THE PENTAGON

A Mathematics Magazine for Students

Volume 53 Number 2 Spring 1994

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The Pentagon (ISSN 0031-4870) is published semiannually in December and May by Kappa Mu Epsilon. No responsibility is assumed for opinions expressed by individual authors. Manuscripts of interest to undergraduate mathematics majors and first year graduate mathematics students are welcome, particularly those written by students. Submissions should be typewritten (double spaced with wide margins) on white paper, standard notation conventions should be respected and special symbols should be carefully inserted by hand in black ink. All illustrations must be submitted on separate sheets and drawn in black ink. Computer programs, although best represented by pseudocode in the main text, may be included as an appendix. Graphs, tables or other materials taken from copyrighted works MUST be accompanied by an appropriate release from the copyright holder permitting further reproduction. Student authors should include the names and addresses of their faculty advisors. Contributors to *The Problem Corner* or *Kappa Mu Epsilon News* are invited to correspond directly with the appropriate Associate Editor. Electronic mail may be sent to (Bitnet) PENTAGON@LIUVAX.

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Search Procedure

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Presented at the 1993 National Convention and
awarded SECOND PLACE by the Awards Committee.

Introduction.

Search theory is the study and development of procedures which optimize the probability that a lost or hidden object will be found. During World War II search theory was used to locate German submarines. In 1966 the U.S. Navy utilized search theory to find an unarmed hydrogen bomb which was lost during a refueling accident off the coast of Spain. When the Challenger space shuttle exploded, analysts used shuttle fragments found in a search to determine the cause of the accident. Searches may be complicated if the object sought is moving or evasive, or if the searcher, whether an individual or some type of equipment, does not have perfect sensory capabilities. We will assume stationary objects and perfect sensors. However, the model may be altered to allow for object movement and imperfections in sensors. We will begin with the simplest case, a two-region search.

Statement of problem.

A key is hidden either in Room A or in Room B. The probability that the key is in Room A is p and that it is in Room B is $1-p$. How should we search for the key if given a limited amount of time?

Solution.

Let $P(t)$ be the probability of finding the key in a room in t units of time. Denote by A the statement that the key is in Room A and denote by B the statement that the key is in Room B. Then $P(t \mid \mathrm{A})$ is the probability of finding the key in Room A in t units of time given that it is in Room A.

Suppose that we are limited to T units of time for the search. Let t be the amount of time spent in Room A (so $0 \le t \le T$) with the remaining $T-t$ units of time spent in Room B.

Finally, denote by $\mathcal{P}(t)$ the probability of finding the key in T units of time when t units of time are spent in Room A and the remainder of time is spent in Room B. We want to maximize the function $\mathcal{P}(t)$. Since we are assuming that the key is either in Room A or in Room B (and not in both), we can use Baye's Theorem to get

$$\begin{aligned} \mathcal{P}(t) &= P(t \mid \mathrm{A}) \cdot P(\mathrm{A}) + P(T-t \mid \mathrm{B}) \cdot P(\mathrm{B}) \\ &= P(t \mid \mathrm{A}) \cdot p + P(T-t \mid \mathrm{B}) \cdot (1-p) \end{aligned}$$

We now have $\mathcal{P}(t)$ for a general two room case. Even if we are given the value for p, we do not know the functions $P(t \mid \mathrm{A})$ and $P(t \mid \mathrm{B})$. These two functions are called "detection functions" since they describe the probability of detecting the object in a certain amount of time. In general, the most difficult part of finding an optimal search strategy is determining the exact characteristics of the detection functions. However, we can determine some of their properties.

Required properties of a detection function.

Let Room A be the room of interest.

(1) $$\lim_{t \to \infty} P(t \mid \mathrm{A}) = 1.$$

If we search long enough in a room where the key is located, we should be able to find the key.

(2) $$P(0 \mid \mathrm{A}) = 0.$$

If we spend no time searching, we will not find the key. (The key is hidden).

(3) If $t_1 > t_2$, then $P(t_1 \mid \mathrm{A}) \geq P(t_2 \mid \mathrm{A})$.

The longer we search in a room where the key is located, the better chance we have of finding it. (At least, the probability does not decrease.)

There are many functions which satisfy these requirements. We will look at some which satisfy all of the above criteria. Consider $P(t \mid \mathrm{A})$ and $P(t \mid \mathrm{B})$ defined as

$$P(t \mid \mathrm{A}) = \begin{cases} t/a & \text{if } 0 \leq t \leq a \\ 1 & \text{if } t \geq a \end{cases}$$

$$P(t \mid \mathrm{B}) = \begin{cases} t/b & \text{if } 0 \leq t \leq b \\ 1 & \text{if } t \geq b \end{cases}$$

The graphs of $P(t \mid \mathrm{A})$ and $P(t \mid \mathrm{B})$ are sketched in Figure 1.

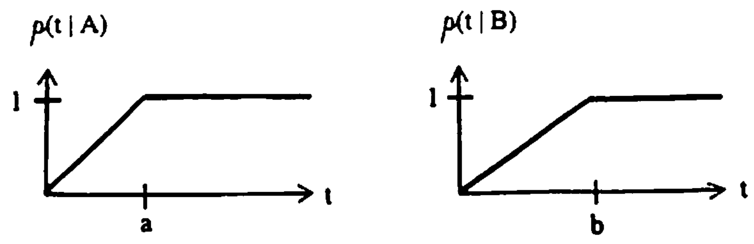

Figure 1.

Assume that $a + b > T > b > a > 0$. Then we do not have enough time to completely search both rooms. Therefore, the problem becomes one of determining how much time should be spent searching in each room. This is equivalent to maximizing $\mathcal{P}(t)$ with respect to how much time is spent in Room A. Notice that in this example the detection function for Room A appears to increase at a faster rate than the detection function for Room B. We can think of this as Room A being less cluttered than Room B.

We have $P(t \mid \mathrm{B})$, but we need $P(T - t \mid \mathrm{B})$, which is

$$P(T - t \mid \mathrm{B}) = \begin{cases} (T - t)/b & \text{if } 0 \leq T - t \leq b \\ 1 & \text{if } T - t \geq b \end{cases}$$

$$= \begin{cases} 1 & \text{if } t \le T-b \\ (T-t)/b & \text{if } T-b \le t \le T \end{cases}$$

Since we have assumed that $a+b>T$, then $a>T-b$ and we have

$$\mathcal{P}(t) = \begin{cases} \frac{t}{a}p + 1(1-p) & \text{if } 0 \le t \le T-b \\ \frac{t}{a}p + \frac{T-t}{b}(1-p) & \text{if } T-b \le t \le a \\ 1 \cdot p + \frac{T-t}{b}(1-p) & \text{if } a \le t \le T \end{cases}$$

$\mathcal{P}(t)$ is piecewise linear. Therefore, the only critical points are $T-b$ and a. Also, since the maximum value of $\mathcal{P}(t)$ occurs at one of the critical points or at an end-point of the interval, we compare the four values

$$\mathcal{P}(0) = 1-p,$$

$$\mathcal{P}(T-b) = \frac{T-b}{a}p + (1-p),$$

$$\mathcal{P}(a) = p + \frac{T-a}{b}(1-p), \text{ and}$$

$$\mathcal{P}(T) = p.$$

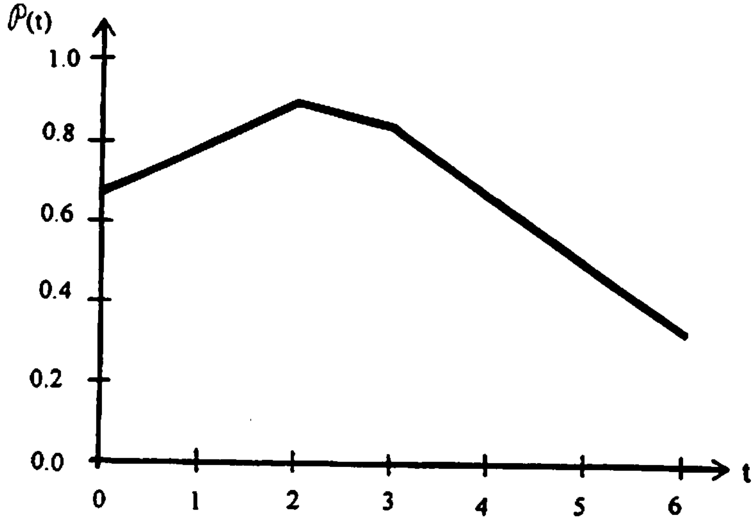

Figure 2.

As an example suppose $a = 3$, $b = 4$, $p = 1/3$ and $T = 6$ (see Figure 2). We get the following critical and end-point values for $\mathcal{P}(t)$: $\mathcal{P}(0) = 2/3$, $\mathcal{P}(T-b) = \mathcal{P}(2) = 8/9$, $\mathcal{P}(a) = \mathcal{P}(3) = 5/6$ and $\mathcal{P}(T) = \mathcal{P}(6) = 1/3$. Therefore, since $\mathcal{P}(t)$ has its maximum value at 2, our search will be optimized by spending 2 units of time in Room A and the remaining 4 units of time in Room B. If we are given 1 hour to search we should spend 20 minutes in Room A and 40 minutes in Room B to maximize the probability of finding the key.

Most search procedures that are used employ an exponential detection function of the form

$$P(t \mid \mathrm{A}) = 1 - e^{-\int_0^t \gamma_{\mathrm{A}}(\tau)\, d\tau}$$

where $\gamma_{\mathrm{A}}(\tau)$ is a positive valued, integrable function. Detection functions of this form are versatile enough to cover most search situations. Consider the special case where $\gamma_{\mathrm{A}}(\tau)$ and $\gamma_{\mathrm{B}}(\tau)$ are constant functions. If $\gamma_{\mathrm{A}}(\tau) = a$ and $\gamma_{\mathrm{B}}(\tau) = b$, we have $P(t \mid \mathrm{A}) = 1 - e^{-at}$ and $P(t \mid \mathrm{B}) = 1 - e^{-bt}$. Notice that these have all of the properties we wanted in our detection functions. In this case,

$$\begin{aligned} \mathcal{P}(t) &= P(t \mid \mathrm{A}) \cdot P(\mathrm{A}) + P(T - t \mid \mathrm{B}) \cdot P(\mathrm{B}) \\ &= (1 - e^{-at}) \cdot p + (1 - e^{-b(T-t)}) \cdot (1 - p) \end{aligned}$$

To maximize $\mathcal{P}(t)$, differentiate with respect to t to find the critical points and then compare the values of the function at these points with the end-point values.

As an example, suppose that $p = 2/3$, $a = 1/2$, $b = 1$ and $T = 3$ (see Figure 3). Then

$$\mathcal{P}(t) = (1 - e^{-t/2})(2/3) + (1 - e^{t-3})(1/3) .$$

The derivative of $\mathcal{P}(t)$ with respect to time is

$$\mathcal{P}'(t) = (1/3)e^{-t/2} - (1/3)e^{t-3} .$$

The only critical point is 2. Comparing the values

$$\begin{aligned} \mathcal{P}(0) &= (1/3)(1 - e^{-3}) \doteq 0.3167 , \\ \mathcal{P}(2) &= 1 - e^{-1} \doteq 0.6321 \text{ and} \\ \mathcal{P}(3) &= (2/3)(1 - e^{-3/2}) \doteq 0.5179 , \end{aligned}$$

we see that the probability of finding the key is maximized by spending 2 units of time in Room A and the remaining unit of time in Room B.

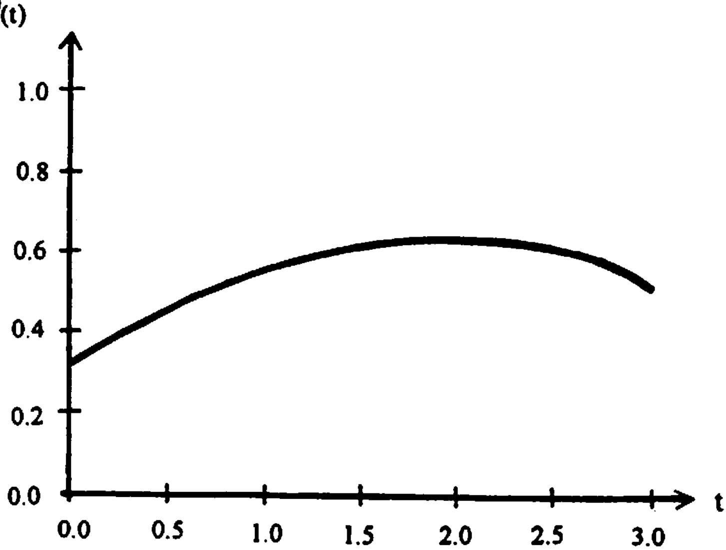

Figure 3.

This exponential detection function model can easily be modified to accommodate different values of a, b, p and T. Of course, not all search problems are restricted to the two-region case. However, the technique that we have used for two rooms can be generalized for any finite number of regions. To show how difficult things become, we present a three-region problem.

Suppose there is a lost object in one of three regions, A_1, A_2 or A_3. The probability that the object is in each region is, respectively, p_1, p_2 and $1 - p_1 - p_2$. Let A_i be the statement that the object is in Region A_i. Then the probability that the object sought is in Region A_i is $P(A_i)$. We will use the exponential detection model so that

$$P(t \mid A_i) = 1 - e^{-a_i t}.$$

Then the probability of finding the key in T units of time when t_i units of time are spent in Region A_i is

$$\begin{aligned}\mathcal{P}(t_1, t_2) &= P(t_1 \mid A_1) \cdot P(A_1) + P(t_2 \mid A_2) \cdot P(A_2) + P(t_3 \mid A_3) \cdot P(A_3) \\ &= (1 - e^{-a_1 t_1}) \cdot p_1 + (1 - e^{-a_2 t_2}) \cdot p_2 \\ &\quad + (1 - e^{-a_3(T - t_1 - t_2)}) \cdot (1 - p_1 - p_2)\end{aligned}$$

To maximize $\mathcal{P}(t_1, t_2)$, find the common zeros of the partial derivative with respect to t_1 and the partial derivative with respect to t_2 and compare the values of the function at these points with the values of $\mathcal{P}(t_1, t_2)$ on the boundary of the triangular region shown in Figure 4.

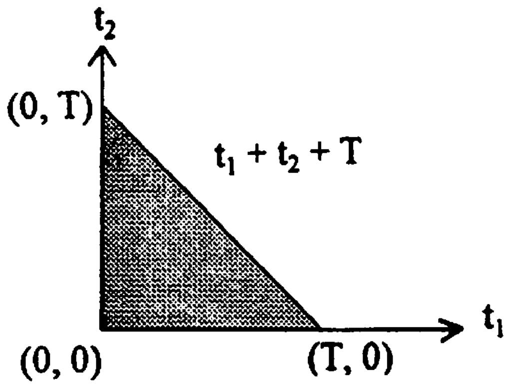

Figure 4.

As an example suppose $p_1 = 1/2$, $p_2 = 1/4$, $a_1 = 1/2$, $a_2 = 1$, $a_3 = 2$ and $T = 5$ (see Figure 5). Then $\mathcal{P}(t_1, t_2)$ is

$$(1 - e^{-t_1/2})(1/2) + (1 - e^{-t_2})(1/4) + (1 - e^{2t_1 + 2t_2 - 10})(1/4)$$

The partial derivatives of $\mathcal{P}(t_1, t_2)$ are

$$\frac{\partial \mathcal{P}(t_1, t_2)}{\partial t_1} = (1/4)e^{-t_1/2} - (1/2)e^{2t_1 + 2t_2 - 10}$$

and

$$\frac{\partial \mathcal{P}(t_1, t_2)}{\partial t_2} = (1/4)e^{-t_2} - (1/2)e^{2t_1 + 2t_2 - 10}.$$

The only critical point is $(\,(20 - \ln 4)/7,\ (10 - \ln 2)/7\,) = (x_0, y_0)$. Using the second derivative test for a real valued function of two variables, we have

$$\frac{\partial^2 \mathcal{P}(x_0, y_0)}{\partial t_1^2} \frac{\partial^2 \mathcal{P}(x_0, y_0)}{\partial t_2^2} - \left(\frac{\partial^2 \mathcal{P}(x_0, y_0)}{\partial t_1 \partial t_2} \right)^2 \doteq 0.015 > 0$$

and

$$\frac{\partial^2 \mathcal{P}(x_0, y_0)}{\partial t_1^2} \doteq -0.165 < 0 \,.$$

Therefore, $\mathcal{P}(x_0, y_0)$ is a local maximum value of $\mathcal{P}$. Since (x_0, y_0) is the only critical point in the region, we know the value of the function at this point is greater than any values attained on the boundary of the triangular region shown in Figure 4. Thus

$$\mathcal{P}\left(\frac{20 - \ln 4}{7}, \frac{10 - \ln 2}{7}\right) \doteq 0.7685$$

is the maximum value of $\mathcal{P}(t_1, t_2)$ in the time interval of interest. The probability of finding the key is maximized by spending $(20 - \ln 4)/7 \doteq 2.659$ units of time in Region A_1, $(10 - \ln 2)/7 \doteq 1.330$ units of time in Region A_2 and the remainder of time in Region A_3.

