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# Contents





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# Editor's Notes

### Two years

This issue marks the end of my first two-year cycle as editor of *The Pentagon*. I sincerely hope that you, the reader, have found many interesting articles in these two years. Of course, any comments and suggestions relating to this journal are welcome! Simply send them to the editor at the address on page 2.

### Submissions

Not all articles in *The Pentagon* come from those presented at KME conventions. Articles of interest to students are always welcome. Such submissions are refereed. Student papers are given priority. One interesting submitted article, "Lengths of Generalized Rose Curves," appears in this issue. Although this article is by a KME student, membership in KME is not a requirement for publishing in this journal. If you or someone you know has done something interesting, please consider publication in *The Pentagon*! Instructions are on page 2.

### End-of-page notes

Much information is contained in the notes at the end of articles in each issue of this journal. This issue contains notes on the new URL of KME, the newest largest known prime, information on back issues and subscription expiration dates, a KME Quiz update and more! Don't forget to look for such information in each issue! Also, submissions for possible inclusion in these notes will be considered; simply send them to the editor.

### Palidromes

It has recently come to the attention of the editor that a series of articles appeared in *Mathematics Magazine* (see vol. 40 (1967) pp. 26–28, vol. 42 (1969) pp. 252–254 and 254–256, and vol. 46 (1973) pp. 96–99) addressing the palindrome conjecture. One of the articles, by the well-known problem solver Charles W. Trigg, raises basically the same questions as those raised in the previous issue of *The Pentagon* by Christopher Brown (vol. 56 no. 1 (Fall 1996), pp. 23–38), although Brown's article is a little more extensive and provides additional information. Another of the articles provides a counterexample to the conjecture for numbers in any base of the form $2^k$. Readers whose curiosity was aroused by Brown's article may find these articles very interesting. The conjecture is apparently still open for base 10.

# Atmospheric Gambling: An Investigation of Rayleigh Scattering in the Atmosphere using Monte Carlo Methods

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Presented at the 1995 National Convention and awarded "top four" status by the Awards Committee.

A well-known example of atmospheric scattering is the blue sky we see on a clear day. This paper will investigate the dynamics of atmospheric scattering of light using Monte Carlo techniques. It will discuss the generation of random numbers by computer algorithms and their use in Monte Carlo techniques. The paper will conclude with a discussion of atmospheric scattering based on a Monte Carlo simulation. My decision to investigate this topic came about because of an interest in the greenhouse effect and the impact on the environment of an increasing amount of atmospheric carbon dioxide.

Random numbers are defined as numbers that occur at random; that is, they occur with no specific pattern. If an event is truly random, then all possible outcomes must have an equal chance of occurring. However, random numbers are often taken for granted. If you were asked to provide a list of random numbers, most likely you would sit down at your computer or with your calculator and call on its random number generator. But, how can a computer with its deterministic program generate a so-called random number? The answer is that it can not. It does, however, give us the next best thing.

Computers generate pseudo-random numbers. These numbers act like random numbers, but are generated from a deterministic algorithm. In order to act like a random number, these pseudo-random numbers must meet several properties, such as to appear to occur independently of one another and to be distributed uniformly over the sample space. Good generators will be able to pass several statistical tests of randomness. The tests would be used to validate uniformity, goodness-of-fit, independence, number of runs, and other properties.

The most commonly used generators today are congruential generators. These generators calculate the residues modulo some integer $m$ of a linear transformation to produce a nonrandom sequence of numbers [4]. A seed or initial value must be provided to the algorithm, which then multiplies it by a constant and adds another constant to that value. That sum is then divided modulo another large constant to produce the next number. This number then becomes the seed value for the next iteration. The following generator is an example of this:

$$X_i = (5243 \cdot X_j + 55397) \mod 262139.$$

In this example, $X_j$ is the seed value and $X_i$ is the pseudo-random number and also the next seed value. Many times it is desired to have numbers occurring between 0 and 1. This is accomplished by dividing the above number by 262139. If proper constants are chosen, this method provides an extremely good set of pseudo-random numbers.

Congruential generators were used in all of the simulations that I performed. To begin my exploration of the Monte Carlo method and to test the congruential generators, my first simulation was to estimate $\pi$. The area under the curve $x^2 + y^2 = 1$ represents $\pi/4$ of the total area of the square ranging from the origin to 1 on both the $x$ and $y$ axes (see figure 1). Probability theory dictates that points placed at random in the square would be on or under the curve $\pi/4$ amount of time. This is simulated by using the random number generator to produce the $x$ and $y$ coordinates of a point in the square. Each coordinate was squared. If the sum of these squares was less than or equal to 1, a counter was incremented by 1. The counter divided by the total number of iterations approximated $\pi/4$. After 100,000 iterations, $\pi$ was estimated at 3.141400099. The estimate depends on the seed value and the number of iterations [2].

As mentioned above, the use of pseudo-random numbers and probability theory to solve problems numerically is termed Monte Carlo methods. The term "Monte Carlo methods" was first coined by mathematicians Stanislaw Ulam and Nicholas Metropolis in the 1940's. The methods were named after the famous gambling casinos of Monaco, since they use games of chance, similar to gambling, to study other interesting phenomena [1].

Today, Monte Carlo methods are a very powerful tool for investigating complex problems from a wide range of fields. They are useful in quantum mechanics to study energies of general quantum systems and ground-state wave functions. They can be used to estimate the area bounded by a curve. Monte Carlo methods have been used in several studies to examine the equilibrium properties of various atomic and molecular clusters [1].

These methods are often quicker and easier to compute than traditional methods. Monte Carlo methods can be very useful in solving problems which are not only difficult but impossible to solve analytically. They

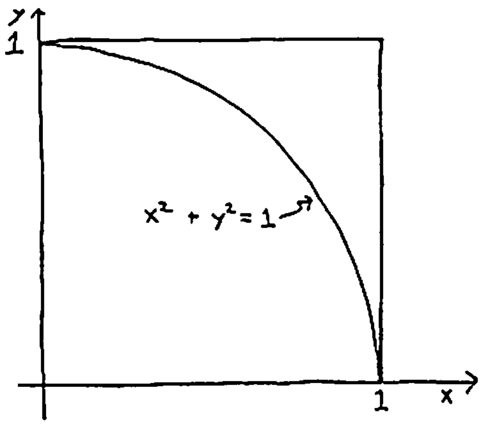


**Figure 1.**

greatly benefit from the fact that repetition of the solution process only refines the accuracy of the solution. This is a major reason that the methods are popular among computational chemists and physicists.

Monte Carlo techniques were employed in my study of atmospheric scattering. To fully understand the benefits of their application, we must first understand the atmosphere. The earth's atmosphere is composed of several different molecules. Nitrogen, which makes up approximately 78% of the atmosphere, is by far the most abundant of these molecules. The other major constituent is oxygen at approximately 21%. Argon is the third largest occurring at almost 1%. The next largest constituent is carbon dioxide at slightly less than .05% [5]. Many other elements are in the atmosphere but occur in such small proportions that they were not considered in our simulations. However, this simulation could be adapted to take into consideration other atmospheric constituents such as water vapor and pollutants.

We used Dalton's law of partial pressure and the ideal gas law to determine the ratio of the volume occupied by nitrogen, oxygen, carbon dioxide, and argon to a volume of space. Dalton's law of partial pressure simply states that the total pressure is the sum of the individual pressures. The individual pressure of a gas is called its partial pressure, which is defined as the pressure of the gas divided by the total pressure of all present gases. The ideal gas law relates pressure to volume by the equation $PV = nRT$, where $P$ represents partial pressure, $V$ represents the volume of occupied space, $n$ is the number of moles of gas, $R$ is a constant, and $T$ is temperature in degrees Kelvin. Rewriting the above equation, we get

$$n/V = P/RT.$$

Using Avogadro's number $N_a = 6.022 \times 10^{23}$ molecules/mole,

$$nN_a/V = PN_a/RT$$

gives the number of molecules/volume. Now, set

$$nN_a = N,$$

where $n$ is the total number of molecules, and

$$V_t = (4/3)\pi r^3 N,$$

where $V_t$ is the volume occupied by molecules.

In order for these equations to hold true, we assumed that the molecules were spherical in shape. While molecules do not occur as spheres in nature, it is a common assumption to make, and it was necessary to make an assumption about their shape in order to calculate their volume. The radii were calculated from the atomic radii of the molecules. Continuing,

$$\frac{V_t}{(4/3)\pi r^3} = N.$$

By substitution,

$$\frac{V_t}{(4/3)\pi r^3 V} = \frac{PN_a}{RT},$$

which can be rewritten as

$$\frac{V_t}{V} = \frac{PN_a(4/3)\pi r^3}{RT}.$$

This equation represents the volume of space occupied by a certain type of molecule per the total volume of space. From this equation, I calculated the percentages of space occupied by each of the four main constituents of the atmosphere.

Realizing that the volumes were dependent on pressure, it was necessary to consider the dependence of pressure on altitude. The atmosphere consists of four layers. The first layer is that of the troposphere. It ranges from the surface of the earth to approximately 10km. From the troposphere to about 50km lies the stratosphere. Above the stratosphere is the mesosphere, which ranges up to almost 80km. The thermosphere goes up to 100km. See figure 2.

I decided that the simulation would consider a small section of the atmosphere that was close to the surface of the earth. This allowed for pressure to be held constant. If we let $P_0$ = pressure at height 0, then

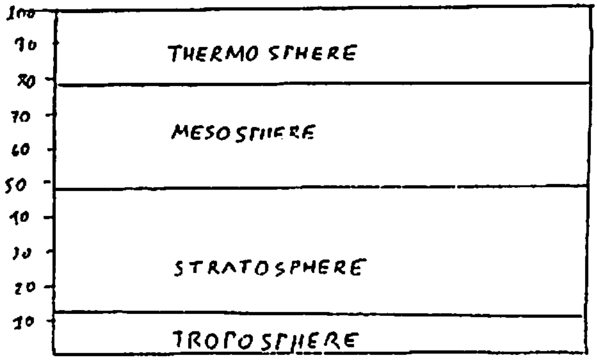


**Figure 2.**

$P = P_0 \cdot \exp(-mgh/RT)$. At a height of $h = 1$ meter, the argument of the exponent is approximately $10^{-4}$, so $P_0 = P$ for heights less than 1 meter.

Light that is incident upon the earth from the sun is really a form of energy. Light is characterized by wavelength or frequency. Energy is inversely related to the wavelength of light. When the wavelength of light is less than the molecular size, light can be scattered by the molecule. This is an example of Rayleigh scattering. Such scattering is insignificant when considering the long wavelength light, such as infrared light, emitted from earth. The greenhouse effect traditionally deals with absorption and emission of this long wavelength light. Short wavelength light is not absorbed but rater scattered by atmospheric gases. Short wavelength light is emitted by the earth's surface as a result of reflection. Figure 3 indicates reflection of short wavelength light as a function of surface type [5].

| Surface Type | Reflectivity (in percent) |
|---|---|
| Sea surface (low angle sun) | 10-70 |
| Sea surface (higher angle sun) | <10 |
| Snow (wet or dirty) | 25-75 |
| Snow (clean or dry) | 75-95 |
| Forest | 10-20 |
| Sand, desert | 25-40 |
| Bare soil | 10-25 |

(Wallace, Hobbs 1977)

**Figure 3.**

Scientists usually report the intensity of radiation, which is the energy of radiation per second per square centimeter. The intensity of scattered light is given by the relationship

$$I = I_0 \alpha^2 \cos\phi / r^2 \lambda^4.$$

The alpha represents a polarizability constant that varies directly with molecule size; $\cos\phi$ represents the direction of the scattered energy and in our case was held constant at 1; $r$ represents the distance from the molecule; and lambda represents the wavelength of the scattered light.

In this equation, only one of the variables is dependent upon the type of molecule being scattered. It quickly became apparent that if I could determine not only where in the atmosphere molecules are being hit but also which molecules are being hit, then I could make some conclusions about atmospheric scattering.

I wrote a FORTRAN program (see table 1) to model the effects of a photon of light reflecting from the earth back up into the atmosphere. The program takes a photon and passes it through several layers of the atmosphere. As the photon reaches each level, the program goes through an algorithm to see if the photon hits a molecule or if it passes through. If it does hit a molecule, it records which molecule is hit and at what level it occurs.

The algorithm used to detect whether or not a photon hits a molecule is based on Monte Carlo methods. While the molecules in each layer of the atmosphere occur in certain percentages, they do not occur in groupings; they are spread randomly throughout the layer. This effect is simulated with the use of random numbers. The total volume of each of the molecules in a layer per the total volume of space in the layer is placed on a number line from 0 to 1. The algorithm then calls on the random number generator for a pseudo-random number. The pseudo-random number is then tested to see if it falls on the nitrogen section of the number line. If it does, it adds one to the nitrogen counter at that level and starts a new photon up from the bottom level. If it does not fall on the nitrogen section, it is then tested to see if it falls on the oxygen section. This is repeated until a molecule has been hit or all of the molecules at one layer have been tested. If no molecules have been hit, the photon is then sent up to the next level. At this level, a new pseudo-random number is generated and the molecules at that level are checked. This process continues until a molecule is hit or all of the levels are completed.

The generation of these pseudo-random numbers is an effective way of randomly mixing the molecules of a layer. By generating a new number at each level tested, we are assured that the molecules at any one level are not in the same position at any other level.

This process is repeated numerous times to simulate the effects of several photons being reflected up into the atmosphere. When the simulation is completed, the program displays the number of times each molecule is hit and at what level they are hit (see table 2). From this information, I made some preliminary conclusions about scattering created by the molecules being hit by the light.

```
*      multiple level scattering program
       double precision rn, v1, v2, v3, v4
       integer seed, seed1, x, I, J, Q, L,n2,o2,ar,co2, nj
       integer nit(11), oxy(11), arg(11), carb(11)

       Print *, 'Enter seed values and # of iterations'
       read *, seed, seed1,x

       Do 50 Q=1, 11
       nit(Q)=0
       oxy(Q)=0
       arg(Q)=0
       carb(Q)=0
  50   continue
       Do 1001 I=1, x
*      setting volumes
       v1=.0002782
       v2=.0000486
       v3=.00000537
       v4=.000000399
       Do 501 J=1,11
       rn=random(seed, seed1)
       If (rn.lt.(.5678987)) goto 500
       If (rn.ge.(.5678987+v1)) goto 100
       nit(J)=nit(J)+1
       goto 1000
 100   If(rn.ge.(.5678987+v1+v2)) goto 200
       oxy(J)=oxy(J)+1
       goto 1000
 200   If (rn.ge.(.5678987 +v1+v2+v3)) goto 300
       arg(J)=arg(J)+1
       goto 1000
 300   If (rn.ge.(.5678987+v1+v2+v3+v4)) goto 500
       carb(J)=carb(J)+1
       goto 1000
 500   Continue
 501   Continue
 1000  continue
 1001  continue
       Print *, 'Level    N2    O2     AR     CO2'
       Do 1050 L=1,11
       Print 2001,L-1,nit(L),oxy(L),arg(L),carb(L)
 2001  Format (1X,I2,4X,I5,1X,I5,2X,I5,3X,I5)
 1050  continue
       end

       function random(ix,jx)
       real*8 random
       integer*4 ix,jx
       ix=ix*65539
       jx=jx*262147
       random=.4656613E-9*FLOAT(IABS(ix+jx))
       return
       end
```

**Table 1. FORTRAN program.**

```
Script command is started on Fri Feb 10 08:4
 Enter seed values and # of iterations^M
34765 8325 500000^M
 Level       N2      O2       AR       CO2^M
  0         135      24        1         0^M
  1         135      22        1         0^M
  2         139      22        1         0^M
  3         131      26        2         0^M
  4         159      21        1         0^M
  5         149      27        3         0^M
  6         151      28        3         0^M
  7         131      14        5         0^M
  8         133      27        7         1^M
  9         145      21        3         0^M
 10         146      18        1         0^M
$ exir      t^M
```

**Table 2. Sample program results.**

My initial conclusion indicates that carbon dioxide does not make a significant contribution to Rayleigh scattering. I am currently working on ways to improve the simulation to see if it would have any effect on the results I recorded. One of these improvements is in the way that the atmospheric gases are described. I am also investigating other geometries to best represent the shape of the atmospheric molecules.

Another refinement could be made in the assumptions used in the equation that measures the intensity of scattered light. In my calculations, $\cos\phi$ was held constant. It is possible this could have had an effect on the results obtained. There are also other types of atmospheric scattering. Mie scattering, which deals with large particle scattering of long wavelength infrared light from the earth, is an example of other possible types of scattering. Other atmospheric constituents should be considered to see their effect as well. It is possible that some of the other larger molecules might produce more intensity from scattering. It is also intriguing to consider the possibility of scattering from successive layers, that is, scattered light being scattered again by molecules in nearby layers.

It must be considered that this is just the first step in studying a fascinating aspect of the atmosphere. There are many difficulties in trying to study the atmosphere. As Thomas Kyle said [3], "Atmospheric studies have always suffered from the atmosphere itself not being reproducible. It is not that something is wrong with the experiments or techniques; it is the atmosphere which differs each time the experiment is repeated. Even computations of atmospheric properties suffer from the complement to this problem; different models assume different atmospheric conditions and get conflicting results."

You can see the difficulty in establishing a feasible model of the atmosphere. I believe that with the proper refinements over time, my simulation could be a useful tool in studying the effects of atmospheric scattering.

References

1. Doll, J. and Freeman, D., "Monte Carlo Methods in Chemistry," *IEEE Computational Science & Engineering* 1 (Spring 1994), 22–31.
2. Kalos, M. and Whitlock, P., *Monte Carlo Methods*, John Wiley and Sons, New York, 1986.
3. Kyle, T., *Atmospheric Transmission, Emission and Scattering*, Pergamon Press, Oxford, 1991.
4. Rubinstein, R., *Simulation and the Monte Carlo Method*, John Wiley and Sons, New York, 1981.
5. Wallace, J. and Hobbs, P., *Atmospheric Science: An Introductory Survey*, Harcourt Brace Jovanovich, San Diego, 1977.

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Please update your existing links! All local chapters with home pages should send their URL's to Arnold Hammel (a.hammel@cmich.edu) if they have not already done so. It is hard to provide a link to a page that no one knows about!

A cumulative subject index for *The Pentagon* may be on-line by the end of 1997. The editor has completed approximately half of the work necessary to create the index. Organization will be by generic course title or other such category (not by AMS subject classifications). It is hoped that this will be a valuable resource for faculty and students alike!

# Higher Order Niven Numbers

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## Introduction

Niven numbers were first introduced in [3] by the following definition.

**Definition.** *A positive integer is said to be Niven if it is divisible by its digital sum.*

Since that time, many discoveries have been made regarding Niven numbers. We note that the digital sum of $x$, which we denote $S(x)$, is computed by the formula

$$S(n) = \sum_{j=0}^{\lfloor \log x \rfloor + 1} \left\lfloor 10 \cdot \left( \frac{x}{10^j} - \left\lfloor \frac{x}{10^j} \right\rfloor \right) \right\rfloor .$$

Some important results concerning Niven numbers follow:

1) There exists an infinite set of Niven numbers [1].
2) Let $N(x)$ denote the number of integers less than or equal to $x$ that are Niven. Then $\lim_{x \to \infty} \frac{N(x)}{x} = 0$ (see [2]).
3) Currently, no asymptotic formula is known for $N(x)$.

In this paper, we will attempt to partition the Niven numbers by the defining the order of a Niven number and answer some of the basic questions (similar to (1) thorough (3)) concerning Niven numbers of various orders. We define an integer $x \geq 0$ to be an $n$th order Niven number, denoted $N_{(n)}(x)$, if $x$ is a Niven number and $S(x)$ is a Niven number of order $n-1$, and that $N_{(1)}(x)$ if and only if $S(x) = x$. All other Niven numbers are said to have no order.

It is easily seen that the Niven numbers of the first order are the integers from 1 to 9. We also see that the set of the first ten Niven numbers of the second order is $\{10, 12, 18, 20, 21, 24, 27, 30, 36, 40\}$. It would seem to follow that the first 3rd order Niven number has a digital sum of 10, but that is incorrect. The first 3rd order Niven number is 48. The first 3rd order Niven number whose digital sum is 10 is 190.

**Results**

Question (1): Do there exist Niven numbers of infinite order? In order to answer this, we will construct a number which is a multiple of a Niven number $x$. The following lemma, derived in [1], will prove useful for our proof.

**Lemma 1.** *$S(mn) = S(m)S(n) - 9c(m,n)$, where $c(m,n)$ denotes the sum of carries which occur when calculating the product of $m$ and $n$.*

We also note the following:

**Lemma 2.** *For any $x$,*

$$c\left(x, \sum_{t=0}^{\frac{x}{S(x)}-1} 10^{t(\lfloor\log x\rfloor+1)}\right) = 0.$$

We see that for each '1' in the sum, it is exactly $\lfloor \log x \rfloor$ digits away from the next '1' in the number. Since the last digit is 1 and the length of $x$ is equal to $\lfloor \log x \rfloor + 1$, the product is $x/S(x)$ consecutive copies of $x$, i.e., the product looks like

$$d_n d_{n-1} \ldots d_1 d_0 \;\ldots\; d_n d_{n-1} \ldots d_1 d_0,$$

where $n$ = length of $x$ and $d_i$ denotes the $i$th digit of $x$. It is easily seen that this product has no carries.

