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Error Correction for Basic Codes

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awarded FOURTH PLACE by the Awards Committee.

In many areas of science and technology, it becomes necessary to transmit information from one source to another, as in the transfer of data from one computer to another. Such was the case with the photographs taken of Mars by Mariner 6. In order to transmit information, it must first be converted into some type of code, which can easily be decoded by the receiver. Upon transmission, however, the code may possibly be modified due to human or random error. Usually, if an error occurs a code can be re-transmitted, and the error corrected. However, many times messages cannot be sent again. Therefore, it is necessary to determine a process for detecting and correcting errors in a code. In 1950, Robert W. Hamming published a paper on error-correction for linear codes, which pioneered the further study of coding theory. The purpose of this paper will be to study the general properties of codes and then proceed to a discussion of simple linear codes and their corresponding methods of error-correction.

This paper will focus on binary block codes, in which all information is transmitted as a string of zeros and ones. A codeword is such a string of n 0's and 1's, which consists of k ($k \leq n$) message digits and r ($r = n - k$) parity check digits. The total number of possible combinations of strings of length n using only 0's and 1's as digits is 2^n. For example, the total number of strings of length 5 is 2^5, or 32. Of these 32 strings, not all are codewords, but only a certain few. Suppose that the previous string contained only 3 message digits and 2 parity check digits. Since the parity check digits are determined by the message digits, the actual number of codewords will only be 2^3, or 8. Hence, not all possible strings of length 5 are codewords, and therefore a method of error-detection must be found that will determine actual codewords from strings of digits. It must also be noted that binary coding is defined over addition modulo 2. Hence, $1 + 1 = 0$.

Before looking at specific types of codes, a general overview of basic properties of codes and error correction must be reviewed. The Hamming distance — named after R. W. Hamming — is defined as the number of digits that are different between two strings. For example, if $d(x,y)$ is the symbol denoting the Hamming distance between x and y, then $d(011010, 000110) = 3$, because the second, third and fourth digits differ. In the first string, these digits are 1, 1 and 0 respectively; in the second string, they are 0, 0 and 1 respectively. The first, fifth and sixth digits in both sets do not differ; they are 0, 1, and 0 respectively in both strings. Therefore, since three digits differ, the Hamming distance is three. This can be rewritten as $d(x,y) = \sum_{i=1}^{n} | x_i - y_i |$ for all strings x and y of length n. This is obvious, because the only time the sum is incremented is if the digits between x and y differ.

The Hamming distance is in fact a metric, in that it satisfies the three basic properties of metrics, as I shall now show.

Property 1: $d(x,y)$ is greater than zero for all x,y and if $d(x,y) = 0$, then $x = y$. Since $d(x,y)$ is defined in terms of the summation of absolute values, then by properties of absolute value $d(x,y)$ is always positive. Also, $d(x,y) = 0$ if and only if $x_i = y_i$ for all i. This is only true if $x = y$.

Property 2: $d(x,y) = d(y,x)$ for all x and y. Now, $d(x,y) = \sum_{i=1}^{n} | x_i - y_i |$. By properties of absolute values, this equals $\sum_{i=1}^{n} | y_i - x_i |$. Therefore, by the definition of distance, this becomes $d(y,x)$. So, $d(x,y) = d(y,x)$.

Property 3: $d(x,z) \le d(x,y) + d(y,z)$. By use of the triangle inequality for real numbers and properties of absolute value, we have

$$\begin{aligned} d(x,z) &= \sum_{i=1}^{n} | x_i - z_i | = \sum_{i=1}^{n} | x_i - y_i + y_i - z_i | \\ &\le \sum_{i=1}^{n} | x_i - y_i | + \sum_{i=1}^{n} | y_i - z_i | = d(x,y) + d(y,z) . \end{aligned}$$

Therefore, since d satisfies all properties of metrics, the Hamming distance is a metric.

Now, in order to detect an error, the error must convert a codeword to a non-codeword. Therefore, there must be a minimum number of digits that are different between each individual codeword. This is called the minimum Hamming distance, or d. If $d = 1$, then code words only differ in one digit, so errors would be impossible to detect. For example, suppose that the digit string 001010 was sent and the string 001011 was received. If $d = 1$, then 001011 would also be a codeword and the error would not be discovered. The greatest number of errors that can be detected in a code is $d - 1$. This is obvious because if there are d number of errors in a code word, then the original word would be received as a different code word.

Detecting errors and correcting them are two very different matters. Although $d - 1$ errors can be detected, even fewer can be corrected. For example, using the code with words 000000 and 111111, $d = 6$ since the minimum Hamming distance between the two "words" is 6. Therefore, up to 5 errors can be detected. However, if the string 010111 is transmitted, errors can be detected but cannot be corrected, because it cannot be determined which of the code words was meant to be sent. In most cases, though, it can be assumed that since $d(000000, 010111) = 4$ and $d(111111, 010111) = 2$, the string to be sent was 111111.

This "error-correcting" method is called the nearest-neighbor rule. Using this rule, all errors which are in fewer than $d/2$ digits can be corrected. If fewer than $d/2$ errors are made, then there is exactly one codeword to which the incorrect word is closest and to which it can therefore be corrected. If there are $d/2$ or more errors, many code words are of equal distance from the incorrect word. Therefore, the string received cannot be corrected. Using the example above, 010111 has less than $d/2$ errors that are detected; it can be corrected to 111111. Suppose instead that 010101 was received. This string has $d/2$ or 3 detectable errors. It is clear that this cannot be corrected, since it is of equal distance from both 000000 and 111111. The number of errors which can be corrected then from the nearest neighbor rule is $\lceil (d/2) - 1 \rceil$ where $\lceil x \rceil$ is the least integer greater than or equal to x. From this principle comes the following result.

Theorem 1. Suppose that d is the minimum Hamming distance between two codewords in the binary code C. Then no error-detecting rule can detect more than $d - 1$ errors and no error-correcting rule can correct more than $\lceil (d/2) - 1 \rceil$ errors.

Proof. From the discussion above, it is clear that no error-detecting rule can detect more than $d - 1$ errors. However, the error-correction conclusion is a bit more difficult to explain. First, the following lemma must be proved.

Lemma. Let C be a binary code with minimum Hamming distance d. If α and β are codewords such that $d(\alpha,\beta) = d$, then there exists a digit string γ such that $d(\alpha,\gamma) \leq \lceil d/2 \rceil$ and $d(\beta,\gamma) \leq \lceil d/2 \rceil$.

Proof. Without loss of generality, assume that α and β differ in the first d digits. Now let γ be the string that matches α in its first $\lceil d/2 \rceil$ digits and matches β in the next $d - \lceil d/2 \rceil$ digits. Now, $d(\alpha,\gamma) = d - \lceil d/2 \rceil \leq \lceil d/2 \rceil$. Also, $d(\beta,\gamma) = \lceil d/2 \rceil$. Therefore, there always exists a γ such that $d(\alpha,\gamma) \leq \lceil d/2 \rceil$ and $d(\beta,\gamma) \leq \lceil d/2 \rceil$.

The second part of the theorem can now be proved. Suppose that a method of error-correction exists which can correct $\lceil d/2 \rceil$ errors. Now let α and β be two codewords where $d(\alpha,\beta) = d$. From the previous lemma, there exists a γ where $d(\alpha,\gamma) \leq \lceil d/2 \rceil$ and $d(\beta,\gamma) \leq \lceil d/2 \rceil$. Now, without loss of generality, suppose α is sent and γ is received. This word could be corrected to either α or β. Since this is not permitted, the error-correction method is not valid, and therefore any error-correction method can correct at most $\lceil (d/2) - 1 \rceil$ errors. Q.E.D.

Now that the groundwork has been established, the paper can proceed to the discussion of different types of codes and their corresponding error-correction methods.

It has already been stated that the parity check digits are determined by the message digits; therefore there must be some rule for ascertaining what these will be. This is known as the encoding problem. Ideally, an encoding technique should send as many message digits as possible, while subsequently limiting the number of parity check digits. The information rate of a code is calculated by dividing the number of message digits by the total number of digits in the string. Obviously, a higher information rate is desired.

The first type of codes to be studied is repetition codes. The parity check digits are simply the message digits repeated a pre-determined number of times. For example, when $k = 1$ and $n = 5$, the two possible codewords would be 00000 and 11111. These codes could generally be easily corrected using the nearest neighbor rule. However, repetition codes have a very low information rate which will never be greater than one-half.

A type of code which has an extremely high information rate is the single-parity-check code. To find the one parity check digit, the message digit string is added modulo 2, and the parity check digit is given the resulting sum. Hence, the sum of the digits in every codeword is 0. (This can also be done having the sum always equal 1.) Because of this trait, one error is extremely easy to detect, by simply adding the digits in the string. If there are two or any even number of errors, though, an error would not be detected, and the string might pass for an intentionally transmitted word. Moreover, it would be impossible to find the error, as all of the digits have an equal probability of error. Despite the high information rate, the single-parity-check code has many disadvantages.

The compromising solution is to find a method of encoding which has both a moderate information rate and reasonable level of correctability. Suppose a message digit string of length k is encoded by multiplying it by a matrix which consists of the $k \times k$ identity matrix augmented by a $k \times (n-k)$ matrix to generate parity check digits. This matrix will be known as the generator matrix **M**. For example, if 010 is a string of message digits and **M** is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix},$$

the codeword received would be

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Obviously, when k message digits are multiplied by a generator matrix, the first k letters of the resulting string are the message digits, because of the presence of the identity matrix in the generator matrix. Hence, to determine the original string, only the parity check digits need to be dropped. However, the parity check digits are extremely useful in locating and correcting errors.

The generator matrix has already been shown to be $[\ \mathbf{I}_k\ \mathbf{G}\ ]$ with **G** as a $k \times (n-k)$ matrix. Now let $\mathbf{G}^{\mathrm{T}}$ be the transpose of the matrix **G**. Also, let **H** be called the parity check matrix where **H** is the transpose of **G** augmented by the $(n-k)$ identity matrix, or $[\ \mathbf{G}^{\mathrm{T}}\ \mathbf{I}_{(n-k)}\ ]$. In the previous example, the transpose of **G** is

$$\mathbf{G}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

which makes the parity check matrix

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

From this comes the following:

Definition. A code is said to be a *linear code* (or a "group code") if and only if its codewords are the set of vectors C which satisfy an equation of the form $\mathbf{Hc}^{\mathrm{T}} = 0$.

In fact, the repetition codes and single-parity-check codes are linear codes with corresponding parity check matrices. For the repetition code of length n, $\mathbf{H} = [11\ldots1]$, where 1 is repeated an n number of times. The single-parity-check codes' parity check matrix is the $k \times n$ matrix consisting of a column of 1's augmented by a $k \times k$ identity matrix. For example, if $n = 4$,

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

The parity check matrix can be used to identify codewords, and therefore determine if an error has been made.

Theorem 2. In a linear code, a block $\mathbf{a} = a_1a_2\ldots a_k$ is encoded as $\mathbf{x} = x_1x_2\ldots x_n$ if and only if $a_i = x_i$ for all i less than or equal to k and $\mathbf{Hx}^{\mathrm{T}} = \mathbf{0}$ (where $\mathbf{0}$ is the row matrix of all zeros).

Proof. By the definition of encoding, **a** is multiplied by the generator matrix to get **x**, so, as explained above, the first k digits of **x** will be the same as the first k digits of **a**. Therefore, $a_i = x_i$ for all i less than or equal to k. Secondly, **H** times the transpose of **x**, or $\mathbf{Hx}^{\mathrm{T}}$, is equal to $\mathbf{H}(\mathbf{a}[\mathbf{I}_k\ \mathbf{G}])^{\mathrm{T}}$, by definition. Then, by properties of transpose of matrices, this equals $\mathbf{H}[\mathbf{I}_k\ \mathbf{G}]^{\mathrm{T}}\mathbf{a}^{\mathrm{T}}$. Now, by substituting the definition of **H** and further properties of transposes, we obtain

$$\mathbf{Hx}^{\mathrm{T}} = [\mathbf{G}^{\mathrm{T}}\ \mathbf{I}_{(n-k)}]\begin{bmatrix} \mathbf{I}_k \\ \mathbf{G}^{\mathrm{T}} \end{bmatrix}\mathbf{a}^{\mathrm{T}}.$$

Since $[\mathbf{G}^{\mathrm{T}}\ \mathbf{I}_{(n-k)}]$ is a $(n-k)\times(k+(n-k)) = (n-k)\times n$ matrix and the second matrix in the product is a $(k+(n-k)\times k = n \times k$ matrix, the two matrices can be multiplied. Since both contain an identity matrix, it can easily be proven that their product results in $(\mathbf{G}^{\mathrm{T}} + \mathbf{G}^{\mathrm{T}})$. Therefore, $\mathbf{Hx}^{\mathrm{T}} = (\mathbf{G}^{\mathrm{T}} + \mathbf{G}^{\mathrm{T}})\mathbf{a}^{\mathrm{T}}$. But, $(\mathbf{G}^{\mathrm{T}} + \mathbf{G}^{\mathrm{T}})$ is equal to $\mathbf{0}$, since the addition is modulo 2. Hence, $\mathbf{Hx}^T = \mathbf{0a}^T = \mathbf{0}$.

Conversely, suppose that $a_i = x_i$ for all i less than or equal to k and $\mathbf{Hx}^T = \mathbf{0}$. Suppose further that $\mathbf{a}$ is encoded as $\mathbf{y} = y_1 y_2 \ldots y_n$. Then as shown above, $a_i = y_i$ for all i less than or equal to k and, further, $\mathbf{Hy}^T = \mathbf{0}$. But $\mathbf{Hx}^T = \mathbf{0}$ is also true. It follows that $\mathbf{x} = \mathbf{y}$. Q.E.D.

Therefore, it has been proved that in order for a string of digits $\mathbf{x}$ to be a codeword, $\mathbf{Hx}^T$ must equal $\mathbf{0}$. Now the syndrome is defined as $\mathbf{s}^T = \mathbf{Hr}^T$, where $\mathbf{r}$ is a word that has been received. Therefore, $\mathbf{x}$ is a codeword if and only if the syndrome of $\mathbf{x}$ is $\mathbf{0}$. To illustrate this, we turn to the previous example.

Case 1: Suppose that the string of digits received using the given generator matrix is 000101. This is $\mathbf{x}$. Then the transpose $\mathbf{x}^T$ is

$$\mathbf{x}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

and when $\mathbf{x}^T$ is multiplied by $\mathbf{H}$, the syndrome is

$$\mathbf{Hx}^T = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Since this is not equal to $\mathbf{0}$, then 000101 must not be a codeword.

Case 2: Now take $\mathbf{x} = [001101]$. The transpose of $\mathbf{x}$ is

$$\mathbf{x}^T = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

and when $\mathbf{x}^T$ is multiplied by $\mathbf{H}$, the syndrome is $\mathbf{0}$, so the string is actually a codeword:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In fact, the set S of strings of length n is actually an Abelian group with respect to addition, and the set C of codewords is a subgroup of that group.

Proof (that the set of strings of length n is an Abelian group). (1) Closure. Let $\mathbf{a}$, $\mathbf{b} \in$ S. Since addition is defined over modulo 2, $\mathbf{a}+\mathbf{b}$ is also an element of S. (2) Associativity. Since addition is associative in modulo 2, then associativity holds over the elements of S. (3) Identity. The identity will be the string of 0's of length n, which is an element of S. (4) Inverse. In modulo 2 addition, each element will be the inverse of itself. Therefore S is a group. (5) Commutativity (Abelian Group). Since addition is commutative in modulo 2, S is also commutative. Hence, S is an Abelian group. Q.E.D.

Proof (that the codewords are a subgroup). The set C of the codewords is a subset of S. Now, C is associative since S is associative. Also, the identity element is contained in C. Since each element of C is its own inverse, it suffices to show that C is closed. Let $\mathbf{a}$, $\mathbf{b} \in$ C. Then, as $\mathbf{a}$ and $\mathbf{b}$ are codewords, $\mathbf{Ha}^T = \mathbf{0}$ and $\mathbf{Hb}^T = \mathbf{0}$. Thus $\mathbf{0} = \mathbf{Ha}^T - \mathbf{Hb}^T = \mathbf{H}(\mathbf{a}^T - \mathbf{b}^T) = \mathbf{H}(\mathbf{a}-\mathbf{b})^T$. By definition, $(\mathbf{a}-\mathbf{b}) \in$ C. Since the subgroup is defined over addition modulo 2, $(\mathbf{a}-\mathbf{b}) = (\mathbf{a}+\mathbf{b})$. Hence, $(\mathbf{a}+\mathbf{b}) \in$ C. Therefore, C is a subgroup of S. Q.E.D.

Now, the following theorem can be easily explained and proved.

Theorem 3. Suppose that the columns of the parity check matrix $\mathbf{H}$ are all nonzero and all distinct. Suppose that a codeword $\mathbf{y}$ is transmitted and a word $\mathbf{x}$ is received. If $\mathbf{x}$ differs from $\mathbf{y}$ only on the i-th digit, then $\mathbf{Hx}^T$ is the i-th column of $\mathbf{H}$.

Proof. Since $\mathbf{y}$ is a codeword, it follows that $\mathbf{Hy}^T = \mathbf{0}$. Since $\mathbf{x}$ differs from $\mathbf{y}$, there is a string $\mathbf{e}$ such that $\mathbf{x} = \mathbf{y}+\mathbf{e}$ (every digit in $\mathbf{x}$ whose corresponding digit is different in $\mathbf{y}$ has a one in that digit in $\mathbf{e}$ and every digit that is the same contains a 0). Then $\mathbf{Hx}^T = \mathbf{H}(\mathbf{y}+\mathbf{e})^T = \mathbf{H}(\mathbf{y}^T+\mathbf{e}^T) = \mathbf{Hy}^T + \mathbf{He}^T = \mathbf{0} + \mathbf{He}^T = \mathbf{He}^T$. Therefore, if exactly one error is made, then, when the string is multiplied by the parity check matrix, the result must be one of the columns of the matrix. The number of the column is the digit which is incorrect in the string. Q.E.D.

Using "Case 1" of the example, since the result was $\begin{bmatrix}1\\0\\1\end{bmatrix}$, then that matches with the third column of the matrix. Therefore, the third digit of the original string is incorrect and can be corrected to 001101, which, as shown by "Case 2," is indeed a codeword.

If, on the other hand, $\mathbf{H}\mathbf{x}^T$ is not one of the i-th columns of $\mathbf{H}$, then more than one error has occurred and the string cannot be corrected using this code. Again using the example, suppose instead of receiving 001101, $\mathbf{x} = 000111$ is received. This obviously has two errors. Then

$$\mathbf{H}\mathbf{x}^T = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0\\0\\0\\1\\1\\1 \end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Since $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is not one of the columns of $\mathbf{H}$, then 000111 has more than two errors and cannot be corrected.

Using the given generator matrix for codewords of length three with three parity digits added, the minimum Hamming distance is also three. By Theorem 1, this means that up to $\lceil (d/2) - 1 \rceil$, or one digit, can be corrected. Hence, the example satisfies Theorem 1 also. The information rate for the example used is 1/2, since the six digit string only had 3 message digits; however, this number will vary depending upon the size of the generator matrix used.

There are several different types of codes and error-correction methods which are much more complex and efficient than the ones presented in this paper; only basic studies have been discussed. New breakthroughs are being found about the relationship error-correcting methods have with genetic coding, and coding is being used by NASA to aid in space exploration. Coding and error-correction are just a few of the relatively "new" areas of mathematics which are proving to be beneficial in many fields.

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Exploring Braess' Paradox

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We have all experienced the frustration of trying to go across town at rush hour. Suppose you need to go to the airport on the other side of the city. The problem is not that the distance to the airport is so great, but that the roads are very congested. There is only a limited amount of time. If you are late, the plane will leave without you. You wish they would put in a new road to the airport since so many cars are trying to use the routes currently available. Surely an additional route would help everyone get to their destination on time.