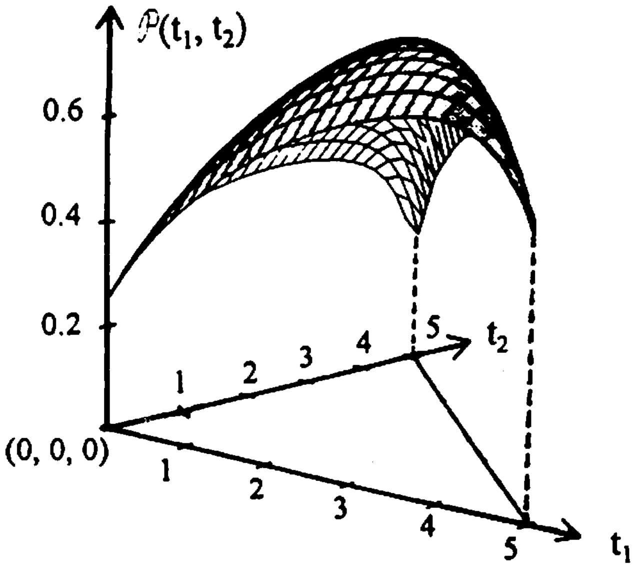

Figure 5.

In many situations an object will be lost in some region which can be divided into n sub-regions. If these sub-regions are labeled $A_1, A_2, \ldots, A_n$, then $\mathcal{P}(t_1, t_2, \ldots, t_n)$ is found for the n regions just as it was for three regions. The partial derivatives of $\mathcal{P}$ will yield an $(n-1) \times (n-1)$ system of equations which one attempts to solve to find the critical points of the probability function. An optimal search procedure is determined after the critical points are tested with partial derivatives.

Conclusion.

Optimal search plans depend not only on the probability that an object is in a certain region, but also on the detection function associated with each region. The models presented demonstrate how a search procedure may be developed to model many situations.

Acknowledgements. I would like to thank my advisor, Dr. Thomas O'Neil, for his support and guidance. Dr. O'Neil and the Cal Poly Mathematics Department encourage and inspire me a great deal. I would also like to thank Stanley Benkoski of Daniel H. Wagner, Associates for providing "A Survey of the Search Theory Literature."

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Folding the Conics

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Presented at the 1992 Region II Convention
under the title "The Hunt for the Red Parabola."

For many years, I have studied the Japanese art of paper folding known as origami. Part of origami's appeal is its simplicity. With a single piece of paper, one can create beautiful artwork. One begins with a square of paper and simply folds it. Folding the paper in various manners results in different forms.

I have always suspected the connection between origami and geometry. Origami involves folding various shapes out of a regular quadrilateral. Simple folds change the paper through an ordered series of steps into a more aesthetically pleasing shape. It is the final result, the square after a geometric transformation, that we call art.

I have often heard that one can find mathematics anywhere. Recently I discovered a construction that convinces me that everything, in its own way has some connection to mathematics. In her books, *The Joy of Mathematics* and *More Joy of Mathematics*, author Theoni Pappas explains how to fold all the conic sections using a piece of paper. In this way, we actually use origami to construct conic sections. The simplest of the constructions is the ellipse.

With a compass, construct a circle on a piece of paper with some center O. Then select any point F inside the circle. If one folds each point on the circumference of the circle to the point F, the creases are the tangent lines to an ellipse. The two foci of the ellipse are the two points

F and O. The proof that the figure formed when OX < OF is an ellipse is fairly straightforward.

Construct a circle with center O, and select a point F inside the circle. Select a point X on the circumference. Draw the line XF and the radius OX. Folding the point X to F creates the perpendicular bisector to XF, which intersects XF in midpoint M. The perpendicular bisector also intersects OX in some point P. Draw in the line FP. Because we know that MP lies on the perpendicular bisector of FX, $\angle XMP = \angle FMP$. Also segment FM is congruent to segment XM, and MP is congruent to itself, so $\triangle XMP$ and $\triangle FMP$ are congruent by SAS (see Figure 1). Then segment FP = segment XP, but because OP+PX is the radius and therefore a constant, OP+PF is constant. The definition of an ellipse is the locus of points whose total distance from the two foci is constant. We have found our ellipse, as all points P are at a constant total distance from the foci F and O.

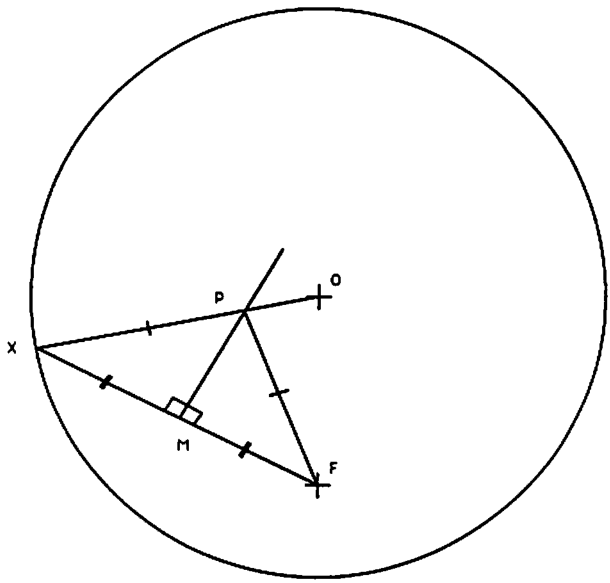

Figure 1.

It is also interesting to note that if point F is placed at the center of the circle (coincident with point O), and each point on the circumference is again folded to point F, the result is a circle with center F and exactly half the radius of the original circle. As point F gets closer to the center of the circle, point M moves closer to point P. M is always located halfway along XF, but P is merely the point in which the perpendicular bisector to XF meets OX. If F is located at the center of the circle, M is exactly coincident with P. If XF lies on OX, and the perpendicular bisector intersects XF in M, then it intersects OX in M also, which we call P. The distance FP is a constant, and the points P now describe a circle with center F with exactly half the radius of the original circle. One would expect the circle as the limiting result of an ellipse as the two foci are brought to be coincident.

Using a very similar technique, one can also fold a hyperbola. Again, begin by constructing a circle with center O. Then select any point F outside the circle. Fold each point on the circumference to the point F. The folds created by this procedure are the tangent lines to a hyperbola, which can be proven similarly.

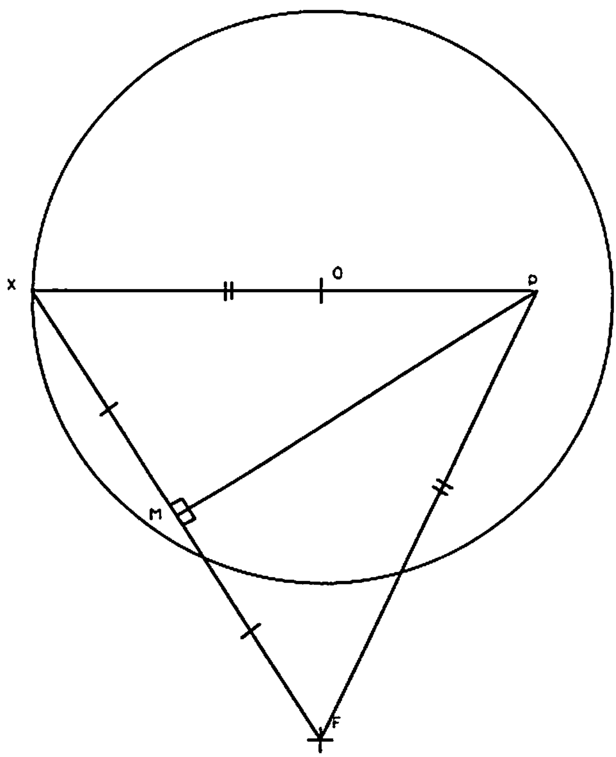

Figure 2.

Construct a circle with center O, and select a point F outside the circle. Choose any point X on the circumference. Draw the line XF and the radius OX. As before, folding the point X to F creates the perpendicular bisector of XF, intersecting XF in midpoint M. Extend the radius until the line coincident with the radius meets the perpendicular bisector in P. Again, draw in FP. Because MP is the perpendicular bisector of XF, the segments XM and FM are congruent, as are angles $\angle XMP$ and $\angle FMP$. Of course, MP is congruent to itself; thus, the triangles ΔXMP and ΔFMP are congruent by SAS (see Figure 2). We now note that XP is merely the sum of XO and OP, but we also know that $FP = XP$, so $FP - OP = XO$, which is the constant radius. Because the difference of the distances FP and OP is constant, the locus of points P is a hyperbola.

Associated with the conic sections is the concept of eccentricity. The midpoint of segment OF is the center of the conic. The eccentricity is defined to be the distance from the center of the conic to a focus divided by the distance from the center to a vertex of the conic. In our notation, the eccentricity is OF/OX. Because the radius is a constant, the eccentricity gets larger as the focus moves farther away from the center of the circle. Ellipses are figures with eccentricities less than one, with the limiting case of the circle having eccentricity zero. Hyperbolas have eccentricities greater than one. The parabola is the conic section whose eccentricity is exactly one. When $OF = OX$, that is, when F lies on the circumference of the circle, the eccentricity is exactly one.

What happens, then, if one tries to place F on the circumference of the circle? As one goes around folding each point to F, the folds get closer to OF. As points very close to F get folded to F, the tangent lines close in on OF, until the conic simply vanishes. We have shown the tangent lines to be tangent at the point where the perpendicular bisector meets the radius. In this case, any segment XF will be a chord of the circle. The perpendicular bisector of the chord of any circle passes through the center of the circle. All points P, then, are actually located at the center of the circle! How then, can we resolve the case of the parabola?

We look to *The Joy of Mathematics* for help. Pappas gives instructions for folding a parabola. Begin by drawing a straight line. Select another point F on the paper to be the focus. Folding each point on the line to the point F forms the tangent lines to a parabola. The proof is straightforward.

Begin by drawing a straight line and select the point F. Choose a point X on the line. Folding X to F creates the perpendicular bisector of

XF through the midpoint M. We extend a perpendicular to the original line through point X to meet the perpendicular bisector of XF in point P. We also draw in FP. As before, we have the two congruent triangles ΔXMP and ΔFMP (see Figure 3). This indicates that FP and XP are equal in length. All points P are equidistant from the focus and the original line, also known as the directrix, but this is merely the definition of the parabola, so the proof is complete.

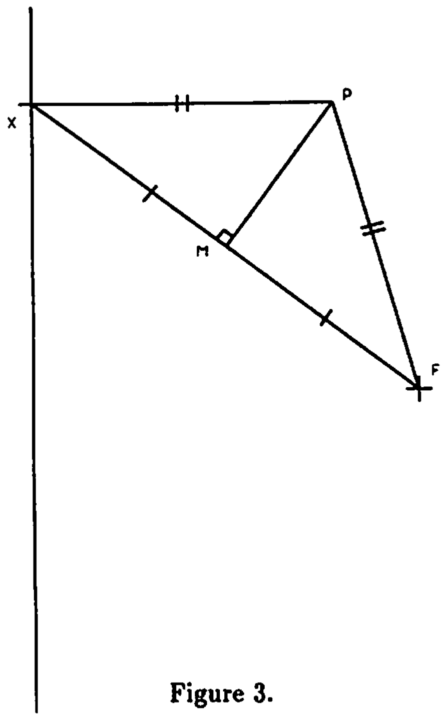

Figure 3.

How can we reconcile the case of the parabola with the notation we have been using? The difficulty lies in using a finite piece of paper. As we let the radius of the circle get arbitrarily large, the circumference locally approximates a straight line. We have shown, however, that if a focus is placed near a straight line, the result of the folding is a parabola. As the focus gets arbitrarily close to the edge of the circle, it seems as if the radius, relative to the distance from F to the circumference, gets arbitrarily large. The parabola is the limit of the ellipse as F gets close to the circumference. Because the model behaves as if the edge of the circle is at infinity, any parabola folded from a small, finite model will tend to be extremely narrow. How, then, can we see the parabola in our construction?

Computer modeling has become integral in mathematics. In our case, we cannot get a sufficiently large piece of paper to fold a parabola. To examine the whole idea of folding conics, I wrote a simple computer program in TrueBasic (see the Appendix). The program was used to illustrate, and in some cases discover, some of the interesting points about folding conics. It has the capability to not only display the conic, but also the tangent lines to the conic. With very little manipulation, the program can be modified to display radial lines, the lines PF and even show the program in color (on a color monitor).

We can use this program to display the case of the parabola. The center of the circle must go to infinity if the parabola is to be formed with the focus F. Because the computer is limited to finite numbers, we must settle for numbers that are large. We make the eccentricity approach 1 by sending O out as far as we can. From the figure, we can see that the program yields nice results (see Figure 4). Of course, we realize that Figure 4 does not describe a parabola, but it describes an ellipse that approaches the limit of a parabola.

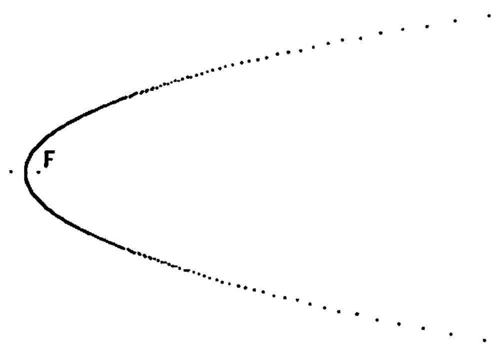

Figure 4. A "parabola" with eccentricity = 0.999999.

Even in the singular case of the parabola, however, the construction is not without merit. All along, we have been using the interesting point M in the construction of the conics. In the case of the parabola, we examine point M. After constructing the circle with center O, select a point F on the circumference and another point X on the circumference. Draw the chord XF and the radii OF and OX. The perpendicular bisector created by the fold will not only pass through the midpoint M, but also the center of the circle. For every point X on the circumference, the triangle FMO is right. Locate the midpoint of segment OF at O′. The segments OO′ and FO′ are equal (see Figure 5). In a right triangle, however, the distance from the vertex containing the right angle to the midpoint of the hypotenuse is equal in length to half the hypotenuse. The length of FO′ is a constant, so the length of MO′ must also be a constant. The locus of points M, then is a circle with half the radius of the original circle around O′.

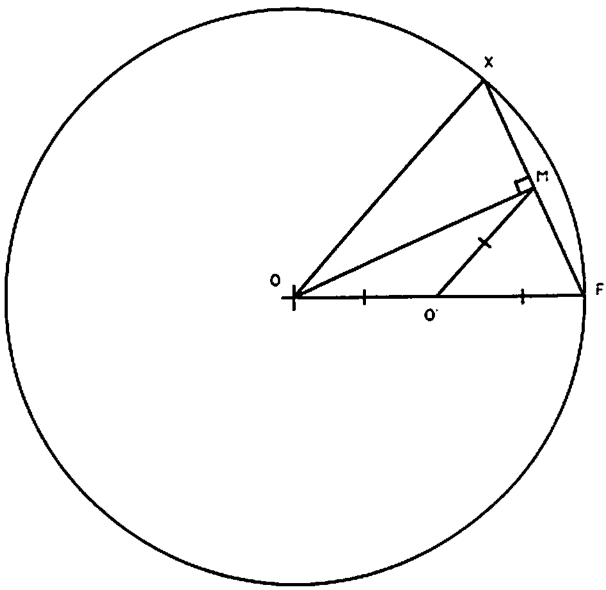

Figure 5.

What does point M do in other cases? We have already shown that the locus of points M in the case of the circle was a circle about the center O with half the radius of the original circle. With only slight modification, we can make the program display point M as well. After a few test cases were run on the program, it seemed that point M always formed a circle with half the radius of the original circle.

The exercise of folding conics has proven to be an interesting and illustrative one. We have located and reconciled the missing parabola to our method by creating a computer model of the situation. We have even described the locus of points M, yet another conic, whose size is independent of the location of F. Perhaps even more interesting is that we have found a very concrete connection between origami and geometry. Plans for future research include making the program more user-friendly and including the asymptotes for the hyperbola case.