**Theorem 1.** *If $x$ is Niven, then there exists a Niven number $n$ such that $S(n) = x$.*

*Proof.* We define

$$n = \sum_{t=0}^{\frac{x}{S(x)}-1} x \cdot 10^{t(\lfloor\log x\rfloor+1)}.$$

Clearly $x|n$, and we see via Lemma 1 that

$$\begin{aligned} S(n) &= S\left(\sum_{t=0}^{\frac{x}{S(x)}-1} x \cdot 10^{t(\lfloor\log x\rfloor+1)}\right) \\ &= S(x)S\left(\sum_{t=0}^{\frac{x}{S(x)}-1} 10^{t(\lfloor\log x\rfloor+1)}\right) - 9c\left(x, \sum_{t=0}^{\frac{x}{S(x)}-1} 10^{t(\lfloor\log x\rfloor+1)}\right) \end{aligned}$$

$$= S(x)\left(\sum_{t=0}^{\frac{x}{S(x)}-1} 1\right) - 9c\left(x, \sum_{t=0}^{\frac{x}{S(x)}-1} 10^{t(\lfloor \log x \rfloor + 1)}\right).$$

Via Lemma 2, we see that the number of carries of the product is 0 and thus

$$S(n) = S(x) \sum_{t=0}^{\frac{x}{S(x)}-1} 1 = \frac{x}{S(x)} \cdot S(x) = x.$$

**Corollary 1.** *If there exists an integer $x$ such that $N_{(n)}(x)$, then there exists an integer $y$ such that $N_{(n+1)}(y)$ and $S(y) = x$.*

**Corollary 2.** *There exist Niven numbers of arbitrarily large order.*

Theorem 1 looks like a convenient function for generating higher order Niven numbers. Given the entire set of $n$th order Niven numbers, could this generate all Niven numbers of higher order? No. This is easily disproved when one looks at the algorithm with respect to the number 12. The algorithm yields 12121212, but the first 3rd order Niven number whose digital sum is 12 is 48.

Question 2: If $N^*_{(n)}(x) = \#\{y : N_{(n)}(y), y \leq x\}$, for what $n$ does $\lim_{x\to\infty} N^*_{(n)}(x) = \infty$? We have already stated that the set of first order Niven numbers is finite. This leads one to believe that the higher orders are finite, but this is not true.

**Theorem 2.** *For $n \geq 2$, $\lim_{x\to\infty} N^*_{(n)}(x) = \infty$.*

*Proof.* We see that $10^j$ is a second order Niven number for any $j \geq 0$. Hence, $\lim_{x\to\infty} N^*_{(2)} = \infty$. For any $x$ such that $N_{(2)}(x)$, one can create an integer $y$ such that $N_{(3)}(y)$ via Theorem 1. Thus, $\lim_{x\to\infty} N^*_{(3)}(x) = \infty$. We easily see that, by induction, the set of Niven numbers of order $n$ is infinite for any $n \geq 2$.

Question 3: What is $\lim_{x\to\infty} \frac{N^*_{(n)}(x)}{x}$? Since $N^*_{(n)}(x) \leq N(x)$, we easily see that $\lim_{x\to\infty} \frac{N^*_{(n)}(x)}{x} = 0$.

Question 4: What is known about the asymptotic formula for $N^*_{(n)}(x)$? Unfortunately, we know little about an asymptotic formula for $N(x)$. Yet, it is very obvious that if there does exist an asymptotic formula for $N(x)$, say $f(x)$, then the asymptotic formula for $N^*_{(n)}(x)$, if it exists, is bounded by $f(x)$.

## Open Questions

The following are some of the open questions regarding higher order Niven numbers. First, what is the largest set of consecutive numbers that belong to the $n$th order? In [2], it was proven that 20 is the largest set of consecutive Niven numbers. The sequence generated each had a digital sum of 2464645030, which is not Niven. By definition, these Niven numbers have no order. Hence, the question of whether there exists 20 consecutive Niven numbers of order $n$ arises. It is my conjecture that this might be true, but the set consists of numbers that have greater than 1300 digits.

Other questions arise, like what is the order which contains the largest set of consecutive integers? For what values of $x$ is $N^*_{(n)}(x) \leq N^*_{(n+1)}(x)$ true for all $n$? For what values is $N^*_{(n)}(x) \geq N^*_{(n+1)}(x)$ true for all $n$? Finally, what is the asymptotic formula for $N^*_{(n)}(x)$?

References

1. Cooper, C. and Kennedy, R., "On Consecutive Niven Numbers," *Fibonacci Quarterly* 8 (1992), 146–151.
2. Kennedy, R., "Digital Sums, Niven Numbers, and Natural Density," *Crux Mathematicorum* 8 (1982), 129–133.
3. Kennedy, R., Goodman, T., and Best, C., "Mathematical Discovery and Niven Numbers," *The MATYC Journal* 14 (1980), 21–25.

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# Lengths of Generalized Rose Curves

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## Introduction / Project Description

This project began by considering the well-known polar "rose curves." These are the polar graphs of $r(\theta) = \cos(m\theta)$ where $m \in \mathbf{N}$. We then generalized the situation by allowing the coefficient of $\theta$ to be rational. In other words, we investigated the graphs of $r(\theta) = \cos\left(\frac{m}{n}\theta\right)$, where $m$ and $n$ are relatively prime, and $n \neq 0$. We realized the need to determine the minimal interval $I = [0,p]$ for which the polar graph of this function on domain $I$ is complete; we define the "polar period" to be this value of $p$. The polar period is discussed in detail in Section 2. It is well known that the polar period for the rose curves is either $\pi$ or $2\pi$, depending on the parity of $m$, and so we expected to observe a similar phenomenon in the case of the "generalized rose curves," $r(\theta) = \cos\left(\frac{m}{n}\theta\right)$. We were not disappointed. Indeed, the parity of $m$ and $n$ are involved, as shown in Lemma 3.

Next we focused on finding the arc length of the generalized rose curve, numerically approximating this length for several values of $m$ and $n$. The resulting data suggested an interesting conjecture, namely that the arc length of $r = \cos\left(\frac{m}{n}\theta\right)$ and that of $r = \cos\left(\frac{n}{m}\theta\right)$ are equal. Section 3 is devoted to arc length, and this result is stated and proved as our main theorem.

## The Polar Period of a Generalized Rose Curve

In what follows, we make use of some basic properties of polar coordinates. Most calculus texts are likely to provide the necessary background on polar coordinates, their graphs, and their arc lengths (c.f. [3]). First of all, when we write $(r_1, \theta_1) = (r_2, \theta_2)$, we mean equality as points in the plane, not as ordered pairs of numbers. In particular, we have the following:

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**Proposition 1.** *The equality* $(r_1, \theta_1) = (r_2, \theta_2)$ *holds if and only if one of the following conditions hold:*

*(i)* $r_1 = r_2 = 0$;

*(ii) there is an* $n \in \mathbf{Z}$ *such that* $r_1 = r_2$ *and* $\theta_1 = \theta_2 + 2n\pi$; *or*

*(iii) there is an* $n \in \mathbf{Z}$ *such that* $r_1 = -r_2$ *and* $\theta_1 = \theta_2 + (2n+1)\pi$.

**Definition 1.** *The* polar period *of a function* $r = r(\theta)$ *is the smallest* $\alpha > 0$ *that satisfies*

$$(r(\theta+\alpha), \theta+\alpha) = (r(\theta), \theta) \text{ for all } \theta \in \mathbf{R}. \tag{2.1}$$

Roughly speaking, the polar period is the smallest positive number $p$ for which the polar graph is completed as $\theta$ varies from 0 to $p$. As an example, we can easily compute the polar period for the standard rose curves $r(\theta) = \cos(m\theta)$:

**Lemma 2.** *The polar period for* $r(\theta) = \cos(m\theta)$ *is* $\begin{cases} \pi & \text{if } m \text{ is odd} \\ 2\pi & \text{if } m \text{ is even} \end{cases}$.

*Proof.* Consider $r(\theta) = \cos(m\theta)$.

Case I: Assume $m$ is odd. To show that $\alpha = \pi$ satisfies (2.1), note that

$$\begin{aligned}
(r(\theta+\pi), \theta+\pi) &= (\cos m(\theta+\pi), \theta+\pi) \\
&= (\cos(m\theta + m\pi), \theta+\pi) \\
&= (-\cos(m\theta), \theta+\pi) \text{ (using well-known identities)} \\
&= (\cos(m\theta), \theta) \text{ (by Proposition 1)} \\
&= (r(\theta), \theta).
\end{aligned}$$

To show that $\pi$ is the smallest $\alpha > 0$ that satisfies (2.1), assume that $\alpha$ satisfies (2.1) with $0 < \alpha < \pi$. Substituting $\theta = 0$ into (2.1) with $r(\theta) = \cos(m\theta)$, we obtain $(\cos(m\alpha), \alpha) = (1, 0)$. Now by Proposition 1, $\alpha$ must be a multiple of $\pi$, which is impossible since $\alpha \in (0, \pi)$. Thus $\pi$ is the polar period.

Case II: Assume $m$ is an even integer. In a manner similar to the above, we first show that

$$\begin{aligned}
(r(\theta+2\pi), \theta+2\pi) &= (\cos(m\theta + 2m\pi), \theta+2\pi) \\
&= (\cos(m\theta), \theta).
\end{aligned}$$

Then we prove that $\alpha = 2\pi$ is the smallest $\alpha$ that satisfies (2.1) by using Proposition 1, as above.

Note that for the rose curves the parity of $m$ necessarily determines the polar period. We now introduce the generalized rose curves $r(\theta) =$

$\cos\left(\frac{m}{n}\theta\right)$ in which the parity of both $m$ and $n$ play a role. For notational convenience in what follows, let

$$\beta_{mn} = \begin{cases} 1 & \text{if } mn \text{ is odd} \\ 2 & \text{if } mn \text{ is even} \end{cases}.$$

Note that $\beta_{mn} = \beta_{nm}$.

**Lemma 3.** *The polar period for* $r(\theta) = \cos\left(\frac{m}{n}\theta\right)$ *is* $\beta_{mn} n\pi$.

*Proof.* Let $r(\theta) = \cos\left(\frac{m}{n}\theta\right)$. We first show that (2.1) holds when $\alpha = 2n\pi$:

$$\begin{aligned}(r(\theta+2n\pi), \theta+2n\pi) &= \left(\cos\left(\frac{m}{n}\theta+2m\pi\right), \theta+2n\pi\right) \\ &= \left(\cos\left(\frac{m}{n}\theta\right), \theta+2n\pi\right) \text{ (using known identities)} \\ &= \left(\cos\left(\frac{m}{n}\theta\right), \theta\right) \text{ (by Proposition 1).}\end{aligned}$$

Now to compute the smallest $\alpha$ that satisfies (2.1), let us assume that $\alpha$ satisfies (2.1). Substituting $\theta = 0$ into (2.1) with $r(\theta) = \cos\left(\frac{m}{n}\theta\right)$, we obtain

$$\left(\cos\left(\frac{m}{n}\alpha\right), \alpha\right) = (1, 0).$$

By Proposition 1, $\cos\frac{m}{n}\alpha = \pm 1$, forcing $\alpha = \frac{n}{m}k\pi$ for some integer $k$. Also by Proposition 1, $\alpha$ is a multiple of $\pi$, so $\frac{nk}{m} \in \mathbf{Z}$. Since $n$ and $m$ are relatively prime, $k$ is a multiple of $m$, and thus $\alpha$ is a multiple of $n\pi$. Thus, the polar period must be either $n\pi$ or $2n\pi$.

Case I: Assume $m$ is odd and $n$ is even ($n \neq 0$). Then the polar period is $2n\pi$, since $\alpha = n\pi$ does not satisfy (2.1):

$$\begin{aligned}\left(\cos\left(\frac{m}{n}(\theta+n\pi)\right), \theta+n\pi\right) &= \left(\cos\left(\frac{m}{n}\theta+m\pi\right), \theta\right) \\ &= \left(-\cos\left(\frac{m}{n}\theta\right), \theta\right) \\ &\neq \left(\cos\left(\frac{m}{n}\theta\right), \theta\right)\end{aligned}$$

in general, as can be easily seen by substituting $\theta = 0$.

Case II: Assume $m$ is even and $n$ is odd. In this case, the polar period is also $2n\pi$ and not $n\pi$, since

$$\begin{aligned}\left(\cos\left(\frac{m}{n}(\theta+n\pi)\right), \theta+n\pi\right) &= \left(\cos\left(\frac{m}{n}\theta+m\pi\right), \theta+n\pi\right) \\ &= \left(\cos\left(\frac{m}{n}\theta\right), \theta+n\pi\right) \\ &\neq \left(\cos\left(\frac{m}{n}\theta\right), \theta\right)\end{aligned}$$

as above.

Case III: Assume $m$ and $n$ are both odd. Then the polar period is $n\pi$. To show that (2.1) holds when $\alpha = n\pi$, we compute as follows:

$$\begin{aligned}(r(\theta+n\pi),\theta+n\pi) &= \left(\cos\left(\frac{m}{n}\theta+m\pi\right),\theta+n\pi\right)\\ &= \left(-\cos\left(\frac{m}{n}\theta\right),\theta+n\pi\right) \text{ (using known identities)}\\ &= \left(\cos\left(\frac{m}{n}\theta\right),\theta\right) \text{ (by Proposition 1).}\end{aligned}$$

Thus, $n\pi$ is the polar period if $m$ and $n$ are odd.

In preparation for the main results, we introduce some notation. We denote the polar period of $r(\theta) = \cos(\frac{m}{n}\theta)$ by $p(m,n)$. Using this notation, we can express an immediate consequence of the previous lemma.

**Corollary 4.** $p(n,m) = \frac{m}{n} \cdot p(m,n)$.

*Proof.* Note that $p(n,m) = \beta_{nm} m\pi = \beta_{mn} \frac{m}{n} n\pi = \frac{m}{n} \cdot p(m,n)$.

One additional lemma required in the next section is the following result concerning integrals of periodic functions.

**Lemma 5.** *If $g(x)$ is periodic with period $p$, then for any $a,b \in \mathbf{R}$, $\int_a^{a+p} g(x)\,dx = \int_b^{b+p} g(x)\,dx$.*

*Proof.*

$$\begin{aligned}\int_a^{a+p} g(x)\,dx &= \int_a^b g(x)\,dx + \int_b^{a+p} g(x)\,dx\\ &= \int_{a+p}^{b+p} g(x)\,dx + \int_b^{a+p} g(x)\,dx\\ &= \int_b^{b+p} g(x)\,dx.\end{aligned}$$

**Corollary 6.** *If $g(x)$ is periodic with period $p$, and $k$ is any positive integer, then $\int_0^{kp} g(x)\,dx = k \cdot \int_0^p g(x)\,dx$.*

*Proof.*

$$\begin{aligned}\int_0^{kp} g(x)\,dx &= \int_0^p g(x)\,dx + \int_p^{2p} g(x)\,dx + \cdots + \int_{(k-1)p}^{kp} g(x)\,dx\\ &= k \cdot \int_0^p g(x)\,dx.\end{aligned}$$

**Main Results: Arc Length**

In this section we will prove our observations about the arc lengths of generalized rose curves. For this purpose we denote the arc length of $r(\theta) = \cos\left(\frac{m}{n}\theta\right)$ by $l(m,n)$.

**Main Theorem.** *For* $m, n \in \mathbb{N}$ , $l(m,n) = l(n,m)$.

*Proof.* Consider $r(\theta) = \cos\left(\frac{m}{n}\theta\right)$. By using the standard arc length formula

$$\text{Arc Length} = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$

for a polar function $r = f(\theta)$, where $\theta \in [\alpha, \beta]$, we obtain

$$l(m,n) = \int_0^{p(m,n)} \sqrt{\cos^2\left(\frac{m}{n}\theta\right) + \frac{m^2}{n^2}\sin^2\left(\frac{m}{n}\theta\right)}\, d\theta.$$

Then, by substituting $\varphi = \frac{m}{n}\theta$ we get

$$l(m,n) = \int_0^{\frac{m}{n}p(m,n)} \sqrt{\cos^2(\varphi) + \frac{m^2}{n^2}\sin^2(\varphi)} \cdot \frac{n}{m}\, d\varphi.$$

Now by Lemma 3,

$$l(m,n) = \frac{n}{m}\int_0^{m\pi\beta_{mn}} \sqrt{\cos^2(\varphi) + \frac{m^2}{n^2}\sin^2(\varphi)\, d\varphi}.$$

Since one can easily show that $\cos^2(\varphi) + \frac{m^2}{n^2}\sin^2(\varphi)$ is periodic with period $\pi$, Corollary 6 yields

$$\begin{aligned} l(m,n) &= (m\beta_{mn})\frac{n}{m}\int_0^\pi \sqrt{\cos^2(\varphi) + \frac{m^2}{n^2}\sin^2(\varphi)\, d\varphi} \\ &= \beta_{mn}\int_0^\pi \sqrt{n^2\cos^2(\varphi) + m^2\sin^2(\varphi)\, d\varphi}. \end{aligned} \tag{3.1}$$

Similarly, $l(n,m) = \beta_{mn}\int_0^\pi \sqrt{m^2\cos^2(\varphi) + n^2\sin^2(\varphi)\, d\varphi}$. Now, substituting $\psi = \frac{\pi}{2} - \varphi$ in (3.1) and using the identity $\cos\left(\frac{\pi}{2} - \varphi\right) = \sin(\varphi)$, we obtain

$$\begin{aligned} l(m,n) &= -\beta_{mn}\int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \sqrt{n^2\sin^2(\psi) + m^2\cos^2(\psi)\, d\psi} \\ &= \beta_{mn}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sqrt{n^2\sin^2(\psi) + m^2\cos^2(\psi)\, d\psi} \\ &= \beta_{mn}\int_0^\pi \sqrt{m^2\cos^2(\psi) + n^2\sin^2(\psi)\, d\psi} \text{ (By Lemma 5)} \\ &= l(n,m). \end{aligned}$$

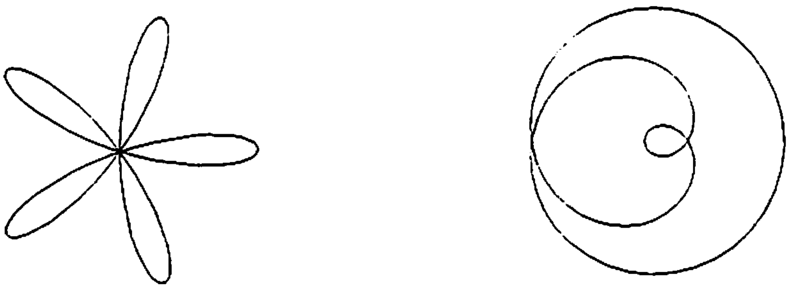

**Figure 1.** Five-leaved rose $r = \cos(5\theta)$ (left) with period $\pi$ and generalized rose curve $r = \cos\left(\frac{1}{5}\theta\right)$ (right) with period $5\pi$. Each has approximate arc length 10.505.

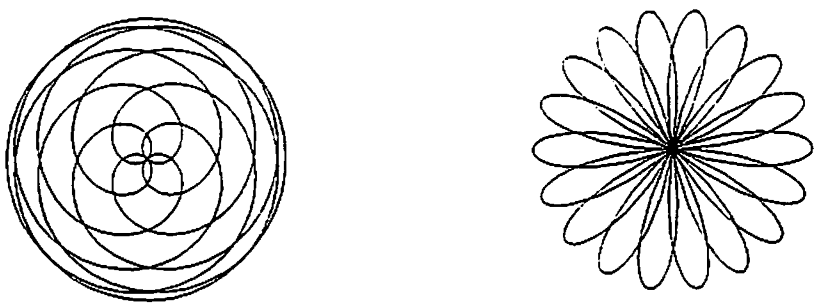

**Figure 2.** Generalized rose curves $r = \cos\left(\frac{2}{9}\theta\right)$ (left) with period $18\pi$ and $r = \cos\left(\frac{9}{2}\theta\right)$ (right) with period $4\pi$. Each has approximate arc length 38.155.

This theorem surprised us. The graphs of $r = \cos\left(\frac{m}{n}\theta\right)$ and $r = \cos\left(\frac{n}{m}\theta\right)$ have very different polar periods, and look dramatically different; see the comparisons in figures 1 and 2.

Applying trigonometric identities to (3.1), one can show that

$$l(m,n) = 2n\beta_{mn} \int_0^{\frac{\pi}{2}} \sqrt{1 - \left(\frac{n^2 - m^2}{n^2}\right) \sin^2\varphi \, d\varphi}.$$

This integral cannot be evaluated as an elementary function. Nevertheless, it can be expressed in terms of a special function known as an elliptic integral. More generally, integrals of the form

$$E(\mu) = \int_0^{\frac{\pi}{2}} \sqrt{1 - \mu^2 \sin^2\theta} \, d\theta$$

are known as *complete elliptic integrals of the second kind* [2], and are often found in the context of physical problems and engineering [1]. Elliptic integral tables were once common computational tools; however, as we discovered, the development of powerful computer algebra systems has rendered these largely obsolete. In the language of elliptic integrals, when $n > m$, the generalized rose curve $r = \cos(\frac{m}{n}\theta)$ has arc length

$$l\,(m,n) = 2n\beta_{mn}E\left(\sqrt{\frac{n^2-m^2}{n^2}}\right).$$

This would enable one to approximate these lengths using the elliptic integral tables. Of course, if $m > n$, the Main Theorem allows us to compute $l\,(n,m)$ to achieve the same result.

