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Kappa Mu Epsilon, mathematics honor society, was founded in 1931. The object of the fraternity is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; to provide a society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; to disseminate the knowledge of mathematics and to familiarize the members with the advances being made in mathematics. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Trisection of An Angle*

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For two thousand years mathematicians and others have been searching for a method of trisecting an angle. However, every construction that was proposed was either inexact or it used something more than a compass and straightedge. Finally the question changed from "How does one trisect an angle?" to "Does a trisection of an angle exist?" A rigorous proof of the nonexistence of an exact trisection using only compass and straightedge was finally found after the development of algebra.

Several attempted trisections are available today and are very interesting even though they do not fit the requirements for construction. It is the intention in this paper to introduce some of the methods the author has encountered in her research. Since there would not be time to give a complete explanation of each, only one is selected to explain in full detail.

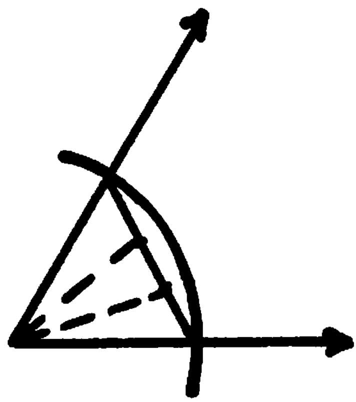
Of those methods of inexact trisections that use only a compass and straightedge, the first one introduced is that which uses the infinitely decreasing geometric progression:

$$1/3 = 1/4 + 1/16 + 1/64 + 1/256 \cdots$$

By using the formula for the sum of the infinite series $1/2^n$ and then subtracting the first two terms from it, we get $1/3$. This result suggests a way of adding on small bits of an angle one by one to approximate $1/3$ of the angle.

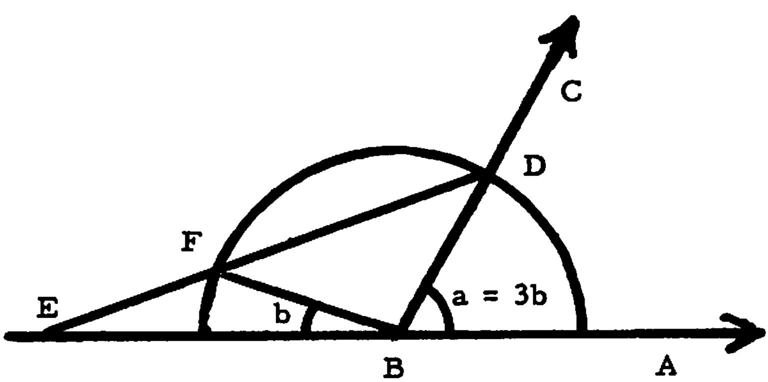
Another inexact method utilizes the trisection of a segment. The angle to be trisected is intersected by an arc with the center at the vertex of the angle. By drawing the chord of that angle and trisecting it, the angle can be divided into three angles by drawing lines from the vertex through the points of division of the segment. (Fig. 1) However, these three angles can be proved to be not equal.

*A paper presented at the regional convention at Warrensburg, Mo., April 25, 1970.



(FIGURE 1)

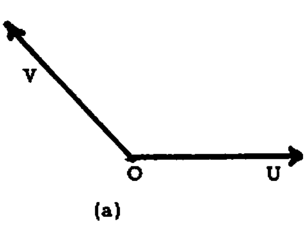
One method of trisecting an angle exactly is done by means of an insert. Actually this method is not exact, but it probably comes as close by trial and error as a real construction would if the human error were taken into account. It goes as such: (Fig. 2) Extend

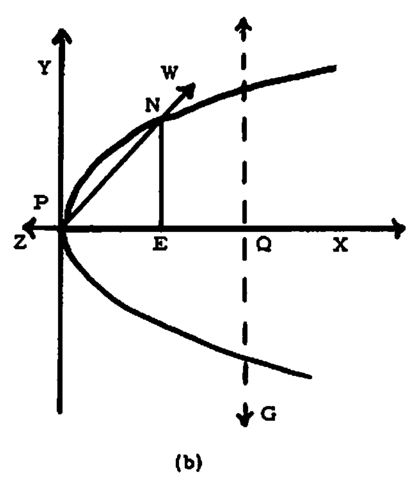


(FIGURE 2)

one of the sides of the given arbitrary angle, $\angle ABC = a$, beyond the vertex B and draw a semicircle with arbitrary radius r and center at B. Let that semicircle intersect the second side of the angle at the point D. Then take a straightedge and make on it two marks E and F at a distance r apart. Place the straightedge in such a way that its arm passes through the point D and that the point E is on the extension of $\overrightarrow{BA}$ and that the point F is on the semicircle. The angle b is $1/3$ of angle a. The proof is left for the reader.

Another exact method of trisecting an angle uses a fixed parabola. This method is the one which has been chosen to construct and prove. First take a parabola with vertex at P and axis $\overleftrightarrow{PZ}$ or $\overleftrightarrow{PX}$. (Fig. 3b) Constructing a perpendicular to $\overleftrightarrow{PZ}$ at P, we get

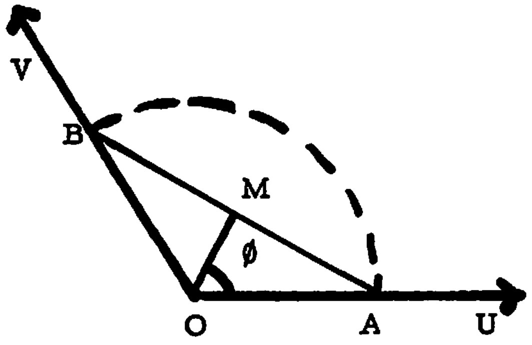




(FIGURE 3)

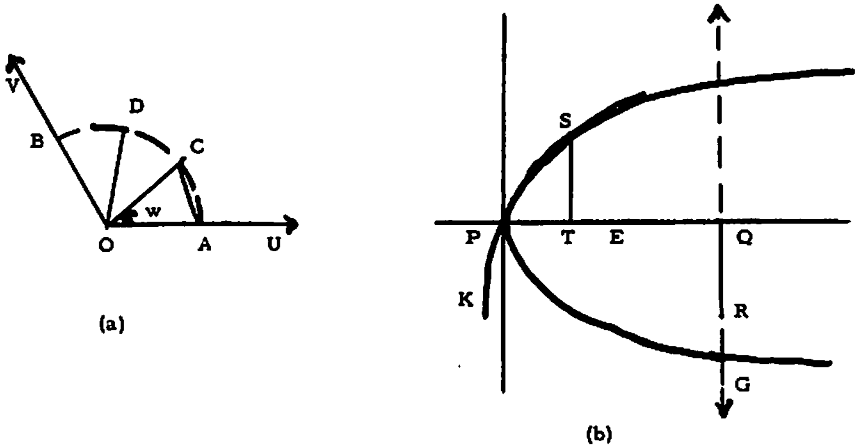
angle YPX. Bisecting YPX, we get a 45 degree angle WPX with the ray $\overrightarrow{PW}$ intersecting the parabola at N. Drop a perpendicular from N to $\overleftrightarrow{PX}$, intersecting $\overleftrightarrow{PX}$ at E. Mark off a segment $\overline{EQ}$ on $\overleftrightarrow{PX}$ equal in length to $\overline{PE}$. At Q construct a line perpendicular to $\overleftrightarrow{PX}$ and call it $\overleftrightarrow{QG}$.

Angle UOV (Fig. 3a) is the angle to be trisected. Using O as center and PE as radius, draw a circle. (Fig. 4) $\overrightarrow{OV}$ intersects the circle at B and $\overrightarrow{OU}$ intersects the circle at A. Draw chord $\overline{AB}$ and bisect it, with M the midpoint. Mark off $\overline{MA}$ on $\overrightarrow{QG}$ of the parabola construction. (Fig. 5b) Call this new segment $\overline{RQ}$. With R as center and RP as radius, draw circle K. The circle cuts the



(FIGURE 4)

parabola in three places, but choose the closest to P and call it S. Drop a perpendicular from S which intersects $\overleftrightarrow{PX}$ at T. Then using the length of $\overline{ST}$, mark off a point C on the arc AB so that $AC = ST$. (Fig. 5a) The angle AOC is exactly a third of angle AOB.



(FIGURE 5)

PROOF

Let $OA = PE = 1$

The equation of a parabola with vertex at (0, 0) and symmetric about the x-axis is:

$$y^2 = cx$$

Angle $XPW = 45°$: then $EN = PE = 1$ and the coordinates of $N = (1, 1)$. Substitute into $y^2 = cx$: then $c = 1$ so that the equation is $y^2 = x$. Let angle $AOM = \phi$: $AM = OA(\sin \phi) = \sin \phi$; $AB = 2AM = 2\sin \phi$.

> (*Lemma*: The length of a chord of an angle is twice the sine of ½ the central angle it determines.)

Let $\phi/3 = w$ so that $2w$ is the angle we want. ($2w = 1/3$ of angle AOB). Then ST must be $2\sin \phi/3$, meaning that the ordinate of S is $y = 2\sin \phi/3$. Now let us start with a point on the parabola whose ordinate is $y = 2\sin w$ and prove that this point is S.

Let (a, b) be the center of circle K. $a = PQ = 2$ and $b = AM = -\sin \phi$. Equation of circle K: $(x - a)^2 + (y - b)^2 = r^2$

$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0.$$

Since the circle passes through $(0, 0)$, this point must satisfy the equation of the circle so that:

$$a^2 + b^2 - r^2 = 0.$$

Then: $x^2 + y^2 - 2ax - 2by = 0$. Substitute $a = 2$ and $b = -\sin\phi$

(H) $\quad x^2 + y^2 - 4x + 2y\sin\phi = 0.$

Now let us work with some trigonometry formulas. By using the fact that $\sin(A + B) = \sin A\cos B + \cos A\sin B$ and $\cos(A + B) = \cos A\cos B - \sin A\sin B$, we can get:

$\sin 3A = 3\sin A\cos^2 A - \sin^3 A$ or since $\cos^2 A = 1 - \sin^2 A$,

$\sin 3A = -4\sin^3 A + 3\sin A$ or

$4\sin^3 A - 3\sin A + \sin 3A = 0.$

(L) Multiplying by 2: $(2\sin A)^3 - 3(2\sin A) + 2\sin 3A = 0.$
Let $A = \phi/3 = w$: $\quad 2\sin 3A = 2\sin\phi = g.$

$$2\sin A = 2\sin w = y.$$

Then we have $y^3 - 3y + g = 0$, where g is AB and y is the chord subtending $1/3$ of the angle.

For each value of y, there is only one point on the parabola.

Let F be the point on the parabola whose ordinate is $y = 2\sin w$. We want to show that F is S. Then from (L), $y = 2\sin w$ satisfies the equation $y^3 - 3y + g = 0$.

Multiplying by y, we have:

$$y^4 - 3y^2 + gy = 0: \; y^4 + y^2 - 4y^2 + gy = 0.$$

Since F is on the parabola, we can set $y^2 = x$ or $y^4 = x^2$ and get:

$$x^2 + y^2 - 4x + gy = 0 \quad (g = 2\sin\phi)$$

$x^2 + y^2 - 4x + 2y\sin\phi = 0$ so that since the point F (on the parabola) whose ordinate $y = 2\sin w$ is on the circle K, it must be the point of intersection of K with the parabola so that $F = S$.

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(continued on p. 75)

Trisection Revisited

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One of the interesting and beautiful characteristics of mathematics is the fact that almost all of the really important and fruitful areas of mathematics had their origins in very concrete and easily visualized concepts. Such a foundation for much of algebra is found in attempts to make various geometric constructions using only a straight-edge and compass. In fact, many high school students become quite intrigued with geometric constructions, and many express disbelief or at least doubt when told that it is impossible to trisect a general angle using only straight-edge and compass. Unfortunately, most of the students never see or hear an explanation of why this is so. The few students who continue in mathematics seldom ever receive such an explanation until their first abstract algebra course, and this is not universal.

The purpose of this paper is to provide an explanation, on a level which can be understood by calculus students, of the reason that the angle cannot be trisected by use of straight-edge and compass alone. Examples of the pertinent theorems are given to aid understanding of these theorems which are merely stated. These theorems are adaptations of standard field theory results which are stated here with as few specialized terms as possible. The general theorems can be found in many abstract algebra texts, two of which are cited in the references.

In this paper we will consider geometric constructions which employ only a straight-edge and compass. Consequently, the words "straight-edge and compass" will often be dropped.

First, we must consider the concept of a field. The definition given below is not the standard one; however, it is equivalent to the standard definition of a field and is more useful for the purposes of this paper.