Bibliography.

Pappas, Theoni. *The Joy of Mathematics.* San Carlos: Wide World Publishing, 1986. (page 50).

Pappas, Theoni. *More Joy of Mathematics.* San Carlos: Wide World Publishing, 1991. (pages 22-23).

Appendix.

```
option nolet
!****************************************************
!**                                               **
!**                CONIC SECTION PLOTTER          **
!**                          by                   **
!**                    Kevin P. Davis             **
!**                                               **
!**              KME conference Spring 1992       **
!**                                               **
!****************************************************

!Program may be displayed on a color monitor by
!deleting the exclamation points before the "set
!color" statements. The program will give very a good
!demonstration if the following values are used. To
!activate the values for a given conic, simply delete
!the exclamation points for each line underneath the
!name of the conic.
```

```
r=0

!Allow user to select a demo conic section

!Ellipse
!b=4
!a=-4
!r=10

!Hyperbola
!b=3
!a=-3
!r=5

!Circle
!b=0
!a=0
!r=10

if r=0 then
!        only executes if no conic selection was made
  print "This program will plot a conic section by"
  print "drawing tangent lines to the conic. The"
  print "tangent lines are based on a given focal"
  print "point and a circle with given radius."
  print
  print "Please input the x coordinate of the focus"
  print "on the x axis."
  print "(values between -10 and +10 are best)"

 input b

  print "Please input the x coordinate of the center"
  print "of the circle on the x axis."
  print "(values between -10 and 10 are best)"

 input a

  print "What value should be assigned to the"
  print "radius? (values between 0 and 10 are best)"

 input r

end if
```

```
!set the graphics environment

open #1: screen 0,1,0,1
window #1
set window -14,14,-10,10

plot points:  0,0; a,0; b,0
plot text, at a+.2,.2: "O"

if a=b then
  plot text, at b-.4,.2: "F"
else
  plot text, at b+.2,.2: "F"
end if

!Draw the circle

set color "magenta"                 !Set color for circle
for x=(a-r) to (a+r) step 2*r/500
  y=sqr(r^2-(x-a)^2)
  plot points: x,y;x,-y
next x

c=(b+a)/2  !center of conic

for Xo=(a-r) to (a+r) step 2*r/300  !Points on circle
  xa=Xo-a
  Yo=sqr(r^2-(Xo-a)^2)
  if xa<>0 then
    l=l+1
    xb=Xo-b
    set color "cyan"  !set color for MP
    X1=(Xo+b)/2       !coordinatess for M
    Y1=Yo/2

X2=(Yo^2*xa+(Xo+b)*xb*xa+2*Yo^2*a)/(2*Yo^2+2*xb*xa)
!coordinates for P

Y2=(Yo*(Yo^2*xa+(Xo+b)*xb*xa+2*Yo^2*a))/(xa*(2*Yo^2+2
*xb*xa))-Yo*a/xa

    ! plot lines: X1,Y1;X2,Y2     !plots MP
    ! plot lines: X1,-Y1;X2,-Y2   !plots mirror of MP
```

```
    ! plot lines: 2*c-X1,Y1;2*c-X2,Y2
    ! plot lines: 2*c-X1,-Y1;2*c-X2,-Y2

    set color "red"                  !set color for M
    plot points: X1,Y1;X1,-Y1        !plots M

    set color "green"                !set color for P
    plot points: X2,Y2;X2,-Y2;2*c-X2,Y2;2*c-X2,-Y2
                     !plots P and mirror of P about c

    if mod(1,50) =0 then
    ! plot lines: b,0;X2,Y2  !plots perpendicular
    ! plot lines: a,0;Xo,Yo  !plots radius
    end if

  end if

next Xo

e=(abs(a-b)/2)/(r/2)

set color "yellow"
print "Eccentricity = ",e

close #1

end
```

Using Mathematica to Find Minimal Length Confidence Intervals

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Early in the summer of 1992, my advisor proposed the following problem to me:

> Can you find a "simple" function which, when given α, the confidence coefficient, and n, the number of sample points, will give the smallest possible confidence interval for the population variance of the chi-squared distribution?

With his help, the following report was made possible.

Introduction.

The first question that needs an answer, is the why of the matter; why a smaller confidence interval than that which is given by accepted methods? The answer is that one can use a smaller confidence interval to make a more accurate estimate of the population variance. That is, say we wanted to check the variability of equipment designed to measure the volume of an audio source. Suppose we had seven independent measurements with the sample variance $s^2 = 10.57$, and we want to estimate σ^2, the population variance, with confidence coefficient $1 - \alpha = 0.90$. The accepted method (see [1]) gives us a confidence interval for σ^2 of:

$$\left(\frac{(n-1)s^2}{\chi^2_{(n-1;\alpha/2)}}, \frac{(n-1)s^2}{\chi^2_{(n-1;1-\alpha/2)}}\right) = \left(\frac{6\cdot 10.57}{12.5916}, \frac{6\cdot 10.57}{1.63539}\right)$$

$$= (5.04, 38.78).$$

Note how wide this interval is: $38.78 - 5.04 = 33.74$. With our method however, the confidence interval would be :

$$\left(\frac{6\cdot 10.57}{19.8739}, \frac{6\cdot 10.57}{2.175}\right) = (3.19, 29.16)$$

which has a length of $29.16 - 3.19 = 25.97$.

So, as can be seen, this shortens the interval by 7.77, or 23%. Now that we have seen why this can be of importance, we need a way to compute these confidence intervals. Then, noticing that our method is quite computer intensive, we try to obtain a "simple" function of α and n which will give us the approximate values for the confidence intervals and that can be computed by a "generic" hand held calculator.

Calculations.

To determine the confidence interval for the population variance, σ^2, with sample size n taken from a normally distributed population with mean μ and variance σ^2, the sample variance, s^2, is computed by

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2.$$

The usual method of computing a $100(1-\alpha)\%$ confidence interval for σ^2 is given by:

$$(1) \qquad \left(\frac{(n-1)s^2}{\chi^2_{(n-1;\alpha/2)}}, \frac{(n-1)s^2}{\chi^2_{(n-1;1-\alpha/2)}}\right).$$

The probability density function $f(t)$ for the chi-square distribution is:

$$(2) \qquad f(t) = \begin{cases} \dfrac{t^{(n-3)/2}\, e^{-t/2}}{2^{(n-1)/2}\,\Gamma((n-1)/2)} & \text{if } t > 0 \\ 0 & \text{if } t \le 0 \end{cases}.$$

Instead of taking the middle $100(1-\alpha)\%$ of the area under the curve $y = f(t)$, we can shorten the length of the interval:

$$(3) \qquad \left(\frac{(n-1)s^2}{x}, \frac{(n-1)s^2}{y}\right)$$

by minimizing

$$(4) \qquad \ell(x,y) = \frac{1}{x} - \frac{1}{y}$$

subject to the condition that

$$(5) \qquad \int_x^y f(t)dt = 1-\alpha .$$

The problem of minimizing (4) subject to (5) is a classic "Lagrange Multiplier" problem. We use the standard method to solve such a problem (see, for instance, [3]), to obtain the non-linear system

$$(6) \qquad x - y - (n+1)\Big(\ln(x) - \ln(y)\Big) = 0$$

$$(7) \qquad \int_x^y t^{(n-3)/2}\, e^{-t/2}\, dt \;-\; (1-\alpha)2^{(n-1)/2}\, \Gamma((n-1)/2) = 0 ,$$

whose solutions we call x_m and y_m.

Using the above equations (6) and (7), a program (listed in Appendix A) written in Mathematica was used to obtain the values of x_m and y_m for $\alpha = 0.01, 0.02, 0.03, \dots, 0.09, 0.10$ with $n = 4, 5, 6, \dots, 49, 50$. The data found can easily be reproduced by running the program listed in Appendix A.

As we said earlier, this method is quite computer intensive, so we seek an easier "hand held calculator" method which will give an approximation for x_m and y_m using those values found in standard statistical tables corresponding to α and n, which we call x_b and y_b, instead of the more cumbersome notation $\chi^2_{(n-1;\alpha/2)}$ and $\chi^2_{(n-1;1-\alpha/2)}$.

Let $c = 1/(2^{(n-1)/2}\, \Gamma((n-1)/2))$. Then, using the fact that the difference in area of the chi-squared distribution between x_b and x_m

(similarly for y_b and y_m) is

$$\Delta = c \int_{x_b}^{x_m} t^{(n-3)/2} e^{-t/2} \, dt < \alpha/2 \, .$$

We use the midpoint rule (see [4]) to approximate this integral, which gives

$$\Delta/c \approx (x_m - x_b) \, \hat{x}^{(n-3)/2} e^{-\hat{x}/2} \, ,$$

where $\hat{x}$ is the midpoint of the interval $[x_b, x_m]$; i.e., $\hat{x} = (x_b + x_m)/2$. From this, we get (and similarly for y_m) that

$$x_m = x_b + \Delta \, (1/c) \, \hat{x}^{-(n-3)/2} e^{\hat{x}/2} \, .$$

Noticing that x_b and x_m are fairly close together (from data generated using the Mathematica program in Appendix A), we can approximate the above by letting $\hat{x} = x_b$ (also, noticing that the slope in the tail end of the chi-square distribution is small, we similarly let $y = y_b$) and $\Delta = \alpha/2$, the largest possible value. With these substitutions, we obtain the approximations

$$\text{(8)} \qquad \left\{ \begin{array}{l} x_m \approx x_b + (\alpha/2)(1/c) \, x_b^{-(n-3)/2} e^{x_b/2} \\ y_m \approx y_b + (\alpha/2)(1/c) \, y_b^{-(n-3)/2} e^{y_b/2} \end{array} \right. .$$

We now have an equation we can use to derive our values for x_m and y_m; namely

$$\text{(9)} \qquad z_m \approx z_b + (\alpha/2)(1/c) \, z_b^{-(n-3)/2} e^{z_b/2} \, .$$

This gives z_m as a function of n and α alone, since z_b depends only on n and α.

Although we may estimate x_m and y_m with equation (9), the differences between actual and estimated values, for a given α, increase as n increases. To improve on this, we seek to obtain x_m (y_m) as a function of α, n and x_b (y_b), since these are the values which are either given or are readily available from tables. Since we already have the data for x_m (and y_m) and x_b (and y_b), we can fit the above equations using the data structure

$$\text{data} = \left\{ z_b \, , \, (\alpha/2)(1/c) \, z_b^{-(n-3)/2} e^{z_b/2} \, , \, z_m \right\}$$

where all our values are used; that is, for each $\alpha = 0.01, \ldots, 0.10$ and $n = 4, \ldots, 50$, the ordered triple $\{z_b,\ (\alpha/2)(1/c)\ z_b^{-(n-3)/2}\ e^{z_b/2},\ z_m\}$ is a row in the data structure, data. We seek a least squares fit approximation for x_m (y_m), as a function of 1, z_b and $(\alpha/2)(1/c)\ z_b^{-(n-3)/2}\ e^{z_b/2}$.

We use Mathematica's "Fit" command (see [2]) and equation (9) to fit the data 1, z_b and $(\alpha/2)(1/c)\ z_b^{-(n-3)/2}\ e^{z_b/2}$ to z_m, the command being

Fit[data , {1 , f , g} , {f , g}] .

By using the Fit command on the data list as shown above, Mathematica will find the function that best fits, in the sense of least squares, the data to our specifications. This will also give us a "simple" function in the sense that it is a linear combination of 1, f, and g. The Fit command produced the following relationships

$$(10) \qquad \begin{cases} x_m &= 0.19749 + 0.985875\ \mathrm{f}_x + 0.559252\ \mathrm{g}_x \\ y_m &= 12.6576 + 1.01694\ \mathrm{f}_y - 1.8871\ \mathrm{g}_y \end{cases}$$

where $\mathrm{f}_x = x_b$, $\mathrm{f}_y = y_b$,

$$\mathrm{g}_x = (\alpha/2)(1/(2^{(n-1)/2}\ \Gamma((n-1)/2)))\ x_b^{-(n-3)/2}\ e^{x_b/2}$$

and

$$\mathrm{g}_y = (\alpha/2)(1/(2^{(n-1)/2}\ \Gamma((n-1)/2)))\ y_b^{-(n-3)/2}\ e^{y_b/2}$$

where

$$\Gamma((n-1)/2) = \begin{cases} \left(\dfrac{n-3}{2}\right)! & \text{if } n \text{ is odd} \\[2ex] \dfrac{1\cdot 3\cdot \cdots \cdot(n-5)\cdot(n-3)\ \sqrt{\pi}}{2^{(n-2)/2}} & \text{if } n \text{ is even} \end{cases}.$$

We can now find x_m and y_m by the "simple" equation (10) using the computational capabilities of any scientific calculator or some business calculators. Or, one may use a version of the program in Appendix A to find the exact values needed. By using the program and Mathematica, one can find the exact values of x_m and y_m fairly easily. By using equation (10) , one can find the values of x_m and y_m without expensive software, but there is some error involved (see Appendix B).

Bibliography.

[1] Mendenhall, Wackerly and Scheaffer, *Mathematical Statistics with Applications, 4th Ed.* PWS-KENT, 1990.

[2] Wolfram, *Mathematica, 2nd Ed.* Addison-Wesley, 1991.

[3] Stewart, *Calculus, 2nd Ed.* Brooks Cole, 1991. (page 774).

[4] Burden, Faires, *Numerical Analysis. 5th Ed.* PWS-KENT, 1993. (page 182).

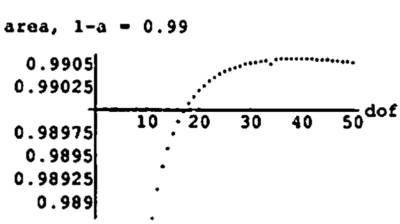
Appendix A.

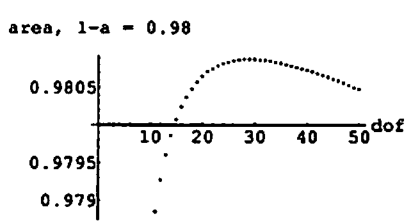
```
(* This statement allows Mathematica to use its statistical capabilities. *)
<<Statistics`NormalDistribution`

flag=1;
While[(flag==1),
    Input["Please enter in the range of n as n={min,max},
          n should be in the range of [4,oo), where min<max."];
    flag=0;
    min=n[[1]]; max=n[[2]];
    If[((min<4)||(min>=max)),flag=1;];];
flag=1;
While[(flag==1),

    Input["Please enter in alpha as a=__,
         a should be in the range of (0,100)"];
    flag=0;
    If[((a<=0)||(a>=100)),flag=1;];];

(* These variables are the arrays of values given for each alpha and n. *)
(* Xmin and ymin are the values found by our method. Xbook and *)
(* ybook are the accepted values found in statistical tables. *)
xmin={}; ymin={}; xbook={}; ybook={};

Do[

(* Sol is used to find the values of xmin and ymin using our method. *)
  sol=FindRoot[{x-y-(n+1)*(Log[x]-Log[y])==0,
           NIntegrate[t^((n-3)/2)*E^(-t/2),{t,x,y}]-
```

```
        (1-a)*Gamma[(n-1)/2]*2^((n-1)/2)==0},
     {x,n/2},{y,n*2},MaxIterations->30];

  AppendTo[xmin,{(n-1),sol[[1]][[2]]}];
  AppendTo[ymin,{(n-1),sol[[2]][[2]]}];

(* These lines are used to obtain the book values. *)
  yb=FindRoot[CDF[ChiSquareDistribution[n-1],x]==
         (1-a/2),{x,n*2}, MaxIterations->30][[1]][[2]];
  xb=FindRoot[CDF[ChiSquareDistribution[n-1],x]==
         (a/2),{x,n*2},MaxIterations->30][[1]][[2]];

  AppendTo[xbook,{(n-1),xb}];
  AppendTo[ybook,{(n-1),yb}];
,{n,min,max}];

(* These lines print out the values in xmin and ymin as (dof , x) *)
Print[xmin];
Print[ymin];
```

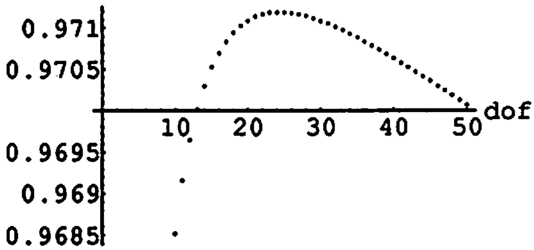
Appendix B.