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## References

1. Arfken, G., *Mathematical Methods For Physicists,* Academic Press, 1985.
2. Beyer, W. (ed.), *CRC Standard Mathematical Tables,* 26th ed., CRC Press, Cleveland, 1981.
3. Stewart, J., *Calculus,* 3rd ed., Brooks/Cole, Pacific Grove, California, 1995.

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## *From the Pages of ...*

"LETHAL ENVELOPE ... The most important—and startling—scientific discovery of the year has been the evidence telemetered back by the instrumented Earth satellites of a region of dangerous radiation extending from outer space into the Earth's atmosphere. Presenting as it does a new obstacle to manned space flight, the "radiation shell" is of major interest to scientists of Systems Corporation of America, who are already at work investigating techniques for protecting men and instruments from this unexpected hazard. Other topflight physicists, engineers and applied mathematicians who would like to specialize in precision solutions to the problems of space travel are invited to address their inquiries to ... SYSTEMS CORPORATION OF AMERICA"

—advertisement in *The Pentagon*, Fall 1958

# Universal Chaos

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I thought the universe was governed by strict, oppressive laws, until I learned about the emerging, amazing science called Chaos. This science has such significance that I believe everyone in the educational world should be familiar with it. I would like to outline some of the basic ideas of chaos and fractal geometry. James Gleick [3] defines chaos as "the obstinate element of disorder within order, of variation where predictability was expected." The chaos theory is reshaping how scientists see the universe. No longer can we blindly follow one set of orderly rules and ignore the disorder found in nature. But in examining this disorder, we may discover the fundamental secrets of the universe.

Chaos was discovered barely a decade ago when various scientists began examining the disorder in nature. Heart oscillations, gypsy moth populations, and stock prices all led to parallels in nature with cloud shapes, lightning paths, blood vessel networks, and star clusters. Scientists from every field noticed the surprising patterns of nature. Chaos has revolutionized science by uniting scientists from every field in a world in which increasing specialization prevented physicists from communicating with biologists or mathematicians.

However, chaos has also sparked controversy in scientific circles. Some call it the greatest overstatement of the century. On the other hand, James Gleick [3] states, "The most passionate advocates of the new science go so far as to say that 20th century science will be remembered for just three things: relativity, quantum mechanics, and chaos." One thing that is unique about the chaos revolution is that it applies to objects we can see and touch. Theoretical physics has not directly affected the average person recently. Chaos has brought physics back into the mainstream. Theorists in chaos look at the big picture, believing the whole is truly greater than the sum of its parts.

Because chaos theory applies to a wide range of disciplines, the individuals who are helping shape it are also quite diverse. One scientist involved in the origin of chaos theory was Edward Lorenz, a meteorologist at MIT. He ran a computer program which simulated weather patterns on earth by obeying a few basic physical laws. He decided to repeat a certain sequence and typed in its initial conditions. The weather surprisingly did not repeat the original sequence; it began the same but quickly became unrecognizable. Was it a computer malfunction? No, the difference was that the computer held six decimal places in the first sequence, while Lorenz only included three in the second sequence. This difference of one ten-thousandth caused a completely different outcome, virtually ruling out long-range forecasting of weather, the stock market, and other complex systems.

The resulting phenomenon, small changes causing complex results, had never before been considered. Scientists had always assumed that simple changes led to simple results and complex results could only be a product of complex initial conditions. This event, termed "sensitive dependence on initial conditions," has become known as the Butterfly Effect. As explained in the hit book and movie *Jurassic Park*, by Michael Crichton, this is the idea that the flap of a butterfly wing in China today can cause storms in New York in a month. Scientists have ignored minor changes, thinking that they were negligible; chaos is saying they were wrong.

The chaotic patterns of nature can be modeled by complex shapes that Benoit Mandelbrot termed "fractals." Fractal patterns are highly irregular and fragmented, like a jagged coastline. Also self-similar, fractals repeat themselves across scale, pattern inside pattern. For example, the Koch snowflake is constructed by beginning with an equilateral triangle, and adding similar triangles one-third of its size to the middle third of the larger triangle's sides. This process is repeated to infinity. The most famous fractal is the Mandelbrot set, named for Benoit Mandelbrot who, in 1980, was among the first to heavily study it. It is a cousin to the Julia sets, studied by French mathematician Gaston Julia in the 1920's. One seemingly simple equation, $f(z) = z^2 + c$, is a basis for the Mandelbrot set and some filled Julia sets.

Though the equations appear simple, the image of a fractal is infinitely complex. For example, let us see how to generate a basic fractal. Beginning with a function, we will examine its "orbit." An orbit is the sequence generated upon iteration of a function, starting with some initial value $x_0$. Consider the function $F(x) = x^2$; perform the iterations $x_{k+1} = x_k^2$. If $|x_0| < 1$, the orbit converges to 0. If $|x_0| > 1$, the orbit diverges to $\pm\infty$. If $|x_0| = 1$, the orbit converges to 1. Thus, there exist three different possibilities for the orbits of $F(x) = x^2$. A fractal is based on orbits of functions.

I wrote a short program to compute the orbit of $f(x) = x^2 - 2$ and surprisingly found that even a simple function like this can produce a chaotic orbit. In fact, it will produce orbits with absolutely no pattern if $|x_0| \leq 2$ and $x_0$ is not an integer. So, choose an initial value of 1.3 (or 0.8, or 1.17, etc.), and the orbit generated does not have any pattern.

In more realistic situations, we can use complex numbers to model systems of two variables. Filled Julia sets contain complex numbers whose orbits do not go to infinity. The boundaries of most Julia sets of complex functions are fractals. To generate a Julia set of the function $f(z) = z^2 + c$, we first choose a complex number, $c = a + bi$, and a grid of points in the complex plane. Then, we compute the first $n$ points on the orbit of each point, $z_0$, in the grid by iterating $z_{k+1} = z_k^2 + c$. If the orbit does not exceed a certain bound after $n$ iterations, we will assume that the original point, $z_0$, is in the Julia set. If the orbit does exceed the bound, it is assumed to diverge to infinity. Points in the set can be colored white and points outside the set are colored according to the number of iterations, $i$, before exceeding the bound. We can let the low range of $i$'s be red, and follow the spectrum on up to the highest $i$'s being violet. Using *Fractint for Windows* [2], I have generated a Julia set (see figure 1 for a greyscale version; the areas which appear speckled are the fractal boundary of the set) based on the constant $c = -0.75978106 - 0.097077101i$. This complex and beautiful image is contained in the grid of the complex plane from $(-2, -1.5)$ to $(2, 1.5)$.

The function $f(z) = z^2 + c$ was just one example of the many types of functions which can be used to form Julia sets. Transcendental functions, like $c \cdot \sin(z)$ and $c \cdot e^z$ also form interesting fractals. You can see some of these in "The Fractal Poster Set" by Robert L. Devaney [1].

The Julia sets utilize several orbits, one orbit for each point in the plane, for a single function. The Mandelbrot set, or $M$-set, only examines one orbit, the zero orbit ($z_0 = 0$), but does so for the family of functions of the form $F(z) = z^2 + c$, where $z$ and $c$ are complex and $c$ varies. Thus the $M$-set is shown in the $c$-plane, while the Julia sets are in the $z$-plane. To graph the $M$-set, calculate the orbit of zero for a grid of different $c$-values. For the Mandelbrot set, it can be shown that any orbit which reaches 2 or $-2$ will diverge. The coloring can be decided in the same manner as for the Julia sets. Generating these infinitely complex images requires the speed of the computer, which has facilitated generating fractal images in the last twenty years.

I have generated, again with *Fractint*, several images of the Mandelbrot set in its entirety as well as tiny, magnified pieces of it. The first image is the entire Mandelbrot set (see figure 2 for a greyscale version); its infinite complexity is contained in a small circle of radius 2, centered about the origin. Each successive image is a blown-up picture of the boxed area in the previous image (figures 3, 4, and 5). The magnification and maximum

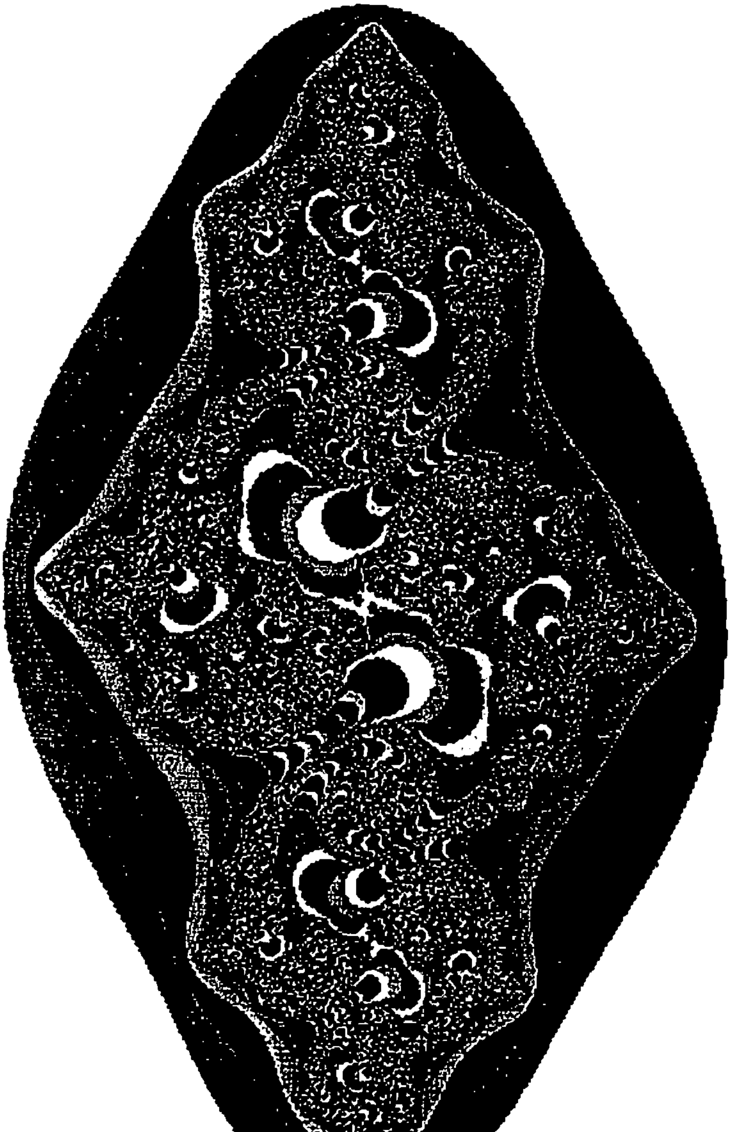

Figure 1. Julia set.

iterations for each image are:

| figure | magnification | iterations |
|---|---|---|
| fig. 2 | 1 | 150 |
| fig. 3 | 6376.52 | 150 |
| fig. 4 | 6975425.12 | 500 |
| fig. 5 | 9783689130.19 | 2000 |

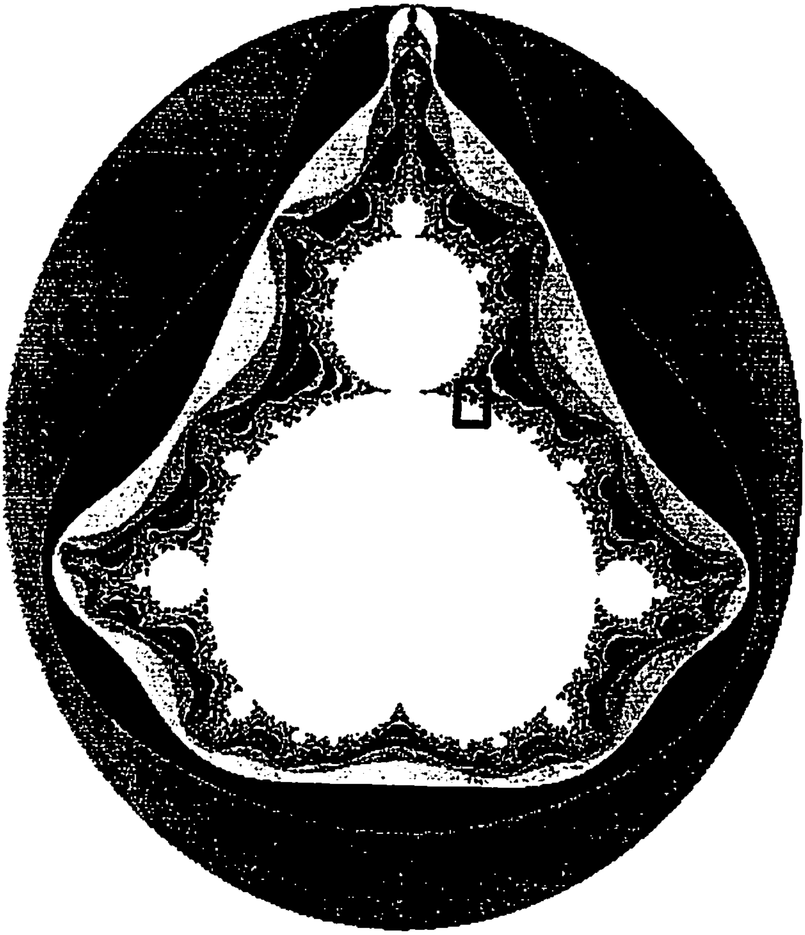

**Figure 2. Mandelbrot set.**

**Figure 3. Portion of the Mandelbrot set.**

I finally arrived at an image that resembles the entire $M$-set (figure 5). However, upon close examination, you may see that it is not an exact replica but has its own variations. In fact, no two parts of this seemingly self-similar fractal are exactly the same. Theoretically, if we wished, we could forever view increasingly higher magnifications of the Mandelbrot set, or any fractal, and all of the images would be infinitely complex. Computer-generated images of fractal geometry are colorful and beautiful, connecting science and art. These amazing images are still far from being understood.

Since fractals are a continuous loop of infinite length contained in a finite amount of space, Gleick illustrates [3], "in the mind's eye, a fractal

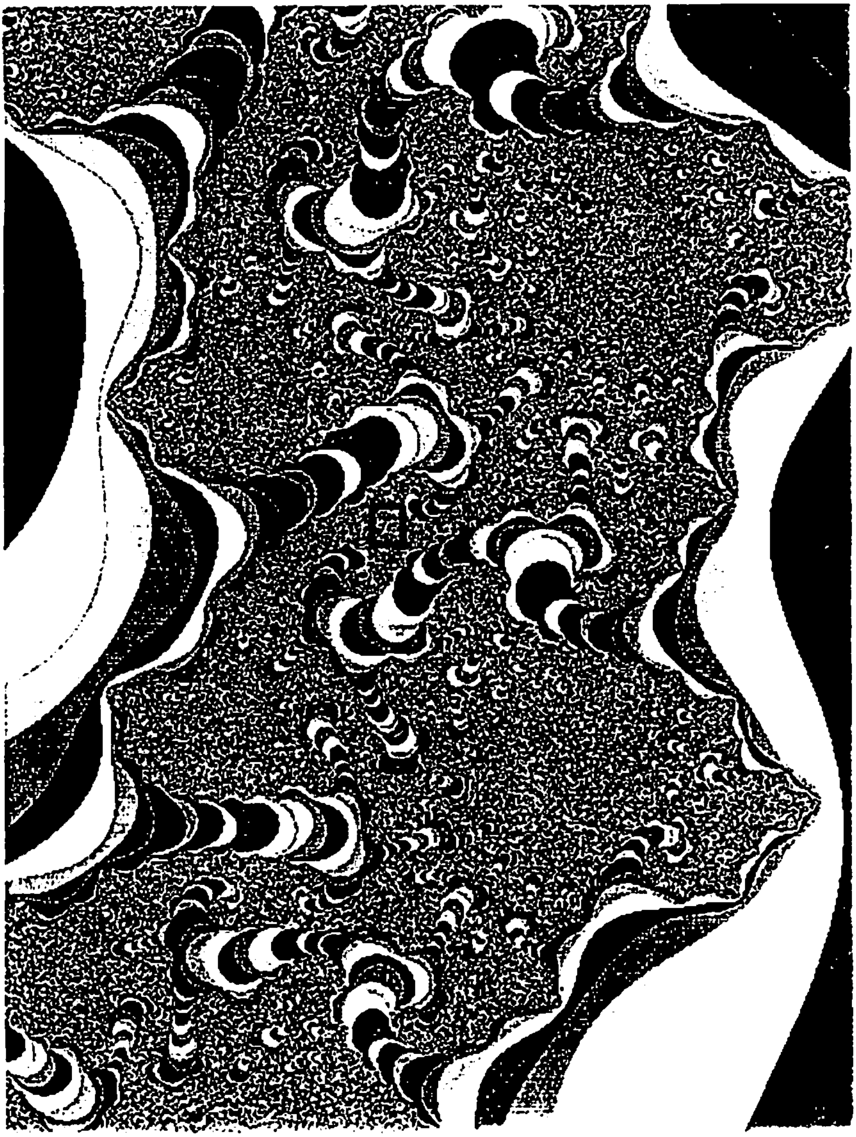

**Figure 4. Portion of the Mandelbrot set.**

is a way of seeing infinity." This aspect of infinite length in finite area was actually examined at the turn of the century, but then discarded because it was so bothersome to most mathematicians. A simple one-dimensional line fills no space. However, a fractal's length does fill some space. More than

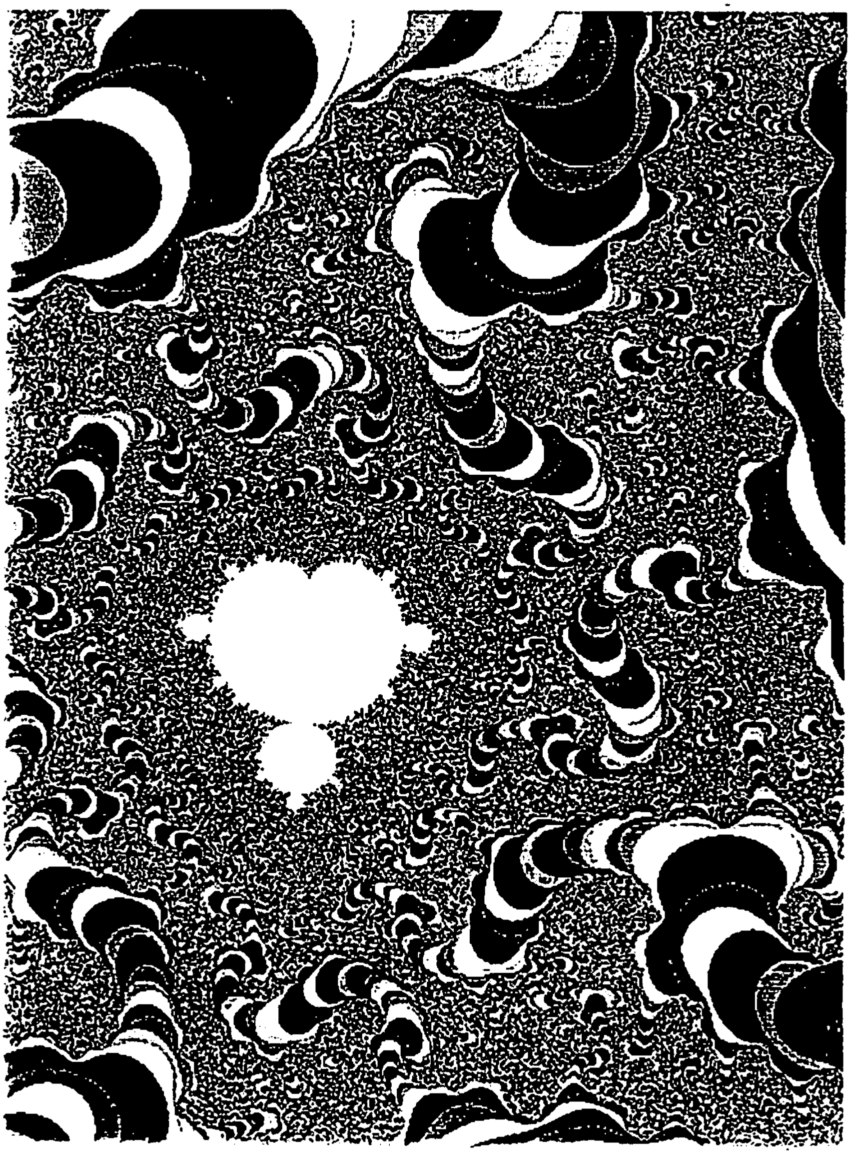

**Figure 5. Portion of the Mandelbrot set.**

a line, but less than a plane, it may have fractional dimension. Mandelbrot use the forgotten techniques of the early 20th century mathematicians to calculate some fractional dimensions. For example, he found the dimension of the Koch snowflake to be approximately 1.2618. This number is a measure of the geometric complexity of a fractal. The fractional dimension of a fractal that models a real-world system can give important information about properties of the system. Fractional dimension is one of the few things that modern science knows about fractals.