Our intuition tells us that adding more routes will make the system flow faster. Unfortunately, a new route added to a congested transportation system may or may not help traffic flow any faster. This paper will examine a transportation system which exhibits paradoxical behavior as well as a mechanical example presented by Cohen and Horowitz [2] to explore the reasons why an additional route added to a congested system may actually increase travel time for everyone.

Background.

In 1968, German operations researcher Dietrich Braess suggested that the addition of an additional route to a congested network could lead to an overall slowdown of the system [1]. This phenomenon, known as "Braess' paradox," has applications beyond traffic flow studies. Other applications can be found in physics, in hydraulics and in electrical network systems.

In general, all applications of the paradox occur on some form of a transportation network. A transportation network is a simple, weighted and directed graph with the following properties (see [5]):

i) exactly one vertex, the source, has no incoming edges;
ii) exactly one vertex, the sink, has no outgoing edges;
iii) the capacity of an edge is a nonnegative number; and
iv) the undirected graph obtained from the network by ignoring the directions of the edges is connected.

In our traffic example, the time it takes to traverse each edge is not simply tied to the distance traveled. Other factors like road condition, number of lanes, type of road surface and number of traffic lights will have an effect on the amount of time a car will spend on a section of roadway. The greater the number of cars on the road, the more the driving time will increase for each section of roadway. We can express driving time as a function of a given number of cars using each edge on the graph. Integer amounts are used in our route time functions. In modeling a particular situation, these functions could be adjusted so that the number of cars would remain an integer amount.

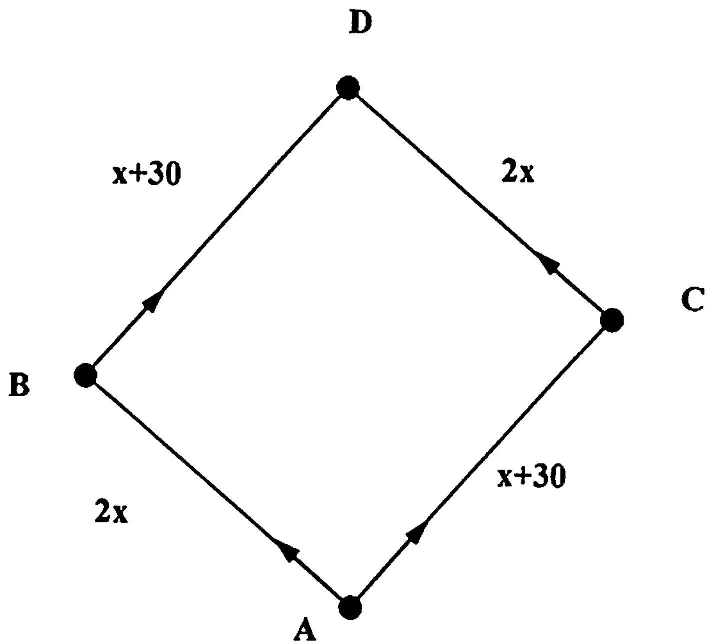

Figure 1.

A Traffic Example.

We begin by adapting the simple network presented by Braess to our airport problem as shown in Figure 1. There are two possible routes from the source at vertex A to the sink at vertex D:

Route 1: AB, BD
Route 2: AC, CD

Travel time is expressed as a function of the number x of cars on the edge. The edges have the following travel time functions:

$$AB = CD = 2x$$
$$AC = BD = x + 30 \; .$$

Traveling alone to the airport, we could easily catch our flight. We could travel on either route in 2(1)+(1+30) = 2+31 = 33 minutes. Unfortunately, our trip must take place during rush hour. We will need to add additional cars to the system to see what really happens when we drive the route in heavy traffic.

Suppose 12 cars are traveling on our road system. Six cars traveling each route will have the following travel times:

Route 1: AB + BD = 2(6) + (6+30) = 12+36 = 48 minutes

Route 2: AC + CD = (6+30) + 2(6) = 36+12 = 48 minutes.

By adding more cars to the system, our travel time has increased by more than 45%.

Now suppose a new road BC is constructed with travel time function x+14 as shown in Figure 2. Now we have the following routes available:

Route 1: AB, BD
Route 2: AC, CD
Route 3: AB, BC, CD .

Traveling alone on the third route, we could reach the airport in

Route 3: AB+BC+CD = 2(1) + (1+14) + 2(1) = 19 minutes!

Traveling alone, the new route would save us 14 minutes. This is a time savings of over 42% from our previous trip alone.

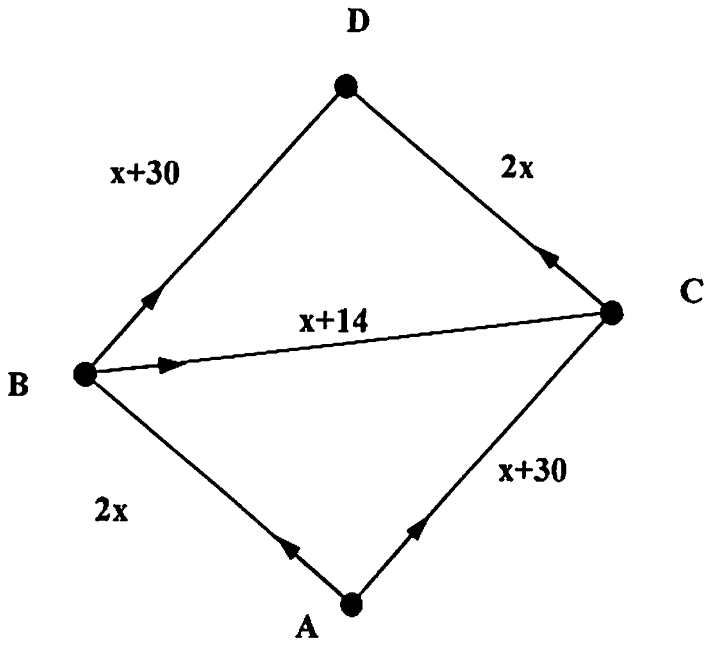

Figure 2.

While the third route saves us a great deal of time when we travel the route alone, consider the congested system where 12 cars are enroute to the airport. Four cars will travel on each of the routes. (We will explain why later.) From our origin point A, the 8 cars taking routes 1 and 2 set out on AB. The four remaining cars, traveling route 3, take edge AC. At vertex B, four of the cars from edge AB take edge BC while the other four cars from edge AB take edge BD. The four cars traveling edge BC on route 2 now join the four cars from route 1 to give us a total of eight cars traveling edge CD to the airport. With the additional edge on our graph, we now have the following travel times for each of the routes:

Route 1: $2(8) + (4+30) = 50$ minutes
Route 2: $(4+30) + 2(8) = 50$ minutes
Route 3: $2(8) + (4+14) + 2(8) = 50$ minutes.

By adding another route to the system and adding enough cars, we have encountered Braess' paradox on the network. Although travel time was greatly decreased when a single user traveled the network alone, travel time on the expanded network was increased for everyone when the network was congested. Instead of the new route decreasing travel time for rush hour users, drivers now experience a 4% increase in travel time. We can see that adding a route may not always be a good idea.

In this example, cars traveled each route in equal numbers. The basis for this can be found in game theory. This transportation network is a model of a non-cooperative game. This means that if a driver manages to shorten his travel time, he does not gain time at the expense of other drivers. It is certainly true that his decision will have an effect on other drivers, but it will not directly cause them to lose an amount of time equal to the time he gains. Since all drivers are aware of all routes and the travel times associated with them, the traffic pattern will settle into an equilibrium. No user will have any incentive to change his choice of route [3].

The distribution of traffic in a transportation network is governed by "Wardrop's principles" as stated in [7]:

> *System Optimization* "The total travel cost is a minimum."
>
> *User Optimization* "The travel cost on all origin-destination paths joining the origin and the destination actually used are equal, and less than those which would be experienced by a single user on any unused OD path."

User optimized flow on a network does not always minimize total travel time as we have seen in this example. In fact, total travel time was increased for all users on the congested system.

A Mechanical Example.

Cohen and Horowitz [2] describe a network of strings and springs which exhibits paradoxical behavior similar to that found in our traffic example. In this mechanical example as shown in Figure 3, we have a spring suspended from a support. A second spring is suspended from the first by a 3/8 meter length of string. Safety strings, each 1 meter long, are suspended from the support to the top of spring 2 and from the bottom of spring 1 to the 1/2 newton weight. In their example, springs are assumed to have zero unstretched length and spring constant k=1. Strings are considered to be inelastic. Initially, both safety strings are slack. The distance from the support to the weight is 11/8 meter, as shown in Figure 3. When the linking string is cut, we would expect the weight to sink. However, cutting the linking string causes the weight to rise after the system settles to equilibrium. The distance between the support and weight now measures 5/4 meter, as shown in Figure 4.

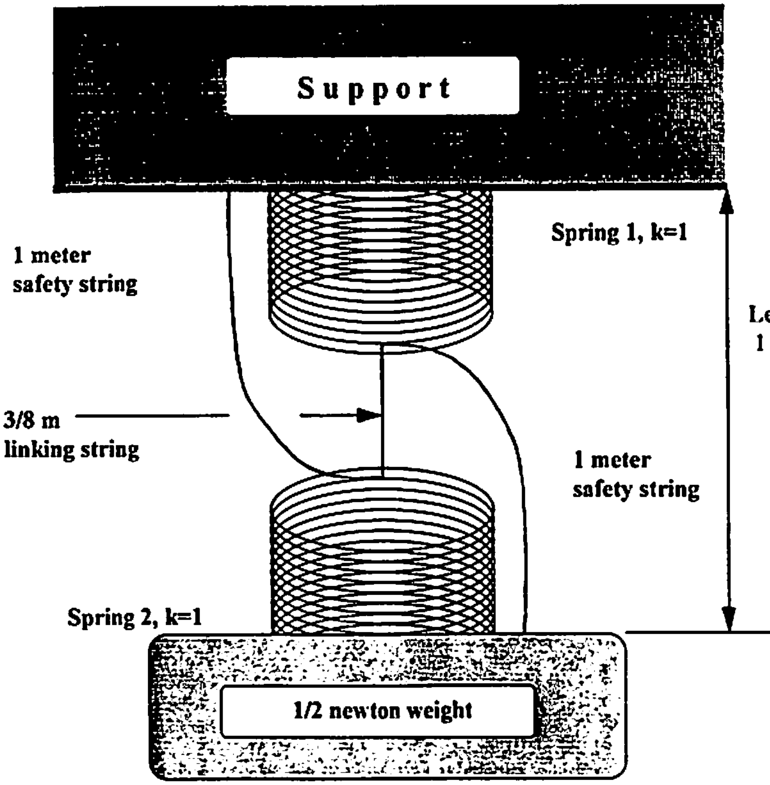

Figure 3.

We can explain this behavior by observing that in Figure 3 the springs act in series to support the 1/2 newton weight. Each spring is supporting 1/2 newton. When the linking string is cut and the system settles once more into equilibrium, each of the springs in Figure 4 is supporting half the weight, or 1/4 newton. This system is analogous to our traffic example. Length, in their mechanical example, corresponds to time in our traffic example. Adding weight to their mechanical example would be the same as increasing the number of cars on our road system (see [2]).

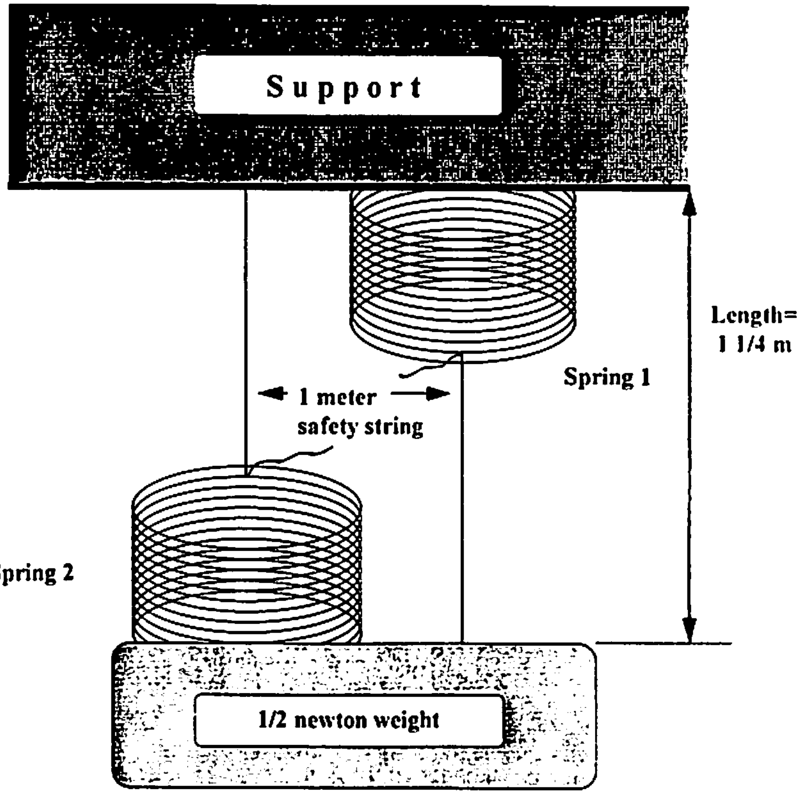

Figure 4.

The graph in Figure 5 shows how the network behavior at equilibrium is analogous to our traffic example. Without loss of generality, we will consider the reverse order where the linking string is being added instead of being eliminated. Our vertices are labeled as in our traffic example. The system is attached to the weight at vertex A and to the support at vertex D. The lengths of the springs are a function of the weight w added to the system. Letting α be a constant, we can describe the spring length where k=1 for both springs as AB = CD = αw. Since we are given that the string is inelastic, we can let the constant β represent the length of the safety strings and let the constant θ represent the length of the linking string. Now our routes in Figure 4 can be represented on the graph in Figure 5 as

$$\text{Route 1: AB + BD} = \alpha w + \beta$$
$$\text{Route 2: AC + CD} = \beta + \alpha w\ .$$

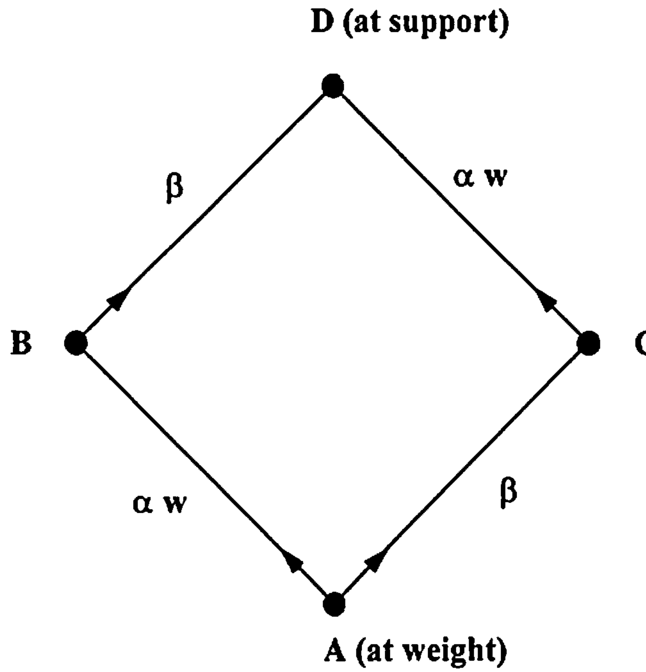

Figure 5.

With the addition of the linking string, a third route is created, as in Figure 6. Thus we will have

$$\text{Route 3: AB+BC+CD} = \alpha w + \theta + \alpha w \; .$$

In their mechanical example, routes 1 and 2 have been abandoned because the lengths AB+BC < AC and BC+CD < BD. This is consistent with Braess behavior since the new route, BC, is used. The mechanical example does fit the criteria for a transportation network.

In this particular example, once the system has reached equilibrium, the weight will rise for values of θ between 1/4 meter and 3/4 meter [2]. With a linking string length of 1/4 meter, the system length would be 1-1/4 meter in both networks. The energy flow on the system plays a non-cooperative game. As Cohen and Horowitz point out in [2], "... at equilibrium, the whole network has less potential energy after the string is cut than before."

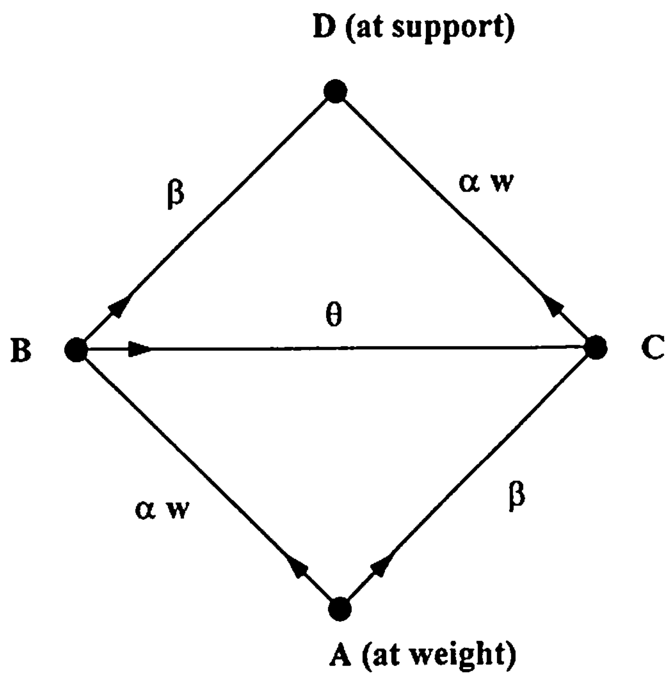

Figure 6.

Conclusion.

Examples of Braess' paradox do exist in the real world. In [1], Cohen states, "I know of at least one real-life example. In Stuttgart they built a new road through the area around the Schlossplatz. But traffic through the city moved even slower than before, so they closed down the road. Nobody has looked at it mathematically, but I wouldn't be surprised if this was a case of Braess' paradox at work." A 1977 study regarding the addition of new and upgraded low quality (earth and gravel) roads in the Awash province of Ethiopia turned up surprising results [4]. Road improvements would be expected to increase efficiency; however, traffic flow efficiency would have been sacrificed had the improvements been made, so the best strategy was to do nothing.

Contrary to its name, Braess' paradox is not, in fact, a paradox. As we can see from the examples, this behavior can be explained using game theory and physical laws. The paradoxical behavior can best be described as counterintuitive. While behavior of this type can be explained, it presents unpleasant consequences for those who fail to plan for its appearance. Steinberg [6] noted in 1988 that Braess' paradox is about "as likely to occur as not occur" on congested networks.

Our examination of Braess' paradox made it clear that making improvements to a network without regard to user behavior may lead to an overall slowdown of the system. By following our feeling that more is better and arbitrarily increasing network capacity, we observed an overall decrease in efficiency. There are many places in life where following your intuition can be fatal. Exploring Braess' paradox is valuable in reminding us of that fact.

Acknowledgment. I would like to thank Mr. Mike Adams for his encouragement in the preparation of this paper.

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A Genetic Algorithm Applied to a Problem in Coding Theory

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There are two main topics that I will be discussing in this paper: genetic algorithms and error-correcting codes. After giving some brief descriptions and important definitions from these topics, I will move on to how the concepts work together. Genetic algorithms are a relatively new search technique that seemed to have a potential for finding the minimum distance of a code. My corresponding professor, Mr. Adams, and I tested this possibility by adapting a program written by David E. Goldberg for our problem in coding theory.

Genetic algorithms are search algorithms based on the mechanics of natural selection and natural genetics. A genetic algorithm is an optimization technique which combines the idea of survival of the fittest with a randomized information exchange. They are different than traditional optimization methods because: they search from a population of points, rather than a single point; they evaluate the fitness of the element, but they do not use auxiliary knowledge; they use probabilistic rules, and are not deterministic (Goldberg, 7).

I will begin with a discussion of the terminology associated with genetic algorithms. It will be helpful to think of strings, or messages, in a genetic algorithm as the chromosomes in biological systems. A string is a finite sequence of the alphabet over which we are working. For instance, if the alphabet is $\{0,1\}$, a string could be 01100. Chromosomes are composed of genes in biological systems. However, when working with artificial systems, strings are composed of features, or positions, which take on different values. For example, given the string above, position two (counting from the left) has a value of 1.