Using the program given in Appendix A, the exact values can be obtained for x_m and y_m. With the "simple" formula given in (10), some error will occur. The following graphs give this error in graphical form. The graphed points show the area under the chi-squared distribution between x_m and y_m and the horizontal axis is at the exact value of $1-\alpha$ that we wished to obtain. In each graph, "area" is the area under the χ^2 distribution between x_m and y_m using equation (10) and "dof" is "degrees of freedom."

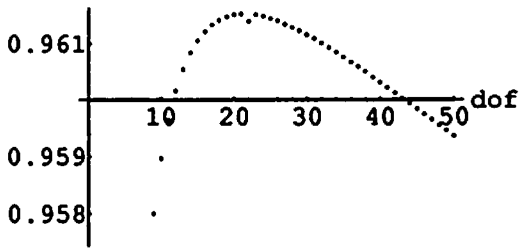




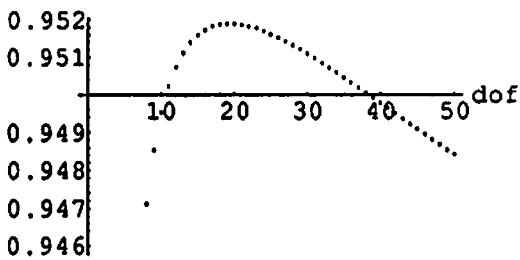
area, 1-a = 0.97



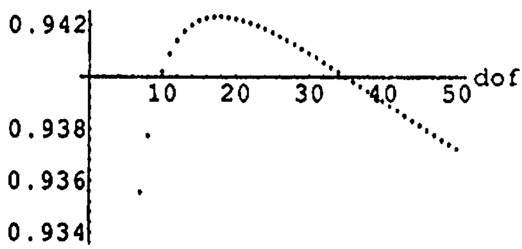
area, 1-a = 0.96



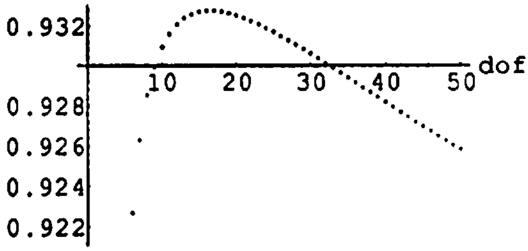
area, 1-a = 0.95



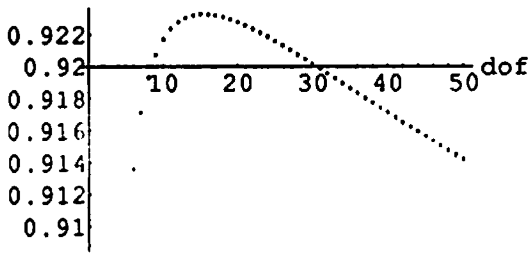
area, 1-a = 0.94



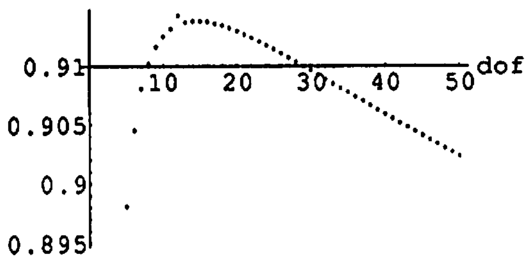
area, 1-a = 0.93



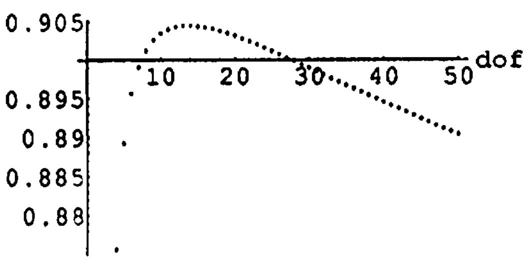
area, 1-a = 0.92



area, 1-a = 0.91



area, 1-a = 0.90



Reciprocal Equations

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Definitions and Examples.

Consider a polynomial of degree n of the form

(1) $$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

where a_j is a complex number for $j = 0, 1, \ldots, n$ and $a_0 \neq 0$. A method will be given to solve $f(x) = 0$ when special conditions are put on the coefficients in (1).

If

$$a_n = a_0,\ a_{n-1} = a_1,\ a_{n-2} = a_2,\ a_{n-3} = a_3,\ \ldots,$$

then the equation

(2) $$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

is a *reciprocal equation of the first class.*

Example 1. Each of the equations $3x^2 - 2x + 3 = 0$, $2x^3 - x^2 - x + 2 = 0$ and $x^4 - 7x^3 + 14x^2 - 7x + 1 = 0$ is a reciprocal equation of the first class.

If

$$a_n = -a_0,\ a_{n-1} = -a_1,\ a_{n-2} = -a_2,\ a_{n-3} = -a_3,\ \ldots,$$

then (2) is a *reciprocal equation of the second class.*

Example 2. Each of the equations $x^2-1=0$, $2x^3+3x^2-3x-2=0$ and $x^4-7x^3+7x-1=0$ is a reciprocal equation of the second class.

Observations.

If n is odd, then there is an even number of terms in (1). If n is even, then there is an odd number of terms in (1). So, if $n=2k$ for some positive integer k, then for a reciprocal equation of the second class $a_k=-a_k$ and so $a_k=0$. Hence, when the terms of the polynomial are written in descending powers of x, the "middle" term is zero.

Let r be a root of a reciprocal equation of either class. If $r=0$, then $f(r)=f(0)=0$ becomes $a_n=0$. So, since either $a_n=a_0$ or $a_n=-a_0$, $a_0=0$ — which contradicts the fact that the degree of $f(x)$ is n. Hence, a reciprocal equation cannot have zero as a root. Since $f(r)=0$,

$$a_0r^n+a_1r^{n-1}+\cdots+a_{n-1}r+a_n = 0 .$$

Dividing this equation by $r^n \neq 0$ gives

$$a_0+a_1(1/r)+\cdots+a_{n-1}(1/r)^{n-1}+a_n(1/r)^n = 0 .$$

Using the relationships among the coefficients yields $f(1/r)=0$. Hence, if r is a root of a reciprocal equation, then $1/r$ is also a root of the equation. This is the reason for calling these equations "reciprocal equations."

Consider a reciprocal equation of the first class of odd degree and let $n=2k+1$ for some nonnegative integer k. Then

$$f(x) = a_0x^{2k+1}+a_1x^{2k}+\cdots+a_1x+a_0 .$$

Hence, $f(-1)=-a_0+a_1-a_2+\cdots+a_2-a_1+a_0=0$. That is, $x=-1$ is a root of the equation and so $x+1$ is a factor of $f(x)$. Then $f(x) =(x+1)g(x)$ where $g(x)$ is a polynomial of degree $2k$ and $g(x)=0$ is also a reciprocal equation of the first class.

As an illustration, with $n=5$, let

$$f(x) = a_0x^5+a_1x^4+a_2x^3+a_2x^2+a_1x+a_0 .$$

Using synthetic division, it is straightforward to show that $f(x)=(x+1)g(x)$ where

$$g(x) = a_0x^4+(a_1-a_0)x^3+(a_2-a_1+a_0)x^2+(a_1-a_0)x+a_0 .$$

Similarly, $x = 1$ is a root of any reciprocal equation of the second class whose degree is odd. In this case, it is possible to write $f(x) = (x-1)g(x)$ where $g(x)$ is a polynomial of even degree and $g(x) = 0$ is a reciprocal equation of the first class.

Now, suppose that n is even and $f(x) = 0$ is a reciprocal equation of the second class. Then $x = -1$ and $x = 1$ are roots of the equation and $f(x) = (x^2-1)g(x)$ where $g(x)$ is a polynomial of even degree and $g(x) = 0$ is a reciprocal equation of the first class.

As an illustration, with $n = 6$, and

$$f(x) = a_0x^6 + a_1x^5 + a_2x^4 - a_2x^2 - a_1x - a_0 ,$$

it can easily be shown that $f(x) = (x^2-1)g(x)$ where

$$g(x) = a_0x^4 + a_1x^3 + (a_2 + a_0)x^2 + a_1x + a_0 .$$

A General Technique.

From the preceding observations, the solution of any reciprocal equation can be reduced to solving a reciprocal equation of the first class with even degree. As a result, a method will be given to solve an equation of the form

$$a_0x^{2k} + a_1x^{2k-1} + \cdots + a_1x + a_0 = 0 . \tag{3}$$

Since $x \neq 0$, divide this equation by x^k and regroup the terms to get

$$a_0\left(x^k + \frac{1}{x^k}\right) + a_1\left(x^{k-1} + \frac{1}{x^{k-1}}\right) + \cdots + a_{k-1}\left(x + \frac{1}{x}\right) + a_k = 0. \tag{4}$$

Now, let $y = x + 1/x$. Then $y^2 = x^2 + 1/x^2 + 2$ and so $y^2 - 2 = x^2 + 1/x^2$. In general, it can be shown by mathematical induction that

$$x^j + 1/x^j = (x^{j-1} + 1/x^{j-1})y - (x^{j-2} + 1/x^{j-2})$$

for $j \geq 2$. As a result, equation (4) can be written as a polynomial in y of degree k, half of the original degree. If one can solve the resulting equation to obtain roots $y_1, \ldots, y_k$, then the roots of (3) can be found by solving $x + 1/x = y_j$ for $j = 1, \ldots, k$.

Further Examples.

Example 3. To solve the equation $x^4-7x^3+14x^2-7x+1=0$, rewrite the equation in the form

$$\left(x^2+\frac{1}{x^2}\right)-7\left(x+\frac{1}{x}\right)+14 = 0 .$$

Letting $y=x+1/x$ leads to $y^2-7y+12=0$ and so $(y-3)(y-4)=0$. Hence $y=3$ or $y=4$. Now, $x+1/x=3$ and $x+1/x=4$ yield $x=(3\pm\sqrt{5})/2$ and $x=2\pm\sqrt{3}$, respectively, as the solutions to the given equation.

Example 4. Consider the equation

$$x^6-x^5-2x^4+3x^3-2x^2-x+1 = 0 .$$

To solve this equation, divide by x^3 and regroup the terms to give

$$\left(x^3+\frac{1}{x^3}\right)-\left(x^2+\frac{1}{x^2}\right)-2\left(x+\frac{1}{x}\right)+3 = 0 .$$

If $y=x+1/x$, then $x^2+1/x^2=y^2-2$ and $x^3+1/x^3=y^3-3y$. Hence, in terms of y, the above equation becomes

$$0 = y^3-y^2-5y+5 = y^2(y-1)-5(y-1) = (y^2-5)(y-1) .$$

So, $y=1$ or $y=\pm\sqrt{5}$. Now, $x+1/x=1$ leads to $x=(1\pm i\sqrt{3})/2$. For $x+1/x=\sqrt{5}$, the solutions are $(\pm 1+\sqrt{5})/2$ while for $x+1/x=-\sqrt{5}$, the solutions are $(\pm 1-\sqrt{5})/2$. Thus the solutions are $(1\pm i\sqrt{3})/2$ and $(\pm 1\pm\sqrt{5})/2$.

The next example illustrates how the techniques in this paper can be used to express certain nonstandard trigonometric values in terms of radicals.

Example 5. To express cos 72° in terms of radicals, let $x = \cos 72^\circ + i \sin 72^\circ$. By De Moivre's formula, $x^5 = \cos 360^\circ + i \sin 360^\circ = 1$. Hence $x^5-1=0$ and so $(x-1)(x^4+x^3+x^2+x+1)=0$. So, $x=1$ or $x^4+x^3+x^2+x+1=0$. The latter equation can be written as

$$\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1 = 0$$

or

$$y^2+y-1 = 0$$

where $y=x+1/x$. Since $y=(-1\pm\sqrt{5})/2$, it suffices to solve

$$x+1/x = (-1+\sqrt{5})/2 \quad \text{and} \quad x+1/x = (-1-\sqrt{5})/2 .$$

These equations have as solutions

$$\left((-1+\sqrt{5}) \pm i\sqrt{10+2\sqrt{5}}\right)/4 \quad \text{and} \quad \left((-1-\sqrt{5}) \pm i\sqrt{10-2\sqrt{5}}\right)/4 ,$$

respectively. So $\Re x = (-1 \pm \sqrt{5})/4$ — which is equivalent to $\cos 72^\circ = (-1 \pm \sqrt{5})/4$. Since $\cos 72^\circ > 0$,

$$\cos 72^\circ = (-1+\sqrt{5})/4 .$$

By equating imaginary parts in the solutions, it is straightforward to show that

$$\sin 72^\circ = \left(\sqrt{10+2\sqrt{5}}\right)/4 .$$

Using the techniques in Example 5, the interested reader can obtain

$$\cos 36^\circ = \sin 54^\circ = (1+\sqrt{5})/4 ,$$

$$\sin 36^\circ = \cos 54^\circ = \left(\sqrt{10-2\sqrt{5}}\right)/4 ,$$

$$\cos \pi/8 = \left(\sqrt{2+\sqrt{2}}\right)/2 \quad \text{and}$$

$$\cos 3\pi/8 = \left(\sqrt{2-\sqrt{2}}\right)/2 .$$

The last example involves a sixth degree polynomial with some nonreal coefficients.

Example 6. The roots of the equation

$$35x^6 - 408ix^5 - 1047x^4 - 816ix^3 - 1047x^2 - 408ix + 35 = 0$$

are i, $-i$, $5i$, $-i/5$, $7i$ and $-i/7$. These roots can be obtained by dividing the equation by x^3, letting $y = x + 1/x$ and solving the resulting equations. It should be mentioned that an alternate method to solve this equation is to let $z = ix$. The resulting equation is

$$-35z^6 - 408z^5 - 1047z^4 + 816z^3 + 1047z^2 - 408z + 35 = 0 .$$

This equation can be solved using the "rational root theorem" to give $z = -1,\ 1,\ -5,\ 1/5,\ -7$ or $1/7$. It is then easy to obtain x.

The roots of the following equations can be found in the same way:

$$-6x^6 + 25ix^5 + 6x^4 + 50ix^3 + 6x^2 + 25ix - 6 = 0$$

and

$$2x^6 + 3ix^5 + 2x^4 - 2x^2 - 3ix - 2 = 0 .$$

For more advanced related material, the reader may consult texts dealing with "symmetric equations" and "cyclotomic polynomials."

Bibliography.

Conkwright, Nelson B. *Introduction to the Theory of Equations.* Ginn and Company, 1941.

Dickson, Leonard E. *New Course in the Theory of Equations.* John Wiley and Sons, 1939.

Lovitt, William V. *Elementary Theory of Equations.* Prentice-Hall, 1939.

$\int_0^\infty exp(-x^2)\,dx = \sqrt{\pi}/2$ is a Limiting Case of Euler's Integral

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The probability integral mentioned in the title is generally evaluated by transforming the problem into a double integral and passing to polar coordinates by the substitution $x = r\cos(\theta)$ and $y = r\sin(\theta)$. However, this transformation is unnecessary as we will show by deducing the given integral from Euler's integral by a limiting process.

We are familiar with Euler's integral (see [1]) given by

$$\int_0^1 x^m(1-x)^n dx = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \qquad (m,n > -1). \tag{1}$$

With the aid of this integral, we will first deduce the value of $\Gamma(1/2)$, which is $\sqrt{\pi}$. To this end, we apply the substitution $x = \sin^2(\theta)$, $dx = 2\sin(\theta)\cos(\theta)d\theta$, and set $m = n = -1/2$ on the left hand side of (1). This yields

$$\int_0^1 x^{-1/2}(1-x)^{-1/2}dx = \int_0^{\pi/2} 2\,d\theta = \pi. \tag{2}$$

Now for $m = n = -1/2$, we equate the the right hand side of (2) and the right hand sides of (1) to get

$$\frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1)} = \pi,$$

which implies that

$$\Gamma(1/2) = \sqrt{\pi}$$

since $\Gamma(1) = 1$.

In the next step, we apply the substitution $x = y/n$, $dx = dy/n$ to the left hand side of (1) and, remembering that $\Gamma(n+1) = n \cdot \Gamma(n)$, we get

$$\int_0^n y^m (1 - y/n)^n dy = n^{m+1} \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}$$

$$= n^{m+1} \frac{\Gamma(m+1)\Gamma(n+1)}{(m+n+1)(m+n)\cdot\dots\cdot(n+1)\cdot\Gamma(n+1)}$$

$$= \frac{\Gamma(m+1)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdot\dots\cdot\left(1+\frac{m+1}{n}\right)}.$$

Now, on taking the limit as n tends to infinity, we obtain

$$\text{(3)} \qquad \int_0^\infty y^m \, exp(-y) \, dy = \Gamma(m+1) ,$$

which is Euler's integral of the second kind. In the left hand side of (3) we set $y = t^2$, $dy = 2 \; t \; dt$ to get

$$\text{(4)} \qquad \int_0^\infty t^{2m+1} \, exp(-t^2) \, dt = \Gamma(m+1)/2$$

which, on setting $m = -1/2$, yields

$$\text{(5)} \qquad \int_0^\infty exp(-t^2) dt = \sqrt{\pi}/2.$$

We can replace the variable t with x and the proof is finished. Thus we see that the probability integral and the Euler's integral belong to the same family.

