

# THE PENTAGON

*A Mathematics Magazine for Students*

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Volume 52 Number 2

Spring 1993

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*The Pentagon* (ISSN 0031-4870) is published semiannually in December and May by Kappa Mu Epsilon. No responsibility is assumed for opinions expressed by individual authors. Manuscripts of interest to undergraduate mathematics majors and first year graduate mathematics students are welcome, particularly those written by students. Submissions should be typewritten (double spaced with wide margins) on white paper, standard notation conventions should be respected and special symbols should be carefully inserted by hand in black ink. All illustrations must be submitted on separate sheets and drawn in black ink. Computer programs, although best represented by pseudocode in the main text, may be included as an appendix. Graphs, tables or other materials taken from copyrighted works MUST be accompanied by an appropriate release from the copyright holder permitting further reproduction. Student authors should include the names and addresses of their faculty advisors. Contributors to *The Problem Corner* or *Kappa Mu Epsilon News* are invited to correspond directly with the appropriate Associate Editor. Electronic mail may be sent to (Bitnet) PENTAGON@LIUVAX.

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## The Jacobi Symbol

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Presented at the 1991 National Convention.

The Jacobi symbol, which was introduced by Carl Jacobi, is useful in eliminating the possibility of a number being prime. Since the Jacobi symbol is a generalization of the Legendre symbol, a discussion of the Legendre symbol will be presented first and then generalized to the Jacobi symbol.

**Definition.** Let  $p$  be an odd prime and  $a$  an integer not divisible by  $p$ . Then the Legendre symbol  $\left(\frac{a}{p}\right)$  is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } x^2 \equiv a \pmod{p} \text{ for some } x \\ -1, & \text{if } x^2 \not\equiv a \pmod{p} \text{ for any } x \end{cases}$$

That is,  $\left(\frac{a}{p}\right) = 1$  if  $a$  is a quadratic residue of  $x$  and  $\left(\frac{a}{p}\right) = -1$  if  $a$  is a quadratic nonresidue of  $x$ .

"Euler's Criterion" can be used to determine the value of a Legendre symbol.

**Euler's Criterion.** Let  $p$  be an odd prime and  $1 \leq a \leq p-1$ . (1) If  $a^{(p-1)/2} \equiv 1 \pmod{p}$ , then  $a$  is a quadratic residue. (2) If  $a^{(p-1)/2} \equiv -1 \pmod{p}$ , then  $a$  is a quadratic nonresidue.

Combining the definition of the Legendre symbol with Euler's Criterion, we have

$$\left(\frac{a}{p}\right) = a^{(p-1)/2} \pmod{p}.$$

The Legendre symbol has the following properties:

$$(1) \quad \text{If } a \equiv b \pmod{p}, \text{ then } \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right).$$

$$(2) \quad \left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right).$$

$$(3) \quad \left(\frac{-1}{p}\right) = \begin{cases} 1, & \text{if } p \equiv 1 \pmod{4} \\ -1, & \text{if } p \equiv -1 \pmod{4} \end{cases}$$

$$(4) \quad \left(\frac{a^2}{p}\right) = 1 \text{ if } a \text{ is not divisible by } p.$$

$$(5) \quad \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} \text{ if } p \text{ is an odd prime.}$$

Let us give a proof for property (2); that is, that the product of two residues will be a residue, the product of two nonresidues will be a residue and the product of a residue and a nonresidue will be a nonresidue. By Euler's Criterion,

$$\left(\frac{a}{p}\right) = a^{(p-1)/2} \pmod{p}, \left(\frac{b}{p}\right) = b^{(p-1)/2} \pmod{p}$$

and

$$\left(\frac{ab}{p}\right) = (ab)^{(p-1)/2} \pmod{p}.$$

Thus

$$\begin{aligned} \left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right) &= a^{(p-1)/2} \cdot b^{(p-1)/2} \pmod{p} \\ &= (ab)^{(p-1)/2} \pmod{p} = \left(\frac{ab}{p}\right). \end{aligned}$$

These properties are useful to evaluate Legendre symbols. For example,

$$\left(\frac{2}{7}\right) = (-1)^{(7^2-1)/8} = (-1)^6 = 1$$

by property (5),

$$\left(\frac{4}{7}\right) = \left(\frac{2^2}{7}\right) = 1$$

by property (4), and

$$\left(\frac{6}{7}\right) = \left(\frac{2}{7}\right) \cdot \left(\frac{3}{7}\right) = 1 \cdot \left(\frac{3}{7}\right) = \left(\frac{3}{7}\right)$$

by property (2) and our first example, and then by Euler's Criterion,

$$\left(\frac{3}{7}\right) = 3^{(7-1)/2} \pmod{7} = 3^3 \pmod{7} = -1 \pmod{7}$$

so that

$$\left(\frac{6}{7}\right) = -1.$$

As stated previously, the Jacobi symbol is a generalization of the Legendre symbol.

**Definition.** Let  $n$  be an odd positive integer with prime factorization

$$n = p_1^{t_1} p_2^{t_2} \cdots p_m^{t_m}$$

and  $a$  an integer relatively prime to  $n$ . Then the Jacobi symbol  $\left(\frac{a}{n}\right)$  is defined by

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1^{t_1} p_2^{t_2} \cdots p_m^{t_m}}\right) = \left(\frac{a}{p_1}\right)^{t_1} \left(\frac{a}{p_2}\right)^{t_2} \cdots \left(\frac{a}{p_m}\right)^{t_m}$$

where the symbols on the right-hand side are Legendre symbols.

For example,

$$\left(\frac{109}{385}\right) = \left(\frac{109}{5 \times 7 \times 11}\right) = \left(\frac{109}{5}\right) \left(\frac{109}{7}\right) \left(\frac{109}{11}\right)$$

and then, using the properties of the Legendre symbols, this is

$$\begin{aligned} &= \left(\frac{4}{5}\right) \left(\frac{4}{7}\right) \left(\frac{10}{11}\right) = \left(\frac{2^2}{5}\right) \left(\frac{2^2}{7}\right) \left(\frac{-1}{11}\right) \\ &= 1 \cdot 1 \cdot \left(\frac{-1}{11}\right) = -1, \end{aligned}$$

since  $11 \equiv -1 \pmod{4}$ .

Note that the value of the Jacobi symbol does not tell us whether or not  $x^2 \equiv a \pmod{n}$  has a solution (the value of the Legendre does give

this information). However, if  $x^2 \equiv a \pmod{n}$  has a solution, the Jacobi symbol  $\left(\frac{a}{n}\right) = 1$ . For example,

$$\left(\frac{2}{15}\right) = \left(\frac{2}{3}\right)\left(\frac{2}{5}\right) = (-1) \cdot (-1) = 1$$

(using Euler's Criterion). If this were a Legendre symbol, we could now say that  $x^2 \equiv 2 \pmod{15}$  has a solution. But this is a Jacobi symbol and we cannot draw this conclusion. In fact,  $x^2 \equiv 2 \pmod{15}$  has no solution since we have that

$$\begin{array}{lll} 1^2 \equiv 1 & 6^2 \equiv 6 & 11^2 \equiv 1 \\ 2^2 \equiv 4 & 7^2 \equiv 4 & 12^2 \equiv 9 \\ 3^2 \equiv 9 & 8^2 \equiv 4 & 13^2 \equiv 4 \\ 4^2 \equiv 1 & 9^2 \equiv 6 & 14^2 \equiv 1 \\ 5^2 \equiv 10 & 10^2 \equiv 10 & \end{array} \pmod{15}.$$

The Jacobi symbol has properties similar to those of the Legendre symbol.

$$(1) \quad \text{If } a \equiv b \pmod{n}, \text{ then } \left(\frac{a}{n}\right) = \left(\frac{b}{n}\right).$$

$$(2) \quad \left(\frac{ab}{n}\right) = \left(\frac{a}{n}\right) \cdot \left(\frac{b}{n}\right).$$

$$(3) \quad \left(\frac{-1}{n}\right) = (-1)^{(n-1)/2}.$$

$$(4) \quad \left(\frac{2}{n}\right) = (-1)^{(n^2-1)/8}.$$

We now prove properties 1, 2 and 3. Let  $n = p_1^{t_1} p_2^{t_2} \cdots p_m^{t_m}$ . For (1), we have that  $a \equiv b \pmod{n}$ ; recall that if  $p$  is a prime divisor of  $n$ , then  $a \equiv b \pmod{p}$  as well (i. e., if  $13 \equiv 1 \pmod{12}$ , then  $13 \equiv 1 \pmod{3}$ ). Therefore,

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{t_1} \left(\frac{a}{p_2}\right)^{t_2} \cdots \left(\frac{a}{p_m}\right)^{t_m} = \left(\frac{b}{p_1}\right)^{t_1} \left(\frac{b}{p_2}\right)^{t_2} \cdots \left(\frac{b}{p_m}\right)^{t_m} = \left(\frac{b}{n}\right)$$

as needed. For (2),

$$\begin{aligned} \left(\frac{ab}{n}\right) &= \left(\frac{ab}{p_1}\right)^{t_1} \left(\frac{ab}{p_2}\right)^{t_2} \cdots \left(\frac{ab}{p_m}\right)^{t_m} \\ &= \left(\frac{a}{p_1}\right)^{t_1} \left(\frac{b}{p_1}\right)^{t_1} \left(\frac{a}{p_2}\right)^{t_2} \left(\frac{b}{p_2}\right)^{t_2} \cdots \left(\frac{a}{p_m}\right)^{t_m} \left(\frac{b}{p_m}\right)^{t_m} \end{aligned}$$

$$= \left( \left( \frac{a}{p_1} \right)^{t_1} \left( \frac{a}{p_2} \right)^{t_2} \cdots \left( \frac{a}{p_m} \right)^{t_m} \right) \left( \left( \frac{b}{p_1} \right)^{t_1} \left( \frac{b}{p_2} \right)^{t_2} \cdots \left( \frac{b}{p_m} \right)^{t_m} \right) = \left( \frac{a}{n} \right) \left( \frac{b}{n} \right).$$

To show (3), we use Euler's Criterion  $\left( \frac{-1}{p} \right) = (-1)^{(p-1)/2}$  to find

$$\begin{aligned} \left( \frac{-1}{n} \right) &= \left( \frac{-1}{p_1} \right)^{t_1} \left( \frac{-1}{p_2} \right)^{t_2} \cdots \left( \frac{-1}{p_m} \right)^{t_m} \\ &= (-1)^{t_1(p_1-1)/2} \cdot (-1)^{t_2(p_2-1)/2} \cdots (-1)^{t_m(p_m-1)/2} \\ &= (-1)^{t_1(p_1-1)/2 + t_2(p_2-1)/2 + \cdots + t_m(p_m-1)/2}. \end{aligned}$$

If we can show that

$$(n-1)/2 \equiv t_1(p_1-1)/2 + t_2(p_2-1)/2 + \cdots + t_m(p_m-1)/2 \pmod{2},$$

then our proof will be complete. By adding and subtracting one to each prime, we may rewrite the factorization of  $n$  as

$$n = \left( 1 + (p_1 - 1) \right)^{t_1} \left( 1 + (p_2 - 1) \right)^{t_2} \cdots \left( 1 + (p_m - 1) \right)^{t_m}$$

Since each  $p_i$  is an odd prime,  $p_i - 1$  is even and by the binomial theorem it follows that

$$\left( 1 + (p_i - 1) \right)^{t_i} \equiv 1 + t_i(p_i - 1) \pmod{4}.$$

Further, since  $p_i$  and  $p_j$  are odd primes,  $p_i - 1$  and  $p_j - 1$  are even and so

$$\begin{aligned} &\left( 1 + t_i(p_i - 1) \right) \cdot \left( 1 + t_j(p_j - 1) \right) \\ &\equiv 1 + t_i(p_i - 1) + t_j(p_j - 1) \pmod{4}. \end{aligned}$$

Combining these two observations,

$$\begin{aligned} n &= \left( 1 + (p_1 - 1) \right)^{t_1} \left( 1 + (p_2 - 1) \right)^{t_2} \cdots \left( 1 + (p_m - 1) \right)^{t_m} \\ &\equiv \left( 1 + t_1(p_1 - 1) \right) \left( 1 + t_2(p_2 - 1) \right) \cdots \left( 1 + t_m(p_m - 1) \right) \pmod{4} \\ &\equiv 1 + t_1(p_1 - 1) + t_2(p_2 - 1) + \cdots + t_m(p_m - 1) \pmod{4}. \end{aligned}$$

So

$$n-1 \equiv t_1(p_1-1) + t_2(p_2-1) + \cdots + t_m(p_m-1) \pmod{4},$$

$$(n-1)/2 \equiv t_1(p_1-1)/2 + t_2(p_2-1)/2 + \cdots + t_m(p_m-1)/2 \pmod{2}$$

and we are done.

We now state the "law of reciprocity," which is also true for the Legendre symbol and which will be used in my last example.

**Reciprocity Law for the Jacobi Symbol.** Let  $n$  and  $m$  be relatively prime odd positive integers. Then

$$\left(\frac{n}{m}\right) = \left(\frac{m}{n}\right)$$

if either  $n \equiv 1 \pmod{4}$  or  $m \equiv 1 \pmod{4}$  or both, or if both  $n \equiv 3 \pmod{4}$  and  $m \equiv 3 \pmod{4}$ .

For example,

$$\left(\frac{1009}{2307}\right) = \left(\frac{1009}{3 \times 769}\right) = \left(\frac{1009}{3}\right) \cdot \left(\frac{1009}{769}\right)$$

and, by property (1) for these Legendre symbols, this is

$$\begin{aligned} &= \left(\frac{1}{3}\right) \cdot \left(\frac{240}{769}\right) = 1 \cdot \left(\frac{2^4 \times 3 \times 5}{769}\right) \\ &= \left(\frac{2}{769}\right)^4 \cdot \left(\frac{3}{769}\right) \left(\frac{5}{769}\right) = 1 \cdot \left(\frac{3}{769}\right) \left(\frac{5}{769}\right). \end{aligned}$$

Since  $769 \equiv 1 \pmod{4}$ , reciprocity gives that this is

$$= \left(\frac{769}{3}\right) \left(\frac{769}{5}\right)$$

and then by property (1) and Euler's Criterion,

$$\left(\frac{1}{3}\right) \cdot \left(\frac{-1}{5}\right) = 1 \cdot (-1)^{(5-1)/2} = 1.$$

In conclusion, one method of using Jacobi and Legendre symbols to eliminate numbers as possible primes will be presented.

Fermat's "Little Theorem" states that if  $p$  is a prime, then  $b^{p-1} \equiv 1 \pmod{p}$ . The contrapositive of this theorem can be used to eliminate numbers as possible primes. That is, if  $b^{n-1} \not\equiv 1 \pmod{n}$ , then  $n$  is composite. However, if  $b^{n-1} \equiv 1 \pmod{n}$ ,  $n$  may or may not be prime. In this case, where  $b^{n-1} \equiv 1 \pmod{n}$ , we can further test  $n$  by using



Jacobi symbols: if the congruence  $b^{(n-1)/2} \equiv \left(\frac{b}{n}\right) \pmod{n}$  fails, then  $n$  is composite (if the congruence holds,  $n$  may or may not be prime). For example, let us try the number 3186821. Trying the contrapositive of Fermat's Little Theorem, we find

$$2^{(3186821-1)} \equiv 1 \pmod{3186821}.$$

Therefore, we cannot determine whether 3186821 is prime or composite. We can now use Jacobi symbols to further test the number. By property (4) of Jacobi symbols,

$$\left(\frac{2}{3186821}\right) = (-1)^{(3186821^2-1)/8} = -1,$$

while by Euler's Criterion

$$2^{(3186821-1)/2} \equiv +1 \pmod{3186821}.$$

Since

$$\left(\frac{2}{3186821}\right) \not\equiv 2^{(3186821-1)/2} \pmod{3186821},$$

the number 3186821 is composite (in fact,  $3186821 = 11 \times 281 \times 1031$ ).

*Acknowledgement.* I would like to thank my faculty advisor, Dr. Patrick J. Costello, for his help with this project.

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## Generalized Inverses of Rectangular Matrices and Applications

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Many students and instructors of mathematics are familiar with inverses of square matrices, as well as numerous uses for those inverses. However, data are not always so kind as to fit into neat little square matrices. Although it is not possible to create an actual inverse, as such, for a rectangular matrix, it is possible to construct several matrices which have a number of the same properties of an actual inverse. These matrices are called generalized inverses, g-inverses, pseudo-inverses or conditional inverses. The term "generalized inverse" will work quite efficiently for our purposes, so it will be used hereafter. The primary difference between a generalized inverse and an "actual" inverse is that an actual inverse  $B^{-1}$  of a square matrix  $B$  must satisfy  $B^{-1}B = I$ , whereas a generalized inverse  $A^{-}$  of a rectangular matrix  $A$  must satisfy  $AA^{-}A = A$ .

In an effort to demonstrate one procedure for determining generalized inverses for rectangular matrices and to illustrate one use of such inverses, let us consider the following hypothetical situation.

Barbara recently graduated from college with a Bachelor of Science degree in mathematics. She was quickly hired by the Hewlett Packard Corporation and placed in charge of three machines. Each of these three machines worked upon four different models of calculators, the first machine did all of the soldiering, the second molded the outside casings, and the third did the final assembly.

On her second day at work, Barb's boss presented her with the following assignment: find a way to keep all three machine busy for eight

hours every day given that each machine must operate on each model of calculator for the length of time specified in the following table.

	Model #1	Model #2	Model #3	Model #4
Machine #1	1	2	1	2
Machine #2	3	1/2	2	0
Machine #3	1	3/2	1/6	4

Time required per calculator.

The table states, for instance, that machine number three must work on each calculator of model number two exactly one and one-half hours.