There is still a lot that I do not know, and still a lot that no one knows, about chaos and fractal geometry. I hope that I have helped illuminate some of the basic ideas behind this revolutionary science. The infinitely complex and repeating patterns of fractals may contain the answers to nature's fundamental questions. Chaos has the potential to unlock the greatest secrets of the universe. Future generations will use chaos theory to solve what classical science could not, to reveal what only Mother Nature has known since the dawn of time.

*Acknowledgements.* I would like to take this opportunity to thank Christopher Brown for sharing with me some of his knowledge of computer-generated imagery and Dr. Timothy Randolph for helping me find some valuable resources. I would also like to thank Dr. Jim Joiner, our KME chapter's faculty advisor. Without his persistent encouragement I would not have completed this paper.

References

1. Devaney, Robert L., *The Fractal Poster Set: An Invitation to Mathematics*, Dale Seymour Publications, 1992.
2. *Fractint for Windows*, version 3.10, The Stone Soup Group, 1990.
3. Gleick, James, *Chaos: Making a New Science*, Penguin Books, 1987.

## *Newest Record Primes*

Since the last time it was reported in *The Pentagon* (Fall 1995), the record for largest known prime has been broken twice! The two new numbers are $2^{1257787}-1$ and the current record of $2^{1398269}-1$. In the Fall 1941 issue, the decimal expansion of the largest known prime was printed using one line of type. The current record has 420921 digits, which would take just over 181 pages to print, or more than 2.26 issues of *The Pentagon*!

Two WWW sites for primes are given below. The first is the site for information on the largest primes. The second is for the Great Internet Mersenne Prime Search, in which you can participate! It is through George Woltman's free software located there that Joel Armengaud found the current record. Maybe you will set the next record!

www.utm.edu/research/primes/largest.html#biggest
ourworld.compuserve.com/homepages/justforfun/prime.htm

# Real Division Algebras and Dickson's Construction

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Many times the discussion of division algebras is limited to the associative cases. However, nonassociative real division algebras do exist. They are often left out because it is assumed that in order to solve linear equations it is enough to require multiplicative inverses. In the following pages, we will show that this is not the case by using Dickson's construction to form an infinite number of real algebras in which multiplicative inverses exist but in which there are linear equations without solutions.

We begin by stating Curtis' definition of an algebra [2]. As a reminder, a field is a commutative ring in which the set of all nonzero elements forms a group with respect to multiplication.

**Definition.** *Let $F$ be a field. Then $D$ is an algebra over a field $F$ provided $D$ is a vector space over $F$ and $D$ has multiplication such that*
*(i) $r(ab) = (ra)b = a(rb)$ for all $r \in F$ and $a, b \in D$, and*
*(ii) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in D$.*

Furthermore, an algebra $D$ is a division algebra when it contains a unity and each linear equation with coefficients in $D$ has a unique solution in $D$.

Division algebras have many of the characteristics of rings. Both are required to be Abelian groups with respect to addition. Also, the multiplication is both left and right distributive in rings and in division algebras. The main difference lies in the associativity of multiplication. Rings must have an associative multiplication. Division algebras, however, only require that scalar multiplication be associative. This difference may seem small but in fact it is quite important.

With the above definition of a division algebra, the following theorem holds.

**Theorem** (Bott and Milnor [1] and Kervaire [7]). *A finite dimensional division algebra over the field of real numbers has dimension 1, 2, 4, or 8.*

These algebras are the reals, the complex, the quaternions, and the Cayley numbers, respectively. The reals and the complex numbers are familiar. However, the quaternions and the Cayley numbers are more obscure so I will describe them briefly. The quaternions are defined as $Q = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \ : \ \alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}\}$ (throughout, $\mathbf{R}$ will be used to denote the real numbers). The addition in the quaternions is componentwise and the multiplication is as defined in table 1 (Durbin [5]):

| * | 1 | i | j | k |
|---|---|---|---|---|
| 1 | 1 | i | j | k |
| i | i | -1 | k | -j |
| j | j | -k | -1 | i |
| k | k | j | -i | -1 |

**Table 1. Multiplication in the quaternions.**

The Cayley numbers are of the same format but with eight components instead of only four. Again, the addition is componentwise and the multiplication is as defined in table 2 (Kleinfeld [8]).

It is obvious from this table that the Cayley numbers are nonassociative. For example, $i * (j * l) = -o$ but $(i * j) * l = o$. Therefore, the Cayley numbers are both the largest example of a division algebra over the real numbers and the nonassociative example of a divison algebra over the reals.

Now we shall construct an infinite number of real algebras in which multiplicative inverses exist for all nonzero elements but in which there are linear equations without solutions. This construction is known as Dickson's construction. First, however, another definition is necessary.

**Definition** (Dickson [3]). *Let $D$ be an algebra with identity over a field $F$. A map $\theta : D \to D$ is called a conjugation if $\theta$ is a vector space homomorphism such that for all $a, b \in D$,*

*(i)* $\theta(\theta(a)) = a$,
*(ii)* $\theta(ab) = \theta(b)\theta(a)$,
*(iii)* $a\theta(a) \in F$, and
*(iv)* $a\theta(a) \neq 0$ for $a \neq 0$.

| * | 1 | i | j | k | l | m | n | o |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | i | j | k | l | m | n | o |
| i | i | -1 | k | -j | m | -l | -o | n |
| j | j | -k | -1 | i | n | o | -l | -m |
| k | k | j | -i | -1 | o | -n | m | -l |
| l | l | -m | -n | -o | -1 | i | j | k |
| m | m | l | -o | n | -i | -1 | -k | j |
| n | n | o | l | -m | -j | k | -1 | -i |
| o | o | -n | m | l | -k | -j | i | -1 |

**Table 2. Multiplication in the Cayley numbers.**

An example of a conjugation is $\theta : \mathbf{R} \to \mathbf{R}$ defined by $\theta(a) = -a$. The properties can be checked easily and will be left to the reader.

The following lemma is also necessary for the construction.

**Lemma** (Dickson [4]). *An algebra with a multiplicative identity and a conjugation has inverses for all nonzero elements.*

*Proof.* Let $a \in D$, where $D$ is an algebra with a unity and a conjugation. Let $a \neq 0$. Then since $a\theta(a) \in F$ and $a\theta(a) \neq 0$, $(a\theta(a))^{-1}$ exists. Now $a * \left(\theta(a)(a\theta(a))^{-1}\right) = 1$, so $\theta(a)(a\theta(a))^{-1}$ is the inverse of $a$.

With these definitions and lemmas in place, the search for an algebra in which each nonzero element has a multiplicative inverse but in which some linear equations do not have solutions may begin.

This search involves Dickson's construction of a division algebra. This construction sets $D$ to be an algebra with a unity and a conjugation. Therefore, $D$ has inverses for all nonzero elements by the above lemma. Dickson proposes that $D^2 = D \times D$ is also an algebra with unity and conjugation. The addition is componentwise and multiplication and conjugation are defined by

$$(a,b)(c,d) = (ac - \theta(d)b, da + b\theta(c)) \text{ and}$$
$$\theta((a,b)) = (\theta(a), -b).$$

Obviously, this can be extended to $D^4 = D^2 \times D^2$, $D^8 = D^4 \times D^4$, $D^{16} = D^8 \times D^8$, etc. Letting $D = \mathbf{R}$, we obtain $\mathbf{R}^2, \mathbf{R}^4, \mathbf{R}^8, \mathbf{R}^{16}, \ldots$.

Now, $\mathbf{R}^{16}$ is an example of an algebra which has inverses for all nonzero elements but which contains linear equations without solutions. Zero divisors are numerous in $\mathbf{R}^{16}$. One such pair is

$$((0,1),(0,j)) * ((0,j),(0,-1)) = ((0,0),(0,0)),$$

where each of the above ordered pairs is a pair of quaternions, that is,

$$((0,1),(0,j)) =$$
$$((0+0i+0j+0k, 1+0i+0j+0k),(0+0i+0j+0k, 0+0i+1j+0k)).$$

These zero divisors lead us to

$$((0,1),(0,j))x = ((0,0),(j,0)),$$

an equation without solutions in $\mathbf{R}^{16}$.

Proving that this equation has no solution involves choosing an arbitrary solution, say $\beta$. Let $\beta = ((a,b),(c,d))$ where $a,b,c,d \in Q$. Putting $\beta$ in for $x$ in the equation simplifies it to

$$((b+jd, -a+jc),(bj-d, c+ja)) = ((0,0),(j,0)).$$

Solving for $b$ yields the equations $b+jd=0$ and $bj-d=j$. Substitution leads to $(-jd)(j)-d=j$. We now write $d = r_1 + r_2 i + r_3 j + r_4 k$ where each $r_i \in \mathbf{R}$. Simplification yields $-2ir_2 - 2kr_4 = j$. Since $i$, $j$, and $k$ are linearly independent over $\mathbf{R}$, there is no solution.

We now state one more related proposition. First, a definition is necessary.

**Definition** (Durbin [5]). *Let $R$ be a commutative ring. Then $a \in R$ is a zero divisor if there exists $b \in R$ such that $b \neq 0$ and $ab = 0$.*

**Proposition** (Hopf [6]). *Let $D$ be a finite dimensional division algebra over $F$. The following are equivalent:*
*(i) $D$ has no zero divisors;*
*(ii) no linear equation in $D$ has more than one solution; and*
*(iii) every linear equation in $D$ has a solution.*

This proposition allows us, in effect, to use the phrase "no zero divisors" to replace the requirement that each linear equation have a unique solution.

In order to completely discuss division algebras, both the associative and the nonassociative cases must be considered. To do this, it must first be realized that the requirement of multiplicative inverses is not enough to be able to solve all linear equations. In fact, we have found an infinite sequence of algebras which have multiplicative inverses for all the nonzero elements but which contain linear equations without solutions. We also mentioned three statements which are equivalent to the statement that linear equations have solutions. Among these was not, however, the requirement of multiplicative inverses alone. For students to completely understand all division algebra cases, they must be exposed to the nonassociative case. In order to do so, they must be taught that the fact that all linear equations have solutions does not always follow from simply requiring multiplicative inverses in division algebras.

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## References

1. Bott, R. and Milnor, J., "On the Parallelizability of the Spheres," *Bull. Amer. Math. Soc.* 64 (1958), 87–89.
2. Curtis, C. W., "The Four and Eight Square Problem and Division Algebras," in *Studies in Algebra,* ed. A. A. Albert, MAA Studies in Mathematics, Vol. II, Prentice Hall, Englewood Cliffs, New Jersey, 1963, 100–125.
3. Dickson, L. E., *Linear Algebras,* Cambridge Univ. Tract 16, Cambridge University Press, Cambridge, 1914.
4. Dickson, L. E., *Algebras and their Arithmetics,* Univ. of Chicago Press, Chicago, 1923.
5. Durbin, J. R., *Modern Algebra,* Wiley, New York, 1992.
6. Hopf, H., "Ein topoligischer Beitrag zur reellen Algebra," *Comment. Math. Helv.* 13 (1941), 219–239.
7. Kervaire, M., "Non-parallelizability of the $n$-sphere for $n = 7$," *Proc. Nat. Acad. Sci. USA* 44 (1958), 280–283.
8. Kleinfeld, Erwin, "A Characterization of the Cayley Numbers," in *Studies in Algebra,* ed. A. A. Albert, MAA Studies in Mathematics, Vol. II, Prentice Hall, Englewood Cliffs, New Jersey, 1963, 126–143.

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## *The Symbol "="*

"When Robert Recorde in *The Whetstone of Wit*, published in 1557, used our familiar symbol "=" for equality he proposed this symbolism in the following words: "And to avoide the tediouse repitition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of parallelles, or Gemowe lines of one lengthe, thus: =, bicause noe .2. thynges can be moare equalle."

"Despite this eloquent appeal to reason, the symbol consisting of a stylized combination of the letters ae from the word "aequalis" was in frequent use (See Descartes, *La Geometrie* for most of the 17th century."

—*The Pentagon*, Vol. 16 No. 1
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# Molecules and Their Symmetries: Determining the Hybridization of a Central Atom using Point Groups

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There are many symmetries evident in nature. Leaves are symmetrical, snowflakes are symmetrical, and crystals are symmetrical. Another very important example of symmetries in nature occurs in most molecules.

There are many uses of finding the symmetries of a given molecule in a chemical setting. Once a set of symmetries has been established, work can be done in quantum mechanics or crystallography, for example. Also, the atomic orbitals that form a given hybrid orbital of a central atom in a molecule can be determined. The goal of this paper is to establish a mathematical method for determining which atomic orbitals form the hybrid orbital of the central atom in a given molecule.

## The Method

One way of defining symmetry is as a geometrical transformation that arranges a body into an equivalent (possibly identical) configuration that is indistinguishable from the original. One goal of this paper is to show that the set of all symmetries of a molecule meets all of the requirements for forming a mathematical group.

A group is a set $G$ together with an operation $\star$ on $G$ such that each of the following axioms is satisfied:

(i) $a \star (b \star c) = (a \star b) \star c$ for all $a, b, c \in G$ (associativity);

(ii) there is an element $e \in G$ such that $a \star e = e \star a = a$ for each $a \in G$ (existence of an identity element); and

(iii) for each $a \in G$ there is an element $b \in G$ such that $a \star b = b \star a = e$ (existence of inverse elements).

In this context, we will see that $G$ is a set of symmetry operations and that the operation on the set is composition. This group is also called a symmetry group or a point group. Note that this definition can also be extended to three dimensions. This is the definition that we will use for the symmetry group of a molecule.

In order to establish a symmetry group for a specific molecule, we must first define what is meant by a symmetry element and a symmetry operation. According to Cotton [2], a symmetry element is a geometrical entity such as a line, a plane, or a point, with respect to which one or more symmetry operations may be carried out, and a symmetry operation is a movement of a body such that, after the movement has been carried out, every point of the body is coincident with an equivalent point (or perhaps the same point) of the body in its original configuration.

There are four different types of symmetry elements: a plane, a center of symmetry (center of inversion), a proper axis of rotation, and an improper axis of rotation. It is highly unlikely that a molecule will possess all four types of symmetry elements. Each of these symmetry elements yields one or more symmetry operations. A plane, for example, yields reflection in the plane; a center of symmetry or center of inversion yields inversion of all the atoms through the center of the molecule. A proper axis of rotation generates one or more rotations about the axis, and an improper axis of rotation generates at least one repetition of the following sequence: rotation followed by reflection in a plane perpendicular to the axis of rotation. There is also another trivial symmetry operation which can be applied to all molecules, and that is the identity operation. It has the effect of doing nothing at all to the molecule. The identity operation is symbolized by $E$.

The first symmetry element is a plane which yields reflection in a plane. A symmetry plane is required to pass through the molecule. The symbol $\sigma$ is used to designate a plane of symmetry and also the operation of reflecting through the plane. Carbon dioxide is an example of a molecule that has a plane of symmetry which, in figure 1, extends out from the page along the dashed line. Figure 1 shows the effect of applying $\sigma$ to carbon dioxide.

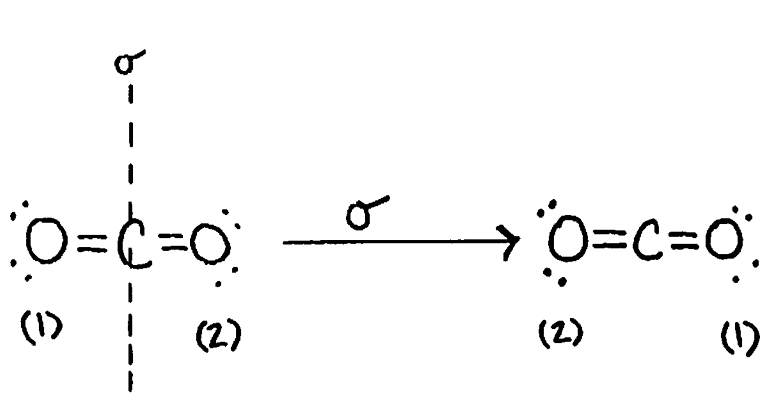


**Figure 1. Reflection in a plane.**

The second symmetry element is a center of inversion, which results in the inversion of all atoms about the center of the molecule. The symbol for the center of inversion and the operation of inverting all atoms through the center of the molecule is $i$. Figure 2 shows an example of inversion.

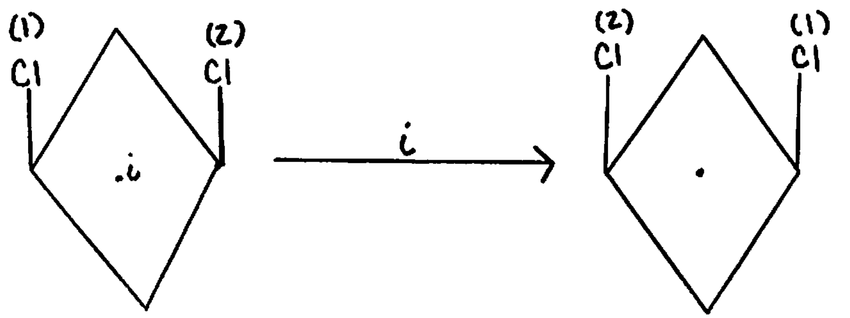


**Figure 2. Inversion.**

The third type of symmetry element mentioned was a proper axis of rotation, which generates at least one rotation about the axis. The symbol for this symmetry element is $C_n$, where $n$ is the order of the axis. Once again, $C_n$ also represents the operation of rotating a molecule about the $C_n$ axis. The order of the axis is the number of times that the smallest rotation capable of giving an equivalent configuration must be repeated in order to give a configuration not merely equivalent to the original but also identical to it. Figure 3 shows an example of rotation about a $C_3$ axis, where the $C_3$ axis cuts through the boron molecule and is perpendicular to the molecular plane.

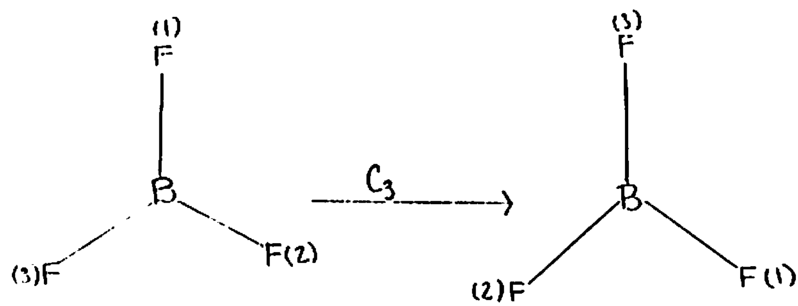


**Figure 3. Rotation about a $C_3$ axis.**

The final symmetry element is an improper axis of rotation, which generates a proper rotation about the axis and then a reflection through a plane perpendicular to the rotation axis. The symbol for an improper axis of rotation is $S_n$, where $n$ denotes the order of the axis. It also designates the corresponding symmetry operation. Figure 4 illustrates an $S_6$ operation (see [2]).

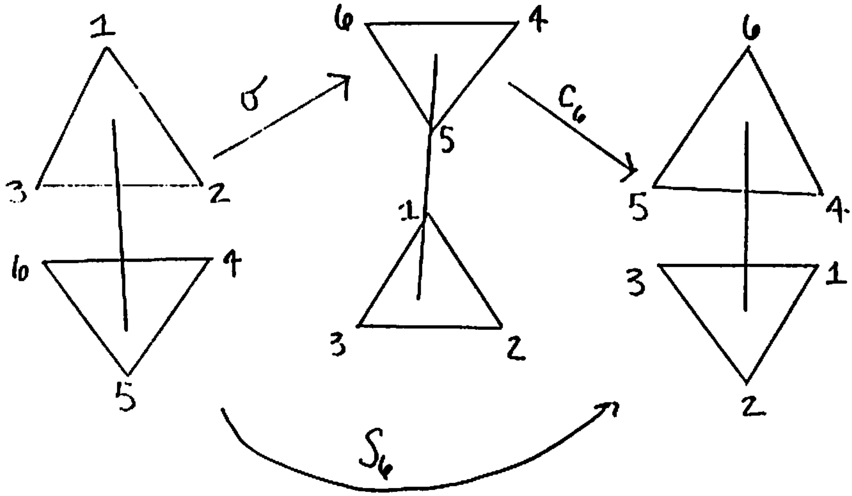


**Figure 4. An $S_6$ operation.**

It is now possible to show that the set of all symmetry operations on a given molecule forms a mathematical group. The first requirement for a mathematical group is that the set under consideration be a non-empty set. The set of symmetry operations for every molecule contains at least one element, the identity operation, $E$. Thus, the first requirement for a group has been satisfied.

We have seen that the second requirement for a mathematical group is that the product of any two elements in the set is a third element in the set. This can be verified for every molecule. As an example, we will consider the molecule $H_2O$. The complete set of all symmetry operations for $H_2O$ is $\{E, C_2, \sigma_v(xz), \sigma_v'(yz)\}$ (see [2]). It can be seen that the product of any two operations in the set yields a third operation in the set. For example, when $C_2$ is applied to $H_2O$ and is followed by $C_2$, the result is $E$ (see figure 5). Similarly, this can be verified for all other combinations of operations in the set. The second requirement for a mathematical group has been satisfied in the case of water. By a similar argument, this can be shown for all existing molecules.