Reference.

1. Murray R. Spiegel, *Theory and Problems of Fourier Analysis.* New York: McGraw-Hill, 1974.

The Problem Corner

Edited by Kenneth M. Wilke

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before 1 January 1995. Solutions received after the publication deadline will be considered also until the time when copy is prepared for publication. The solutions will be published in the Spring 1995 issue of *The Pentagon*, with credit being given to student solutions. Affirmation of student status and school should be included with solutions. Address all communications to Kenneth M. Wilke, Department of Mathematics, 275 Morgan Hall, Washburn University, Topeka, Kansas 66621.

PROBLEMS 475-479.

Problem 475. Proposed by Francis E. Masat, Glassboro State College, Glassboro, New Jersey. Let n and $n+2$ be positive integers. Prove that n and $n+2$ are both prime numbers if and only if

(a) $$\sigma(n) = \phi(n+2)$$
or
(b) $$\sigma(n(n+2))\phi(n(n+2)) = (n^2+2n+1)(n^2+2n+3).$$

In this problem, $\phi(n)$ denotes Euler's Phi function which gives the number of integers less than n and which are relatively prime to n. Two integers a and b are relatively prime if $\gcd(a,b)=1$. $\sigma(n)$ denotes the Sigma function which denotes the sum of all the divisors (including 1 and n), of the integer n.

Problem 476. Proposed by J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin. Around equilateral triangle ABC, circumscribe rectangle PBQR (as shown in the figure below). In general each side of triangle ABC cuts off a right triangle from the rectangle. Prove that the sum of the areas of the two smaller right triangles equals the area of the larger right triangle.

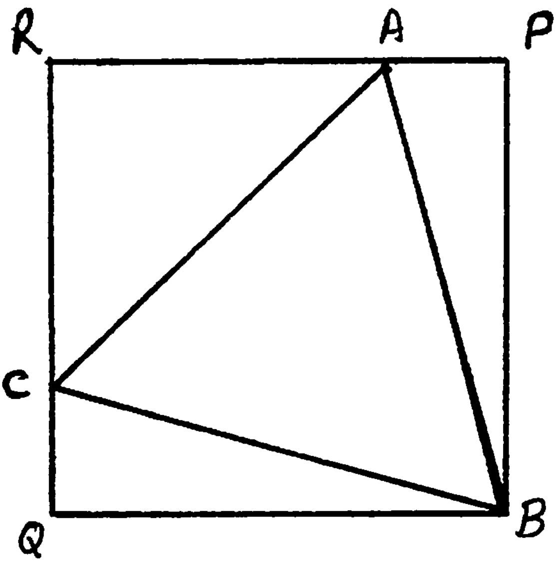

Figure for Problem 476.

Problem 477. Proposed by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin. Let n be an integer ≥ 2. Express

$$\sum_{k=2}^{n} \binom{n}{k-2}\binom{n}{k}$$

as a binomial coefficient and prove that your equality is correct.

Problem 478. Proposed by J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin. The adjacent pairs of the trisectors of the angles of equilateral triangle ABC meet at the vertices of triangle A′B′C′ as shown in the figure below. Find the ratio between the areas of triangles ABC and A′B′C′.

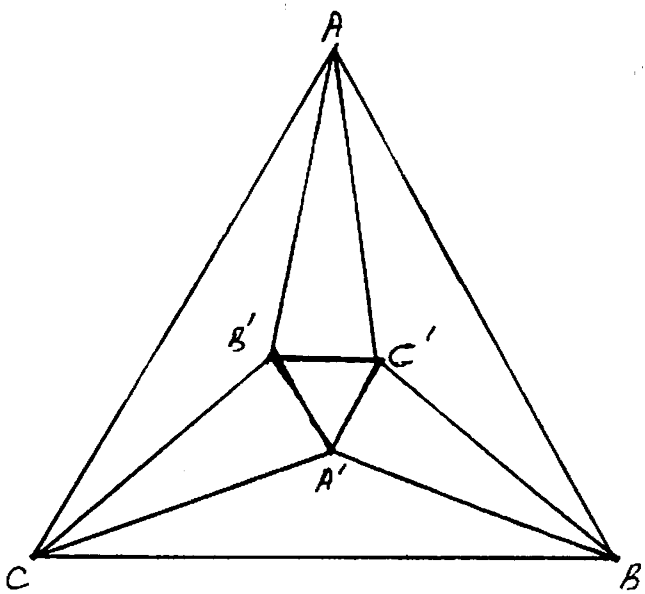

Figure for Problem 478.

Problem 479. Proposed by the Editor. Prove that

$$(\cot 63^\circ)(\cot 132^\circ) + (\cot 132^\circ)(\cot 165^\circ) + (\cot 165^\circ)(\cot 63^\circ) = 1.$$

Find all other such instances, if any exist, in which

$$(\cot A)(\cot B) + (\cot B)(\cot C) + (\cot C)(\cot A) = 1.$$

Please help your editor by submitting problem proposals.

SOLUTIONS 465-469.

Problem 465. Proposed by Stanley Rabinowitz, Westford, Maine. Three circles with centers A, B and C are mutually tangent externally (see the figure below). Circles A and C touch at X. Circles B and C touch at Y. Prove that the line through X and Y passes through the point where the common external tangents to the circles A and B meet.

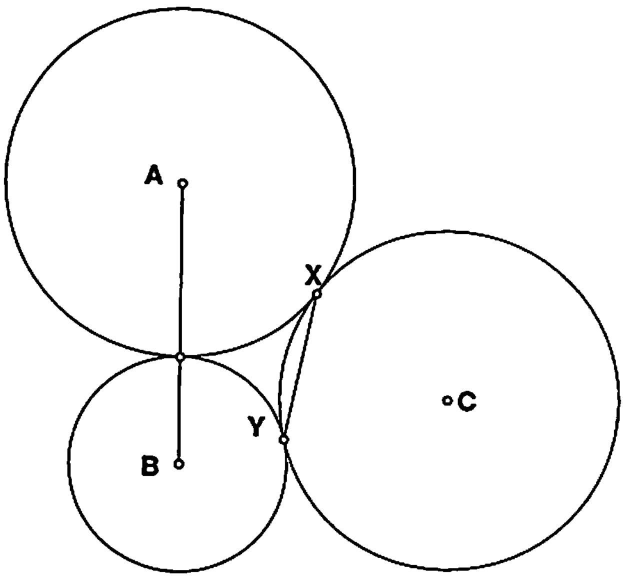

Figure for Problem 465.

Solution by Sammy Yu and Jimmy Yu, jointly, both special students at the University of South Dakota, Vermillion, South Dakota.

Referring to the figure below, let r_a and r_b be the radii of circles A and B, respectively. Let D be the intersection of ray $\overrightarrow{XY}$ and circle B, P be the intersection of ray $\overrightarrow{XY}$ and ray $\overrightarrow{AB}$, and P′ be the intersection of ray $\overrightarrow{AB}$ and one of the common external tangents $\overline{EF}$, which is tangent to circle A at E and circle B at F. Also, assume that the point P′ lies between the points P and B. We shall show that P′ and P coincide, or that the common external tangent will meet ray $\overrightarrow{XY}$ at the intersection of ray $\overrightarrow{XY}$ and the line of centers of circles A and B. Both triangles Δ BDY and Δ CXY are isosceles. Hence $m\angle 1 = m\angle 2 = m\angle 3 = m\angle 4$. It follows that AC $\parallel$ BD and that triangles Δ PBD and Δ PAX are similar.

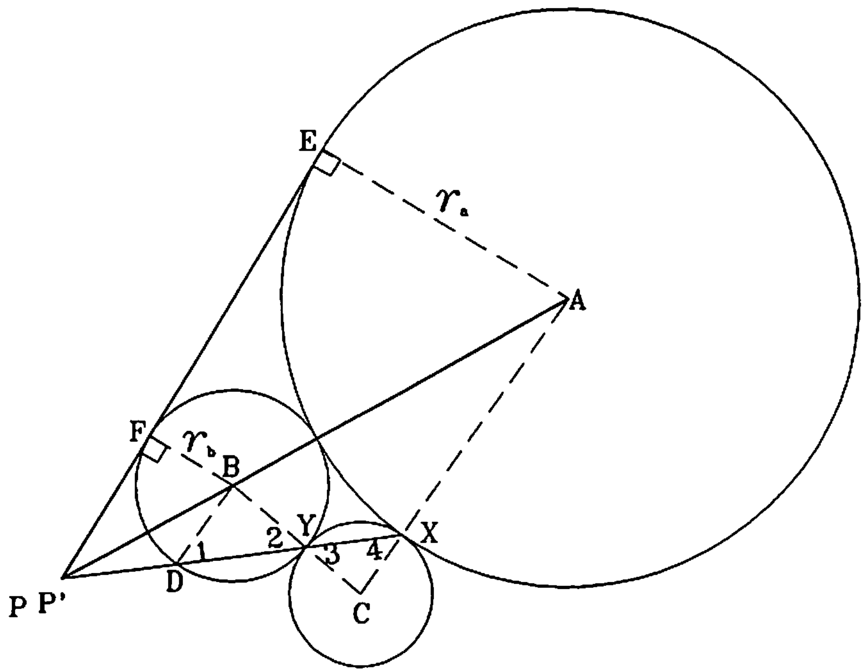

Figure for solution of Problem 465.

Thus

(1) $$\frac{\overline{PB}}{\overline{PA}} = \frac{\overline{BD}}{\overline{AX}} = \frac{r_b}{r_a}$$

Now $\overline{AE} \perp \overline{P'E}$ and $\overline{BF} \perp \overline{P'E}$, hence triangles Δ P'BF and Δ P'AE are similar. Thus

(2) $$\frac{\overline{P'B}}{\overline{P'A}} = \frac{\overline{BF}}{\overline{AE}} = \frac{r_b}{r_a}$$

Comparing (1) and (2), we obtain

$$\text{(3)} \qquad \frac{\overline{\text{P'B}}}{\overline{\text{P'A}}} = \frac{\overline{\text{PB}}}{\overline{\text{PA}}}$$

Now, since P′ lies between P and B, we have $\overline{\text{PB}} = \overline{\text{P'B}} + \overline{\text{PP'}}$ and $\overline{\text{PA}} = \overline{\text{P'A}} + \overline{\text{PP'}}$. Substituting these values into (3) and rearranging terms, we obtain

$$\overline{\text{PP'}}\left(\overline{\text{P'A}} - \overline{\text{P'B}}\right) = 0.$$

Since A and B are the centers of circles A and B respectively, $\overline{\text{P'A}} - \overline{\text{P'B}} \neq 0$. Hence $\overline{\text{PP'}} = 0$ so that the points P and P′ coincide. If the point P lies between the points P′ and B, similar reasoning leads to the same result that $\overline{\text{PP'}} = 0$; i.e., that the points P and P′ coincide. Since only the fact that $\overline{\text{EF}}$ was a common external tangent to the circles A and B was used without regard to the choice of its location, both of the common external tangents to circles A and B will meet ray $\overrightarrow{\text{XY}}$ at the point P.

Also solved by Sammy Yu and Jimmy Yu, jointly, both special students at the University of South Dakota, Vermillion, South Dakota (second solution) and the proposer.

Problem 466. Proposed by the Editor. It can be shown that the standard deviation s of any three consecutive integers is itself an integer. Characterize those integers n which have the property that the standard deviation s of n consecutive integers is also an integer. Find two values of $n > 3$ which have this property. (For a related problem involving the mean and variance — but not the standard deviation — of n consecutive integers, see *Crux Mathematicorum* 18 (November 1992), problem 1786).

Composite of solutions submitted by Al White, St. Bonaventure University, St. Bonaventure, New York and Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Assume that $x_1, x_2, \ldots, x_n$ are consecutive integers. First consider the case where $n = 2k+1$. Coding these consecutive integers, we obtain

$$\text{(1)} \qquad \begin{aligned} & x_1 - x_k, \ldots, x_k - x_k, \ldots, x_{2k+1} - x_k \\ &= -k, -k+1, \ldots, 0, \ldots, k-1, k. \end{aligned}$$

Here the standard deviation s^2 is given by

$$(2) \qquad \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n(n-1)} = \frac{(k+1)(2k+1)}{6} = \frac{n(n+1)}{12}.$$

Next consider the case where $n = 2k$. Coding these consecutive integers, we obtain

$$(3) \qquad \begin{aligned} &x_1 - x_k, \ldots, x_k - x_k, \ldots, x_{2k} - x_k \\ &= -k, -k+1, \ldots, 0, \ldots, k-1. \end{aligned}$$

Here the standard deviation s^2 is given by

$$(4) \qquad \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n(n-1)} = \frac{k(2k+1)}{6} = \frac{n(n+1)}{12}.$$

Equations (2) and (4) can be combined into the Pell equation

$$(5) \qquad (2n+1)^2 - 3(4s)^2 = 1.$$

All solutions of the Pell equation $X^2 - 3Y^2 = 1$ are given by the relations

$$(6) \qquad x_k + y_k\sqrt{3} = (2+\sqrt{3})^k$$

and

$$(7) \qquad \begin{aligned} x_{k+1} + y_{k+1}\sqrt{3} &= (x_k + y_k\sqrt{3})(2+\sqrt{3}) \\ &= (2x_k + 3y_k) + (x_k + 2y_k)\sqrt{3}. \end{aligned}$$

Hence

$$(8) \qquad x_{k+1} = (2x_k + 3y_k) \quad \text{and} \quad y_{k+1} = (x_k + 2y_k)\sqrt{3}.$$

For (6), (7) and (8), $k = 1, 2, \ldots$. Thus we can now construct the following table of solutions:

k	x_k	y_k	n	s
2	7	4	3	1
4	97	56	97	14
6	1351	780	675	195
8	18817	10864	9408	2716
10	262087	151316	131043	37829

12	3650401	2107560	1825200	526890
14	50843527	29354524	25421763	7338631
16	708158977	408855776	354079488	102213944
18	9863382151	5694626340	4931691075	1423656585

Also solved by Charles D. Ashbacher, Cedar Rapids, Iowa; Paul R. Coe, Rosary College, Wake Forest, Illinois; J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin; and Sammy Yu and Jimmy Yu, jointly, both special students at the University of South Dakota, Vermillion, South Dakota.

Editor's Comment. Applying relation (6) to (7) we obtain

$$(9) \qquad x_{k+2} + y_{k+2}\sqrt{3} = (x_{k+1} + y_{k+1}\sqrt{3})(2+\sqrt{3}) = (7x_k + 12y_k) + (4x_k + 7y_k)\sqrt{3}.$$

Note that $x_1 = 2$, $y_1 = 1$ and $x_2 = 7$, $y_2 = 4$. We seek k such that y_k is divisible by 4. Given the values of $x_2 = 7$, $y_2 = 4$ and (9), y_k is even whenever k is even and the corresponding value of x_k is odd. This explains the choices for k in the table of solutions.

Paul R. Coe noticed that the values of $n/12$ are perfect squares whenever $k \equiv 0 \pmod 4$ and that the values of $n+1$ are perfect squares whenever $k \equiv 2 \pmod 4$. This latter property holds for all even values of k.

Problem 467. Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri. Let m and n be the lengths of the chords of arcs AB and BC, respectively, on a circle with a radius of length r. If p is the length of the chord of arc AC, prove that

$$p = \frac{m\sqrt{4r^2 - n^2} + n\sqrt{4r^2 - m^2}}{2r}.$$

Solution by Tom Chen, student, Albion College, Albion, Michigan.

In the figure below, let O be the center of the circle. Let A′ be the midpoint of AB, let B′ be the midpoint of BC and let C′ be the midpoint of AC. Then OA′, OB′ and OC′ are the perpendicular bisectors of AB, BC and AC respectively.