Barb realized that since she had four products, three machines and each machine was to work exactly eight hours, she could call the products  $n_1$ ,  $n_2$ ,  $n_3$  and  $n_4$  and write the following equations:

$$1n_1 + 2n_2 + 1n_3 + 2n_4 = 8$$

$$3n_1 + (1/2)n_2 + 2n_3 + 0n_4 = 8$$

$$1n_1 + (3/2)n_2 + (1/6)n_3 + 4n_4 = 8$$

This system of equations was of the form  $Ax = y$  where

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1/2 & 2 & 0 \\ 1 & 3/2 & 1/6 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix} \text{ and } y = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}$$

so that  $Ax = y$  was the matrix equation

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1/2 & 2 & 0 \\ 1 & 3/2 & 1/6 & 4 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}.$$

Barbara then recalled that if she could determine an inverse for the matrix of linear coefficients, she could also determine a solution to the system of linear equations. She noticed, however, that the data formed a

rectangular matrix instead of a square matrix. After pondering over this complication for a while, Barb suddenly remembered that she needed to find a generalized inverse for the rectangular matrix — since it was not possible to find a true inverse. She used the following algorithm.

**Step 1.** In matrix  $A_{m \times n}$ , find any nonsingular submatrix  $W$  of order  $r_A$ , where  $r_A$  is the rank of matrix  $A$ .

Note that it is not necessary for  $W$  to come from  $r_A$  adjacent rows and columns of  $A$ .

Given the original equations, the matrix  $A$  Barb used was

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1/2 & 2 & 0 \\ 1 & 3/2 & 1/6 & 4 \end{bmatrix}$$

and  $r_A = 3$ . The rank of  $A$  was determined by reducing  $A$  to row echelon form and then counting the number of nonzero rows. (In the interest of time, these computations have not been included.) Thus, Barbara simply needed to find a  $3 \times 3$  submatrix  $W$  of the original matrix. She chose

$$W = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1/2 & 0 \\ 1 & 3/2 & 4 \end{bmatrix}.$$

The algorithm stipulated, however, that  $W$  must be nonsingular and thus Barb had to test her choice of  $W$ . One method of checking for nonsingularity of a matrix is to determine if  $W$  has an inverse (if an inverse exists, then  $W$  is definitely nonsingular; if no inverse exists, then  $W$  may or may not be nonsingular). Barbara proceeded as follows.

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 1/2 & 0 & 0 & 1 & 0 \\ 1 & 3/2 & 4 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \text{row } a \\ \text{row } b \\ \text{row } c \end{array}$$

Using elementary row operations,

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -4 & -12 & 0 & 1 & -3 \\ 0 & -1/2 & 2 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} \\ (b-3c) \\ (c-a) \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & -1/4 & 3/4 \\ 0 & 0 & 7/2 & -1 & -1/8 & 11/8 \end{array} \right] \begin{array}{l} (-1/4)b \\ c + (1/2)b \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 1/2 & -3/2 \\ 0 & 1 & 0 & 6/7 & -1/7 & -3/7 \\ 0 & 0 & 1 & -2/7 & -1/28 & 11/28 \end{array} \right] \begin{array}{l} (a-2b) \\ (b-3c) \\ (2/7)c \end{array}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & -1/7 & 5/14 & 1/14 \\ 0 & 1 & 0 & 6/7 & -1/7 & -3/7 \\ 0 & 0 & 1 & -2/7 & -1/28 & 11/28 \end{array} \right] (a+4c)$$

So she had obtained an identity matrix  $I_3$  for  $W$ . In an attempt to make sure no arithmetic errors had been made, she checked to see if  $W^{-1}$  satisfied the condition that  $W^{-1}W = I_3$ .

$$\left[ \begin{array}{ccc} -1/7 & 5/14 & 1/14 \\ 6/7 & -1/7 & -3/7 \\ -2/7 & -1/28 & 11/28 \end{array} \right] \left[ \begin{array}{ccc} 1 & 2 & 2 \\ 3 & 1/2 & 0 \\ 1 & 3/2 & 4 \end{array} \right] = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

and so  $W^{-1}$  was equal to

$$\left[ \begin{array}{ccc} -1/7 & 5/14 & 1/14 \\ 6/7 & -1/7 & -3/7 \\ -2/7 & -1/28 & 11/28 \end{array} \right].$$

This showed that  $W$  was nonsingular and that Barb was ready for step number two.

**Step 2.** Invert and transpose  $W$  to find  $(W^{-1})^T$ .

If  $B = (b_{ij})$  is an  $m \times n$  matrix, the transpose  $B^T$  of  $B$  is the  $n \times m$  matrix  $(c_{ij})$  where  $c_{ij} = b_{ji}$ . From Step 1, Barb already had  $W^{-1}$  and so

$$(W^{-1})^T = \left[ \begin{array}{ccc} -1/7 & 6/7 & -2/7 \\ 5/14 & -1/7 & -1/28 \\ 1/14 & -3/7 & 11/28 \end{array} \right].$$

Now, on to

**Step 3.** Form a new matrix  $A^*$  as follows. In the original matrix  $A$ , replace each element used in the submatrix  $W$  with the corresponding element of  $(W^{-1})^T$  and replace all other elements of  $A$  with zeros.

In Barb's case, the matrices  $A$ ,  $W$  and  $(W^{-1})^T$  were

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1/2 & 2 & 0 \\ 1 & 3/2 & 1/6 & 4 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1/2 & 0 \\ 1 & 3/2 & 4 \end{bmatrix}$$

and

$$(W^{-1})^T = \begin{bmatrix} -1/7 & 6/7 & -2/7 \\ 5/14 & -1/7 & -1/28 \\ 1/14 & -3/7 & 11/28 \end{bmatrix}.$$

Thus,

$$A^* = \begin{bmatrix} -1/7 & 6/7 & 0 & -2/7 \\ 5/14 & -1/7 & 0 & -1/28 \\ 1/14 & -3/7 & 0 & 11/28 \end{bmatrix}.$$

At last, Barb was ready for the final step!

**Step 4.** Transpose  $A^*$  to find  $A^-$ , a generalized inverse for the original matrix  $A$ .

Transposing  $A^*$ , Barbara found

$$A^- = \begin{bmatrix} -1/7 & 5/14 & 1/14 \\ 6/7 & -1/7 & -3/7 \\ 0 & 0 & 0 \\ -2/7 & -1/28 & 11/28 \end{bmatrix}.$$

But Barb wanted to be very careful; with so many computations, it would have been extremely easy to make a mistake. So she decided to check her work. She remembered that any generalized inverse  $A^-$  of a matrix  $A$  must satisfy the condition  $AA^-A = A$ . In this particular case,  $AA^-$  is

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 1/2 & 2 & 0 \\ 1 & 3/2 & 1/6 & 4 \end{bmatrix} \begin{bmatrix} -1/7 & 5/14 & 1/14 \\ 6/7 & -1/7 & -3/7 \\ 0 & 0 & 0 \\ -2/7 & -1/28 & 11/28 \end{bmatrix},$$

which equals  $I_3$ . So,  $AA^-A = I_3A$ . But this equals  $A$  by definition, so  $A^-$  is a valid generalized inverse for  $A$ .

Now Barb was extremely happy! Since the original equations were of the form  $Ax = y$ , she knew that  $A^-y$  would provide one solution to the system. So she multiplied

$$\begin{bmatrix} -1/7 & 5/14 & 1/14 \\ 6/7 & -1/7 & -3/7 \\ 0 & 0 & 0 \\ -2/7 & -1/28 & 11/28 \end{bmatrix} \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix}$$

and got

$$\begin{bmatrix} 16/7 \\ 16/7 \\ 0 \\ 4/7 \end{bmatrix}$$

Thus, one solution to Barb's problem was to let  $n_1 = 16/7$ ,  $n_2 = 16/7$ ,  $n_3 = 0$  and  $n_4 = 4/7$ . Suddenly, Barbara was totally crushed. She could not suggest to her supervisor that Hewlett Packard no longer make any model number three calculators. Model number three was the best selling calculator last year!

Sullenly, she started to find another solution. She went back to step number one and chose a different submatrix of order three and proceeded as follows.

$$W = \begin{bmatrix} 2 & 1 & 2 \\ 1/2 & 2 & 0 \\ 3/2 & 1/6 & 4 \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} 48/49 & -22/49 & -24/49 \\ -12/49 & 30/49 & 6/49 \\ -5/14 & 1/7 & 3/7 \end{bmatrix},$$

$$(W^{-1})^T = \begin{bmatrix} 48/49 & -22/49 & -5/14 \\ -22/49 & 30/49 & 1/7 \\ -24/49 & 6/49 & 3/7 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 0 & 48/49 & -12/49 & -5/14 \\ 0 & -22/49 & 30/49 & 1/7 \\ 0 & -24/49 & 6/49 & 3/7 \end{bmatrix}$$

$$A^- = \begin{bmatrix} 0 & 0 & 0 \\ 48/49 & -22/49 & -24/49 \\ -12/49 & 30/49 & 6/49 \\ -5/14 & 1/7 & 3/7 \end{bmatrix}$$

Finally, she had determined a second generalized inverse. Now a second solution could be found.

$$\begin{bmatrix} 0 & 0 & 0 \\ 48/49 & -22/49 & -24/49 \\ -12/49 & 30/49 & 6/49 \\ -5/14 & 1/7 & 3/7 \end{bmatrix} \begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 16/49 \\ 192/49 \\ 12/7 \end{bmatrix}$$

Unfortunately, this solution also demanded that one model of calculator be eliminated from production. Feelings of total defeat washed over her.

Abruptly, in a whirl of excitement, Barbara remembered that certain linear combinations of two solutions to the system also yield solutions to that system. Thus an infinite number of solutions could be determined as

$$\alpha \begin{bmatrix} 16/7 \\ 16/7 \\ 0 \\ 4/7 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 16/49 \\ 192/49 \\ 12/7 \end{bmatrix}$$

where  $\alpha + \beta = 1$  and  $\alpha, \beta \geq 0$ , and the two matrices are the two solutions determined earlier. So Barbara substituted  $\alpha = 1/2$  and  $\beta = 1/2$ , and found:

$$\begin{aligned} & (1/2) \begin{bmatrix} 16/7 \\ 16/7 \\ 0 \\ 4/7 \end{bmatrix} + (1/2) \begin{bmatrix} 0 \\ 16/49 \\ 192/49 \\ 12/7 \end{bmatrix} \\ &= \begin{bmatrix} 8/7 \\ 8/7 \\ 0 \\ 2/7 \end{bmatrix} + \begin{bmatrix} 0 \\ 8/49 \\ 96/49 \\ 6/7 \end{bmatrix} = \begin{bmatrix} 8/7 \\ 64/49 \\ 96/49 \\ 8/7 \end{bmatrix}. \end{aligned}$$

Thus a third solution had been determined, and this solution accounted for the production of all four models of calculators.

The applications of generalized inverses extend well beyond the aforementioned, simple example. Generalized inverses are also used in regression analysis, scientific experimentation, economic and ecological predictions, and numerous other areas. Similarly, many different methods for determining generalized inverses also exist. The general conclusion, however, is that generalized inverses are extremely useful and powerful tools in a multitude of fields.

As for Barbara? After several hours of grueling work and brimming with pride and accomplishment, she presented her final solution to her



employer. Her employer was impressed with her thoroughness and mathematical skill, but was quite baffled as to why Barbara had not used her computer to solve the problem; the software was available, and the computer was capable of finding the same solution in about ten minutes. As a result, her supervisor found her lacking in resourcefulness and demoted her to "machine operator." She calculates, however, that she should be CEO of the company within twelve years — provided her initial equations are correct and she has made no errors while finding the generalized inverse of her linear system ...

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## Mysterious Modeling

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Presented at the 1992 Region IV Convention and  
awarded FIRST PLACE by the Awards Committee.

The human body is governed by a delicate balance of processes. The question has been set forth of whether the body's functions can be modeled. Significant advances have been made in modeling the mechanical motion of the limbs. Likewise, modeling of many of the body's internal organs has greatly improved in recent years. However, the inner complexities of the brain have remained, for the most part, a mystery.

Can the activities of the human mind be modeled? To be specific, consider the sense of sight. To model this process, we require a thorough understanding not only of how the eyes transmit information to the brain, but also of how the brain interprets, analyzes and categorizes that information. The process that must be examined is not only sight, but optical perception.

This paper addresses only a portion of the task of understanding optical perception. Consider the so called "Mach bands effect." This phenomenon occurs as a result of neuron processing in the brain and is, in fact, an enhancement (rather than a mere reception) of the optical input from the eye. The received image is altered so that a given surface appears darker when next to a lighter surface and lighter when next to a darker surface. Green looks greener next to red, etc. [4] In effect, the eye provides an optical illusion for the body's benefit. For instance, picture a cliff, a staircase or even the side of a bed: perception of these edges is enhanced for safety's sake. This paper focuses on the process of locating edges.

The image must be described as a numerical display before the edges in an image can be found. For example, consider the image of a mountain range in Figure 1. This image can be given a numerical representation such as that given in Figure 2. The numbers in the boxes denote values assigned for the intensity of reflected light at each point in the image. On a scale of 1 to 10, 1 is the highest intensity and 10 the lowest. On a TV screen the number would represent the number of electrons fired at a given position on the screen, in the eye they would represent the number of photons imposed upon the retina, etc. These numerical representations will be referred to as images. The next step is to actually find the edges within these images.

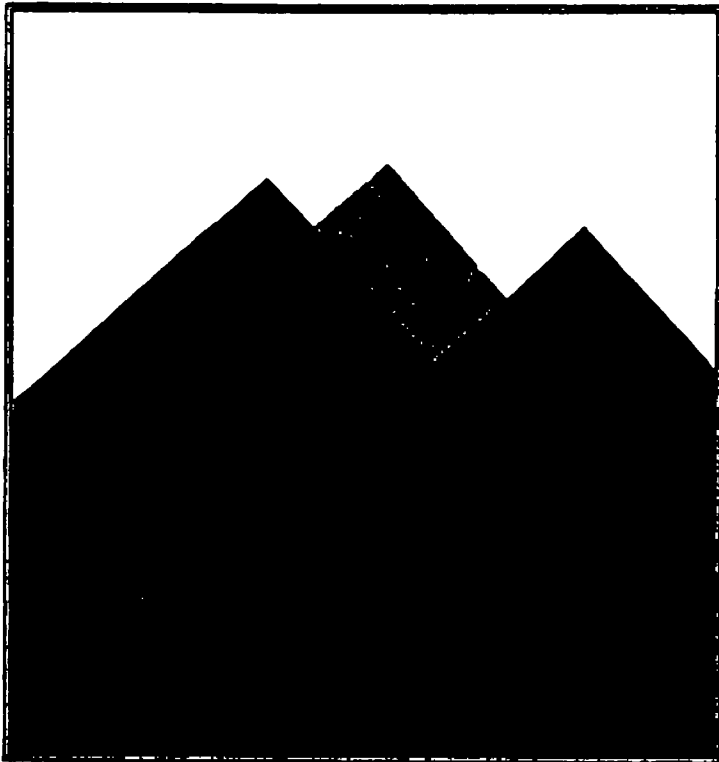


Figure 1.

10	10	10	10	10	10	10	10	10
10	10	10	10	10	10	10	10	10
10	10	10	1	7	10	10	4	10
10	1	1	1	1	7	4	4	4
1	1	1	1	1	1	4	4	4
1	1	1	1	1	1	1	4	4
1	1	1	1	1	1	1	1	4
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1

Figure 2.

1	7	10
1	1	7
1	1	1

Figure 3.

An edge detector is a device based upon characterizing an edge as a local intensity discontinuity. The intensities in a local region of the image are examined and an edge value and orientation is assigned to each point in the image based upon the change of intensity within that local region [2]. Take a  $3 \times 3$  section of the image, as shown in Figure 3, and consider the center point in relation to the others immediately surrounding it. Notice the areas with minimal differences between numbers as opposed to those areas with greater differences between numbers. In this way, the location of edges in the image can be determined. Think of sliding a

square window across the image looking for these patterns in each  $3 \times 3$  area. That is, in effect, an edge detector. The brain has its own built-in edge detector for which two models will be presented, both based upon the gradient operator.

Ritter, Wilson and Davidson [6] describe a new algebra system developed by the Air Force Armament Laboratory of the Air Force Systems Command in conjunction with the Defense Advanced Research Projects Agency for the Department of Defense. Their paper "Image Algebra: An Overview" is an effort to unify and organize current techniques in the development and implementation of image processing.

The fundamental concept upon which the image algebra rests is the notion of an image. An image is basically a function from  $\mathbb{R}^n$  into a specific set of numbers. Several definitions are needed to formalize this concept. A *coordinate set*  $X$  is any subset of  $\mathbb{R}^n$  describing location or position in a spatial reference. A *value set*  $F$  is any one of the following sets:  $\mathbb{R}$  (the real numbers),  $\mathbb{C}$  (the complex numbers),  $\mathbb{I}$  (the integers),  $(-\infty, \infty)$  (the extended real numbers), or binary numbers of fixed length  $k$ . An *image*  $a$  is  $a = \{(x, a(x)): x \in X\}$  where  $a(x) \in F$ . For example, suppose  $F = \mathbb{I}$  and  $X = \{(0,0), (1,0), (0,1), (1,1)\}$  and then choose

$$a = \{((0,0), 8), ((1,0), 4), ((0,1), 5), ((1,1), 9)\}$$

or, for a clearer visualization,

$$a = \begin{bmatrix} 5 & 9 \\ 8 & 4 \end{bmatrix}.$$

Henceforth, the word "image" will be used only with this definition whereas in preceding paragraphs it was used less formally with the implied definition of a picture or scene.