A *field* is an ordered triple, $(F, +, \cdot)$, where $+$ and $\cdot$ are two binary operations on F which satisfy:

(1) $+$ and $\cdot$ are both commutative and associative;

(2) there is a $0 \in F$ such that $a + 0 = a$ for all $a \in F$;

(3) there is a $1 \varepsilon F$ such that $1 \cdot a = a$ for $a \varepsilon F$;
(4) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all a, b, and $c \varepsilon F$;
(5) each equation of the form $a \cdot x + b = c$, $a \neq 0$, has a unique solution in F.

We will usually denote $a \cdot b$ by ab.

Examples of fields are the real numbers, R, with the usual $+$ and $\cdot$, and the rational numbers, Q, with the usual $+$ and $\cdot$. In fact, all of the fields which we will discuss in this paper contain Q and are contained in R.

To understand why certain geometric constructions are impossible, it is necessary to understand the concept of dimension of a field. We will present three examples which illustrate the principles involved. These principles will be enunciated as theorems.

Example 1. Let $S = \{a + b\sqrt{2} \mid a, b \varepsilon Q\}$, and let S be endowed with the usual addition and multiplication of real numbers. $(S, +, \cdot)$ forms a field, and we check some of the postulates.

(a) To see that multiplication is indeed a binary operation on S consider $(a + b\sqrt{2})(c + d\sqrt{2})$.

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} \varepsilon S.$$

(b) Let $a + b\sqrt{2} \neq 0$. The equation

$$(a + b\sqrt{2})x + (c + d\sqrt{2}) = e + f\sqrt{2}$$

can be solved by elementary algebra, and the solution is

$$x = \frac{a(e - c) - 2b(f - d)}{a^2 - 2b^2} + \frac{a(f - d) - b(e - c)}{a^2 - 2b^2}\sqrt{2}.$$

There are three principal things we wish to notice about $(S, +, \cdot)$.

(i) For each $x \varepsilon S$ there are unique a, $b \varepsilon Q$ such that $x = a \cdot 1 + b\sqrt{2}$. Hence, we say that 1 and $\sqrt{2}$ form a *basis* for S over Q. We say the *dimension* of S over Q is 2, and write $[S: Q] = 2$.

(ii) Consider the equation $x^2 - 2 = 0$. It cannot be factored over Q although the coefficients are in Q. The roots of the equation are $\sqrt{2}$ and $-\sqrt{2}$, and both of these roots are in S.

(iii) Let F be a field containing Q. Let $\sqrt{2}$ and $-\sqrt{2}$ be elements of F. Then for all a and b in Q, $a + b\sqrt{2} \in F$. So $S \subset F$. That is, S is the smallest field which contains Q and all roots of $x^2 - 2 = 0$.

Example 2. Let $T = \{a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2 \mid a, b, c \in Q\}$. Let T be endowed with the usual real number addition and multiplication. $(T, +, \cdot)$ forms a field, and we give an example of a multiplicative inverse.

$$(1 + \sqrt[3]{2})(1/3 - 1/3\sqrt[3]{2} + 1/3(\sqrt[3]{2})^2) = 1.$$

As with Example 1, we notice three principal things.

(i) For each $x \in T$ there are unique $a, b, c, \in Q$ such that $x = a \cdot 1 + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$. Thus a basis for T is formed by 1, $\sqrt[3]{2}$, and $(\sqrt[3]{2})^2$, and $[T: Q] = 3$.

(ii) All real roots of the equation $x^3 - 2 = 0$ are contained in T. This equation cannot be factored over Q, but all of the coefficients are elements of Q.

(iii) If F is a field which contains Q and $\sqrt[3]{2}$, then it contains each number $a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$ where $a, b, c, \in Q$. Thus $T \subset F$. So T is the smallest field which contains Q and all roots of the equation $x^3 - 2 = 0$.

Example 3. Let $C = \{a + b\sqrt[6]{2} + c(\sqrt[6]{2})^2 + d(\sqrt[6]{2})^3 + e(\sqrt[6]{2})^4 + f(\sqrt[6]{2})^5\}$. Let C be endowed with the usual real number addition and multiplication. $(C, +, \cdot)$ forms a field. We observe four principal things about this field.

(i) For $x \in C$ there are unique a, b, c, d, e and $f \in Q$ such that $x = a + b\sqrt[6]{2} + c(\sqrt[6]{2})^2 + d(\sqrt[6]{2})^3 + e(\sqrt[6]{2})^4 + f(\sqrt[6]{2})^5$. Hence, 1, $\sqrt[6]{2}$, $(\sqrt[6]{2})^2$, $(\sqrt[6]{2})^3$, $(\sqrt[6]{2})^4$, and $(\sqrt[6]{2})^5$ forms a basis for C. Also, $[C: Q] = 6$.

(ii) C contains all real roots of the equation $x^6 - 2 = 0$, which cannot be factored over Q, yet has all of its coefficients in Q.

(iii) As in the previous two examples, C is the smallest field which contains Q and all real roots of $x^6 - 2 = 0$.

(iv) We see that $[C: S] = 3$, by the following:

$$a + b\sqrt[6]{2} + c(\sqrt[6]{2})^2 + d(\sqrt[6]{2})^3 + e(\sqrt[6]{2})^4 + f(\sqrt[6]{2})^5 =$$
$$(a + d\sqrt{2}) \cdot 1 + (b + e\sqrt{2})\sqrt[6]{2} + (c + f\sqrt{2})(\sqrt[6]{2})^2,$$

and all of the numbers $a + d\sqrt{2}$, $b + e\sqrt{2}$, $c + f\sqrt{2}$ are elements of S. It can be shown that these coefficients are uniquely determined by the element of C.

Also, we see that $[C: T] = 2$, as follows:

$$a + b\sqrt[6]{2} + c(\sqrt[6]{2})^2 + d(\sqrt[6]{2})^3 + e(\sqrt[6]{2})^4 + f(\sqrt[6]{2})^5 =$$
$$(a + c\sqrt[3]{2} + e(\sqrt[3]{2})^2)\cdot 1 + (b + d\sqrt[3]{2} + f(\sqrt[3]{2})^2)\sqrt[6]{2},$$

and the coefficients of 1 and $\sqrt[6]{2}$ elements of T. It can be shown that these coefficients are uniquely determined by the element of C.

Collecting these facts, we have:

(a) $Q \subset S \subset C$ and $[C: Q] = [C: S][S: Q]$;

(b) $Q \subset T \subset C$ and $[C: Q] = [C: T][T: Q]$.

These three examples illustrate the following theorems:

Theorem 1. Let F be a field, $F \subset R$, and $p(x)$ a polynomial with coefficients in F which cannot be factored over F. Suppose there is a real number, r, such that $p(r) = 0$. Let the degree of $p(x)$ be n. Then there is a field E which contains F and all real roots of $p(x) = 0$. The elements of E have the form

$$a_0 + a_1r + a_2r^2 + \cdots + a_{n-1}r^{n-1}$$

and $[E: F] = n$.

Theorem 2. Let F, E, B be three fields with $F \subset E \subset B$ for which $[E: F]$ and $[B: E]$ exist. Then $[B: F] = [B: E][E: F]$.

Now we consider the manner in which points may be constructed by use of a straight-edge and compass. First, notice that a point with coordinates (x, y) can be constructed if and only if the points $(x, 0)$ and $(0, y)$ can be constructed. So we will call a real number *constructible* if either of the points $(a, 0)$ or $(0, a)$ are constructible. In our discussion we will often use the terms "constructible point" and "constructible number" interchangeably.

It should be recalled that if a and b are constructible numbers, the procedures for constructing $a + b$, $a - b$, ab, and $a \div b$ for $b \neq 0$, are well known. Thus, if $a \neq 0$, b, and c are constructbile numbers, the solution to the equation $ax + b = c$ is constructible.

Points may be constructed in a sequence of stages.

Stage 1. The points (x, y) with rational coordinates can be constructed. Thus, the rational number field, Q, consists of constructible numbers.

Stage. 2. These points are constructed using Stage 1 points as a beginning. They arise in three ways.

(i) Intersections of lines joining Stage 1 points.

Since equations of straight lines are linear equations, the coordinates of these points are simultaneous solutions to systems of the form

$$ax + by + c = 0$$
$$dx + ey + f = 0$$

where a, b, c, d, e, and f are rational numbers. Thus, by field property 5, x and y are rational. So no new points are obtained in this manner.

(ii) Intersections of lines joining Stage 1 points and circles with centers from Stage 1 and radii which are Stage 1 numbers.

These points have coordinates which are simultaneous solutions to systems of the form

$$x^2 + y^2 + ax + by + c = 0$$
$$dx + ey + f = 0$$

where a, b, c, d, e, and f are rational numbers. This procedure may yield new points. In solving this system, (provided $d \neq 0$), we obtain

$$((-e/d)y - f/d)^2 + y^2 + a((-e/d)y - f/d) + by + c = 0$$

which is a quadratic equation in y or a linear equation in y. Also, the coefficients are rational numbers. Hence, the new constructible numbers are obtained from quadratic equations with rational coefficients which do not factor over the rational numbers.

(iii) Intersections of circles with Stage 1 centers and Stage 1 radii.

These points have coordinates which are simultaneous solutions to systems of the form

$$x^2 + y^2 + ax + by + c = 0$$

$$x^2 + y^2 + dx + ey + f = 0$$

where a, b, c, d, e, and f are rational numbers.

These solutions are the same as solutions to the system

$$x^2 + y^2 + ax + by + c = 0$$
$$(a - d)x + (b - e)y + (c - f) = 0.$$

Solutions to these systems are discussed in (ii) above.

Hence, we have the following conclusion: If x is a number which is constructible in Stage 2 but not in Stage 1, then it is a solution to a quadratic equation coefficients in Stage 1 which does not factor over Stage 1. Let F be the smallest field containing Q and x, then $[F: Q] = 2$. Again, this is illustrated by Example 1.

Stage 3. Points may be constructed in three ways, exactly as in Stage 2 with the following substitutions:

Stage 3 for Stage 2

Stage 2 points or numbers for

Stage 1 points or numbers, or rational numbers.

Continuing in this way a sequence of stages can be built up, such that if x is constructible, but $x \notin Q$, and F is the smallest field containing Q and x, then $[F: Q] = 2^n$ for some n. This follows from the above discussion and Theorem 2.

Finally, we turn to a consideration of the impossibility of trisecting the general angle by using only a straight-edge and compass.

Consider the angle $\Theta = 60°$. Then $\Theta/3 = 20°$. From trigonometry, we have

$$\cos \Theta = 4 \cos^3(\Theta/3) - 3 \cos (\Theta/3).$$

If we let $\alpha = \cos 20°$, then we have

$$1/2 = 4\alpha^3 - 3\alpha$$

since $\cos 60° = 1/2$. Thus, α is a solution of the equation

$$8x^3 - 6x - 1 = 0.$$

This equation has rational coefficients and cannot be factored over Q. To see this, apply the rational roots theorem. Consequently, by Theorem 1, if F is the smallest field containing Q and α, then $[F: Q] = 3$. But if α were constructible, $[F: Q] = 2^n$. This is a contradiction. So α is not constructible.

Let β be any angle, then β is constructible if $\cos \beta$ is constructible. Lay off on a line, l, through a point, P, a segment, PQ of length $\cos \beta$. Construct a circle of radius one with center P. Erect a line m, perpendicular to l and passing through Q. Denoting an intersection point of l and the circle of B, the angle BPQ equals β.

Hence, a 60° angle cannot be trisected by use of straight-edge and compass, because cos 20° cannot be constructed, and hence a 20° angle cannot be constructed.

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Linear Elements of A Vector Space

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This note intends to explain some geometric concepts which have motivated some of the ideas of a vector space over a field. First we discuss vector equations of straight lines and planes in a Euclidean three-dimensional space, then we give some generalizations.

1. *A geometric model of a space:* Let us consider a three-dimensional Euclidean space. We choose a fixed point 0 and call it the origin. Every vector has the same beginning 0. Thus a vector will be denoted by its end point such as vector A. The zero vector O ends at 0 and has any desired direction. We accept the algebra of vectors, i.e., addition and scalar multiples of vectors [1]. Scalars will be real numbers and denoted by small letters.