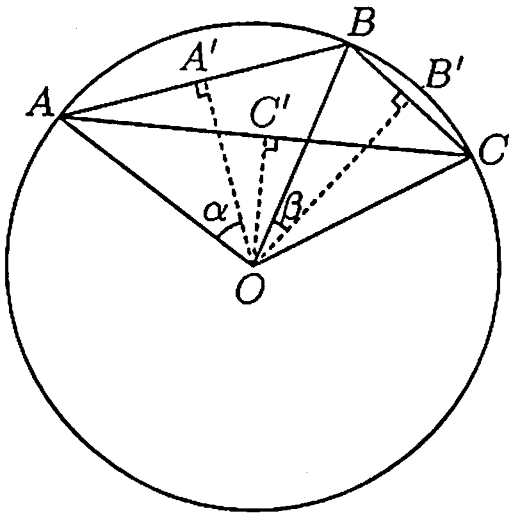

Figure for solution of Problem 467.

Let $\alpha = m\angle AOA'$ and let $\beta = m\angle BOB'$. Note that OA′ bisects AOB, OB′ bisects BOC, and OC′ bisects AOC. Then $m\angle AOB = 2\alpha$ and $m\angle BOC = 2\beta$. Thus, $m\angle AOC = 2\alpha + 2\beta$. Therefore, $m\angle COC' = \alpha + \beta$.

Now,

$$\sin\alpha = \frac{m}{2r}, \cos\alpha = \sqrt{1 - \frac{m^2}{4r^2}}$$

and

$$\sin\beta = \frac{n}{2r}, \cos\beta = \sqrt{1 - \frac{n^2}{4r^2}}.$$

Also,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$= \frac{m}{2r}\sqrt{1 - \frac{n^2}{4r^2}} + \frac{n}{2r}\sqrt{1 - \frac{m^2}{4r^2}} = \frac{m\sqrt{4r^2 - n^2} + n\sqrt{4r^2 - m^2}}{4r^2}$$

But

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha = \frac{p}{2r}$$

and thus

$$p = \frac{m\sqrt{4r^2-n^2}+n\sqrt{4r^2-m^2}}{2r}.$$

Also solved by Scott H. Brown, Auburn University, Montgomery Alabama; Sean Forbes, Drake University, Des Moines, Iowa; Rhonda McKee, Central Missouri State University, Warrensburg, Missouri; Fred A. Miller, Elkins, West Virginia; Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Sammy Yu and Jimmy Yu, jointly, both special students at the University of South Dakota, Vermillion, South Dakota; and the proposer.

Problem 468. Proposed by the Editor. Young Leslie Morely shuffled a standard deck of playing cards (not a pinochle deck) which contained no jokers. When he finished, he started turning over cards one at a time from the top of the deck until he found a jack after turning over the sixteenth card. Assuming that the deck contained four jacks and that young Leslie repeated this experiment several times, what would be the average number of cards which he would have to turn over before finding a jack?

Solution by Mark R. Snavely, Carthage College, Kenosha, Wisconsin.

The probability that Leslie draws the first jack on the nth card is $4/52 = 1/13$ for $n = 1$ and

$$\left(\frac{48}{52}\times\frac{47}{51}\times\frac{46}{50}\times\cdots\times\frac{48-(n-2)}{52-(n-2)}\right)\times\frac{4}{52-(n-1)}$$

for $2 \le n \le 52$.

Note that the probability of Leslie drawing the first jack on cards 50, 51 or 52 is zero. Hence the expected number of cards needed to draw a jack is given by the expression

$$E = \sum_{n=1}^{52} n \times (\text{probability of getting the first jack on the } n\text{th card})$$

For $n = 52$, a computer evaluation of the value for E yields $E = 53/5 = 10.6$.

Also solved by Charles D. Ashbacher, Cedar Rapids, Iowa.

Editor's Comment. Our featured solver supplied and proved by mathematical induction the following generalization:

If Leslie is using a deck of n cards which contains j jacks, $n \geq j \geq 1$, then the average number of cards he would turn up before finding a jack is $(n+1)/(j+1)$.

The same result can be obtained economically by considering symmetry in the following manner. The j jacks divide the remaining pack of $n-j$ cards into $j+1$ segments of size from 0 to $n-j$. We interpret two consecutive jacks as a segment of length zero. Also, if a jack is at the front of the pack or if a jack is at the end of the pack, we treat the pack as beginning or ending with a segment of length zero. Then the Principle of Symmetry states that the $j+1$ segments should average $(n-j)/(j+1)$ cards. Then, since the next card is a jack, the average number of cards one would expect to turn up before getting a jack is $(n-j)/(j+1)+1 = (n+1)/(j+1)$, as found by our featured solver.

Problem 469. Proposed by the Editor. After finishing the statistics experiment described in the previous problem, young Leslie Morely discovered an interesting number while playing with his computer. The number which he discovered has a cube which ends in 0987654321, the string of digits in reverse order with the zero having been moved to the front. Find the smallest positive integer n which has this property.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

We use the following results from [1].

Theorem 68. Let p be an odd prime, let α be a positive integer not divisible by p, and let $n = p^{\alpha}$ where α is a positive integer and $\delta = \gcd(q, \phi(n))$ where q is a positive integer. Then the congruence

$$x^q \equiv a \pmod{n}$$

has exactly δ incongruent solutions modulo n if the index of a is divisible by δ. Otherwise, it has no solution.

Corollary. Let p be an odd prime, let a be a positive integer not divisible by p, and let $n = p^\alpha$ where α is a positive integer and $\delta = \gcd(q, \phi(n))$ where q is a positive integer. If $\delta = 1$, then the congruence

$$x^q \equiv a \pmod{n}$$

has exactly 1 solution modulo n.

Theorem 72. If q is an odd positive integer and α is a positive integer, then the congruence

$$x^q \equiv a \pmod{2^\alpha}$$

has exactly one solution modulo 2^α.

Based upon these known results, the solution to the problem reduces to the following set of equivalent statements in which n is a positive integer throughout:

(1) $$n^3 \equiv 0987654321 \pmod{10^{10}}$$

(2) $n^3 \equiv 0987654321 \pmod{2^{10}}$ and $n^3 \equiv 0987654321 \pmod{5^{10}}$

(3) $n^3 \equiv 177 \pmod{2^{10}}$ and $n^3 \equiv 1326196 \pmod{5^{10}}$

(4) $n \equiv 145 \pmod{2^{10}}$ and $n^3 \equiv 6283466 \pmod{5^{10}}$

Note that by the corollary given above, these solutions are unique for their respective moduli.

(5) $n \equiv 1871517841 \pmod{2^{10}}$ and $n \equiv 1871517841 \pmod{5^{10}}$

(6) $$n \equiv 1871517841 \pmod{10^{10}}$$

This result also follows from applying the Chinese Remainder Theorem to the congruences in (5). Thus the smallest positive integer satisfying the conditions of the problem is 1871517841.

[1] T. Nagell, *Introduction to Number Theory.* Chelsea Publishing Company, New York (1964), 115 and 118-119.

Also solved by Charles D. Ashbacher, Cedar Rapids, Iowa; Tom Chen, student, Albion, College, Albion, Michigan; Sean Forbes, student, Drake University, Des Moines, Iowa; and Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin (2nd solution).

Editor's Comment. All of the other solutions essentially "built up" the solution by considering possible last digits of cubes starting with 1^3 ends in 1 and proceeding similarly or by using equivalent systems of congruences modulo 10^n for $1 \leq n \leq 10$.

Kappa Mu Epsilon News

Edited by Mary S. Elick, Historian

News of chapter activities and other noteworthy *KME* events should be sent to Mary S. Elick, Historian, Kappa Mu Epsilon, Mathematics Department, Missouri Southern State College, Joplin, Missouri 64801.

INSTALLATION OF NEW CHAPTERS

Kentucky Beta

Cumberland College, Williamsburg, Kentucky

The installation of the Kentucky Beta Chapter of Kappa Mu Epsilon was held on May 3, 1993, in the Gatliff Chapel on the campus of Cumberland College. Dr. Pat Costello, National President-Elect of Kappa Mu Epsilon, acted as the installing officer of the new chapter. Seventeen students and eight faculty constituted the charter members of the new chapter at Cumberland College. Those initiated were:

> *Students:* Loay Ackleh, Dawn Allen, Donna Anderson, Andrea Brown, Peggy Burke, Veronica Carmical, Lynn Engle, Lisa Farrish, Joseph Frencl, Stephanie Hammack, Jeffrey Harris, Shannon Mahurin, Roy Markham, Donald Poynter, Moris Shorrosh, Kenneth Siler, and Charles Waits.
>
> *Faculty:* Joseph Early, John Hymo, Diane Jamison, James Manning, Lawrence Newquist, Jonathan Ramey, Lolan Redden, and Jennifer Sexton.

Following the installation ceremony, a banquet was held in Corbin, KY, at the Western Steer Steak House. During the banquet, Dr. Costello was presented with a souvenir from Cumberland College.

Officers installed during the ceremony were Veronica Carmical, president; Charles Waits, vice president; Stephanie Hammack, secretary; and Shannon Mahurin, treasurer. Faculty members John Hymo and

James Manning accepted the responsibilities of faculty sponsor and corresponding secretary, respectively.

CHAPTER NEWS

AL Beta Chapter President - Vicky Locker
University of North Alabama, Florence 19 actives, 10 associates

Other 1993-94 officers: James Killingsworth, vice president; Rita Taylor, secretary; Eddy Brackin, corresponding secretary; Patricia Roden, faculty sponsor.

AL Gamma Chapter President - Amanda Hopkins
University of Montevallo, Montevallo 12 actives, 3 associates

Other 1993-94 officers: Steven Price, vice president; Lucretia Weeks, secretary; James Raymond, treasurer; Charles Coats, corresponding secretary; Donald Alexander, faculty sponsor.

CA Delta Chapter President - Margie Ye
California State Polytechnic University, Pomona 26 actives

Other 1993-94 officers: Brian Anderson, vice president; Tracy Baughn, secretary; Mirela Ciocan, treasurer; Jim McKinney, faculty sponsor.

CO Gamma Chapter President - Susan Clinkenbeard
Fort Lewis College, Durango 29 actives, 9 associates

In addition to the fall initiation ceremony, the chapter held two other meetings. One of these, a pizza party, featured a hexaflexagon construction program. The chapter also submitted an invitation to the National Council to host the 1995 Biennial Convention. Having learned that their invitation has been accepted, they are now busily making plans for a great meeting in April 1995. Other 1993-94 officers: Jason Briggs, vice president; Noreen Frost, secretary; Stevan Scott, treasurer; Richard Gibbs, corresponding secretary; Deborah Berrier, faculty sponsor.

GA Alpha Chapter President - Debbie Ingle
West Georgia College, Carrollton 30 actives

The annual *KME* Food and Clothing Drive was held in November; items received were delivered to the Interfaith Help Center. On

November 20 the organization enjoyed a social at a local restaurant. Other 1993-94 officers: Michael Boleman, vice president; Gregg Dennis, secretary; Polly Quertermus, treasurer; Joe Sharp, corresponding secretary/faculty sponsor.

IL Beta Chapter President - Angie Cothrn/Laura Tougaw
Eastern Illinois University, Charleston 42 actives

Fall activities for IL Beta included business meetings, a MathClub/*KME* picnic in October, and a Christmas Party. The group also attended the 45th Annual ICTM Meeting in Springfield, IL and hosted a trip to Wolfram Research. Other 1993-94 officers: Jane Geier, vice president; Stephanie Seiler, secretary; Jennifer Brennan, treasurer; Lloyd Koontz, corresponding secretary/faculty sponsor; Paul Tougaw, faculty sponsor.

IL Eta Chapter President - Scott Clawson
Western Illinois University, Macomb

The IL Eta Chapter, in conjunction with the WIU Student MAA Chapter, held three meetings during the 1993 Fall Semester. The program for the September meeting featured mathematical games and exercises presented by student members of ICME and MAA. In October, presentations were made by faculty describing the content of various upper-division mathematics courses to be offered during the 1994 Spring Semester. Mr. Al Waters, Director of the Occupational Information and Placement Office, spoke in November, informing students of qualities prospective employers are looking for when interviewing students. He also discussed services the Placement Office can be expected to provide, and had students present complete a survey questionnaire for his office. Other 1993-94 officers: Brian Dalpaiz, vice president/treasurer; Eric Kies, secretary; Larry Morley, corresponding secretary/faculty sponsor.

IA Alpha Chapter President - Karen Brown
University of Northern Iowa, Cedar Falls 36 actives

The annual *KME* Homecoming Coffee, held October 9 at the home of Professors Emeriti Carl and Wanda Wehner, was well attended by faculty, students, and *KME* alums. Students presenting papers at IA Alpha *KME* meetings include: Ann Klaessy on "Hadamard Matrices," Lisa Gaskell on "Fractals, the Geometry of Nature," and Christopher Dix on "The Business of Calculus." On December 7, six new members were initiated into Iowa Alpha Chapter: Andrew Christianson, Jack Dostal, John Hamman, Maura Mast, Michelle Ruse, and Amy Wiemerslage. Student *KME* member Thomas Oleson, Jr., gave the address at the initiation banquet held at the Broom Factory on "Examples of Matrix Multiplication for the High School." Troy Meyers was selected to receive a membership in the Mathematical Association of America. Other 1993-94 officers: Jason Sash, vice president; Jennifer Puffett, secretary; Emily Eckman, treasurer; John S. Cross, corresponding secretary/faculty sponsor.

IA Beta
Drake University, Des Moines — 12 actives

The IA Beta Chapter met once during Fall, 1993. Professor Dan Alexander gave a talk on "The Monty Hall Problem and Marilyn vos Savant." H. K. Krishnapriyan is corresponding secretary.

IA Gamma
Chapter President - Brad Bock

Morningside College, Sioux City — 3 actives, 7 associates

New members were initiated and plans for fund raisers for Spring '94 were discussed. Other 1993-94: Dean Stevens, vice president; Taylor Guo, secretary; Mike Murray, treasurer; Steve Nimmo, corresponding secretary/faculty sponsor.

IA Delta
Chapter President - Van Beach

Wartburg College, Waverly — 32 actives

The program for the September meeting consisted of a report on the National Convention held in Niagara Falls in April of 1993. Pictures taken at the Convention were shared with those who were unable to attend and a science hall scavenger hunt served as a fun mixer to begin the year. Tom Hausmann, the Computer Center Manager, presented "Problems in Geometric Modeling and Collision Detection" as the program for the October meeting. Wartburg alumnus, Andrew Roquet, Corporate Project Manager at Century Companies of America, presented the November program entitled "Effectiveness in the Workplace." In December, a cookie decorating and eating party was held at the home of Dr. Glenn Fenneman. Other 1993-94 officers: Rebecca Hertenstein, vice president; Wendy Ahrensen, secretary; Jodie Harper, treasurer; August Waltmann, corresponding secretary; Glenn Fenneman, faculty sponsor.

KS Alpha
Chapter President - Mitch Richling

Pittsburg State University, Pittsburg — 60 actives

Fall semester activities focused on initiation of nine new members in October; a pizza party was held preceding the initiation. The chapter sponsored a guest speaker for the November meeting. Dr. John Wolfe, from Oklahoma State University, gave a presentation on "Geometry of Patterns." Other 1993-94 officers: Eddy Kuo, vice president; Kristi Simone, secretary; Sherry Brennon, treasurer; Harold Thomas, corresponding secretary; Bobby Winters, faculty sponsor.

KS Gamma Chapter President - Michael McGuire

Benedictine College, Atchison 12 actives, 16 associates

KS Gamma officers informed new students of *KME* activities at "Club Night." Throughout the semester planning sessions have been held every two weeks in the cafeteria. In mid-October the group enjoyed a Mexican dinner. In late October *KME* sponsored an Open House to share the new facility with parents during Parents' Weekend. An earlier Open House was held for students. The group participated in fund-raising activities, selling tickets for the Development Office at Homecoming and on Casino Night of Parents' Weekend. In November alum Jill Weigand returned to campus to share information on how she looked for and obtained a job. She gave the group many helpful hints and talked about coursework at Benedictine that she felt prepared her well for her job. The Christmas Wassail party was enjoyed at Jim Ewbank's home. Other 1993-94 officers: Chris Enyeart, vice president; Mary Kay Heideman, secretary; Tiffany Opsahl, treasurer; Shelly Kerwin, Stugo Representative; Jo Ann Fellin, OSB, corresponding secretary; Ann Petrus, CDP, faculty sponsor.

KS Delta Chapter President - Shelley Bauman

Washburn University, Topeka 32 actives

In October KS Delta held a meeting in conjunction with the Mathematics Club. The program was given by Dr. Aaron Stucker, the new Chair of the Washburn Mathematics Department, who spoke on "The Six Color Theorem." Other 1993-94 officers: Kirk Drager, vice president; David Brady, secretary; Kyndra Graybeal, treasurer; Allan Riveland, corresponding secretary; Gary Schimdt and Ron Wasserstein, faculty sponsors.