Their paper further describes the binary and unitary operations on these images. Basic binary operations (addition, multiplication and dot product) are defined point-wise; that is, at a particular point in the coordinate set operations are made on the values at that point in the respective images. Note that the dot product of two images yields a number, not another image.

$$a + b = \{(x, c(x)): c(x) = a(x) + b(x), x \in X\}$$

$$a \times b = \{(x, c(x)): c(x) = a(x) \cdot b(x), x \in X\}$$

$$a \cdot b = \sum_{x \in X} a(x) \cdot b(x)$$

For example, if  $a$  is as before and

$$b = \{((0,0),3), ((1,0),6), ((0,1),7), ((1,1),1)\},$$

or rather

$$b = \begin{array}{|c|c|} \hline 7 & 1 \\ \hline 3 & 6 \\ \hline \end{array},$$

then these images operations are:

$$a + b = \{((0,0),11), ((1,0),10), ((0,1),12), ((1,1),10)\}$$

$$a \times b = \{((0,0),24), ((1,0),24), ((0,1),35), ((1,1),9)\}$$

$$a \cdot b = 8 \cdot 3 + 4 \cdot 6 + 5 \cdot 7 + 9 \cdot 1 = 92,$$

or rather

$$a + b = \begin{array}{|c|c|} \hline 12 & 10 \\ \hline 11 & 10 \\ \hline \end{array} \quad \text{and} \quad a \times b = \begin{array}{|c|c|} \hline 35 & 9 \\ \hline 24 & 24 \\ \hline \end{array}.$$

"Templates" can now be defined in terms of images. Given  $X$  and  $Y$  as coordinate sets and  $F$  as a value set, then for each  $y \in Y$ ,  $t(y) \in F^X$  where the template  $t$  is defined as

$$t(y) = \{(x, t_y(x)): x \in X\}.$$

Note that  $t$  is a function from  $Y$  to the set of images on  $X$ .  $Y$  is the domain of  $t$ , called the target domain and  $X$  is called the range space of  $t$ . The values  $t_y(x)$  are called the weights of the template at  $y$ . According to the authors, templates and template operations are the most powerful tools of the image algebra. The local edge detector is really just a specialized template in that it takes a portion of the original images and "blows it up" in order to look at the image on a very localized scale.

The foundation for defining and constructing an edge detector must be laid before the operations on one may be discussed [3]. Refer again to the definition of an image. Note that an image is a function. The maximum rate of change in the intensity values of an image is known to occur along a line perpendicular to the edge. To find this maximum rate of change, the gradient operator must be applied to the function. In calculus texts (see, for instance, [7]), a theorem may be found which states: "Suppose that  $f$  is a differentiable function of two or three variables. The maximum value of the directional derivative  $D_u f(x)$  is  $|\nabla f(x)|$  and it occurs when  $u$  has the same direction as the gradient vector  $\nabla f(x)$ ."

The first edge detector considered is taken straight from the gradient's definition,

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}.$$

Moreover, the partial derivatives are defined as

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and

$$\frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Letting  $h = 1$ , the partial derivatives are approximately

$$\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = \frac{f(1,0) - f(0,0)}{1}$$

and

$$\left. \frac{\partial f}{\partial y} \right|_{(0,0)} = \frac{f(0,1) - f(0,0)}{1}.$$

$f(0,1)$	$f(1,1)$
$f(0,0)$	$f(1,0)$

Figure 4.

Consider the image shown in Figure 4 and let  $f(0,0) = a$ ,  $f(1,0) = b$ ,  $f(0,1) = c$  and  $f(1,1) = d$ . Then it is easy to see that

$$\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = b - a \quad \text{and} \quad \left. \frac{\partial f}{\partial y} \right|_{(0,0)} = c - a.$$

However, the partial derivatives may also be given by the dot product of the image and a mask, where this mask is the vertical (or horizontal) component of the edge detector. So

$$\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \cdot \begin{bmatrix} g & h \\ e & f \end{bmatrix} = b - a$$

and

$$\left. \frac{\partial f}{\partial y} \right|_{(0,0)} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \cdot \begin{bmatrix} g' & h' \\ e' & f' \end{bmatrix} = c - a.$$

Therefore, the masks must be

$$\begin{bmatrix} g & h \\ e & f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} g' & h' \\ e' & f' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

These are the horizontal and vertical masks which, when applied to the image, yield the gradient approximated at  $(0,0)$ . This is a simple method, but helpful for understanding the technique.

The second edge detector follows from Prewitt [5], who suggests, "A more precise estimate is obtained by fitting a quadratic surface over a  $3 \times 3$  neighborhood by least squares and then computing the gradient for the fitted surface." Consider the  $3 \times 3$  region shown in Figure 5. Now suppose the image is given by  $f(x,y)$  and define a quadratic function

$$F(x,y) = ax^2 + by^2 + cxy + dx + ey + f.$$

Using the least squares method to fit  $F(x,y)$  to  $f(x,y)$ , the function  $S(a,b,c,d,e,f)$  is given by



$f(0,2)$	$f(1,2)$	$f(2,2)$
$f(0,1)$	$f(1,1)$	$f(2,1)$
$f(0,0)$	$f(1,0)$	$f(2,0)$

Figure 5.

$$S = \sum_{\substack{x=0,1,2 \\ y=0,1,2}} (F(x,y) - f(x,y))^2.$$

Let  $f(0,0) = g$ ,  $f(1,0) = h$ ,  $f(2,0) = i$ ,  $f(0,1) = j$ ,  $f(1,1) = k$ ,  $f(2,1) = l$ ,  $f(0,2) = m$ ,  $f(1,2) = n$  and  $f(2,2) = o$ . Substituting these values into the equation for  $S$  yields

$$\begin{aligned} S = & (f - g)^2 + ((a + d + f) - h)^2 + ((4a + 2d + f) - i)^2 \\ & + ((b + e + f) - j)^2 + ((a + b + c + d + e + f) - k)^2 \\ & + ((4a + b + 2c + 2d + e + f) - l)^2 + ((4b + 2e + f) - m)^2 \\ & + ((a + 4b + 2c + d + 2e + f) - n)^2 \\ & + ((4a + 4b + 4c + 2d + 2e + f) - o)^2. \end{aligned}$$

$S$  is minimized by taking the partial derivatives of  $S$ , setting each equal to 0 and solving for the variables  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $e$  and  $f$ . The partial derivatives are:

$$\begin{aligned} \frac{\partial S}{\partial a} = & 2((a + d + f) - h) + 8((4a + 2d + f) - i) \\ & + 2((a + b + c + d + e + f) - k) + 8((4a + b + 2c + 2d + e + f) - l) \\ & + 2((a + 4b + 2c + d + 2e + f) - n) \\ & + 8((4a + 4b + 4c + 2d + 2e + f) - o) \end{aligned}$$

$$\begin{aligned} \frac{\partial S}{\partial b} = & 2((b + e + f) - j) + 2((a + b + c + d + e + f) - k) \\ & + 2((4a + b + 2c + 2d + e + f) - l) + 8((4b + 2e + f) - m) \end{aligned}$$

$$\begin{aligned}
& + 8 \left( (a + 4b + 2c + d + 2e + f) - n \right) \\
& \quad + 8 \left( (4a + 4b + 4c + 2d + 2e + f) - o \right) \\
\frac{\partial S}{\partial c} &= 2 \left( (a + b + c + d + e + f) - k \right) + 4 \left( (4a + b + 2c + 2d + e + f) - l \right) \\
& \quad + 4 \left( (a + 4b + 2c + d + 2e + f) - n \right) \\
& \quad + 8 \left( (4a + 4b + 4c + 2d + 2e + f) - o \right) \\
\frac{\partial S}{\partial d} &= 2 \left( (a + d + f) - h \right) + 4 \left( (4a + 2d + f) - i \right) \\
& \quad + 2 \left( (a + b + c + d + e + f) - k \right) + 4 \left( (4a + b + 2c + 2d + e + f) - l \right) \\
& \quad + 2 \left( (a + 4b + 2c + d + 2e + f) - n \right) \\
& \quad + 4 \left( (4a + 4b + 4c + 2d + 2e + f) - o \right) \\
\frac{\partial S}{\partial e} &= 2 \left( (b + e + f) - j \right) + 2 \left( (a + b + c + d + e + f) - k \right) \\
& \quad + 2 \left( (4a + b + 2c + 2d + e + f) - l \right) + 4 \left( (4b + 2e + f) - m \right) \\
& \quad + 4 \left( (a + 4b + 2c + d + 2e + f) - n \right) \\
& \quad + 4 \left( (4a + 4b + 4c + 2d + 2e + f) - o \right) \\
\frac{\partial S}{\partial f} &= 2 \left( f - g \right) + 2 \left( (a + d + f) - h \right) + 2 \left( (4a + 2d + f) - i \right) \\
& \quad + 2 \left( (b + e + f) - j \right) + 2 \left( (a + b + c + d + e + f) - k \right) \\
& \quad + 2 \left( (4a + b + 2c + 2d + e + f) - l \right) + 2 \left( (4b + 2e + f) - m \right) \\
& \quad + 2 \left( (a + 4b + 2c + d + 2e + f) - n \right) \\
& \quad + 2 \left( (4a + 4b + 4c + 2d + 2e + f) - o \right).
\end{aligned}$$

Simplifying yields the following:

$$\begin{aligned}
\frac{\partial S}{\partial a} &= 102a + 50b + 54c + 54d + 30e + 30f \\
& \quad - 2(h + 4i + k + 4l + n + 4o) \\
\frac{\partial S}{\partial b} &= 50a + 102b + 54c + 30d + 54e + 30f \\
& \quad - 2(j + k + l + 4m + 4n + 4o) \\
\frac{\partial S}{\partial c} &= 54a + 54b + 50c + 30d + 30e + 18f \\
& \quad - 2(k + 2l + 2n + 4o) \\
\frac{\partial S}{\partial d} &= 54a + 30b + 30c + 30d + 18e + 18f \\
& \quad - 2(h + 2i + k + 2l + n + 2o)
\end{aligned}$$

$$\frac{\partial S}{\partial e} = 30a + 54b + 30c + 18d + 30e + 18f - 2(j + k + l + 2m + 2n + 2o)$$

$$\frac{\partial S}{\partial f} = 30a + 30b + 18c + 18d + 18e + 18f - 2(g + h + i + j + k + l + m + n + o).$$

Each of these partial derivatives was set equal to 0 and two methods were used to solve the resulting system of equations: Gaussian elimination (by hand) and *DERIVE* [1], a mathematical computer program. The solution of these equations is

$$a = \frac{1}{6} (g - 2h + i + j - 2k + l + m - 2n + o)$$

$$b = \frac{1}{6} (g + h + i - 2j - 2k - 2l + m + n + o)$$

$$c = \frac{1}{4} (g - i - m + o)$$

$$d = \frac{1}{12} (-9g + 8h + i - 6j + 8k - 2l - 3m + 8n - 5o)$$

$$e = \frac{1}{12} (-9g - 6h - 3i + 8j + 8k + 8l + m - 2n - 5o)$$

$$f = \frac{1}{36} (29g + 8h - i + 8j - 4k - 4l - m - 4n + 5o).$$

These values are then substituted into the equation for  $F$  and the gradient of  $F$  is given by

$$\nabla F = \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j}$$

while the partial derivatives are found to be

$$\frac{\partial F}{\partial x} = 2ax + cy + d \quad \text{and} \quad \frac{\partial F}{\partial y} = 2by + cx + e.$$

Hence,

$$\nabla F = (2ax + cy + d) \mathbf{i} + (2by + cx + e) \mathbf{j}$$

Evaluated at (1,1), the center point in the region under consideration, this gives

$$\nabla F = (2a + c + d) \mathbf{i} + (2b + c + e) \mathbf{j}$$

and substituting the above values for  $a$ ,  $b$ ,  $c$ ,  $d$  and  $e$ , we find

$$2a + c + d = \frac{1}{6} (-g + i - j + l - m + o)$$

$$2b + c + e = \frac{1}{6}(-g - h - i + m + n + o).$$

As before, the horizontal and vertical masks are obtained by the following steps.

$$\frac{\partial F}{\partial x} \Big|_{(1,1)} = \begin{array}{|c|c|c|} \hline m & n & o \\ \hline j & k & l \\ \hline g & h & i \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline v & w & x \\ \hline s & t & u \\ \hline p & q & r \\ \hline \end{array} \\ = \frac{1}{6}(-g + i - j + l - m + o)$$

and

$$\frac{\partial F}{\partial y} \Big|_{(1,1)} = \begin{array}{|c|c|c|} \hline m & n & o \\ \hline j & k & l \\ \hline g & h & i \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline v' & w' & x' \\ \hline s' & t' & u' \\ \hline p' & q' & r' \\ \hline \end{array} \\ = \frac{1}{6}(-g - h - i + m + n + o).$$

These equations are solved to obtain

$$6 \cdot \begin{array}{|c|c|c|} \hline v & w & x \\ \hline s & t & u \\ \hline p & q & r \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline -1 & 0 & 1 \\ \hline -1 & 0 & 1 \\ \hline -1 & 0 & 1 \\ \hline \end{array}$$

and

$$6 \cdot \begin{array}{|c|c|c|} \hline v' & w' & x' \\ \hline s' & t' & u' \\ \hline p' & q' & r' \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline -1 & -1 & -1 \\ \hline \end{array}.$$

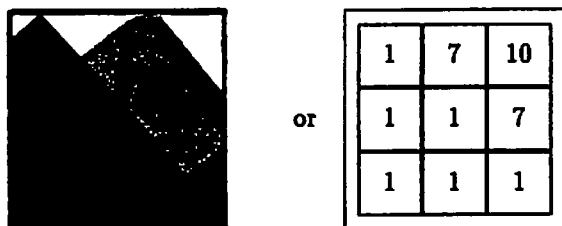


Figure 6.

The true test of the edge detectors is in their application. Choose a local region from the original mountain range, as shown in Figure 6. Now apply the two edge detectors to this local region. The first edge detector yields:

$$\begin{array}{|c|c|} \hline 7 & 10 \\ \hline 1 & 7 \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline 0 & 0 \\ \hline -1 & 1 \\ \hline \end{array} = -1 + 7 = 6$$

and

$$\begin{array}{|c|c|} \hline 7 & 10 \\ \hline 1 & 7 \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline 1 & 0 \\ \hline -1 & 0 \\ \hline \end{array} = 7 - 1 = 6.$$

Both the  $x$  and  $y$  components of the gradient are 6 and yield a line at  $45^\circ$  from the  $x$ -axis. The edge lies perpendicular to this line and is thus located at  $135^\circ$  from the  $x$ -axis (see Figure 7). Clearly, this edge matches the edge in the region under consideration.

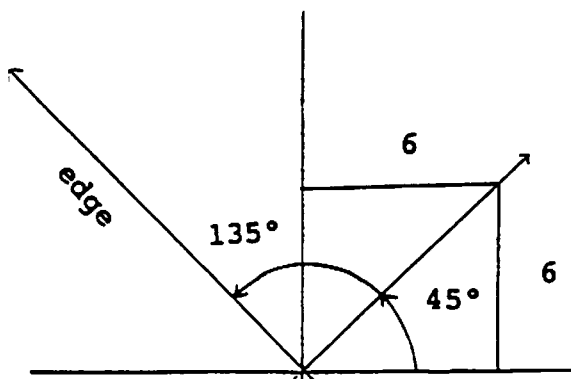


Figure 7.

The second edge detector yields:

$$\begin{array}{|c|c|c|} \hline 1 & 7 & 10 \\ \hline 1 & 1 & 7 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline -1 & 0 & 1 \\ \hline -1 & 0 & 1 \\ \hline -1 & 0 & 1 \\ \hline \end{array}$$

$$= -1 + 10 - 1 + 7 - 1 + 1 = 15$$

and

$$\begin{array}{|c|c|c|} \hline 1 & 7 & 10 \\ \hline 1 & 1 & 7 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \cdot \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline -1 & -1 & -1 \\ \hline \end{array}$$

$$= 1 + 7 + 10 - 1 - 1 - 1 = 15.$$

Both masks give 15 and so again yield a line at  $45^\circ$  from the  $x$ -axis and the edge  $135^\circ$  from the  $x$ -axis. For the image of the mountains given in Figure 1, the two edge detectors accurately determine the edge at the given point. However, this image is quite simplified in comparison to an actual physical image. Increasing the number of points within an image will improve accuracy, but at this point data can no longer be recorded and calculated by hand. Instead, a computer of sufficient memory is needed.

The next step in modeling optical perception is the linking of point edges to form actual edges. Modeling of such phenomena as the Mach bands effect will require additional techniques of edge enhancement. Optical perception can be, and is being, modeled. The future will continue to produce improved techniques as the human mind is understood more and more.

*Acknowledgement.* A sincere "thank you" is extended to Dr. Allen, my mentor, as well as to Dr. Moss, Dr. Weber, Dr. Smith, Dr. Flikkema, Dave Liss, Lucien Swerdloff and Bonny Langston.

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## Introduction to Automata

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Presented at the 1992 Region IV Convention and  
awarded SECOND PLACE by the Awards Committee.

There are many mathematical structures in the field of computer science. One such structure is the automaton. This paper will provide an elementary definition and explanation of an automaton and some of the terms and concepts associated with it.

An automaton  $A$  is defined to be a triple  $A = (S, \Sigma, \delta)$  where  $S$  is a set of states,  $\Sigma$  is a non-empty set of inputs and  $\delta: S \times \Sigma^* \rightarrow S$  is a transition function such that if  $s \in S$  and  $x, y \in \Sigma^*$ , then  $\delta(s, xy) = \delta(\delta(s, x), y)$  and  $\delta(s, \epsilon) = s$ . ( $\Sigma^*$  is defined to be the set of all sequences formed from the elements of  $\Sigma$ , including the empty string  $\epsilon$ .) The automaton  $A$  is called the empty automaton  $\langle \emptyset \rangle$  if and only if  $S = \emptyset$ .

An example of an automaton is given in Figure 1. In this example,  $S = \{A, B, C, D\}$  and  $\Sigma = \{0, 1\}$ . The explicit definition of the transition function  $\delta$  restricted to  $S \times \Sigma$  is given in Table 1.