A genetic algorithm is composed of a reproduction operator, a crossover operator, and a mutation operator. These are all important components of the process of natural selection. A genetic algorithm starts with a randomly generated population of strings, and it generates the successive patterns of strings. The randomly generated population is called the initial population. It can be produced by using a coin toss, where heads = 1 and tails = 0, to generate any number of strings of a given length. After this point, all subsequent strings are generated by the algorithm.

The reproduction operator is the process by which individual messages are copied according to their fitness values. These values are determined by a function, called the fitness function, which is positive valued. The messages that produce higher fitness values have a higher probability of contributing one or more offspring to the next generation. The probability is found by dividing the fitness value of a given string by the sum of the total population. For example, given the strings x_1, x_2, x_3, x_4, and their fitness values: $f(x_1)$, $f(x_2)$, $f(x_3)$, $f(x_4)$, the probability of x_1 reproducing is $f(x_1)/(f(x_1)+f(x_2)+f(x_3)+f(x_4))$. This is an artificial simulation of natural selection.

The crossover operator is a process in which members of the newly reproduced messages in the "mating pool" are "mated" at random. Two new strings are created by randomly choosing a position along the two messages and swapping all the characters between that position and the ends of the messages, including the last position (see Figure 1). Crossover allows the messages with the highest fitness values to combine information and possibly produce strings with higher fitness values.

```
                 Before                              After
String 1        AAA|AAAA       New string 1         AAABBBB
                   |
                   |    ---------
                   |    Crossover->
                   |    ---------
                   |
String 2        BBB|BBBB       New string 2         BBBAAAA
                   |
                crossing site
```

Figure 1.

The mutation operator is the final step, providing occasional random alteration of the value of a string position. This alteration has a small probability of natural occurrence. Mutation is important because the reproduction and crossover search "may become overzealous and lose some potentially useful genetic material" (Goldberg, 14), or messages in the space. By introducing a random factor, we ensure that the search will not fixate solely on one part of the set of strings.

The nature of genetic algorithms makes them useful in problems that require searching through large sets of strings. A problem that arises in coding theory is that of finding minimum weight vectors in a code, and we decided to study the effectiveness of genetic algorithms for this purpose (as of this date, we have not found any articles that have been published which link genetic algorithms to finding minimum weight vectors in a code). We used a genetic algorithm designed for Turbo Pascal by David E. Goldberg. The original program was modified by me to meet our needs.

Error-correcting codes originated in response to problems in the reliable communication of digitally encoded information. Their objective is to "... add redundancy so that the original messages can be correctly decoded" (Pless, 6). They are used in many aspects of everyday life. Some of these uses include communications channels, compact discs, and bar code scanners. I will next give some definitions involving error-correcting codes.

A binary linear code can be defined as the set of all linear combinations of k independent vectors in V, where V is the space of all n-tuples of 0's and 1's, with addition of vectors done componentwise mod 2. A linear code C is called cyclic if whenever the codeword $x_1 = (a_0, a_1, \ldots, a_{n-1})$ is in C, then $x_2 = (a_{n-1}, a_0, a_1, \ldots, a_{n-2})$ is in C (Pless, 6). We dealt with binary cyclic codes in our program.

There is a nice correspondence between a codeword and a polynomial. Consider the codeword x_1 from above. The polynomial $f_{x_1}(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$ can represent x_1. Cyclic codes are nice to work with because they have generator polynomials which make encoding a message simple. When a message is received, it is first expressed as a polynomial and then by multiplying it with the generator polynomial we obtain the polynomial corresponding to the codeword.

An important function in the study of cyclic linear codes is the Hamming weight, or simply the weight. The weight of a vector x is the number of nonzero components it contains and is denoted $wt(x)$. The minimum weight of a code is the weight of the nonzero vector of smallest weight in the code.

A code C is called an $[n,k,d]$ code, where n, k and d are the parameters of C. The length of the code is given by n, and the dimension of the code (as a subspace of V) is given by k. The third parameter, d, is the minimum distance of the code. The distance between two vectors u and v is defined as the number of positions in which they differ and is written $d(u,v)$. Minimum distance is defined to be the minimum $d(u,v)$ for all $u,v \in C$, where $u \neq v$.

One advantage of linear codes is that their minimum weight and minimum distance are equal values (MacWilliams and Sloane, 9). This is true for all linear codes because for all $u,v \in C$, $u-v \in C$, so $wt(u-v) = d(u,v)$. This made it possible for us to have our program search for a message which encoded to a minimum weight codeword in order to obtain the minimum distance of the code.

For an example illustrating the parameters of a code, we will look at the Hamming $[7,4,d]$ code (given below) and find d. Notice that there are $2^4 = 16$ codewords in the code and each is of length 7. In order to find d, we will simply list the 16 possible codewords (see Figure 2), using {1011000, 0011000, 0010110, 0001011} as a basis, and find d by inspection. Because there are no nonzero codewords with weight less than 3, the minimum weight is 3. Also, because we know the minimum distance equals the minimum weight, the minimum distance is 3. Thus the Hamming code is a $[7,4,3]$ code.

1011000	1000101	1110100	0100111
0101100	1100010	0111010	1010011
0010110	0110001	0011101	1101001
0001011	0000000	1001110	1111111

Figure 2.

The minimum distance d is an important parameter because of the information about the code that can be obtained from d. For example, there is a theorem which states that a code C can correct $t = [(d-1)/2]$ or fewer errors, where $[x]$ denotes the greatest integer less than or equal to x (Pless, 12). So, the $[7,4,3]$ Hamming code can correct $t = [(3-1)/2] = [2/2] = [1] = 1$ or fewer errors.

The reader has now been exposed to some basic definition from error-correcting codes, as well as the basic form of a genetic algorithm. I will continue with a description of the program and our results. First, however, I must note that I modified the program given by Goldberg so that it would correctly encode a message by using the generator polynomial. The message was changed to a polynomial and then multiplied by the generator polynomial in order to obtain the codeword. We also changed the fitness function as well as the appearance of the output.

In our program, we used the messages from the vector space as our chromosomes. This allowed reproduction, crossover and mutation to occur without the possibility of creating a new vector that was not in the code. The messages were only encoded to determine their Hamming weight and fitness values.

We applied several different fitness functions to the problem. When we were testing the program, we used the [7,4,3] Hamming code with an almost linear, non-negative fitness function. The function was the weight ($wt(x)$) subtracted from the length of the code (n), unless the weight was zero, in which case the function returned a zero. So, the fitness function was (see Figure 3):

$$f(x) = \begin{cases} n - wt(x) & \text{if } wt(x) \neq 0 \\ 0 & \text{if } wt(x) = 0 \end{cases}$$

This function gave higher fitness values for lower weight codewords, except for the zero codeword (which we did not want to have a high fitness value). The algorithm found a message which encoded as a minimum weight vector in the first test run. Remember, however, that this code has only 16 vectors.

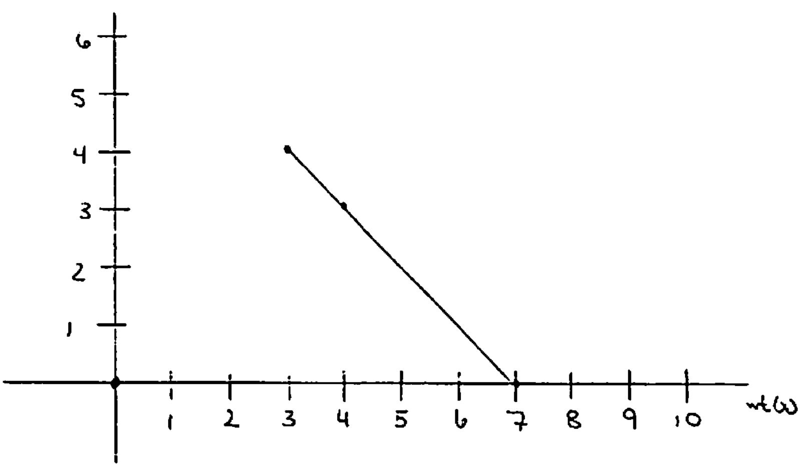

Figure 3.

Bose-Chaudhuri-Hocquenghem (BCH) codes are linear, cyclic codes defined in terms of the roots of their generator polynomials and designed to correct a certain number of errors (Pless, 114). There is a specific function of BCH codes that we are interested in, called the BCH bound. This says that the minimum weight d of a BCH code C of designed distance δ is at least δ (Pless, 115). The BCH bound is a lower bound, and not all BCH codes have their minimum distance equal to δ. Our goal was still to find a message which encoded to a minimum weight codeword. However, applying the algorithm to a BCH code, a message which encoded to a codeword of weight δ would then verify the BCH bound for that code.

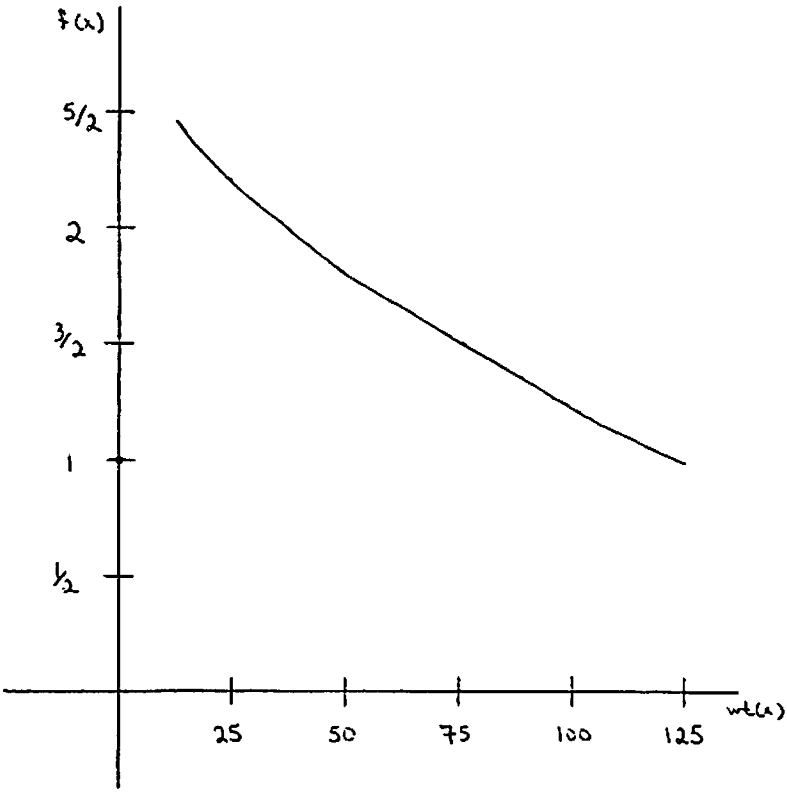

Figure 4.

Our original fitness function was not giving values which would enable the population to approach the minimum weight vectors. It appeared that a small change in the weight of a vector did not create a big enough change in the fitness value. So, I designed a new function. The function I decided on involved an exponential function of the Hamming weight, namely the following (see Figure 4):

$$f(x) = \begin{cases} e^{(n - wt(x))/n} & \text{if } wt(x) \neq 0 \\ 1 & \text{if } wt(x) = 0 \end{cases}$$

Using this function, a slight decrease in the weight corresponds to a large percentage of increase in the fitness value. However, a weight of zero gives us the lowest possible fitness value.

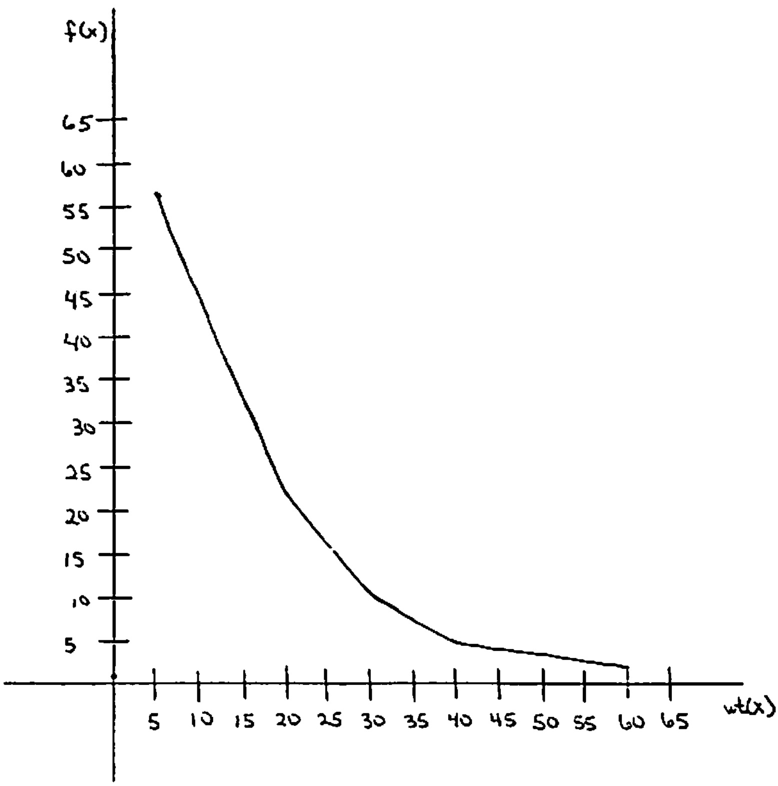

Figure 5.

We then ran the program on the BCH [127, 92, d] (where $d \geq 11$ by the BCH bound) code. The smallest weight vector we found was $wt(x)$

$= 16$. Keep in mind that 2^{42} is extremely large and we were searching a very small percentage of those codewords. We did, however, gain some useful information. Namely, that once again, we needed to examine the curve of our fitness function and try to make it steepest around low weight codewords. We also realized that a large initial population helped the algorithm in its search for low weight vectors. So, we changed our fitness function to the following (see Figure 5):

$$f(x) = \begin{cases} e^{k(n - wt(x))} & \text{if } wt(x) \neq 0 \\ 1 & \text{if } wt(x) = 0 \end{cases}$$

where $k = [\ln(n-\delta)/(n-\delta)]$.

This function worked well, and using an initial population of 250 we were able to locate a minimum weight codeword for a $[63,42,7]$ BCH code in 49 generations. It is worth noting that the probability of finding a minimum weight codeword by a single random selection in the $[63,42,7]$ code is

$$\#\text{ of minimum weight codewords} = 9 \times 5 \times 31 = 1395$$
$$\#\text{ of codewords} = 2^{42} = 4.398 \times 10^{12}$$

giving us

$$\frac{1395}{4.398 \times 10^{12}} = 3.172 \times 10^{-10}$$

(MacWilliams and Sloane, 282). We found the codeword in generation number 49. Because we started with an initial population of 250 vectors, we examined at most $49 \times 250 = 12{,}250$ codewords. This is not very many (proportionately only 2.000×10^{-9}) of the 2^{42} codewords possible. Had we examined 12,250 randomly selected codewords, the probability that we would not select a minimum weight codeword is $((2^{42} - 1395)/2^{42})^{12250} = 9.9999611 \times 10^{-1}$. Thus, the probability that we would select one or more minimum weight codewords is $1 - ((2^{42} - 1395)/2^{42})^{12250} = 3.8855243 \times 10^{-6}$.

The genetic algorithm was successful at finding a minimum weight vector when the possibility of finding one was extremely low. The search methods it employed found a minimum weight codeword relatively quickly. We would like to continue our research by experimenting with different fitness functions and crossover operators, as well as mutation and crossover probabilities. We would like to run the program on some BCH codes for which the minimum distances are unknown. Combining genetic algorithms with coding theory appears to be worthwhile. In conclusion, we were successful in applying a genetic algorithm to the problem of finding a minimum weight vector in a cyclic binary BCH code.

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Volleyball: The Effects of Spin on the Serve

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Introduction.

Volleyball, like most things in life, can be analyzed by the study of mathematics. The aspect of volleyball that we are going to focus on is the serve, but first let us concentrate on some of the basic rules, goals and restrictions that we will need to consider.

Volleyball serves are actually more than just hitting the ball over the net. The most effective serve is one that either travels extremely fast or one that drops as it floats through the air. One serve that accomplishes both of these is the "top-spin" serve. By contacting the ball toward the top of the surface instead of directly in the middle, a downward spin is put on the ball. Now the ball is not only affected by gravity but also by a spin factor. Our goal is to compare a range of successful serves with spin to a range of successful serves without spin and see which is more effective.

Explanation of acceleration and spin.

The ball's acceleration through the air is given by $A = A_s - A_g$ where A_s is the acceleration due to spin and A_g is the acceleration due to gravity. The acceleration due to spin is found using the equation $A_s = cS_a \times V$, where c is the surface constant that takes into consideration the surface of the ball, S_a is the spin vector and V is the velocity vector, and is found by setting up the cross product. Its matrix will look like this:

$$\begin{vmatrix} i & j & k \\ 0 & cS & 0 \\ V_x & V_y & V_z \end{vmatrix} = (cSV_z)i + (-cSV_x)k$$

The first row represents the i, j and k vectors defined as $i = (1,0,0)$, $j = (0,1,0)$ and $k = (0,0,1)$. The second row represents the contribution of the spin vector, where S is measured in revolutions per second. The third row represents the velocity at which the ball is traveling through the air. Since we are working with three dimensions, we have velocities in each of the x-, y- and z-directions.

From this cross product, we can conclude that the acceleration in the x-direction (that is, horizontal) is given by $dV_x/dt = cSV_z(t)$, the acceleration in the z-direction (vertical) is $dV_z/dt = -cSV_x(t) - g$, and the acceleration in the y-direction (side to side) is $dV_y/dt = 0$. We subtract gravity in the second equation because only vertical acceleration is affected by it.

Serves without spin.

Using our equation for acceleration, we know that $dV_x/dt = cSV_z(t)$ and $dV_z/dt = -cSV_x(t) - g$. In this case, our equations are going to become very simple because considering the fact that there is no spin, S is 0, the derivative of V_x is 0 and the derivative of V_z is $-g$. We know that this is true because gravity is the only factor affecting acceleration and it is only in the vertical direction.

To solve for our velocities in both directions, we must integrate their two equations, add in their constants, and solve for V_x and V_z. In the case of this real life problem, the constants are going to be the initial conditions. Integrating, we find that $V_x =$ constant and $V_z = -gt +$ constant. The initial conditions then give $V_x = V_0(\cos\theta)$ and $V_z = V_0(\sin\theta) - gt$, where θ is the angle the ball is served at and V_0 is the initial velocity the ball is served at.

To solve for the positions in both directions, we integrate the two equations for velocity and add in the initial conditions for position to get that $P_x = V_0(\cos\theta)t$ and $P_z = V_0(\sin\theta)t - gt^2/2$.

Serves with spin.

Figuring out the equations for the serves with spin is a bit more complicated because the spin constant is not zero anymore. To solve for these equations, we will use differential equations instead. What we are going to do is take another derivative of our original equation for vertical velocity: $dV_z/dt = -cSV_x(t) - g$. The second derivative is $d^2V_z/dt^2 = -cS \cdot dV_x/dt = -cS \cdot cSV_z(t) = -c^2S^2V_z(t)$.

Using this equation and the following initial conditions, we are able to solve for our velocity functions. Our initial conditions for this portion are (a) $V_x(0) = X_{init}$, the horizontal velocity the ball is serve at, which is given by $X_{init} = V_0 \cos\theta$ where V_0 is the initial velocity and θ is the angle the ball is served at; (b)$V_z(0) = Z_{init}$, the vertical velocity the ball is served at, which is given by $Z_{init} = V_0 \sin\theta$; (c) $dV_x(0)/dt = cSV_z(0) = cSZ_{init}$, the initial conditions for the horizontal acceleration; and (d) $dV_z(0)/dt = -cSV_x(0) = -cSX_{init}$, the initial conditions for the vertical acceleration. Solving the differential equations, the two new functions for the horizontal and vertical velocities are

$$V_x(t) = -\frac{g}{cS} + \frac{(g + cSX_{init})\cos(cSt)}{cS} + Z_{init}\sin(cSt)$$

and

$$V_z(t) = Z_{init}\cos(cSt) - \frac{(g + cSX_{init})\sin(cSt)}{cS}.$$

The initial conditions for the position are (a) $P_x(0) = 0$, the ball's initial distance traveled and (b) $P_z(0) = h_0$, the ball's initial height at the time of the serve. We integrate the velocity functions to find the equations for the horizontal and vertical position:

$$P_x(t) = \frac{Z_{init} - gt - Z_{init}\cos(cSt) + X_{init}\sin(cSt)}{cS} + \frac{g\sin(cSt)}{c^2S^2}$$

and

$$P_z(t) = h_0 + \frac{-X_{init} + X_{init}\cos(cSt) + Z_{init}\sin(cSt)}{cS} + \frac{g\cos(cSt) - g}{c^2S^2}.$$

Physical conditions.