KS Epsilon Chapter President - Dale Brungardt

Fort Hays State University, Fort Hays 29 actives, 6 associates

In addition to monthly meetings, KS Epsilon enjoyed a fall picnic, a Halloween Party, and a Christmas Party. Other 1993-94 officers: Mark Pahls, vice president; Jason Purdy, secretary/treasurer; Charles Votaw, corresponding secretary; Mary Kay Schippers, faculty sponsor.

KY Alpha

Chapter President - Paula Christian

Eastern Kentucky University, Richmond — 20 actives, 7 associates

The work of the chapter's officers began even before the classes did. The officers manned a table displaying information about *KME* at the University's Organization Fair on the day before classes started. Sales of floppy disks to students in both the computer literacy class and the Mathematica class proceeded briskly during the beginning of the semester. At the September meeting Dr. Amy King spoke on "Hypatia, Hopper, Hyperbole and Hearsay." At the end of September a picnic with the faculty was held at the Costello house. Many students got their first exposure to Round-Robin Ping Pong at this event. The October meeting featured a panel discussion on various aspects of graduate school. At the November meeting Dr. Marijo LeVan spoke on "The Magical TI-85." For Thanksgiving, Sue Mattingly helped coordinate a food drive that collected food both on campus and at a local supermarket. The food was greatly appreciated by the Cardinal House. The last activity of the semester was the Christmas Party with the white elephant gift exchange. A great deal of hoopla was made about the gift brought by Dr. Mary Fleming, a very gaudy necklace. Other 1993-94 officers: Kaye Black, vice president; Andrea Warren, secretary; Angela Smith, treasurer; Pat Costello, corresponding secretary.

MD Alpha

Chapter President - Shawne Fischer

College of Notre Dame of Maryland, Baltimore — 13 actives, 6 associates

During the fall semester the members of the Math Society voted to change its name to The Hypatian Society. Programs for the meetings were given by the following members who reported on the opportunities offered them this past summer: Noreen Kurieshy, on a McNair Scholarship, worked at the University of Maine in Orono doing research; Rebecca Thompson, with a research grant from the National Science Foundation, worked in New Mexico studying the Intermediate Diaphran Connections on Steel Bridges; Sharon Pesto did research at the Space Telescope Science Institute; and Shawne Fischer and Debbie Riney both did research for the Maryland State Highway Administration. Dr. DiRienzi also contributed, telling the students about his summer at the Goddard Space Center. In December the chapter sponsored a talk by John Powers entitled "Pythagoras and His Academy." Other 1993-94 officers: Laure Saffran, vice president; Debbie Riney, secretary; Rebecca Thompson, treasurer; Sister Marie A. Dowling, corresponding secretary; Joseph DiRienzi, faculty sponsor.

MD Beta Chapter President - Todd Wisotzkey
Western Maryland College, Westminster 17 actives, 2 associates

The Fall Induction of two new members featured talks by two recent graduates, now in graduate school in mathematics. Other 1993-94 officers: Robert Brown, vice president; Emily Snyder, secretary; Sin Yee Wu, treasurer; James Lightner, corresponding secretary; Linda Eshleman, faculty sponsor.

MD Delta Chapter President - John Hughes
Frostburg State University, Frostburg 30 actives

MD Delta Chapter held monthly meetings. At the September meeting, Dr. Mark Hughes spoke on "Some Achievements of Asian Mathematics;" in October, the group enjoyed a "Pizza and Problem-Solving" session; and in November, Dr. Edward White presented "Some Mathematical Fallacies, or Ten Proofs That I Am Michael Jordan." The chapter also staffed the Mathematics Table at a Majors Expo sponsored by the University. Other 1993-94 officers: Kileen Baker, vice president; Karl Streaker, secretary; Melissa Thomas, treasurer; Edward White, corresponding secretary; John P. Jones, faculty sponsor.

MI Beta Chapter President - Erica Hall
Central Michigan University, Mount Pleasant 12 actives

Meetings were held every two weeks during the fall semester. During the first week of school, the chapter conducted a computer disc and mathematics textbook sale. This was a combined effort with the Actuarial Club, Gamma Iota Sigma, of the CMU Mathematics Department. The two organizations also cohosted a Homecoming picnic for faculty, local members, and *KME* and GIS alumni. Near the end of the semester, *KME* hosted an open house for students who possibly would be interested in joining *KME* in spring 1994 or later. Those in attendance enjoyed pizza while learning about the purposes and activities of the organization. MI Beta is eagerly looking forward to meeting the other chapters of Region 2 when hosting the Regional Convention in March. Other 1993-94 officers: Sara Meese, vice president; Jenny Blake, secretary; Mark Anderson, treasurer; Arnold Hammel, corresponding secretary/faculty sponsor.

MS Alpha Chapter President - Veda Fritch
Mississippi University for Women, Columbus 13 actives, 4 associates

Other 1993-94 officers: Daniel Fondren, vice president; Wanda Camill, secretary; Daniel Fondren, treasurer; Jean Parra, corresponding secretary; Shaochen Yang, faculty sponsor.

MS Gamma Chapter President - Tracy Thiel
University of Southern Mississippi, Hattiesburg 25 actives

Other 1993-94 officers: David Sitton, vice president; Stephanie Hart, secretary; Alice W. Essary, treasurer/corresponding secretary; Barry Piazza, Lida McDowell, and Karen Thrash, faculty sponsors.

MO Alpha Chapter President - Rhonda Fuge
Southwest Missouri State University, Springfield 25 actives, 8 associates

In addition to regular monthly meetings, MO Alpha, in conjunction with the Student Chapter of MAA, held the annual fall picnic and end of semester pizza party. Other 1993-94 officers: Chris Newland, vice president; Sarah Meiers, secretary; Randall Wakefield, treasurer; Ed Huffman, corresponding secretary/faculty sponsor.

MO Beta Chapter President - Kerry Baumgarth
Central Missouri State University, Warrensburg 25 actives, 6 associates

MO Beta chapter held three monthly meetings during the fall 1993 semester. At the September meeting, students reported on their summer internships and last year's National Convention. Initiation was held in October and the video "Chaos, Fractals, and Dynamics" was shown. At the November meeting, the group was given a demonstration of the WeMET two-way interactive television network. A book sale was also held in November. Social activities for the semester included a fall picnic and a Christmas party. Other 1993-94 officers: Moss Prewitt, vice president; Jan Van Leer, secretary; Paula Peters, treasurer; Rhonda McKee, corresponding secretary; Scotty Orr, Larry Dilley, and Phoebe Ho, faculty sponsors.

MO Epsilon Chapter President - Holly Toler
Central Methodist College, Fayette 13 actives

Other chapter officers: Audrey Hathaway, vice president; Jason Graves, secretary; Heather Warren, treasurer; William McIntosh, corresponding secretary/faculty sponsor; Linda Lembke, faculty sponsor.

MO Eta Chapter President - Jason Lott
Northeast Missouri State University, Kirksville 20 actives, 8 associates

In addition to monthly meetings, MO Eta is hosting a Math Expo for high school students, a reception in celebration of the chapter's 25th

Anniversary, and an annual picnic. A Division Card Party and Christmas Party were jointly hosted by the organization. Other 1993-94 officers: Deanne Reber, vice president; Judy Allen, secretary; Jeff Denzin, treasurer; Michelle Heupel, historian; Mary Sue Beersman, corresponding secretary; Shelle Palaski, faculty sponsor.

MO Iota Chapter President - Kim Tarnowieckyi
Missouri Southern State College, Joplin 8 actives, 10 associates

The chapter held three regular meetings during the semester. Former Chapter President Robert Stokes discussed with members his work as Mathematics Curriculum Specialist with Synergistic Systems. A second program was given by faculty member Charles Curtis, who conducted a demonstration on the "envy-free" division of a cake for the three-person case. Refreshments for this meeting were bits and pieces of cake. The group sponsored a picnic/hike/cave search at Roaring River State Park. As in previous Fall Semesters, chapter members, assisted by Math Club members, worked football concession stands as a fund raiser and a service to the college. The semester ended with a Christmas Party held at the home of Mrs. Elick. Other 1993-94 officers: Jim Boyer, vice president; Chante Artherton, secretary/treasurer; Mary Elick, corresponding secretary; Chip Curtis, faculty sponsor.

MO Kappa Chapter President - Cindy Schwab
Drury College, Springfield 9 actives, 5 associates

The first activity of the semester was an ice cream social for freshmen students. Throughout the semester, Math Club/*KME* ran a tutoring service for both the day school and the Continuing Education Division (Drury Evening College). In October the chapter made a trip to Argonne National Laboratory for The Graduate Fair. The three *KME* chapters in Springfield, MO Alpha, MO Theta, and MO Kappa, joined forces for an evening of volleyball at a local establishment. In conjunction with luncheon meetings, programs were given by Mr. Ted Nickle of the Mathematics Department and by student Cindy Schwab, who reported on her undergraduate research project. Winners of the Annual Math Contest were announced at a pizza party held for all contestants. Prize money was awarded to Michelle Biggers, winner of the Calculus I and Below Division, and to Cindy Schwab, winner of the Calculus II and Above Division. Other 1993-94 officers: Bill Davis, vice president; Kurtis Gann, secretary; Jeanie Wisdom, treasurer; Charles Allen, corresponding secretary; Don Moss, faculty sponsor.

MO Lambda Chapter President - Tracy Schemmer
Missouri Western State College, St. Joseph 38 actives, 15 associates

The semester got underway with a picnic and softball game in September. The chapter sponsored a flood relief effort in October and also held initiation ceremonies on October 17. Members shared a covered dish dinner in November. Other 1993-94 officers: Lee Napravnik, vice president; Dawn Powell, secretary; Denise Fuller, treasurer; John Atkinson, corresponding secretary; Jerry Wilkerson, faculty sponsor.

NE Alpha Chapter President - Leslie Iwai
Wayne State College, Wayne 23 actives

Throughout the semester club members monitored the Math-Science building in the evenings as a money making project. In conjunction with other science organizations on campus, the club made a banner and built a float for the Homecoming Parade. Both placed first in the campus competition. This year the club-sponsored book scholarships for members were awarded to Susan Sorenson and Jeff Schneider. Social activities included a fall picnic with the Math-Science faculty and other student organizations in the building, and a movie/pizza party at the home of Dr. Paige. Other 1993-94 officers: Susan Sorenson, vice president; Pat Spieler, secretary/treasurer; Jennie Hartman, historian; Fred Webber, corresponding secretary; Jim Paige and John Fuelberth, faculty sponsors.

NE Gamma Chapter President - Amy Wiese
Chadron State College, Chadron 15 actives, 5 associates

Other 1993-94 officers: Todd Ziettlow, vice president; Brandon Herdt, secretary; Kenneth Schultz, treasurer; James Kaus, corresponding secretary; Monty Fickel, Faculty Sponsor.

NE Delta Chapter President - George Wahle
Nebraska Wesleyan University, Lincoln 20 actives

Other 1993-94 officers: Shawn Clymer, vice president; Allison Hurt, secretary; Michael Dempsey, treasurer; Muriel Skoug, corresponding secretary/faculty sponsor.

NM Alpha Chapter President - Terry Lynn Vigil
University of New Mexico, Albuquerque 40 actives, 7 associates

Other 1993-94 officers: Jason Sanchez, vice president; David Black, secretary; Javier Armendariz, treasurer; Richard C. Metzler, corresponding secretary/faculty sponsor.

NY Eta Chapter President - Kenneth Krawczyk
Niagara University, Niagara University 15 actives

Other 1993-94 officers: Christine D'Angelo, vice president; Rebecca Bauer, secretary/treasurer; Robert Bailey, corresponding secretary; Kenneth Bernard, faculty sponsor.

NY Kappa Chapter President - Paula Murray
Pace University, New York 20 actives, 12 associates

Other 1993-94 officers: Mary Jo Curcio, vice president; Eileen Lawrence, secretary; Andrea Marchese, treasurer; Geraldine Taiani, corresponding secretary; Blanche Abramov and John Kennedy, faculty sponsors.

NY Lambda Chapter President - Suzanne Hecker
C. W. Post Campus of Long Island University, Brookville 17 actives

Other 1993-94 officers: Khaled Amleh, vice president; Concetta Vento, secretary/treasurer; Andrew Rockett, corresponding secretary; Sharon Kunoff, faculty sponsor.

NC Gamma Chapter President - Miguel Johnston
Elon College, Elon College 33 actives, 2 associates

At an organizational meeting in September, it was decided to offer associate membership to qualified students who had been studying in London at the time of the chapter's induction.The chapter shared some recreational mathematics with the school's Student MAA Chapter at a joint meeting in late September. Pranab Das of the Physics Department at Elon spoke to the group about "Chaos" in late October. Other 1993-94 officers: Charles Touron, vice president; Amy Hill, secretary; Emily Viverette, treasurer; Jeff Clark, corresponding secretary; David Nawrocki, faculty sponsor.

OH Alpha Chapter President - Jonathon Mitchell
Bowling Green State University, Bowling Green

Other 1993-94 officers: Mark Schumm, vice president; Marci Glavic, secretary/treasurer; Waldemar Weber, corresponding secretary; Truc Nguyen, faculty sponsor.

OK Alpha

Chapter President - Denise Sturgeon

Northeastern State University, Tahlequah

31 actives, 5 associates

Fall initiation ceremonies for nine students were held in the banquet room of a local restaurant. Chapter activities continue to be shared with NSU's student chapter of the MAA. Joint activities of the fall semester have included a monthly math contest, a fund raiser book sale, and the purchase of a video, "Life After Math," which has been shared with local junior and senior high math instructors. OK Alpha also sponsored a lecture by the nationally known book publisher and mathematician, John Saxon, which was attended by close to 100 students, faculty, and local teachers. In observance of Christmas, the chapter made a donation to a local charity to help buy food baskets for the poor and held a Christmas Party with good food, good games, and good times enjoyed by all. Other 1993-93 officers: Allison Mohr, vice president; Joni Nichols, secretary; Jennifer Beals, treasurer; Joan E. Bell, corresponding secretary/faculty sponsor.

OK Gamma

Chapter President - Joe Shepard

Southwestern Oklahoma State University, Weatherford

20 actives

Other 1993-94 officers: Cathy Lair, vice president; Stephanie Hill, secretary/treasurer; Wayne Hayes, corresponding secretary; Radwan Al-Jarrah, faculty sponsor.

OK Delta

Chapter President - Brian Augenstein

Oral Roberts University, Tulsa

10 actives

The chapter is attempting to organize a project to recycle mark-sense cards used campus wide for examinations. Other 1993-94 officers: Stephanie Wall, vice president; Amy Amsler, secretary; Lisa Brecheisen, treasurer; Debra Oltman, corresponding secretary/faculty sponsor.

PA Alpha

Chapter President - Julie Presecan

Westminster College, New Wilmington

21 actives

In addition to holding regular meetings, chapter members provided tutoring in the Learning Center for any Westminster student in need of help with mathematics. Other 1993-94 officers: Brian Wilson, vice president; Rhonda Witkowski, secretary; Melissa Napolean, treasurer; J. Miller Peck, corresponding secretary; Carolyn Cuff and Warren Hickman, faculty sponsors.

PA Beta Chapter President - Danielle Ambrosini
La Salle University, Philadelphia 10 actives

Daniel Madden, University of Arizona and the National Science Foundation, visited the University on November 11 and spoke on Public Encryption. Reenie Rojewski and Danielle Ambrosini accompanied Dr. Marijke Wijsmuller to a conference at Lebanon Valley College on careers in mathematics. Reenie and Danielle, together with Angela Rowbottom gave a multimedia presentation on careers in mathematics and computer science to an interested audience. Other 1993-94 officers: Richard Wojnar, vice president; Angela Rowbottom, secretary; Anne Hofmann, treasurer; Hugh Albright, corresponding secretary; Carl McCarty, faculty sponsor.

PA Gamma Chapter President - D. Scott Knee
Waynesburg College, Waynesburg 10 actives, 3 associates

Other 1993-94 officers: Jenifer L. Bowman, vice president; Kelly Sethman, secretary; Robert McNulty, treasurer; A. B. Billings, corresponding secretary/faculty sponsor.

PA Delta Chapter President - Kathleen Hanlon
Marywood College, Scranton 5 actives

Other 1993-94 officers: Robert Andrews, vice president; Marsha Galgon, secretary/treasurer; Sr. Robert Von Ahnen, corresponding secretary/faculty sponsor.