$\delta(A, 0) = B$	$\delta(A, 1) = C$
$\delta(B, 0) = B$	$\delta(B, 1) = C$
$\delta(C, 0) = D$	$\delta(C, 1) = D$
$\delta(D, 0) = A$	$\delta(D, 1) = B$

Table 1.

It is easy to see from this example that the transition is a function from all pairs of  $S \times \Sigma^*$  to  $S$ . Note that the set  $S$  has a finite number of



elements. Thus, this example is called a finite automaton. Finite automata will be the basis for this paper.

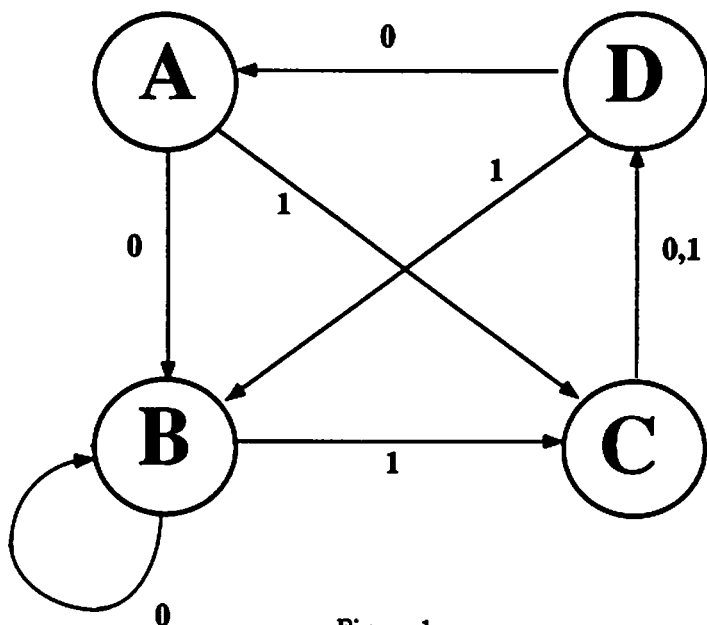


Figure 1.

We have now seen two different representations of an automaton, the state diagram and the explicit method. There is a third representation known as a transition table. For the previous example, the transition table is given in Table 2. Of these three representations, the transition table and the state diagram are the most frequent representations used. This is because both types offer better visual assistance in understanding the automaton than the explicit transition representation. However, it should be noted that the transition table and the state diagram are only applicable to finite automata and not infinite automata.

$\delta$	0	1
A	B	C
B	B	C
C	D	D
D	A	B

Table 2.

An important aspect of the study of automata is the subautomaton.  $B = (T, \Sigma, \delta')$  is a subautomaton  $B \ll A$  of the automaton  $A = (S, \Sigma, \delta)$  if and only if (1)  $T \subseteq S$ , (2)  $\delta'$  is the restriction of  $\delta$  to  $T \times \Sigma^*$  and (3)  $\delta(t, x) \in T$  for  $t \in T$  and all  $x \in \Sigma^*$ . In other words,  $B$  must be closed under the set of transitions. A subautomaton is a proper automaton if and only if  $B \neq A$  and  $B \neq \emptyset$ . Note that although  $T \subseteq S$ , the set  $\Sigma$  of inputs remains the same for any subautomaton.

Consider the automaton  $A = (S, \Sigma, \delta)$  given in Figure 2. The transition table for this automaton is given in Table 3.

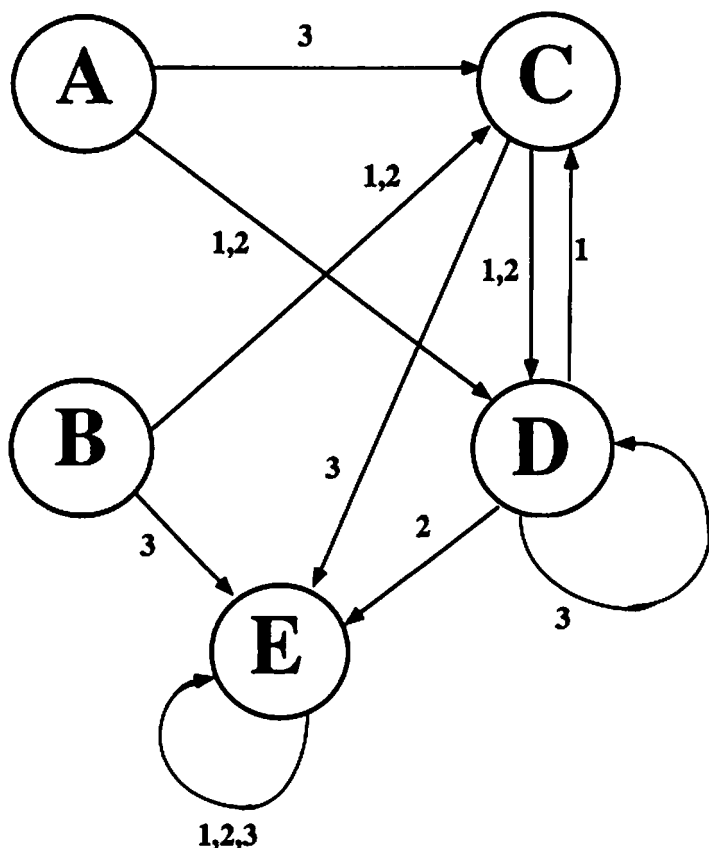


Figure 2.

$\delta$	1	2	3
A	D	D	C
B	C	C	E
C	D	D	E
D	C	E	D
E	E	E	E

Table 3.

We can generate all of the subautomata of  $A$  by using the specially developed algorithm given in [Bavel, 137-139]. The transition tables achieved by the use of this algorithm and all of the subautomata are given in Table 4.

$\delta$	1	2	3	Successor Set
A	D	D	C	{A,D,C,E}
B	C	C	E	{B,C,E,D}
C	D	D	E	{C,D,E}
D	C	E	D	{D,C,E}
E	E	E	E	{E}

$\delta$	Successor Set
$\delta(A) \cup \delta(B)$	{A,B,C,D,E}

Table 4.

In these transition tables, the successor set for  $s \in S$  is denoted by  $\delta(s)$ . This represents the set of all states that can be reached from the state  $s$  by any and every input sequence. For this example, there are six subautomata, four of which are proper. The set of states for each of the proper subautomata and for the improper subautomaton that is the automaton itself are exactly the successor sets given in the previous transition tables. The state sets for all the subautomata are {A,D,C,E}, {B,C,E,D}, {C,D,E}, {E}, {A,B,C,D,E} and  $\emptyset$ .

There are several different ways to classify automata. One of these classifications is the retrievable automaton. An automaton  $A$  is retrievable if and only if for every  $s \in S$  and for every  $\sigma \in \Sigma$ , there is an  $x \in \Sigma^*$  such that  $\delta(s, \sigma x) = s$ . In other words, an automaton is retrievable

if from any state, you can begin with each input and find a non-empty input string such that the transition of the input together with this string is the original state. The automaton given in Figure 1 is a retrievable automaton since

$$\begin{array}{ll} \delta(A, 0100) = A & \delta(A, 100) = A \\ \delta(B, 00) = B & \delta(B, 111) = B \\ \delta(C, 001) = C & \delta(C, 111) = C \\ \delta(D, 011) = D & \delta(D, 110) = D \end{array}$$

For an example of a non-retrievable automaton, consider the automaton  $R$  given in Figure 3. This is clearly not retrievable since for any non-empty input string  $x$  we have  $\delta(A, x) = C$  yet  $C \neq A$ .

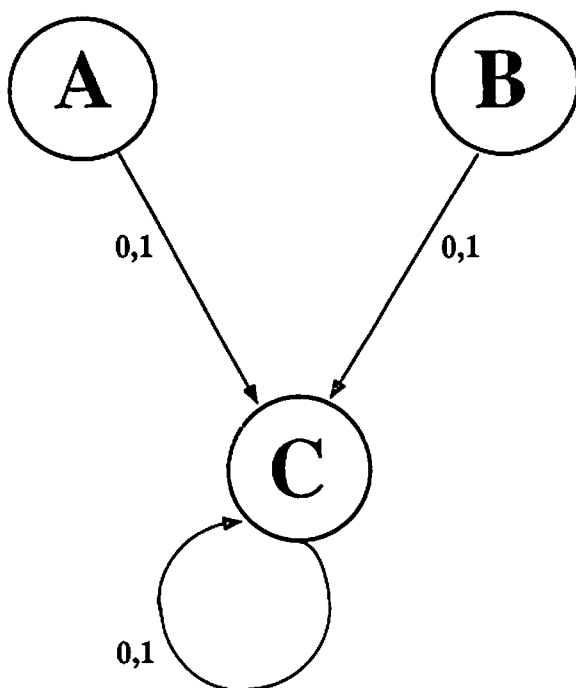


Figure 3.

Another type of classification is that of a strongly connected automaton. An automaton  $A$  is strongly connected if and only if  $S \neq \emptyset$  and  $t \in \delta(s)$  for every  $s, t \in S$ . Thus, any state can be reached from any

of the states. Here again, the automaton in Figure 1 can be used as an example. This automaton is easily shown to be strongly connected by considering the following transitions:

$$\begin{array}{llll}
 \delta(A, 0110) = A & \delta(A, 0) = B & \delta(A, 1) = C & \delta(A, 010) = D \\
 \delta(B, 110) = A & \delta(B, 0) = B & \delta(B, 1) = C & \delta(B, 11) = D \\
 \delta(C, 10) = A & \delta(C, 01) = B & \delta(C, 001) = C & \delta(C, 1) = D \\
 \delta(D, 0) = A & \delta(D, 1) = B & \delta(D, 01) = C & \delta(D, 111) = D
 \end{array}$$

For an example of an automaton that is not strongly connected, refer again to Figure 3. Note that this is not strongly connected since  $\delta(C, x) = C$  for  $x \in \Sigma^*$ . Hence, the state A can not be reached from the state C.

Note that the strongly connected automaton in Figure 1 was also retrievable. We shall now see that this is true for every strongly connected automaton.

**Theorem.** If  $A = (S, \Sigma, \delta)$  is a strongly connected automaton, then  $A$  is retrievable.

**Proof.** Since  $A$  is strongly connected,  $S \neq \emptyset$ . Thus, let  $s \in S$ . Since  $\Sigma \neq \emptyset$ , let  $\sigma \in \Sigma$ . There are two cases to consider: (i)  $S = \{s\}$  and (ii)  $S \neq \{s\}$ .

(i) Let  $x \neq \epsilon \in \Sigma^*$ . Now, since  $\delta: S \times \Sigma^* \rightarrow S$  and  $\sigma x \in \Sigma^*$ ,  $\delta(s, \sigma x) \in S$ . However,  $S = \{s\}$  so  $\delta(s, \sigma x) = s$ . Thus,  $A$  is retrievable.

(ii) Since  $S \neq \{s\}$ ,  $\exists r \in S$  such that  $r \neq s$ . Since  $A$  is strongly connected,  $\exists x, y \in \Sigma^*$  such that  $\delta(\delta(s, \sigma), x) = \delta(s, \sigma x) = r$  and  $\delta(\delta(r, \sigma), y) = \delta(r, \sigma y) = s$ . Thus,  $\delta(s, \sigma x \sigma y) = s$  and  $\delta(r, \sigma y \sigma x) = r$ . Hence,  $A$  is retrievable and the proof is finished.

Thus, every strongly connected automaton is retrievable. However, it is not the case that every retrievable automaton is strongly connected. Consider the automaton given in Figure 4 as an example of a retrievable but non-strongly connected automaton. This automaton is clearly not strongly connected since there is no input string where the transition of the string on the state A will yield the state C.

I would like to point out that the study of automata involves a unique combination of pure and applied mathematics. The automaton is used in the study of formal language theory, compiler design and in the study of Turing machines and "busy beavers." Yet there is research currently being done that is based more on the theoretical side of automaton.

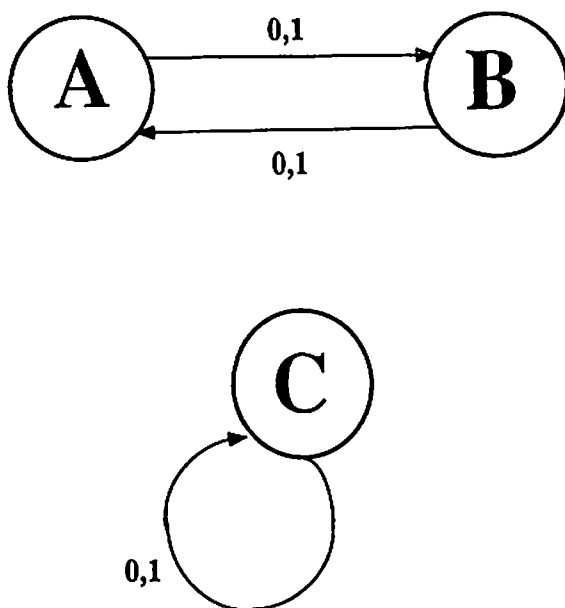


Figure 4.

This concludes my introduction to automata. It is my hope that this paper has given some insight into the mathematical side of computer science. It is also my hope that you will be able to relate the study of automata to that of group theory, graph theory or other areas of mathematics.

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## On Asymptotes of the Graphs of Algebraic Functions

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Although an asymptote is not a part of the graph of a function, it helps to visualize how the graph behaves in distant regions of the coordinate plane and, consequently, it helps to sketch a more accurate graph. The purpose of this article is to offer an alternative technique for finding asymptotes (if any) of the graph of an algebraic function. We begin with the following definition of an asymptote.

**Definition 1.** A nonvertical line  $L$  with equation  $y = mx + n$  is an asymptote for the graph of a function  $f$  if  $f(x) - (mx + n)$ , the vertical separation between the line and the graph, tends to zero as  $|x|$  approaches infinity (see Figure 1).

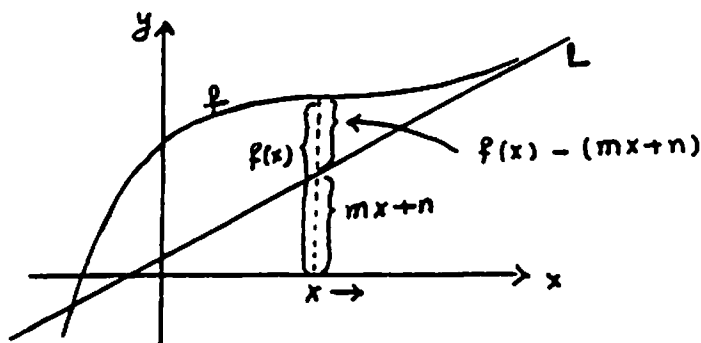


Figure 1.

Before we discuss our proposed method, let us consider the standard method.

**Standard technique.** From the definition it follows that a nonvertical line  $y = mx + n$  is an asymptote for the graph of a function  $f$  if and only if

$$m = \lim_{x \rightarrow \infty} \frac{f(x)}{x} \quad \text{and} \quad n = \lim_{x \rightarrow \infty} (f(x) - mx)$$

or

$$m = \lim_{x \rightarrow -\infty} \frac{f(x)}{x} \quad \text{and} \quad n = \lim_{x \rightarrow -\infty} (f(x) - mx).$$

We need the following two lemmas to prove our first theorem that plays an important role in developing our proposed method. We state and prove these lemmas for the case in which  $x \rightarrow \infty$ , but they are valid when  $x \rightarrow -\infty$ , too. First, we observe that if  $f$  is an algebraic function defined on an interval  $(a, \infty)$ , then either  $\lim_{x \rightarrow \infty} f(x)$  exists or  $\lim_{x \rightarrow \infty} f(x) = \pm \infty$ .

**Lemma 1.** If  $\lim_{x \rightarrow \infty} xf(x) = L < \infty$ , then  $\lim_{x \rightarrow \infty} f(x) = 0$ .

**Proof.** Suppose that  $\lim_{x \rightarrow \infty} f(x) \neq 0$ . Then, as  $x \rightarrow \infty$ , either  $f(x)$  has an infinite limit or a nonzero finite limit. In either case,  $xf(x)$  has an infinite limit, which is a contradiction. Therefore,  $\lim_{x \rightarrow \infty} f(x) = 0$ .

**Lemma 2.** If  $f$  and  $g$  are defined on an interval  $(a, \infty)$  and  $\lim_{x \rightarrow \infty} (f(x) - g(x)) = 0$ , then  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x)$ .

**Proof.** Since  $f$  and  $g$  are algebraic functions, we have

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} ((f(x) - g(x)) + g(x)) \\ &= \lim_{x \rightarrow \infty} (f(x) - g(x)) + \lim_{x \rightarrow \infty} g(x) \\ &= 0 + \lim_{x \rightarrow \infty} g(x) = \lim_{x \rightarrow \infty} g(x). \end{aligned}$$

Now we are in position to prove our theorem.

**Theorem 1.** The line  $y = mx + n$  is an asymptote of the graph of a function  $f$  if and only if  $m = \lim_{x \rightarrow \infty} f'(x)$  and  $n = \lim_{x \rightarrow \infty} (f(x) - xf'(x))$  or  $m = \lim_{x \rightarrow -\infty} f'(x)$  and  $n = \lim_{x \rightarrow -\infty} (f(x) - xf'(x))$ .



