and no further extension in this direction is anticipated in the immediate future. It seems wise to point out however that there is a difference in the structure and thus the use of the three constants. The Finagle Constant changes the universe to fit the equation. The fudge factor changes the equation to fit the universe. Finally the diddle factor changes both just enough to insure an adequate fit somewhere about half way between. This sacrifices both reality and theory and is known as statistical fence sitting."

—*The Pentagon*, Fall 1960 (pp. 45–46)

# The Problem Corner

Edited by Kenneth M. Wilke

*The Problem Corner* invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before January 1, 1999. Solutions received after the publication deadline will be considered also until the time when copy is prepared for publication. The solutions will be published in the Spring 1999 issue of *The Pentagon*. Address all communications to Kenneth M. Wilke, Department of Mathematics, 275 Morgan Hall, Washburn University, Topeka, Kansas 66621 (e-mail: xxwilke@acc.wuacc.edu).

## PROBLEMS 515–519

*Problem 515.* Proposed by Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin.

Consider the following equation:

$$(*) \qquad 4(x^2 - x + 1)^3 - 27(x-1)^2x^2 = (x-2)^2(2x-1)^2(x+1)^2.$$

Either (a) prove that the equation (*) holds for each real number $x$ using elementary algebra or (b) find a real number $x$ such that the left side of (*) does not equal the right side of (*).

*Problem 516.* Proposed jointly by Underwood Dudley, DePauw University, Greencastle, Indiana and Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri.

For every positive integer $n$, prove that there exists a prime $p$ with $n$ digits.

*Problem 517.* Proposed by the editor.

Consider a rectangular piece of paper $ABCD$ where $AB = CD = 24$ inches and $BC = DA = 10$ inches. Next bring point $A$ into coincidence with point $C$ and fold the sheet, creating a crease from $AB$ to $CD$. How long is the crease?

*Problem 518.* Proposed by Russell Euler and Jawad Sadek, jointly, Northwest Missouri State University, Maryville, Missouri.

Let $x$ be a positive integer greater than 1. Prove that $x^{20k+4}+x^{10k+2}+1$ is composite for all nonnegative integers $k$.

*Problem 519.* Proposed by the editor.

Define a sequence of integers $a_1, a_2, a_3, \ldots$ where $a_1$ is an arbitrarily chosen positive integer and for $k > 1$ , $a_k = (3a_{k-1}/2) + 1$. Can one find a value for $a_1$ such that $a_{1001}$ is odd and $a_k$ is even for all integers $k < 1001$?

*Please help your editor by submitting problem proposals.*

## SOLUTIONS 503, 505–509

*Editor's comment.* The following names were inadvertently omitted from lists of solvers in previous columns: Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri (problems 490, 491, 493, and 494) and Carl Libis, University of Alabama, Tuscaloosa, Alabama (problem 499). The editor apologizes for these oversights. Also, late solutions for problem 504 were received from Scott H. Brown, Montgomery, Alabama and James Dunn and Julie Dunn (jointly), Fresno Problem Solving Group, California State University, Fresno, California.

*Problem 503.* Proposed by C. Bryan Dawson, Emporia State University, Emporia, Kansas.

Using only a compass and an unmarked straightedge, construct the orthocenter, circumcenter, centroid, and the nine-point circle of an arbitrary triangle using the compass six or fewer times. The drawing of the nine-point circle is included as one of the uses of the compass.

*Solution* by the proposer.

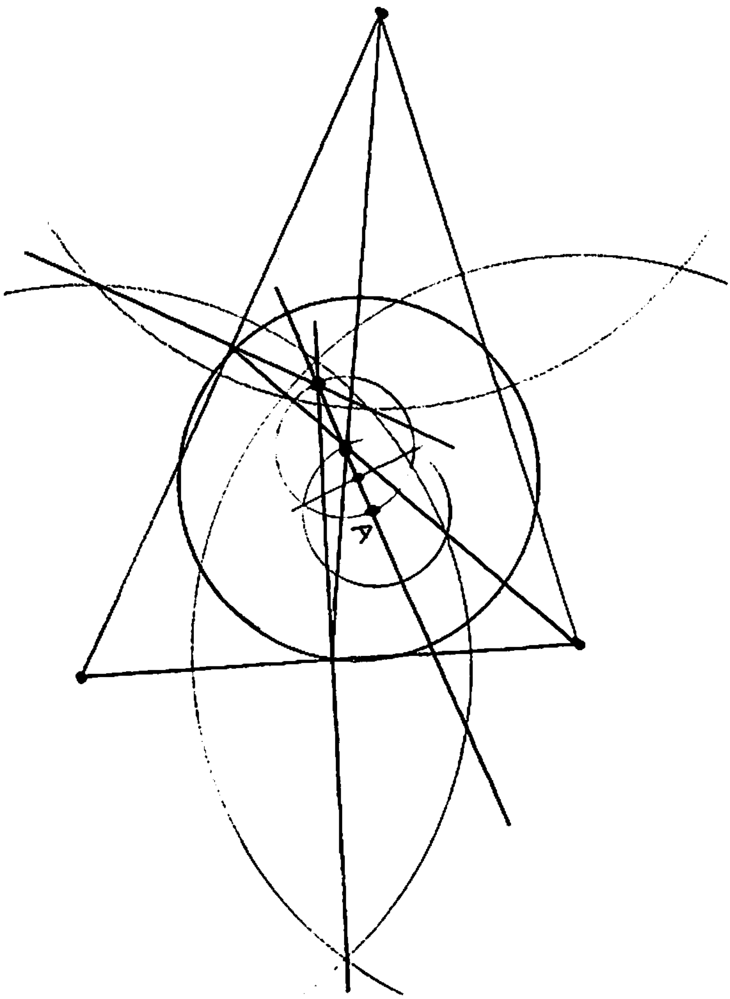
The solution can be achieved by following these steps. See the figure for an illustration.

1. Using the vertices of a triangle as centers, construct a circle at each vertex. The radius of each circle shall be the same and the radius shall be large enough so that the circles intersect each other in pairs. (This accounts for three uses of the compass.) Now the straightedge is used to find the perpendicular bisector of each side of the triangle, the circumcenter of the triangle, the median to each side of the triangle, the centroid of the triangle, and the Euler Line which passes through the centroid and the circumcenter.

2. Using the centroid as a center, construct the circle which passes through the circumcenter and label the second intersection of this circle with the Euler line as $A$. (This accounts for the fourth use of the compass.)

3. Using the point $A$ as a center, construct the circle which passes through the centroid. (This accounts for the fifth use of the compass.) This circle intersects the Euler line at the triangle's orthocenter since the centroid trisects the line segment joining the circumcenter and the orthcenter. Also the line connecting the points of the two circles constructed in steps 2 and 3 intersects the Euler line at the center of the nine-point circle.

4. Construct the nine-point circle using the point found in step 3 as a center and another point on this circle, such as a midpoint on a side of the triangle, to determine the radius of the nine-point circle.



*Editor's comment.* Clayton W. Dodge, University of Maine—Orono, Orono, Maine, points out that the Poncelet-Steiner construction theorem states that any Euclidean construction, insofar as the given and required elements are points, may be accomplished with straightedge alone in the presence of a given circle and its center [H. Eves, College Geometry, Boston, Jones and Bartlett, 1995, p. 185]. Thus one must use the compass just twice, the first time to draw an arbitrary circle to satisfy the hypothesis of the Poncelet-Steiner theorem, and the second time to draw the nine-point circle after locating its center and any of the points on its circumference. Dodge also notes that if one can use both sides of the straightedge, so that one can draw a pair of parallel lines, one does not need the arbitrary circle to perform the constructions. In this case just one use of the compass suffices, the actual drawing of the nine-point circle.

*Problem 505.* Proposed by J. Sriskandarajah, University of Wisconsin Center—Richland, Richland Center, Wisconsin.

If $a+b+c=abc$, prove that

$$\frac{2a}{1-a^2}+\frac{2b}{1-b^2}+\frac{2c}{1-c^2}=\frac{8abc}{(1-a^2)(1-b^2)(1-c^2)}.$$

*Solution* by Donna K. Wilkinson, Pittsburg State University, Pittsburg, Kansas. (Revised by the editor.)

Combining terms, the left side of the given relation becomes

$$\frac{2a(1-b^2)(1-c^2)+2b(1-a^2)(1-c^2)+2c(1-a^2)(1-b^2)}{(1-a^2)(1-b^2)(1-c^2)}.$$

Hence it suffices to show that

$$(1)\quad 2a(1-b^2)(1-c^2)+2b(1-a^2)(1-c^2)+2c(1-a^2)(1-b^2)=8abc$$

given that $a+b+c-abc=0$.

Expanding the left side of (1) and collecting terms, we have

$$\begin{aligned}&2a(1-b^2)(1-c^2)+2b(1-a^2)(1-c^2)+2c(1-a^2)(1-b^2)\\&\quad=2(a+b+c)+2ac(-a-c+abc)+2bc(-b-c+abc)\\&\qquad+2ab(-a-b+abc)\\&\quad=2abc+2abc+2abc+2abc=8abc.\end{aligned}$$

This completes the proof.