The third requirement is that there be an element $E$ in the group such that for every other element $X$ in the group, $EX = XE = X$. We

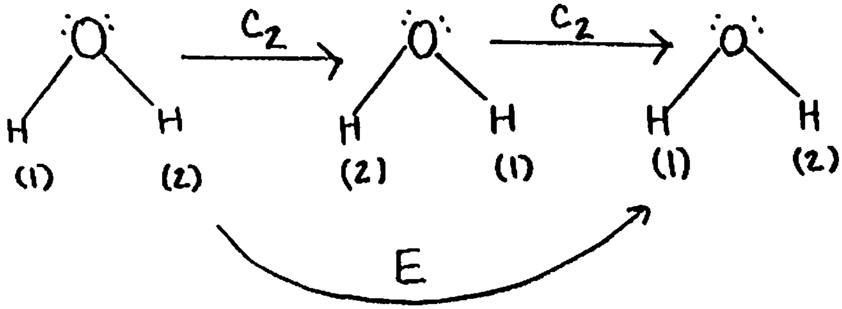


**Figure 5. Example of composition of operations on $H_2O$.**

have already shown that the identity is $E$, which is the operation of doing absolutely nothing at all to the molecule. Thus, the third requirement for a group has been satisfied.

The fourth requirement is that the associative law hold for the set of symmetry operations. According to Durbin [3], the associative law always holds whenever compositions can be formed provided that the functions are well-defined.

The fifth and final requirement is that every element in the group has an inverse that is also in the group. Since our group is a group of symmetry operations, the inverse of an operation is one that will undo what the first operation does. Mathematically, an operation $R$ has an inverse $S$ if the following is true: $RS = SR = E$. This can be easily verified. For example, if a molecule is transformed into a similar molecule through reflection in a plane, it may be transformed into the original molecule by reflecting that image in the same plane. Thus, the final requirement for a mathematical group has been satisfied. Therefore, it has been shown that the set of all symmetry operations of a molecule forms a mathematical group which is called a symmetry group, or point group.

The next step is to establish an exhaustive list of these symmetry groups of molecules so that we will then be able to determine what symmetry group, or point group, a given molecule falls into, and thus, determine what are its atomic orbitals and hybridization. In his book *Chemical Applications of Group Theory* [2], Cotton lists nineteen symmetry groups likely to be encountered in ordinary molecules. He groups them according to the symmetry elements and operations present in the molecule. The symbols for the point groups are named Schonflies symbols after their inventor.

Once it has been decided which point group a molecule belongs to, a character table can be consulted. A character table for a specific point group contains a list of all the symmetry operations in the group, the characters of the irreducible representations of the group, the representations of a group, and symmetry properties of the group, which can be used to decide which atomic orbitals (AO's) constitute a given hybrid orbital. According to [4], the numbers in a character table are simply the representations of the symmetry operations in a group. There are $n$ irreducible representations for a group consisting of $n$ classes of symmetry operations [2]. An irreducible representation is simply a representation in which it is not possible to find a symmetry operation that will reduce the matrix into two matrices of smaller dimension. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

represents an irreducible representation for the identity operation, $E$, in a given group.

An $n \times n$ matrix is then constructed where $n$ represents the number of bonds between the atoms in the molecule. The matrix is obtained by placing the number 1 in the $a_{jj}$ position of the matrix if that axis is unchanged by the symmetry operation for axis $j$ and placing the number 0 in that position if that axis is changed by the symmetry operation. A "1" is placed in the $a_{ij}$ position if the $i$th axis moves into the $j$th position. The rest of the matrix is then filled in with zeros.

The character or trace of any symmetry operation in a group is simply the number of axes (or atoms) in a molecule left unshifted by the geometric transformation. When represented in a matrix, the character is the sum of the diagonal elements in a square matrix (this only applies to calculations involving the characters of the symmetry operations, not to calculating the characters of the irreducible representations, which are obtained from a character table). Thus, for a square matrix $A$,

$$\chi_A = \sum_j a_{jj}.$$

We now consider the following example.

**Example: Boron Trifluoride**

The formula for boron trifluoride is $BF_3$. It exists in nature as a trigonal planar molecule, which can be represented as in figure 6.

We see that $BF_3$ has one $C_3$ axis, three $C_2$ axes perpendicular to the $C_3$ axis, and one $S_3$ axis. Also, $BF_3$ has a horizontal plane of reflection and three vertical planes of reflection. Therefore, $BF_3$ must belong to the symmetry group $D_{3h}$ according to the chemists' notation or $D_6$ according to the mathematicians' notation. This is a dihedral group of order twelve which is non-Abelian. According to the $D_{3h}$ character table, boron trifluoride has twelve symmetry operations, namely $E$, two $C_3$'s, three $C_2$'s, $\sigma_h$, two $S_3$'s, and three $\sigma_v$'s.

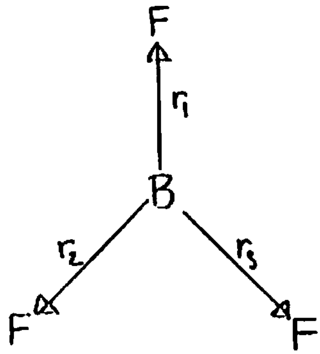


**Figure 6. Boron triflouride.**

Now, the characters of an overall representation must be determined by examining one symmetry operation from each class. We have

$$E = \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix},$$

with $\chi = 3$;

$$C_3 = \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix},$$

with $\chi = 0$;

$$C_2 = \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix},$$

with $\chi = 1$;

$$\sigma_h = \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix},$$

with $\chi = 3$;

$$S_3 = \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix},$$

with $\chi = 0$; and

$$\sigma_v = \begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix},$$

with $\chi = 1$. Therefore, the characters of the overall reducible representation are

$$\begin{array}{cccccc} E & 2C_3 & 3C_2 & \sigma_h & 2S_3 & 3\sigma_v \\ 3 & 0 & 1 & 3 & 0 & 1 \end{array}.$$

The formula to use for finding the number of each irreducible representation needed in the irreducible representation of the symmetry group $D_{3h}$ is given to be

$$\text{irreducible representations needed } = (1/n)\sum \chi_R \chi_I N,$$

where $n$ is the number of symmetry operations in the point group, $\chi_R$ is the character of the reducible representation, $\chi_I$ is the character of the irreducible representation, and $N$ is the number of symmetry operations in the class (c.f. [4]). Recall that this formula is only used to determine the character of a reducible representation. A partial character table for $D_{3h}$ is as shown in table 1.

| $D_{3h}$ | E | $2C_3$ | $3C_2$ | $1\sigma_h$ | $2S_3$ | $3\sigma_v$ |
|---|---|---|---|---|---|---|
| $A_1'$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $A_2'$ | 1 | 1 | -1 | 1 | 1 | -1 |
| E' | 2 | -1 | 0 | 2 | -1 | 0 |
| $A_1''$ | 1 | 1 | 1 | -1 | -1 | -1 |
| $A_2''$ | 1 | 1 | -1 | -1 | -1 | 1 |
| E" | 2 | -1 | 0 | -2 | 1 | 0 |

**Table 1. Partial character table for $D_{3h}$.**

Using the formula of the previous paragraph, an irreducible representation for $D_{3h}$ can be found. We obtain

$$\begin{aligned}
(1/12)[3\cdot1\cdot1+0\cdot1\cdot2+1\cdot1\cdot3+3\cdot1\cdot1+0\cdot1\cdot2+1\cdot1\cdot3] &= 1A_1' \\
(1/12)[3\cdot1\cdot1+0\cdot1\cdot2+1\cdot-1\cdot3+3\cdot1\cdot1+0\cdot1\cdot2+1\cdot-1\cdot3] &= 0A_2' \\
(1/12)[3\cdot2\cdot1+0\cdot-1\cdot2+1\cdot0\cdot3+3\cdot2\cdot1+0\cdot-1\cdot2+1\cdot0\cdot3] &= 1E' \\
(1/12)[3\cdot1\cdot1+0\cdot1\cdot2+1\cdot1\cdot3+3\cdot-1\cdot1+0\cdot-1\cdot2+1\cdot-1\cdot3] &= 0A_1'' \\
(1/12)[3\cdot1\cdot1+0\cdot1\cdot2+1\cdot-1\cdot3+3\cdot-1\cdot1+0\cdot-1\cdot2+1\cdot1\cdot3] &= 0A_2'' \\
(1/12)[3\cdot2\cdot1+0\cdot-1\cdot2+1\cdot0\cdot3+3\cdot-2\cdot1+0\cdot1\cdot2+1\cdot0\cdot3] &= 0E''.
\end{aligned}$$

Therefore, this reduces to $A_1' + E'$. This means that the atomic orbitals must have the same symmetry properties as $A_1'$ and $E'$.

| | | |
|---|---|---|
| $A_1'$ | | $x^2+y^2$; $z^2$ |
| E' | (x,y) | $(x^2-y^2,xy)$ |

**Table 2**

Consulting a character table for $D_{3h}$ again, we can determine which atomic orbitals contribute to the hybrid orbital. A partial character table for $D_{3h}$ is as given in Table 2.

The above information tells us that one orbital must be perfectly symmetrical with respect to the $x$ and $y$ axes and also with respect to the $z$ axis. Therefore, it must be an $s$ orbital or a $d_{z^2}$ orbital. Also, there must be two orbitals that collectively have the same symmetry properties as $E'$. According to the character table, these orbitals could be a $p_x$ orbital and a $p_y$ orbital, or they could be a $d_{x^2-y^2}$ orbital and a $d_{xy}$ orbital. Therefore, $BF_3$ must be $sp^2$, $sd^2$, $d^3$, or $p^2d$ hybridized. Using knowledge of chemical bonding energies, we know that the boron atom must be $sp^2$ hybridized because that hybridization is the most stable hybridization. Therefore the hybrid orbital is made from a $2s$ orbital, a $2p_x$ orbital, and a $2p_y$ orbital. According to [1], $BF_3$ is indeed an $sp^2$ hybridized molecule.

*Acknowledgements.* I would like to thank the following people for their gracious assistance in helping me to complete my paper: Dr. Kate Doan, for suggesting that I study the hybridization of atoms; Dr. Timothy Hardy, for (locally) refereeing my paper; and Prof. John Cross. Your help is greatly appreciated.

## References

1. Bodner, George M. and Pardue, Harry L., *Chemistry: An Experimental Science*, John Wiley and Sons, New York, 1995.
2. Cotton, F. Albert, *Chemical Applications of Group Theory*, Third ed., John Wiley and Sons, New York, 1990.
3. Durbin, John R., *Modern Algebra: An Introduction*, Third ed., John Wiley and Sons, New York, 1992.
4. Vincent, Alan, *Molecular Symmetry and Group Theory: A Programmed Introduction to Chemical Applications*, John Wiley and Sons, London, 1977.

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"As a boy I knew the logarithms of thirty or fifty numbers."

—Napoleon

# The Hidden Profession

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Presented at the 1995 National Convention

On any given campus, if a random group of students were approached and asked if they knew what actuarial science was, some might groan and say something about too much math, but most would think you just created a new major. Yet, if a group of students in the math department were approached and asked the same question, you might hear some of the same groans as before, but the majority would have some idea about the subject in mind. Most believe actuarial science to be an overbearing amount of number crunching in the insurance industry. That's to be expected. Even *Webster's New World* defines an actuary as "a person whose work is to calculate statistically risks, premiums, life expectancies, and so forth for insurance." However, actuarial science is branching out, and the once hidden profession is becoming known.

The study of actuarial science evolved in England in 1792 during the primitive beginnings of life insurance. At that time, much was to be discovered about the intricacies of insurance, and mathematicians led the way. Upon the onset of the twentieth century, more technical problems involving insurance emerged in the United States, thus stressing the need for some control and regulation. Meeting this demand, the Society of Actuaries and the Casualty Actuarial Society formed. Presently, these two organizations head the series of exams administered to aspiring actuaries.

Until recently, the exam process eventually led an actuary candidate down one of two avenues, the first being the study of life insurance offered by the Society of Actuaries, and the second the study of casualty insurance offered by the Casualty Actuarial Society. Each of the two choices consisted of a series of ten examinations, with the first four jointly given by both organizations, and the final six given separately by each organization. The exams are worth a set amount of credits. An associateship may be obtained after completing 200 credits, and a fellowship after 350. However, due to the changing needs of today's world, the Society of Actuaries is in the process of relinquishing its set limitations of life insurance and expanding its horizon (Society of Actuaries [4]).

Once limited to life insurance, the Society of Actuaries is now offering an alternative exam series. As of July 31, 1995, the proposed series of exams available to actuaries will be finance/investment, group and health benefits, individual life and annuity, and pension plans. However, along with these modifications comes new requirements for a candidate's attainment of both associateship and fellowship status. Associateship status cannot be achieved until 300 credits have been successfully completed. Upon completing those credits, 90 credits from the candidate's specialty field, 60 elective credits, and attendance of the Fellowship Admissions Course are required before fellowship status may be earned (Society of Actuaries [4]). The new changes to the examination process are in hopes of building a stronger foundation and a broader knowledge base to better exercise the diverse skills of an actuary.

To further illustrate the assorted nature of problems of which an actuary is capable beyond the insurance industry, I have chosen two examples to demonstrate. In the first example, the operations research technique of building a linear programming model is useful in finding a solution. In the second, financial theory of interest is utilized.

## EXAMPLE 1

This example has been condensed and simplified from its original form found in the *Introduction to Operations Research* text ([1], pp. 43–46) to serve the purpose of this paper. Realistically, more variables would be needed.

Mary is a modern day success story. She is highly regarded in her career, active in her community, and she has an adoring husband and children. However, tragedy has fallen upon Mary. She has just been diagnosed with a malignant tumor in the abdominal region. Radiation therapy is her best possible chance for survival. This treatment involves using two external beams which will pass ionizing radiation through the patient's body, damaging both cancerous and healthy tissues. Due to the problems of attenuation and scatter, the procedure is quite complex. The goal is to select the combination of beams to be used, and the intensity of each one, in order to generate the best possible dose distribution.

Once the treatment design has been developed, it is then administered in may installments spread over several weeks. After a thorough anatomical analysis of the intensity of each beam, a medical team came up with the following dosage requirements. If the dose level at the entry point for beam 1 is 1 kilorad, then an average of .4 kilorad will be absorbed by the entire healthy anatomy in the two dimensional plane, an average of .3 kilorad will be absorbed by the nearby critical tissues, an average of .5 kilorad will be absorbed by the various parts of the tumor, and .6 kilorad will be absorbed by the center of the tumor. If the dose entry level for beam 2 is 1 kilorad, the healthy anatomy, critical tissues, tumor region, and center of the tumor will absorb .5, .1, .5, and .4 kilorads, respectively. Also, concerning the total dosage from combined beams 1 and 2, that for the healthy anatomy must be as small as possible, critical tissues must not exceed 2.7 kilorads, the average of the tumor region must equal 6 kilorads, and the center of the tumor must be at least 6 kilorads.

Simultaneously satisfying all of these requirements is a difficult task. However, it is possible with a linear programming model. First, it is necessary to review the goal in mind and set up a table. We are given the area affected by the beam, the fraction of each beam absorbed, and restrictions on the total average dosage. Therefore, the following table can be produced:

| Area | Beam 1 | Beam 2 | Restrictions/dosage |
|---|---|---|---|
| healthy anatomy | .4 | .5 | minimize |
| critical tissue | .3 | .1 | $\leq 2.7$ |
| tumor region | .5 | .5 | $= 6$ |
| center of tumor | .6 | .4 | $\geq 6$ |

From the table, we can form a linear programming model. Let $x_1$ and $x_2$ be the two decision variables representing the entry points for beam 1 and beam 2. Because the total dosage reaching the healthy anatomy is to be minimized, let $Z$ represent this quantity. We want to minimize

$$Z = .4x_1 + .5x_2$$

subject to

$$\begin{aligned} .3x_1 + .1x_2 &\leq 2.7 \\ .5x_1 + .5x_2 &= 6 \\ .6x_1 + .4x_2 &\geq 6 \\ x_1 &> 0 \\ x_2 &> 0. \end{aligned}$$

Solving graphically (see figure 1), the feasible region consists of points on the line segment from (6,6) to (7.5,4.5). Because all of these points simultaneously satisfy all of the constraints, we must find the minimum value to arrive at the optimal solution. Filling in the points for $x_1$ and $x_2$ in the original $Z$ equation, we have

$$(.4 * 7.5) + (.5 * 4.5) = 5.25$$

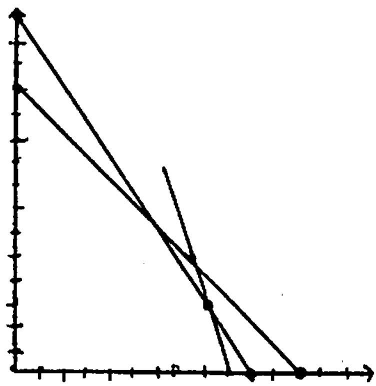

**Figure 1**

and

$$(.4 * 6) + (.5 * 6) = 5.4.$$

Thus, the point $(7.5, 4.5)$ yields the minimum value. Seven and one half kilorads of beam 1 and 4.5 kilorads of beam 2 must be administered. This example clearly shows other paths an actuary is capable of taking.

Example two will demonstrate the difference between simple and compound interest rates using the financial theory of interest.

EXAMPLE 2

It is known that \$600 invested for 2 years will earn \$264 in interest. What is the accumulated value of \$2000 invested at the same rate of compound interest for three years? We have

$$\begin{aligned} 600(1+i)^2 &= 864 \\ (1+i)^2 &= 1.44 \\ 1+i &= 1.2 \\ i &= 1.2 - 1 \\ i &= .2. \end{aligned}$$

Then

$$2000(1+.20)^3 = 3456.$$

However, if we took that same \$2000 and invested it at 20% simple interest for three years, our investment would yield

$$2000(1+.2(3)) = 3200.$$

Therefore, the compounded interest yields a larger return. This knowledge is not only beneficial for companies, but for personal investments and loans as well.

These examples have shown that the knowledge gained through the study of actuarial science truly is not a process of relentless number crunching in insurance. An actuary's knowledge and understanding of the many facets of mathematics and related areas could benefit many businesses in their future economic endeavors. However, the "hidden" secret of an actuary's knowledge must first be revealed.

In an effort to reveal this secret, in 1993 the Society of Actuaries organized an "Ask an Actuary" campaign to encourage interest in the actuary profession. The Society passed out 26,000 buttons with the inscription "Ask an Actuary" to launch the campaign. In doing so, the Society hoped to uplift and motivate morale, show pride in the profession, and increase awareness of actuarial science. A glimpse of these buttons was seen on the David Letterman Show and through the halls of the U.S. Congress. Former Society of Actuaries President Walter Rugland remarked, "I see the day in the twenty-first century that whenever a business or government decision maker has a question about risk, her or his first thought will be 'Ask an Actuary.'" His goal for the twenty-first century is becoming a reality more and more each day due to the constant changes and flexibility of the actuarial profession ([3]).

This once "hidden" secret is no longer standing in the shadows of the insurance industry. It is now making its mark in the business world, and the possibilities are endless. From being listed as the number one career by the *Jobs Rated Almanac* [2] to building mathematical models, actuarial science has no bounds.

References

1. Hillier, Frederick and Lieberman, Gerald, *Introduction to Operations Research,* McGraw-Hill, New York, 1990.
2. Krantz, Les, *The Jobs Rated Almanac,* World Almanac, New York, 1988.
3. Society of Actuaries, "Ask An Actuary," *The Actuary* (December 1993), 8–9.
4. Society of Actuaries and Casualty Actuarial Society, *Associateship and Fellowship Catalog,* Spring 1995.

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"Two sons of Carl Friederick Gauss came to America and settled in Missouri. Eugene lived on a farm a short distance from Columbia where he fied in 1896 and William lived in St. Louis where he died in 1879."

—*The Pentagon,* Vol. 9 No. 1 (Fall 1949)

# The Problem Corner

Edited by Kenneth M. Wilke

*The Problem Corner* invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before January 1, 1998. Solutions received after the publication deadline will be considered also until the time when copy is prepared for publication. The solutions will be published in the Spring 1998 issue of *The Pentagon*, with credit being given to student solutions. Affirmation of student status and school should be included with solutions. Address all communications to Kenneth M. Wilke, Department of Mathematics, 275 Morgan Hall, Washburn University, Topeka, Kansas 66621 (e-mail: xxwilke@acc.wuacc.edu).

PROBLEMS 504 (corrected) and 505-509

*Problem 504 (corrected).* Proposed by Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin.

If $A$, $B$, and $C$ are the angles of a triangle, prove that

$$2\cos A\cos B\cos C = 1 - \cos^2 A - \cos^2 B - \cos^2 C.$$

*Problem 505.* Proposed by J. Sriskandarajah, University of Wisconsin Center—Richland, Richland Center, Wisconsin.

If $a + b + c = abc$, prove that

$$\frac{2a}{1-a^2} + \frac{2b}{1-b^2} + \frac{2c}{1-c^2} = \frac{8abc}{(1-a^2)(1-b^2)(1-c^2)}.$$

*Problem 506.* Proposed by Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin.

Find all of the positive integer values of $n$ for which the expression $4n^2 + 21n$ is a perfect square.