Proof. We prove the theorem for the case in which  $x \rightarrow \infty$ . A similar proof can be given for the case in which  $x \rightarrow -\infty$ . First, suppose that  $y = mx + n$  is an asymptote for the graph of  $f$ . Then  $m = \lim_{x \rightarrow \infty} f(x)/x$  and  $n = \lim_{x \rightarrow \infty} (f(x) - mx)$ . The second limit can be written as

$$n = \lim_{x \rightarrow \infty} (f(x) - mx) = \lim_{x \rightarrow \infty} x \left( \frac{f(x)}{x} - m \right) = \lim_{x \rightarrow \infty} \frac{\frac{f(x)}{x} - m}{\frac{1}{x}}.$$

Since the limits of both the numerator and the denominator in the last limit are zero, we apply L'Hospital's Rule and obtain

$$n = \lim_{x \rightarrow \infty} \frac{\frac{xf'(x) - f(x)}{x^2}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} (f(x) - xf'(x)).$$

Now, to show  $m = \lim_{x \rightarrow \infty} f'(x)$ , we note that

$$n = \lim_{x \rightarrow \infty} (f(x) - xf'(x)) = \lim_{x \rightarrow \infty} x \left( \frac{f(x)}{x} - f'(x) \right).$$

Therefore, it follows from Lemma 1 that

$$\lim_{x \rightarrow \infty} \left( \frac{f(x)}{x} - f'(x) \right) = 0.$$

Hence, Lemma 2 implies that

$$\lim_{x \rightarrow \infty} f'(x) = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = m.$$

Conversely, let us assume that

$$m = \lim_{x \rightarrow \infty} f'(x) \quad \text{and} \quad n = \lim_{x \rightarrow \infty} (f(x) - xf'(x)).$$

Since

$$n = \lim_{x \rightarrow \infty} (f(x) - xf'(x)) = \lim_{x \rightarrow \infty} x \left( \frac{f(x)}{x} - f'(x) \right),$$

it follows from Lemma 1 that

$$\lim_{x \rightarrow \infty} \left( \frac{f(x)}{x} - f'(x) \right) = 0.$$

Therefore, Lemma 2 implies that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} f'(x) = m.$$

To complete the proof, we must show that  $n = \lim_{x \rightarrow \infty} (f(x) - mx)$ . We observe that

$$\lim_{x \rightarrow \infty} (f(x) - mx) = \lim_{x \rightarrow \infty} x \left( \frac{f(x)}{x} - m \right) = \lim_{x \rightarrow \infty} \frac{\frac{f(x)}{x} - m}{\frac{1}{x}}.$$

Again, we apply L'Hospital's Rule and obtain

$$\begin{aligned} \lim_{x \rightarrow \infty} (f(x) - mx) &= \lim_{x \rightarrow \infty} \frac{\frac{xf'(x) - f(x)}{x^2}}{-\frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} (f(x) - xf'(x)) = n. \end{aligned}$$

This shows that  $y = mx + n$  is an asymptote for the graph of  $f$  and the theorem is proved.

We know that the line  $y = mx + n$  is tangent to the graph of  $f$  at the point  $(c, f(c))$  if and only if  $m = f'(c)$  and  $n = f(c) - cf'(c)$ . This suggests the following definition, which is needed for our proposed method.

**Definition 2.** Suppose  $f$  is a function defined on an interval  $(a, \infty)$  or  $(-\infty, b)$ . If  $\lim_{x \rightarrow \infty} f'(x) = m$  and  $\lim_{x \rightarrow \infty} (f(x) - xf'(x)) = n$ , or  $\lim_{x \rightarrow \infty} f'(x) = m$  and  $\lim_{x \rightarrow \infty} (f(x) - xf'(x)) = n$ , then we say that the line  $y = mx + n$  is tangent to the graph of  $f$  at infinity.

In view of Theorem 1 and Definition 2, the line  $y = mx + n$  is an asymptote for the graph of  $f$  if and only if it is tangent to the graph of  $f$  at infinity. We will make use of the following definition (see [2]) to establish our next theorem.

**Definition 3.** Consider the polynomial equation

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0.$$

The nonzero number  $p$  is a root of this equation if and only if  $1/p$  is a root of the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

obtained from the first equation by replacing  $x$  by  $1/x$ . When  $a_n \rightarrow 0$ , at least one of the solutions of the original equation approaches zero and, consequently, at least one of the solutions of the second equation approaches infinity. In particular, when  $a_n = 0$ , the first equation has a zero root and in this case we say that the second equation has a root at infinity.

**Theorem 2.** The nonvertical line  $L$  with equation  $y = mx + n$  is an asymptote of the graph of an algebraic function  $f$  if and only if the equation

$$(1) \quad f(x) = mx + n$$

has multiple roots at infinity.

**Proof.** It was shown in [1] that  $L$  is tangent to the graph of  $f$  at the point  $(c, f(c))$  if and only if  $c$  is a root of (1) with multiplicity greater than one. By the observation made right after Definition 2,  $L$  is an asymptote for the graph of  $f$  if and only if it is tangent to the graph of  $f$  at infinity. Therefore, it follows that  $L$  is an asymptote of the graph of  $f$  if and only if (1) has multiple roots at infinity. This completes the proof of the theorem.

We note that the equation

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0$$

has  $k$  repeated zero roots if and only if  $a_n = a_{n-1} = \cdots = a_{n-k+1} = 0$  but  $a_{n-k} \neq 0$ . Therefore, it follows that the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

has  $k$  roots at infinity if and only if  $a_n = a_{n-1} = \cdots = a_{n-k+1} = 0$  but  $a_{n-k} \neq 0$  (see [2]).

We now present the alternative method.

**Alternative method.** To find the equations of the nonvertical asymptotes (if any) of the graph of an algebraic function  $f$ , proceed as follows:

- (i) Form the equation

$$(1) \quad f(x) = mx + n.$$

If (1) is a rational or a radical equation, convert it to a polynomial equation

$$(2) \quad P(x) = 0$$

arranged in order of descending powers of  $x$ .

- (ii) Form the system of equations obtained by setting the coefficients of the two highest powers of  $x$  in (2) equal to zero and solve this system for  $m$  and  $n$ . If this system has no solution, then stop here and the graph of  $f$  has no nonvertical line as an asymptote. If the system has a solution, go to the next step.
- (iii) If  $f(x)$  does not contain a radical expression with an even index, then  $y = mx + n$  is an asymptote; otherwise, since we raised both sides of (1) to an even power to obtain (2), we must check for extraneous results. To determine if  $y = mx + n$  is actually an asymptote,  $f(x) - (mx + n)$  must be approximately zero when  $|x|$  is large.

The following examples illustrate this procedure.

**Example 1.** Find the equations of the asymptotes of the graph of

$$f(x) = \sqrt[3]{x^3 + 1} + 1.$$

**Solution.** (i) Equation (1) is

$$\sqrt[3]{x^3 + 1} + 1 = mx + n.$$

This implies

$$\sqrt[3]{x^3 + 1} = mx + (n - 1).$$

Equation (2), obtained by cubing both sides of this equation and regrouping the terms, is

$$(m^3 - 1)x^3 + 3m^2(n - 1)x^2 + 3m(n - 1)^2x + (n - 1)^3 - 1 = 0.$$

(ii) Setting the coefficients of  $x^3$  and  $x^2$  equal to zero results in

$$\begin{cases} m^3 - 1 = 0 \\ 3m^2(n-1) = 0 \end{cases}$$

The solution of this system is  $m = 1$ ,  $n = 1$ . Therefore, the line  $y = x + 1$  is the slant asymptote for the graph of  $f$  (see Figure 2).

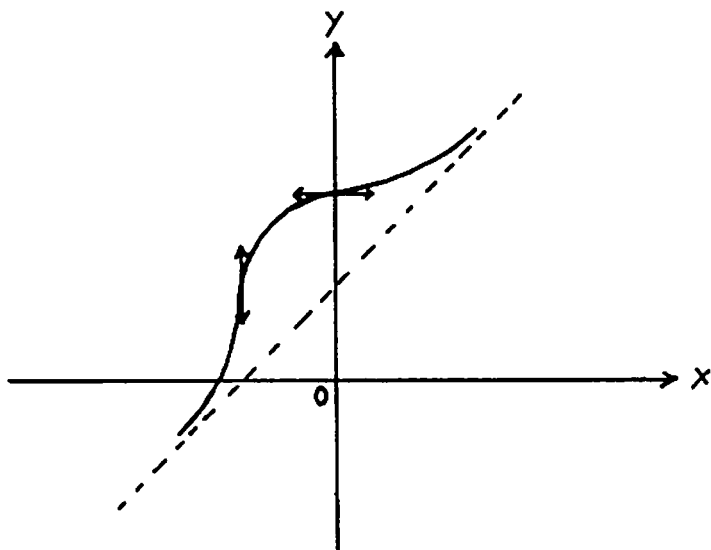


Figure 2.

Applying the limit method to this example, we see that

$$m = \lim_{|x| \rightarrow \infty} \frac{f(x)}{x} = \lim_{|x| \rightarrow \infty} \frac{\sqrt[3]{x^3+1} + 1}{x} = \dots = 1$$

and

$$\begin{aligned} n &= \lim_{|x| \rightarrow \infty} (f(x) - mx) = \lim_{|x| \rightarrow \infty} (\sqrt[3]{x^3+1} + 1 - x) \\ &= \lim_{|x| \rightarrow \infty} \frac{2 - 3x + 3x^2}{\sqrt[3]{(x^3+1)^2} - (1-x)\sqrt[3]{x^3+1} + (1-x)^2} = \dots = 1. \end{aligned}$$

Therefore,  $y = x + 1$  is the slant asymptote for the graph of  $f$  and this confirms the result obtained using our proposed method.

If in step (ii) of the above procedure the solution to one of the equations in the system makes the other equation an identity, then we set the next highest leading coefficient equal to zero and use this equation in finding  $m$  and  $n$ . This is shown in the next example.

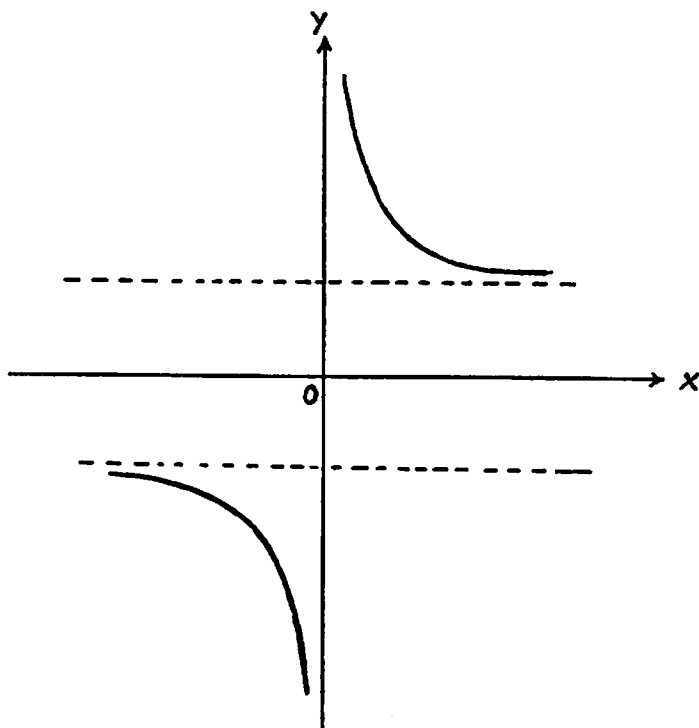


Figure 3.

**Example 2.** Find the nonvertical asymptotes of the graph of

$$f(x) = \frac{\sqrt{x^2 + 1}}{x}.$$

**Solution.** (i) Equation (1) is

$$\frac{\sqrt{x^2 + 1}}{x} = mx + n,$$

which is equivalent to

$$\sqrt{x^2 + 1} = mx^2 + nx.$$

To obtain (2), we square both sides of this equation and regroup the terms. Then we have

$$m^2x^4 + 2mnx^3 + (n^2 - 1)x^2 - 1 = 0.$$

(ii) Setting the coefficients of  $x^4$  and  $x^3$  equal to zero yields

$$\begin{cases} m^2 = 0 \\ mn = 0 \end{cases}$$

The solution of the first equation makes the second equation an identity. Thus, we replace  $mn = 0$  with  $n^2 - 1 = 0$  and we obtain  $m = 0$ ,  $n = \pm 1$ . Therefore,  $y = 1$  and  $y = -1$  are the equations of the horizontal asymptotes for the graph of  $f$  (see Figure 3). The graph of this function does not have a slant asymptote.

As we mentioned earlier, if  $f(x)$  contains a radical expression with an even index, then we may obtain extraneous results, as shown in the next example.

**Example 3.** Find the asymptotes of the graph of

$$f(x) = \sqrt{x^2 + 1} - \sqrt{x^2 - 1}.$$

**Solution.** (i) Following the pattern of the previous examples, we find that equation (2) is

$$(m^4 - 4m^2)x^4 + 4mn(m^2 - 2)x^3 + (6m^2n^2 - 4n^2)x^2 + 4mn^3x + n^4 + 4 = 0.$$

(ii) The solutions of the system

$$\begin{cases} m^4 - 4m^2 = 0 \\ 4mn(m^2 - 2) = 0 \end{cases}$$

are  $m = 0, n = 0$ ,  $m = 2, n = 0$  and  $m = -2, n = 0$ . Therefore, we obtain  $y = 0$ ,  $y = 2x$  and  $y = -2x$ .

(iii) It is clear that  $(\sqrt{x^2 + 1} - \sqrt{x^2 - 1}) \pm 2x$  is not close to zero when

$|x|$  is large. Thus, neither  $y = 2x$  nor  $y = -2x$  is an asymptote. However,  $y = 0$  is the horizontal asymptote of the graph of  $f$  (see Figure 4).

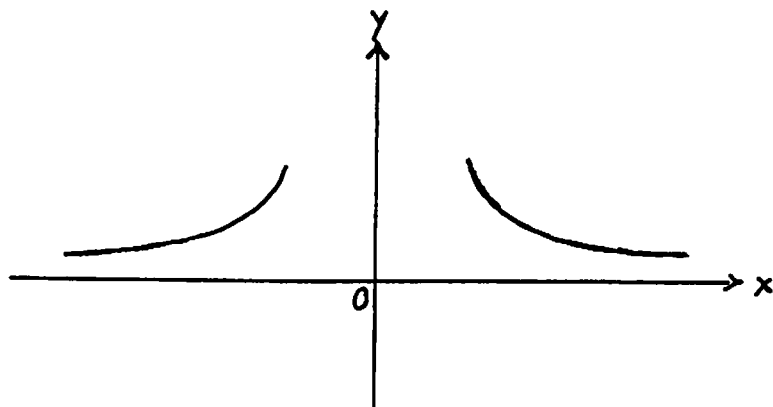


Figure 4.

This method can be applied to algebraic functions defined implicitly. This is demonstrated in the next example.

**Example 4.** Find the asymptotes of the hyperbola  $y^2 + 2xy - 2x = 0$ .

**Solution.** (i) Equation (2), obtained by substituting  $mx + n$  for  $y$  in the equation of the hyperbola and regrouping the terms, is

$$(m^2 + 2m)x^2 + 2(mn + n - 1)x + n^2 = 0.$$

(ii) By setting the coefficients of  $x^2$  and  $x$  equal to zero, we obtain

$$\begin{cases} m^2 + 2m = 0 \\ mn + n - 1 = 0 \end{cases}$$

The solutions of this system are  $m = 0, n = 1$  and  $m = -2, n = -1$ . Therefore,  $y = 1$  and  $y = -2x - 1$  are the equations of the asymptotes of the hyperbola (see Figure 5).

We conclude by using our method to find the equations for the asymptotes of a general hyperbola.



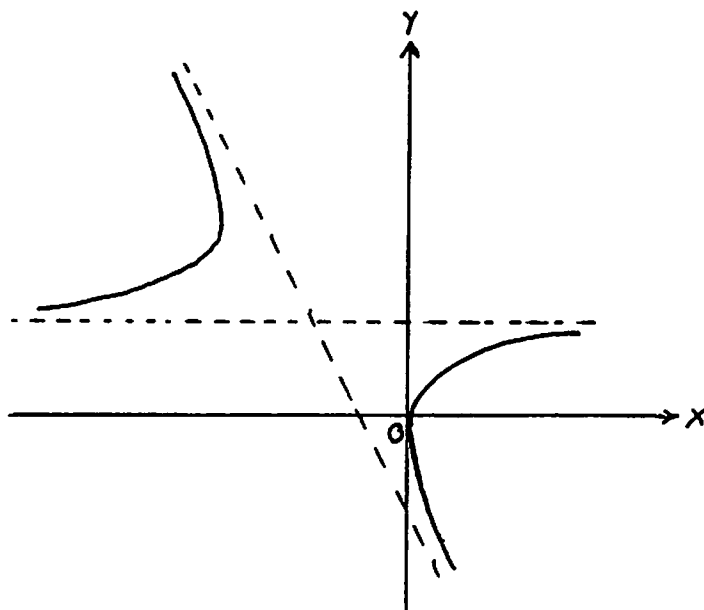


Figure 5.

**Example 5.** Find the asymptotes of the hyperbola

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where  $B^2 - 4AC > 0$ .

**Solution.** (i) Equation (2), obtained by substituting  $mx + n$  for  $y$  in the equation of the hyperbola, is

$$(Cm^2 + Bm + A)x^2 + (Bn + 2Cmn + Em + D)x + (Cn^2 + En + F) = 0.$$

(ii) If  $C \neq 0$ , the solutions of the system

$$\begin{cases} Cm^2 + Bm + A = 0 \\ Bn + 2Cmn + Em + D = 0 \end{cases}$$

are

$$m = \frac{-B \pm \sqrt{B^2 - 4AC}}{2C} \quad \text{and} \quad n = -\frac{Em + D}{B + 2Cm}.$$

Therefore, in this case the equations of the asymptotes of the hyperbola are given by  $y = mx + n$ , where  $m$  and  $n$  are as above. If  $C = 0$ , then  $B \neq 0$  and the equation of the hyperbola becomes

$$y = -\frac{Ax^2 + Dx + F}{Bx + E}.$$

In this case, the vertical line  $x = -E/B$  is the vertical asymptote and

$$y = -\frac{A}{B}x + \frac{EA - BD}{B^2}$$

is the slant asymptote.