2. *Straight lines:* Let A and B be two distinct vectors ending on a line which does not contain O (Fig. 1). Then $B - A$ is a

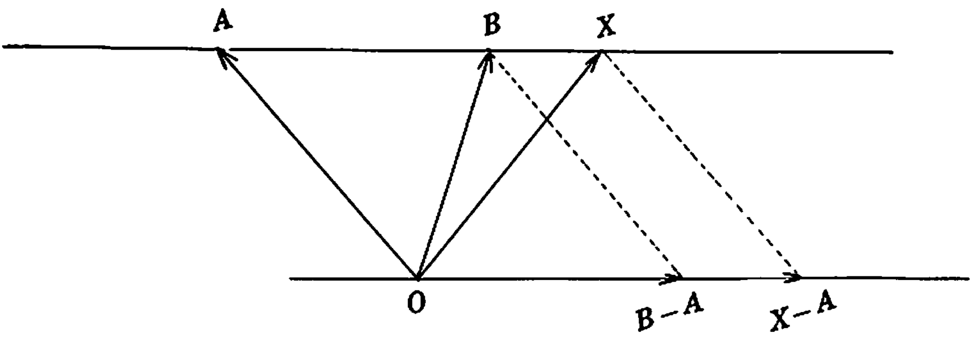

FIGURE 1

vector parallel to the line AB. For any vector X ending on this line we also observe that $X - A$ is a vector parallel to this line. Thus $X - A$ and $B - A$ are linearly independent; therefore, there is a real number t such that $X - A = t(B - A)$. If we solve this equation for X we get $X = (1 - t)A + tB$. Sometimes this equation is written as

$$X = aA + bB\,, \qquad a + b = 1\,.$$

We observe that what is said so far is independent of the dimension of the space. That is, a line in a plane or in the space has the same vector equation.

3. *Planes:* Let $\{A, B, C\}$ be a set of linearly independent vectors ending on a plane $\mathcal{P}$ (Fig. 2). We observe that $B - A$

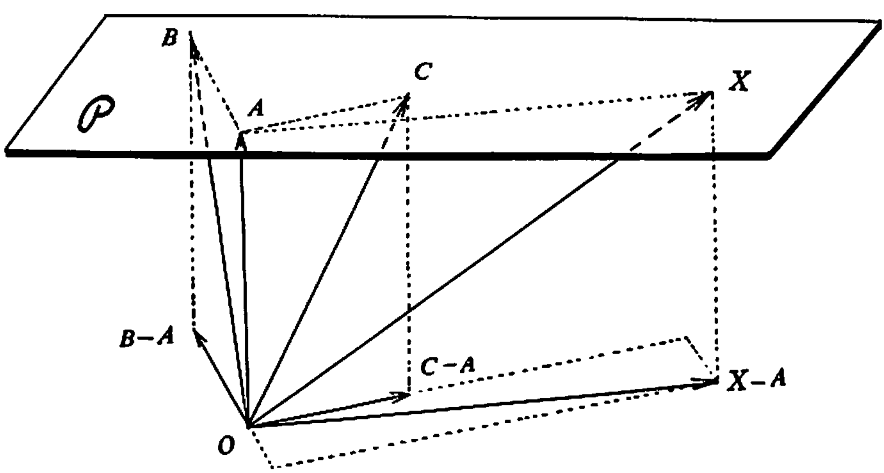

FIGURE 2

and $C - A$ are two vectors which are in a plane parallel to $\mathcal{P}$. We easily see that $B - A$ and $C - A$ are linearly independent. For if $B - A = t(C - A)$, $t \neq 0$, then $B = tC + (1 - t)A$ which contradicts the fact that $\{A, B, C\}$ is linearly independent. So all linear combinations of $B - A$ and $C - A$ will give a plane through the origin. Now for any vector X ending on the plane $\mathcal{P}$ we have

$X - A$ parallel to $\mathcal{P}$ and thus in the subspace generated by $B - A$ and $C - A$. Therefore, there are real numbers t and s such that

$$X - A = t(B - A) + s(C - A) \quad \text{or} \quad X = (1 - t - s)A + tB + sC.$$

This equation is often written as

$$X = aA + bB + cC, \qquad a + b + c = 1.$$

4. *Hyperplanes:* Vector equations of straight lines and planes in 2 and 3, described sets of vectors. For example, for a line we should have written

$$\{X : X = aA + bB, \quad a + b = 1\}.$$

There is a one-to-one correspondence between this set and the set of points on the line. The usual inaccuracy of logic comes in—a set is identified by another set which is isomorphic to it—and we call the above set a straight line.

To generalize the ideas of sections 2 and 3, we choose a vector space V over a field $\mathcal{F}$ [3]. Here scalars, i.e., elements of $\mathcal{F}$ will be denoted by small letters $a, b, \cdots$, and vectors will be denoted by Greek letters. A subspace of V may be called a hyperplane through the origin. As we saw in section 3, a plane could be described as a set of vectors P such that for $X, Y \in P$, we get $X - Y$ in a plane through the origin. So we shall define a hyperplane in V to be a set of vectors H such that for $\beta, \gamma \in H$ the vector $\beta - \gamma$ is an element of a subspace S of V. The dimension of H is defined to be the one of S. We may say that H is parallel to S.

5. *Equation of a hyperplane:* Let $\{\alpha_1, \cdots, \alpha_k\}$ be linearly independent. Then

$$H = \left\{ \xi : \xi = \sum_{i=1}^{k} a_i\alpha_i, \ \sum_{i=1}^{k} a_i = 1 \right\}$$

is a hyperplane of dimension $k - 1$.

Proof: We observe that $\{\alpha_i - \alpha_1, i = 2, \cdots, k\}$ is linearly independent. For, if for example,

$$\alpha_2 - \alpha_1 = \sum_{i=3}^{k} t_i(\alpha_i - \alpha_1), \text{ some } t_i \neq 0,$$

then

$$\alpha_2 = (1 - t_3 - \cdots - t_k)\alpha_1 + t_3\alpha_3 + \cdots + t_k\alpha_k$$

which contradicts the fact that $\{\alpha_1, \cdots, \alpha_k\}$ is linearly independent. Thus $\{\alpha_i - \alpha_1, i = 2, \cdots, k\}$ generates a $(k - 1)$-dimensional subspace S of V. Let $\xi, \zeta \varepsilon H$. Then

$$\xi = \sum_{i=1}^{k} a_i\alpha_i, \quad \sum_{i=1}^{k} a_i = 1, \zeta = \sum_{i=1}^{k} b_i\alpha_i, \quad \sum_{i=1}^{k} b_i = 1.$$

We observe that

$$\begin{aligned} \xi - \alpha_1 = (a_1 - 1)\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k = \\ (-a_2 - \cdots - a_k)\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k = \\ a_2(\alpha_2 - \alpha_1) + \cdots + a_k(\alpha_k - \alpha_1). \end{aligned}$$

Similarly $\zeta - \alpha_1 = b_2(\alpha_2 - \alpha_1) + \cdots + b_k(\alpha_k - \alpha_1)$. Thus

$$\begin{aligned} \xi - \zeta = (\xi - \alpha_1) - (\zeta - \alpha_1) = (a_2 - b_2)(\alpha_2 - \alpha_1) \\ + \cdots + (a_k - b_k)(\alpha_k - \alpha_1) \end{aligned}$$

which implies that $\xi - \zeta \varepsilon S$ and the dimension of H is $k - 1$.

Here H was given and S was constructed. Now let us give a subspace S and construct H. First we study a geometrical model. Consider a line $\mathcal{L}$ through the origin in a plane (Fig. 3). Let A be a vector not on the line. Then

$$\{X: X = A + B, B \text{ is on } \mathcal{L}\}$$

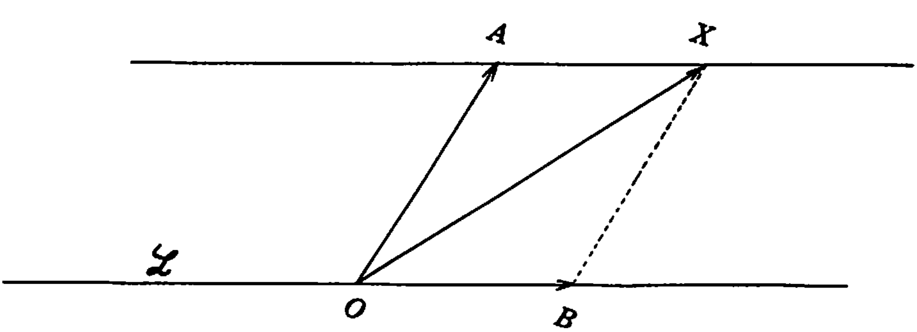

FIGURE 3

describes a line parallel to $\mathcal{L}$. That is, as the point B moves on $\mathcal{L}$ the end point of X moves on the line through O parallel to $\mathcal{L}$.

Now we shall generalize this idea for a vector space V over a field $\mathcal{F}$. Let S be a $(k - 1)$-dimensional subspace of V. Let α be a fixed vector, $\alpha \notin S$. Then

$$H = \{\xi : \xi = \alpha + \beta \, , \beta \varepsilon S\}$$

is a hyperplane of dimension $k - 1$ parallel to S. To show this, let $\{\beta_2, \cdots, \beta_k\}$ be a basis in S. Then the set

$$\{\alpha_1 = \alpha, \alpha_i = \alpha + \beta_i, i = 2, \cdots, k\}$$

is linearly independent and H can be written as

$$H = \left\{ \xi : \xi = \sum_{i=1}^{k} a_i\alpha_i \, , \sum_{i=1}^{k} a_i = 1 \right\} .$$

We leave the proof to the reader.

Indeed one should look into more general cases [4]. For example, when for the subspace S of V dimension cannot be defined, still

$$H = \{\xi : \xi = \alpha + \beta \, , \alpha \notin S, \text{fixed}, \beta \varepsilon S\}$$

is a hyperplane. The reader may examine the fact that ξ, $\zeta \varepsilon H$ implies $\xi - \zeta \varepsilon S$.

6. *Unitary spaces:* Let V be a vector space over the field of complex numbers. Let an inner product of ξ, $\zeta \varepsilon V$ be defined and be denoted by (ξ, ζ) [2]. Then V is called a unitary space. Thus one can combine the inner product with ideas of previous sections for generalizations of many geometric theorems. For example, we generalize the normal equation of a straight line. Let P be a fixed vector perpendicular to a given line. Suppose A is a vector ending on the line (Fig. 4). If W is a vector of norm one on P, then $(A, W) = p$, where $p = \pm| P |$. If a coordinate system is given, then we let $A = (x, y)$ and $W = (\cos \Theta, \sin \Theta)$. Thus we get the well-known equation

$$x\cos \Theta + y\sin \Theta = p.$$

In a unitary space the normal equation of a hyperplane is obtained similarly. Let α be a fixed vector of norm one orthogonal to a subspace S of V. Then $(\xi, \alpha) = c$, where c is a complex num-

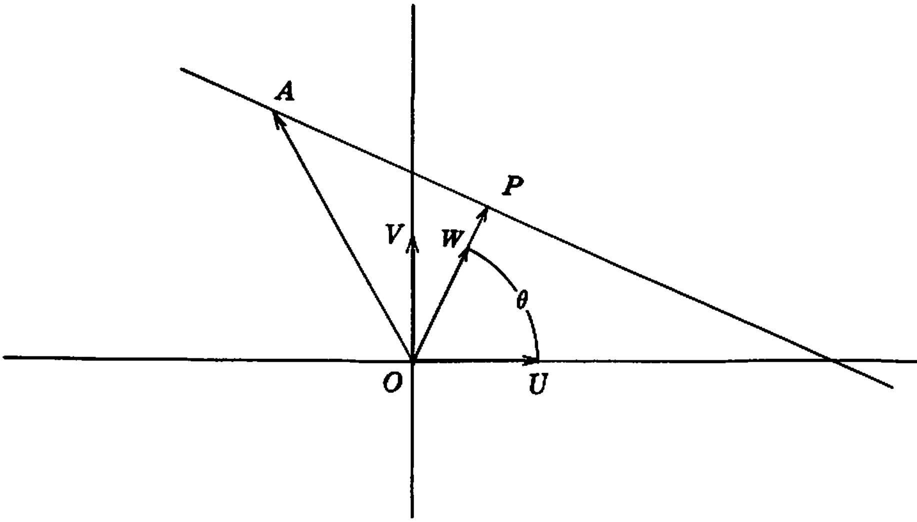

FIGURE 4

ber, is the equation of a hyperplane parallel to S. Here we can say

$$H = \{\xi : (\xi, \alpha) = c, \alpha \perp S, \alpha \text{ fixed}\}$$

is a hyperplane orthogonal to α. The reader may examine the fact that $\xi, \zeta \in H$ implies $(\xi - \zeta, \alpha) = 0$ which means $\xi - \zeta \in S$.

7. *Creating problems:* Here we would like to suggest generalizations of geometrical problems. For example:

In the right triangle ABC, where C is the vertex of the right angle and CH is the altitude corresponding to AB, show that

$$\frac{1}{CH^2} = \frac{1}{CA^2} + \frac{1}{CB^2} .$$

We translate the problem into the language of vectors: let C be at the origin. Thus the hypotheses will be

$$(A, B) = 0, H = aA + bB, a + b = 1, (H, A - B) = 0.$$

The conclusion will be

$$\frac{1}{|H|^2} = \frac{1}{|A|^2} + \frac{1}{|B|^2} .$$

We leave the solution to the reader.