*Also solved by:* Scott H. Brown, Montgomery, Alabama; James Dunn, Julie Dunn and Anthony Leyba (jointly), Fresno Problem Solving Group, California State University, Fresno, California; Mark Maxwell, Missouri Southern State College, Joplin, Missouri; Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin; Tran van Thuong, Missouri Southern State College, Joplin, Missouri; and the proposer.

*Problem 506.* Proposed by Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin.

Find all of the positive integer values of $n$ for which the expression $4n^2 + 21n$ is a perfect square.

*Solution* by Casey Leavitt, student, Missouri Southern State College, Joplin, Missouri.

Let $A^2 = 4n^2 + 21n$ where $A$ and $n$ are positive integers. Then since $21n > 0$, $A^2 > 4n^2$ or $A > 2n$. Let $A = 2n+C$ for some positive integer $C$. Then $(2n+C)^2 = 4n^2 + 21n$ and by our choice of $n$ and $C$, we must have $21n - 4nC = C^2 > 0$. Hence $21 > 4C$ so that $C$ must be 1, 2, 3, 4 or 5. By testing each of these values of $C$ in the equation $(2n+C)^2 = 4n^2+21n$ and looking for integer values of $n$, only $C = 3$ corresponding to $n = 1$ and $C = 5$ corresponding to $n = 25$ are solutions. Hence the only possible values for $A^2$ are 25 corresponding to $n = 1$, and 3025 corresponding to $n = 25$.

*Also solved by:* Alexander Shaumyan, Eastern Kentucky University, Richmond, Kentucky; Linda Obeid, Alejandro Munguia and Mike Fuller (jointly), Fresno Problem Solving Group, California State University, Fresno, California; and the proposer.

*Problem 507.* Proposed by Kenichiro Kashihara, Sagamihara, Kanagawa, Japan.

Given any integer $n \geq 1$, the value of the pseudo-Smarandache function $Z(n)$ is the smallest integer $m$ such that $n$ evenly divides $\sum_{k=1}^{m} k$. Let $p$ be a positive prime and $s$ be an integer $\geq 2$. Show that

$$Z(p^s) = \begin{cases} p^{s+1} - 1 \text{ if } p \text{ is even} \\ p^s - 1 \text{ if } p \text{ is odd.} \end{cases}$$

*Solution* by Carl Libis, University of Alabama, Tuscaloosa, Alabama.

We know that $\sum_{k=1}^{m} k = m(m+1)/2$ for some integer $m > 0$. Furthermore, $\gcd(m, m+1) = 1$. Now for $p$ an odd prime, $p^s \mid m(m+1)/2$ implies that either $m = jp^s$ or $m+1 = jp^s$ for some positive integer $j$. By taking $j = 1$, and taking the smaller of the two values, we have $Z(p^s) = p^s - 1$, if $p$ is odd.

Now for $p = 2$, $p^s \mid m(m+1)/2$ implies that either $m = j2^{s+1}$ or

$m+1=j2^{s+1}$. By taking $j=1$, and taking the smaller of the two values, we have $Z(2^s)=2^{s+1}-1$.

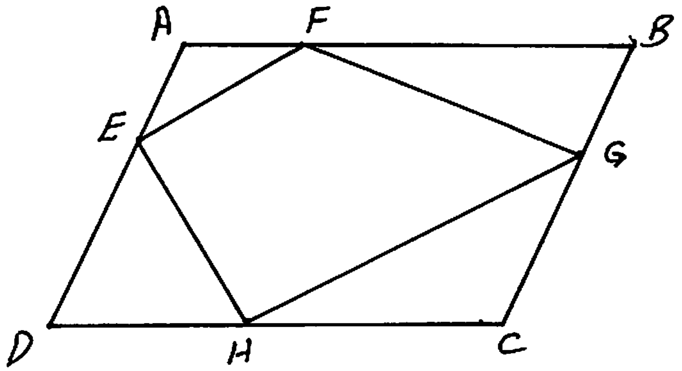
Thus

$$Z(p^s)=\begin{cases} p^{s+1}-1 \text{ if } p \text{ is an even prime } (p=2); \\ p^s-1 \text{ if } p \text{ is an odd prime.} \end{cases}$$

*Also solved by:* Charles Ashbacher, Hiawatha, Iowa; Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh,Wisconsin; Alexander Shaumyan, Eastern Kentucky University, Richmond, Kentucky; Tran van Thuong, Missouri Southern State College, Joplin, Missouri and the proposer.

*Problem 508.* Proposed by the editor.

Let $ABCD$ be a parallelogram. Let $EFGH$ be a quadrilateral inscribed in parallelogram $ABCD$ such that the area of $EFGH$ is exactly half the area of parallellogram $ABCD$. Show that at least one diagonal of $EFGH$ is parallel to a side of $ABCD$ (see figure below).



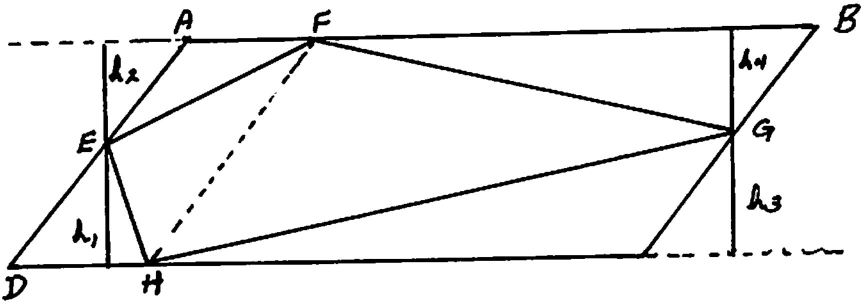

*Solution* by Michael Robert Kleinhenz, Missouri Southern State College.

We shall use the figure below (see p. 56) with the quantities designated as marked. Let $AB=CD=a$, $AF=y$, $FB=a-y$, $DH=x$ and $HC=a-x$. Note that $h_1$ is the line which passes through the point $E$ and is perpendicular to the line $DC$. Similarly $h_2$ is the line which passes through the point $E$ and is perpendicular to the extension of the line $AB$. Similarly $h_3$ is the line which passes through the point $G$ and is perpendicular to the extension of line $DC$. Similarly $h_4$ is the line which passes through the point $G$ and is perpendicular to line $AB$.

If $h_1=h_3$, then the diagonal $EG$ is parallel to both of the sides $CD$ and $AB$ and we are done. Now suppose that $EG$ is not parallel to the line $DC$. Then $h_1\neq h_3$ and $h_1-h_3\neq 0$. Then by our assumption, the sum of the areas of the triangles $EDH$, $AEF$, $FBG$ and $GHC$ equals one half of the area of parallelogram $ABCD$. Hence

$$(1) \qquad x \cdot h_1 + y \cdot h_2 + (a - x) \cdot h_3 + (a - y) \cdot h_4 = a \cdot (h_3 + h_4).$$

Now since $h_1 + h_2 = h_3 + h_4$ or equivalently, $h_1 - h_3 = h_4 - h_2$, equation (1) simplifies to $x(h_1 - h_3) = y(h_4 - h_2) = y(h_1 - h_3)$, or $x = y$. Hence $FH$ is parallel to $AD$.




*Also solved by:* Charles Ashbacher, Hiawatha, Iowa; Alejandro Munguia, Linda Obeid and Mike Fuller (jointly), Fresno Problem Solving Group, California State University, Fresno, California and Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin. One incorrect solution was received.

*Problem 509.* Proposed by Kenichiro Kashihara, Sagamihara, Kanagawa, Japan.

Given any integer $n \geq 1$, the value of the pseudo-Smarandache function $Z(n)$ is the smallest integer $m$ such that $n$ evenly divides $\sum_{k=1}^{m} k$.

(a) Solve the diophantine equation $Z(x) = 8$.

(b) Show that for any positive integer $p$ the equation $Z(x) = p$ has solutions.

(c*) Show that the equation $Z(x) = Z(x + 1)$ has no solutions.

(d*) Show that for any given positive number $r$ there exists an integer $s$ such that $|Z(s) - Z(s + 1)| > r$.

*Solution* to (a), (b), and (d) by Tran van Thuong, Missouri Southern State College, Joplin, Missouri.