If 
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is the standard equation of a hyperbola with horizontal transverse axis, then we have

$$b^2x^2 - a^2y^2 - 2b^2hx + 2a^2ky + b^2h^2 - a^2k^2 - a^2b^2 = 0.$$

Comparing this equation to the general form of the equation of a hyperbola, we see that  $A = b^2$ ,  $B = 0$ ,  $C = -a^2$ ,  $D = -2b^2h$ ,  $E = 2a^2k$  and  $F = b^2h^2 - a^2k^2 - a^2b^2$ . Therefore,  $m = \pm b/a$ ,  $n = k \mp bh/a$  and the equations of the asymptotes are  $y = k \pm b(x-h)/a$ . If the hyperbola has a vertical transverse axis, then a similar calculation shows that  $y = k \pm a(x-h)/b$  are the equations of the asymptotes.

#### References.

1. Memaury, H. "An alternative method for solving tangent line problems involving algebraic functions," *Alabama J. Math.* 14 (1990), 35-40.
2. Turnbull, H. W. *Theory of Equations*. New York: Interscience Publishers, Inc., 1947.

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## Notice

"The Hexagon" section has been discontinued. Papers intended for that section should be submitted directly to the Editor.

## The Problem Corner

Edited by Kenneth M. Wilke

*The Problem Corner* invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before 1 January 1994. Solutions received after the publication deadline will be considered also until the time when copy is prepared for publication. The solutions will be published in the Spring 1994 issue of *The Pentagon*, with credit being given to student solutions. Affirmation of student status and school should be included with solutions. Address all communications to Kenneth M. Wilke, Department of Mathematics, 275 Morgan Hall, Washburn University, Topeka, Kansas 66621.

### PROBLEMS 465-469.

**Problem 465.** Proposed by Stanley Rabinowitz, Westford, Maine. Three circles with centers A, B and C are mutually tangent externally (see Figure 1 on page 52). Circles A and C touch at X. Circles B and C touch at Y. Prove that the line through X and Y passes through the point where the common external tangents to the circles A and B meet.

**Problem 466.** Proposed by the Editor. It can be shown that the standard deviation  $s$  of any three consecutive integers is itself an integer. Characterize those integers  $n$  which have the property that the standard deviation  $s$  of  $n$  consecutive integers is also an integer. Find two values of  $n > 3$  which have this property. (For a related problem involving the mean and variance — but not the standard deviation — of  $n$  consecutive integers, see *Cruz Mathematicorum* 18 (November 1992), problem 1786).

**Problem 467.** Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri. Let  $m$  and  $n$  be the lengths of the chords

of arcs AB and BC, respectively, on a circle with a radius of length  $r$ . If  $p$  is the length of the chord of arc AC, prove that

$$p = \frac{m\sqrt{4r^2 - n^2} + n\sqrt{4r^2 - m^2}}{2r}.$$

**Problem 468.** Proposed by the Editor. Young Leslie Morely shuffled a standard deck of playing cards (not a pinochle deck) which contained no jokers. When he finished, he started turning over cards one at a time from the top of the deck until he found a jack after turning over the sixteenth card. Assuming that the deck contained four jacks and that young Leslie repeated this experiment several times, what would be the average number of cards which he would have to turn over before finding a jack?

**Problem 469.** Proposed by the Editor. After finishing the statistics experiment described in the previous problem, young Leslie Morely discovered an interesting number while playing with his computer. The number which he discovered has a cube which ends in 0987654321, the string of digits in reverse order with the zero having been moved to the front. Find the smallest positive integer  $n$  which has this property.

*Please help your editor by submitting problem proposals.*

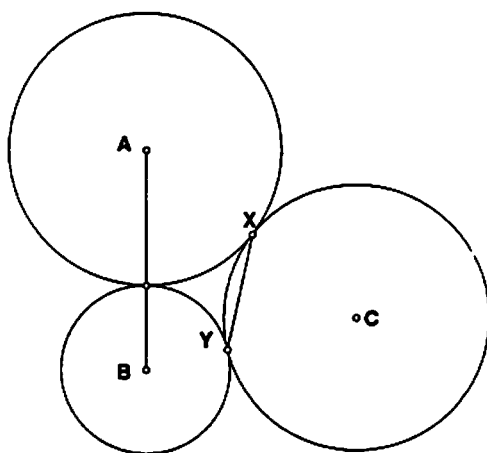


Figure 1 (see *Problem 465*).

## SOLUTIONS 451 (Corrected) and 455-459.

**Problem 451.** Proposed by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin. Find the value of the following limit.

$$\lim_{n \rightarrow \infty} e^{-n} \left(1 + \frac{1}{n}\right)^{n^2}$$

**(Corrected) Solution** by Sean Forbes, student, Drake University, Des Moines, Iowa.

Let

$$y = e^{-n} \left(1 + \frac{1}{n}\right)^{n^2}$$

Then

$$\ln y = -n + n^2 \ln\left(1 + \frac{1}{n}\right)$$

Thus we can rewrite  $\ln y$  as

$$\ln y = \frac{-\frac{1}{n} + \ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n^2}}$$

which is a "0/0" form to which L'Hospital's rule applies. Applying L'Hospital's rule and simplifying, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \ln y &= \lim_{n \rightarrow \infty} \frac{n^{-2} + (-n^{-2})(1 + (1/n))^{-1}}{-2n^{-3}} \\ &= \lim_{n \rightarrow \infty} \frac{-n(1/n)}{2(1 + (1/n))} = -\frac{1}{2} \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} y = e^{-1/2} = \frac{1}{\sqrt{e}}.$$

*Also solved* by The Alma College Problem Solving Group, Alma College, Alma, Michigan; Russell Euler, Northwest Missouri State University, Maryville, Missouri; and the proposer. Two incorrect solutions were received.

*Editor's comment.* Russell Euler's sharp eyes noticed that the  $y$  in line 7 of the published solution should be replaced with  $\ln y$  as shown above. The Editor apologizes to the featured solver for this proof reading error. The Editor also apologizes for inadvertently omitting Russell Euler's name from the list of solvers in the original publication of this solution.

**Problem 455.** Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri. Rays  $r_1$  and  $r_2$  are concurrent at  $O$ . Let  $\{a_i\}$  and  $\{b_i\}$  be increasing sequences of points on  $r_1$  and  $r_2$  respectively, such that  $d(O, a_i) = d(O, b_i)$  for  $i = 1, 2, 3, \dots$ . If  $M_i$  is the midpoint of the line segment  $a_i b_i$ , prove that the points  $\{M_i\}$  are collinear.

*Solution by Michael White, University of Chicago, Chicago, Illinois.*

We may assume that  $r_1$  is the  $x$ -axis and that  $r_2$  has the equation  $y = mx$ . On  $r_1$  we have the points  $(a_i, 0)$  and on  $r_2$  we have the points  $(b_i, mb_i)$ . Thus

$$a_i = \sqrt{b_i^2 + m^2 b_i^2} = b_i \sqrt{1 + m^2}. \quad (1)$$

Also

$$M_i = \left( \frac{a_i + b_i}{2}, m \frac{b_i}{2} \right). \quad (2)$$

We shall show that the slope  $m_{12}$  of the line connecting the points  $M_1$  and  $M_2$  equals the slope  $m_{23}$  of the line connecting the points  $M_2$  and  $M_3$ . We have

$$\begin{aligned} m_{12} &= \frac{(mb_2 - mb_1)/2}{(b_2 - b_1)/2 + (a_2 - a_1)/2} \\ &= \frac{m(b_2 - b_1)}{(b_2 - b_1) + (a_2 - a_1)} = \frac{m}{1 + \sqrt{1 + m^2}}. \end{aligned}$$

Similarly,

$$m_{23} = \frac{m}{1 + \sqrt{1 + m^2}}.$$

Thus, since the slopes are equal and each line segment contains the point  $M_2$ , we have  $M_1$ ,  $M_2$  and  $M_3$  are collinear. It follows that the points  $\{M_i\}$  are collinear.

*Also solved by Charles Ashbacher, Cedar Rapids, Iowa; and the proposer.*

**Problem 456.** Proposed by the Editor. Hy Potenuse, president of the Society of Pythagoreans, announced that starting this year all members would celebrate certain special days as "Pythagorean Days." By definition, a Pythagorean Day occurs when the numerical value of the month and day are the legs of a right triangle whose sides are all integers. How many Pythagorean Days are there in a year and when do they occur?

*Solution by Scott H. Brown, Stuart Middle School, Stuart, Florida.*

Let  $a$  be an integer from 1 to 12. By [1], we can let the sides  $b$  and  $c$  be written as  $b = (a^2 - d^2)/2d$  and  $c = (a^2 + d^2)/2d$ , where  $d$  is an arbitrary integer. Using these formulas for  $b$  and  $c$  with  $d < a$ , the following twelve Pythagorean triples were found which meet the conditions of the problem:  $(a, b, c) = (3, 4, 5), (4, 3, 5), (5, 12, 13), (6, 8, 10), (7, 24, 25), (8, 6, 10), (8, 15, 17), (9, 12, 15), (10, 24, 26), (12, 5, 13), (12, 9, 15)$  and  $(12, 16, 20)$ . Thus the twelve Pythagorean days are March 4th, April 3rd, May 12th, June 8th, July 24th, August 6th, August 15th, September 12th, October 24th, December 5th, December 9th and December 16th.

*Also solved by* Matthew Amoroso, St. Bonaventure University, Saint Bonaventure, New York; Charles Ashbacher, Cedar Rapids, Iowa; Wanda G. Cahill, Mississippi University for Women, Columbus, Mississippi; Agostino Iallouardo, Baruch College, New York, New York; Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Tim Rittenbush, Chadron State College, Chadron, Nebraska; J. Sriskandarajah, University of Wisconsin Center, Richland Center, Wisconsin; Nandor Szentkiralyi, Bowling Green State University, Bowling Green, Ohio; and Michael White, University of Chicago, Chicago, Illinois.

**Problem 457.** Proposed by Albert White, St. Bonaventure University, Saint Bonaventure, New York. In a standard bridge deck, assign the value 11, 12, and 13 to the jack, queen and king, respectively. Aces may assume the value 1 or 14. If four cards are selected, what is the probability that the cards are of the same suit and the numbers of the cards are consecutive with the first card having the smallest value? What is the probability if the cards do not have to be of the same suit?

*Solution by Charles Ashbacher, Cedar Rapids, Iowa.*

First, we assume that all four cards are in the same suit. Suppose that the first card is an ace. Assign it the value of 1. The probability that the next card is a two in the same suit as the ace is  $1/51$ . The probability that the next card is a three in the same suit is  $1/50$ . The probability that the next card is a four in the same suit is  $1/49$ . But there are four suits from which the first ace can be chosen. Then the probability of the four cards being in the same suit and in consecutive ascending order is

$$\frac{4}{52} \cdot \frac{1}{51} \cdot \frac{1}{50} \cdot \frac{1}{49} = \frac{1}{1624350}. \quad (1)$$

The result in (1) holds whenever the first card is any card from the ace to the jack. Finally, since the first card can be anything from the ace to a jack, this represents 11 mutually exclusive events depending only upon the value of the first card chosen. Hence the desired probability is  $11/1624350 = 0.00000677$ .

Now we remove the requirement that all cards be in the same suit. Proceeding as before, if the value of the first card is in the range from ace to jack, the probability for the first card is  $4/52$ . Then the probability of the second card is  $4/51$ . The probability for the third card is  $4/50$  and the probability for the fourth card is  $4/49$ . Then the probability that the four cards are in serial ascending order is

$$\frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} \cdot \frac{4}{49} = \frac{32}{812175}.$$

As before, there are 11 mutually exclusive ways to do this and the desired probability is  $(11 \cdot 32)/812175 = 0.0004334$ .

*Also solved by the proposer.*

**Problem 458.** Proposed by Michael White, Portville, New York and Albert White, St. Bonaventure University, Saint Bonaventure, New York. Find

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{k}{k!} \left( 1 - \frac{(-1)^{n-k}}{(n-k)!} \right).$$

**Solution by** Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Let  $L$  denote the value of the given limit. In this solution, the following well known results will be used:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e \tag{1}$$

and

$$\sum_{k=1}^n (-1)^{k+1} k \binom{n}{k} = 0 \quad \text{for } n \geq 2. \tag{2}$$

We have that

$$\sum_{k=0}^n \frac{k}{k!} \left( 1 - \frac{(-1)^{n-k}}{(n-k)!} \right) = \sum_{k=0}^n \frac{k}{k!} - \sum_{k=0}^n \frac{(-1)^{n-k}}{(n-k)!}$$



$$\begin{aligned}
&= \sum_{k=1}^n \frac{k}{k!} - \frac{(-1)^n}{n!} \sum_{k=0}^n \frac{(-1)^{-k} k n!}{k!(n-k)!} \\
&= \sum_{k=1}^n \frac{1}{(k-1)!} - \frac{(-1)^n}{n!} \sum_{k=0}^n (-1)^k k \binom{n}{k} \\
&= \sum_{k=0}^{n-1} \frac{1}{k!} - \frac{(-1)^{n-1}}{n!} \sum_{k=1}^n (-1)^{k+1} k \binom{n}{k} \\
&= \sum_{k=0}^{n-1} \frac{1}{k!} - \frac{(-1)^{n-1}}{n!} \cdot 0 = \sum_{k=0}^{n-1} \frac{1}{k!},
\end{aligned}$$

by (2), assuming  $n \geq 2$ . Thus by (1)

$$L = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{k}{k!} \left( 1 - \frac{(-1)^{n-k}}{(n-k)!} \right) = e.$$

(For (2), see 0.154.2 on page 4 of Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products*, New York: Academic Press, Inc., 1980.)

Also solved by Charles Ashbacher, Cedar Rapids, Iowa; and the proposer.

**Problem 459.** Proposed by the Editor. A Heronian triangle has integral sides and an integral area. Find an infinite family of such triangles which have two consecutive integers and a third odd integer for sides and such that the sides are not in arithmetic progression. Are there any Heronian triangles whose sides include two primes and two consecutive integers? For the purpose of this problem, right triangles also are excluded from consideration. An example of a right triangle which satisfies the problem is (11, 60, 61).

**Solution by** Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

Let

$$2j^2 + 1, 2j^2 + 2 \text{ and } 4j^2 + 1, \quad (*)$$

where  $j > 1$  is an integer, be the sides of the triangle. Since  $(2j^2 + 2) + 1 = 4j^2 + 1$ , where  $j$  is a positive integer, holds if and only if  $j = 1$ , the sides of the triangle of type (\*) is not a right triangle.

From Heron's Formula, the area of the triangle of type (\*) is

$$A = \sqrt{(4j^2 + 2)(2j^2 + 1)(2j^2)(1)} = 2j(2j^2 + 1),$$

where  $A$  is an integer  $\geq 36$ .

Hence, if  $j$  is an integer  $> 1$ , then  $2j^2 + 1$ ,  $2j^2 + 2$  and  $4j^2 + 1$  are the sides of a Heronian triangle which is not a right triangle and such that the sides are not in arithmetic progression. Setting  $j = 3$  yields the Heronian triangle with the sides 19, 20 and 37. The sides of this Heronian triangle include two consecutive integers and two primes.

*Also solved by* Charles Ashbacher, Cedar Rapids, Iowa.

*Editor's comment.* Ashbacher supplied the numerical solutions  $(a, b, c, A) = (3, 148, 149, 210)$ ,  $(4, 193, 195, 336)$ ,  $(5, 29, 30, 72)$  and  $(5, 509, 510, 1248)$ . These solutions, and many like them, can be obtained from Pell-type equations. For example, one can arbitrarily choose 3 as one of the sides of a triangle and take  $p - 1$  and  $p$  as the other two sides. Applying Heron's formula for the area of the triangle, as suggested in our featured solution, one obtains the Pell type equation

$$\left(\frac{x}{3}\right)^2 - 2\left(\frac{A}{3}\right)^2 = 1,$$

where  $x = 2p - 1$  and  $A$  is the area of the triangle. All solutions of this equation are given by the relations

$$X_n + Y_n\sqrt{2} = (3 + 2\sqrt{2})^n$$

for  $n$  a positive integer. Here  $x = 3X_n$  and  $A = 3Y_n$ . Ashbacher's  $(3, 148, 149)$  corresponds to  $n = 3$ .

## Kappa Mu Epsilon News

Edited by Mary S. Elick, Historian

News of chapter activities and other noteworthy *KME* events should be sent to Mary S. Elick, Historian, Kappa Mu Epsilon, Mathematics Department, Missouri Southern State College, Joplin, Missouri 64801.

### INSTALLATION OF NEW CHAPTERS

#### *New Hampshire Alpha*

Keene State College, Keene, New Hampshire

The installation of the New Hampshire Alpha Chapter of Kappa Mu Epsilon was held on February 16, 1993, in the Great Hall of Holloway Hall on the campus of Keene State College. Prof. Charles E. Brusard, corresponding secretary of Massachusetts Alpha, conducted the installation ceremony. Dr. Charles A. Riley, of the Department of Mathematics at Keene State, served as Conductor during the ceremony. Eighteen students and six faculty constituted the founding group of the new chapter at Keene State College. Those initiated were:

*Students:* Bethany Andrews, Erik Barbere, Eileen Depecol, Cathleen J. Farnsworth, Daniel Fischer, Daniel Grummon, Matthew Gwinn, Robert J. Hastings, Elise Lachance, Pamela Moore, Shayne Noyes, Stephen Rack, Kevin Roderick, Stephanie G. Rogers, Tracy Smith, James Stewart, Tracey Thibeault, and Emily J. Weber.

*Faculty:* Dr. Vincent Ferlini, Dr. Stuart Goff, Dr. Ockle Johnson, Dr. Charles A. Riley, Dr. Joseph Witkowski, and Dr. Edwin Wolf.