A generalization of the problem will be: let $\{\alpha_1, \cdots, \alpha_k\}$ be a set of non-zero orthogonal vectors in a unitary space. Let ζ end on the hyperplane defined by $\{\alpha_1, \cdots, \alpha_k\}$ and orthogonal to it. This means that

$$\zeta = \sum_{i=1}^{k} a_i \alpha_i \,, \quad \sum_{i=1}^{k} a_i = 1,$$

and $(\zeta, \alpha_i - \alpha_j) = 0$. The last equality indicates that ζ is orthogonal to every vector in the subspace parallel to the hyperplane. Then the conclusion will be

$$\frac{1}{|\eta|^2} = \sum_{i=1}^{k} \frac{1}{|\alpha_i|^2} \,.$$

In this problem a_i has to be real; otherwise, one runs into difficulties of algebra. The reader may translate other geometric problems into the language of vectors and generalize.

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Neglect of mathematics works injury to all knowledge since he who is ignorant of it cannot know the other sciences or the things of this world. —Roger Bacon

Building a Group, a Ring, and a Field

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We have all heard the expression "experience is the best teacher." However trite this expression may be, it is still relatively true. Some times the only way we can become familiar with a concept is through working with it. Through experience in using this concept we gain confidence and anchor our ideas.

In mathematics the same is often true. Concepts such as group, ring, and field may be hazy to us, but by working with them, perhaps only constructing them, we gain understanding of their definitions. The purpose of this paper is simply to gain familiarity with the concepts of group, ring, and field by constructing them. Insight may be gained by designing our own.

Let us first construct a group. The definition of a group is as follows: A group G is a set, together with a rule (called a law of composition) which to each pair of elements x, y in G associates an element denoted by xy in G, having the following properties:

GR 1. for all x, y, z in G we have associativity, namely $(xy)z = x(yz)$;

GR 2. there exists an element i of G such that $ix = xi = x$ for all x in G;

GR 3. if x is an element of G, then there exists an element y of G such that $xy = yx = i$.

To construct a group, according to the definition we must first pick a set. For now, we will use the set of two-by-two matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are all rational numbers. Denote this set by letter G.

Next, we have to have a rule which associates two elements of the set with a single third element of G. Our rule will be denoted by + and defined as follows: Given $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ elements of G,

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$. In other words normal matrix addition is our operation. Here we notice that this operation takes two elements of G and associates them with a third element of G. This we call closure.

Now we must check properties GR 1 through GR 3.

GR 1. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$, and $\begin{bmatrix} k & l \\ m & n \end{bmatrix}$ be elements of G.

Then $\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) + \begin{bmatrix} k & l \\ m & n \end{bmatrix} =$

$\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right) + \begin{bmatrix} k & l \\ m & n \end{bmatrix} =$

$\begin{bmatrix} a+e+k & b+f+l \\ c+g+m & d+h+n \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} e+k & f+l \\ g+m & h+n \end{bmatrix}\right)$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} k & l \\ m & n \end{bmatrix}\right)$. GR 1 holds.

GR 2. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of G. Let $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ be element i of G. What must e, f, g, and h be to satisfy this condition?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

if GR 2 holds.

Therefore, $e = f = g = h = 0$.

Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$+ \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. GR 2 holds.

GR 3. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any element of G. If GR 3 holds,

there must exist some $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$$= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $a + e = 0, b + f = 0, c + g = 0, d + h = 0$. Therefore, $e = -a, f = -b, g = -c, h = -d$. Any element $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse of the form $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. GR 3 holds. By our definition G is a group.

Next we define an abelian group. If x, y are elements of G, and $xy = yx$, then G is abelian. Is our G abelian? Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $y = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then $x + y = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} =$

$$\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} = \begin{bmatrix} e+a & f+b \\ g+c & h+d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$= y + x$, since the rational numbers are commutative under addition. Therefore, our G is abelian. As an abelian group it has the properties of all other abelian groups.

Now some sets are not just groups, but may also be rings. Perhaps our set is one of these rings.

A ring is defined as follows: A ring G is a set, whose objects can be added and multiplied (i.e. we are given associations (x, y) with $x + y$ and (x, y) with xy from pairs of elements of G into G) satisfying the following conditions:

RI 1. under addition, G is an additive abelian group;

RI 2. for all x, y, z elements of G we have $(xy)z = x(yz)$;

RI 3. for all x, y, z elements of G we have $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$.

A ring is called a ring with unity if there exists an element, u, in G, such that $ux = xu = x$ for all x an element of G. A commutative ring with a unit element is a ring with unity with the following condition holding—given x, y elements of G, then $xy = yx$ for all x, y elements of G.

If the above set G is to be one of these rings, we need another operation besides addition. We will call it multiplication and define it in the following way. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ be elements of G. Denote multiplication by $\times$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ $= \begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix}$. Here we again notice that this operation gives us a third element of the set G. Therefore, the set is closed under the operation. Now we must again check the conditions and determine if G is indeed a ring.

RI 1. Under the above addition we have already seen that G is an abelian group. RI 1 holds.

RI 2. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ and $\begin{bmatrix} k & l \\ m & n \end{bmatrix}$ be elements of G. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \times \begin{bmatrix} k & l \\ m & n \end{bmatrix}\right) =$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} ek & fl \\ gm & hn \end{bmatrix} = \begin{bmatrix} aek & bfl \\ cgm & dhn \end{bmatrix} = \begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix} \times$ $\begin{bmatrix} k & l \\ m & n \end{bmatrix} = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) \times \begin{bmatrix} k & l \\ m & n \end{bmatrix}$. RI 2 holds.

RI 3. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ and $\begin{bmatrix} k & l \\ m & n \end{bmatrix}$ be elements of G. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} k & l \\ m & n \end{bmatrix}\right) =$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e+k & f+l \\ g+m & h+n \end{bmatrix} = \begin{bmatrix} ae+ak & bf+bl \\ cg+cm & dh+dn \end{bmatrix} =$$

$$\begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix} + \begin{bmatrix} ak & bl \\ cm & dn \end{bmatrix} = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) +$$

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} k & l \\ m & n \end{bmatrix}\right).$$

Also $\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} + \begin{bmatrix} k & l \\ m & n \end{bmatrix}\right) \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$

$\begin{bmatrix} e+k & f+l \\ g+m & h+n \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea+ka & fb+lb \\ gc+mc & hd+nd \end{bmatrix} =$

$\begin{bmatrix} ea & fb \\ gc & hd \end{bmatrix} + \begin{bmatrix} ka & lb \\ mc & nd \end{bmatrix} = \left(\begin{bmatrix} e & f \\ g & h \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) +$

$\left(\begin{bmatrix} k & l \\ m & n \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$. RI 3 holds.

Therefore, G is a ring.

Now we wish to ask if G is a ring with unity, and if it is a commutative ring with unity. We seek the answers below.

By the above definition of a ring with unity, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is any element of G there must be an element, u, $\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ This implies that $ae = a$, $bf = b$, $cg = c$, and $dh = d$ which in turn implies that $e = 1$, $f = 1$, $g = 1$, and $h = 1$. The u of G then is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and G is a ring with unity.

If G is also commutative, then the following must hold. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ be elements of G. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae & bf \\ cg & dh \end{bmatrix} = \begin{bmatrix} ea & fb \\ gc & hd \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. G is therefore, a commutative ring with unity.

Now we seem to have accomplished one thing at least, we have built a ring. We might be tempted to stop here and be satisfied with the results of our work. But, as mathematicians, we must go forward with our work until we can advance no farther; therefore, let us see if G is a field.

The definition of a field is as follows. A commutative ring G with a unit element is called a field (where the unit element is the multiplicative identity) if for every a, an element of G (where a is not the additive identity) there exists an element a^{-1}, an element of G such that $aa^{-1} = a^{-1}a = 1$ (where 1 is the multiplicative identity). The element, a^{-1}, is called the multiplicative inverse.

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an element of G. Then its inverse in general must be such that the following holds for all a, b, c, and d.

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This implies that $ae = 1$, $bf = 1$, $cg = 1$, and $dh = 1$ which implies $e = 1/a$, $f = 1/b$, $g = 1/c$, and $h = 1/d$. At first glance then, G may appear to be a field. Upon reflection though, we realize that should any element of the two-by-two matrices be 0 there would be no inverse since $1/0$ is undefined. Therefore, G is not a field.

Again as mathematicians, we cannot be satisfied. What can be done to make G a field? We might try a different operation on G and thus hope to build a field, but in this case let's weaken the hypothesis. Let us use the subset of G called G' which consists of all the two-by-two matrices of the form $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$. Then the only element of G' which will have a 0 in it will be of the form $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which does not have to have an inverse. Since this is a subset of G then all other properties of the field have been shown except for closure. If the set be closed, then it is a field.

Let $\begin{bmatrix} a & a \\ a & a \end{bmatrix}$ and $\begin{bmatrix} e & e \\ e & e \end{bmatrix}$ be elements of G'. then $\begin{bmatrix} a & a \\ a & a \end{bmatrix} + \begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} a+e & a+e \\ a+e & a+e \end{bmatrix}$ and $\begin{bmatrix} a & a \\ a & a \end{bmatrix} \times \begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} ae & ae \\ ae & ae \end{bmatrix}$, all elements of G'. G' is closed and is a field, since G' has all the properties of a field.

Still the mathematician is not satisfied. He might go ahead

(continued on p. 124)

A New Look At the Pythagorean Theorem

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The Pythagoreans began the serious study of the right triangle in the sixth century B.C. with the discovery that the sum of the squares of the legs equals the square of the hypotenuse, or

$$c^2 = a^2 + b^2 . \qquad (1)$$

Diophantus of Alexandria advanced the study in the third century A.D. with a general solution of the right triangle, as represented by equation (1). The Pythagoreans had had no general solution although they and their successors had known the solutions of certain special cases. Diophantus, in his book *Arithmetica,* set forth the classical solution: Let the hypotenuse equal $m^2 + n^2$; base equal $m^2 - n^2$; altitude equal $2mn$. The parameters m and n are positive integers chosen such that m is greater than n. To exclude multiples, i.e., similar triangles, let m and n be relatively prime and of different parity. (Throughout this paper, "triangle" will mean "right triangle" unless otherwise specified.)

Each triad of positive integers which satisfies equation (1) is called a pythagorean triplet. If multiples are excluded, they are called "primitive" triangles. (Any such solution set is called a triangle, even though it strictly represents only the dimensions of a triangle.) The largest list ever published, at least until 1941, was 864 triads arranged according to m and n, with $n < m \leqq 65$. It included the area of every triangle. Another list gave the acute angles to the nearest tenth of a second, but the list was much shorter than 864. Still another publication gave all 477 primitive triangles in which the hypotenuse does not exceed 3000. Yet another list gave the triangles according to area, up to 934,000 square units. Various arrangements have been made, but the most common one is according to increasing values of the hypotenuse. The author of one list included auxiliary tables of all his triangles in which $c = b + 1$ and $c = b + 2$ [1]. His paper was not available to this student, but the supposition must be that he pursued this clue no further, contenting himself with the observation that they could

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be so arranged. C. A. W. Berkhan gave nineteen methods for solving the right triangle in a publication "Die Merkwundigen Eigenschaften der Pythagorean Zahlen," Eisleben, 1853. In 1911 a French mathematician, Fitting, published tables in which $c = b + n$ and $a^2 = 2bn + n$, where n is an odd number squared, in a French mathematics journal, "L'Intermediaire des Math.," 28, 1911, 87-90. Another German, E. Meyer, compared many known ways of solving $c^2 = a^2 + b^2$ in a mathematics journal, "Zeitschrift Math. Natur. Unterricht," 43, 1912, 281-287 [2]. It is unlikely that scholars such as these would have overlooked any solution to this intriguing theorem. Yet because of the relative inaccessibility of the papers referred to and the dearth of such literature in recent times, it seems worthwhile to continue this study.

The purpose of this investigation is to discover the formula or formulas by which one can quickly and easily compute the values of the other two sides of a right triangle when the first side is given; multiples of these "pythagorean triplets" being excluded. A single formula will be sought to logically relate all formulas discovered or developed.

The simplest group of triangles is that in which $c = b + 1$, such as the familiar 3, 4, 5 or 5, 12, 13. By substituting $c = b + 1$ in equation (1), it follows that

$$(b + 1)^2 = a^2 + b^2, \text{ whence}$$

$$b = (a^2 - 1)/2. \tag{2}$$

Clearly, if b is to be an integer, a must be an odd number; but there is no other limitation on a.