Now that we have our equations ready to apply to a real life situation, we must consider the physical conditions in which we are experimenting. The most important factors to concentrate on are the physical restrictions of the volleyball court and net. After it is served, the first thing the ball must do is clear the net, which is 7.33 feet high. However, we must also take the ball's size into consideration. The ball is approximately 8.28 inches in diameter and, because we want it to go over the net without touching it, we will require the center of the ball to go at least 7 feet 9 inches off the ground when it passes over the net (or it would hit the net and the serve would be a failure). The second point we must consider is the ceiling height. Although outdoor players do not necessarily need to worry about this, the indoor players do. For our purposes, we are going to assume that the ceiling is 22 feet 9 inches high (15 feet above the net). The third thing to consider is the length of the court. The court size is 29.5 feet by 29.5 feet on either side, giving the serve a restriction of 59.0 feet in horizontal length. The last point, though a minor one, is the constraint that the serve peaks in at most 2 seconds, because if it takes any longer than that, the ball is either going to hit the ceiling or land out of bounds. In general, we are looking for serves that are between 7 feet 9 inches and 22 feet 9 inches high when they reach the net and land in the court (within the court's length of 59 feet).

Our constant c (the surface constant) is derived using the fact that it would probably be proportional to the surface constant of a baseball, which is estimated at 0.005. Taking into consideration the different texture of the surface, due to softer leather and the lack of laces, we estimate the surface area constant c to be 0.015. We will estimate our spin constant S to be 15 revolutions per second. This spin constant will determine how much pressure will be put on the ball due to the fact that the ball is spinning in the opposite direction that it is traveling. In our problem, the ball will be at an initial height of 6 feet when served.

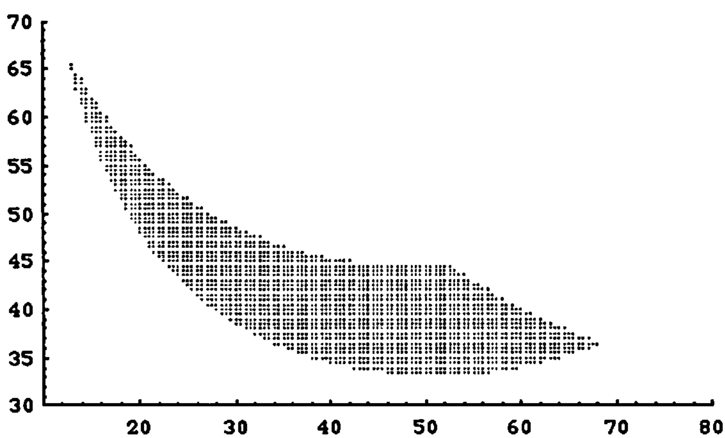

Figure 1.

To find our range of successful serves we develop two graphs by taking a very large range of velocities and angles and computing where they will land on the ground and at what height they will cross the net. We then take these values and eliminate those serves which either hit the net, hit the ceiling, land out of bounds, or do not reach their maximum height within 2 seconds. Figure 1 shows those successful serves with spin where the x-axis is the angle (in degrees) and the y-axis is the initial velocity at which the ball was served. Figure 2 is a similar plot of successful serves without spin.

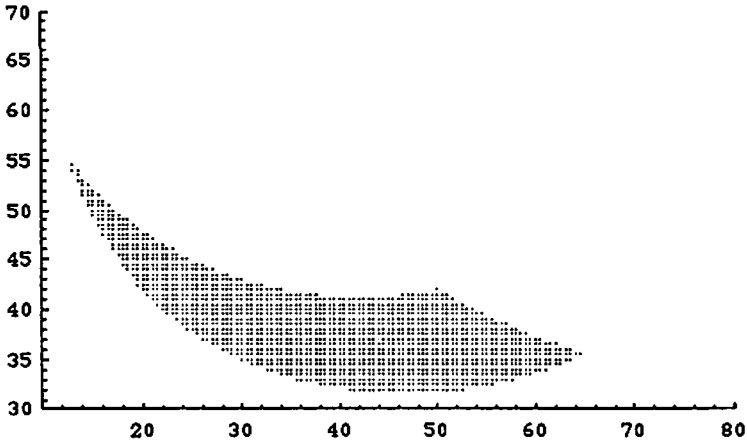

Figure 2.

Conclusions from the graphs.

The two graphs seem very similar at first glance because of their shapes. However, the differences they have are just what we would expect and do mean a great deal in the sport of volleyball. Some of the conclusions that can be made from the graphs are as follows.

(1) By comparing the two graphs, it is obvious that the serves with spin can be served at much higher speeds than those without spin. With spin, we can have serves as fast as 65.5 ft/sec while without spin the fastest serve is 54.5 ft/sec. This is true because the added spin on the ball makes the serve drop faster as it travels through the air. In effect, a serve that would have landed out of bounds without spin drops much faster with spin and now lands in bounds. In volleyball, the faster the serve the better because it gives the opponent less time to react. A serve with an initial angle of 13 degrees can be served 11 ft/sec faster when spin is added, which is a 20% improvement and is very significant given that we are only playing on a 59 foot court. However, because we are adding spin, we do have to hit the ball harder. The slowest serve we could hit

with spin had a velocity of 33.5 ft/sec, but without spin we could hit one as slow as 32 ft/sec. From these observations, it is obvious that adding spin to a serve is a definite improvement.

(2) By finding the area of each graph (in effect counting the successful serves), we can see which type of serve is less sensitive. The serve with the greatest number of serves in its graph is less sensitive because it is less likely that the serve will fail. In the case of our two figures, Figure 2 had only 1461 successful serves while Figure 1 had 1880 successful serves, which is more than a 28% improvement. We are able to conclude that the serve with spin is less sensitive.

(3) We are also able to use the graphs to see which specific velocities and angles account for the greatest number of successful serves. Examining serves without spin, we see that balls served at an angle of 50 degrees have the widest range of successful serves (horizontal) or lowest sensitivity to initial conditions, which means that if someone were to serve the ball at this angle, they could serve it anywhere between 32 and 42 ft/sec and it would still land in the court. Using a similar analysis, we determined that the initial velocities which have the largest range of acceptable angles were 35.5 ft/sec and 36 ft/sec.

(4) Examining serves with spin in the same manner, we see that balls served between 46 and 53 degrees have the widest range of success with velocities (a range of 33.5 to 43.5 ft/sec) and that serves at 36.5 or 37 ft/sec have the largest range of successful angles.

Consistency plays a major role in volleyball. A volleyball player who is able to consistently serve successfully is much more valuable than a player who occasionally serves successfully but usually fails. By serving within our optimal range of serves, we are guaranteed a greater chance that the serve will be successful, which in turn will make a more valuable volleyball player. Because the range of successful serves is smaller for the serve without spin, it makes sense that a player would want to serve using spin.

This project took a real life situation, applied mathematical tools and theory so that we could study the results and make conclusions to improve the situation. In this specific case, we concentrated on taking a normal serve and changing it by adding spin. From this change we were able to see if, in the long run, it was actually going to be an improvement. Using differential equations, we took a situation that was dependent on velocity, angle, and time and put it into a mathematical model. From there, we analyzed the model's acceleration, velocity, and position functions, and were then able to experiment and calculate the serves that were most successful according to the constraints that were set by the playing rules, restrictions, and conditions.

There are many opportunities for further study using this project. For example, we concentrate on the most successful range of serves, but what is the range of serves, or the serve, that takes the least time to reach the other side of the court? What are the equations for the curves on the graphs and why do they have that shape? What effect will air friction have on these serves? If we vary and test different spins put on the serves, what would the optimal spin be? We are assuming that the serve is straight into the other court, but what would happen if the server turned 26.56 degrees and was now facing the opposite corner of the court? What would our optimal serves be? These are only a few of the questions that we could raise. The point is that using mathematical tools such as differential equations, we are able to answer these questions and make even more conclusions.

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Some Comments on the Field Extension $Q(\sqrt[n]{\alpha})$.

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In this note we concern ourselves with a simplification of some results from field extensions. Our inspiration was problems of everyday mathematics. The problem we began with was: the equality $\lambda+\sqrt{\alpha} = \lambda'+\sqrt{\alpha'}$ (where α, α', λ, λ' are positive rational numbers and $\sqrt{\alpha}$, $\sqrt{\alpha'}$ are irrational) implies $\alpha=\alpha'$ and $\lambda=\lambda'$. This result is easily generalized when the rank of the radicals is 3 or 4; but there is a question of whether such equalities are concluded in the general case of $\lambda+\sqrt[n]{\alpha}=\lambda'+\sqrt[n]{\alpha'}$. We think that the subject is of interest to secondary-school teachers as well as college students as they often meet problems of this kind; for instance, solving polynomial equations or studying irrational expressions.

The so-called Abel's theorem and its generalization by Vahlen-Capelli's theorem as well as the theorem of the "simple algebraic field extensions" (see [2], theorems 295, 427, 428 or [1], theorem 3 on page 215) may give an answer. Nevertheless it is more convenient for us to use elementary tools to the extent that this is possible. We are familiar with the tools of European secondary education and presume the definitions of elementary structures such as group, ring, field, vector space, the elementary properties of determinants, and the Euclidean algorithm for integers and polynomials in one variable x. In particular, we use the fact that if the polynomials $a(x)$ and $b(x)$ are prime to each other, there are polynomials $c(x)$ and $d(x)$ such that $a(x)\cdot c(x)+b(x)\cdot d(x)=1$.

As usual, Q is the set of rational numbers (Q^+ the set of positive rational numbers), $Q[x]$ is the ring of polynomials in one variable x and $Q(\vartheta)$ is the field generated by ϑ over Q. We call *primitive n-th root of unity* every root of unity which does not satisfy any equation $x^k-1=0$

for $1 \le k < n$. A *monic polynomial* is a polynomial in which the coefficient of the greatest x's power is 1.

§1. The equality $\lambda + \sqrt[n]{\alpha} = \lambda' + \sqrt[n]{\alpha'}$ with n prime.

Throughout this section, we symbolize by ∂ the radical $\sqrt[n]{\alpha}$.

Theorem 1. If α, α', λ, λ' belong to Q^+, n is a prime and α, α' are not n-powers of rationals, then the equation $\lambda + \sqrt[n]{\alpha} = \lambda' + \sqrt[n]{\alpha'}$ implies $\alpha = \alpha'$ and $\lambda = \lambda'$.

We begin the proof with the following lemmas.

Lemma 1. The numbers ∂^k, with $k \in \{1, 2, \ldots, n-1\}$, are irrational.

Proof. Since $(n, k) = 1$, there exist integers s, t such that $sn + tk = 1$. Thus $\partial = \partial^1 = \partial^{sn} \cdot \partial^{tk} = (\partial^n)^s \cdot (\partial^k)^t$. If $\partial^k \in Q$, then $\partial \in Q$, which is a contradiction.

Lemma 2. The polynomial $x^n - \alpha$, where n is a prime, $\alpha \in Q^+$ and it is not an n-power of a positive rational, is a monic polynomial over Q which has ∂ as a root and is of minimum degree.

Proof. The polynomial $h(x) = x^n - \alpha$ is irreducible over Q. In fact, its zeros are the numbers $j^k\partial$, where j is a primitive nth root of unity and $k \in \{0, 1, 2, \ldots, n-1\}$. If $h(x) = p(x)q(x)$, where $p(x)$ and $q(x)$ belong to $Q[x]$, then each of them would be factorized into

$$(x - j^{\lambda_1}\partial) \cdot (x - j^{\lambda_2}\partial) \cdot \cdots \cdot (x - j^{\lambda_\tau}\partial)$$

where $\lambda_i < n$, $i \in \{1, 2, \ldots, \tau\} \subset \{0, 1, 2, \ldots, n-1\}$. But such an expression is not a polynomial over Q, since at least the constant term is not rational because of Lemma 1. So, if $r(x) \in Q[x]$ has ∂ as a zero and is of degree $< n$, then it is relatively prime to $h(x)$ and there exist two polynomials $f_1(x)$ and $f_2(x)$ such that $f_1(x)h(x) + f_2(x)r(x) = 1$, which fails at $x = \partial$. Contradiction.

Lemma 3. If n is a prime, the elements $1, \partial, \ldots, \partial^{n-1}$ constitute a basis of $Q(\partial)$.

Proof. Let E be the set of all the sums of the form $\sum_{k=0}^{n-1} \alpha_k \partial^k$ where $\alpha_k \in \mathbb{Q}$. We prove that $E = \mathbb{Q}(\partial)$.

Clearly, $E \subseteq \mathbb{Q}(\partial)$. Thus it remains to be proved that E is a field and that $\mathbb{Q} \subseteq E$. First, we see that every element of the form $\sum_{k=0}^{n} \alpha_k \partial^k$, $\partial \in E$, can be written as $\sum_{k=0}^{n-1} \alpha_k \partial^k$ since $\partial^n = \alpha$. Note that for any $A, B \in E$, we have $A + B \in E$ and $A \cdot B \in E$. It remains to be shown that $A^{-1} \in E$.

Let $A = \sum_{k=0}^{n-1} \alpha_k \partial^k$ and $g(x) = \sum_{k=0}^{n-1} \alpha_k x^k$. By Lemma 2, $g(x)$ and $h(x) = x^n - \alpha$ are prime to each other over $\mathbb{Q}$. So there exist polynomials $f_1(x)$ and $f_2(x)$ such that

$$f_1(x)g(x) + f_2(x)h(x) \;=\; 1. \tag{1}$$

Since $g(\partial) = A$, by (1) we have $f_1(\partial) \cdot A = 1$. Since $f_1(\partial) \in E$, then $A^{-1} = f_1(\partial)$. Hence $\mathbb{Q}(\partial) \subseteq E$.

Corollary to Lemma 3. If n is a prime, then each element of $\mathbb{Q}(\partial)$ can be written uniquely in the form $g(\partial)$ where $g(x) \in \mathbb{Q}[x]$.

Proof of the theorem. According to the proof of Lemma 3, every element of $\mathbb{Q}(\partial)$ can be written in a unique way as

$$g(\partial) \;=\; \sum_{k=0}^{n-1} \alpha_k \partial^k \qquad (\text{where } \alpha_k \in \mathbb{Q}). \tag{2}$$

Suppose $\lambda + \sqrt[n]{\alpha} = \lambda' + \sqrt[n]{\alpha'}$. Then $\sqrt[n]{\alpha'} = (\lambda - \lambda') + \sqrt[n]{\alpha}$ and so

$$\begin{aligned} \alpha' \;=\; & (\lambda - \lambda')^n + n(\lambda - \lambda')^{n-1}\partial + \frac{n(n-1)}{2}(\lambda - \lambda')^{n-2}\partial^2 + \\ & \cdots + \frac{n(n-1)}{2}(\lambda - \lambda')^2 \partial^{n-2} + n(\lambda - \lambda')\partial^{n-1} + \alpha \end{aligned}$$

where $\partial = \sqrt[n]{\alpha}$. Now, by (2) and the fact that α and α' are rational, $\alpha' = (\lambda - \lambda')^n + \alpha$ (constant terms) and $\lambda - \lambda' = 0$ (radical terms). Thus, $\lambda = \lambda'$ and $\alpha = \alpha'$.

Corollary 1. If n is a prime and ∂ (as above) is a zero of a polynomial $p(x)$ over $\mathbb{Q}$, then the polynomial is divisible by $h(x) = x^n - \alpha$.

Proof. The polynomial $h(x)$ has the $j^k \partial$ as zeros, where j is a primitive root of unity and $k \in \{0, \ldots, n-1\}$. If $p(x)$ is the polynomial, then

$$p(x) \;=\; h(x) \cdot g(x) + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \cdots + A_1 x + A_0 \,,$$

where $A_k \in \mathbb{Q}$, $0 \le k \le n-1$. Since $p(\vartheta) = 0$, $A_k = 0$ for all k and it follows that $p(j^k\vartheta) = 0$.

Corollary 2. If n is prime and $\mu + \vartheta$ ($\mu \in \mathbb{Q}$ and ϑ as above) is a zero of a polynomial over $\mathbb{Q}$, then the polynomial is divisible by $(x-\mu)^n - \alpha$.

Proof. We apply Corollary 1 for $p(x-\mu)$. The polynomial has the $\mu + j^k\vartheta$ as zeros.

We give another sufficient condition for $\sqrt[n]{\alpha} = \vartheta$ in order for the set $\{1, \vartheta, \ldots, \vartheta^{n-1}\}$ to constitute a basis and for the equation $\lambda + \sqrt[n]{\alpha} = \lambda' + \sqrt[n]{\alpha'}$ to imply $\alpha = \alpha'$ and $\lambda = \lambda'$.

Definition 1. A radical $\sqrt[n]{\alpha}$ ($\alpha \in \mathbb{Q}^+$ and $\sqrt[n]{\alpha} \notin \mathbb{Q}$) is called an *irreducible radical* if it cannot be written in the form $\sqrt[m]{\beta}$ with $\beta \in \mathbb{Q}^+$ and $m < n$.

Proposition 1. If $\sqrt[n]{\alpha} = \vartheta$ (where $\alpha \in \mathbb{Q}^+$) is irreducible, the numbers $1, \vartheta, \ldots, \vartheta^{n-1}$ are linearly independent.

Proof. By Lemmas 2 and 3, the only thing that has to be proved is that ϑ^k, $k \in \{1, \ldots, n-1\}$, are irrational.

Suppose that $n = p^c$ where p is a prime and $c \in \mathbb{N}$.

If $(k, p^c) = 1$, then the proof is as in Lemma 1.

If $k = tp^{c_1}$, $(t, p) = 1$ and $c_1 < c$, we have

$$\sqrt[p^c]{\alpha^{tp^{c_1}}} = \sqrt[p^q]{\alpha^t}$$

where $q = c - c_1$. For $\sqrt[p^q]{\alpha}$ irrational, $(t, p^q) = 1$, there are integers A, B such that $tA + Bp^q = 1$ and

$$\sqrt[p^q]{\alpha} = \left(\sqrt[p^q]{\alpha^t}\right)^A \left(\sqrt[p^q]{\alpha^{p^q}}\right)^B$$

For $\left(\sqrt[p^c]{\alpha}\right)^k$ rational, $\alpha^{tp^{c_1}/p^c} = \alpha^{t/p^{c_1}} = \sqrt[p^{c_1}]{\alpha^t}$ is rational, which is absurd.

If $\sqrt[p^q]{\alpha} = \beta \in \mathbb{Q}$, then $\alpha = \beta^{p^q}$ and $\sqrt[p^c]{\alpha} = \beta^{p^q/p^c} = \beta^{1/p^{c_1}} = \sqrt[p^{c_1}]{\alpha}$, and the element $\sqrt[p^c]{\alpha}$ would not be irreducible.

Inductively the proposition is proved if $n = p_1^{c_1} \cdot \cdots \cdot p_\tau^{c_\tau}$ with p_i prime, $i \in \{1, 2, \ldots, \tau\}$, and c_i nonzero natural numbers.

Thus Proposition 1 and Corollaries 1 and 2 may be generalized by replacing the supposition "n is prime" by the supposition "$\sqrt[n]{\alpha}$ is an irreducible radical." Moreover, we remark that Vahlen-Capelli's condition (see [2], theorem 428) fulfills the above condition.

§2. The case of many radicals.