First we shall establish the following:

Lemma: *For any positive integer $n$, $Z(n) = (-1 + \sqrt{1 + 8nt})/2$ where $t$ is the smallest positive integer no greater than $2n - 1$ for which the expression $1 + 8nt$ is an odd perfect square.*

*Proof:* Let $Z(n) = m$ where $m$ is a positive integer. Then by the definition of the pseudo-Smarandache function, $2n$ divides the product $m(m+1)$. Then $m(m + 1) = 2nt$ for some positive integer $t$. Solving the quadratic equation $m^2 + m - 2nt = 0$ and taking the positive root, we have

$$m = Z(n) = \frac{-1 + \sqrt{1 + 8nt}}{2}. \tag{1}$$

By taking $t = 2n - 1$, $1 + 8nt = 1 + 8n(2n - 1) = (4n - 1)^2$ which is an odd perfect square. Noting that equation (1) shows that $m$ decreases as the value of $t$ decreases, the minimality of $m$ can be guaranteed by choosing $t \leq 2n - 1$ and such that $1 + 8nt$ is an odd perfect square. Choosing $t = 1$ shows that $(-1 + \sqrt{1 + 8n})/2$ is the lower bound for $Z(n)$. Choosing $t = 2n - 1$ shows that $2n - 1$ is an upper bound for $Z(n)$. This proves the lemma.

As a result of the lemma, we obtain the following results: if $n$ is an even integer, we can take $t = 2n - 1$ so that

$$\frac{-1 + \sqrt{1 + 8n}}{2} \leq Z(n) \leq 2n - 1; \tag{2a}$$

if $n$ is an odd integer, we can take $t = (n - 1)/2$ so that

$$\frac{-1 + \sqrt{1 + 8n}}{2} \leq Z(n) \leq n - 1. \tag{2b}$$

We now solve part (a). Since $Z(x) = 8$, $x$ must be a divisor of $(8 \cdot 9)/2 = 36$. Then since there is no integer $r < 8$ for which $x$ divides $r(r + 1)/2$, we need only check $x = 9$, 12, 18 and 36. One can easily verify that each of the values $x = 9$, 12, 18 and 36 produces a solution.

Solution to part (b). Equation (2b) establishes that a solution for the equation $Z(n) = p$ exists for each odd prime $p$. Since $Z(3) = 2$, then there is always a solution to the equation $Z(n) = p$ for each prime $p$. [Editor's comment: The proposer notes that since $\sum_{k=1}^{m} k = p(p + 1)/2$, $Z(p(p + 1)/2) = p$. This corresponds to taking $t = (p + 1)/2$ in equation (1).]

Solution to part (d). Let $r$ be a positive integer and let $n$ be a positive integer greater than $r$. Then let $s = 2^n$; thus $s + 1 = 2^n + 1$ which is odd. Then by equation (2b), $Z(2^n + 1) \leq 2^n$. By Problem 507, $Z(2^n) = 2^{n+1} - 1$. Finally $Z(2^n) - Z(2^n + 1) \geq 2^{n+1} - 2^n - 1 > n > r$.

*Solution* to part (c) by Bryan Dawson, Emporia State University, Emporia, Kansas.

Suppose $Z(x) = Z(x+1) = m$, where $m$ is the smallest integer for which $x$ divides $m(m+1)/2$. Then we have $x \mid m(m+1)/2$ and $x+1 \mid m(m+1)/2$. Since $x$ and $x + 1$ are relatively prime, we see that $x(x + 1) \mid m(m + 1)/2$. Therefore $m(m + 1)/2 = kx(x + 1)$ for some integer $k$, and $m(m + 1) = 2kx(x + 1)$. Thus $x < m$, i.e., $x < m = Z(x) = Z(x + 1)$.

Note that if $x$ is odd, then $x \mid x(x+1)/2$. Alternatively, if $x$ is even, then $x+1$ is odd and $x+1 \mid x(x+1)/2$. In either case, $Z(x) = Z(x+1) \leq x$. This contradicts the last statement of the previous paragraph and our choice of $m$. Hence there is no solution to $Z(x) = Z(x+1)$.

*Also solved by:* Charles Ashbacher, Hiawatha, Iowa (part (d) only) and the proposer (parts (a) and (b) only).

---

## *Thank You, Referees!*

The following individuals have refereed one or more papers submitted to *The Pentagon* during the last two years. The editor is indebted to these individuals for graciously volunteering their time.

*Joseph Alfano*
Assumption College
Worcester, Massachusetts

*Charles Allen*
Drury College
Springfield, Missouri

*V. Sagar Bakhshi*
Virginia State University
Petersburg, Virginia

*Duane Broline*
Eastern Illinois University
Charleston, Illinois

*Jeffrey Clark*
Elon College
Elon College, North Carolina

*Mary Beth Dever*
Benedictine University
Lisle, Illinois

*Helmut Doll*
Bloomsburg University
Bloomsburg, Pennsylvania

*Martin Erickson*
Truman State University
Kirksville, Missouri

*Chenglie Hu*
Fort Hays State University
Hays, Kansas

*P. Catherine Kong*
Carson-Newman College
Jefferson City, Tennessee

*Lisa Townsley Kulich*
Benedictine University
Lisle, Illinois

*William R. Livingston*
Missouri Southern State College
Joplin, Missouri

*Rhonda McKee*
Central Missouri State University
Warrensburg, Missouri

*Rosalie Nichols*
Fort Hays State University
Hays, Kansas

*Mohammad Riazi-Kermani*
Fort Hays State University
Hays, Kansas

*Andrew M. Rockett*
C. W. Post Campus of Long Island University
Brookville, New York

*Ronald Sandstrom*
Fort Hays State University
Hays, Kansas

*Andy Talmadge*
Delta Teachers Academy
New Orleans, Louisiana

*Don Tosh*
Evangel College
Springfield, Missouri

*Troy D. VanAken*
University of Evansville
Evansville, Indiana

# Kappa Mu Epsilon News

Edited by Don Tosh, Historian

News of chapter activities and other noteworthy KME events should be sent to Don Tosh, Historian, Kappa Mu Epsilon, Mathematics Department, Evangel College, 1111 N. Glenstone, Springfield, MO 65802, or to toshd@evangel.edu.

## CHAPTER NEWS

***AL Gamma*** Chapter President — Cheryl Roberson
University of Montevallo, Montevallo 20 actives, 5 associates

Other 1997–98 officers: Ava Putman, vice president; Carrie Chickerell, secretary; David Taylor, treasurer; Larry Kurtz, corresponding secretary; Karolyn Morgan, faculty sponsor.

***AL Zeta*** Chapter President — Melissa Boren
Birmingham Southern College, Birmingham 17 actives, 13 associates

Other 1997–98 officers: Melanie Styers, vice president; Shelley Moor, secretary/treasurer; Mary Jane Turner, corresponding secretary; Shirley Branan, faculty sponsor.

***AR Alpha*** Chapter President — Danielle Morris
Arkansas State University, State University 4 actives, 1 associate

Other 1997–98 officers: Melissa Dubois, secretary; Rusty Jones, treasurer; William Paulsen, corresponding secretary/faculty sponsor.

***CA Delta*** Chapter President — Caroline Sabol
California State Polytechnic University, Pomona 15 actives, 6 associates

Other 1997–98 officers: Mark Walker, vice president; Holly Lam and Jason Ramirez, secretaries; Duy Pham, treasurer; Richard Robertson, corresponding secretary.

***CO Gamma*** Chapter President — Heather Duncan
Fort Lewis College, Durango 25 actives, 9 associates

The chapter met to hold an initiation ceremony at the beginning of the semester. We met once again to discuss the Regional KME Convention, which our school is hosting. Topics of discussion were holding a poster session as well as student presentations of papers, and the various arrangements that need to be made in order to put this event on. Other 1997–98 officers: Travis Kirkpatrick, vice president; Cynthia Hilliker, secretary; David Crawford, treasurer; Richard Gibbs, corresponding secretary; Debbie Berrier, faculty sponsor.

***CO Delta*** Chapter President — Adam Furst

**Mesa State College, Grand Junction** 20 actives

Forty-two members, initiates, and guests attended the eighth annual initiation banquet and ceremony on April 23, 1997. Eleven students and three faculty members were initiated, bringing the chapter membership to 90. The fall semester began with a "bagel party" in Lincoln Park on Sept. 4th. Twenty-five members and guests attended. Key pins and certificates were presented to the April initiates and possible activities for the year were discussed. Several members are currently working on a web page for our chapter; Kenneth J. Simler was a co-presenter of "The Development of an IPX Delphi Component" at a Brown Bag Seminar in October, and two members are part of a team planning to compete in the upcoming COMAP Mathematical Contest in Modeling. Other 1997–98 officers: Christopher Day, vice president; Michelle McGarry, secretary; Saori Okamura, treasurer; Donna Hafner, corresponding secretary; Kenneth Davis, faculty sponsor.

***FL Beta*** Chapter President — Cindy Chastain

**Florida Southern College, Lakeland** 20 actives, 8 associates

Other 1997–98 officers: Chris Scofield, vice president; Danny Koury, secretary; Gayle Kent, corresponding secretary; Susan Rinker, faculty sponsor.

***GA Alpha*** Chapter President — Tonja Davis

**State University of West Georgia, Carrollton** 25 actives

At our first meeting on October 15, membership certificates and pins were distributed to the new initiates and a Fall Social was planned at a local Mexican restaurant for later in the quarter. For the eleventh consecutive year, Georgia Alpha held its annual Food and Clothing Drive for the Needy with the proceeds being delivered to the Salvation Army at the end of the quarter. Our Fall Social on November 6 was well attended and a good time was had by all. Other 1997–98 officers: Stephanie Parker, vice president; Lisa Farmer, secretary; Marta Valentic, treasurer; Joe Sharp, corresponding secretary; Joe Sharp and Mark Faucette, faculty sponsors.