A second group is comprised of those triangles in which $c = b + 2$, such as 8, 15, 17 or 12, 35, 37. Substituting this value in (1) above,

$$b = (a^2 - 4)/4 \tag{3}$$

or

$$b = (a/2)^2 - 1. \tag{4}$$

An inspection of (4) shows that a must be an even number if b is to be an integer. However, if $a/2$ is an odd number, then b is an even number (along with a), and c is also an even number (since an even number squared is even, and the sum of even numbers is also even). Thus all three variables are even if $a/2$ is an odd number: They are multiples of some other set of numbers;

namely, those in the first group, when $c = b + 1$. It is, therefore, required that $a/2$ be an even number in this group; hence a is four times any number.

If $c = b + 3$,

$$b = (a^2 - 9)/6. \tag{5}$$

It can be seen by careful inspection that b is an integer only when a is three times some odd number, say x. Thus $a = 3x$, and

$$b = ((3x)^2 - 9)/6 \tag{6}$$

$$b = (9x^2 - 9)/6$$

$$b = 3(x^2 - 1)/2 = 3((x^2 - 1)/2). \tag{7}$$

But the coefficient $(x^2 - 1)/2$ is equivalent to $(a^2 - 1)/2$ in (2) above, since both x and a must be odd numbers. Therefore, when $c = b + 3$, values for the three variables are exactly three times the corresponding values when $c = b + 1$.

Suppose $c = b + 5$. Substituting in (1) and simplifying:

$$b = (a^2 - 25)/10. \tag{8}$$

If b is an integer, $a = 5x$, where x is an odd number. Therefore,

$$b = (25x^2 - 25)/10 \tag{9}$$

or

$$b = 5((x^2 - 1)/2). \tag{10}$$

Here x may be given the same values as a in (2) above, odd numbers. Values are exactly five times the corresponding sets when $c = b + 1$.

A similar argument would certainly hold true for $c = b + k$, where k is an odd prime number:

$$c = b + k \tag{11}$$

$$b^2 + 2bk + k^2 = a^2 + b^2 \tag{12}$$

$$b = (a^2 - k^2)/2k. \tag{13}$$

If b is to be an integer in (13), then of necessity $a = kx$ where x is an odd prime number; and

$$b = (k^2x^2 - k^2)/2k \tag{14}$$

and

$$b = k((x^2 - 1)/2). \tag{15}$$

The conclusion is that when the increment n, in the expression $c = b + n$, is an odd prime number, only multiples of the triads from (2) will be found.

Let c take on other increments, which are even numbers. If $c = b + 4$,

$$b = (a^2 - 16)/8. \qquad (16)$$

By inspection, it is clear that b will be an integer only when a is a multiple of four (four times one being excluded because it would result in a triangle having one side zero, a trivial case).

If $c = b + 6$,

$$b = (a^2 - 36)/12. \qquad (17)$$

The pattern here is similar to that obtained from (16) in which the sets are multiples, alternating between the results of (2) and (4). Thus there are no primitive triangles here.

If $c = b + 10$,

$$b = (a^2 - 100)/20. \qquad (18)$$

The resulting pythagorean triplets are alternately three times those obtained from (4) or ten times those of (2).

If $c = b + 12$,

$$b = (a^2 - 144)/24. \qquad (19)$$

The results of (19) are alternately multiples of (2) and (4). No primitive triangles are indicated.

If $c = b + 8$, b is an integer only when a is a multiple of 4. Moreover, when $a/4$ is an even number, as when $a = 20, 28, 36, \cdots$, primitive triangles are indicated.

It is now necessary to examine a few other composite increments. If $c = b + 9$, b is an integer when a is three times any odd number, but odd numbers which are themselves divisible by three produce multiples.

If $c = b + 18$, b is an integer when a is any multiple of six. When $a/18$ is odd ($a = 54, 90, \cdots$), triads are multiples of (2), and when $a/18$ is even ($a = 36, 72, \cdots$), triads are multiples of (4). When a is not divisible by eighteen (but is still divisible by six), triads are either primitive triangles or multiples of $b = (a^2 - 81)/18$.

A summary might be in order at this point. Primitive triangles have been discovered when $c = b + n$, where $n = 1, 2, 8, 9$, or 18. It is obvious that the odd numbers here belong to the sequence of the odd squares, and the even numbers belong to the sequence which is twice the square of any number. This, then, is the complete solution of the Pythagorean Theorem, as represented by

$$c^2 = a^2 + b^2 \tag{1}$$

$$\begin{aligned} c^2 &> b^2 \\ c &> b \\ c &= b + n \\ (b + n)^2 &= a^2 + b^2 \\ b^2 + 2bn + n^2 &= a^2 + b^2 \\ 2bn + n^2 &= a^2 \\ 2bn &= a^2 - n^2 \\ b &= (a^2 - n^2)/2n. \end{aligned} \tag{20}$$

$n =$ the square of any odd number,

$n =$ twice the square of any odd number, or

$n =$ twice the square of any even number. To insure integral solutions and to avoid multiples, it is necessary to place restrictions on a according to the following charts:

CHART I, $c = b + x^2$, where $x =$ an odd number

$c = b + 1$	$a =$ any odd number.
$c = b + 9$	$a = 3$ times any odd number, except those divisible by 3
$c = b + 25$	$a = 5$ times any odd number, except those divisible by 5
$c = b + 49$	$a = 7$ times any odd number, except those divisible by 7
$c = b + 81$	$a = 9$ times any odd number, except those divisible by 3
$c = b + 121$	$a = 11$ times any odd number, except those divisible by 11
$c = b + 169$	$a = 13$ times any odd number, except those divisible by 13

$c = b + 225$	$a = 15$ times any odd number, except those divisible by 5
$c = b + 289$	$a = 17$ times any odd number, except those divisible by 17
$c = b + 361$	$a = 19$ times any odd number, except those divisible by 19
$c = b + 441$	$a = 21$ times any odd number, except those divisible by 3

CHART II, $c = b + 2x^2$, where $x =$ an odd number

$c = b + 2$	$a =$ any even multiple of 2
$c = b + 18$	$a = 6$ times any even number, except those divisible by 6
$c = b + 50$	$a = 10$ times any number, except those divisible by 10
$c = b + 98$	$a = 14$ times any even number, except those divisible by 14
$c = b + 162$	$a = 18$ times any even number, except those divisible by 18
$c = b + 242$	$a = 22$ times any even number, except those divisible by 22
$c = b + 338$	$a = 26$ times any even number, except those divisible by 26
$c = b + 450$	$a = 30$ times any even number, except those divisible by 6 or 5
$c = b + 578$	$a = 34$ times any even number, except those divisible by 34
$c = b + 722$	$a = 38$ times any even number, except those divisible by 38
$c = b + 882$	$a = 42$ times any even number, except those divisible by 7

CHART III, $c = b + 2x^2$, where $x =$ an even number

$c = b + 8$	$a =$ any odd multiple of 4
$c = b + 32$	$a =$ any odd multiple of 8
$c = b + 72$	$a =$ any odd multiple of 12, except numbers divisible by 3

$c = b + 128$	a = any odd multiple of 16
$c = b + 200$	a = any odd multiple of 20, except numbers divisible by 5
$c = b + 288$	a = any odd multiple of 24, except numbers divisible by 3
$c = b + 392$	a = any odd multiple of 28, except numbers divisible by 7
$c = b + 512$	a = any odd multiple of 32
$c = b + 648$	a = any odd multiple of 36, except numbers divisible by 9
$c = b + 800$	a = any odd multiple of 40, except numbers divisible by 5
$c = b + 968$	a = any odd multiple of 44, except numbers divisible by 11

Space forbids a more detailed analysis of the charts, but the developing patterns can be clearly seen although they must be carefully explored. For example, in Chart I, the restriction expected in the case of $c = b + 81$ was ". . . except those divisible by 9" rather than by 3, as was found experimentally.

To relate this approach to that of Diophantus, it is necessary to solve for hypotenuse less altitude $(c - b)$ and hypotenuse less base $(c - a)$. Since m and n are of different parity, it follows that the quantity $(m - n)$ is an odd number. Let $N = m - n$, and compute $(c - a)$:

$$c - a = (m^2 + n^2) - 2mn$$

$$c - a = m^2 - 2mn + n^2$$

$$c - a = (m - n)^2$$

or $\quad c = a + N^2$, N being an odd number, $(m - n)$.

If $N^2 = k$, then $c = a + k$. Eliminating c from (1) above by substituting $(a + k)$ for c, $c^2 = a^2 + b^2$ becomes

$$(a + k)^2 = a^2 + b^2$$

$$a^2 + 2ak + k^2 = a^2 + b^2.$$

Hence

$$a = (b^2 - k^2)/2k, \tag{21}$$

which should be compared to (13) and (20) above.

Now compute $(c - b)$:

$$c - b = (m^2 + n^2) - (m^2 - n^2).$$

Simplifying, $c - b = 2n^2$

and $\qquad c = b + 2n^2$, n being any positive integer.

Again, eliminating c from (1):

$$(b^2 + 2n^2)^2 = a^2 + b^2$$

$$b^2 + 4bn^2 + (2n^2)^2 = a^2 + b^2$$

$$4bn^2 = a^2 - (2n^2)^2$$

$$b = (a^2 - (2n^2)^2)/2(2n^2). \tag{22}$$

Or, if $2n^2 = k$, then

$$b = (a^2 - k^2)/2k^2, \tag{23}$$

which should be compared to (13) and (20) above.

Referring again to the historical notes above, it is seen that several mathematicians have been close to this solution, but none enunciated it clearly. One author, after tabulating his primitive triangles, separated and tabulated those in which $c = b + 1$ and $c = b + 2$, but he did not generalize. Another, Fitting, generalized to $c = b + n$, where n is an odd number squared, but he did not define n to include twice the square of any number, although such is the case.

REFERENCES

1. Henry Lehmer Derrick. *Guide to Tables in the Theory of Numbers.* Washington, D.C.: National Research Council, National Academy of Sciences, 1941.

2. Leonard E. Dickson. *History of the Theory of Numbers,* Vol. II. Washington, D.C.: Carnegie Institution of Washington, 1941.

There was more imagination in the head of Archimedes than in that of Homer. —Voltaire

Installation of New Chapters

EDITED BY LORETTA K. SMITH

MARYLAND GAMMA CHAPTER

Saint Joseph College, Emmitsburg, Maryland

Maryland Gamma Chapter was installed on December 6, 1970, in the DuBois Lounge following a business meeting. Professor William R. Smith, Kappa Mu Epsilon's Vice-President, was to be the installing officer but a severe snowstorm in northwestern Pennsylvania prevented his attending the installation. Dr. James E. Lightner, Corresponding Secretary of the Maryland Beta Chapter, was kind enough to take Dr. Smith's place as the installing officer. Dr. Lightner was assisted by Sr. Marie Augustine Dowling, Corresponding Secretary of the Maryland Alpha Chapter. Following the installation of the chapter and initiation of charter members and officers, Dr. Lightner spoke on "The History of Honor Societies." A tea and reception were held after his lecture. Three faculty members and twelve students are charter members:

Faculty Members:

Sister John Frances Gilman
Donald F. Shriner
Frank Wu

Students:

Frances Boscia	Susan Jonas
Sister Ann Mary Dougherty	Catherine Lisson
Lynn Gloeckler	Kathleen McNaney
Karen Haggerty	Linda Raudenbush
Patricia Hemler	Sister Joan Rowe
Sister Mary Frances Hildenberger	Jane Sweeney

Maryland Gamma Chapter's officers are:

President: Lynn Gloeckler
Vice-President: Sister Mary Frances Hildenberger
Recording Secretary: Karen Haggerty
Treasurer: Linda Raudenbush
Corresponding Secretary: Sister John Frances Gilman
Faculty Sponsor: Donald F. Shriner

MISSISSIPPI DELTA CHAPTER

William Carey College, Hattiesburg, Mississippi

The Mississippi Delta Chapter of Kappa Mu Epsilon was installed at William Carey College on December 17, 1970. Professor Jack D. Munn, Corresponding Secretary of the Mississippi Gamma Chapter of KME at the University of Southern Mississippi, was the installing officer. A reception in honor of the initiates was given immediately following the ceremony.

The following people comprise the charter membership of Mississippi Delta Chapter:

Phil Barnette	Dennis Knight
Bill Breland	Tim Rayborn
Craig Christopher	Mike Richards
Betty Crocker	Dr. Gaston Smith
Charles Ernest	Mary Lynn Stampley
Warner Fellabaum	Nancy Wilson
Charles Gambrell	Nancy Wise

The chapter officers for Mississippi Delta are:

President: Phil Barnette
Vice-President: Charles Gambrell
Secretary: Nancy Wise
Treasurer: Bill Breland
Corresponding Secretary: Professor Warner Fellabaum
Faculty Sponsor: Dr. Gaston Smith.