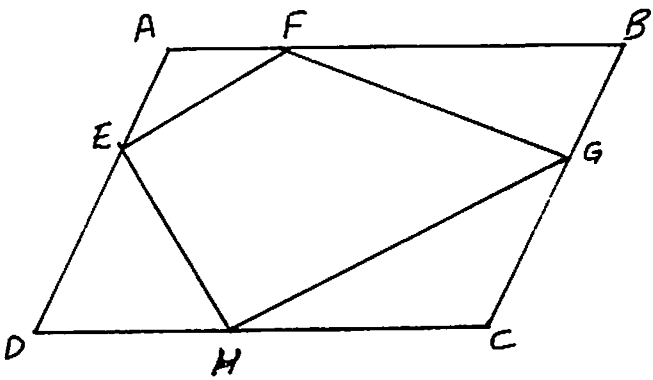
*Problem 507.* Proposed by Kenichiro Kashihara, Sagamihara, Kanagawa, Japan.

Given any integer $n \geq 1$, the value of the pseudo-Smarandache function $Z(n)$ is the smallest integer $m$ such that $n$ evenly divides $\sum_{k=1}^{m} k$. Let $p$ be a positive prime and $s$ be an integer $\geq 2$. Show that

$$Z(p^s) = \begin{cases} p^{s+1} - 1 \text{ if } p \text{ is even} \\ p^s - 1 \text{ if } p \text{ is odd.} \end{cases}$$

*Problem 508.* Proposed by the editor.

Let $ABCD$ be a parallelogram. Let $EFGH$ be a quadrilateral inscribed in parallelogram $ABCD$ such that the area of $EFGH$ is exactly half the area of parallellogram $ABCD$. Show that at least one diagonal of $EFGH$ is parallel to a side of $ABCD$ (see figure below).




*Problem 509.* Proposed by Kenichiro Kashihara, Sagamihara, Kanagawa, Japan.

Given any integer $n \geq 1$, the value of the pseudo-Smarandache function $Z(n)$ is the smallest integer $m$ such that $n$ evenly divides $\sum_{k=1}^{m} k$.

(a) Solve the diophantine equation $Z(x) = 8$.

(b) Show that for any positive integer $p$ the equation $Z(x) = p$ has solutions.

(c*) Show that the equation $Z(x) = Z(x+1)$ has no solutions.

(d*) Show that for any given positive number $r$ there exists an integer $s$ such that $|Z(s) - Z(s+1)| > r$.

*Editor's comment.* Parts (c) and (d) of problem 509 were submitted without solution. Partial solutions are welcome.

*Please help your editor by submitting problem proposals.*

SOLUTIONS 482, 495–499

*Problem 482 (corrected).* Proposed by Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin.

Evaluate

$$I = \int_0^{\pi} \ln(\sin x)\, dx.$$

*Solution* by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

First consider $f(x) = \ln(\sin x)$ on the interval $(0, \pi)$. Since $f(\pi - x) = f(x)$, the graph of $y = f(x)$ is symmetric about the line $x = \pi/2$. Thus if $I$ exists, then

$$I = 2 \cdot \int_0^{\pi/2} \ln(\sin x)\, dx. \tag{1}$$

To show that this integral converges, it suffices to show that

$$\lim_{x \to 0+} [\sqrt{x} \ln(\sin x)]$$

is a finite number. Rewriting this expression and using L'Hospital's rule gives

$$\begin{aligned}
\lim_{x \to 0+} \left(\sqrt{x} \ln(\sin x)\right) &= \lim_{x \to 0+} \frac{\ln(\sin x)}{x^{-1/2}} \\
&= \lim_{x \to 0+} \frac{(\cos x)/(\sin x)}{(-1/2)x^{-3/2}} \\
&= -2 \lim_{x \to 0+} \frac{x^{3/2} \cos x}{\sin x} \\
&= -2 \lim_{x \to 0+} \frac{-x^{3/2} \sin x + (3/2)x^{1/2} \cos x}{\cos x} \\
&= 0 < \infty.
\end{aligned}$$

Thus, the integral converges.

Using $y$ as the variable of integration in (1) and making the substitution $x = (\pi/2) - y$ gives

$$I = 2 \cdot \int_0^{\pi/2} \ln(\cos x)\, dx. \tag{2}$$

Adding equations (1) and (2) yields

$$2I = 2 \cdot \int_0^{\pi/2} (\ln(\sin x) + \ln(\cos x))\, dx,$$

and thus

$$\begin{aligned} I &= \int_0^{\pi/2} \ln(\sin x \cos x)\, dx \\ &= \int_0^{\pi/2} \ln((\sin 2x)/2)\, dx \\ &= \int_0^{\pi/2} \ln(\sin 2x) - \ln 2\, dx \\ &= -(\pi/2)\ln 2 + \int_0^{\pi/2} \ln(\sin 2x)\, dx. \end{aligned} \tag{3}$$

In the integral in (3), let $\Theta = 2x$. Then

$$\int_0^{\pi/2} \ln(\sin 2x)\, dx = \frac{1}{2}\int_0^{\pi} \ln(\sin\Theta)\, d\Theta = \frac{1}{2}I.$$

Then from (3),

$$I = \frac{1}{2}I - (\pi/2)\ln 2,$$

and $I = -\pi \ln 2$.

*Also solved by:* the proposer.

*Problem 495.* Proposed jointly by Sammy and Jimmy Yu, students at the University of South Dakota, Vermillion, South Dakota.

Evaluate

$$\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{b-c}{2}\right)\cos\left(\frac{c-a}{2}\right)$$

if $\sin a + \sin b + \sin c = \cos a + \cos b + \cos c = 0$.

*Solution* by the proposers.

Let $w_1 = \cos a + i\sin a$, $w_2 = \cos b + i\sin b$ and $w_3 = \cos c + i\sin c$, respectively. Then $w_1 + w_2 + w_3 = 0$ and $w_1w_2w_3 = \cos(a+b+c) + i\sin(a+b+c)$, which does not equal zero. We use the following relations repeatedly: if $w_x = \cos r + i\sin r$ and $w_y = \cos s + i\sin s$, then $w_xw_y = \cos(r+s) + i\sin(r+s)$; $\cos r + \cos s = 2\cos((r+s)/2))\cos((r-s)/2)$; and $\sin r + \sin s = 2\sin((r+s)/2))\cos((r-s)/2)$. Now,

$$w_1w_2w_3 = [-(w_2+w_3)][-(w_1+w_3)][-(w_1+w_2)]$$

$$
\begin{aligned}
&= -(w_1 + w_2)(w_1 + w_3)(w_2 + w_3) \\
&= -\left((\cos a + \cos b) + i(\sin a + \sin b)\right) \cdot \left((\cos b + \cos c)\right. \\
&\qquad \left. + i(\sin b + \sin c)\right) \cdot \left((\cos c + \cos a) + i(\sin c + \sin a)\right) \\
&= (-2\cos[(a-b)/2])\,\{\cos[(a+b)/2] + i\sin[(a+b)/2]\} \\
&\qquad \cdot (-2\cos[(b-c)/2])\,\{\cos[(b+c)/2] + i\sin[(b+c)/2]\} \\
&\qquad \cdot (-2\cos[(c-a)/2])\,\{\cos[(c+a)/2] + i\sin[(c+a)/2]\} \\
&= -8\,\{\cos[(a-b)/2]\cos[(b-c)/2]\cos[(c-a)/2]\} \\
&\qquad \cdot \{\cos(a+b+c) + i\sin(a+b+c)\} \\
&= -8\,\{\cos[(a-b)/2]\cos[(b-c)/2]\cos[(c-a)/2]\}\, w_1 w_2 w_3.
\end{aligned}
$$

Hence,

$$\cos[(a-b)/2]\cos[(b-c)/2]\cos[(c-a)/2] = -1/8.$$

*Also solved by:* Clayton W. Dodge, University of Maine, Orono, Maine and Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin. One incorrect solution was received.

*Editor's comment.* Both Prielipp and Dodge used straightforward trigonometric identities in their lengthy solutions. Prielipp also showed that if $A$ is any arbitrary angle measured in degrees, then $\sin A + \sin(A + 120°) + \sin(A + 240°) = \cos A + \cos(A + 120°) + \cos(A + 240°) = 0$.

*Problem 496.* Proposed by Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin.

Find the smallest positive integer that can be increased by 50% by moving the digit on the extreme right to the extreme left.

*Solution* by Bryan Crissinger, Messiah College, Grantham, Pennsylvania.

Let $a_n a_{n-1} a_{n-2} \ldots a_0$ be the desired number where each $a_i$ is a digit and $a_0$ is not zero. Then $a_0 a_n a_{n-1} \ldots a_1$ is the number formed after shifting the rightmost digit to the extreme left. Then the problem requires that

$$10^n a_0 + 10^{n-1} a_n + \cdots + 10^0 a_1 = (3/2)\left(10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10^0 a_0\right)$$

which simplifies to

$$(2 \cdot 10^n - 3)\, a_0 = 28 \sum_{i=1}^{n} 10^{i-1} a_i.$$

Since 4 divides $a_0$, we must have either $a_0 = 4$ or $a_0 = 8$. Furthermore, 7 divides $(2 \cdot 10^n - 3)$. The smallest positive integer for which this occurs is $n = 5$. Now if $a_0 = 4$, then

$$28571 = \sum_{i=1}^{n} 10^{i-1} a_i$$

and the desired number is 285714, since $(3/2) * (285714) = 428571$. Proceeding similarly yields the solution 571428, which corresponds to $a_0 = 8$ and which is the next largest number having the desired property.

*Also solved by:* Charles Ashbacher, Cedar Rapids, Iowa; Clayton W. Dodge, University of Maine, Orono, Maine; Danny L. Stansbury Jr., University of Southern Mississippi, Ocean Springs, Mississippi; Aran D. Stubbs, Colorado Springs, Colorado and the proposer. One incorrect solution was received.

*Problem 497.* Proposed by Charles Ashbacher, Cedar Rapids, Iowa.

The Smarandache function $S(n)$ is defined in the following way: $S(n) = m$ is the smallest integer such that $n$ evenly divides $m!$. The Euler phi function $\phi(n)$ is defined by letting $\phi(n)$ be the number of positive integers less than or equal to $n$ that are relatively prime to $n$. Prove the following:

(a) The equation $S(\phi(n)) = n$ has no solution.

(b) The equation $n - S(\phi(n)) = 1$ has an infinite number of solutions.

*Solution* by Clayton Dodge, University of Maine, Orono, Maine.

Part (a). Since $\phi(n)$ is the number of integers less than or equal to $n$ that are relatively prime to it, and since $n$ is relatively prime to itself only for $n = 1$, then $\phi(n) < n$ except that $\phi(1) = 1$. Furthermore, $\phi(p) = p - 1$ when $p$ is prime and $\phi(n) < n - 1$ for composite $n$. Similarly, $S(n) = n$ for $n = 4$ or when n is prime; otherwise $S(n) \leq 2n/3$ by problem 486. In any case $S(n) \leq n$. Thus $S(\phi(1)) = S(1) = 0$ since $0! = 1$. If $p$ is prime, then $S(\phi(p)) = S(p - 1) \leq p - 1$. Finally, if $n$ is composite, then $S(\phi(n)) \leq \phi(n) < n - 1$.

Part (b). We solve the equation $S(\phi(n)) = n - 1$. Clearly $n = 1, 3$ and 5 are solutions since $S(\phi(3)) = S(2) = 2$ and $S(\phi(5)) = S(4) = 4$. The last line of the proof in part (a) shows that $n$ cannot be composite. Finally, if $n$ is a prime larger than 5, then $\phi(n) = n - 1$ is a composite number larger than 4, so $S(n - 1) \leq 2(n - 1)/3 < n - 1$. Thus there are not infinitely many solutions, but only the three solutions listed above.

*Solution* by Alex Shaumyan, Eastern Kentucky University, Richmond, Kentucky (revised by the editor).

Part (a). Consider $\phi(n)$. If $n = 1$, then $\phi(1) = 1$, and since $0! = 1$, $S(\phi(1)) = S(1) = 0$. If $n > 1$, then $\phi(n) \leq n - 1$. So $S(\phi(n)) \leq n - 1$ and the equation $S(\phi(n)) = n$ has no solution.

Part (b). The given equation is equivalent to

$$S(\phi(n)) = n - 1. \tag{1}$$

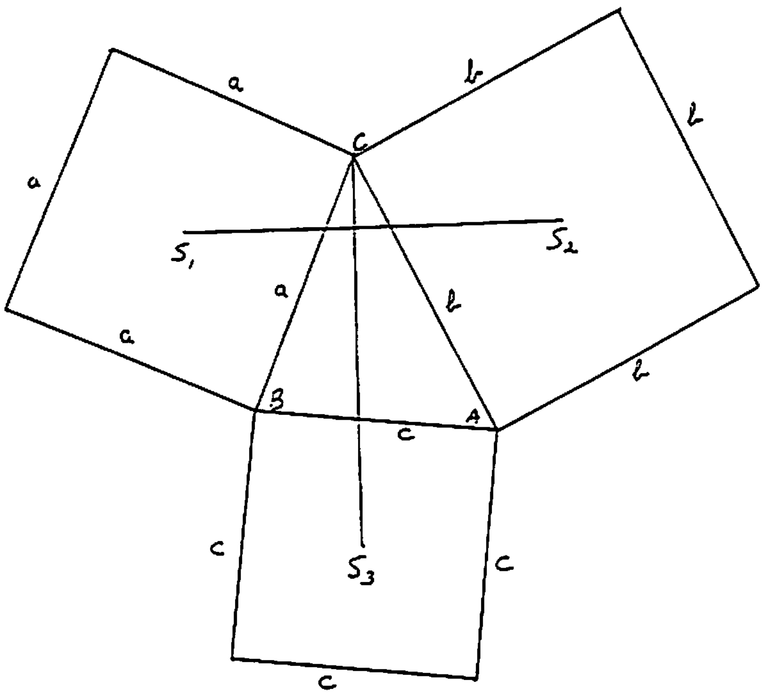
Equation (1) holds for $n = 1$ since $S(\phi(1)) = S(1) = 0$. For $n > 1$, $S(2) = 2$ and $S(4) = 4$ and the equation $S(\phi(n)) = n - 1$ implies that $\phi(n) = n - 1$.

Therefore $n$ must be prime if $n > 1$ since for composite $n$, $\phi(n) \leq n - 2$. There are only two primes which satisfy equation (1). These are 3 and 5 since $S(\phi(3)) = S(2) = 2$ and $S(\phi(5)) = S(4) = 4$. For any prime $n > 5$, consider $((n-1)/2)!$, which contains the two disinct integers 2 and $(n-1)/2$ in its product. Then $(n-1)$ is a divisor of $((n-1)/2)!$ for every prime $n > 5$. Thus, for each prime $n > 5$, $S(\phi(n)) = S(n-1) \leq (n-1)/2$, which does not equal $n-1$. Hence the only solutions for part (b) are $n = 1$, 3 and 5.

*Also solved by:* the proposer (part (a) only).

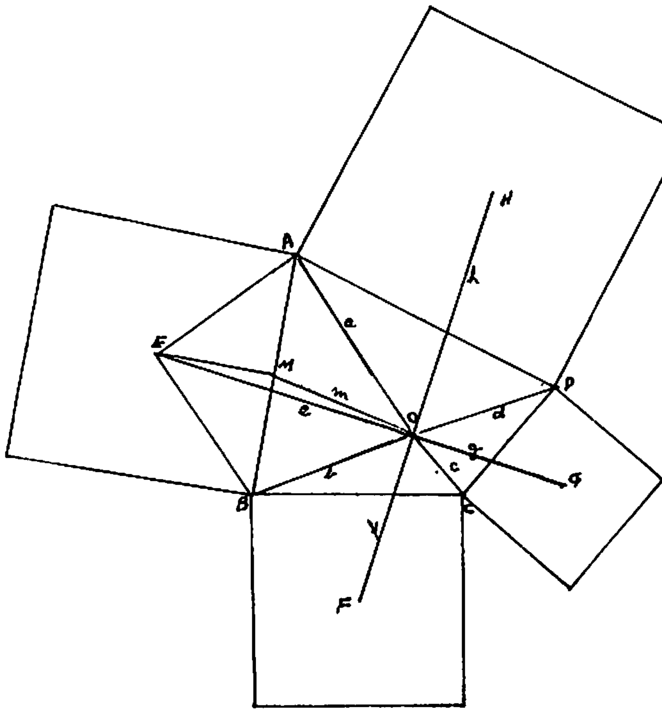
*Problem 498.* Proposed by Oscar R. Casteneda, Southwest High School, San Antonio, Texas.

Let $ABC$ be an arbitrary triangle with sides of lengths $a$, $b$ and $c$. Contruct squares facing outward on each of the sides of the triangle. Prove that the length of the line segment $S_1S_2$ connecting the centers of two adjacent squares equals the length of the line segment $CS_3$ connecting the center of the third square with the common point of the other two squares. Also prove that these two line segments are perpendicular.

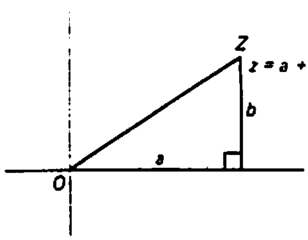



*Solution* by Clayton W. Dodge, University of Maine, Orono, Maine.

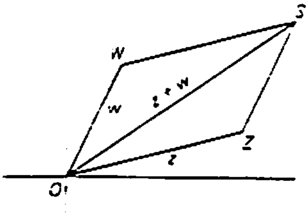
This result is a special case of Van Aubel's theorem: If squares are erected all outwardly or all inwardly on the sides of a quadrilateral, then the centers of the squares are the vertices of a quadrilateral whose diagonals are perpendicular and equal in length. See the figure below.



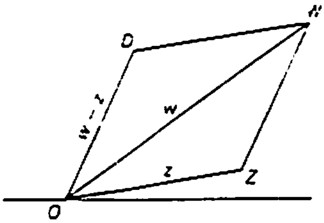
This proof is based upon complex numbers. Let the vertices of the quadrilateral be $A$, $B$, $C$, and $D$ in counterclockwise order, and let $E$, $F$, $G$, and $H$ be the centers of the squares erected outwardly on sides $AB$, $BC$, $CD$ and $DA$, respectively. We let the corresponding lowercase letters represent the complex affixes of the upper case points. By [1], if $z = a + bi$ corresponding to the point $Z$ in the complex plane, the point $Z$ is the image of the complex number $z = a + bi$ and $z$ is the affix of the point $Z$ (see figure below).



Geometrically, $z+w$ is the affix of the fourth vertex of $S$, the parallelogram having affixes $z$, 0, and $w$ for three consecutive vertices (see figure below).




To construct $w-z$, take $w-z$ as the affix of the fourth vertex $D$ of the parallelogram $OZWD$ (see figure below).




For ease in visualization, let the origin be located at the intersection $O$ of the lines $EG$ and $FH$. If $M$ is the midpoint of side $AB$, then $m=(a+b)/2$ and $e-m=i(a-m)$, since a counterclockwise rotation about point $M$ maps segment $MA$ to $ME$. Letting $\omega=(1+i)/2$, we obtain $e=(a/2)(1+i)+(b/2)(1-i)=a\omega+b(1-\omega)$. Similarly, $f=b\omega+c(1-\omega)$, $g=c\omega+d(1-\omega)$ and $h=a(1-\omega)$. It is easy to verify that $i\omega=\omega-1$ and $i\omega-i=-\omega$. Then $i(g-e)=i\omega c-i\omega a+(i-i\omega)d-(i-i\omega)b=\omega c-c-\omega a+a+\omega d-\omega b=h-f$, so $FH$ and $EG$ are perpendicular and equal in length. For the case where the squares are erected inwardly, label the vertices of the quadrilateral $A$, $B$, $C$, $D$ in clockwise order. Then the above proof holds. The desired result follows by letting vertices $C$ and $D$ coincide so that the square on $CD$ reduces to the single point $C$.

[1] Dodge, Clayton W., *Euclidean Geometry and Transformations*, Addison Wesley, 1972, pp. 152, 156.

*Also solved by:* J. Sriskandarajah, University of Wisconsin Center—Richland, Richland Center, Wisconsin (part (b) only) and the proposer.

*Editor's comment.* Fred A. Miller, Elkins, West Virginia found the stated problem as Theorem 4.81 on pp. 96–97 of *Geometry Revisited* by H. S. M. Coxeter and S. L. Greitzer. Clayton Dodge also provided the following reference to a nice proof of Van Aubel's theorem by isometries in the article by R. L. Finney entitled "Dynamic Proofs of Euclidean Theorems" in *Mathematics Magazine* 43 (1970), pp. 177–185. For other references to Van Aubel's theorem, see the solution to *Pentagon* problem 308 (Vol. 39 No. 1, Fall 1979, pp. 34–35).

*Problem 499.* Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n = 0, 1, 2, \ldots$. Evaluate the following expression for all integers $n \geq 1$:

$$F_{2n+1}F_{2n-1} - {F_{2n}}^2 - {F_{n+1}}^2{F_{n-1}}^2 + 2F_{n+1}{F_n}^2F_{n-1} - {F_n}^4.$$

*Solution* by Scott H. Brown, Auburn University, Montgomery, Alabama.