## *IL Delta* Chapter President — Tom Wiese

University of St. Francis, Joliet 18 actives

Other 1997–98 officers: Toni Dactilidis, vice president; David Daly, secretary; Elizabeth Mastin, treasurer; Rick Kloser, corresponding secretary/faculty sponsor.

## *IL Eta* Chapter President — Bethany Webb

Western Illinois University, Macomb 10 actives

In September, we held a new student reception with pizza and soda. Two guest speakers were from the spring of '97 "Writing in The Mathematical Sciences" class. The students presented their papers to introduce new students to what this recently-developed course is all about. In October, two Western Illinois University alumni came back for an evening to speak to our group about their teaching experiences. One has taught in a relatively large high school while the other is at a small rural school. They provided application and interviewing tips, first year suggestions, and words about how times have changed in recent years. In January, another WIU alumnus returned to speak about the differences he has encountered between the public middle school he taught at last year and the private Muslim academy where he currently has a position. He also provided resume tips and guidelines to make the first few years productive and enjoyable. Other 1997–98 officers: Thomas Johnson, vice president; Katherine Fijalkowski, secretary; Andrea Crary, treasurer; Larry Morley, corresponding secretary/faculty sponsor.

## *IL Theta* Chapter President — Donna Snaidauf

Benedictine University, Lisle 12 actives, 6 associates

Our activities for the fall semester included a "Pi" sale (brownies, cookies, and of course pi) and we watched the Nova program on Andrew Wiles and his proof of Fermat's Last Theorem. Other 1997–98 officers: Julie Deroche, vice president; Tracii Friedman, corresponding secretary/faculty sponsor.

## *IN Beta* Chapter President — Jessica Goldsand

Butler University, Indianapolis 19 actives

Other 1997–98 officers: Lee Duncan, vice president; Catherine Tischio, secretary; Yuzhen Ge, corresponding secretary.

## *IN Delta* Chapter President — David Zimmer

University of Evansville, Evansville 40 actives, 23 associates

The chapter offered tutoring free of charge for students at the University of Evansville. Other 1997–98 officers: Dennis Goodman, vice president; Jaclyn Cron, secretary; Mohammad Azarian, treasurer/corresponding secretary/faculty sponsor.

### *IA Alpha* Chapter President — Suzanne Shontz

University of Northern Iowa, Cedar Falls 36 actives

Students presenting papers at local KME meetings included Amy Grotjohn on "Research on mathematics learning by first graders," Erin Blaine reporting on "Summer Actuarial Internship on Investment and Finance," Suzanne Shontz on "Summer Undergraduate Research in Applied Linear Algebra," Jesse Connell on "The Geometry of Involute Gears," and Gary Spieler on "Strassen's Algorithm for Matrix Multiplication." Beth Kock addressed the fall initiation banquet on "Markov Chains." Five new KME members were initiated. Sarah Lacox was awarded the MAA student membership. The annual KME Homecoming Coffee was hosted by Emeritus Professors Carl and Wanda Wehner. Five KME alums attended along with current KME members and faculty. Other 1997–98 officers: Amy Grotjohn, vice president; Sarah Lacox, secretary; Erin Blaine, treasurer; John Cross, corresponding secretary/faculty sponsor.

### *IA Delta* Chapter President — Shilah Lybeck

Wartburg College, Waverly 46 actives

Our year began with a pizza party at the home of Dr. Glenn Fenneman, faculty sponsor. Problem Corner problems were distributed and members were encouraged to solve them and submit solutions. October's meeting centered around making plans for a float in the Renaissance Fair. The November meeting program was the "Marriage Knot Problem" enacted by two members and presented by Dr. Fenneman. Initial plans were also made for attending the regional meeting in Macomb, Illinois. Some additional Sunday evening meals were also arranged as socials during the term. Other 1997–98 officers: Christopher Judson, vice president; Joshua Nelson, secretary; Emily Bailey, treasurer; August Waltmann, corresponding secretary; Glenn Fenneman, faculty sponsor.

### *KS Alpha* Chapter President — Mark Albert

Pittsburg State University, Pittsburg 50 actives, 8 associates

The fall semester activities began with a pizza party and initiation in October. Eight new members were initiated. The initiation ceremony was followed by an interesting and entertaining presentation by Dr. Yaping Liu, called MathemaTricks. In November, KME sponsored a panel discussion, "The Classroom Experience — What Do I Need to Know?", which featured recent Pittsburg State mathematics graduates providing advice for future math educators. Fall semester activities concluded with a holiday party at the home of Dr. Elwyn Davis, PSU Math Dept. chair, at which members watched the video "The Tunnel of Samos." Other 1997–98 officers: Kathy Denney, vice president; Kari Hamm, secretary; Lisa Swaim, treasurer; Cynthia Woodburn, corresponding secretary; Yaping Liu, faculty sponsor.

***KS Beta*** Chapter Co-Presidents — Rae Ann LeValley, Kristen Goetz
Emporia State University, Emporia 20 actives, 4 associates

Our KME chapter started out the semester with a booth at the annual Activities Fair on campus. The purpose of this fair is to let incoming students know about campus organizations. Demonstrations on how to use graphing calculators were given at our booth. We are getting more involved on campus this year, and participated in the campus-wide clean-up for the first time. Our initation was held in November where we initiated four new members. Along with the event we had a student presentation demonstrating how to construct tetrahedral kites. We hope to continue increasing our activities throughout this semester. Other 1997–98 officers: Megan Little, treasurer; Connie Schrock, corresponding secretary; Larry Scott, faculty sponsor.

***KS Gamma*** Chapter President — Chad Eddins
Benedictine College, Atchison 7 actives, 13 associates

Early in September Kansas Gamma joined with the Physics Club in sponsoring an alum visitor, Adam Taff, who spoke to the group on his career in navy aviation. In early October the group gathered for a picnic evening on the patio at faculty member Richard Farrell's home. In mid-October Jeffrey Blanchard shared with the group on his experience as an NSF summer undergraduate researcher. His topic was "Beam Variation Problems." Prior to his talk Kansas Gamma initiated two new members and nine associate members. On December 7, the group enjoyed the traditional Christmas wassail at Marywood, home of Sister Jo Ann Fellin who returned to the department this fall after the completion of a sabbatical. The final event sponsored by the chapter for the semester was a presentation given by three students on their computer science project. They described their efforts in creating a home page with information on some rare books contained in the Benedictine Library. Other 1997–98 officers: Jeff Blanchard, vice president; Jo Ann Fellin, corresponding secretary/faculty sponsor.

***KS Delta*** Chapter President — Douglas Appenfeller
Washburn University, Topeka 22 actives

We had two afternoon meetings with the Washburn Math Club, Mathematica. The first was a picnic with hamburgers, hot dogs, etc. Outdoor games were played. The second was a pizza luncheon and a mathematics presentation was given. Other 1997–98 officers: Laurie Payeur, vice president; Chung-Fei Tang, secretary; Justin Freeby, treasurer; A. Allan

Riveland, corresponding secretary; Donna Lalonde and Ron Wasserstein, faculty sponsors.

## *KY Alpha*

Chapter President — Elizabeth Barrett

**Eastern Kentucky University, Richmond** 25 actives

The fall semester began with floppy disk sales (together with the ACM chapter) to students in the computer literacy class and the Mathematica class. At the first meeting in the fall, the officers were elected and tenative plans were made for the year. A student/faculty picnic was held at Lake Reba Park in late September. Fifty-five people came to the picnic. Volleyball and lots of good food were available. The October meeting included information about the Virginia Tech Math Exam and graduate programs in math. In November, Lynne Brosius gave a presentation on Smith numbers (composite positive integers with digit sum equal to the digit sum of the prime factors). She described one Smith number having 13,614,513 digits. The fall semester ended with the Christmas party and the White Elephant Gift Exchange. The Puff the Magic Dragon tape surfaced again this year and ended up in Ray Tennant's possession. Other 1997–98 officers: David Curd, vice president; Tina Jordan, secretary; Jeremy Miller, treasurer; Patrick Costello, corresponding secretary.

## *KY Beta*

Chapter President — Story Robbins

**Cumberland College, Williamsburg** 28 actives

On September 9, the Kentucky Beta chapter officers helped to host an ice cream party for the freshmen math and physics majors. Along with the Mathematics and Physics Club, the chapter had a picnic at Briar Creek Park in the rain on September 23. On October 24 and November 5, seniors Mindy Hazelwood and Story Robbins presented their summer research. The chapter had an informal dinner at Pizza Hut on November 13. On the last day of classes, December 12, the entire department, including the Math and Physics Club and the KY Beta chapter, had a Christmas party with about 45 people in attendance. The fall semester also saw the continued improvement of the chapter WEB page http://cc.cumber.edu/acad/math/kme.htm. Other 1997–98 officers: Candace Osborne, vice president; Laura Thompson, secretary; Melynda Hazelwood, treasurer; Jonathan Ramey, corresponding secretary; John Hymo, faculty sponsor.