At the installation Dr. Smith welcomed back two recent graduates in mathematics—Miss Nancy Wilson and Miss Mary Lynn Stampley. Miss Wilson is currently enrolled in graduate studies in mathematics at the University of Southern Mississippi and Miss Stampley is currently enrolled in graduate studies in mathematics at Virginia Polytechnic Institute. The two graduates returned to become charter members of Mississippi Delta.

MISSOURI THETA CHAPTER

Evangel College, Springfield, Missouri

The Mathematics Club at Evangel College was installed as the Missouri Theta Chapter of Kappa Mu Epsilon on January 12,

1971. Professor Eddie W. Robinson, the National Historian, was the installing officer. Former National President, Carl V. Fronabarger, assisted in the ceremony. The following are the charter members of Missouri Theta:

Roger B. Baker
Theron J. Blount
Melinda K. Boyles
Evelyn C. Bryant
Linda L. Cilke
Donald E. Draper
Richard H. Gloff
Faythe J. Herman
Barbara J. Lawrence
Donald P. Matthews
David W. Mayfield
Victor Ng
Anthony W. Siders
Kenneth A. Smith (faculty)
Albert Wong

The Chapter officers of Missouri Theta are:

President: David Mayfield
Vice-President: Donald Draper
Recording Secretary and Treasurer: Faythe Herman
Corresponding Secretary and
Faculty Sponsor: Glenn Bernet

The ceremony was followed by a brief history of Kappa Mu Epsilon given by Professor Robinson. After refreshments, Professor Robinson presented the first paper, "A Curious Field Extension," to be addressed to Missouri Theta.

In most sciences one generation tears down what another has built, and what one has established another undoes. In mathematics alone each generation builds a new story to the old structure.

—Hermann Hankel

The Problem Corner

EDITED BY ROBERT L. POE

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before September 1, 1971. The best solutions submitted by students will be published in the Fall 1971 issue of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor Robert L. Poe, Department of Mathematics, Berry College, Mount Berry, Georgia 30149.

241. *Proposed by the Editor.*

Prove or disprove that $f(x) = \sin x^2$, for x any nonnegative real number, is uniformly continuous.

242. *Proposed by the Editor.*

1,804,229,351 is the fifth power of a positive integer. Find the integer without extracting roots or using logarithms.

243. *Proposed by the Editor.*

Find all three-digit numbers each of which is the sum of all possible permutations of its three digits taken two at a time.

244. *Proposed by the Editor.*

A man's advice concerning women's fashions had better add up. Check the advice below by addition by replacing each letter with a digit. (The same letter for the same digit throughout.)

$$\begin{array}{r} W\ E\ A\ R \\ A \\ M\ I\ N\ I \\ \hline H\ O\ N\ E\ Y \end{array}$$

Further, if there is not a unique solution avoid the maxi completely and the midi if possible; that is, look for the mini.

245. *Proposed by the Editor.*

Find the smallest positive integer which ends with the digit 9 such that if this 9 is moved from the last place to the first place the number formed is three times as large as the original.

236. *Proposed by John Caffrey, American Council on Education, Washington, D.C.*

Beginning in the upper left corner of the table below consider the inverse of any square matrix whose elements are listed. Prove that the inverse matrix has elements all of which are integers and define a function which generates the elements of the inverse.

	0	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	10	15	21	28	36	45
4	1	4	10	20	35	56	84	120	165
5	1	5	15	35	70	126	210	330	495
6	1	6	21	56	126	252	462	792	1287
7	1	7	28	84	210	462	924	1716	3003

If the array were tilted 45° clockwise, it would appear as Pascal's triangle.

Solution. (No solution was received. The hints listed below are provided by the Editor.)

1. Use the "sweep-out" process as outlined in *Elementary Matrix Algebra,* second edition, by Franz E. Hohn. It may be shown by mathematical induction that for each n the $n \times n$ upper triangular matrix obtained by the "sweep-out" process achieves a principle diagonal of which each element is one in $n - 1$ steps. This gives the value of the determinant to be $n - 1$ times one, or one.

2. Since the inverse of a nonsingular square matrix A may be written as $A^{-1} = (\text{Co}A)^T/\det A$ we have $A^{-1} = (\text{Co}A)^T$. Hence, A^{-1} will always have integral elements.

3. But A (as a square of the table under discussion) is symmetric. This implies that for any cofactor A_{ij} of det A we have $A_{ij} = A_{ji} = A_{ij}{}^T$ and therefore $\text{Co}A = (\text{Co}A)^T$.

4. Hence, the a_{ij}th element of A^{-1} is just its cofactor A_{ij} which defines a real-valued function for determining the elements of A^{-1}.

Problems 237, 238, 239, and 240 are considered to be problems of antiquity whose proposers are unknown to the Editors.

237. Consider three noncollinear points taken at random on an infinite plane. Determine the probability of these points being the vertices of an obtuse-angled triangle.

238. Consider the angle determined by two rays with a common initial point as the vertex and a given interior point of the angle. Construct the line through the given point which with the two rays forms a triangle with the least area.

239. Solve the system $x/y = x - z$; $x/z = x - y$; and determine the limiting values of all real solutions.

240. A bag contains two marbles of which nothing is known except that each is either black or white. Determine their colors without taking them out of the bag or looking into the bag.

Solutions to 237, 238, 239, and 240.

No solutions were received. With a considerable amount of investigation the Editor finally located problems very similar to 237, 238, 239, and 240 in a book written by Lewis Carroll (Charles L. Dodgson), the 19th century English mathematician who wrote "Alice in Wonderland." The book, PILLOW PROBLEMS AND A TANGLED TALE, is in its fourth edition and may be obtained in paperback from Dover Publications, Inc., New York. Problems 237, 238, 239, and 240 are solved in this book. Therefore, the solutions will not be reprinted here. Instead the reader is urged to purchase this book for his own mathematics library ($1.50). See what "Pillow Problems" you can create and solve.

There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.

—Lobachevsky

The Book Shelf

EDITED BY JAMES BIDWELL

This department of THE PENTAGON brings to the attention of its readers recently published books (textbooks and tradebooks) which are of interest to students and teachers of mathematics. Books to be reviewed should be sent to Dr. Elizabeth T. Wooldridge, Department of Mathematics, Florence State University, Florence, Alabama 35630.

Algebra, Jacob K. Goldhaber and Gertrude Ehrlich, The Macmillan Company, Collier-Macmillan Limited, London, 1970, 432 pp., $11.95.

Professors Goldhaber and Ehrlich have succeeded in producing an algebra textbook that is carefully organized and masterfully written. Although the book is intended primarily as a text for a year course at the first-year graduate level, it could be read profitably by an upper division honor student. Because the pace of the book is lively, and elementary examples are scarce, a course based upon this book should, in general, be preceded by an introductory undergraduate course in algebra.

The material is self-contained, and the reader is led skillfully from the classical to the modern, and from simple definitions to sophisticated theorems. The authors expressed their guiding philosophy when they said, "A problem which appears as a recurrent theme in algebra is the following: Given an algebraic structure, to what extent does knowledge about the homomorphisms of this structure yield information about the structure itself?" To help the student perceive more clearly these homomorphisms, they have included many commutative diagrams in the text.

The book consists of nine chapters, each chapter being divided into anywhere from three to fourteen sections. There are exercises to accompany each section, except those of Chapter 0; the exercises are listed, by sections, at the end of the chapter to which they pertain. The book contains a total of 410 well chosen exercises, most of which would serve to intrigue and challenge the student, but would not discourage him. A four-page Glossary of Symbols is a helpful feature of the textbook.

Chapter 0 takes care of the background preliminaries, such as, for instance, the uniqueness of a two-sided identity in a monoid, the concept of embedding one structure in another (by means of an injective mapping), principle of induction, fundamental theorem of arithmetic, the algebraic structure of residue classes modulo m, Zorn's lemma, cardinal and ordinal numbers, etc. Chapter 1, "Groups," moves rapidly to the concept of groups with operators, and then takes up the homomorphism and isomorphism theorems, solvable groups, Sylow theorems, direct sums, indecomposable groups, free groups, and finitely generated abelian groups. The chapter concludes with an informal treatment of categories. The first semester of a year course would probably end with Chapter 2, "Rings and Integral Domains," which contains the usual homomorphism and isomorphism theorems, and work on ideals, embeddings, principal ideal domains, and quotient fields.

The second semester of work should include all or most of Chapter 3, "Modules," and of Chapter 4, "Finite-dimensional Vector Spaces." At the heart of the course is Chapter 5, "Field Theory," which develops elegantly the topics of field extensions, algebraically closed fields, splitting fields, Galois theory of both finite extensions and infinite extensions, roots of unity, cyclic and radical extensions of a field, and solvability by radicals. The preference of the professor would dictate which sections of the last three chapters he might wish to include. Chapter 6, "Fields with Real Valuations," requires the use of some topological concepts, and treats the subject of valuations in greater detail than do most algebra courses. This chapter, as well as Chapter 7, "Noetherian and Dedekind Domains," were included mainly because of their importance in algebraic number theory and algebraic geometry. The book concludes with Chapter 8, "The Structure of Rings," in which are introduced the basic building blocks of ring theory; included are the main structure theorems for simple and semisimple rings.

In the Preface, the authors state: "There will be those who feel that our book does not provide precisely the right number of arrows; may they refrain from using their bows." This reviewer, for one, will use her bow only for the purpose of sending a swift message of congratulation to Goldhaber and Ehrlich for a job well done!

Violet Hachmeister Larney
State University of New York
at Albany

Calculus with Analytic Geometry, Burton Rodin, Prentice Hall, Inc., Engelwood Cliffs, N.J., 1970, 751 pp. $13.75.

Most of us will remember that our attention was called to this text before publication by a clever advertising brochure which illustrated the teaching of calculus from Newton to the present day and proclaimed that this text would be "an entirely different book that will make everybody happy, especially students." In most respects, the author has succeeded. In each section of the first eight chapters care has been taken to give the student many examples. In fact, about fifty percent of these examples employ a Socratic dialogue so that, as he reads through a section, the student can monitor his understanding (answers are given in the back). Also, at the end of each section, many routine and challenging exercises are provided, with answers to odd-numbered problems.

Also, the text is designed to interest the student in calculus the very first day of class: Chapter One treats integration. Thus the student with average preparation in high school will not be bored with inequalities, analytic geometry, etc., yet the student is led carefully through this chapter. Limit of a sequence is the only limit concept used in Chapter One and limits are introduced gradually throughout the early chapters. In spite of this quick plunge into calculus, there is much precalculus review throughout the book; it is done when it is needed. Topics from analytic geometry and trigonometry are introduced when appropriate. Chapter Two treats the derivative. Thus the student will find that he has acquired the tools necessary for his science course more quickly than when using most calculus texts.

The text is carefully written; early concepts are well motivated. For example, a nice job is done with the geometric motivation for the definition of first derivative. There is a careful, rigorous, yet clear treatment of the theory; however, the author has written with the realization that the instructor has the option to do certain theorems heuristically.

One criticism might be the early treatment of the calculus of several variables. After a chapter on vectors, partial differentiation is done. Definitions and the treatment of limits and continuity are done immediately in n-dimensions, instead of gradually through two and three dimensions. Geometric interpretations are postponed until later. It is questionable that the average student has gained the maturity to handle this treatment.

The applications in the text are mostly physical. The log function is defined as an integral and exp function as its inverse. The chapter on infinite series can be done before or after the calculus of several variables. Vector treatment is used extensively.

There is a chapter of differential equations: first order and linear with constant coefficients. The last chapter treats line integrals, Green's theorem, surface integrals with the wedge product notation, Stokes' theorem and differential forms.

This text should be considered seriously for a single variable calculus course and for a multivariable course made up of good students.

Milton D. Cox
Miami University

Arithmetic, A Semi-Programmed Text, Keith W. Wilkins, Prentice Hall, Inc., Englewood Cliffs, N.J., 269 pp., $4.95 (paper).

It is necessary to understand the individual for whom the book is designed to evaluate the book. The author indicates the book was written for the individuals "who have been exposed to arithmetic and perhaps general mathematics but for some reason still has not attained a proficiency in computational mathematics."

The form of the book is somewhat unique. Each unit consists of three parts: (1) a discussion or explanation section, (2) a programmed section with multiple choice answers, and (3) an exercise section. Answers to the odd-numbered exercises are provided at the end of the book. The programmed section is different from the usual programmed text in that each possible answer to a question has a reference number to a portion of the explanatory material or to another question in the programmed section. The reader is instructed to turn to the reference number to determine if the correct answer has been selected. This reviewer noticed that if a student were at all astute he would soon be able to pick the correct answer by selecting the reference number which refers the student to another question in the programmed section. This is invariably the correct answer.