Following the 6:00 p.m. installation ceremony, Prof. Brusard gave a brief history of honor societies in colleges and universities and, in particular, the founding of Kappa Mu Epsilon. A banquet was then held at 7 p.m.

Officers installed during the ceremony were Eileen Depecol, president; Tracey Thibeault, vice-president; Bethany Andrews, secretary; Elise Lachance, treasurer. Faculty members Charles A. Riley and Ockle Johnson accepted the responsibilities of the corresponding secretary and faculty sponsor, respectively.

## CHAPTER NEWS

### *Alabama Beta*

University of North Alabama, Florence

Chapter President - Kim Weems

34 actives

Other 1992-93 officers: Rachel Powers, vice president; Vicky Locker, secretary; Eddy Joe Brackin, corresponding secretary; Patricia Roden, faculty sponsor.

### *Alabama Zeta*

Birmingham-Southern College, Birmingham

Chapter President - Heath Gatlin

35 actives

Sixteen new members were initiated. The chapter heard Dr. Arthur C. Segal speak on "Demography and Mathematics." Several *KME* members conducted mathematics review sessions for students preparing for the Graduate Record Examination. Other 1992-93 officers: Erica Taylor, vice president; Kelly Elliott, secretary/treasurer; Lola F. Kiser, corresponding secretary; Shirley Branan, faculty sponsor.

### *Arkansas Alpha*

Arkansas State University, State University

Chapter President - Leslie Mitchell

20 actives, 10 associates

Other 1992-93 officers: Melinda Luehrs, secretary; Gary Austin, treasurer; Andy Talmadge, corresponding secretary/faculty sponsor.

### *California Gamma*

California Polytechnic State University, San Luis Obispo

Chapter President - Eric Bauer

30 actives

Early in the fall quarter officers of California Gamma gave their unqualified support to assist with the Cal Poly Mathematics

Department's Annual Math Contest should the Department decide to revive the contest which was cancelled in the spring of '92. Due to the cancellation of last year's contest, the chapter was left with a large number of sweatshirts which the club had hoped to sell during the contest. As a result, the annual sale evolved into an on-going sale of tee shirts and sweatshirts. The chapter continued to be career oriented, inviting representatives from business and industry to give presentations to the club. Doug Rosenfeld of Anderson Consulting addressed the October 1, 1992, meeting of *KME*. He spoke about the role majors in the mathematical sciences play in the world of industry, and in particular at Anderson Consulting. His presentation was followed by a question period and a pizza buffet luncheon. Later that month interested *KME* members were invited to a special information session to be hosted by Anderson Consulting in early November. In other activities, the group assisted the College of Science and Mathematics with its phon-a-thon and sponsored a canned food drive for the needy during the holiday season. Other 1992-93 officers: Jennifer Courter, vice president; Michael Bailey, secretary; Henry Mesa, treasurer; Eric Gordon and Sabrina Hale, pledgemasters; Sabrina Hale, representative to the School Council; Jeff Goldstein, representative to the Mathematics Department curriculum committee; Raymond D. Terry, corresponding secretary/faculty sponsor.

### *California Delta*

California State Polytechnical University, Pomona

Chapter Presidents - Tracy Baughn and Patti Chamroonrat

15 actives, 10 associates

Chapter activities included pizza parties, on-campus service for academic meetings, and the annual trip to Las Vegas to study laws of probability. Other 1992-93 officers: Eric Laszlo, treasurer; Jim McKinney, corresponding secretary/faculty sponsor.

### *Georgia Alpha*

West Georgia College, Carrollton

Chapter President - Debbie Ingle

25 actives

Once again, Georgia Alpha sponsored a Food and Clothing Drive for the needy. Items collected were taken to the Community Shelter for distribution. Fifteen people attended a fall social held at a local restaurant. Other 1992-93 officers: Denise Askins, vice president; William Pottorf, secretary; Joy McCallie, treasurer; Thomas J. Sharp, corresponding secretary/faculty sponsor.

**Illinois Beta**

Eastern Illinois University, Charleston

Chapter President - Laura Tougaw

38 actives

In addition to a fall picnic and a Christmas party, the chapter held regular meetings and attended the 44th Annual ICTM Meeting at Peoria Civic Center. Fall speakers included Peter Andrews who spoke on "Images of the Chaos Game," and William Slough. Other 1992-93 officers: Rodney Johnson, vice president; Wendy Coplea, secretary; Andrew Rice, treasurer; Lloyd Koontz, corresponding secretary; Rosemary Schmalz, faculty sponsor.

**Illinois Delta**

College of St. Francis, Joliet

Chapter President - Mark Mitchell

14 actives, 5 associates

Highlight of the semester was the December meeting featuring three *KME* graduates: a secondary school teacher, a sports statistician, and an employee of Arthur Anderson Corporation. A taffle apple fund raiser was held in October. Other 1992-93 officers: Molly Sullivan, vice president; Jennifer Hoffman, secretary; Carrie Briscoe, treasurer; Sister Virginia McGee, corresponding secretary/faculty sponsor.

**Iowa Alpha**

University of Northern Iowa, Cedar Falls

Chapter President - Julie Beck

37 actives

The annual *KME* Homecoming Coffee, held October 3 in the faculty lounge of the newly refurbished Wright Hall, was well attended by students, faculty, and alumni. Students presenting papers at local *KME* meetings included Jennifer Puffett on "A Mathematical Description of Aluminum Toxicity in Rice," Julie Beck on "Computer Assisted Instruction—A Calculus Student's Friend," and Kevin Hesner on "The Fibonacci Sequence." Jason Sash addressed the *KME* initiation banquet held at the Broom Factory on December 8 on "Computed Tomography: Algebraic Reconstruction Technique." Steven Walk was awarded a one year membership in the Mathematics Association of America. Other 1992-93 officers: Ted Juhl, vice president; Jennifer Puffett, secretary; Karen Brown, treasurer; John S. Cross, corresponding secretary/faculty sponsor.

*Iowa Gamma*

Morningside College, Sioux City

Chapter President - Doug Rants

10 actives

Other 1992-93 officers: Vimal Kumar, vice president; Taylor Guo, secretary; Mike Murray, treasurer; Steve Nimmo, corresponding secretary/faculty sponsor.

*Iowa Delta*

Wartburg College, Waverly

Chapter President - Nancy Wirth

32 actives, 11 associates

The program for the September meeting, "What an MAA Governor Does," was presented by Dr. Lynn Olson, Wartburg faculty member and new governor for the Iowa Section of MAA. Other fall programs included a video, "Mathematics in a New Era," and a presentation by Bret Hoyer entitled "Discrete Mathematics and Cooperative Learning Strategies." Hoyer, an alumnus, teaches high school mathematics. The traditional Christmas Party was held in December. Projects during the fall included the Roy's Egg-Cheese Sandwich Stand at homecoming and plans for the 1993 Wartburg Math Field Day. Other 1992-93 officers: Jeffrey Isaacson, vice president; Melissa Dodd, secretary; Nicole Lang, treasurer; August Waltmann, corresponding secretary/faculty sponsor.

*Kansas Alpha*

Pittsburg State University, Pittsburg

Chapter President - Barry Smith

60 actives

Fall semester activities focused on initiation of nine new members in October. A pizza party was held preceding the initiation. Kansas Alpha chapter sponsored a guest speaker, Tim Flood, for the November meeting. Tim is a PSU alumnus currently working on a PhD at Oklahoma State University. He spoke about his research on "The Hyperbolic Geometry of the Upper Half-plane." The final fall semester meeting was an ice cream and cake social held at the home of faculty member, Dr. Gary McGrath. The program included viewing the Chaos videotape. Barry Smith was also elected president for the spring semester due to the resignation of Ed Morris. Other 1992-93 officers: Thein Maung, secretary; Jerri Lott, treasurer; Harold Thomas, corresponding secretary; Bobby Winters, faculty sponsor.

**Kansas Beta**

Emporia State University, Emporia

Chapter President - Dave Herrs

14 actives

Other 1992-93 officers: Sheila Nutter, vice president; Christel Meyer, secretary/treasurer; Connie S. Schrock, corresponding secretary; Larry Scott, faculty sponsor.

**Kansas Gamma**

Benedictine College, Atchison

Chapter President - David Klenke

12 actives, 21 associates

Kansas Gamma welcomed new students at a pizza party in September. In October, student members presented information on the lives of various mathematicians. Faculty member Jim Ewbank discussed with the group some of the mathematical shapes, such as the involute of the circle, that he uses in making stained glass windows. In November former faculty member Larry Schultz spoke to the chapter about his current research and software development involving math/computer interface. Pamela Clearwater, December graduate, was honored at the annual, traditional Wassail Party at the home of Jim Ewbank in December. Other 1992-93 officers: Pamela Clearwater, vice president; Tiffany Opsahl, secretary/treasurer; Jo Ann Fellin, corresponding secretary/faculty sponsor.

**Kansas Delta**

Washburn University, Topeka

Chapter President - Jennifer Hudson

24 actives

Other 1992-93 officers: Jessica Dyck, vice president; Michelle Reed, treasurer; Allan Riveland, corresponding secretary; Ron Wasserstein, faculty sponsor.

**Kansas Epsilon**

Fort Hays State University, Hays

Chapter President - Donna Weninger

Fall activities enjoyed by Kansas Epsilon included monthly meetings, a fall picnic, a Halloween party, and a Christmas party. Other 1992-93 officers: Dale Brungardt, vice president; Anita Lessor, secretary/treasurer; Charles Votaw, corresponding secretary; Mary Kay Schippers, faculty sponsor.



**Kentucky Alpha**

Eastern Kentucky University, Richmond

Chapter President - Eddie Robinson

22 actives, 3 associates

The highlight of the fall semester picnic, held at the Costello house, was the student versus faculty volleyball game. Crystal Pendencygraft was nominated in September for homecoming queen. The October meeting included a panel discussion on graduate school. The November program was a tasty talk by Professor Mary Fleming entitled "Have Your Data and Eat It Too!" The semester ended with a Christmas party featuring good food, music and a wild White Elephant exchange. Other 1992-93 officers: Mike Mattingly, vice president; Susan Popp, secretary; Crystal Pendencygraft, treasurer; Pat Costello, corresponding secretary; Kirk Jones, faculty sponsor.

**Maryland Beta**

Western Maryland College, Westminster

Chapter President - Brenton Squires

18 actives

Fall chapter activities included an induction meeting and several planning meetings. Plans were made to sponsor a career night in the spring, as well as a picnic and spring induction. Other 1992-93 officers: Sin Yee Wu, vice president; Todd Wizotsky, secretary; G. William Yankosky, treasurer; James Lightner, corresponding secretary; Linda Eshleman, faculty sponsor.

**Maryland Delta**

Frostburg State University, Frostburg

Chapter President - Steven Smith

33 actives

Maryland Delta Chapter enjoyed a pizza/puzzle party in September and made plans to induct new members in February. Other 1992-93 officers: Christine Bittinger, vice president; Thomas Currier, secretary; Diana Beisel, treasurer; Edward White, corresponding secretary; John Jones, faculty sponsor.

**Michigan Beta**

Central Michigan University, Mt. Pleasant

Chapter President - Matt Ayotte

20 actives

The Michigan Beta Chapter celebrated its 50th Anniversary during the 1992 year. The group hosted a pre-game picnic for alumni, faculty, and present members before the CMU Homecoming Football Game in October. Activities at meetings included a viewing of the MAA video "The Story of Pi," a talk by member Matt Ayotte on the Pigeonhole

Principle, and a discussion with mathematics students about mathematics course offerings for spring semester. The semester closed with a Christmas party at the home of advisor Arnie Hammel. Other 1992-93 officers: Dave Koester, vice president; Jenny Blake, secretary; Tami Hanson, treasurer; Arnold Hammel, corresponding secretary/faculty sponsor.

### *Missouri Alpha*

Southwest Missouri State University, Springfield

Chapter President - Susan Gibiser

35 actives, 5 associates

Missouri Alpha met monthly during the fall semester. In a cooperative agreement, *KME* meets jointly with the SMSU student chapter of MAA. Highlights of the semester included the *KME/MAA* fall picnic attended by approximately 80 students, faculty and staff, and the end of semester *KME/MAA* pizza party. Other 1992-93 officers: Mike Jones, vice president; Chrissy Hixon, secretary; Mae Rivera, treasurer; Ed Huffman, corresponding secretary; Mike Awad, faculty sponsor.

### *Missouri Beta*

Central Missouri State University, Warrensburg

Chapter President - Jay Rowland

15 actives, 10 associates

The school year started off with a chapter welcome-back picnic and barbecue. At the first meeting several students talked about internships they had held during the previous summer. The fall initiation was held in September. In October the chapter got together for a Halloween Party and a very cold night-time barbecue. A visiting professor from Budapest, Hungary, presented a program on Simulation Modeling for the November meeting. The chapter also held a semi-annual book sale and volunteered hours for the mathematics department's clinic. The semester ended with a bowling and pizza Christmas Party. Other 1992-93 officers: Russell Savage, vice president; Jennifer Ritzo, secretary; Tracy Rouchka, treasurer; Rhonda McKee, corresponding secretary; Larry Dilley, Homer Hampton, Phoebe Ho, Debbie Detrick, faculty sponsors.

### *Missouri Gamma*

William Jewell College, Liberty

Chapter President - Mark Decker

12 actives

Other 1992-93 officers: Reggie Hoog, vice president; Scott O'Neill, secretary; Joseph Mathis, treasurer/corresponding secretary/faculty sponsor.

*Missouri Epsilon*

Central Methodist College, Fayette

Chapter President - Mary Ann Neal

10 actives

Other 1992-93 officers: Roselyn Magosha, vice president; Ed La Valle, secretary; Holly Toler, treasurer; William D. McIntosh, corresponding secretary; Linda O. Lembke, faculty sponsor.

*Missouri Eta*

Northeast Missouri State University, Kirksville

Chapter President - Jason Lott

28 actives, 3 associates

In addition to monthly meetings, the chapter held a Christmas Party for its members, sponsored a faculty-student softball game, and visited the local nursing home to play cards with the residents. Other 1992-93 officers: Scott Niemeyer, vice president; Deanne Reber, secretary; Angela Hahn, treasurer; Mary Sue Beersman, corresponding secretary; Shelle Palaski, faculty sponsor.

*Missouri Iota*

Missouri Southern State College

Chapter President - Jeannie Cambers

10 actives, 12 associates

Missouri Iota again worked football concessions to raise revenue. A volleyball game/cookout was held at the home of Mrs. Mary Elick in September. In October the group heard Mr. Chip Curtis speak on "Some Interesting Matrices," and in November *KME* alumnus Tom Bartkowiak spoke concerning his work at Eagle Picher. The semester activities ended with a Christmas pizza party and white elephant exchange at the home of Dr. Linda Noel. Other 1992-93 officers: Laura Jay and Diane Hoch, vice presidents; Kim Tarnowiecky, secretary/treasurer; Mary Elick, corresponding secretary; Linda Noel, faculty sponsor.

*Missouri Lambda*

Missouri Western State College, St. Joseph

Chapter President - Joseph Shawn Crawford

28 actives, 8 associates

The organization enjoyed a picnic and softball game early in the semester, and sponsored a booth at Family Day September 19. Eight new members were initiated in October. Other activities included business meetings and a bake sale. Other 1992-93 officers: Tammy Resler, vice president; Tracy Schemmer, secretary; Denise Fuller, treasurer; John Atkinson, corresponding secretary; Jerry Wilkerson, faculty sponsor.

**Missouri Kappa**

Drury College, Springfield

Chapter President - Bill Davis

11 actives, 7 associates

The first activity of the semester was a rush party; potential *KME* members were treated to pizza and a movie. The social activities continued with a bonfire - weiner roast held at Dr. Allen's house. The winners of the Annual Campus Math Contest were Ingo Schranz (Calculus I and below) and Bill Davis (Calculus II and above). Prize money was awarded to the winners at a pizza party held for all contestants. At a luncheon for the chapter, Bill Davis gave a report on his undergraduate research project. The organization ran a tutoring service for both the day school and the evening college as a money making project. The end of the semester was celebrated with a Christmas Party. Other 1992-93 officers: Cindy Schwab, secretary/treasurer; Charles Allen, corresponding secretary; Don Moss, faculty sponsor.

**Nebraska Alpha**

Wayne State College, Wayne

Chapter President - Amy Anderson

22 actives

Throughout the semester club members monitored the Math-Science Building in the evening to earn money for the club. The club participated in the college homecoming activities by cooperatively building a float for the homecoming parade. Amy Anderson who was enrolled in the Honors Program gave her paper at a faculty-student seminar in December. The title of the paper was "Complex Number System And Its Use In Computer Software." Social activities included a fall picnic with the Math-Science faculty and other clubs in the building, a pizza-movie party at Dr. Paige's home and a bowling party. Other 1992-93 officers: Susan Sorensen, vice president; Jaime Tiller, secretary/treasurer; Wendy Stanley, historian; Fred Webber, corresponding secretary; Jim Paige, faculty sponsor.

**Nebraska Beta**

University of Nebraska, Kearney

Chapter President - Anita Lutz

20 actives, 3 associates

Nebraska Beta assisted an area educational service unit with Math Fun Day, a competition for high school students. December math and statistics graduates were honored at a holiday reception hosted at the school's Alumni House by chapter members. Other 1992-93 officers: Brooke Bernhardt, vice president; Cara Ullerich, secretary; Mark

Schnitzler, treasurer; Charles Pickens, corresponding secretary; Lutfi Lutfiyya, faculty sponsor.

**Nebraska Gamma**

Chadron State College, Chadron

Chapter President - James Collins

17 actives

Other 1992-93 officers: Jereme Patterson, vice president; Brandon Herdt, secretary; Todd Zitlow, treasurer; James Kaus, corresponding secretary; Monty Fickel, faculty sponsor.