In this section we attempt a generalization of the previous results referring to two or more radicals. Throughout this section we consider the symbols $\partial = \sqrt[n]{\alpha}$ and $h = \sqrt[m]{\beta}$, where α and β are positive rational numbers, ∂ and h irrational, and i, j are the primitive nth and mth roots of unity, respectively. $Q(\partial, h)$ is the smallest field containing Q and the radicals ∂ and h. We use (m, n) for the gcd (greatest common divisor) of m and n.

Proposition 2. If $(m,n) = 1$ and all $\partial^\sigma h^\tau$ (where $0 \le \sigma \le n-1$, $0 \le \tau \le m-1$ and $(\sigma,\tau) \neq (0,0)$) are irrationals, then $Q(\partial, h) = Q(\partial h)$.

Proof. Our first step is to prove that every element of the field $Q(\partial, h)$ of the form $A_{\sigma\tau}\partial^\sigma h^\tau$ can be written in the form $A_k(\partial h)^k$ for some $k \in \{0, 1, \dots, mn-1\}$. In fact, we define integers y and z such that

$$P = A_{\sigma\tau}\partial^\sigma h^\tau = A_{\sigma\tau}\sqrt[mn]{\alpha^{\sigma m}\beta^{\tau n}} = A\sqrt[mn]{\alpha^{\sigma m + mny}\beta^{\tau n + mnz}}$$

where $A \in Q$. It is enough to define y and z such that $P = A(\partial h)^k$, or

$$\sigma m + mny = mk \text{ and } \tau n + mnz = nk$$

or

$$\sigma + ny = k = \tau + mz.$$

Because $(m,n) = 1$, the equation $ny - mz = \tau - \sigma$ has an integral solution, so k may be defined. Thus $Q(\partial, h) \subseteq Q(\partial h)$, while the relation $Q(\partial h) \subseteq Q(\partial, h)$ is obvious.

Proposition 3. If $(m,n) = 1$ and the numbers $\partial^\sigma h^\tau$ are irrational for $0 \le \sigma \le n-1$, $0 \le \tau \le m-1$ and $(\sigma,\tau) \neq (0,0)$, then the set $B = \{(\partial h)^k : 0 \le k \le nm-1\}$ constitutes a basis for $Q(\partial, h)$.

Proof. By Proposition 2 every element $A_{\sigma\tau}\partial^\sigma h^\tau$ can be written in the form $A_k(\partial h)^k$, $k \in N$. According to the method we have already developed, we prove the following:

(1) for every $1 \leq k \leq nm-1$, $(\partial h)^k$ is irrational,

(2) $x^{mn} - \alpha^m \beta^n$ is the minimal polynomial, and

(3) the field $\mathbb{Q}(\partial h)$ coincides with the ring $\mathbb{Q}[\partial h]$.

Relation (1) is evident by the supposition, while the proof of (2) goes as in Lemma 2. The elements $(i^\tau \partial j^k h)^{mn}$, $1 \leq \tau \leq n-1$ and $1 \leq k \leq m-1$, are zeros of the polynomial $h(x) = x^{mn} - \alpha^m \beta^n$, hence if $h(x)$ is reducible (that is, $h(x) = p(x)q(x)$ with $p(x)$ and $q(x)$ in $\mathbb{Q}[x]$), then each of these polynomials on the right side will be factorized (in a proper extension) in the form

$$(x - i^{\lambda_1}\partial)(x - i^{\lambda_2}\partial)\cdots(x - i^{\lambda_r}\partial)(x - j^{\mu_1}h)(x - j^{\mu_2}h)\cdots(x - j^{\mu_q}h)$$

where $1 \leq \lambda_\alpha \leq n$ for $1 \leq \alpha \leq n$, $1 \leq \mu_\beta \leq m$ for $1 \leq \beta \leq m$ and $(\lambda_\alpha, \mu_\beta) \neq (n,m)$. In such a polynomial at least the constant term is not rational.

Relation (3) is proved as in Lemma 3. The number k satisfies $1 \leq k \leq nm-1$ and all the elements of $\mathbb{Q}(\partial h)$ are of the form

$$\sum_{k=0}^{nm-1} \alpha_k (\partial h)^k$$

since for $k = nm$ we have $(\partial h)^{mn} = \alpha^m \beta^n$ and the elements for $k > mn$ are formed respectively. So, every element of the form $\sum \alpha_{\sigma\tau} \partial^\sigma h^\tau$, where $0 \leq \sigma \leq n$, $0 \leq \tau \leq m$, $(\sigma,\tau) \neq (0,0)$ and $(\sigma,\tau) \neq (n,m)$, is an element of the form $\sum_{k=0}^{nm-1} A_k (\partial h)^k$ and if $\sum_{k=0}^{nm-1} A_k (\partial h)^k = 0$, then $A_k = 0$ for all k; hence the set B constitutes a basis.

Proposition 4. If $\lambda + \sqrt[n]{\alpha} + \sqrt[m]{\beta} = \lambda' + \sqrt[n]{\alpha'} + \sqrt[m]{\beta'}$, where all the letters are rational positive numbers, the radical expressions are irrational, $(n,m) = 1$ and $(\sqrt[nm]{\alpha^m \beta^n})^k$ for $0 < k \leq nm-1$ are also irrational, then $\alpha = \alpha'$, $\beta = \beta'$ and $\lambda = \lambda'$.

The proof is evident.

Corollary 3. If a polynomial has the element $\sqrt[n]{\alpha}\,\sqrt[m]{\beta}$ as a zero (with the restrictions of Proposition 2), then it is divisible by the polynomial $x^{nm} - \alpha^m \beta^n$.

The proof is as in Corollary 1 of §1.

Proposition 5. If $(m,n) = 1$, ∂ and h have the restrictions of Proposition 2, and $\partial + h$ is a zero of a polynomial p over $\mathbb{Q}$, then it has the elements $i^\sigma \partial + j^\tau h$, where $0 \leq \sigma \leq n$ and $0 \leq \tau \leq m$, as zeros.

Proof. Consider the polynomial

$$\varphi(x) = (x-\vartheta-h)(x-\vartheta-jh)\cdots(x-\vartheta-j^{m-1}h)$$
$$\times(x-i\vartheta-h)(x-i\vartheta-jh)\cdots(x-i^{n-1}\vartheta-j^{m-1}h)$$

We suppose for an instant that $\varphi(x)\in Q[x]$ and then by the division algorithm

$$p(x) = \varphi(x)p_1(x)+A_{nm-1}x^{nm-1}+A_{nm-2}x^{nm-2}+\cdots+A_1x+A_0$$

where $p_1(x)\in Q[x]$ and $A_0, A_1,\ldots, A_{nm-1}\in Q$. The equation $\varphi(x)$ for $x=\vartheta+h$ gives

$$A_{nm-1}(\vartheta+h)^{nm-1}+A_{nm-2}(\vartheta+h)^{nm-2}+\cdots+A_1(\vartheta+h)+A_0 = 0$$

and thus $A_i=0$ for all i. So the problem has been reduced to proving that the polynomial $\varphi(x)$ has rational coefficients. As the calculation is very long, for the sake of the continuity of the subject we prove it in Appendix 1 (at the end of this article).

We come now to the natural conclusion.

Theorem 2. If α_i, $i\in\{1,\ldots,k\}$, are rational numbers, every $h_i=\sqrt[n_i]{\alpha_i}$ is an irrational number, all n_i's are prime to each other and the products $h_1^{\tau_1}\cdot h_2^{\tau_2}\cdots h_k^{\tau_k}$ are irrational numbers, for $0\le\tau_i\le n_i$, $(\tau_1,\tau_2,\ldots,\tau_k)\ne(n_1,n_2,\ldots,n_k)$ and $(\tau_1,\tau_2,\ldots,\tau_k)\ne(0,0,\ldots,0)$, then the set $(h_1h_2\cdots h_k)^\tau$, with $0\le\tau\le n_1n_2\cdots n_k-1$, constitutes a basis for the vector space $Q(h_1,\ldots,h_k)$ over Q.

This theorem is an inductive conclusion of Proposition 3 since we have $Q(h_1,h_2,\ldots,h_k)=Q(h_1,\ldots,h_{k-1})(h_k)$ and finally $Q(h_1,h_2,\ldots,h_k) = Q(h_1h_2\cdots h_k)$.

§3. Applications.

1. Proposition 5 gives us a great number of new zeros of a polynomial over Q. For example, if $\sqrt[3]{2}+\sqrt[5]{2}$ is a zero of $p(x)\in Q[x]$, then we know 14 more zeros of it; and conversely we can find the minimum polynomial having the above number as a zero.

2. Likewise, by Proposition 4, a radical cannot be a linear combination of other radicals of positive rational numbers provided that

the indices of all the radicals are prime to one another. (We assume that the radicals are not rational numbers.) So $\sqrt[3]{2}$ is not a linear combination of square radicals of positive rational numbers.

3. Proposition 2 implies that the natural numbers α, whose n-th roots $\sqrt[n]{\alpha}$, for n prime, are irrational numbers, cannot have the same decimal part. Proposition 4 says that this holds even in the case of more radicals with the supposition that the indices of the radicals are prime to one another. In the latter case, if $(m,n)=1$ and $\sqrt[n]{\alpha}$, $\sqrt[m]{\beta}$ are irrational numbers, the number $\sqrt[n]{\alpha}-\sqrt[m]{\beta}$ cannot be an integer.

4. Theorem 1 permits us to write the fractions $p(\vartheta)/q(\vartheta)$, where $\vartheta=\sqrt[n]{\alpha}$, n prime, $\alpha \in \mathbb{Q}^+$, $\sqrt[n]{\alpha}\notin \mathbb{Q}$, and $p(x)$, $q(x)\in \mathbb{Q}[x]$, free of radicals in the denominator. In fact in such a case, let $h(x)=x^n-\alpha$. Because ϑ is not a zero of $q(x)$, $(h(x),q(x))=1$, so there exist $f_1(x)$, $f_2(x)\in \mathbb{Q}[x]$ such that $h(x)f_1(x)+q(x)f_2(x)=1$. Hence $q(\vartheta)f_2(\vartheta)=1$ and $p(\vartheta)/q(\vartheta)=p(\vartheta)f_2(\vartheta)$. Of course, we would say that every fraction of the form $p(\vartheta)/q(\vartheta)$ that is an element of $\mathbb{Q}(\vartheta)$ is always written as a polynomial of the form $\sum_{k=0}^{n-1}\alpha_k\vartheta^k$.

5. We now refer to a determinant having many similarities with the so-called "cyclic determinant." A cyclic determinant is a determinant of the form

$$\begin{vmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n-1} & \alpha_1 & \cdots & \alpha_{n-3} & \alpha_{n-2} \end{vmatrix}.$$

It is used (c.f. [2], pages 567 and 733) in the finding of normal bases in the algebraic extensions of a field. The new determinant is related to the zeros of $x^n-\alpha$ as the former was to the zeros of x^n-1.

Since the calculations here are also extensive, we prove the propositions in Appendix 2 (at the end of this article).

As always, n is a prime, $\alpha\in\mathbb{Q}$, $\sqrt[n]{\alpha}\notin\mathbb{Q}$ and $\sqrt[n]{\alpha}=\vartheta$.

We turn our attention to this determinant in order to prove with elementary tools the independence of the elements $1,\vartheta,\ldots,\vartheta^{n-1}$. We consider the equation

$$\sum_{k=0}^{n-1}\alpha_k\vartheta^k = 0 \qquad (1)$$

with $\alpha_k\in\mathbb{Q}$. By multiplying successively the two sides of (1) by ϑ, ϑ^2, $\ldots$, ϑ^{n-1}, there results a system with ϑ^0, ϑ^1, $\ldots$, ϑ^{n-1} as unknowns, whose determinant is the determinant D which we are working on below.

Let $\alpha_0, \ldots, \alpha_{n-1} \in \mathbf{Q}$, ϑ as above. Put

$$D = \begin{vmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & \alpha_0 \\ \alpha_{n-2} & \alpha_{n-3} & \cdots & \alpha_0 & \alpha_{n-1}\alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_1 & \alpha_0 & \cdots & \alpha_3\alpha & \alpha_2\alpha \\ \alpha_0 & \alpha_{n-1}\alpha & \cdots & \alpha_2\alpha & \alpha_1\alpha \end{vmatrix}.$$

a) Let $i_1, i_2, \ldots, i_n$ denote all the n-th roots of unity. Then

$$D = \prod_{k=1}^{n} (\alpha_0 + i_k\vartheta\alpha_1 + \cdots + i_k^{n-1}\vartheta^{n-1}\alpha_{n-1}) \tag{2}$$

or

$$D = \prod_{k=1}^{n} (\alpha_1 + \alpha_2 i_k\vartheta + \cdots + \alpha_0 i_k^{n-1}\vartheta^{n-1})$$

and generally

$$D = \prod_{k=1}^{n} (\alpha_{\tau-1} + \alpha_\tau i_k\vartheta + \cdots + \alpha_{\tau-2} i_k^{n-1}\vartheta^{n-1}) \tag{2'}$$

for all $\tau \in \{0, \ldots, n-1\}$.

b)

$$D = \sum_{k=0}^{n-1} \alpha_k\vartheta^k \cdot \begin{vmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & 1 \\ \alpha_{n-2} & \alpha_{n-3} & \cdots & \alpha_0 & \vartheta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_1 & \alpha_0 & \cdots & \alpha_3\alpha & \vartheta^{n-1} \\ \alpha_0 & \alpha_{n-1}\alpha & \cdots & \alpha_2\alpha & \vartheta^n \end{vmatrix}.$$

c) If ϑ_k is one of the n-th roots of α,

$$f_0(\vartheta_k) = \alpha_0 + \alpha_1\vartheta_k + \alpha_2\vartheta_k^2 + \cdots + \alpha_{n-1}\vartheta_k^{n-1},$$

$$f_1(\vartheta_k) = \alpha_{n-1}\alpha + \alpha_0\vartheta_k + \cdots + \alpha_{n-2}\vartheta_k^{n-1}, \ldots, \text{ and}$$

$$f_{n-1}(\vartheta_k) = \alpha_1\alpha + \alpha_2\alpha\vartheta_k + \cdots + \alpha_0\vartheta_k^{n-1},$$

then

$$f_\tau(\vartheta_k) = \vartheta_k^\tau f_0(\vartheta_k) \text{ and } \sum_{k=0}^{n-1} f_\tau(\vartheta_k) = n\alpha_{n-1}\alpha \text{ for } 0 \le \tau \le n-1.$$

Appendix 1.

If $(m,n)=1$, α and β in $\mathbb{Q}^+$, $\sqrt[n]{\alpha}=\partial$ and $\sqrt[m]{\beta}=h$ are not rational and i and j are primitive n-th and m-th respectively roots of unity, then the polynomial

$$\varphi(x)=(x-\partial-h)(x-\partial-jh)\cdots(x-\partial-j^{m-1}h)$$
$$\times(x-i\partial-h)(x-i\partial-jh)\cdots(x-i^{n-1}\partial-j^{m-1}h)$$

belongs to $\mathbb{Q}[x]$.

Proof. Since

$$(x-i^k\partial-h)(x-i^k\partial-jh)\cdots(x-i^k\partial-j^{m-1}h) = (x-i^k\partial)^m-\beta,$$
$$\varphi(x) = \prod_{k=0}^{n-1}\Big((x-i^k\partial)^m-\beta\Big), \quad m<n. \tag{1}$$

We develop the right hand side of (1) and write it in the form $\alpha_0+\alpha_1\beta+\cdots+\alpha_{n-1}\beta\alpha^{n-1}$; the coefficient α_0 is evidently rational, while the coefficient $\alpha_{n-\tau}$ of $\beta^{n-\tau}$, $0<\tau<n$, is given by

$$\alpha_{n-\tau}=\sum_{\sigma_k\in\widetilde{\sigma}}(-1)^\tau(x-i^{k_1}\partial)^m(x-i^{k_2}\partial)^m\cdots(x-i^{k_\tau}\partial)^m \tag{2}$$

where $(k_1,k_2,\ldots,k_\tau)=\sigma_k$ is a combination of τ numbers from the set $N_n=\{0,1,\ldots,n-1\}$ and $\widetilde{\sigma}$ is the set of all these combinations.

We prove that all the powers of ∂ in the right side of (2), with the exception of the term $(x^m)^\tau$, equal zero. To this end we consider a term containing, say, the number ∂^τ, $0<\tau<nm$. More precisely, we take the summand

$$(x-i^{k_1}\partial)^m(x-i^{k_2}\partial)^m\cdots(x-i^{k_\tau}\partial)^m \tag{3}$$

and we consider that ∂^τ arises by multiplying the numbers $\partial^{\tau_1},\ldots,\partial^{\tau_\rho}$, that is

$$\partial^\tau = \partial^{\tau_1}\cdots\partial^{\tau_\rho}, \tag{4}$$

each of the latter numbers having been picked up from one parenthesis of (2). The corresponding coefficient of ∂^τ contains a power of i while the rest is irrelevant to the process we propose and thus we omit it.

Let us now assume that the coefficient of ∂^τ contains the number

$$i^{\lambda_1\tau_1+\lambda_2\tau_2+\cdots+\lambda_\rho\tau_\rho}. \tag{5}$$

In such a case we say that the number ∂^τ defined by (4) corresponds to the ρ-ple $(\lambda_1, \lambda_2, \ldots, \lambda_\rho)$. We will show that if we consider all the summands of (2) we can pick up ∂^k's such that the sum of the powers of the corresponding i is zero. All ∂^τ are defined by (4). To systematize the notation, put for every $k \in \mathbb{N}$

$$k = \overline{k} \pmod n,\ 0 \le \overline{k} < n. \tag{6}$$

Assume that corresponding to the ∂^τ ρ-ple is $(\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_\rho)$. We take the ρ-ples

$$(\overline{1+\lambda_1}, \overline{1+\lambda_2}, \ldots, \overline{1+\lambda_\rho}),\ (\overline{2+\lambda_1}, \overline{2+\lambda_2}, \ldots, \overline{2+\lambda_\rho}),$$
$$\ldots,\ (\overline{n-1+\lambda_1}, \overline{n-1+\lambda_2}, \ldots, \overline{n-1+\lambda_\rho}) \tag{7}$$

and we form the respective i's powers

$$i^{(\overline{1+\lambda_1})\tau_1 + (\overline{1+\lambda_2})\tau_2 + \cdots + (\overline{1+\lambda_\rho})\tau_\rho},$$
$$i^{(\overline{2+\lambda_1})\tau_1 + (\overline{2+\lambda_2})\tau_2 + \cdots + (\overline{2+\lambda_\rho})\tau_\rho}, \ldots,$$
$$i^{(\overline{n-1+\lambda_1})\tau_1 + (\overline{n-1+\lambda_2})\tau_2 + \cdots + (\overline{n-1+\lambda_\rho})\tau_\rho} \tag{8}$$

It is evident that from the summands of (2) we could form the numbers ∂^τ corresponding to the ρ-ples given by (7). In order to prove that the sum of (8) is zero, we only have to prove that all these powers of i are different mod n. In fact, let us suppose that

$$\left((\overline{k+\lambda_1})\tau_1 + (\overline{k+\lambda_2})\tau_2 + \cdots + (\overline{k+\lambda_\rho})\tau_\rho\right)$$
$$-\left((\overline{k'+\lambda_1})\tau_1 + (\overline{k'+\lambda_2})\tau_2 + \cdots + (\overline{k'+\lambda_\rho})\tau_\rho\right) = n\sigma,$$

where $0 \le k < n$, $0 \le k' < n$ and σ is an integer. Then

$$(\overline{k+k'})(\tau_1 + \tau_2 + \cdots + \tau_\rho) = n\sigma.$$

Since n is a prime and $\tau < n$ because of (1), the last relation is absurd.

Appendix 2.

Let D be the determinant of application 5 in §3, $\partial = \sqrt[n]{\alpha}$, n prime, and $i_1, i_2, \ldots, i_n$ denote all the n-th roots of unity. We now demonstrate statements (a), (b) and (c).