By [1], we have the well-known identity for Fibonacci numbers

$$F_{k-1}F_{k+1} - F_k^2 = (-1)^k, \tag{1}$$

where $k$ is any integer $\geq 1$. Let $k = 2n$. Then by (1), $F_{2n-1}F_{2n+1} - F_{2n}^2 = (-1)^{2n} = 1$. Then using (1) with $k = n$, the given expression can be rewritten as $1 - F_{n+1}^2F_{n-1}^2 + 2F_{n+1}F_n^2F_{n-1} - F_n^4 = 1 - (F_{n-1}F_{n+1} - F_n^2)^2 = 1 - [(-1)^n]^2 = 0$.

[1] Hoggatt, Vernon E., Jr., *Fibonacci and Lucas Numbers*, 1969, $I_{13}$ on pp. 57–58.

*Also solved by:* Clayton W. Dodge, University of Maine, Orono, Maine; Bob Prielipp, University of Wisconsin—Oshkosh, Oshkosh, Wisconsin and the proposer.

## *Golden Anniversaries*

Five KME chapters have their fiftieth anniversaries this spring! They are Washburn University, Topeka, Kansas (KS Delta), March 29; William Jewell College, Liberty, Missouri (MO Gamma), May 7; Texas Woman's University, Denton, Texas (TX Gamma), May 7; Mount Mary College, Milwaukee, Wisconsin (WI Alpha), May 11; and Baldwin-Wallace College, Berea, Ohio (OH Gamma), June 6. Congratulations! By the way, the names of KS Delta and TX Gamma at the time of installation were Washburn Municipal University and Texas State College for Women, respectively.

# Kappa Mu Epsilon News

Edited by Mary S. Elick, Historian

News of chapter activities and other noteworthy KME events should be sent to Mary S. Elick, Historian, Kappa Mu Epsilon, Mathematics Department, Missouri Southern State College, Joplin, Missouri 64801.

## CHAPTER NEWS

### *AL Gamma*
Chapter President — Terra Jones

University of Montevallo, Montevallo — 18 actives, 2 associates

Other 1996-97 chapter officers: Cheryl Coley, vice president; Ava Putman, secretary; David Taylor, treasurer; James Ochoa, corresponding secretary; Karolyn Morgan, faculty sponsor.

### *AL Zeta*
Chapter President — James Blizard

Birmingham-Southern College, Birmingham — 12 actives, 18 pledges

AL Zeta initiated 18 new members into the chapter on December 5, 1996. Dr. Clyde Stanton, Professor of Chemistry, presented the program for the event. His presentation, entitled "Making Waves," dealt with the mathematics of Quantum Mechanics and Confirmation Stability in Organic Molecules. A reception followed the ceremonies. Other 1996-97 chapter officers: James Corder, vice president; Melissa Boren, secretary/treasurer; Mary Jane Turner, corresponding secretary; Raju Sriram, faculty sponsor.

### *AR Alpha*
Chapter President — Michael Von Dran

Arkansas State University, State University — 10 actives, 3 associates

Other 1996-97 chapter officers: Bobby Peppers, vice president/treasurer; Melissa DuBois, secretary; William Paulsen, corresponding secretary/faculty sponsor.

### *CA Delta*
Chapter President — Steve Guertin

California State Polytechnic University, Pomona — 10 actives, 4 associates

Other 1996-97 chapter officers: Maria Nuñiez, vice president; Jennifer Sommers, secretary; Carol Sabol, treasurer; Richard Robertson, corresponding secretary; Jim McKinney, faculty sponsor.

### *CO Delta* Chapter President — Natisha R. Littlejohn

**Mesa State College, Grand Junction** 10 actives

CO Delta Chapter held two meetings during the Fall semester of 1996. The first, a business meeting, was held in early September to organize and plan activities for the year. Members were encouraged to write papers which might serve as honors theses and also as presentations for the upcoming KME National Convention. Such papers could additionally be given at the Spring MAA Section Meeting and at the Friday Brown Bag Seminars in Mathematics. The second meeting of the semester, held the first of November, featured a potluck dinner and provided an opportunity to socialize, discuss graduate school plans, and report on the papers being written by the members. Scott Davis, past president, is working on a home page for the chapter. The faculty sponsor and the corresponding secretary are busy trying to line up enough funding for all interested members to attend the Thirty-First Biennial Convention in April. Other 1996–97 chapter officers: Deborah J. McCurley, vice president; Robin L. O'Connor, secretary; Tassie S. Medlin, treasurer; Donna K. Hafner, corresponding secretary; Kenneth Davis, faculty sponsor.

### *GA Alpha* Chapter President — Tonja Davis

**State University of West Georgia, Carrollton** 30 actives

Once again, GA Alpha sponsored a November-December Food and Clothing Drive for the needy. Proceeds were taken to the Salvation Army Office. On Thursday, November 21, the annual Fall Social, held at a local Mexican restaurant, was enjoyed by all those in attendance. Other 1996–97 chapter officers: Stephanie Parker, vice president; Michael Jumper, secretary; Kristy Williams, treasurer; Thomas J. Sharp, corresponding secretary/faculty sponsor.

### *IL Delta* Chapter President — Mike Mravle

**College of St. Francis, Joliet** 20 actives

Other 1996–97 chapter officers: Heather McNulty, vice president; Toni Dactilidis, secretary; Meg McAleer, treasurer; Rick Kloser, corresponding secretary/faculty sponsor.

### *IN Beta* Chapter President — Audrey Purmort

**Butler University, Indianapolis** 15 actives

Other 1996–97 chapter officers: Melissa Kolarik, vice president; Stephen Sanders, secretary/treasurer; Yuzhen Ge, corresponding secretary/faculty sponsor.

### *IA Alpha*

Chapter President — Matthew Schafer

**University of Northern Iowa, Cedar Falls** 35 actives

Fall meetings featured student presentations by Sarah Lacox and Suzanne Shontz entitled "Mathematics is Golden" and "Molecules and Their Symmetries," respectively. Cynthia Ohm provided the program for the December initiation banquet, held at Tally's Restaurant. Six new members joined the chapter at that time. The annual KME Homecoming Coffee was held on October 12 at the home of Professor and Mrs. Greg Dotseth. Among those attending was KME alumna Cherie Cobet Schafer. She and her husband were visiting their son, Matt, who is currently serving as the IA Alpha chapter president. KME members assisted with the Fall Phonathon for the Mathematics Department; they also helped with the Mathematics-Science Symposium in November. Other 1996–97 chapter officers: Mary Pittman, vice president; Suzanne Shontz, secretary; Amber Grotjohn, treasurer; John S. Cross, corresponding secretary/faculty sponsor.

### *IA Gamma*

Chapter President — Heather Schott

**Morningside College, Sioux City** 9 actives

Other 1996–97 chapter officers: James Nicolaisen, vice president; Jared Ellwein, secretary; Heather Kelly, treasurer; Douglas Swan, corresponding secretary/faculty sponsor.

### *IA Delta*

Chapter President — Joy Trachte

**Wartburg College, Waverly** 34 actives, 5 associates

Freshmen students were welcomed at an organizational meeting in September at which time an overview of plans for the year were discussed. A Problem Solving Group was formed with the intention of meeting one hour weekly during the evening meal. The October meeting agenda included a planning session for the chapter's homecoming fund-raising booth. A committee was also formed at that time to plan a trip to the Science and Industry Museum in Chicago. The program for the November meeting was given by members of the Problem Solving Group. The chapter's picture was taken for the year book in December and a Christmas Party was planned with the Physics and Computer Science Clubs. The organization has also begun planning for Math Awareness Week in April. Other 1996–97 chapter officers: Shilah Lybeck, vice president; Richard Kloster, secretary; Christopher Judson, treasurer; August Waltmann, corresponding secretary; Lynn Olson, faculty sponsor.

### *KS Alpha*

Chapter President — Kathleen Denney

**Pittsburg State University, Pittsburg** 54 actives, 8 associates

Fall semester activities began with a pizza party and the initiation of eight new members in October. Following the initiation, guest speaker

Troy Goodsell of Brigham Young University presented an interesting talk entitled "Shadows of Cantor Sets." Graduate student Andrew Buchholz, a former KS Alpha president, enthralled the audience at the regular November meeting by relating his experiences during a Budapest Semester in Mathematics. In December, the chapter hosted a road trip to Joplin, Missouri, to view the latest Star Trek movie, *First Contact.* Other 1996–97 chapter officers: Matthew Jackson, vice president; Brian Coots, secretary; Kari Hamm, treasurer; Cynthia Woodburn, corresponding secretary; Bobby Winters, faculty sponsor.

### *KS Beta* — Chapter President — Brenda Sloop

**Emporia State University, Emporia** — 36 actives, 10 associates

The KS Beta Chapter of Kappa Mu Epsilon made a concerted effort during the fall semester to become more active and visible on campus. To this end, members sponsored a table at both the Campus Activities Fair the beginning of the semester and at Family Day in late September. Bryan Dawson, faculty member and editor of *The Pentagon*, presented the program at the first meeting of the year on September 11. Ten new members were initiated during the fall semester and two chapter members attended the NCTM Regional Conference in Kansas City. As a service project, the chapter assisted with Brownie Day in mid November, setting up and manning various stations for the Brownie members. The club also held a Christmas Decoration Creation Party in December. The spring semester is looking extremely promising; new members are enthusiastic and all are looking forward to the annual math day. Other 1996–97 chapter officers: Andrew Applegarth, vice president; Ruth Dale, secretary; Shannon Decker, treasurer; Jason Manhart, historian; Connie Schrock, corresponding secretary; Larry Scott, faculty sponsor.

### *KS Gamma* — Chapter President — Erik Kurtenback

**Benedictine College, Atchison** — 13 actives, 8 associates

KS Gamma was invited to host a display at the MAA-sponsored Math Mania held at Kansas City Kansas Community College on October 3. Erik Kurtenback, Dawn Weston, Seth Spurlock, Christie Englebert, and faculty sponsor Linda Herndon, OSB, set up an "Amazing Bubbles" display for the 300+ high school and college students in attendance. On October 5, Linda Herndon, OSB, presented a workshop "WWW + Math = A World of Resources" as part of the Greater Kansas City Mathematics Technology Expo held at KCKCC. Dawn Weston attended the day's events. Dr. Tim Miller from the MO Lambda Chapter at Missouri Western State College spoke to chapter members on "The Mathematics of Voting" at the October meeting. A fall chili supper was held in Westerman Hall the evening of November 5. The annual Wassail Party was hosted by Jo Ann Fellin, OSB, at Marywood on December 8. Other 1996–97 officers: Chad Eddins, vice president; Dawn Weston, secretary; Christie Engelbert, treasurer; Linda Herndon, OSB, corresponding secretary/faculty sponsor.

## *KS Delta* — Chapter President — Mandy Chester

**Washburn University, Topeka** — 31 actives

Fall activities included two picnics held in conjunction with the departmental Math Club. Other 1996–97 chapter officers: Kevin Hennessy, vice president; Jim Stinson, secretary/treasurer; Allan Riveland, corresponding secretary; Donna LaLonde and Ron Wasserstein, faculty sponsors.

## *KY Alpha* — Chapter President — Lynne Brosius

**Eastern Kentucky University, Richmond** — 17 actives

In conjunction with the ACM Chapter, as a fund raiser, KY Alpha members once again provided floppy disks to students in the computer literacy class and the *Mathematica* class. A September picnic held at Lake Reba Park for KME, ACM, and the departmental faculty featured volleyball games and plenty of good food. At the October meeting members viewed the videotape on Paul Erdös entitled *N is a Number*. The November meeting included a panel discussion on graduate schools: Professors Kirk Jones, Ray Tennant, and Pat Costello served as panelists, sharing their thoughts and respective views on the topic. The highlight of the Christmas Party was, as always, the white elephant gift exchange. One participant was required, before he opened his chosen gift, to pay a penny to the provider of that particular gift. According to the old superstition, this would guarantee that the gift, a knife set, would not sever the friendship of the giver and the receiver. Other 1996–97 chapter officers: Kevin Zachary, vice president; Heather Sadler, secretary; Elizabeth Barrett, treasurer; Pat Costello, corresponding secretary/faculty sponsor.

## *KY Beta* — Chapter President — Timothy David Wilson

**Cumberland College, Williamsburg** — 20 actives

On September 17, 1996, the KY Beta Chapter officers assisted with an ice cream party honoring freshmen math and physics majors. Members heard a presentation by actuary Dr. Virginia Young in October and held an informal dinner get-together at Mi Pueblo, a Mexican restaurant, in November. Other activities shared with the Mathematics and Physics Club included a picnic at Briar Creek Park in late September and an end-of-semester Christmas Party on December 10. About 55 people attended the Christmas Party. The fall semester also saw the introduction of the KY Beta Chapter WWW page, including pictures and activities, at http://q.cumber.edu/math/kme.htm. Other 1996–97 chapter officers: Story Anne Robbins, vice president; Jennette Arlene Adamczak Dees, secretary; Melynda Kay Hazelwood, treasurer; Jonathan Ramey, corresponding secretary; John Hymo, faculty sponsor.

***MD Beta*** Chapter President — Leslie Huffer
Western Maryland College, Westminster 24 actives

Other 1996–97 chapter officers: Toni Smith, vice president; Julie Brown, secretary; Lori Mowen, treasurer; James Lightner, corresponding secretary/faculty sponsor.

***MD Delta*** Chapter President — Joseph Palardy
Frostburg State University, Frostburg 31 actives

MD Delta Chapter kicked off the fall semester with an organizational meeting in September, followed by a picnic at Rocky Gap State Park in October. November saw a Mathematical Treasure Hunt, with cash prizes for the winner. Other 1996–97 chapter officers: Heidi Femi, vice president; Brian Azzi, secretary; Carla White, treasurer; Edward T. White, corresponding secretary; John P. Jones, faculty sponsor.

***MI Beta*** Chapter President — Carrie Rickabaugh
Central Michigan University, Mount Pleasant 20 actives

MI Beta Chapter, along with the CMU Math Department and the Actuarial Club, hosted a Homecoming Alumni Picnic in mid October. Guest speakers for the fall were student teachers Kristen Williams and Dan Rothe. Kristen discussed projects she had used as a student teacher in her trig class and Dan demonstrated the TI-92 calculator. Other 1996–97 chapter officers: Kevin Zajac, vice president; Norma Reynolds, secretary; Debbie Sink, treasurer; Arnold Hammel, corresponding secretary/faculty sponsor.

***MS Alpha*** Chapter President — Karen Chandler
Mississippi University for Women, Columbus 13 actives, 2 associates

The chapter held a general meeting on October 11 and an initiation ceremony on November 1. Other 1996–97 chapter officers: Amee Jo Miles, vice president/treasurer; Karen Chandler, secretary; Jean Ann Parra, corresponding secretary; Shaochen Yang, faculty sponsor.

***MS Beta*** Chapter President — Huoy Jii Khoo
Mississippi State University, Mississippi State 20 actives, 7 associates

Other 1996–97 chapter officers: Brandon Butler, vice president; Christin McCloskey, secretary; Michael Pearson, corresponding secretary/faculty sponsor.

### *MS Gamma*

Chapter President — Chuck Fleming

**University of Southern Mississippi, Hattiesburg** 20 actives

Other 1996–97 chapter officers: Mary Bassinger, vice president; Leigh Lynn, secretary; Alice W. Essary, treasurer/corresponding secretary; Barry Piazza, faculty sponsor.

### *MS Delta*

Chapter President — John Miller

**William Carey College, Hattiesburg** 8 actives

Other 1996–97 chapter officers: Shaun Selman, vice president; Angela Tillman, secretary; Vivian Anderson, treasurer; Charlotte McShea, corresponding secretary/faculty sponsor.

### *MS Epsilon*

Chapter President — Robert East

**Delta State University, Cleveland** 12 actives

Other 1996–97 chapter officers: Kim Grimes, vice president; Alec Roehm, secretary/treasurer; Paula Norris, corresponding secretary; Rose Strahan, faculty sponsor.

### *MO Beta*

Chapter President — Lynn Graves

**Central Missouri State University, Warrensburg** 25 actives, 5 associates

MO Beta Chapter held monthly meetings during the Fall 1996 semester. Six new members were initiated at the October meeting. Tammy Surfus and Derek Beals traveled to Kansas City Kansas Community College to work at a KME booth at Math Mania. In other semester activities, members volunteered at the Math Clinic, held a book sale, adopted a family through the Christmas Store, and sponsored a Christmas Party. Orders for KME sweatshirts were taken in December. Speakers for the semester were Terry Brown, who spoke about the Eudora e-mail software; Charles Pace, the 1996 CMSU Distinguished Alumnus, who recently retired from NASA, having worked with the Apollo missions and space shuttle operations; and Richard Delaware from UMKC, whose topic was "Morley's Trisector Theorem." Other 1996–97 chapter officers: Cassie Young, vice president; Carla Brown, secretary; Barbara Hart, treasurer; Tammy Surfus, historian; Rhonda McKee, corresponding secretary; Larry Dilley, Phoebe Ho, and Scotty Orr, faculty sponsors.

### *MO Gamma*

Chapter President — Amy Fifer

**William Jewell College, Liberty** 10 actives, 5 associates

Other 1996–97 chapter officers: Lori Cantrall, vice president; Allison Cooper, secretary; Joseph T. Mathis, treasurer/corresponding secretary/faculty sponsor.

## *MO Epsilon* — Chapter President — Gary Smith

Central Methodist College, Fayette — 13 actives

Other 1996–97 chapter officers: Michele Niemczyk, vice president; Victoria Vahle, secretary/treasurer; William D. McIntosh, corresponding secretary; Linda O. Lembke, faculty sponsor.

## *MO Iota* — Chapter President — Vicki Nelson

Missouri Southern State College, Joplin — 10 actives, 5 associates

Officers were elected at an organizational meeting held early in the semester at Tarzan's Pizza. In an effort to accommodate everyone's schedules, meeting time was set at 7:00 a.m. This proved to have a negative effect on attendance. Members, however, did a great job of working concession stands at the five home football games, consequently adding $350 to the chapter's treasury. Preliminary plans were made to attend the national convention in the spring. The Problem Solving Group met frequently and a number of members competed on the national Putnam Exam. A Christmas Party Open House was held at the home of Mrs. Mary Elick. Other 1996–97 chapter officers: Shan Brand, vice president; Jennifer Schumaker, secretary; Jerl Simpson, treasurer; Agden Brister, historian; Mary Elick, corresponding secretary; Charles Curtis, faculty sponsor.

## *MO Kappa* — Chapter President — Patrick Hentges

Drury College, Springfield — 11 actives, 3 associates

The first activity of the semester was a pizza rush party for potential KME members held at the home of Dr. Carol Collins. Michelle Biggers and Aaron Wilson tied for first place in the annual Math Club Contest. Prize money was awarded to the winners at a pizza party held for all contestants. The annual fall bonfire party was held at the home of Dr. Charles Allen. Michelle Biggers presented a preliminary report on her undergraduate research at a luncheon meeting. Throughout the semester, Math Club has been running a tutoring service for both the day school and the Continuing Education Division (Drury Evening College) as a money-making project. The semester ended with a Christmas Party at the home of Aaron Wilson. Other 1996–97 chapter officers: Jon Adams, vice president; Michelle Biggers, secretary; Edyta Blasyczuk, treasurer; Charles Allen, corresponding secretary/faculty sponsor.

## *MO Lambda* — Chapter President — Tanya Griffin

Missouri Western State College, St. Joseph — 42 actives

Fall 1996 activities for MO Lambda Chapter of KME included a Welcome Back Picnic in August, a booth at Family Day in September, and Fall Initiation Ceremonies for 12 new members in mid October. The group also sponsored a float in the Homecoming Parade and participated in the window decorating contest in October. In November members enjoyed a Thanksgiving carry-in meal and a Calculator Resale Project provided a fund raiser in December. Other 1996–97 chapter officers: Cynthia Ready, vice president; Devon Kerns, secretary; Stacy Cabeen, treasurer; John Atkinson, corresponding secretary; Jerry Wilkerson, faculty sponsor.

### *NE Alpha* Chapter President — Rick Pongratz

Wayne State College, Wayne 21 actives

Fall activity centered on redecorating the club bulletin board and planning a trip to a hockey game. Other 1996–97 chapter officers: Rusty Slaughter, vice president; Becky Proskocil, secretary/treasurer; John Fuelberth, corresponding secretary; Jim Paige, faculty sponsor.

### *NE Gamma* Chapter President — J. J. Fernandez

Chadron State College, Chadron 14 actives, 10 associates

Other 1996–97 chapter officers: Chris Mundt, vice president; Julie Steinbach, secretary; Erin Johnson, treasurer; Jim Kaus, corresponding secretary; Monty Fickel, faculty sponsor.

### *NE Delta* Chapter President — Justin Rice

Nebraska Wesleyan University, Lincoln 12 actives

Other 1996–97 chapter officers: Dusten Olds, vice president; Christin Cordes, secretary; J.P. Johnson, treasurer; Gavin Larose, corresponding secretary/faculty sponsor.

### *NY Alpha* Chapter President — Norbet Lis

Hofstra University, Hempstead

Activities for the semester included a bowling event and the annual holiday party. Other 1996–97 chapter officers: Adam Katz, vice president; Ophir Feldman, secretary; Lisa Fontana, treasurer; Aileen Michaels, corresponding secretary/faculty sponsor.