## *MD Alpha*

Chapter President — Judith Simon

**College of Notre Dame of Maryland, Baltimore** 7 actives, 3 associates

Other 1997–98 officers: Marie Morrow, vice president; Michelle Yeager, secretary; Laura Bopp, treasurer; Sister Marie Augustine Dowling, corresponding secretary; Joseph DiRienzi, faculty sponsor.

### MD Beta
Chapter President — Jason Barr

Western Maryland College, Westminster — 28 actives

We had an induction of our seven new members in October and invited an alumnus, Tony Sager, who works at NSA (at Fort Meade) to talk informally about his work there and about getting government jobs. We also planned an exam break party for math majors. Spring plans underway include our annual career night dinner (co-sponsored by KME and the WMC Career Services Office), and we will also have a booth at the College's Spring Fling as a fundraiser. Other 1997–98 officers: Robert Newman, vice president; Julie Brown, secretary; Fred Butler, treasurer; Linda Eshleman, corresponding secretary; James Lightner, faculty sponsor.

### MD Delta
Chapter President — Heidi Femi

Frostburg State University, Frostburg — 29 actives

Maryland Delta Chapter opened the year with a donut and juice planning session in September. In October, Dr. Frank Barnet gave a talk to the group entitled "The Eyes of an IMP on Mars," in which he analyzed some of the data from the Mars Pathfinder mission. Later in October the chapter held a bike ride and picnic along the historic C&O canal, whose western terminus is only 12 miles from the university. The November program consisted of a video about Andrew Wiles and his proof of Fermat's Last Theorem. Other 1997–98 officers: Steven Fairgrieve, vice president; Sean Carley, secretary; Andrew Adam, treasurer; Edward T. White, corresponding secretary; John Jones, faculty sponsor.

### MS Alpha
Chapter President — Patricia DiBlasi

Mississippi University for Women, Columbus — 8 actives

On October 8 we held our general meeting. Other 1997–98 officers: Lani R. Crowder, vice president/treasurer; Patricia DiBlasi, secretary; Shaochen Yang, corresponding secretary/faculty sponsor.

### MS Beta
Chapter President — Shelley Hebert

Mississippi State University, Mississippi State — 10 actives

The Mississippi Beta chapter prepared the Interschool Test for the Mu Alpha Theta high school mathematics honorary state convention, to be held in the spring of 1998. This was a very nice opportunity to get together as a group, do some mathematics, and have some pizza, too. Other 1997–98 officers: Janet Waldrop, vice president; Michael Pearson, corresponding secretary.

### MS Gamma
Chapter President — Craig Collier

University of Southern Mississippi, Hattiesburg — 16 actives, 2 associates

Other 1997–98 officers: Paula Thigpen, vice president; Michelle Hill, secretary; Alice W. Essary, treasurer/corresponding secretary; Barry Piazza, Jeff Stuart, Bill Horner, faculty sponsors.

### *MS Epsilon* Chapter President — Ashley Riley

Delta State University, Cleveland 12 actives

Other 1997–98 officers: Ken Byars, vice president; Chad Huff, secretary/treasurer; Paula Norris, corresponding secretary; Rose Strahan, faculty sponsor.

### *MO Alpha* Chapter President — Lisa Burger

Southwest Missouri State University, Springfield 23 actives, 6 associates

The fall activities of the MO Alpha chapter included hosting a departmental picnic in September, a presentation of the history of and mathematics involved with the St. Louis Gateway Arch, initiating 6 new members, and a presentation on cooperative education opportunities for mathematics majors. Other 1997–98 officers: Katie Puetz, vice president; Jessica McDonnell, secretary; Miriam Ligon, treasurer; John Kubicek, corresponding secretary; Yungchen Cheng, faculty sponsor.

### *MO Beta* Chapter President — Dennis Wise

Central Missouri State University, Warrensburg 20 actives, 7 associates

MO Beta had monthly meetings in the fall. In September, Dr. Sue Sundberg spoke on tesselations. Nine new members and seven associates were initiated in October. In addition, Dr. Shing So gave a talk titled "Triangular and Oblong Numbers," and the revised bylaws were approved. Dr. Phoebe Ho spoke at the November meeting on the topic "Emulating the Development of the Real Number System with Finite Sets." At the KME Christmas party in December, food was collected for a Christmas dinner for a needy family. Other events for the semester included a book sale and volunteering in the Math Clinic. Other 1997–98 officers: Tammy Surfus, vice president; Aaron Shaefer, secretary; Cassie Young, treasurer; Melissa Elliott, historian; Rhonda McKee, corresponding secretary; Scotty Orr, Larry Dilley and Phoebe Ho, faculty sponsors.

### *MO Gamma* Chapter President — Jennifer Puls

William Jewel College, Liberty 16 actives

The MO Gamma chapter was active this fall with helping students in a Mathematics Help-Session held on Tuesday evenings. Several of the members assisted in this effort, and signed up on a sheet posted in the Mathematics Department. We have spent much of our time this fall preparing for the regional convention which we will host in April of 1998. Other 1997–98 officers: Allison Cooper, vice president; James Brochtrup, secretary; Joseph Mathis, treasurer/corresponding secretary/faculty sponsor.

## *MO Eta* Chapter President — Laurel Berner

Truman State University, Kirksville 10 actives, 7 associates

Meetings were held on the first Monday of every month. Activities included a pledge class fundraiser, a Christmas market (bake sale), student/faculty spades night, and movie night. Our main concentration was preparing for the February Math Expo. Other 1997–98 officers: Christine Stone, vice president; Angela Kell, secretary; Ann Herberholt, treasurer; Jay Belanger, corresponding secretary/faculty sponsor.

## *MO Theta* Chapter President — Christie Tosh DeArmond

Evangel College, Springfield 10 actives, 4 associates

Meetings were held monthly. The fall social was held at the home of Don Tosh. Other 1997–98 officers: Amy Lee and Stan Roberts, vice presidents; Don Tosh, corresponding secretary/faculty sponsor.

## *MO Iota* Chapter President — Shan Brand

Missouri Southern State College, Joplin 22 actives

Fall semester activites commenced with a dinner and organizational meeting at the home of Mrs. Mary Elick. Officers for the year were elected at this meeting. Throughout the semester regular meetings were held with programs often featuring student solutions to problems from the Problem Corner of *The Pentagon*. Meetings, held jointly with Math Club, also featured free pizza. Attendance at meetings was greatly improved! The organization once again worked concessions at home football games as a money-making venture. The Christmas Tree Decorating Party was held at the new home of Dr. Chip Curtis. A live tree was furnished by the organization and those attending each brought a decoration, some featuring a math theme. Dr. Curtis will long remember the event, as the tree was subsequently planted in his yard. Other 1997–98 officers: Agdon Brister, vice president; Chris Baker, secretary; Amanda Harrison, treasurer; Megan Radcliff, publicity; Mary Elick, corresponding secretary/faculty sponsor.

## *MO Kappa* Chapter President — Edyta Blaszczuk

Drury College, Springfield 9 actives, 5 associates

The first activity of the semester was a bonfire party held at Dr. Allen's house. The winner of the annual Math Contest this year was Ben Ingrum for the Calculus II and above division and Adrena Pearcy for the Calculus I and below division. Prize money was awarded to the winners at a pizza party held for all the contestants. Another pizza party was held at the house of Dr. Carol Collins for the potential KME members (freshmen). The Math Club has also been running a tutoring service for both the day school and the continuing education division (Drury Evening College) as a money-making project. Other 1997—98 chapter officers: Billy Kimmons, vice president; Carol Collins, treasurer; Charles Allen, corresponding secretary; Pam Reich, faculty sponsor.

***MO Lambda*** Chapter President — Perriann McCoppin
Missouri Western State College, St. Joseph 35 actives

The MO Lambda chapter initiated five new members on September 21. Guest speaker for the program was Dr. David John, who spoke on fractal geometry. Other fall 1997 activities included a Welcome Back Picnic, a booth at Family Day, homecoming activities, a Thanksgiving covered dish dinner, a planetarium show by KME sponsor Jerry Wilkerson, and attendance at several departmental colloquia. Other 1997—98 chapter officers: Stephanie Tingler, vice president; William Slabaugh, secretary; Sean Hutto, treasurer; John Atkinson, corresponding secretary; Jerry Wilkerson, faculty sponsor.

***NE Alpha*** Chapter President — Rustin Slaughter
Wayne State College, Wayne 30 actives

For the fall we have been holding our regular meetings. In them we have discussed ways to increase our funds and also increase publicity for Homecoming 1997. During Homecoming week, our chapter had a local business' windows painted. We had a great turnout for this event and it gave everyone a chance to get to know each other. We are continuing efforts for everyone to meet, through our plans of a Sioux City hockey game outing, scheduled for shortly after the beginning of the new semester. We are also currently selling pizza punch cards for the local Pizza Hut, in order to raise funds. Other 1997–98 officers: Karl Laursen, vice president; Renee Fuhr, secretary/treasurer; Ann Boes, historian; John Fuelberth, corresponding secretary; Jim Paige, faculty sponsor.