PA Eta Chapter President - Denise Good
Grove City College, Grove City 33 actives, 9 associates

Dr. McIntyre provided the program for the initiation of new members on October 27. He talked about math puzzles and offered prizes for solutions. Cider and doughnuts were served after the meeting. The annual *KME* Christmas party was held at Mr. Schlossnagel's residence in early December. Those in attendance sang Christmas carols and enjoyed refreshments. Other 1993-94 officers: Caroline Lucheta, vice president; Lara Skirpan, secretary; Heather Menzies, treasurer; Marvin Henry, corresponding secretary; Dan Dean, faculty sponsor.

PA Iota Chapter President - Jason Baker
Shippensburg University of Pennsylvania, Shippensburg 27 actives, 3 associates

The annual *KME* sponsored fall picnic for all math and computer science students and faculty members took place this year after a morning of Highway Cleanup. *KME* and Math Club have adopted a two mile stretch of Route 11 as part of the Adopt-a-Highway program in Pennsylvania and members are doing a great job looking after it. Also, the chapter sold sweatshirts; members had a choice of color and fabric for the logo. Other 1993-94 officers: Angela Foltz, vice president; Christm Tipa, secretary; Frederick Nordai, treasurer; Michael D. Seyfried; corresponding secretary/faculty sponsor.

PA Kappa Chapter President - Dawn McDermond
Holy Family College, Philadelphia 8 actives

PA Kappa is currently undergoing a period of restructuring with the goal of becoming a more visible and viable campus organization. Initiation of possibly six new members will take place during the spring semester. An MAA speaker, Dr. Sanora Fillebrown, has been invited to speak to the membership in March, 1994, on "Fractals, Chaos, and Dynamic Systems." At present, several members staff the Math Tutoring Center during their lunch periods and offer assistance in everything from college algebra to calculus IV. Other 1993-94 officers: Mary Beth Emery, vice president; Sr. Marcella Louise Wallowicz, corresponding secretary/ faculty sponsor.

PA Mu Chapter President - Patricia D. George
Saint Francis College, Loretto 17 actives

Seven students and Dr. Peter Skoner attended the National Council of Teachers of Mathematics (NCTM) Eastern Regional Conference in October. Members and their guests enjoyed volleyball and softball at the annual fall picnic held at Lake Saint Francis. In December, the mathematics and computer science seminar students presented their semester work. Their presentations followed a luncheon and brief meeting. The chapter is anxiously awaiting the installation of new inductees, which will occur in the spring semester. Also, many members are looking forward to attending the regional conference which is to be held in April. Other 1993-94 officers: Scott Beers, vice president; Gerry Albright, secretary; Paul Schiele, treasurer; Peter Skoner, corresponding secretary; Adrian Baylock, faculty sponsor.

PA Nu Chapter President - Dana Fosbenner
Ursinus College, Collegeville 14 actives

PA Nu sponsored a lecture in early October by Professor Sandra Fillebrown of St. Joseph's University. Her talk, entitled "Fractal Images," was well attended by both students and faculty. Refreshments were served before the talk and an interesting discussion period followed. Other 1993-94 officers: Sarah Lee, vice president; Kara Raiguel, secretary; Ryan Savitz, treasurer; Jeff Neslen, corresponding secretary; Richard Bremiller, faculty sponsor.

PA Xi Chapter President - Cynthia Sturtevant
Cedar Crest College, Allenton 8 actives, 2 associates

On November 13, 1993, the fall meeting of Eastern Pennsylvania and Delaware Section of the Mathematical Association of America was held at Cedar Crest College. Guest lecturers for the event were Joseph A. Gallian from University of Minnesota, whose presentation focused on error detection of identification numbers; Polya Lecturer, Carl Pomerance, University of Georgia, who discussed the applications of Fermat's Little Theorem, including public key cryptography; JoAnne Growney of Bloomsburg University, who gave a luncheon talk on mathematics in poetry; Kenneth A. Brakke, of Susquehanna University, who talked on "Soap Films and Covering Spaces;" and Mary Ellen Rudin, University of Wisconsin-Madison, who lectured on "The Rationals and Irrationals." Following the meeting, PA Xi held induction ceremonies for two new members. Other 1993-94 officers: Aileen Gula, vice president/secretary/ treasurer; Regina Brunner, corresponding secretary; Charles Chapman, faculty sponsor.

SC Beta Chapter President - Stephanie Clarkson
South Carolina State College, Orangeburg 16 actives, 10 associates

Other 1993-94 officers: Alphine Bradley, vice president; Yolanda Williams, secretary; Patrick Belton, treasurer; Cynthia Davis, corresponding secretary; G. R. Viswanath, faculty sponsor.

SC Gamma Chapter President - Rhonda Crisp
Winthrop University, Rock Hill 17 actives, 4 associates

Two meetings were held during the fall. The *KME* induction ceremony on November 14 was a great success. The chapter members, in conjunction with the mathematics faculty, sponsored an End-of-the-Year Bash for all math majors. Plans were made to hold a review for elementary education majors who need help with mathematics in February '94. Other 1993-94 officers: Meridth English, vice president; Danna Shaver, secretary; Christina Sanford, treasurer; Donald Aplin, corresponding secretary; Edward Guettler, faculty sponsor.

TN Beta Chapter President - Marvin Johnson
East Tennessee State University, Johnson City 41 actives

Fall activities included electing officers for the academic year, hosting a social at Ryan's Steakhouse, and meeting to have a yearbook picture taken. In addition, plans were made to attend the annual spring meeting of the Mathematical Association of America. Other 1993-94 officers: Jennifer Shupe, vice president; Michele Cooke, secretary; Kristen Cooper, treasurer; Lyndell Kerley, corresponding secretary/faculty sponsor.

TN Delta Chapter President - John C. Knight
Carson-Newman College, Jefferson City 18 actives

The chapter held their annual opening school picnic in defiance of a "frog-strangling downpour." They also made plans for a biennial department newsletter for spring, 1994. Other 1993-94 officers: Brenda Bleavins, vice president; Rebecca Sowder, secretary; Melissa Simpson, treasurer; Verner Hansen, corresponding secretary; Carey R. Herring, faculty sponsor.

TX Alpha Chapter President - Angela May
Texas Tech University, Lubbock 25 actives

Other 1993-94 officers: Curt Bourne, vice president; Jason Nichols, secretary; Doug Stevens, treasurer; Robert Moreland, corresponding secretary; Edward Allen, faculty sponsor.

TX Eta Chapter President - Kristen Hieronymus
Hardin-Simmons University, Abilene

The chapter sponsored a Mathematics Career Seminar in November; soft drinks and pizza were provided for all those who attended. Other 1993-94 officers: Kimberly Sarles, vice president; Robyn Eads, secretary; Amy Garrison, treasurer; Frances Renfroe, corresponding secretary; Charles Robinson and Ed Hewett, faculty sponsors.

TX Kappa Chapter President - Thomas Ross
University of Mary Hardin-Baylor, Belton 15 actives, 5 associates

The Texas Kappa Chapter of Kappa Mu Epsilon conducted its Fall Forum on November 11, 1993. It included a lively discussion on the scope and purpose of Mathematics in today's world. The organization also decided to hold a "Spring Symposium" on April 21, 1994. The topic of the Spring Symposium will be "Scholarships and Awards in Mathematics." Other 1993-94 officers: Elizabeth Barlett, vice president;

Kori Whatley, secretary; Nathan Hagemann, treasurer; Peter H. Chen, corresponding secretary; Maxwell Hart, faculty sponsor.

VA Alpha Chapter President - Kevin Taylor
Virginia State University, Petersburg 20 actives

Other 1993-94 officers: Vaughn McMullin, vice president; Yolanda Pierce, secretary; M. D. Tewari, treasurer; Emma B. Smith, corresponding secretary; V. Sagar Bakhshi, faculty sponsor.

WI Alpha Chapter President - Silvia Navarro
Mount Mary College, Milwaukee 5 actives, 4 associates

On the Saturday before Thanksgiving, Wisconsin Alpha Chapter, along with the Mathematics Department of Mount Mary College, sponsored a mathematics contest for high school junior and senior women. A partial scholarship to Mount Mary College was the top award. Other 1993-94 officers: Erin Hein, vice president/treasurer; Sister Adrienne Eickman, corresponding secretary/faculty sponsor.

WI Beta Chapter President - Suzanne Langer
University of Wisconsin-River Falls, River Falls 7 actives, 2 associates

During the fall semester the chapter shared four major events with the ACM Chapter on campus: they held a fall picnic and a Christmas/ Senior Sendoff Party, set up a table during Homecoming, and sold Halloween bags as a fund-raiser for charity. Other 1993-94 officers: Janett Pugh, vice president; Laura Andrews, secretary; Doug Lindee, treasurer; Robert Coffman, corresponding secretary/faculty sponsor.

WI Gamma Chapter President - Traci Saari
University of Wisconsin-Eau Claire, Eau Claire 12 actives, 10 associates

Other 1993-94 officers: Jennifer Jahnke, vice president; Deb Bifano, secretary; Bryce Rudolph, treasurer; Tom Wineinger, corresponding secretary; Marc Goulet, faculty sponsor.

Announcement of the Thirtieth Biennial Convention of Kappa Mu Epsilon

The Thirtieth Biennial Convention of Kappa Mu Epsilon will be hosted by the Colorado Gamma Chapter and will be held 20-22 April 1995 at Fort Lewis College in Durango, Colorado. Each attending chapter will receive the usual travel expense reimbursement from the national funds as described in Article VI, Section 2, of the Kappa Mu Epsilon Constitution.

A significant feature of this convention will be the presentation of papers by student members of Kappa Mu Epsilon. The mathematical topic selected by each student speaker should be of interest to the author and of such scope that it can be given adequate treatment in a timed oral presentation. Student speakers will be chosen by the Selection Committee on the basis of written papers submitted prior to the convention. At the convention, the Awards Committee (composed of four students and four faculty members representing as many chapters as possible) will judge the speakers on both content and presentation. The rankings of the Selection and Awards Committees will determine the top four papers.

Who may submit a paper?

Any undergraduate or graduate student member of Kappa Mu Epsilon may submit a paper for use on the convention program. A paper may be co-authored. If selected for presentation at the convention, the paper must be presented by one (or more) of the authors. Graduate students will not compete for prizes with undergraduates.

Presentation topics.

Papers submitted for presentation at the convention should discuss material understandable by undergraduate mathematics majors, preferably those who have completed differential and integral calculus. The Selection Committee will favor papers that satisfy this criteria and which can be presented with reasonable completeness within the time allotted.

Presentation time limits.

The presentation of the paper must take at least 15 minutes and no more than 20 minutes.

How to prepare a paper.

Five copies of your paper, together with a description of any charts, models or other visual aids you plan to use during the presentation, must be submitted. The paper should be typewritten in the standard form of a term paper. It should be written as it will be presented, including length. A long paper (such as an honors thesis) must not be submitted with the idea that it will be shortened later when you present it! Appropriate references and bibliography are expected.

The first page of your paper must be a "cover sheet" giving the following information: (1) title, (2) author (your name must not appear elsewhere in the paper), (3) your student status ("undergraduate" or "graduate"), (4) both your permanent and school addresses, (5) the name of your *KME* Chapter and school, (6) a signed statement giving your approval that your paper be considered for publication in *The Pentagon*, and (7) the signed statement of your Chapter's Corresponding Secretary that you are indeed a member of Kappa Mu Epsilon.

How to submit a paper.

You must send the five copies of your paper to:

Dr. Patrick Costello, *KME* National President-Elect
Department of Mathematics, Statistics and Computer Science
Eastern Kentucky University
Richmond, Kentucky 40475

no later than 10 February 1995.

Selection of papers for presentation.

The Selection Committee will review all papers submitted to the National President-Elect and will choose approximately fifteen papers for presentation at the convention; all other papers will be listed by title and author in the convention program and will be available as "alternates." The National President-Elect will notify all authors of the status of their papers after the Selection Committee has completed its deliberations.

Criteria used by the Selection and Awards Committees.

The paper will be judged on (1) topic originality, (2) appropriateness to the meeting and audience, (3) organization, (4) depth and significance of the content, and (5) understanding of the material. The presentation will be evaluated on (1) style of presentation, (2) maintenance of interest, (3) use of audio-visual materials (if applicable), (4) enthusiasm for the topic, (5) overall effect, and (6) adherence to the time limits.

Prizes.

All authors of papers presented at the convention will be given two-year extensions of their *Pentagon* subscriptions. Authors of the four best papers presented by undergraduate students, as determined by the Selection and Awards Committees, will each receive a cash prize of $100. If several papers are presented by graduate students, then one or more prizes will be awarded in this category.

Publication.

All papers submitted to the convention are considered as submitted for publication in *The Pentagon* (see page 2 of *The Pentagon* for further information). Prize winning papers will be published after any necessary revisions have been completed and all other papers will be considered for publication. All authors are expected to schedule brief meetings with the Editor during the convention to review their manuscripts.

ACKNOWLEDGEMENT

The Editor thanks the following individuals for their service during the past two years as referees of papers submitted to *The Pentagon*.

Radwan Al-Jarrah
Southwestern Oklahoma State University
Weatherford, Oklahoma

V. Sagar Bakhshi
Virginia State University
Petersburg, Virginia

Duane M. Broline
Eastern Illinois University
Charleston, Illinois

Mary Beth Dever
Illinois Benedictine College
Lisle, Illinois

Sister Marie Augustine Dowling
College of Notre Dame of Maryland
Baltimore, Maryland

Donald Gelman
C. W. Post Campus of Long Island University
Brookville, New York

Dennis R. Harmon
Missouri Southern State College
Joplin, Missouri

Lyndell M. Kerley
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Johnson City, Tennessee

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Lisle, Illinois

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Mississippi State University
Mississippi State, Mississippi

Andrew V. Talmadge
Arkansas State University
State University, Arkansas

Elizabeth G. Yannik
Emporia State University
Emporia, Kansas

Kappa Mu Epsilon National Officers

Kappa Mu Epsilon, Mathematics Honor Society, was founded in 1931. The object of the Society is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics due to its demands for logical and rigorous modes of thought; to provide a Society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; and to disseminate the knowledge of mathematics and familiarize the members with the advances being made in mathematics. The official journal of the Society, *The Pentagon*, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the Chapters.

Active Chapters of Kappa Mu Epsilon

Listed by date of installation.

Chapter	Location	Installation Date
OK Alpha	Northeastern Oklahoma State University, Tahlequah	18 April 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Southwest Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
MS Beta	Mississippi State University, Mississippi State College	14 Dec 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
NM Alpha	University of New Mexico, Albuquerque	28 March 1935
IL Beta	Eastern Illinois University, Charleston	11 April 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 April 1937
OH Alpha	Bowling Green State University, Bowling Green	24 April 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	Central Missouri State University, Warrensburg	10 June 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
TX Beta	Southern Methodist University, Dallas	15 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 June 1941
NY Alpha	Hofstra University, Hempstead	4 April 1942
MI Beta	Central Michigan University, Mount Pleasant	25 April 1942
NJ Beta	Montclair State College, Upper Montclair	21 April 1944
IL Delta	College of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 March 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 June 1947
CO Alpha	Colorado State University, Fort Collins	16 May 1948
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950

IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 April 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	Kearney State College, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri - Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
IL Epsilon	North Park College, Chicago	22 May 1963
OK Beta	University of Tulsa, Tulsa	3 May 1964
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood College, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 April 1965
AL Epsilon	Huntingdon College, Montgomery	15 April 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
AR Alpha	Arkansas State University, State University	21 May 1965
TN Gamma	Union University, Jackson	24 May 1965
WI Beta	University of Wisconsin - River Falls, River Falls	25 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	Western Maryland College, Westminster	30 May 1965
IL Zeta	Rosary College, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Northeast Missouri State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel College, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 March 1971

KY Alpha	Eastern Kentucky University, Richmond	27 March 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 April 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 April 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State College, Joplin	8 May 1975
GA Alpha	West Georgia College, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin - Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sept 1978
IL Theta	Illinois Benedictine College, Lisle	18 May 1979
PA Mu	St. Francis College, Loretto	14 Sept 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C. W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury College, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 March 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 April 1986
TX Iota	McMurry College, Abilene	25 April 1987
PA Nu	Ursinus College, Collegeville	28 April 1987
VA Gamma	Liberty University, Lynchburg	30 April 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 April 1990
CO Delta	Mesa State College, Grand Junction	27 April 1990
NC Gamma	Elon College, Elon College	3 May 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 April 1991
SD Alpha	Northern State University, Aberdeen	3 May 1992
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 March 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993