(a) Put

$$p_k = \alpha_0 + i_k\partial\alpha_1 + \cdots + i_k^{n-1}\partial^{n-1}\alpha_{n-1}$$

for $1 \le k \le n$ and add to the last column of D, $i_k^{n-1}\partial^{n-1}$ times the first column, $i_k^{n-2}\partial^{n-2}$ times the second column, ... , and $i_k\partial$ times the last but one column. Then the last column consists of the polynomials

$$p_k,\ i_k\partial p_k,\ i_k^2\partial^2 p_k,\ \ldots,\ i_k^{n-1}\partial^{n-1}p_k\,.$$

Hence p_k is a factor of D; furthermore $p_1, p_2, \ldots, p_n$ are factors of D. Since each p_k is linear with respect to $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, and thus irreducible, $p_1p_2\cdots p_n$ divides D. Hence $D = cp_1p_2\cdots p_n$ and by comparing the terms of the two sides, $c = 1$.

We change the above process into the following: the addition takes place not with respect to the last column but with respect to another column, e.g. the last but one, and we add to this column $i_k^{n-2}\partial^{n-2}$ times the first, $i_k^{n-3}\partial^{n-3}$ times the second, ... and $i_k^{n-1}\partial^{n-1}$ times the last column. Then the determinant D changes into the form

$$D = \prod_{k=1}^{n}(\alpha_1 + \alpha_2 i_k\partial + \cdots + \alpha_0 i_k^{n-1}\partial^{n-1})$$

as desired.

b) Let $t = \sum_{k=0}^{n-1}\alpha_k\partial^k$. Then multiplication by $\partial, \partial^2, \ldots, \partial^{n-1}$ gives the system of equations

$$\begin{aligned} \alpha_{n-1}x_{n-1} + \cdots + \alpha_1 x_1 + \alpha &= t \\ \alpha_{n-2}x_{n-1} + \cdots + \alpha_0 x_1 + \alpha_{n-1}\alpha &= t\partial \\ &\vdots \\ \alpha_0 x_{n-1} + \cdots + \alpha_2\alpha x_1 + \alpha_1\alpha &= t\partial^{n-1} \end{aligned}$$

with the $(n-1)$-ple $(\partial, \partial^2, \ldots, \partial^{n-1})$ as a solution. Hence the augmented determinant $\widetilde{D}$ of this system of equations equals zero. Write $\widetilde{D}$ in the form

$$\widetilde{D} = D - t\cdot\begin{vmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_1 & 1 \\ \alpha_{n-2} & \alpha_{n-3} & \cdots & \alpha_0 & \partial \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_1 & \alpha_0 & \cdots & \alpha_3\alpha & \partial^{n-2} \\ \alpha_0 & \alpha_{n-1}\alpha & \cdots & \alpha_2\alpha & \partial^{n-1} \end{vmatrix}.$$

and the proof is over.

(c) The first result is evident. On the other hand, since

$$\sum_{\tau=1}^{n} \vartheta_k^{\tau} = 0$$

for $0 \leq \tau \leq n$, the first part of (1) implies that

$$\sum_{k=1}^{n} f_0(\vartheta_k) = \sum_{k=1}^{n}\left(\alpha_0 + \alpha_1\vartheta_k + \alpha_2\vartheta_k^2 + \cdots + \alpha_{n-1}\vartheta_k^{n-1}\right)$$
$$= n\alpha_0 + \alpha_1 \sum_{k=1}^{n} \vartheta_k + \alpha_2 \sum_{k=1}^{n} \vartheta_k^2 + \cdots + \alpha_{n-1} \sum_{k=1}^{n} \vartheta_k^{n-1} = n\alpha_0$$

and for $1 \leq \tau \leq n-1$ that

$$\sum_{k=1}^{n} f_\tau(\vartheta_k) = \sum_{k=1}^{n}\left(\alpha_{n-\tau}\alpha + \alpha_{n-\tau+1}\alpha\vartheta_k + \cdots + \alpha_{n-\tau-1}\vartheta_k^{n-1}\right)$$
$$= n\alpha_{n-\tau}\alpha.$$

This means that if the values of one of f_τ are known for all ϑ_k, then all the values of all the other f_τ's are also known and our last equations give the values of $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$.

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[1] Goldstein, L. *Abstract Algebra*, Prentice-Hall International, London, 1973.

[2] Redei, L. *Algebra*, Vol. I, Pergamon Press, Oxford, 1967.

[3] Stewart, I. *Galois Theory*, Chapman and Hall, London, 1973.

in memoriam

Sister Adrienne Eickman

1937-1994

The Problem Corner

Edited by Kenneth M. Wilke

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before 1 January 1996. Solutions received after the publication deadline will be considered also until the time when copy is prepared for publication. The solutions will be published in the Spring 1996 issue of *The Pentagon*, with credit being given to student solutions. Affirmation of student status and school should be included with solutions. Address all communications to Kenneth M. Wilke, Department of Mathematics, 275 Morgan Hall, Washburn University, Topeka, Kansas 66621.

PROBLEMS 485-489.

Problem 485. Proposed by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin. Find the sum of the following infinite series.

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m\binom{2m}{m}}$$

Problem 486. Proposed by T. Yau, Pima Community College, Tucson, Arizona. Consider the Smarandache function $S(n)$ which is defined as the smallest integer such that $S(n)!$ is divisible by n. Find the maximum of $S(n)/n$ over all positive composite integers $n \neq 4$.

Problem 487. Proposed by the editor. Suppose that the sides of triangle ABC are all integers. If the measure of angle A is four times the measure of angle B, find the smallest possible integer lengths for the sides of triangle ABC.

Problem 488. Proposed by the editor. Prove or disprove that

$$\sqrt{5}+\sqrt{26+2\sqrt{17}}=\sqrt{13+2\sqrt{38}}+\sqrt{18-2\sqrt{38}+2\sqrt{65-10\sqrt{38}}}\,.$$

Problem 489. Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri. The Pell numbers P_n and their associated numbers Q_n satisfy the relations $P_{n+2}=2P_{n+1}+P_n$, $P_0=0$, $P_1=1$; and $Q_{n+2}=2Q_{n+1}+Q_n$, $Q_0=1$, $Q_1=1$. Show that (a) $P_{n+1}=(Q_n+Q_{n+1})/2$ and (b) $Q_{n+1}=P_n+P_{n+1}$.

Please help your editor by submitting problem proposals.

SOLUTIONS 475, 476, 478 and 479.

Problem 477 remains open.

Editor's Comment. The Alma Problem Solving Group, Alma College, Alma, Michigan was inadvertently omitted from the list of solvers of Problems 471, 473 and 474 in the preceding column. The editor apologizes for this oversight.

Problem 475. Proposed by Francis E. Masat, Glassboro State College, Glassboro, New Jersey. Let n and $n+2$ be positive integers. Prove that n and $n+2$ are both prime numbers if and only if

(a) $$\sigma(n) = \phi(n+2)$$
or
(b) $$\sigma(n(n+2))\phi(n(n+2)) = (n^2+2n+1)(n^2+2n+3).$$

In this problem, $\phi(n)$ denotes Euler's Phi function which gives the number of integers less than n and which are relatively prime to n. Two integers a and b are relatively prime if $\gcd(a,b)=1$. $\sigma(n)$ denotes the Sigma function which denotes the sum of all the divisors (including 1 and n), of the integer n.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Part (a) (1). If n and $n+2$ are both prime numbers, then $\sigma(n) = \phi(n+2)$. By hypothesis, n and $n+2$ are both prime numbers, thus 1 and n are the only positive integer divisors of n and $1,2,\ldots,n+1$ are the only positive integers less than or equal to $n+2$ which are also relatively prime to $n+2$. Therefore, $\sigma(n) = n+1 = \phi(n+2)$.

Part (a) (2). If $\sigma(n) = \phi(n+2)$, then both n and $n+2$ are prime numbers. By hypothesis, $\sigma(n) = \phi(n+2)$. Since $\sigma(1) = 1 \neq 2 = \phi(3)$, $n > 1$. Thus $n+1 \leq \sigma(n) = \phi(n+2) \leq (n+2)-1 = n+1$, so $\sigma(n) = n+1$ and $\phi(n+2) = n+1$. Because $\sigma(n) = n+1$, n is a prime number. Since $\phi(n+2) = n+1$, $n+2$ is also a prime number. Thus both n and $n+2$ are prime numbers.

Part (b) (1). If n and $n+2$ are both prime numbers, then $\sigma(n(n+2))\phi(n(n+2)) = (n^2+2n+1)(n^2+2n-3)$. By hypothesis, both n and $n+2$ are prime numbers and, as such, n and $n+2$ are also relatively prime. Then since σ is a multiplicative function, $\sigma(((n)(n+2)) = \sigma(n)\sigma(n+2) = (n+1)(n+3)$. Also since ϕ is a multiplicative function, $\phi((n)(n+2)) = \phi(n)\phi(n+2) = (n-1)(n+1)$. Combining these, we have $\sigma(n(n+2))\phi(n(n+2)) = (n^2+2n+1)(n^2+2n-3)$.

Part (b) (2). If $\sigma(n(n+2))\phi(n(n+2)) = (n^2+2n+1)(n^2+2n-3)$, then both n and $n+2$ are prime numbers. By hypothesis, $\sigma(n(n+2))\phi(n(n+2)) = (n^2+2n+1)(n^2+2n-3)$ and n is a positive integer. Since $\sigma(3)\phi(3) = 4\cdot 2 = 8 \neq 0 = (1^2+2\cdot 1+1)(1^2+2\cdot 1-3)$, $n \neq 1$ and so $n > 1$. We shall need the following lemmas.

Lemma 1. Let $n > 1$ be an integer. Then $\sigma(n)\phi(n) < n^2$.

Proof. See Theorem 329 on page 267 of Hardy and Wright's *An Introduction to the Theory of Numbers*, Fourth Edition, Oxford University Press, 1960.

Lemma 2. Let $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ where $p_1 < p_2 < \cdots < p_k$ are prime numbers and $a_1, a_2, \ldots, a_k$ and k are positive integers. If $\sigma(n)\phi(n) = n^2 - 1$ then $n = p_1 p_2 \cdots p_k$.

Proof. Since σ and ϕ are multiplicative functions and since $\sigma(n)\phi(n) = n^2 - 1$, we have that the product

$$\left(\frac{p_1^{a_1+1}-1}{p_1-1}\cdot\frac{p_2^{a_2+1}-1}{p_2-1}\cdot\ldots\cdot\frac{p_k^{a_k+1}-1}{p_k-1}\right)$$

$$\cdot\left(p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}.\frac{p_1-1}{p_1}.\frac{p_2-1}{p_2}\cdots\frac{p_k-1}{p_k}\right) = p_1^{2a_1}p_2^{2a_2}\cdots p^{2a_k}-1.$$

Hence

$$(p_1^{a_1+1}-1)(p_2^{a_2+1}-1)(p_k^{a_k+1}-1)p_1^{a_1-1}p_2^{a_2-1}p_k^{a_k-1}$$

$$= p_1^{2a_1}p_2^{2a_2}\cdots p^{2a_k}-1$$

If $a_i > 1$ for some integer i, $1 \le i \le k$, then p_i divides the left side of this equality but not the right, which is impossible. Thus $a_1 = a_2 = \cdots = a_k = 1$, making $n = p_1p_2\cdots p_k$.

Lemma 3. Let $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ where $p_1 < p_2 < \cdots < p_k$ are prime numbers and $a_1, a_2, \ldots, a_k$ and k are positive integers. If $\sigma(n)\phi(n) = n^2 - 1$ then n is a prime number.

Proof. It follows from Lemma 2 that $n = p_1p_2\cdots p_k$. Then since $\sigma(n)\phi(n) = n^2-1$, we have $(p_1+1)(p_2+1)\cdots(p_k+1)(p_1-1)(p_2-1)\cdots(p_k-1) = p_1^2p_2^2\cdots p_k^2-1$. Thus $(p_1^2-1)(p_2^2-1)\cdots(p_k^2-1) = p_1^2p_2^2\cdots p_k^2 - 1$. We shall show that if k is an integer ≥ 2, then $(p_1^2-1)(p_2^2-1)\cdots(p_k^2-1) < p_1^2p_2^2\cdots p_k^2-1$. Since $p_2 > p_1 \ge 2$, $p_1^2+p_2^2 > 2$ makes $-(p_1^2+p_2^2) < -2$. Hence $p_1^2p_2^2 - (p_1^2+p_2^2)+1 < p_1^2p_2^2 - 2 + 1 = p_1^2p_2^2 - 1$. It follows that $(p_1^2-1)(p_2^2-1) < p_1^2p_2^2 - 1$, so the claimed result holds for $k = 2$ and for $k > 2$ it now follows by mathematical induction. Therefore $k = 1$ and n is a prime number.

Lemma 4. If n is a prime number, then $\sigma(n)\phi(n) = n^2-1$.

Proof. Since n is a prime number, $\sigma(n) = n+1$ and $\phi(n) = n-1$ and the result follows by multiplication.

An easy corollary is "Let $n > 1$ be a positive integer. Then $\sigma(n)\phi(n) \le n^2-1$ with equality if and only if n is a prime number."

Lemma 5. If $\sigma(n(n+2))\phi(n(n+2)) = (n^2+2n+1)(n^2+2n-3)$, then $\gcd(n, n+2) = 1$.

Proof. Let $d = \gcd(n, n+2)$. Then $d \mid (n+2-n)$ so $d \mid 2$. Thus either $d = 1$ or $d = 2$. If n is even, then $2 \mid n$ and $2 \mid n+2$ so that $4 \mid n(n+2)$. Then since $\phi(n)$ is even for all integers $n > 2$ and $n(n+2) \ge 4$ whenever n is an even positive integer, we have $\phi(4) = 2$ and $\phi(4) \mid \phi(n(n+2))$. Thus $2 \mid \phi(n(n+2))$. But $(n^2+2n+1)(n^2+2n-3) = (n-1)(n+1)^2 \cdot (n+3)$ and if n is even, then each of $(n-1)$, $(n+1)$ and $(n+3)$ is odd. This contradiction shows that n must be odd and thus $\gcd(n, n+2) = 1$.

We are now ready to establish the final result. By hypothesis, $\sigma(n(n+2))\phi(n(n+2)) = (n^2+2n+1)(n^2+2n-3) = (n-1)(n+1)^2(n+3)$. From Lemma 5, $\gcd(n, n+2) = 1$. Then since both σ and ϕ are multiplicative functions, we have $\sigma(n(n+2))\phi(n(n+2)) = \sigma(n)\sigma((n+2) \cdot \phi(n)\phi(n+2) = (n-1)(n+1)^2(n+3)$. Now if either n or $n+2$ is not a prime number, then correspondingly either $\sigma(n)\phi(n) < n^2-1 = (n-1)\cdot(n+1)$ or $\sigma(n+2)\phi(n+2) < (n+2)^2-1 = (n+1)(n+3)$. This contradiction shows that both n and $n+2$ must be prime numbers.

Also solved by Charles Ashbacher, Cedar Rapids, Iowa; Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin (second solution); and the proposer.

Editor's Comment. Prielipp's second solution was much shorter due to its reliance on the following known result "A positive integer m is the product of two primes differing by 2 if and only if $\sigma(m)\phi(m) = (m+1)(m+3)$." This is Problem 317 in the *Canadian Mathematical Bulletin* and was proposed by W.A. Mullin; see Rabinowitz, *Index to Mathematical Problems 1980-1984*, Math Pro Press, Westford, Massachusetts (1992), page 210.

Problem 476. Proposed by J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin. Around equilateral triangle ABC, circumscribe rectangle PBQR (as shown in Figure 1 below). In general each side of triangle ABC cuts off a right triangle from the rectangle. Prove that the sum of the areas of the two smaller right triangles equals the area of the larger right triangle.

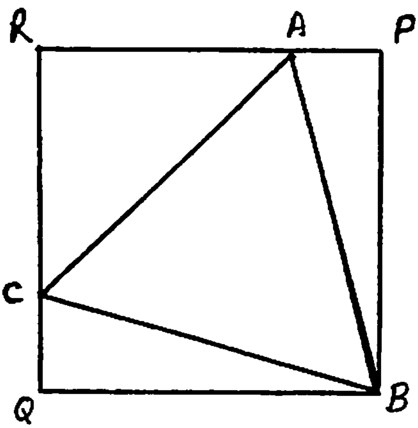


Figure 1.

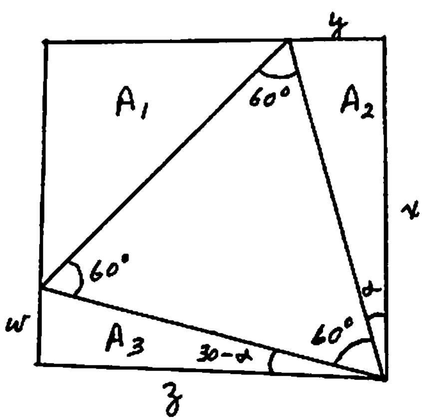


Figure 2.

Solution by Fred Horein, Albion College, Albion, Michigan.

In Figure 2 above, let the length of the sides of the equilateral triangle be one. Let A_1, A_2 and A_3 denote the areas of the respectively designated right triangles. Now $x = \cos\alpha$, $y = \sin\alpha$, $z = \cos(30^\circ - \alpha)$ and $w = \sin(30^\circ - \alpha)$. Then $2A_1 = (\cos(30^\circ - \alpha) - \sin\alpha)(\cos\alpha - \sin(30^\circ - \alpha))$ $= (\cos 30^\circ \cos\alpha + \sin 30^\circ \sin\alpha - \sin\alpha)(\cos\alpha - \sin 30^\circ \cos\alpha + \cos 30^\circ \cos \alpha)$ so that

$$2A_1 = \frac{1}{2}\cos\alpha\sin\alpha + \frac{\sqrt{3}}{4}\left(\cos^2\alpha - \sin^2\alpha\right) \tag{1}$$

Similarly, we have $2A_2 = \cos(30^\circ - \alpha)\sin(30^\circ - \alpha)$ and $2A_3 = \cos\alpha\sin\alpha$. Then adding and combining, we have

$$\begin{aligned} 2(A_2 + A_3) &= \cos(30^\circ - \alpha)\sin(30^\circ - \alpha) + \cos\alpha\sin\alpha \\ &= \frac{1}{2}\cos\alpha\sin\alpha + \frac{\sqrt{3}}{4}\left(\cos^2\alpha - \sin^2\alpha\right) = 2A_1 . \end{aligned}$$

The desired result follows immediately.

Also solved by Charles Ashbacher, Cedar Rapids, Iowa; Tom Chen, Albion College, Albion, Michigan; Russell Euler, Northwest Missouri State University, Maryville, Missouri; Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; and the proposer. One incorrect solution was received in which the rectangle was assumed to be a square.

Editor's Comment. For a pretty solution of this problem using rotations, see the item entitled "Mrs. Dijkstra" on pages 19-21 in Ross Honsberger's *Mathematical Gems III* published by the Mathematical Association of America (1985).

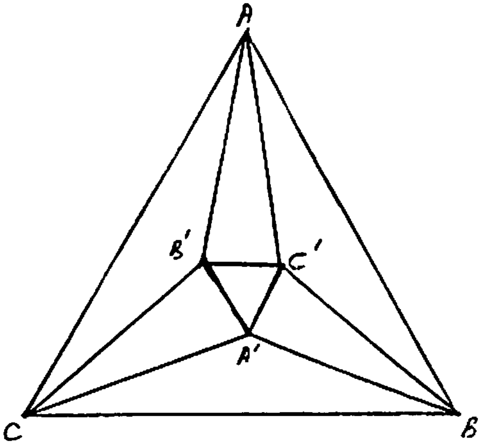
Problem 477. Proposed by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin. Let n be an integer ≥ 2. Express

$$\sum_{k=2}^{n} \binom{n}{k-2}\binom{n}{k}$$

as a binomial coefficient and prove that your equality is correct.

Since no solution has been received, this problem will remain open for another issue.

Problem 478. Proposed by J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin. The adjacent pairs of the trisectors of the angles of equilateral triangle ABC meet at the vertices of triangle A'B'C' as shown in the figure below. Find the ratio between the areas of triangles ABC and A'B'C'.