***NE Gamma*** Chapter President — Jennifer Praeuner
Chadron State College, Chadron 15 actives, 4 associates

Other 1997–98 officers: Otis Pierce, vice president; Julie Steinbach, secretary; Erin Johnson, treasurer; James Kaus, corresponding secretary; Monty Fickel, faculty sponsor.

***NM Alpha*** Chapter President — Jason Strauch
University of New Mexico, Albuquerque 70 actives, 14 associates

Other 1997–98 officers: Jennifer Gill, vice president; Walter Kehowski, secretary/treasurer; Merlin Decker, webmaster; Archie Gibson, corresponding secretary/faculty sponsor.

***NY Alpha*** Chapter President — Vinod Gulani
Hofstra University, Hempstead 15 actives, 2 associates

We cleaned up and decorated the math lounge. We sponsored a luncheon with the math faculty where the math program was discussed. Other 1997–98 officers: Angela Boccio, vice president; JoAnne Taormina, secretary; Sacha DaCosta, treasurer; Aileen Michaels, corresponding secretary/faculty sponsor.

### *NY Eta* — Chapter President — Stacey Lauricella

Niagara University, Niagara — 15 actives

For a fund-raiser, our chapter co-sponsored a silent movie night in October featuring *The Goldrush* with Charlie Chaplin, including live musical accompaniment. Other 1997–98 officers: Jennifer Egan, vice president; Lara Brown, secretary; Leslie Good, treasurer; Robert Bailey, corresponding secretary; Kenneth Bernard, faculty sponsor.

### *NY Nu* — Chapter President — Carol Mattice

Hartwick College, Oneonta — 24 actives

Other 1997–98 officers: Willard Bradner VanderVoort, III, vice president; Brandon Cheely, secretary; Matthew Jones, treasurer; Gary Stevens, corresponding secretary/faculty sponsor.

### *OH Gamma* — Chapter President — Amy Booth

Baldwin-Wallace College, Berea — 39 actives

Two students travelled with David Calvis to the Miami University Fall Mathematics Conference. Other 1997–98 officers: Cassandra Kirby, vice president; Margot Mailloux, secretary; Anila Xhunga, treasurer; David Calvis, corresponding secretary/ faculty sponsor.

### *OK Alpha* — Chapter President — Josh Baker

Northeastern State University, Tahlequah — 35 actives, 1 associate

Our fall initiation ceremonies for twelve students were held in the banquet room of Roni's Pizza. It was well attended by many faculty, their families, and other students.The annual book sale did extremely well. We made over $200. We are grateful to the faculty who donated old texts to us for this sale. In September, we provided refreshments for the college's monthly Science & Technology Series. Dr. Mark Arnold from the University of Arkansas discussed "Who's Modeling Whom?". In October, the chapter sponsored a talk by Dr. Larry Claypool, professor of statistics from Oklahoma State University. His presentation on how statistics is used in business and industry was excellent. He also talked about the graduate program in statistics at OSU. We sold T-shirts and sweatshirts with a design created by several of the KME members. This project did extremely well. We sold the T-shirts at "cost + epsilon" as more of a service project than a money-maker. We continue to have joint activities with NSU's student chapter of the MAA and participate in "The Problem Solving Competition," sponsored by the MAA. Our Christmas "pizza party" was again a success! The games pitted the faculty against the students. The students won, but it was close! Other 1997–98 officers: Dan Sisk, vice president; Tera McGrew, secretary; Tracey McCutchen, treasurer; Joan Bell, corresponding secretary/faculty sponsor.

### *OK Gamma* — Chapter President — Joe Antunez

Southwestern Oklahoma State University, Weatherford — 20 actives

Other 1997–98 officers: Marina Ramirez, vice president; Linda Coley, secretary; Christine Robben, treasurer; Wayne Hayes, corresponding secretary; Gerard East, faculty sponsor.

### *PA Gamma* — Chapter President — Amanda Beisel

Waynesburg College, Waynesburg — 19 actives, 5 associates

Other 1997–98 officers: Jennifer Baltes, vice president; Kristien Fox, secretary; Angela Colinet, treasurer; Anthony Billings, corresponding secretary/faculty sponsor.

### *PA Delta* — Chapter President — Jennifer Snyder

Marywood University, Scranton — 4 actives

The chapter prepared for participation in Moravian College's Student Math Conference. Two members were preparing papers to present at the February Pi Mu Epsilon program. Other 1997–98 officers: Maura Regan, vice president; Brenda Rudzinski, secretary/treasurer; Sr. Robert Ann Von Ahnen, corresponding secretary/faculty sponsor.

### *PA Eta* — Chapter President — Frederick Lam

Grove City College, Grove City — 32 actives

Induction of new members was held October 20. Dr. Dale McIntyre gave a talk on the derivation of a set of parametric equations describing the path of a dog during its pursuit of a rabbit. The group enjoyed donuts and cider after the meeting. Other 1997–98 officers: Greta Kessler, vice president; Sarah Lawhon, secretary; Diane Schnellbach, treasurer; Marvin Henry, corresponding secretary; Dan Dean, faculty sponsor.

### *PA Iota* — Chapter President — Abby Todd

Shippensburg University, Shippensburg — 20 actives, 3 associates

This semester KME along with our Math Club were co-sponsors of a weekly seminar series. They also had a hand in our second annual student math conference, which took place on November 18. Professor Sandra Fillebrown from St. Joseph's University gave the featured talk on fractal attractors and the chaos game. In December, Dr. and Mrs. Doug Ensley allowed us to use their home as the site of our fall initiation. We inducted three new members into KME. Many thanks to Doug and Amy for their hospitality. Other 1997–98 officers: Peter Burnett, vice president; Nycole Miller, secretary; Mike Seyfried, treasurer; Stacey Lytle, historian; Mike Seyfried, corresponding secretary; Gene Fiorini, faculty sponsor.

***PA Kappa*** Chap. Pres.—Paul O'Connor, Cheryll Stone-Schwendimann
Holy Family College, Philadelphia 5 actives, 2 associates

The chapter met Wednesday afternoons to work on problems from *The Pentagon*, *Math Horizons*, and the *College Mathematics Journal*. The chapter also sponsored a math competition for students from local area high schools on October 20. Plaques were awarded to the top school and top student competitor. Seniors Paul O'Connor and Cheryll Stone-Schwendimann presented their senior research papers at a division symposium attended by division faculty and KME members. Cheryll and Paul also presented their research at a poster session held at the college. Other 1997–98 officers: Brian Minster, secretary/treasurer; Sr. Marcella Louise Wallowicz, corresponding secretary/faculty sponsor.

***PA Mu*** Chapter President — Jen Gibbons
Saint Francis College, Loretto 19 actives

KME members picked litter in October along a two-mile stretch of highway near the college as part of Pennsylvania's Adopt-A-Highway program. KME members served in leadership roles as session moderators and as judges, score keepers and time keepers for the Science Bowl in the Fourth Annual Science Day. More than 300 students from 19 high schools participated on November 21. Several KME members also attended the NCTM Regional meeting in Cleveland, Ohio, in November. Other 1997–98 officers: Brad Offman, vice president; Ernie Pagliaro, secretary; Ryan Howard, treasurer; Pete Skoner, corresponding secretary; Adrian Baylock, faculty sponsor.

***PA Omicron*** Chapter President — Daniel Coleman
University of Pittsburgh at Johnstown, Johnstown 37 actives

Other 1997–98 officers: Melissa Owens, vice president; Lori Duncan, secretary; Nikki Gerba, treasurer; Sarah Leach, historian; Nina Girard, corresponding secretary/faculty sponsor.

***SD Alpha*** Chapter President — Kristy Schuster
Northern State University, Aberdeen 12 actives

Other 1997–98 officers: Margo Maynard, vice president; Rebecca Hanson, secretary; Stacey Garrels, treasurer; Lu Zhang, corresponding secretary; Raj Markanda, faculty sponsor.

### *TN Alpha*

Chapter President — Jonathan M. Sprinkle

Tennessee Technological University, Cookeville — 7 actives, 30 associates

Other 1997–98 officers: Russell Edward Watts, vice president; Andy Adams, secretary; Deborah A. Watkins, treasurer; Frances E. Crawford, corresponding secretary; Allen Mills, faculty sponsor.

### *TN Gamma*

Chapter President — Jennifer Murrah

Union University, Jackson — 13 actives

Other 1997–98 officers: Lori Davis, vice president; Mandy Davidson, secretary/treasurer; Matt Lunsford, corresponding secretary; Troy Riggs, faculty sponsor.

### *TN Delta*

Chapter President — Michael Kelley

Carson-Newman College, Jefferson City — 10 actives

Other 1997–98 officers: Rebecca Gritman, secretary; Catherine Kong, corresponding secretary/faculty sponsor.