### *NY Eta* Chapter President — Stacey Lauricella

Niagara University, Niagara University 20 actives

NY Eta Chapter and the Mathematics Department sponsored the third annual "Career Day" on November 22. The purpose of the program was to acquaint undergraduates with the wide range of career opportunities available in the field of mathematics. Five mathematics alumni of Niagara University provided perspectives about career choices and how their mathematical training had helped them achieve their goals. Students gained further information during a question/answer session following the presentations. Careers discussed were in the computing, engineering, and financial fields. Other 1996–97 chapter officers: Jennifer Egan, vice president; Amy

Maar, secretary; Lara Brown, treasurer; Robert Bailey, corresponding secretary; Kenneth Bernard, faculty sponsor.

***NY Lambda*** Chapter President — Kelli Ann Polotaye

C. W. Post Campus of Long Island University, Brookville 20 actives

Other 1996–97 chapter officers: Joseph D. Sprague, vice president; Robin M. Cancellieri, secretary; Jason R. Rand, treasurer; Andrew M. Rockett, corresponding secretary; Sharon Kunoff, faculty sponsor.

***OH Zeta*** Chapter President — Chetan Kandhari

Muskingum College, New Concord 12 actives, 4 associates

Fall activity has been primarily focused on planning for the April trip to the national convention in Springfield. Other 1996–97 chapter officers: Christopher Luzier, vice president; Melissa Frutig, secretary/treasurer; David L. Craft, corresponding secretary; Richard Daquila, faculty sponsor.

***OK Alpha*** Chapter President — Carrie O'Leary

Northeastern State University, Tahlequah 36 actives, 3 associates

Chapter initiation ceremonies for twelve new members, held in a local restaurant, were well attended by faculty, students, and families of the initiates. John Callaway has finished setting up a database containing information about past and current chapter membership. John, who also designed the OK Alpha WWW page, graduated from NSU this past December and will be greatly missed. Members continue to sponsor a monthly math contest and to hold an annual book sale. Mrs. Linda Collins retired in December from NSU's mathematics department. KME presented her with a gift, thanking her for planning the games for KME for these past ten years. The Christmas Pizza Party was again a great success. Those attending played Family Feud. Other 1996–97 chapter officers: Laura Cole, vice president; Lisa Eidson, secretary; Peter Butz, treasurer; Joan E. Bell, corresponding secretary/faculty sponsor.

***OK Delta*** Chapter President — Brock Leach

Oral Roberts University, Tulsa

After a period of inactivity, OK Delta is striving to reorganize. Debra Oltman is corresponding secretary and Roy Rakestraw is faculty sponsor.

***PA Alpha*** Chapter President — Laura Williams

Westminster College, New Wilmington 15 actives

Fall activities centered on planning a high school problem-solving competition to be held in April during Math Awareness Week. Six high schools in the area will participate in the competition. The winner will receive a graphing calculator with *DERIVE* capabilities. Other 1996–97 chapter officers: Laurel Scaff, vice president; Heather Carson, secretary; Jill Schuller, treasurer; J. Miller Peck, corresponding secretary; Carolyn Cuff and Warren Hickman, faculty sponsors.

### *PA Gamma* — Chapter President — Erin Korns

Waynesburg College, Waynesburg — 22 actives, 6 associates

Other 1996–97 chapter officers: Etta Nethken, vice president; Linda Smitley, secretary; Amanda Beisel, treasurer; Anthony Billings, corresponding secretary/faculty sponsor.

### *PA Delta*

Marywood College, Scranton — 1 active

PA Delta is planning a spring semester induction, at which time officers will be named. Sr. Robert Ann von Ahnen, IHM, is corresponding secretary/faculty sponsor.

### *PA Eta* — Chapter President — Ronna Matich

Grove City College, Grove City — 26 actives

Other 1996–97 chapter officers: Suzette Cramer, vice president; Lori Young, secretary; Eric Blum, treasurer; Marvin C. Henry, corresponding secretary; Dan Dean, faculty sponsor.

### *PA Iota* — Chapter President — Rebecca Shubert

Shippensburg University of Pennsylvania, Shippensburg — 20 actives, 7 associates

Other 1996–97 chapter officers: Mary Wenrich, vice president; Cynthia Hefty, secretary; Vicki Shanahan, treasurer; Michael Seyfried, corresponding secretary; Gene Fiorini, faculty sponsor.

### *PA Kappa* — Chapter President — Nicholas J. Gross

Holy Family College, Philadelphia — 6 actives, 2 associates

The chapter organized a math competition for high school students to be held on December 7, 1996. Unfortunately, the event was canceled due to insufficient registration. Chapter members currently enrolled in the Senior Seminar presented their research papers to division faculty on December 12. Nick Gross spoke on "Efficiency in Sports Gambling," Tom Feldmann addressed "The Aerodynamics of Golf Ball Design," and Lisa Esposito reported her research on the use of mathematics in computer tomography. Chapter members are presently making plans for the annual installation ceremony which will take place on April 3, 1997. Members are also planning college-wide activities for Math Awareness Month as well as for the 3rd Annual Grade School Math Competition. Other 1996–97 chapter officers: Thomas Feldmann, vice president; Lisa Esposito, secretary; Cheryll Stone-Schwendiman, secretary; Sister Marcella Louise Wallowicz, corresponding secretary/faculty sponsor.

***PA Mu*** Chapter President — Colleen Connors

Saint Francis College, Loretto 22 actives, 2 associates

The PA Mu Chapter of KME participated in a variety of activities during the 1996–97 year. The annual KME induction ceremony was held on February 4, 1997. The evening began with a mass celebrated by KME member Father Joseph Chancler, T.O.R. Following the mass, a dinner was held in the Maurice Stokes room for the 20 inductees, members, and guests. After the dinner, two students were inducted, bringing the total chapter membership to 157. Several KME members participated in the Third Annual Science Day. The senior mathematics majors presented a poster session featuring their projects. Other KME majors served as moderators, judges, scorekeepers, and timers for the Science Bowl. A total of 256 high school students from 19 schools attended. KME members also participated in the Adopt-A-Highway program on October 13, 1996. Approximately 15 people picked up litter from the highway near the college. Other 1996–97 chapter officers: Jennifer Ropp, vice president; Jennifer Gibbons, secretary; Jaysn Voshell, treasurer; Peter Skoner, corresponding secretary; Adrian Baylock, faculty sponsor.

***SD Alpha*** Chapter President — Stacy Garrels

Northern State University, Aberdeen 11 actives

Other 1996–97 chapter officers: Kristi Schuster, vice president; Margo Maynard, secretary; Lu Zhang, corresponding secretary; Raj Markandz, faculty sponsor.

***TN Gamma*** Chapter President — Kyle Brown

Union University, Jackson 23 actives

Other 1996–97 chapter officers: Elizabeth Morgan, vice president; Sherry Lin, secretary; Rachel Wright, treasurer; Matt D. Lunsford, corresponding secretary; Troy D. Riggs, faculty sponsor.

***TN Delta*** Chapter President — Deron C. Walraven

Carson-Newman College, Jefferson City 11 actives

The primary activity for TN Delta thus far for 1996–97 has been the planning and implementing of the fall picnic. Other 1996–97 chapter officers: Michael D. Kelley, vice president; Jana L. Taylor, secretary/treasurer; Catherine Kong, corresponding secretary/faculty sponsor.

***WI Gamma*** Chapter President — Steve Wall

University of Wisconsin—Eau Claire, Eau Claire 20 actives, 10 associates

Other 1996–97 chapter officers: Kady Hickman, vice president; Kendra

Zillmer, secretary; Jeremy Eppler, treasurer; Marc Goulet, corresponding secretary/faculty sponsor.

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## *The Math Student Blues*

(Tune: *I've Been Working on the Railroad.*)

I've been working on a problem
All the livelong night,
I've been working on a problem
To hit that quiz just right.
Don't you hear the teacher raving,
"Get up early in the morn,
Study hard or you'll be flunking
Sure as you were born."

I've been wondering about that formula
And how it's ever got,
I've been wondering about that answer
And all that tom-e-rot,
Cause I hear the teacher raving,
"Get up early in the morn,
Study hard or you'll be flunking
Sure as you were born."

I've been working all this winter,
All this livelong term;
I've been listening to professors
And for rest my heart does yearn.
I've been goin' to lab'ratory,
To gym and math class too—
If I go to many other things
I don't know what I'll do.

—NEW YORK ALPHA CHAPTER

—Reprinted from *The Pentagon*, Fall 1949

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## *KME Quiz Update*

No attempted solutions to the KME Quiz have been received. If we receive no complete solutions, the submission with the greatest number of correct answers will be declared the winner. See page 50 of the previous issue (Fall 1996) for the quiz.

## *Ballad of the Octagon*

Once there was an Octagon
In love, so it would seem;
For when the Circle girl came 'round
His face, oh, it would beam!

The Circle girl was curved all over
Uniformly, everywhere!
No rough corners could be found;
If there were he wouldn't care.

Then one day he saw the girl
walking with a Square!
"How can you do that?" he said,
He said, "How could you dare?"

"Look," she said, "for it is simple.
A regular polygon is he.
His sides are great, his angles right,
He's full of symmetry."

"But you," she said, "just look at you!
Your sides are all the same.
Your angles, though, are all messed up —
Go hang your head in shame!"

As he left that fateful day
He said, "That Square will flop!
I'll be a regular Octagon
So at me he'll have to stop!"

He pumped, he pushed, he stretched, he pulled,
He walked, he jogged, he ran;
He got himself in shape again
To carry out his plan.

Then he saw her once again,
He thought, "This will be fun!"
"Look at me now," he said with pride,

Look at what I've done!"

"You are much better," said the girl,
"Much better than the Square!
But you're still not the best around,
To Parallelogram you can't compare."

"Beating him," she said, "you cannot do,
To try is of no use;
He has two angles, so, a, cute,
But yours are so obtuse."

He walked away even before
The words had left her lips;
But disappointment didn't last long —
That night he met Ellipse.

—The editor.

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## *KME Election Results*

The election of national officers was held Saturday, April 5, 1997, during the Thirty-First Biennial Convention of Kappa Mu Epsilon in Springfield, Missouri. The winners and their addresses are:

Robert Bailey, President-Elect
Mathematics Department
Niagara University
Niagara University, NY 14109
rbail@niagara.edu

Don Tosh, Historian
Evangel College
1111 N. Glenstone Ave.
Springfield, MO 65802
toshd@evangel.edu

Patrick J. Costello was also installed as President; his address is on page 77. The George R. Mach Distinguished Service Award was presented to Harold L. Thomas. The full report of the convention will be printed in the Fall issue of this journal. For a preview of the papers presented, see the paper by Suzanne Shontz on pages 38–46, which was awarded "top four" status at this convention. The version printed here is the one presented at the 1996 Region IV Convention, but is essentially the same in content.

# Kappa Mu Epsilon National Officers



*Kappa Mu Epsilon*, Mathematics Honor Society, was founded in 1931. The object of the Society is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics due to its demands for logical and rigorous modes of thought; to provide a Society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; and to disseminate the knowledge of mathematics and familiarize the members with the advances being made in mathematics. The official journal of the Society, *The Pentagon*, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the Chapters.

# Active Chapters of Kappa Mu Epsilon

*Listed by date of installation.*

| Chapter | Location | Installation Date |
|---|---|---|
| OK Alpha | Northeastern State University, Tahlequah | 18 April 1931 |
| IA Alpha | University of Northern Iowa, Cedar Falls | 27 May 1931 |
| KS Alpha | Pittsburg State University, Pittsburg | 30 Jan 1932 |
| MO Alpha | Southwest Missouri State University, Springfield | 20 May 1932 |
| MS Alpha | Mississippi University for Women, Columbus | 30 May 1932 |
| MS Beta | Mississippi State University, Mississippi State | 14 Dec 1932 |
| NE Alpha | Wayne State College, Wayne | 17 Jan 1933 |
| KS Beta | Emporia State University, Emporia | 12 May 1934 |
| NM Alpha | University of New Mexico, Albuquerque | 28 March 1935 |
| IL Beta | Eastern Illinois University, Charleston | 11 April 1935 |
| AL Beta | University of North Alabama, Florence | 20 May 1935 |
| AL Gamma | University of Montevallo, Montevallo | 24 April 1937 |
| OH Alpha | Bowling Green State University, Bowling Green | 24 April 1937 |
| MI Alpha | Albion College, Albion | 29 May 1937 |
| MO Beta | Central Missouri State University, Warrensburg | 10 June 1938 |
| TX Alpha | Texas Tech University, Lubbock | 10 May 1940 |
| TX Beta | Southern Methodist University, Dallas | 15 May 1940 |
| KS Gamma | Benedictine College, Atchison | 26 May 1940 |
| IA Beta | Drake University, Des Moines | 27 May 1940 |
| TN Alpha | Tennessee Technological University, Cookeville | 5 June 1941 |
| NY Alpha | Hofstra University, Hempstead | 4 April 1942 |
| MI Beta | Central Michigan University, Mount Pleasant | 25 April 1942 |
| NJ Beta | Montclair State University, Upper Montclair | 21 April 1944 |
| IL Delta | College of St. Francis, Joliet | 21 May 1945 |
| KS Delta | Washburn University, Topeka | 29 March 1947 |
| MO Gamma | William Jewell College, Liberty | 7 May 1947 |
| TX Gamma | Texas Woman's University, Denton | 7 May 1947 |
| WI Alpha | Mount Mary College, Milwaukee | 11 May 1947 |
| OH Gamma | Baldwin-Wallace College, Berea | 6 June 1947 |
| CO Alpha | Colorado State University, Fort Collins | 16 May 1948 |
| MO Epsilon | Central Methodist College, Fayette | 18 May 1949 |
| MS Gamma | University of Southern Mississippi, Hattiesburg | 21 May 1949 |
| IN Alpha | Manchester College, North Manchester | 16 May 1950 |
| PA Alpha | Westminster College, New Wilmington | 17 May 1950 |
| IN Beta | Butler University, Indianapolis | 16 May 1952 |
| KS Epsilon | Fort Hays State University, Hays | 6 Dec 1952 |
| PA Beta | LaSalle University, Philadelphia | 19 May 1953 |
| VA Alpha | Virginia State University, Petersburg | 29 Jan 1955 |
| IN Gamma | Anderson University, Anderson | 5 April 1957 |
| CA Gamma | California Polytechnic State University, San Luis Obispo | 23 May 1958 |
| TN Beta | East Tennessee State University, Johnson City | 22 May 1959 |
| PA Gamma | Waynesburg College, Waynesburg | 23 May 1959 |
| VA Beta | Radford University, Radford | 12 Nov 1959 |
| NE Beta | University of Nebraska—Kearney, Kearney | 11 Dec 1959 |

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|---|---|---|
| IN Delta | University of Evansville, Evansville | 27 May 1960 |
| OH Epsilon | Marietta College, Marietta | 29 Oct 1960 |
| MO Zeta | University of Missouri—Rolla, Rolla | 19 May 1961 |
| NE Gamma | Chadron State College, Chadron | 19 May 1962 |
| MD Alpha | College of Notre Dame of Maryland, Baltimore | 22 May 1963 |
| IL Epsilon | North Park College, Chicago | 22 May 1963 |
| OK Beta | University of Tulsa, Tulsa | 3 May 1964 |
| CA Delta | California State Polytechnic University, Pomona | 5 Nov 1964 |
| PA Delta | Marywood College, Scranton | 8 Nov 1964 |
| PA Epsilon | Kutztown University of Pennsylvania, Kutztown | 3 April 1965 |
| AL Epsilon | Huntingdon College, Montgomery | 15 April 1965 |
| PA Zeta | Indiana University of Pennsylvania, Indiana | 6 May 1965 |
| AR Alpha | Arkansas State University, State University | 21 May 1965 |
| TN Gamma | Union University, Jackson | 24 May 1965 |
| WI Beta | University of Wisconsin—River Falls, River Falls | 25 May 1965 |
| IA Gamma | Morningside College, Sioux City | 25 May 1965 |
| MD Beta | Western Maryland College, Westminster | 30 May 1965 |
| IL Zeta | Rosary College, River Forest | 26 Feb 1967 |
| SC Beta | South Carolina State College, Orangeburg | 6 May 1967 |
| PA Eta | Grove City College, Grove City | 13 May 1967 |
| NY Eta | Niagara University, Niagara University | 18 May 1968 |
| MA Alpha | Assumption College, Worcester | 19 Nov 1968 |
| MO Eta | Truman State University, Kirksville | 7 Dec 1968 |
| IL Eta | Western Illinois University, Macomb | 9 May 1969 |
| OH Zeta | Muskingum College, New Concord | 17 May 1969 |
| PA Theta | Susquehanna University, Selinsgrove | 26 May 1969 |
| PA Iota | Shippensburg University of Pennsylvania, Shippensburg | 1 Nov 1969 |
| MS Delta | William Carey College, Hattiesburg | 17 Dec 1970 |
| MO Theta | Evangel College, Springfield | 12 Jan 1971 |
| PA Kappa | Holy Family College, Philadelphia | 23 Jan 1971 |
| CO Beta | Colorado School of Mines, Golden | 4 March 1971 |
| KY Alpha | Eastern Kentucky University, Richmond | 27 March 1971 |
| TN Delta | Carson-Newman College, Jefferson City | 15 May 1971 |
| NY Iota | Wagner College, Staten Island | 19 May 1971 |
| SC Gamma | Winthrop University, Rock Hill | 3 Nov 1972 |
| IA Delta | Wartburg College, Waverly | 6 April 1973 |
| PA Lambda | Bloomsburg University of Pennsylvania, Bloomsburg | 17 Oct 1973 |
| OK Gamma | Southwestern Oklahoma State University, Weatherford | 1 May 1973 |
| NY Kappa | Pace University, New York | 24 April 1974 |
| TX Eta | Hardin-Simmons University, Abilene | 3 May 1975 |
| MO Iota | Missouri Southern State College, Joplin | 8 May 1975 |
| GA Alpha | State University of West Georgia, Carrollton | 21 May 1975 |
| WV Alpha | Bethany College, Bethany | 21 May 1975 |
| FL Beta | Florida Southern College, Lakeland | 31 Oct 1976 |
| WI Gamma | University of Wisconsin—Eau Claire, Eau Claire | 4 Feb 1978 |
| MD Delta | Frostburg State University, Frostburg | 17 Sept 1978 |
| IL Theta | Benedictine University, Lisle | 18 May 1979 |
| PA Mu | St. Francis College, Loretto | 14 Sept 1979 |
| AL Zeta | Birmingham-Southern College, Birmingham | 18 Feb 1981 |
| CT Beta | Eastern Connecticut State University, Willimantic | 2 May 1981 |
| NY Lambda | C.W. Post Campus of Long Island University, Brookville | 2 May 1983 |
| MO Kappa | Drury College, Springfield | 30 Nov 1984 |

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|---|---|---|
| CO Gamma | Fort Lewis College, Durango | 29 March 1985 |
| NE Delta | Nebraska Wesleyan University, Lincoln | 18 April 1986 |
| TX Iota | McMurry University, Abilene | 25 April 1987 |
| PA Nu | Ursinus College, Collegeville | 28 April 1987 |
| VA Gamma | Liberty University, Lynchburg | 30 April 1987 |
| NY Mu | St. Thomas Aquinas College, Sparkill | 14 May 1987 |
| OH Eta | Ohio Northern University, Ada | 15 Dec 1987 |
| OK Delta | Oral Roberts University, Tulsa | 10 April 1990 |
| CO Delta | Mesa State College, Grand Junction | 27 April 1990 |
| NC Gamma | Elon College, Elon College | 3 May 1990 |
| PA Xi | Cedar Crest College, Allentown | 30 Oct 1990 |
| MO Lambda | Missouri Western State College, St. Joseph | 10 Feb 1991 |
| TX Kappa | University of Mary Hardin-Baylor, Belton | 21 Feb 1991 |
| SC Delta | Erskine College, Due West | 28 April 1991 |
| SD Alpha | Northern State University, Aberdeen | 3 May 1992 |
| NY Nu | Hartwick College, Oneonta | 14 May 1992 |
| NH Alpha | Keene State College, Keene | 16 Feb 1993 |
| LA Gamma | Northwestern State University, Natchitoches | 24 March 1993 |
| KY Beta | Cumberland College, Williamsburg | 3 May 1993 |
| MS Epsilon | Delta State University, Cleveland | 19 Nov 1994 |

## *Starting a KME Chapter*

Complete information on starting a chapter of KME may be obtained from the National President (see address on p. 77). Some information is given below.

An organized group of at least ten members may petition through a faculty member for a chapter. These members may be either faculty or students; student members must meet certain coursework and g.p.a. requirements.

The financial obligation of new chapters to the national organization includes the cost of the chapter's charter and crest (approximately \$50) and the expenses of the installing officer. The individual membership fee to the national organization is \$20 per member and is paid just once, at that individual's initiation. Much of this \$20 is returned to the new members in the form of membership certificates and cards, keypin jewelry, a two-year subscription to the society's journal, etc. Local chapters are allowed to collect semester or yearly dues as well.

The petition itself, which is the formal application for the establishment of a chapter, requests information about the petitioning group, the academic qualifications of the eligible petitioning students, the mathematics faculty, mathematics course offerings and other facts about the institution. It also requests evidence of faculty and administrative approval and support of the petition. Petitions are subject to approval by the National Council and ratification by the current chapters.