The book covers the usual mathematical skills for which proficiency might be expected of an eighth grade student. The units covered are whole numbers, integers, rational numbers, irrational numbers, and real numbers. Although the author usually attempts to rationalize the rules of arithmetic, the book is somewhat rule oriented. For example: A rule can be stated for finding the G.C.D.

(1) Factor each number into prime factors.
(2) Find the factors that are common.
(3) Then take the lowest exponent of the common factor or factors.
(4) Multiply these factors to the least powers.

Another example: When multiplying whole numbers "the units place in the second partial product is placed under the tens place of the partial product above."

An unusual feature is the presentation of a procedure to find the cube root by algorithm. A rather cursory discussion of e is included.

The book would be of value for the student who has not obtained a proficiency in computational mathematics. It is perhaps of more value than a strictly programmed text since the discussion is not broken down into such small segments. On the other hand, the book is more cumbersome to use than a programmed text since much referring to previous pages of reading material is required. If one were teaching a general mathematics course in which computational skill is required, the book could be used in an accompanying "skills" laboratory or as necessary drill work. If a general mathematics class were being presented in which the student was his own tutor, this book could be used.

Wilbur Waggoner
Central Michigan University

Elementary Algebra, Lee A. Stevens, Wadsworth Publishing Company, Inc., Belmont, California, 1970, 319 pp., $7.95.

This book is designed for college students who either have had no previous work in algebra or who need a review of the material. The material covered is essentially a major portion of ninth grade algebra.

Recently there have been several such texts placed on the market all very alike. We have the following contents: 1. Logic and Sets 2. Real Numbers 3. First-Degree Equations and Inequalities 4. Polynomials 5. Rational Expressions 6. Radicals 7. Second-Degree Equations and Inequalities 8. Relations, Functions and Graphs 9. Systems of Linear Equations and Inequalities.

The author claims the admirable goal of motivating the course "on the finding of solutions to various equations, inequalities, and word problems." He has made a successful attempt to accomplish this objective. The spirit of the text is that the students should be able to apply the algebraic ideas to problem-solving. It appears that the problems the author has in mind are mostly computational algebra problems and there are over 2000 of them in the book.

There are very few formal proofs in the main part of the book, and most of the usual theorems are stated without proof. Even though the author briefly and clearly explains the idea of a deductive system in Chapter 1, the text may be criticized for its lack of proofs. Some of the stated theorems are proved in an appendix and each instructor may choose to prove more theorems, if the class is receptive.

Overall, the book should be considered one of the better ones at this level. The writing is well done and the explanations are in general very good. Most of the theorems and definitions are followed by numerical examples which illustrate the ideas involved. The notation is not excessive and the vocabulary is minimal.

The publishers have done an excellent job in packaging the text. Each page is headed by the chapter or section. Red ink is used so that definitions and theorems stand out as well as crucial steps in some solutions. Inside the front and back covers there appear the Key Theorems for easy reference. The text should be well received by the students.

It is not appropriate to end this review without pointing out some faults. Symbols like "{the days of the week}" (page 7) are ambiguous. The symbol "=" is defined on page 13 as the identity relation and used differently in Chapter 3 when dealing with equations. There is the (often quoted) vague and imprecise formulation of "The axiom of substitution for equality." The "Real-Number Axioms" (Section 2.2) are simply field axioms—algebraic ordering and completeness are omitted. Factoring occurs in Chapter 4 without explicit reference to integral coefficients. There are no "prime factors" (page 109) over the real number field.

Alan R. Hoffer
University of Montana

MINIREVIEWS

Modern Mathematics for Business Students, Ruric E. Wheeler and W. D. Peeples, Jr., Brooks/Cole Publishing Co., Belmont, California, 1969, 601 pp.

This text is adaptable to various course requirements and student backgrounds. It prepares the student for advanced study of statistics, decision theory, and operations research. It can be adapted for emphasis on accounting, insurance, and marketing. After introductory chapters, the book contains chapters on systems of equations, vectors, mathematics of finance, probability, linear programming, game theory, calculus, and statistics. It contains tables and the answers to problems.

Introduction to Numerical Methods, Peter A. Stark, The Macmillan Co., New York, 1970, 348 pp., $9.95.

This text is designed for undergraduates and follows the recommendation of the ACM for course B4, Numerical Calculus. Knowledge of calculus and FORTRAN is required. It contains chapters on power series calculation, roots operations, simultaneous equations (Newton-Raphson and matrix methods), numerical integration, ordinary differential equations, and curve fitting. No solutions to problems are included.

Analytic Geometry, Third Edition, Paul K. Rees, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1970, 297 pp., $8.95.

First published in 1956, this text contains standard material on analytics. The approach is traditional. Included are chapters on algebraic and transcendental curves, parametric equations. A chapter on vectors (20 pages) and two on solid geometry complete the book. Answers to most problems are included. Text is designed for a three-semester hour course.

College Algebra, Steven Bryant, Jack Karush, Leon Nower, Daniel Saltz, Goodyear Publishing Co., Pacific Palisades, California, 1970, 390 pp., $9.95.

This book is a modern one with a traditional title. It contains the usual precalculus material, emphasizing functions and their graphs. A chapter on sequences precedes topics using continuity and irrational exponents. In addition to the chapter on functions, work is included on vectors, analytic geometry, linear systems, matrices, and probability. Odd answers are included (with many graphs shown).

Intermediate Algebra for College Students, H.S. Bear, Cummings Publishing Company, Menlo Park, California, 1970, 399 pp.

This book contains no surprises. It covers the normal topics in the normal ways. No work with matrices is included, although systems of equations are solved by Cramer's Rule. There is a chapter on complex numbers. Selected answers are included. Answer booklet is available for the other problems.

The Mathematical Scrapbook

EDITED BY RICHARD LEE BARLOW

Readers are encouraged to submit Scrapbook material to the Scrapbook editor. Material will be used where possible and acknowledgment will be made in THE PENTAGON. If your chapter of Kappa Mu Epsilon would like to contribute the entire Scrapbook section as a chapter project, please contact the Scrapbook editor.

In set theory, one many times wishes to prove the equivalence of various sets. The diagram of these sets often becomes tedious when several sets are involved. A useful diagram is the Veitch diagram.

First we will consider a statement involving only two sets A and B, both subsets of a universe U. A square representing the universe U is divided into four equal parts (see figure 1). The left half will represent set A and the upper half is set B.

	A	A'
B	1	2
B'	3	4

Figure 1

One will note that the subsquare numbered 1 represents those elements which are in both A and B and hence in $A \cap B$, which shall also be denoted as the product AB. The subsquare numbered 2 represents those elements in B but not in A, and hence is $B \cap A'$ or using the product notation BA'. Similarly subsquare 3 represents $A \cap B'$ or AB' and subsquare 4 is $A' \cap B'$ or $A'B'$. One will note that every element in universe U is in one of these four subsquares. For notational purposes, by placing an x in the proper subsquare we will denote a particular case. The empty set ϕ will not have an x anywhere and universe U will have x's in all four subsquares. Also, the sum $A + B$ will be used to represent $A \cup B$. Hence, one obtains

$$\underset{A}{\begin{array}{|c|c|}\hline x & \\ \hline x & \\ \hline\end{array}}\,,\quad \underset{B}{\begin{array}{|c|c|}\hline x & x \\ \hline & \\ \hline\end{array}}\,,\quad \underset{A'}{\begin{array}{|c|c|}\hline & x \\ \hline & x \\ \hline\end{array}}\,,\quad \underset{B'}{\begin{array}{|c|c|}\hline & \\ \hline x & x \\ \hline\end{array}}\,,$$

x	

$A \cap B$ or AB ,

x	x
x	

$A \cup B$ or $A + B$,

	x
x	x

$A' \cup B'$ or $A' + B'$, etc.

Consider the De Morgan Theorems:

(1) The complement of a sum of two elements is the product of their complements. That is, $(A + B)' = A'B'$

(2) The complement of a product of two elements is the sum of their complements. That is, $(AB)' = A' + B'$.

To prove (1), one will note

x	x
x	

$A + B$

so

	x

$(A + B)'$

which is identical to

	x

$A'B'$.

For (2),

x	

AB

so

	x
x	x

$(AB)'$

which is identical to

	x
x	x

$A' + B'$.

Similarly, other statements involving two sets can be proved using the Veitch diagrams.

For three sets, say A, B, and C, we shall use the following Veitch diagram (see figure 2). Let A be the left half, B the upper half, and C the middle half.

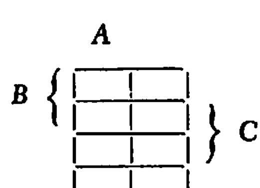

FIGURE 2

Hence, for example

x	x
x	x

C

x	
x	

AC

x	x
x	x
x	x

$B + C$

x	x

BC', etc.

To verity $A(B + C) = AB + AC$, we have the following:

x	
x	
x	
x	

A

and

x	x
x	x
x	x

$B + C$

so

x	
x	
x	

$A(B + C)$

Also,

x	
x	

AB

and

x	
x	

AC

so

x	
x	
x	

$AB + AC$.

One will note that the above end results are identical and hence the statement is verified.

For four sets, say A, B, C, and D, the following represents the Veitch diagram (See figure 3).

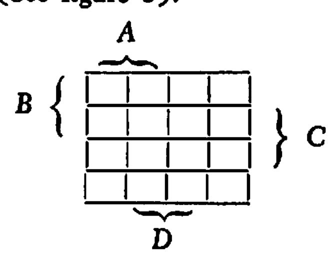

FIGURE 3

Can you verify that $AD' + BC' + A'D' + DC' = (DC)'$?

— Δ —

Another variation of the usual logic and truth set concepts is the following procedure. Let 1 represent "true" and 0 represent "false." Define the two operations which we shall call "addition" and "multiplication" by two tables below:

+	0	1
0	0	1
1	1	1

x	0	1
0	0	0
1	0	1

One will note that these two operations are closed and that they satisfy the commutative, associative, and distributive properties and have identities.

They satisfy the following additional property:

Idemopotent Law: For every element of the set the operation of element a by a yields a.

Also, one notes $0' = 1$ and $1' = 0$. In this system, we have only two elements to consider and so almost all theorems can be proved by enumeration.

Let us prove the first De Morgan Theorem for this system, that is, prove $(a + b)' = a'b'$. By enumeration, we get the following four possibilities:

a	b	$(a + b)$	$(a + b)'$	a'	b'	$a'b'$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

The theorem is therefore proved since the columns for $(a + b)'$ and $a'b'$ are identical.

Using this method, can you prove the second De Morgan Theorem?

— Δ —

One often learns all the truth tables for the basic connectives but fails to apply them in many situations where they are important to the outcome of a problem. For example, consider the following problem from probability: A fair coin is tossed twice and the results recorded. Find the probability that if the first toss resulted in heads then the second was a tail.

Examining the problem and its conditions, one might say that the required probability is ½ since the probability of a tail on the second toss is ½. But by considering the truth table for the situation above, one finds there is much more to this problem. Consider,

Case No.	Statement p(1st toss head)	Statement q(2nd toss tail)	$p \rightarrow q$
1	T	T	T
2	T	F	F
3	F	T	T
4	F	F	T

Upon examination, one sees these are really three cases to consider (namely 1, 3 and 4) where $p \rightarrow q$ is true. Since the probability for each case is ¼ (equally likely), the probability required for the problem is ¾.

Can you find the probability for the first toss is a head if and only if the second toss is a tail?

Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half.

—Leibniz

Kappa Mu Epsilon News

EDITED BY EDDIE W. ROBINSON, *Historian*

CHAPTER NEWS

Alabama Gamma, University of Montevallo

Chapter President—James M. Tuck, Jr.

20 members—10 pledges

Initiation of new members accrued in December and chapter activities included a party for freshmen mathematics students.

Colorado Alpha, Colorado State University

Chapter President—Nila Hobbs

21 members—16 pledges

Meetings were held in the homes of mathematics professors and included a demonstration of the Hewlett-Packard table top computer, a discussion of student-teaching experiences, a program of placement opportunities in mathematics, and a talk on groups and fields. A display at Activities Night told students about KME. Initiation and a potluck dinner were other activities.

Florida Alpha, Stetson University

Chapter President—Bruce Rose

21 actives—5 pledges

Activities included a Christmas party, a beach party and an initiation banquet.

Illinois Beta, Eastern Illinois University

Chapter President—Roy McKittrick

Seventeen new members were initiated in November, 1970, making a total of 658 for the thirty-seven years of the existence of the chapter. The chapter is composed of sixty undergraduates, twenty-seven faculty members and four graduate students.