### *TX Kappa*

Chapter President — Carrie Tucker

University of Mary Hardin-Baylor, Belton — 15 actives, 8 associates

A KME Christmas party was held at Dr. Harding's home on December 12. Other 1997–98 officers: Mellissa Schexnayider, vice president; Cheyanna Orsag, secretary; Jennifer Murphy, treasurer; Peter Chen, corresponding secretary; Maxwell Hart, faculty sponsor.

---

## *Former National President Receives Award*

Harold Thomas of Kansas Alpha, a former KME national president (1989–1993), has received a Certificate for Meritorious Service for service to the Kansas Section of the Mathematical Association of America. The award was presented during the Joint Prize Session at the annual Joint Mathematics Meetings in Baltimore in January. Only seven of these certificates were awarded this year.

Readers might also be interested to know that one of the regular contributors to the Problem Corner, Jegenathan Sriskandarajah, also received a Certificate for Meritorious Service, for service to the Wisconsin Section.

---

## *Largest Known Prime Update*

The newest largest known prime was announced January 27. It is $2^{3021377} - 1$, and has 909,526 digits. To join the Great Internet Mersenne Prime Search, see this WWW site:

ourworld.compuserve.com/homepages/justforfun/prime.htm

# Announcement of the Thirty-Second Biennial Convention of Kappa Mu Epsilon

The Thirty-Second Biennial Convention of Kappa Mu Epsilon will be hosted by the Florida Beta chapter located at Florida Southern College in Lakeland, Florida. The convention will take place April 15–17, 1999. Each attending chapter will receive the usual travel expense reimbursement from the national funds as described in Article VI, Section 2, of the Kappa Mu Epsilon Constitution.

A significant feature of our national convention will be the presentation of papers by student members of Kappa Mu Epsilon. The mathematical topic selected by each student speaker should be of interest to the author and of such scope that it can be given adequate treatment in a timed oral presentation. Student talks to be judged at the convention will be chosen prior to the convention by the selection committee on the basis of submitted written papers. At the convention, the awards committee will judge the selected talks on both content and presentation. The rankings of both the selection and awards committees will determine the top four papers.

## Who may submit a paper?

Any undergraduate or graduate student member of Kappa Mu Epsilon may submit a paper for consideration as a talk at the national convention. A paper may be co-authored. If selected for presentation at the convention, the paper must be presented by one (or more) of the authors.

## Presentation topics

Papers submitted for presentation at the convention should discuss material understandable by undergraduates who have completed only differential and integral calculus. The selection committee will favor papers that satisfy this criterion and which can be presented with reasonable completeness within the time allotted. Papers may be original research by the student(s) or exposition of interesting but not widely known results.

## Presentation time limits

Papers presented at the convention should take between 15 minutes and 25 minutes. Papers should be designed with these time limits in mind.

### How to prepare a paper

The paper should be written in the standard form of a term paper. It should be written much as it will be presented. A long paper (such as an honors thesis) must not be submitted with the idea that it will be shortened at presentation time. Appropriate references and a bibliography are expected. Any special visual aids that the host chapter will need to provide (such as a computer and overhead projection system) should be clearly indicated at the end of the paper.

The first page of the paper must be a "cover sheet" giving the following information: (1) title, (2) author or authors (these names should not appear elsewhere in the paper), (3) student status (undergraduate or graduate), (4) permanent and school addresses and phone numbers, (5) name of the local KME chapter and school, (6) signed statement giving approval for consideration of the paper for publication in *The Pentagon* (or a statement about submission for publication elsewhere) and (7) a signed statement of the chapter's corresponding secretary attesting to the author's membership in Kappa Mu Epsilon.

### How to submit a paper

Five copies of the paper, with a description of any charts, models, or other visual aids that will be used during the presentation, must be submitted. The cover sheet need only be attached to one of the five copies. The five copies of the paper are due by February 10, 1999. They should be sent to:

Dr. Robert Bailey, KME President-Elect
Depaprtment of Mathematics
Niagara University
Niagara University, NY 14109

### Selection of papers for presentation

A selection committee will review all papers submitted by undergraduate students and will choose approximately fifteen papers for presentation and judging at the convention. Graduate students and undergraduate students whose papers are not selected for judging will be offered the opportunity to present their papers at a parallel session of talks during the convention. The president-elect will notify all authors of the status of their papers after the selection committee has completed its deliberations.

### Criteria used by the Selection and Awards Committees

Each paper will be judged on (1) topic originality, (2) appropriateness to the meeting and audience, (3) organization, (4) depth and significance of the content, and (5) understanding of the material. Each presentation will be judged on (1) style of presentation, (2) maintenance of interest, (3) use of audio-visual materials (if applicable), (4) enthusiasm for the topic, (5) overall effect, and (6) adherence to the time limits.

### Prizes

All authors of papers presented at the convention will be given two-year extensions of their subscription to *The Pentagon*. Authors of the four best papers presented by undergraduates, as decided by the selection and awards committees, will each receive a cash prize of $100.

### Publication

All papers submitted to the convention are generally considered as submitted for publication in *The Pentagon*. Unless published elsewhere, prize-winning papers will be published in *The Pentagon* after any necessary revisions have been completed (see page 2 of *The Pentagon* for further information). All other papers will be considered for publication. The Editor of *The Pentagon* will schedule a brief meeting with each author during the convention to review his or her manuscript.

---

## *Thou SHALT Steal?*

Yes, Really! KME chapters should feel free to "steal" logos, graphics files, information, links, etc. from the national KME web site! Besides making your web pages look "cool," they're easier to construct that way. You could also consider adding links to portions of the national website, such as to general information about KME or to the cumulative subject index of *The Pentagon*. The URL for the national homepage is:

www.cst.cmich.edu/org.kme/

When you design a chapter homepage, please remember to make it clear that your page is for your chapter, and not for the national organization. Also, please include a link to the national homepage and submit your local chapter webpage's URL to the national webmaster. By doing so, other chapters can explore activities of your chapter and get some great ideas!

By the way, this exception to the eighth commandment only applies to the national KME web page, and not to your roommate's sports car. Sorry!

# Kappa Mu Epsilon National Officers

Patrick J. Costello *President*
Department of Mathematics, Statistics and Computer Science
Eastern Kentucky University, Richmond, Kentucky 40475
matcostello@acs.eku.edu

Robert Bailey *President-Elect*
Mathematics Department
Niagara University, Niagara University, New York 14109
rlb@niagara.edu

Waldemar Weber *Secretary*
Department of Mathematics and Statistics
Bowling Green State University, Bowling Green, Ohio 43403
kme-nsec@mailserver.bgsu.edu

A. Allan Riveland *Treasurer*
Department of Mathematics and Statistics
Washburn University, Topeka, Kansas 66621
zzrive@acc.wuacc.edu

Don Tosh *Historian*
Department of Science and Technology
Evangel College, 1111 N. Glenstone Ave., Springfield, Missouri 65802
toshd@evangel.edu

*Kappa Mu Epsilon*, Mathematics Honor Society, was founded in 1931. The object of the Society is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics due to its demands for logical and rigorous modes of thought; to provide a Society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; and to disseminate the knowledge of mathematics and familiarize the members with the advances being made in mathematics. The official journal of the Society, *The Pentagon*, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the Chapters.