Solution by Tom Chen, Albion College, Albion, Michigan.

Let s denote a side of equilateral triangle ABC. Let q denote a side of equilateral triangle A'B'C'. Let r denote the length of each of the remaining segments in the diagram. Computing the area of triangle A'BC in two ways, the area is given by $(1/2)(r^2 \sin 140°)$ and also by $(1/2)(rs \sin 20°)$. Therefore $s = r(\sin 140°/\sin 20°)$. Similarly, the area of triangle AB'C' is given by $(1/2)(qr \sin 80°)$ and also by $(1/2)(r^2 \sin 20°)$. Therefore $q = r(\sin 20°/\sin 80°)$. The area of triangle ABC is given by $(1/2)(s^2 \sin 60°)$ and the area of triangle A'B'C' is given by $(1/2)(q^2 \sin 60°)$. The ratio of these areas is

$$\frac{s^2}{q^2} = \left(r\,\frac{\sin 140°}{\sin 20°}\right)^2 \Big/ \left(r\,\frac{\sin 20°}{\sin 80°}\right)^2 = \left(\frac{\sin 140°}{\sin 20°}\right)^2\left(\frac{\sin 80°}{\sin 20°}\right)^2 = \left(\frac{\sin 40°}{\sin 20°}\right)^2\left(\frac{\cos 10°}{\sin 20°}\right)^2$$

$$= \left(\frac{2\sin 20°\cos 20°}{\sin 20°}\right)^2\left(\frac{\cos 10°}{2\sin 10°\cos 10°}\right)^2 = \left(\frac{\cos 20°}{\sin 10°}\right)^2.$$

Also solved by Charles Ashbacher, Cedar Rapids, Iowa; Scott H. Brown, Auburn University, Montgomery, Alabama; Donovan Diede, Northern State University, Aberdeen, South Dakota; Rita Kroytor, Pace University, New York, New York; Fred A. Miller, Elkins, West Virginia;

Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; and the proposer.

Problem 479. Proposed by the Editor. Prove that

$$(\cot 63^\circ)(\cot 132^\circ) + (\cot 132^\circ)(\cot 165^\circ) + (\cot 165^\circ)(\cot 63^\circ) = 1.$$

Find all other such instances, if any exist, in which

$$(\cot A)(\cot B) + (\cot B)(\cot C) + (\cot C)(\cot A) = 1.$$

Solution by Jimmy and Sammy Yu, jointly, special students, University of South Dakota, Vermillion, South Dakota.

Let A, B and C not be multiples of π so that cot A, cot B and cot C are well defined. We shall show that if and only if $A+B+C = n\pi$, where n is an integer, then (cot A)(cot B)+(cot B)(cot C)+(cot C)(cot A)=1. In this proof we shall use the formula $\cot(B+C) = ((\cot B)(\cot C) - 1)/(\cot B + \cot C)$ where $\cot B + \cot C \neq 0$.

Necessity. If (cot A)(cot B)+(cot B)(cot C)+(cot C)(cot A) = (cot A)(cot B+cot C)+(cot B)(cot C) = 1, then $\cot A = (1-(\cot B)(\cot C))/(\cot B+\cot C) = -\cot(B+C) = \cot(-(B+C))$ and so $A = n\pi - (B+C)$, so that $A+B+C=n\pi$. Note that if cot B+cot C=0, then (cot B)(cot C)=1 so that $\cot^2 B = \cot^2 C = -1$, which is impossible. Thus $\cot B + \cot C \neq 0$.

Sufficiency. First we shall prove that $\cot B + \cot C \neq 0$. If cot B+cot C=0, then $B = m\pi - C$, where m is an integer. But $A+B+C=n\pi$ so that $A=(n-m)\pi$, which makes cot A undefined. Thus we must have $\cot B + \cot C \neq 0$. Now we have that

$$\cot A = \cot(n\pi - (B+C)) = -\cot(B+C) = -\frac{(\cot B)(\cot C) - 1}{\cot B + \cot C}$$

or

$$(\cot A)(\cot B) + (\cot B)(\cot C) + (\cot C)(\cot A) = 1$$

and this proves the result. Finally, since $A+B+C = 63^\circ+132^\circ+165^\circ = 360^\circ = 2\pi$, the desired numerical result follows.

Also solved by Rita Kroytor, Pace University, New York, New York, and Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Kappa Mu Epsilon News

Edited by Mary S. Elick, Historian

News of chapter activities and other noteworthy *KME* events should be sent to Mary S. Elick, Historian, Kappa Mu Epsilon, Mathematics Department, Missouri Southern State College, Joplin, Missouri 64801.

INSTALLATION OF NEW CHAPTERS

Mississippi Epsilon

Delta State University, Cleveland

The installation of the Mississippi Epsilon Chapter of Kappa Mu Epsilon was held on November 19, 1994, in the University Chapel on the campus of Delta State University. Dr. Arnold Hammel, National President of Kappa Mu Epsilon, conducted the installation ceremony. Professor Louise Rodgers, Kappa Mu Epsilon member initiated by Missouri Beta and currently on the faculty at Delta State served as the Conductor. Eleven students and eight faculty, in addition to Rodgers, constituted the founding group of the new chapter at Delta State. Those initiated were:

> *Students:* Elizabeth Billingslea, Danny Carpenter, David James, Debra Joel, Susan Meriwether, Kathy Mosley, Krystal Spealman, Bobbie Thompson, Jimmy Tullos, Renee Upton, and Robin Varner.
>
> *Faculty:* Diane Blansett, Kathy Griffith, Linda Miller, Paula Norris, James Potts, Beth Rogers, Rose Strahan, and Robert Waller.

Also in attendance at the 6:30 p.m. installation ceremony were: Dr. Wayne Blansett, Vice President for Student Affairs, Dr. W. Frank McArthur, Vice President for Academic Affairs, and Dr. Richard S. Myers, Dean of School of Arts and Sciences. A banquet was held following the ceremony in the Nowell Union following which Dr. Hammel gave a brief history of honor societies in colleges and universities and, in particular, the founding, development, and history of Kappa Mu Epsilon.

Officers installed during the ceremony were: Susan Meriwether, President; Holly Billingslea, Vice President; Danny Carpenter, Secretary; and Robin Varner, Treasurer. Faculty members Paula Norris and Rose Strahan accepted the responsibilities of the corresponding secretary and faulty sponsor, respectively.

CHAPTER NEWS

AL Gamma Chapter President - Jamie Tally

University of Montevallo, Montevallo

Other chapter officers: Timo Langerworf, vice president; Melissa Ellison, secretary; Aleksis Langerwolf, treasurer; Larry Kurtz, corresponding secretary; Don Alexander, faculty sponsor.

AR Alpha Chapter President - Cindy Nicholson

Arkansas State University, State University 14 actives, 9 associates

Other chapter officers: Sandy Jett, vice president; Nicole Nelson, secretary; Odis Cook, treasurer; William Paulsen, corresponding secretary/faculty sponsor.

CA Gamma Chapter President - Eric Emerton

California Polytechnic State University, San Luis Obispo 13 actives, 2 associates

The chapter sponsored an orientation meeting for new math majors and hosted a guest presentation on careers. The group also planted trees as a community project. Other chapter officers: Dylan Retsek, vice president; Eric Bauer, secretary; Emily Fisher, treasurer; John Van Eps, corresponding secretary/faculty sponsor.

CA Delta Chapter President - Gillian Robbins

California State Polytechnic University, Pomona 17 actives

Other chapter officers: Deborah Garcia, vice president; Jennifer Baird, secretary; Kevin Lum, treasurer; Richard Robertson, corresponding secretary; Jim McKinney, faculty sponsor.

CO Gamma

Chapter President - Ben Moore

Fort Lewis College, Durango 28 actives, 7 associates

CO Gamma held several planning meetings in the fall in preparation for hosting the 1995 Biennial Convention. A pizza party was held in conjunction with the induction of seven new members in November. Other chapter officers: Jody Davis, vice president; Faith Ward, secretary; Stevan Scott, treasurer; Richard A. Gibbs, corresponding secretary; Deborah Berrier, faculty sponsor.

CO Delta

Chapter President - Scott Davis

Mesa State College, Grand Junction 20 actives

Keys and certificates were presented at the fall meeting to the nine members initiated at the April 1994 initiation ceremony. Members were encouraged to plan for the 1995 Biennial Convention in Durango and a video of student presentations at the 1993 national convention was viewed. Chapter T-shirts were sold as a fund-raising activity for convention expenses. Other chapter officers: Joy E. Rayside, vice president; April R. Galyean, secretary; William J. Haworth, treasurer; Donna K. Hafner, corresponding secretary; Cliff Britton, faculty sponsor.

FL Beta

Chapter President - Tammy Causey

Florida Southern College, Lakeland

Activities during the fall semester included business meetings and a pizza party. Other chapter officers: Dane Mooney, vice president; Jennifer Harris, secretary/treasurer; Gayle Kent, corresponding secretary/faculty sponsor.

GA Alpha

Chapter President - Polly Quertermus

West Georgia College, Carrollton 25 actives

On November 18 the chapter held its Fall Social at a local restaurant. Twenty five MAT-CSC faculty, *KME* members, and guests enjoyed the event. For the seventh consecutive year, the organization sponsored a Food and Clothing Drive for the needy. The contributions obtained during the drive were delivered to the local Interfaith Help and Service Center to be distributed to those in need. Other chapter officers: Chad Bean, vice president; Chris Flournoy, secretary; Kristie Hannah, treasurer; Joe Sharp, corresponding secretary/faculty sponsor; Mark Faucette, faculty sponsor.

IL Beta Chapter President - Steve Trepachko
Eastern Illinois University, Charleston 42 actives

The first meeting of the year was held on September 20th, the program being given by Dr. Rosemary Schmalz. Her topic was "Wise Words From this Century's Mathematicians." Others who spoke at chapter meetings included Dr. Gregory Galperin and Dr. Charles Delman. The organization also enjoyed a faculty sponsored picnic in September and a Christmas Party in early December. Other chapter officers: Brittney Zupan, vice president; Jenny Wilhelmsen, secretary; Curtis Price, treasurer; Lloyd L. Koontz, corresponding secretary/faculty sponsor.

IL Zeta Chapter President - Michele Rogalski
Rosary College, River Forest 20 actives

Other chapter officers: Karen Klimara, vice president; Cheri Smith, secretary; Paul R. Coe, corresponding secretary/faculty sponsor.

IN Delta Chapter President - Chris Thielman
University of Evansville, Evansville 53 actives

Other chapter officers: Dyan Struckmeier, vice president; Denise Lynam, secretary/treasurer; Troy D. VanAken, corresponding secretary; Mohammad Azarian, faculty sponsor.

IA Alpha Chapter President - Lisa Gaskell
University of Northern Iowa, Cedar Falls 34 actives

Students presenting papers at local *KME* meetings include: Jack Dostal on "The Fast Fourier Transform," Kristina Herbers on "Motivation for the Semantic Interpretation of Material Implication," and Karen Brown on "Introduction to Dynamical Systems." Iowa Alpha is experimenting with sending its notices of events to members and faculty by e-mail. It turns out that some don't read their e-mail at all! At the initiation of three new members in early December, Andrew J. Schafer addressed those assembled on "Fundamental Applications of Hill Ciphers." The initiation ceremony was accompanied by a 10 inch snow fall, so getting back to campus was an adventure!! Other chapter officers: Dan Gruman, vice president; Jack Dostal, secretary; Chris Dix, treasurer; John S. Cross, corresponding secretary/faculty sponsor.

IA Gamma
Chapter President - Dean Stevens

Morningside College, Sioux City 8 actives

IA Gamma held one organizational meeting and continued with their tutoring program in the area high schools. Most of the year's activities are planned for the spring semester. Other chapter officers: Cara Kern, vice president; Jason Shriver, secretary; Denise Anderson, treasurer; Steven D. Nimmo, corresponding secretary/faculty sponsor.

IA Delta
Chapter President - Wendy Ahrendsen

Wartburg College, Waverly 34 actives

Program for the September meeting, entitled "Statistical Studies of Chaotic Systems," was presented by Dr. Dan Black. Dr. Lynn Olson gave the program in October; it was entitled "Learning Using Geometer's Sketchpad." Student officers Wendy Ahrendsen and Gretchen Roth presented "Math Recreations - Ideas from the Math Horizons Magazine" at the November 22nd meeting. A Christmas Party with games and refreshments was held at Dr. Glenn Fenneman's home in early December. Other chapter officers: Kelly Berkeland, vice president; Gretchen Roth, secretary; Adam Sanford, treasurer; August Waltmann, corresponding secretary; Lynn Olson, faculty sponsor.

KS Alpha
Chapter President - Andrew Buchholz

Pittsburg State University, Pittsburg 45 actives, 10 associates

The chapter held monthly meetings in October, November, and December. Ten new members were initiated at the October meeting. The meeting was preceded by a pizza party. The October program included the viewing of Mitch Richling's first place paper presentation given at the 29th Biennial Convention at Niagara University. The chapter began preliminary plans for attending the 30th Biennial Convention to be held at Ft. Lewis College in Durango, Colorado, in 1995. Dr. Cynthia Woodburn, PSU faculty member, gave the November program. Her topic was "Beyond Gaussian Elimination." In December, a special pre-final exam and pre-Christmas social was held at the home of Dr. Harold Thomas. The group viewed the presentation of the second place paper presented at the Niagara University convention. They also enjoyed several culinary delights prepared by faculty members or spouses. Other chapter officers: Bethany Schnackenberg, vice president, Zoeann Michel, secretary; Sherry Brennon, treasurer; Harold Thomas, corresponding secretary; Bobby Winters, faculty sponsor.

KS Beta

Chapter President - Jason Henry

Emporia State University, Emporia

27 actives, 7 associates

Other chapter officers: Stacey Walker, vice president; Michelle Martling, secretary; Sherry Drummond, treasurer; Connie Schrock, corresponding secretary; Larry Scott, faculty sponsor.

KS Gamma

Chapter President - Mary Kay Heideman

Benedictine College, Atchison

12 actives, 16 associates

KS Gamma officers informed new students of *KME* activities on "Club Night" at the beginning of the semester. In mid-September the group welcomed new students with a Mexican dinner in Schroll Center. Throughout the semester members have met regularly in the cafeteria to plan activities. KS Gamma again sold tickets this fall during the Casino Night of Parents' Weekend as a fund-raising project. In late November the chapter sponsored a video showing of "The Alhambra." The Christmas Wassail party was enjoyed at Jim Ewbank's home on the fourth of December. Other chapter officers: Gregory Boucher, vice president; Jodie Muhlbauer, secretary; Gerard Pineda, treasurer; Jo Ann Fellin, OSB, corresponding secretary/faculty sponsor.

KS Delta

Chapter President- Jeffrey Brown

Washburn University, Topeka

30 associates

During the fall semester the chapter members joined with the mathematics club for several activities. An afternoon picnic was held in October with volleyball and other games being played. In November Mary Wilson and Jennifer Oldham (Hudson), two past presidents of KS Delta, discussed life after Washburn University. Mary is a graduate student in statistics at Kansas State University; Jennifer is an actuary trainee at a local insurance company. Other chapter officers: Vincent Davis, vice president; Daniel Wessel, secretary; Karen Richard, treasurer; Allan Riveland, corresponding secretary; Gary Schmidt and Ron Wasserstein, faculty sponsor.

KS Epsilon

Chapter President - Jason Purdy

Fort Hays State University, Fort Hays

26 actives, 7 associates

KS Epsilon held monthly meetings and sponsored three social events: a fall picnic in September, a Halloween Party, and a Christmas Party. Other chapter officers: Jerry Braun, vice president; Sherry Kinderknecht, secretary/treasurer; Charles Votaw, corresponding secretary; Mary Kay Schippers, faculty sponsor.

KY Alpha
Chapter President - Paula Christian
Eastern Kentucky University, Richmond
20 actives, 10 associates

As a fund raiser, KY Alpha sold floppy disks to students in the computer literacy class and the Mathematica class. Chapter members, along with faculty, enjoyed an early October picnic at the Costello house. The event featured volleyball and round-robin ping-pong. In late October, Dallas Graves and Jason Nichols took the Virginia Tech Regional Math Exam, reporting it to be a tough exam this year. A panel discussion on various aspects of graduate school served as the program for the October meeting; a videotape of one of the student presentations at the 1993 National *KME* Convention was viewed at the November meeting. The Christmas Party and white elephant gift exchange was held in December. Dr. Mary Fleming brought her freshman orientation class and everyone had a great time. Dr. Fleming was especially happy because she managed to threaten students into making sure she got the "Puff the Magic Dragon" tape. Other chapter officers: John Ward, vice president; Andrea Warren, secretary; Andrea McCreary, treasurer; Pat Costello, corresponding secretary/faculty sponsor.

MD Beta
Chapter President - Robert W. Brown
Western Maryland College, Westminster
16 actives, 3 associates

The chapter sponsored a meeting for all mathematics majors featuring a well-attended talk by faculty member Dr. Robert Boner. Members also served as presiders, registrars, parking attendants, etc., for the fall meeting of the Maryland/Virginia Section of the Mathematical Association of America held on the Western Maryland campus in November. Plans are under way for a special thirtieth anniversary Mathematics Homecoming to be held on April 22; all former MD Beta members (about 235) have been invited to the event. The afternoon will feature a talk by Dr. Howard Eves, noted mathematician and historian, as well as short talks by alumni who have made careers in mathematics. The day will close with a birthday dinner and celebration. The chapter is very alive and well and quite active this year. Other chapter officers: Emily Snyder, vice president; Kari Dunn, secretary; Kathy Gaston, treasurer; James E. Lightner, corresponding secretary/faculty sponsor.

MD Delta
Chapter President - John Hughes
Frostburg State University, Frostburg
33 actives

MD Delta Chapter opened the fall semester with a picnic held jointly with the Computer Science Club at Dan's Mountain State Park. In October the group enjoyed the video, "N is a Number: A Portrait of Paul

Erdös." In November members viewed the Nova program, "Mathematical Mystery Tour." As a fund raising activity, the organization served as intermediary for students wishing to purchase Casio 7700GE calculators for their mathematics courses. Other chapter officers: Kileen Baker, vice president; Karl Streaker, secretary; Melissa Thomas, treasurer; Edward T. White, corresponding secretary; John P. Jones, faculty sponsor.

MI Beta Chapter President - Nicole Zakrajsek

Central Michigan University, Mount Pleasant 20 actives

Meetings were held every two weeks throughout the fall semester. As a fund raiser, the chapter co-sponsored a mathematics textbook sale. Several members of *KME* contributed monetarily and helped swing hammers, push wheelbarrows, etc., during the community volunteer construction of a Playscape playground in Mt. Peasant. Besides co-hosting a homecoming picnic for faculty, local members, and alumni, *KME* joined with the Actuarial Club, Gamma Iota Sigma, in riding in a float at the homecoming parade. The farm wagon of math faculty member, Bill Miller, was pulled by the 1938 John Deere B tractor owned by department chair, Rich Fleming. Professor Miller also spoke at one of our meetings on Golden Rectangles. Professors Tom Miles and Shu Ping Hodgson spoke on Careers in Mathematics and Statistics at an open house meeting. Several members are planning to attend the national convention in April. Other chapter members: Christine Riggs, vice president; Tara Kelly, secretary; Rich Lamb, treasurer; Arnold Hammel, corresponding secretary/faculty sponsor.

MS Alpha Chapter President - Nancy Piper

Mississippi University for Women, Columbus 12 actives, 2 associates

Other chapter officers: Laura Pendergest, vice president; Jill Whites, secretary; Mary-Margaret Wolff, treasurer; Jean Parra, corresponding secretary; Shaochen Yang, faculty sponsor.

MS Beta Chapter President - Larry Gariepy

Mississippi State University, Mississippi State 25 actives

Other chapter officers: Brian Parks, vice president; Bobby Jarrell, secretary; Karen Petersen, treasurer; Michael Pearson, corresponding secretary; Seth Oppenheimer, faculty sponsor.