The formal initiation ceremony and reception had Dr. Sukrit Dey as speaker. He told of his experiences as a student and teacher in Calcutta.

Illinois Eta, Western Illinois University

Chapter President—Roger Eickman

25 actives—5 pledges

Monthly meetings consist of business meetings and guest speakers from the university and businesses. The chapter plans to take several field trips to area businesses to tour their computer systems and to participate in "Science Day for High School Students."

Indiana Alpha, Manchester College

Chapter President—David Warrick

14 actives—0 pledges

Programs at meetings have been on the following topics: actuarial science, model schools programs in mathematics, different bases for numeration systems and mathematical recreations.

Indiana Gamma, Anderson College

Chapter President—Ronald E. Whittom

18 actives—5 pledges

Among the activities were the installation of new members and the revision of the chapter constitution.

Indiana Delta, University of Evansville

Chapter President—Wayne Ruell

90 actives—20 pledges

Films, panel discussions, election of officers and informative talks have comprised the programs for chapter meetings. One such meeting was a talk on Emmy Noether. Activities have included an informal get-together with mathematics majors and minors, an initiation banquet and a picnic honoring graduating seniors.

Iowa Alpha, University of Northern Iowa

Chapter President—Ken Cox

27 active members—0 pledges

Student papers were presented at the first three meetings of this school year. A Christmas party was held in December and the January initiation was cancelled because of the "Blizzard of '71."

Iowa Gamma, Morningside College

Chapter President—Donald Schouten

34 actives—0 pledges

"Mathematics, Retrospect and Prospect," and "Two Methods of Simplification of Functions in Boolean Algebra," were two papers presented at meetings. Two guest lecturers were Dr. James Cornette of Iowa State and Dr. Grace Wahba of the University of Wisconsin. A Mathematics Colloquium was held in March in conjunction with two neighboring liberal arts colleges.

Kansas Alpha, Kansas State College of Pittsburg

Chapter President—Catherine Peterson

46 active members—0 pledges

Dr. Elwyn Davis of the mathematics staff presented the September program on "Moulton's Non-desarguesian Plane." Pat Kuhel, the chapter treasurer, presented a program about Non-Euclidean Geometry and Mark Davis discussed "The Probability Distribution Associated with Shooting at a Bullseye." Activities included the fall picnic, an event sponsored by KME which traditionally brings together all students and faculty of the mathematics and physics departments.

Recipients of the annual Robert Miller Mendenhall Award for scholastic achievement were Kathy Peterson and James Ciardullo. Each received a KME pin in recognition of this achievement.

Kansas Gamma, Mount St. Scholastica College

Chapter President—Margaret Hoehl

6 actives—13 pledges

Regular meetings were held with student members presenting papers to the group. One guest lecturer was Dr. Heatherington of the Computer Science Department at the University of Kansas, who spoke on "Computer Science: Its Past, Present, and Possible Future."

Six new members were initiated and four new pledges were inducted in January. Four members submitted papers to the selection committee for the national convention in April. Sister Helen Sullivan is on a sabbatical leave this academic year teaching at the University of Galway, Ireland.

Maryland Alpha, College of Notre Dame of Maryland

Chapter President—Julia Haffler

8 actives—5 pledges

The fall program included a tour of the computer facilities of the Social Security Building and attendance at the installation of the Maryland Gamma Chapter at St. Joseph College in Emmitsburg.

Maryland Beta, Western Maryland College

Chapter President—Raymond D. Brown

25 active members—0 pledges

Dr. Benjamin Tepping of the Bureau of the Census met with the chapter and discussed statistics and its applications. The chapter attended the installation of Maryland Gamma Chapter in December.

Michigan Alpha, Albion College

Chapter President—Bob Flaherty

10 actives—6 pledges

Program topics have been "Fun with the Geoboard," "Mathematics in Industry," and "Games on the Cantor Set." Activities included pledge paper presentation, a guest speaker from Oakland University, the annual picnic and election of officers.

Michigan Beta, Central Michigan University

Chapter President—Theodora Perreault

33 active members—0 pledges

Meetings are held on the second Wednesday of each month with programs given by faculty and students. The chapter activities include: tutoring program, senior papers, visitations to local high schools concerning KME and general college life and student representation on faculty and departmental committees. Michigan Beta hosted the 1970 Regional Convention for the North Central area.

Mississippi Alpha, Mississippi State College for Women

Chapter President—Martha C. Pope

14 actives—0 pledges

1971 pledging and initiations were held during the second semester. Programs included a talk by Mr. Gil Harris, Chief Engineer of Mitchell Engineering Company who talked about mathematical and computer applications.

Missouri Alpha, Southwest Missouri State College

Chapter President—Peggy Turnbough

32 actives—0 pledges

Dr. James O'Brien spoke to the chapter on applications of mathematics to chemistry. At the initiation banquet for eight new members, Dr. John Hatcher spoke about the integral as applied to finding maximum values of a continuous function.

Missouri Gamma, William Jewell College

Chapter President—Don Page

12 active members

Programs at meetings were the following: "The Number π," "Color Problems," "Methods of Solving Third and Fourth Degree Equations," and "A Study of Ptolemy."

Missouri Epsilon, Central Methodist College

Chapter President—Chris Binggeli

5 actives—2 pledges

The senior members of the chapter present papers at the meetings which are held on the first Tuesday of each month. New members are welcomed at a picnic in the spring.

Missouri Zeta, University of Missouri at Rolla

Chapter President—Joe McBride

10 actives—11 pledges

Speakers and topics have been: Dr. Charles Hatfield—"Mathematical Puzzles," Mr. John Garrett—"The Four-Color Problem," Dr. Charles Johnson—"The Teaching Equation."

The chapter conducts help sessions for courses through calculus and has distributed 3000 copies of "conversion factor tables."

Missouri Eta, Northeast Missouri State College

Chapter President—Nancy Wood

25 actives—8 pledges

Meetings include student paper presentations and guest speakers. Activities included a tutoring session and the preparation of a chapter flag.

Nebraska Beta, Kearney State College

Chapter President—Robert Rutar

37 actives—8 pledges

Programs included talks by Dean Joe McFadden on academic excellence, Paul Wilmont on the KSC Placement Bureau, student paper presentations and the annual Christmas party arranged by the first semester pledge class.

Activities include the tutoring program—the Mathematics Booster Hour, the sponsorship of a MAA visiting lecturer and the awarding of a $50 scholarship to a member.

Nebraska Gamma, Chadron State College

Chapter President—Ron Green

28 actives—12 pledges

Events of the chapter were a dinner meeting with faculty members, a bowling night with the physics-chemistry honor societies, and a formal initiation at Camp Norwesca with skating, tobaggoning and other winter sports.

New Jersey Beta, Montclair State College

Chapter President—Carol Suscreba

39 actives—0 pledges

Students Janice Garner and Jacques Caillult presented papers; Dr. Al Chai spoke on "What Applied Mathematicians Do;" and Dr. K. G. Janardan spoke on "A New Derivation of the Binomial Distribution." Used mathematics books were sold as a chapter project.

New York Eta, Niagara University

Chapter President—Ken Kerr

20 actives—10 pledges

Two programs were discussions of career opportunities by invited speakers from industrial firms. Activities were student paper presentations, faculty lectures, a joint faculty-student Christmas party and attendance at the national convention.

New York Theta, St. Francis College

Chapter President—Kevin Westley

24 active members—0 pledges

Organizational meetings, participation in freshman orientation, determination of the fall and spring schedule and induction of new members have been chapter programs. Students spoke on "Polyhedra," "A Note on Teaching" and "Konigsberg Bridge." Films shown were "Mathematics of the Honeycomb" and "Elliptical Orbits." The chapter had a field trip to Brookhaven Atomic Energy Laboratories and conducted a Mathematics Bowl between St. Francis College and Malloy College.

Ohio Alpha, Bowling Green State University

Chapter President—Martha Barnes

Chapter business and computers have been programs this year. The chapter has set up a display and acted as guides at the dedication of the new mathematics-science building.

Ohio Gamma, Baldwin-Wallace College

Chapter President—Lawrence Meklemburg

17 active members—0 pledges

The fall program included a guest speaker who lectured on the applications of mathematics in the business and management fields.

Ohio Zeta, Muskingum College

Chapter President—Harold J. Rouster

21 active members—0 pledges

The programs consist of selected outside speakers, research from various faculty members from the college or research conducted by various students.

Oklahoma Alpha, Northeastern State College

Chapter President—Joe Morris

30 active members—0 pledges

Presentations by students have been: vector analysis, computer language, squaring the circle and the Dappler Effect. The fall initiation was combined with a Christmas party and the spring initiation will be combined with the annual Founders Day Banquet.

Pennsylvania Gamma, Waynesburg College

Chapter President—Gail Hindman

9 active members—12 pledges

Meetings are held monthly. One program was a talk on environment problems by a research chemist. The main activity is a tutoring program.

Pennsylvania Delta, Marywood College

Chapter President—Noelle Acculto

Most members of the chapter are student teachers. The fall schedule consisted of three business meetings. Film loops on wave motion with emphasis on "Tacoma Bridge" and a demonstration on the "Illusions of Color" were second semester programs.

Pennsylvania Epsilon, Kutztown College

Chapter President—Charles Gerhart

18 active members—5 pledges

Visiting speakers and student speakers have been programs. The chapter hosted a freshman tea and helped with the mathematics conference.

Pennsylvania Zeta, Indiana University of Pennsylvania

Chapter President—Donald Laughery

50 active members—0 pledges

Twenty-eight new members were initiated in October. "The Integers, Modulo 3," was the topic of a talk by Raymond Gibson. Miss Ida Arms, the corresponding secretary, spoke on "Highlights in the History of Mathematics." Mr. Arlo Davis presented a talk on "Some Unsolved Problems in Mathematics."

Student members conducted HELP sessions for mathematics students who are having difficulties in their courses.

Pennsylvania Theta, Susquehanna University

Chapter President—Elizabeth Varner

18 active members

Professor Fladmark spoke to the chapter on "Inventory Systems" and Mr. James Handlan spoke on ecology from a mathematical standpoint. Doreen Bolton was awarded the Stine Mathematics Award, which is given annually to the mathematics major having the highest mathematics grades for the freshman and sophomore years.

Pennsylvania Iota, Shippensburg State College

Chapter President—Jane Becker

39 actives—11 pledges

Graduate school was the topic of a talk by Dr. James Sieber, Chairman of the Mathematics Department. Another program was presented by Dr. Carl Kerr, entitled "Algebra is Geometry and Geometry is Algebra." The chapter has added a winter term initiation to its previous fall and spring term initiation schedule. New requirements were begun for pledges to be initiated.

Tennessee Beta, East Tennessee State University

Chapter President—John Drake

30 active members—0 pledges

Five new members were initiated at the fall meeting in the faculty lounge of the Student Center. The guest speaker, Dr. G. K. Ginnings, showed several slides on UFO's. Michael Brooks, an active member of the chapter, was invited to attend the launch of Apollo 14.

Texas Beta, Southern Methodist University

Chapter President—Katherine Green

50 active members—21 pledges

Besides presentations by faculty members of the mathematics department, one presentation was given by a professor of organic chemistry, using mathematics in solving chemical equations. The vice president of the chapter, Brint Morris, demonstrated the use of mathematical permutations in parlor card tricks.

Texas Zeta, Tarleton State College

Chapter President—Larry Snider

13 active members—1 pledge

Programs were: Professor Timothy Flinn—"Properties of Infinity," Professor Conley Jenkins—"Distance and Imagination," Professor Tom Bohannon—"Whatzit."

Virginia Alpha, Virginia State College

Chapter President—Sylvia Dixon

27 active members—0 pledges

Students and faculty regularly present papers at meetings.

The chapter hosts visiting scholars who visit the campus under the sponsorship of the Department of Mathematics. The chapter voted to invest funds so that accrued interest can be used as an award to a KME member.

Wisconsin Alpha, Mount Mary College

Chapter President—Sister Catherine Yekenevicz

9 active members—5 pledges

The programs were talks given by the following pledges: "Graphing Line and Cosine Functions Using the Unit Circle"—Patricia Ross, "Polygonal Numbers"—Linda Hilgendorf, "The Nine-Point Circle"—Catherine Starck, "A Glimpse of Topology"—Geri-Lynn O'Boyle, "Permutations and Combinations"—Margaret Wyszynski. The chapter sponsors a mathematics contest for high school students.

(continued from p. 88)

first to extend this to n-by-n matrices, showing that they form a ring, a group, and a field. He might develop theorems and applications. In working through the preceding we should gain more knowledge of the concepts, but perhaps just as important, we should also realize the methods by which the mathematician works. As beginning mathematicians we must realize that we cannot be satisfied with mathematics as it now stands, but we must build and improve upon the work of those whom we have studied and are now studying.

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