# Active Chapters of Kappa Mu Epsilon

*Listed by date of installation.*

| Chapter | Location | Installation Date |
|---|---|---|
| OK Alpha | Northeastern State University, Tahlequah | 18 April 1931 |
| IA Alpha | University of Northern Iowa, Cedar Falls | 27 May 1931 |
| KS Alpha | Pittsburg State University, Pittsburg | 30 Jan 1932 |
| MO Alpha | Southwest Missouri State University, Springfield | 20 May 1932 |
| MS Alpha | Mississippi University for Women, Columbus | 30 May 1932 |
| MS Beta | Mississippi State University, Mississippi State | 14 Dec 1932 |
| NE Alpha | Wayne State College, Wayne | 17 Jan 1933 |
| KS Beta | Emporia State University, Emporia | 12 May 1934 |
| NM Alpha | University of New Mexico, Albuquerque | 28 March 1935 |
| IL Beta | Eastern Illinois University, Charleston | 11 April 1935 |
| AL Beta | University of North Alabama, Florence | 20 May 1935 |
| AL Gamma | University of Montevallo, Montevallo | 24 April 1937 |
| OH Alpha | Bowling Green State University, Bowling Green | 24 April 1937 |
| MI Alpha | Albion College, Albion | 29 May 1937 |
| MO Beta | Central Missouri State University, Warrensburg | 10 June 1938 |
| TX Alpha | Texas Tech University, Lubbock | 10 May 1940 |
| TX Beta | Southern Methodist University, Dallas | 15 May 1940 |
| KS Gamma | Benedictine College, Atchison | 26 May 1940 |
| IA Beta | Drake University, Des Moines | 27 May 1940 |
| TN Alpha | Tennessee Technological University, Cookeville | 5 June 1941 |
| NY Alpha | Hofstra University, Hempstead | 4 April 1942 |
| MI Beta | Central Michigan University, Mount Pleasant | 25 April 1942 |
| NJ Beta | Montclair State University, Upper Montclair | 21 April 1944 |
| IL Delta | University of St. Francis, Joliet | 21 May 1945 |
| KS Delta | Washburn University, Topeka | 29 March 1947 |
| MO Gamma | William Jewell College, Liberty | 7 May 1947 |
| TX Gamma | Texas Woman's University, Denton | 7 May 1947 |
| WI Alpha | Mount Mary College, Milwaukee | 11 May 1947 |
| OH Gamma | Baldwin-Wallace College, Berea | 6 June 1947 |
| CO Alpha | Colorado State University, Fort Collins | 16 May 1948 |
| MO Epsilon | Central Methodist College, Fayette | 18 May 1949 |
| MS Gamma | University of Southern Mississippi, Hattiesburg | 21 May 1949 |
| IN Alpha | Manchester College, North Manchester | 16 May 1950 |
| PA Alpha | Westminster College, New Wilmington | 17 May 1950 |
| IN Beta | Butler University, Indianapolis | 16 May 1952 |
| KS Epsilon | Fort Hays State University, Hays | 6 Dec 1952 |
| PA Beta | LaSalle University, Philadelphia | 19 May 1953 |
| VA Alpha | Virginia State University, Petersburg | 29 Jan 1955 |
| IN Gamma | Anderson University, Anderson | 5 April 1957 |
| CA Gamma | California Polytechnic State University, San Luis Obispo | 23 May 1958 |
| TN Beta | East Tennessee State University, Johnson City | 22 May 1959 |
| PA Gamma | Waynesburg College, Waynesburg | 23 May 1959 |
| VA Beta | Radford University, Radford | 12 Nov 1959 |
| NE Beta | University of Nebraska—Kearney, Kearney | 11 Dec 1959 |

| | | |
|---|---|---|
| IN Delta | University of Evansville, Evansville | 27 May 1960 |
| OH Epsilon | Marietta College, Marietta | 29 Oct 1960 |
| MO Zeta | University of Missouri—Rolla, Rolla | 19 May 1961 |
| NE Gamma | Chadron State College, Chadron | 19 May 1962 |
| MD Alpha | College of Notre Dame of Maryland, Baltimore | 22 May 1963 |
| IL Epsilon | North Park College, Chicago | 22 May 1963 |
| OK Beta | University of Tulsa, Tulsa | 3 May 1964 |
| CA Delta | California State Polytechnic University, Pomona | 5 Nov 1964 |
| PA Delta | Marywood University, Scranton | 8 Nov 1964 |
| PA Epsilon | Kutztown University of Pennsylvania, Kutztown | 3 April 1965 |
| AL Epsilon | Huntingdon College, Montgomery | 15 April 1965 |
| PA Zeta | Indiana University of Pennsylvania, Indiana | 6 May 1965 |
| AR Alpha | Arkansas State University, State University | 21 May 1965 |
| TN Gamma | Union University, Jackson | 24 May 1965 |
| WI Beta | University of Wisconsin—River Falls, River Falls | 25 May 1965 |
| IA Gamma | Morningside College, Sioux City | 25 May 1965 |
| MD Beta | Western Maryland College, Westminster | 30 May 1965 |
| IL Zeta | Rosary College, River Forest | 26 Feb 1967 |
| SC Beta | South Carolina State College, Orangeburg | 6 May 1967 |
| PA Eta | Grove City College, Grove City | 13 May 1967 |
| NY Eta | Niagara University, Niagara University | 18 May 1968 |
| MA Alpha | Assumption College, Worcester | 19 Nov 1968 |
| MO Eta | Truman State University, Kirksville | 7 Dec 1968 |
| IL Eta | Western Illinois University, Macomb | 9 May 1969 |
| OH Zeta | Muskingum College, New Concord | 17 May 1969 |
| PA Theta | Susquehanna University, Selinsgrove | 26 May 1969 |
| PA Iota | Shippensburg University of Pennsylvania, Shippensburg | 1 Nov 1969 |
| MS Delta | William Carey College, Hattiesburg | 17 Dec 1970 |
| MO Theta | Evangel College, Springfield | 12 Jan 1971 |
| PA Kappa | Holy Family College, Philadelphia | 23 Jan 1971 |
| CO Beta | Colorado School of Mines, Golden | 4 March 1971 |
| KY Alpha | Eastern Kentucky University, Richmond | 27 March 1971 |
| TN Delta | Carson-Newman College, Jefferson City | 15 May 1971 |
| NY Iota | Wagner College, Staten Island | 19 May 1971 |
| SC Gamma | Winthrop University, Rock Hill | 3 Nov 1972 |
| IA Delta | Wartburg College, Waverly | 6 April 1973 |
| PA Lambda | Bloomsburg University of Pennsylvania, Bloomsburg | 17 Oct 1973 |
| OK Gamma | Southwestern Oklahoma State University, Weatherford | 1 May 1973 |
| NY Kappa | Pace University, New York | 24 April 1974 |
| TX Eta | Hardin-Simmons University, Abilene | 3 May 1975 |
| MO Iota | Missouri Southern State College, Joplin | 8 May 1975 |
| GA Alpha | State University of West Georgia, Carrollton | 21 May 1975 |
| WV Alpha | Bethany College, Bethany | 21 May 1975 |
| FL Beta | Florida Southern College, Lakeland | 31 Oct 1976 |
| WI Gamma | University of Wisconsin—Eau Claire, Eau Claire | 4 Feb 1978 |
| MD Delta | Frostburg State University, Frostburg | 17 Sept 1978 |
| IL Theta | Benedictine University, Lisle | 18 May 1979 |
| PA Mu | St. Francis College, Loretto | 14 Sept 1979 |
| AL Zeta | Birmingham-Southern College, Birmingham | 18 Feb 1981 |
| CT Beta | Eastern Connecticut State University, Willimantic | 2 May 1981 |
| NY Lambda | C.W. Post Campus of Long Island University, Brookville | 2 May 1983 |
| MO Kappa | Drury College, Springfield | 30 Nov 1984 |

| | | |
|---|---|---|
| CO Gamma | Fort Lewis College, Durango | 29 March 1985 |
| NE Delta | Nebraska Wesleyan University, Lincoln | 18 April 1986 |
| TX Iota | McMurry University, Abilene | 25 April 1987 |
| PA Nu | Ursinus College, Collegeville | 28 April 1987 |
| VA Gamma | Liberty University, Lynchburg | 30 April 1987 |
| NY Mu | St. Thomas Aquinas College, Sparkill | 14 May 1987 |
| OH Eta | Ohio Northern University, Ada | 15 Dec 1987 |
| OK Delta | Oral Roberts University, Tulsa | 10 April 1990 |
| CO Delta | Mesa State College, Grand Junction | 27 April 1990 |
| NC Gamma | Elon College, Elon College | 3 May 1990 |
| PA Xi | Cedar Crest College, Allentown | 30 Oct 1990 |
| MO Lambda | Missouri Western State College, St. Joseph | 10 Feb 1991 |
| TX Kappa | University of Mary Hardin-Baylor, Belton | 21 Feb 1991 |
| SC Delta | Erskine College, Due West | 28 April 1991 |
| SD Alpha | Northern State University, Aberdeen | 3 May 1992 |
| NY Nu | Hartwick College, Oneonta | 14 May 1992 |
| NH Alpha | Keene State College, Keene | 16 Feb 1993 |
| LA Gamma | Northwestern State University, Natchitoches | 24 March 1993 |
| KY Beta | Cumberland College, Williamsburg | 3 May 1993 |
| MS Epsilon | Delta State University, Cleveland | 19 Nov 1994 |
| PA Omicron | University of Pittsburgh at Johnstown, Johnstown | 10 April 1997 |
| MI Delta | Hillsdale College, Hillsdale | 30 April 1997 |

## *Starting a KME Chapter*

Complete information on starting a chapter of KME may be obtained from the National President. Some information is given below.

An organized group of at least ten members may petition through a faculty member for a chapter. These members may be either faculty or students; students must meet certain coursework and g.p.a. requirements.

The financial obligation of new chapters to the national organization includes the cost of the chapter's charter and crest (approximately $50) and the expenses of the installing officer. The individual membership fee to the national organization is $20 per member and is paid just once, at that individual's initiation. Much of this $20 is returned to the new members in the form of membership certificates and cards, keypin jewelry, a two-year subscription to the society's journal, etc. Local chapters are allowed to collect semester or yearly dues as well.

The petition itself, which is the formal application for the establishment of a chapter, requests information about the petitioning group, the academic qualifications of the eligible petitioning students, the mathematics faculty, mathematics course offerings and other facts about the institution. It also requests evidence of faculty and administrative approval and support of the petition. Petitions are subject to approval by the National Council and ratification by the current chapters.