**Nebraska Delta**

Nebraska Wesleyan University, Lincoln

Chapter President - Shawn Clymer

21 actives

Other 1992-93 officers: Matt Meyer, vice president; Chris Roth, secretary; Rachel Bunting, treasurer; Muriel Skoug, corresponding secretary/faculty sponsor.

**New York Alpha**

Hofstra University, Hempstead

Chapter President - Jason Lieberman

10 actives, 5 associates

New York Alpha enjoyed a fall volleyball game. Other 1992-93 officers: Lee Ann Molten and Susanne Morscher, vice presidents; John Veniger and Elizabeth Connolly, secretaries; Theresa Vecchiarelli, treasurer; Aileen Michaels, corresponding secretary/faculty sponsor.

**New York Eta**

Niagara University, Niagara University

Chapter President - Paul Schreiner

12 actives, 18 associates

Fall activities have been focused on planning for the national convention which the chapter will host in April. Other 1992-93 officers: Richard Inserra, vice president; Lisa Maselli, secretary/treasurer; Robert Bailey, corresponding secretary; Kenneth Bernard, faculty sponsor.

**New York Kappa**

Pace University, New York

Chapter President - Paula Murray

40 actives

Other 1992-93 officers: Ricky Gocool, vice president; Eileen Lawrence, secretary; Geraldine Taiani, corresponding secretary; John W. Kennedy, faculty sponsor.

**New York Lambda** C. W. Post Campus/Long Island University, Brookville  
Chapter President - Myleen Rojano  
16 actives, 3 associates

The primary efforts of New York Lambda focused on working with a student science club to encourage more associate members and to encourage interaction with the other sciences. Other 1992-93 officers: Suzanne Hecker, vice president; Nicholas Ramer, secretary; Lisa Evans, treasurer; Sharon Kunoff, corresponding secretary; Andrew Rockett, faculty sponsor.

**New York Nu** Hartwick College, Oneonta  
Chapter President - Eric P. DeJager  
13 actives, 5 associates

Ten *KME* members and associates accompanied by two faculty *KME* members, attended the Seaway Section Meeting of MAA at Cornell University in November. Other 1992-93 officers: Jacalyn M. O'Connor, vice president; Timothy C. French, secretary; John A. Pape, treasurer; Gary Stevens, corresponding secretary/faculty sponsor.

**North Carolina Gamma** Elon College, Elon College  
Chapter President - Varun Rao

The chapter held two joint meetings with Elon's MAA student chapter in October and November. These meetings featured short talks on math and math-related games. Plans were made for more frequent informal gatherings in the spring. Other 1992-93 officers: Miguel Johnston, vice president; Kristie Collins, secretary; Charles Tonron, treasurer; Jeffrey Clark, corresponding secretary; Rosalind Reichard, faculty sponsor.

**Ohio Alpha** Bowling Green State University, Bowling Green  
Chapter President - Kevin P. Davis  
70 actives, 5 associates

Other 1992-93 officers: Diana Nietz, vice president; Holly McDaniel, secretary/treasurer; Waldemar Weber, corresponding secretary; Neal Carothers, faculty sponsor.

*Ohio Zeta*

Muskingum College, New Concord

Chapter President - Janet Gongola

19 actives

The program for the September meeting was a presentation by Jim Buddenberg entitled "Nearest Scalar Matrix." Induction of new members was held in October. The chapter heard Drs. Joe Kennedy and Dave Groggel, visiting speakers from Miami University, in December. Other 1992-93 officers: Steve Miller, vice president; Sabrina Fuller, secretary; Jim Buddenberg, treasurer; James L. Smith, corresponding secretary; Russell Smucker, faculty sponsor.

*Oklahoma Alpha*

Northeastern State University, Tahlequah

Chapter President - Stephanie Monks

39 actives, 4 associates

Oklahoma Alpha continues to have joint activities with NSU's student MAA chapter. The group sponsored a welcome back party in September featuring subway sandwiches and the game "Mathematical Jeopardy." Fall initiation ceremonies for 14 students were held in the banquet room of a local restaurant. Other activities included the annual book sale, a monthly math contest, and a Christmas Party. Other 1992-93 officers: S. Kalen Dodson, vice president; Okcha Cockrum, secretary; Donna Baughman, treasurer; Joan E. Bell, corresponding secretary/faculty sponsor.

*Oklahoma Delta*

Oral Roberts University, Tulsa

Chapter President - Brian Augenstein

19 actives

Oklahoma Delta sponsored a team to compete on the National Putnam Exam. Other 1992-93 officers: Stephanie Wall, vice president; Amy Amsler, secretary; Lisa Brecheisen, treasurer; Debra Oltman, faculty sponsor; Roy Rakestraw, faculty sponsor.

*Pennsylvania Alpha*

Westminster College, New Wilmington

Chapter President - Monica Mundo

23 actives

Pennsylvania Alpha continued to provide tutoring as a service to math students through the Learning Center and Westminster's Life Long Learning Program. The chapter sponsored an ice cream social for faculty and perspective math students and majors. A career night is going to be organized to inform members of career opportunities in mathematics and plans are being made for participation in Math Awareness Week. Other

1992-93 officers: Amy Shannon, vice president; Jennifer Peelor, secretary; Kelly Hughes, treasurer; J. Miller Peck, corresponding secretary; Warren Hickman, faculty sponsor.

***Pennsylvania Beta***

La Salle University, Philadelphia

Chapter President - Michael Scafidi

15 actives

Other 1992-93 officers: Angela Rowbottom, vice president; Joseph Evangelist, secretary; Richard Wojnar, treasurer; Hugh N. Albright, corresponding secretary; Carl McCarty, faculty sponsor.

***Pennsylvania Gamma***

Waynesburg College, Waynesburg

Chapter President - Christy Barclay

6 actives, 4 associates

Other 1992-93 officers: Michelle Armbrust, vice president; Bob McNulty, secretary; Pete Massung, treasurer; A. B. Billings, corresponding secretary/faculty sponsor.

***Pennsylvania Delta***

Marywood College, Scranton

Chapter President - Kelly Curtin

9 actives

Other 1992-93 officers: Alice Ward, vice president; Marsha Galgon, secretary; Kathleen Hanlon, treasurer; Sister Robert Ann von Ahnen, corresponding secretary/faculty sponsor.

***Pennsylvania Epsilon***

Kutztown University, Kutztown

Chapter President - Laura Perola

10 actives, 3 associates

Other 1992-93 officers: Amy Catalano, vice president; Cheryl Kilpatrick, secretary; Margaret Reodinger, treasurer; Cherry C. Mauk, corresponding secretary; Randy Schaeffer, faculty sponsor.

***Pennsylvania Eta***

Grove City College, Grove City

Chapter President - Kristi Kowalski

36 actives

Fall semester activities included the initiation of new members in October and the annual Christmas Party at Jack Schlossnagel's residence in December. Other 1992-93 officers: Vajeera Dorabawila, vice president; Tracy Plieninger, secretary; Steve Swartzlander, treasurer; Marvin Henry, corresponding secretary; Dan Dean, faculty sponsor.



*Pennsylvania Zeta*

Indiana University of Pennsylvania, Indiana

Chapter President - Mark Rayha

15 actives, 3 associates

Other 1992-93 officers: Laurie Valowe, secretary; Steve Spratt, treasurer; Arlo Davis, corresponding secretary; Dan Burkett, faculty sponsor.

*Pennsylvania Iota*

Shippensburg University of Pennsylvania, Shippensburg

Chapter President - Marnie Paul

24 actives

Pennsylvania Iota Chapter, along with Math Club, sponsored a successful fall picnic for students and faculty members. On November 15, six new members were initiated at the home of Dr. and Mrs. Rick Ruth. Plans are underway to send representatives to the National Convention at Niagara University in April. Other 1992-93 officers: Jeff Rady, vice president; Lisa Nesbitt, secretary; Susan Waltmyer, historian; Michael Seyfried, corresponding secretary/faculty sponsor.

*Pennsylvania Kappa*

Holy Family College, Philadelphia

Chapter President - Kevin Carsley

7 actives, 10 associates

The major focus of the meetings was on problem solving. Tutoring of underprivileged children was initiated in October. Plans for a March induction of new members were discussed. Other 1992-93 officers: David McCabe, vice president/secretary/treasurer; Sister M. Grace Kuzawa, corresponding secretary/faculty sponsor.

*Pennsylvania Lambda*

Bloomsburg University, Bloomsburg

Chapter President - Kathleen Szymczak

15 actives

Math tutoring sessions were held on Tuesday and Wednesday evenings. Programs included a talk by faculty member Helmut Doll on Knot Theory. A Halloween party was hosted by the chapters for all math students and faculty. Other 1992-93 officers: Thaddea Puzio, vice president; Katie Yarrington, secretary; Todd Rider, treasurer; Jim Pomfret, corresponding secretary; John Riley, faculty sponsor.

**Pennsylvania Nu**

Ursinus College, Collegeville

Chapter President - Deborah Collinge

15 actives, 2 associates

Pennsylvania Nu sponsored two lectures during the fall semester. The first was a presentation in October by Professor Louise Berard of Wilkes University entitled "Turing Machines and Decidability." The second, entitled "Uh Oh! My Diagnostic Test is Positive," was given in November by Professor Kay Somers of Moravian College. Other 1992-93 officers: Beth Carkner, vice president; Reid Gilbert, secretary; Kara Raiguel, treasurer; Jeff Neslen, corresponding secretary; Richard Bremiller, faculty sponsor.

**South Carolina Delta**

Erskine College, Due West

Chapter President - Jodi Dixon Long

11 actives, 2 associates

The group met three times during the fall. Activities were somewhat limited as two of the officers were practice teaching. A more active spring is expected. Other 1992-93 officers: Dawn Alison Smith, vice president; Lindi Latham, secretary/treasurer; Ann Bowe, corresponding secretary/faculty sponsor.

**South Dakota Alpha**

Northern State University, Aberdeen

Chapter President - Ann Vidoloff

10 actives

South Dakota Alpha had a fall initiation and gained three new members. Their first semester as a new chapter was spent getting established on campus, making plans for the spring national convention, and organizing for spring activities. Student members were Secret Santas for the entire math faculty during the end of the semester and revealed their identities at a *KME* student-faculty Christmas Party. Other 1992-93 officers: Joe Brooks, vice president; Marci Leberman, secretary; Brenda Rook, treasurer; Abid Elkhader, corresponding secretary; Raj Markanda, faculty sponsor.

**Tennessee Alpha**

Tennessee Technological University, Cookeville

Chapter President - Molly Slaughter

15 actives, 1 associate

Other 1992-93 officers: Leanne Link, vice president; Jennifer Kite, secretary; Lori Robbins, treasurer; Frances Crawford, corresponding secretary; Jake Beard, faculty sponsor.

*Tennessee Delta*

Carson-Newman College, Jefferson City

Chapter President - Laurie Plunk

19 actives

Tennessee Delta enjoyed a picnic at Panther Creek Park in September. In November, Dr. Don Hinton of University of Tennessee visited to discuss graduate school opportunities. A Christmas Party was held at Lisa Fox's house in December. Other 1992-93 officers: Chris Knight, vice president; Rebecca Sowder, secretary; Lora Brogan, treasurer; Verner Hansen, corresponding secretary; Carey Herring, faculty sponsor.

*Texas Alpha*

Texas Tech University, Lubbock

Chapter President - Troy R. Smith

48 actives

Other 1992-93 officers: Chris Norden, vice president; Nina Nelson, secretary; Brian D. Ashcraft, treasurer; Robert Moreland, corresponding secretary; Gary Harris, faculty sponsor.

*Texas Kappa*

University of Mary Hardin-Baylor, Belton

Chapter President - Becky Hunt

15 actives, 10 associates

Ms. Sherrie Kivlighn, manager of SSC education programs, spoke to the club in the fall about the ongoing superconducting super collider project located at Waxahachie, TX, and its associated employment opportunities. She also presented an audio-visual demonstration and performed several simple physics demonstrations involving superconducting magnetism. Other 1992-93 officers: Tim Collins, vice president; Shirley Feild, secretary; Scott Callaway, treasurer; Peter H. Chen, corresponding secretary; Maxwell M. Hart, faculty sponsor.

*Texas Eta*

Hardin-Simmons University, Abilene

Chapter President - Louis Revor

12 actives, 5 associates

Texas Eta Chapter decided in September to form a student Math Club which would include those who do not meet *KME* requirements but are interested in involvement. In October the chapter, in conjunction with Math Club, held a hamburger cookout and volleyball game. The group assisted with the UIL Math Meet for high school students in November. Other 1992-93 officers: Jill Sims, vice president; Kristen

Hieronymus, secretary; Amy Garrison, treasurer; Frances Renfroe, corresponding secretary, Charles Robinson, Edwin Hewett, and Dan Dawson, faculty sponsors.

*Wisconsin Alpha*

Mount Mary College, Milwaukee

Chapter President - Jill Rogahn

5 actives, 5 associates

Chapter members, along with the Mathematics/Computer Science Department, sponsored a Mathematics Contest for junior and senior high school women. The top individual was awarded a \$2000 renewable scholarship to Mount Mary College. Other 1992-93 officers: Jill Rogahn, secretary; Sister Adriene Eickman, corresponding secretary/faculty sponsor.

*Wisconsin Beta*

University of Wisconsin, River Falls

Chapter President - Dixie Carroll

15 actives

Wisconsin Beta, in conjunction with the computer science club on campus, sponsored the annual fall picnic. Special guests were the freshman math majors/minors. Despite the rain, those attending enjoyed volleyball and frisbee. A fund raising bake/weinie sale was held in October as part of Science Day activities. Members also assisted in other ways with the Science Day event. In addition to tutoring on campus, members did volunteer tutoring at local junior high schools. Other activities included designing and ordering *KME* sweatshirts and teeshirts, an outing to the Science Museum of Minnesota in St. Paul, and the annual Christmas Party at a local establishment, followed by some rousing games of darts. Other 1992-93 officers: Greg Redding, vice president; Michael Weber, secretary; Timothy Stroth, treasurer; Robert Coffman, corresponding secretary/faculty sponsor.

*Wisconsin Gamma*

University of Wisconsin-Eau Claire, Eau Claire

Chapter President - Lara Whitehead

21 actives, 22 associates

Meetings were held regularly twice a month, each featuring one or two student speakers. At several of the meetings mixers of mathematical games and puzzles were utilized. Popcorn and used books were sold for revenue. Other 1992-93 officers: Jeff Ion, vice president; Jacqueline Hoffman, secretary; Jodi Hanson, treasurer; Tom Wineinger, corresponding secretary/faculty sponsor.

## Kappa Mu Epsilon National Officers

Harold L. Thomas	<i>President</i>
Department of Mathematics Pittsburg State University, Pittsburg, Kansas 66762	
Arnold D. Hammel	<i>President-Elect</i>
Department of Mathematics Central Michigan University, Mt. Pleasant, Michigan 48859	
Robert L. Bailey	<i>Secretary</i>
Department of Mathematics Niagara University, Niagara University, New York 14109	
Jo Ann Fellin	<i>Treasurer</i>
Mathematics and Computer Science Department Benedictine College, Atchison, Kansas 66002	
Mary S. Elick	<i>Historian</i>
Department of Mathematics Missouri Southern State College, Joplin, Missouri 64801	

Kappa Mu Epsilon, Mathematics Honor Society, was founded in 1931. The object of the Society is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics due to its demands for logical and rigorous modes of thought; to provide a Society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; and to disseminate the knowledge of mathematics and familiarize the members with the advances being made in mathematics. The official journal of the Society, *The Pentagon*, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the Chapters.

## Active Chapters of Kappa Mu Epsilon

*Listed by date of installation.*

Chapter	Location	Installation Date
OK Alpha	Northeastern Oklahoma State University, Tahlequah	18 April 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Southwest Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
MS Beta	Mississippi State University, Mississippi State College	14 Dec 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
NM Alpha	University of New Mexico, Albuquerque	28 March 1935
IL Beta	Eastern Illinois University, Charleston	11 April 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 April 1937
OH Alpha	Bowling Green State University, Bowling Green	24 April 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	Central Missouri State University, Warrensburg	10 June 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
TX Beta	Southern Methodist University, Dallas	15 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 June 1941
NY Alpha	Hofstra University, Hempstead	4 April 1942
MI Beta	Central Michigan University, Mount Pleasant	25 April 1942
NJ Beta	Montclair State College, Upper Montclair	21 April 1944
IL Delta	College of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 March 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 June 1947
CO Alpha	Colorado State University, Fort Collins	16 May 1948
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949

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IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 April 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	Kearney State College, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri - Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
IL Epsilon	North Park College, Chicago	22 May 1963
OK Beta	University of Tulsa, Tulsa	3 May 1964
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood College, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 April 1965
AL Epsilon	Huntingdon College, Montgomery	15 April 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
AR Alpha	Arkansas State University, State University	21 May 1965
TN Gamma	Union University, Jackson	24 May 1965
WI Beta	University of Wisconsin - River Falls, River Falls	25 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	Western Maryland College, Westminster	30 May 1965
IL Zeta	Rosary College, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Northeast Missouri State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel College, Springfield	12 Jan 1971

PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 March 1971
KY Alpha	Eastern Kentucky University, Richmond	27 March 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 April 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 April 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State College, Joplin	8 May 1975
GA Alpha	West Georgia College, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin - Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sept 1978
IL Theta	Illinois Benedictine College, Lisle	18 May 1979
PA Mu	St. Francis College, Loretto	14 Sept 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C. W. Post Center of Long Island University, Brookville	2 May 1983
MO Kappa	Drury College, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 March 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 April 1986
TX Iota	McMurry College, Abilene	25 April 1987
PA Nu	Ursinus College, Collegeville	28 April 1987
VA Gamma	Liberty University, Lynchburg	30 April 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 April 1990
CO Delta	Mesa State College, Grand Junction	27 April 1990
NC Gamma	Elon College, Elon College	3 May 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 April 1991
SD Alpha	Northern State University, Aberdeen	3 May 1992
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993