MS Gamma Chapter President - Joong Lee

University of Southern Mississippi, Hattiesburg 27 actives, 9 associates

A video presentation, "Fractals: The Colors of Infinity," was sponsored by the chapter. The event was open to everyone on campus. Approximately thirty *KME* members and faculty attended the fall initiation and picnic, held October 25, at USM's Lake Bryon. Members and faculty competed in determining the design for the chapter's first ever chapter T-shirts. The winning design: *KME* on the front and a choice of two fractals on the back. Other chapter officers: Davor Cubranic, vice president; Kathy Jones, secretary; Alice Essary, treasurer/corresponding secretary; Barry Piazza, Jeff Stuart, Karen Thrash, Lida McDowell, faculty sponsors.

MO Beta Chapter President - Steve Shattuck
Central Missouri State University, Warrensburg 25 actives, 5 associates

MO Beta held monthly meetings during the fall semester. The September program was a presentation on "Chaos, Fractals and Dynamics - What are They?" The October initiation of eight new members featured a talk on prime numbers by one of the graduate students. In November members watched the video, "Donald in Mathmagic Land." The semester ended with a Christmas party at the home of Dr. McKee. Other chapter officers: Ann Scheffing, vice president; Mindy Eder, secretary; Chad Doza, treasurer; Rhonda McKee, corresponding secretary; Scotty Orr, Larry Dilley, Phoebe Ho, faculty sponsors.

MO Gamma Chapter President - Stephanie Pauls
William Jewell College, Liberty 10 actives

Other chapter officers: Ashley Sherman, vice president; Leann Lotz, secretary; Joseph T. Mathis, treasurer/corresponding secretary/faculty sponsor.

MO Epsilon Chapter President - Heather Warren
Central Methodist College, Fayette 15 actives

Other chapter officers: Jason Graves, vice president; Audrey Heidekreuger, secretary; Mitu Bajpayee, treasurer; William D. McIntosh, corresponding secretary; Linda O. Lembke, faculty sponsor.

MO Eta Chapter President - Douglas Cutler
Northeast Missouri State University, Kirksville 25 actives, 8 associates

MO Eta hosted a Spades tournament night for math students and division faculty. Plans were also made for hosting the 1995 Mathematics Exposition for high school students in the spring. Other chapter officers: Sarah Schwab, vice president; Joshua Aldrich, secretary; Tanya Walter, treasurer; Mary Sue Beersman, corresponding secretary; Joseph Hemmeter, faculty sponsor.

MO Theta

Chapter President - Kelly Godzwa

Evangel College, Springfield

11 actives, 3 associates

Other chapter officers: Dan Brewer, vice president; Don Tosh, corresponding secretary/faculty sponsor.

MO Iota

Chapter President - Jolena Gilbert

Missouri Southern State College, Joplin

20 actives

New officers for the year were elected at an organization meeting held in September. Meetings were held monthly featuring presentations by faculty or students. Vice president Tom Wofford gave two talks on recreational math topics; faculty member Tim Flood gave a presentation entitled "Why 1 or 58?" Once again chapter members worked concessions at home football games as a fund raising activity. The club sponsored Linette Vazquez as a homecoming queen candidate. During the Christmas shopping season members assisted Salvation Army by manning one of the bellringing stations. Also, plans were laid to provide tutoring two hours a week at a local junior high to students in the Hammonds Program. A Christmas party and white elephant exchange were held the end of the semester at the home of Dr. Juan Vazquez. Other chapter officers: Tom Wofford, vice president; Linette Vazquez, secretary; Jennifer Schumaker, treasurer; Dave Hunter, historian; Mary Elick, corresponding secretary; Charles Curtis, faculty sponsor.

MO Kappa

Chapter President - Mark Garton

Drury College, Springfield

11 actives, 6 associates

Semester activities began with a pizza and movie rush party for potential *KME* members (freshmen). The chapter made a trip to the Argonne National Laboratory for the Graduate Fair held in early October. Mark Garton won the Annual Math Contest, Calculus II and above division. Prize money was awarded to the winners at a pizza party held for all contestants. Members enjoyed a bonfire party held at the home of Dr. Allen. In conjunction with a luncheon, Kate Good gave a report on her undergraduate research project at the University of Missouri last summer. The Math Club ran a tutoring service for both the day school and the Continuing Education Division (Drury Evening College) as a money-making project. End of semester activities included a retirement party for *KME* member Dr. Stephen Rutan, who had been a member of the Drury College math faculty for over thirty years. Other chapter officers: Kate Good, vice president; Pat Roper, secretary; Jeanie Allen, treasurer; Charles Allen, corresponding secretary; Don Moss, faculty sponsor.

MO Lambda Chapter President - Dawn Powell
Missouri Western State College, St. Joseph 39 actives

Fall semester began with a get-acquainted picnic in early September. Seven students and one faculty member were initiated in October. The initiation talk was given by Dr. Keith Brandt, the new faculty initiate. Chapter participation in homecoming activities included a float in the parade and an entry in the window decorating contest. In other activities the club held a bake sale fund raiser and sponsored a booth at Family Day. Two social events, a Thanksgiving covered dish dinner and a Christmas Party, brought the semester to a close. Other chapter officers: Tracy Schemmer, vice president; Ryoko Tamoto, secretary; Henry Trammell, treasurer; John Atkinson, corresponding secretary; Jerry Wilkerson, faculty sponsor.

NE Alpha Chapter President - Michelle Roberts
Wayne State College, Wayne 26 actives

Club members once again monitored the Math-Science building evenings as a money-making project. A fall welcoming picnic was held jointly with the Math-Science faculty and with other clubs in the science building. Several of the chapter members are planning to attend the National Convention in Durango, Colorado. Other chapter officers: Darrin Brumbaugh, vice president; Robert Schultz, secretary/treasurer; Todd Koehler, historian; Fred Webber, corresponding secretary; Jim Paige and John Fuelberth, faculty sponsors.

NE Beta Chapter President - Traci Focke Elwood
University of Nebraska-Kearney, Kearney 10 actives

In mid-September the chapter sponsored a Mathematics and Medicine Symposium. The event featured a panel discussion composed of members from the medical community at the local hospital and, also, an evening banquet with a speaker from University of Nebraska Medical Center. Both events centered on the importance and necessity of having a good mathematics background. In November members helped with a Mathematics Fun Day sponsored by the local Educational Service Unit. A Food Drive for the local Food Pantry was held in December. Other chapter officers: Jennifer Sonnefeld, vice president; Beth Jorgesen, secretary; Daniel Parkison, treasurer; Charles Pickens, corresponding secretary; Peggy Miller, faculty sponsor.

NH Alpha Chapter Presidents - Sue Letendre and Tracey Thibeault
Keene State College, Keene 24 actives

NH Alpha took the initiative in revitalizing a math club to include students not qualified for *KME* membership. It is expected that this chapter and the club will act together on many projects. *KME* activities included a fall picnic and an end-of-semester social. The chapter participated in Keene's annual pumpkin festival, contributing pumpkins carved with mathematical symbols. Other chapter officers: Tammy Spearrin, vice president; Tina Haggett, secretary; Shayne Noyes, treasurer; Charles Riley, corresponding secretary; Ockle Johnon, faculty sponsor.

NY Alpha Chapter President - Martha Chong
Hofstra University, Hempstead

Chapter activities included volleyball games and an induction dinner. Other chapter officers: Jeanette Jones, vice president; Christine Kalos, secretary; Brandi York, treasurer; Aileen Michaels, corresponding secretary/faculty sponsor.

NY Eta Chapter President - Kenneth Krawczyk
Niagara University, Niagara University 15 actives

Other chapter officers: Emily Hulbert, vice president; Rebecca Bauer, secretary/treasurer; Robert Bailey, corresponding secretary; Kenneth Bernard, faculty sponsor.

NY Kappa Chapter President - Andrea Marchese
Pace University, New York 20 actives, 6 associates

Other chapter officers: Teresa Lester, vice president; Geraldine Taiani, corresponding secretary; John W. Kennedy, faculty sponsor.

OH Alpha Chapter President - Alisha Reesh
Bowling Green State University, Bowling Green

Other chapter officers: Leah Breckstein and Leah Walden, vice presidents; Kevin Kundert, secretary/treasurer; Waldemar Weber, corresponding secretary; Stephen McCleary, faculty sponsor.

OK Alpha — Chapter President - Ryan Swank

Northeastern Oklahoma State University, Tahlequah — 32 actives, 5 associates

The chapter continues to hold joint activities with the NSU student chapter of MAA. Fall initiation of nine students was held in the banquet room of a local restaurant; it was well attended by faculty and students. Receipts from the annual book sale totaled 71 dollars. The group met several times to work on problems from The Pentagon, and continues to sponsor a monthly math contest. A highlight of the semester was the Christmas "pizza party" featuring a Math Jeopardy game created by *KME* member Jana Cole. Other chapter officers: Allison Selby, vice president; Jennifer Beals, secretary/treasurer; Joan Bell, corresponding secretary/faculty sponsor.

OK Gamma — Chapter President - Terry Price

Southwestern State University, Weatherford — 20 actives

Other chapter officers: Kris Kessinger, vice president; Lori Ordway, secretary/treasurer; Wayne Hayes, corresponding secretary; Radwan Al-Jarrah, faculty sponsor.

PA Alpha — Chapter President - Melissa Napoleon

Westminster College, New Wilmington — 18 actives

The chapter hosted an ice cream social for new freshmen. All members tutored in the Learning Center, a service much appreciated by students needing help with their math courses. The chapter also sponsors a Reading Day prior to final exams. On this day pizza and soft drink study breaks are provided for all math majors. A monthly Problem Contest sponsored by the chapter features a new problem the start of each month with three weeks to come up with a solution. The prize is a large pizza at the local pizza shop. Other chapter officers: Kara Sheets, vice president; Susan Shaffer, secretary; Karin Speer, treasurer; J. Miller Peck, corresponding secretary; Carolyn Cuff and Warren Hickmann, faculty sponsors.

PA Beta — Chapter President - Rose Anne Hofmann

La Salle University, Philadelphia — 10 actives

Other chapter officers: Mary McAvoy, vice president; Jennifer Bostak, secretary; Janet Munyan, treasurer; Hugh N. Albright, corresponding secretary; Carl McCarty, faculty sponsor.

PA Gamma Chapter President - Gwen Nicklow
Waynesburg College, Waynesburg 11 actives, 5 associates

Other chapter officers: Laura Marquis, vice president; Crystal Thomas, secretary; Paul Gacek, treasurer; A. B. Billings, corresponding secretary/faculty sponsor.

PA Delta Chapter President - Ann Conflitti
Marywood College, Scranton 7 actives

Much of fall activities centered on preparation for the Annual Math Contest provided for high school students in the spring. Other chapter officers: Abigail Brace, vice president; Kim Fisher, secretary; Melissa Mang, treasurer; Sr. Robert Ann von Ahnen, IHM, corresponding secretary/faculty sponsor.

PA Epsilon Chapter President - Sheri Smucker
Kutztown University of Pennsylvania, Kutztown 9 actives

Other chapter officers: Michelle Wiley, vice president; Brandy Thiele, secretary; Karen Biesecker, treasurer; Cherry C. Mauk, corresponding secretary; Randy Schaeffer, faculty sponsor.

PA Eta Chapter President - Kristin Gieringer
Grove City College, Grove City 19 actives, 10 associates

The Spring Picnic which was rained out last spring was rescheduled for September 18. The picnic featured volleyball, hamburgers, and hotdogs. The Schlossnagels hosted the annual *KME* Christmas party at their home on December 13. The group sang carols and enjoyed refreshments provided by Mrs. Schlossnagel. Other chapter officers: Claudine Desjardins, vice president; Danielle Miller, secretary; Bryan Weet, treasurer; Marvin C. Henry, corresponding secretary; Dan Dean, faculty sponsor.

PA Iota Chapter President - Jason Baker
Shippensburg University of Pennsylvania, Shippensburg 16 actives, 14 associates

The annual Math Picnic, held in October, was jointly sponsored by Math Club and *KME*. Two talks, both well attended, were also cosponsored. In conjunction with the Adopt a Highway Program, members met one Saturday morning to take care of their part of US 11. Fall initiation was held in November at the home of Dr. and Mrs. Douglas Ensley. Their gracious hospitality is greatly appreciated by chapter members and sponsors. Other chapter officers: Angela Foltz, vice president; Todd Bittinger, secretary; Jennie Hooper-Gardner, historian; Fred Mordai, treasurer; Michael Seyfried, corresponding secretary/faculty sponsor.

PA Kappa Chapter Presidents - Joshua Wagner and Sarah Iskra
Holy Family College, Philadelphia 7 actives, 3 associates

During the fall semester, the PA Kappa members met bi-weekly for problem solving session with solved problems being submitted for publication in Math Horizons. Senior members prepared for comprehensive exams by tackling problems in the GRE math test practice booklets. A TI-85 graphing calculator session was open to the entire student body/faculty/staff, with *KME* members providing refreshments. Several meetings were also devoted to the viewing and discussion of two math videos, NCTM's "Supercalculators in the College Classroom" and "Women in Mathematics." Chapter members also staff the college math tutoring center. Other chapter officers: Leanne Majors, secretary/treasurer; Sr. Marcella Louise Wallowicz, corresponding secretary/faculty sponsor.

PA Mu Chapter President - Gerald Albright, Jr.
Saint Francis College, Loretto 33 actives, 11 associates

Seven students and two faculty attended the NCTM Regional Conference in Charleston, West Virginia, in November. Several *KME* members volunteered to help with Science Day. On this day 256 high school students from 19 high schools were on campus attending presentations, competing in science bowl, and learning about science careers. In October the chapter collected road-side litter as part of the Pennsylvania Adopt-a-Highway Program. Inductions are scheduled for February. Other chapter officers: Lisa Smith, vice president; Edward Steinbugl, secretary; Scott Beers, treasurer; Peter R. Skoner, corresponding secretary; Adrian Baylock, faculty sponsor.

SC Gamma Chapter President - Tiffany Allen
Winthrop University, Rock Hill 11 actives, 2 associates

In conjunction with the Mathematics Club, the chapter entered the Christmas decorating contest, winning first place for their decorating efforts in the math department building. This is the third consecutive year they have won. Other chapter officers: Candace Rogers, vice president; Jamie Pittman, secretary; Ronald Knox, treasurer; Donald Aplin, corresponding secretary; James Bentley, faculty sponsor.

TN Beta

Chapter President - Becky Sweeney

East Tennessee State University, Johnson City — 17 actives

The first order of business for the fall was the election of chapter officers for the 1994-95 school year. The group met to have a photo taken for the college yearbook and also sponsored the fall social at Ryan's Steakhouse. Other chapter officers: Brian Heaton, vice president; Michele Swoager, secretary; Carl Menako, treasurer; Lyndell Kerley, corresponding secretary/faculty sponsor.

TN Delta

Chapter President - Brenda Bleavins

Carson-Newman College, Jefferson City — 12 actives

Fall 1994 activities included a fall picnic and a Christmas get-together. Other chapter officers: Alexander J. Mutterspaugh, vice president; Amy Smith, secretary/treasurer; Catherine Kong, corresponding secretary/faculty sponsor.

TX Alpha

Chapter President - Curt Bourne

Texas Tech University, Lubbock — 60 actives

Other chapter officers: Nora Chang, vice president; Wes Kirk, secretary; Chuck Steed, treasurer; Edward J. Allen, corresponding secretary/faculty sponsor.

TX Eta

Chapter President - Ann Meuret

Hardin-Simmons University, Abilene — 18 actives

TX Eta Chapter members held a get-together on December 2, at which time new members present were presented their pins and shingles. Following the presentations, members watched the movies "Sneakers" and "Stand and Deliver" and enjoyed snacks and cold drinks. Other chapter officers: Jeremy Fitch, vice president; Robyn Eads, secretary; Carmen Turner, treasurer; Frances Renfroe, corresponding secretary; Charles Robinson, Ed Hewett, and Dan Dawson, faculty sponsors.

TX Kappa

Chapter President - James Davidson

University of Mary Hardin-Baylor, Belton — 15 actives, 10 associates

TX Kappa Fall Forum, held in November, featured discussions on prospective jobs and graduate schools for mathematics students. Other chapter officers: Eric Madsen, vice president; Mary Cook, secretary; Rachel McWha, treasurer; Peter H. Chen, corresponding secretary; Maxwell M. Hart, faculty sponsor.

VA Gamma Chapter President - Gerri Stultz
Liberty University, Lynchburg 25 actives

Fall activities got underway with an organizational meeting held in September. The October meeting featured a guest speaker, LU graduate Mike McLeery, who spoke on operations research. Also in October, members, alumni, and faculty enjoyed a Chili Luncheon prepared by math faculty. In December the chapter sponsored Christmas caroling at a local nursing home. The group also caroled at various math faculty homes, ending up at Dr. Rumore's home for good food and "White Christmas." Other chapter officers: Angela Bolis, vice president; Tricia Muscato, secretary; John Harrell, treasurer; Glyn Wooldridge, corresponding secretary; Sandra Rumore, faculty sponsor.

WI Gamma Chapter President - Debbie Bauer
University of Wisconsin-Eau Claire, Eau Claire 9 actives, 7 associates

Monthly meetings were held featuring one or two student speakers. The group also got together for a pizza party. Other chapter officers: Lisa Vander Missen, vice president; Bryan Kilian, secretary; Bryce Rudolph, treasurer; Tom Wineinger, corresponding secretary; Marc Goulet, faculty sponsor.

New Editor and Business Manager

The National Council is pleased to announce the appointments, effective 1 June 1995, of the next Editor and Business Manager of *The Pentagon.*

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Kappa Mu Epsilon, Mathematics Honor Society, was founded in 1931. The object of the Society is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics due to its demands for logical and rigorous modes of thought; to provide a Society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; and to disseminate the knowledge of mathematics and familiarize the members with the advances being made in mathematics. The official journal of the Society, *The Pentagon*, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the Chapters.

Active Chapters of Kappa Mu Epsilon

Listed by date of installation.

Chapter	Location	Installation Date
OK Alpha	Northeastern Oklahoma State University, Tahlequah	18 April 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Southwest Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
MS Beta	Mississippi State University, Mississippi State College	14 Dec 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
NM Alpha	University of New Mexico, Albuquerque	28 March 1935
IL Beta	Eastern Illinois University, Charleston	11 April 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 April 1937
OH Alpha	Bowling Green State University, Bowling Green	24 April 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	Central Missouri State University, Warrensburg	10 June 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
TX Beta	Southern Methodist University, Dallas	15 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 June 1941
NY Alpha	Hofstra University, Hempstead	4 April 1942
MI Beta	Central Michigan University, Mount Pleasant	25 April 1942
NJ Beta	Montclair State College, Upper Montclair	21 April 1944
IL Delta	College of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 March 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 June 1947
CO Alpha	Colorado State University, Fort Collins	16 May 1948
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952

KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 April 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	Kearney State College, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri - Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
IL Epsilon	North Park College, Chicago	22 May 1963
OK Beta	University of Tulsa, Tulsa	3 May 1964
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood College, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 April 1965
AL Epsilon	Huntingdon College, Montgomery	15 April 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
AR Alpha	Arkansas State University, State University	21 May 1965
TN Gamma	Union University, Jackson	24 May 1965
WI Beta	University of Wisconsin - River Falls, River Falls	25 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	Western Maryland College, Westminster	30 May 1965
IL Zeta	Rosary College, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Northeast Missouri State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel College, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 March 1971
KY Alpha	Eastern Kentucky University, Richmond	27 March 1971

TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 April 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 April 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State College, Joplin	8 May 1975
GA Alpha	West Georgia College, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin - Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sept 1978
IL Theta	Illinois Benedictine College, Lisle	18 May 1979
PA Mu	St. Francis College, Loretto	14 Sept 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury College, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 March 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 April 1986
TX Iota	McMurry College, Abilene	25 April 1987
PA Nu	Ursinus College, Collegeville	28 April 1987
VA Gamma	Liberty University, Lynchburg	30 April 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 April 1990
CO Delta	Mesa State College, Grand Junction	27 April 1990
NC Gamma	Elon College, Elon College	3 May 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 April 1991
SD Alpha	Northern State University, Aberdeen	3 May 1992
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 March 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994