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Kappa Mu Epsilon, mathematics honor society, was founded in 1931. The object of the fraternity is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; to provide a society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; to disseminate the knowledge of mathematics and to familiarize the members with the advances being made in mathematics. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Egyptian Fractions*

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For some reason, the early Egyptians thought it best to replace a fraction, such as $\frac{4}{9}$, by a finite sum of distinct fractions of the form $\frac{1}{x}$ where x is a positive integer. The purpose of this paper is to show that this replacement is always possible. For example, $\frac{4}{9}$ might be written as $\frac{1}{3} + \frac{1}{9}$. Since $\frac{4}{9}$ may also be written as $\frac{1}{4} + \frac{1}{9} + \frac{1}{12}$, we readily see that the representation is not necessarily unique. This notation may be easily accounted for by the identity

$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)} . \qquad (1)$$

Hence any fraction of the form $\frac{1}{x}$ may always be expressed as the sum of two fractions of the same form.

This identity suggests a method of replacing any fraction by a sum of distinct fractions $\frac{1}{x}$. Express the fraction as $\frac{a}{b} = \frac{1}{b} + \frac{1}{b} + \cdots + \frac{1}{b}$ and treat the last $a - 1$ terms with the identity (1). Remove any duplications by further applications of the identity. For example, we could write $\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{1}{5} + \frac{1}{6} + \frac{1}{30} + \frac{1}{6} + \frac{1}{30} = \frac{1}{5} + \frac{1}{6} + \frac{1}{30} + \frac{1}{7} + \frac{1}{42} + \frac{1}{31} + \frac{1}{930}$;

*A paper presented at the regional convention at Warrensburg, Mo., April 25, 1970, and awarded first place by the Awards Committee.

however, the problem here is to guarantee that we can always remove all duplications in a finite number of steps. We can do much better because we can prove that any fraction $\frac{a}{b}$ with $1 \leq a < b$ can always be written as the sum of s distinct fractions of the form $\frac{1}{x}$ where $s \leq a$.

THEOREM 1: *If $1 \leq a < b$, then $\frac{a}{b}$ has a representation of the form*

$$\frac{a}{b} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_s} \tag{2}$$

with $s \leq a$.

Proof: The proof is by mathematical induction on a. First prove that the theorem is true for the case $a = 1$. If $a = 1$, the theorem obviously holds since $\frac{a}{b} = \frac{1}{b}$ and $s = 1$ so that $s = a$.

Next, assume that the theorem is true for all $\frac{a'}{b'}$ with $s' \leq a'$, where $1 \leq a' < a$. The Euclidean Algorithm tells us that there exist q and r, $0 \leq r < a$ such that $b = qa + r$. If $r = 0$ then $\frac{a}{b} = \frac{1}{q}$ and we are through. If $0 < r < a$, then $b = qa + r + a - a = (q + 1)a - (a - r) = xa - x'$ where $x = q + 1$ and $x' = a - r$. Since $a > r$ and $r > 0$, then $x' > 0$ and $x' < a$. Hence $1 \leq x' < a$. Dividing through by bx, we have $\frac{1}{x} = \frac{a}{b} - \frac{x'}{bx}$ or $\frac{a}{b} = \frac{1}{x} + \frac{x'}{bx}$. But by the induction hypothesis, $\frac{x'}{bx}$ can be written in the form (2) with $s' \leq x' < a$. Hence $\frac{a}{b}$ may be written in form (2) with $s = s' + 1 < a + 1$. Thus $s \leq a$. The theorem follows by mathematical induction.

In order to consider the more general case where we do not make the restriction $a < b$ we must consider the harmonic series

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

It is easily shown in calculus and elsewhere that this series is divergent; that is, given any integer m, there exists some n such that $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \geqq m$. We can now prove the following theorem.

THEOREM 2: *Every positive rational fraction may be expressed in the form (2) where the x_i's are all distinct and s is finite.*

Proof:

(i) If $0 < \frac{a}{b} < 1$ we may use Theorem 1.

(ii) If $1 \leqq \frac{a}{b}$, then since the harmonic series is divergent, we can find an n such that $S_n \leqq \frac{a}{b} < S_{n+1}$. Let $\frac{x}{y} = \frac{a}{b} - S_n$. Then $0 \leqq \frac{a}{b} - S_n < S_{n+1} - S_n = \frac{1}{n+1}$. Thus $0 \leqq \frac{x}{y} < \frac{1}{n+1}$. If $\frac{x}{y} = 0$, then $\frac{a}{b} = S_n$ and $\frac{a}{b}$ is expressible in the given form. If $0 < \frac{x}{y} < \frac{1}{n+1}$, we have a proper fraction $\frac{x}{y}$ and can use Theorem 1 to expand it. Also, since each fraction $\frac{1}{x'}$ in the expansion of $\frac{x}{y}$ is less than $\frac{1}{n+1}$, it follows that for each x', $x' > n + 1$. Thus, the combined representation $\frac{a}{b} = S_n + \frac{x}{y}$ will contain no duplications.

As an example, consider the representation of $\frac{7}{3}$. Since $S_5 < \frac{7}{3} < S_6$, and $S_5 = \frac{137}{60}$, then $\frac{7}{3} = \frac{137}{60} + \frac{x}{y}$ from which $\frac{x}{y} = \frac{1}{20}$. Since $\frac{1}{20}$ is a fraction of the desired form our work is through. Thus, $\frac{7}{3}$ may be represented as

$$\frac{7}{3} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20} \quad .$$

It is interesting to ask under what conditions a fraction can be expressed as the sum of exactly two distinct fractions of the form $\frac{1}{x}$. The following theorem states the conditions:

THEOREM 3: *The equation* $\frac{a}{b} = \frac{1}{x} + \frac{1}{y}$ *with* a *and* b *given positive relatively prime integers (that is, there is no integer* d*, greater than one, which divides both* a *and* b*) is solvable for distinct positive integers* x *and* y *if and only if there exist distinct positive, relatively prime integers* P *and* Q *such that* P *and* Q *divide* b*, and* a *divides* $P + Q$.

Proof: First assume that distinct, positive, relatively prime integers P and Q exist such that P and Q divide b and a divides $P + Q$. Then $P + Q = ka$ and we have that $\frac{a}{b} = \frac{ka}{kb} = \frac{P+Q}{kb}$ $= \frac{P}{kb} + \frac{Q}{kb} = \frac{1}{kP'} + \frac{1}{kQ'}$. Since $P \neq Q$ then $P' \neq Q'$, and thus kP' and kQ' are the solutions for x and y in $\frac{a}{b} = \frac{1}{x} + \frac{1}{y}$. The converse may be proved by using a combination of theorems on divisibility. Thus the theorem is established.

To illustrate this theorem, consider the representation of $\frac{19}{280}$ with $s = 2$. If we choose $P = 56$ and $Q = 1$, then P and Q both divide 280, and $P + Q = 57$ which is $3 \cdot 19$. Hence $\frac{19}{280} =$ $\frac{3 \cdot 19}{3 \cdot 280} = \frac{56 + 1}{840} = \frac{1}{15} + \frac{1}{840}$.

Conjectures have been made about the representations of two special kinds of fractions. Erdös speculated that every fraction of the form $\frac{4}{n}$ with $n \geqq 3$ may be written as the sum of exactly three fractions of the form $\frac{1}{x}$. Sierpinski has made the same conjecture about fractions of the form $\frac{5}{n}$. To date neither of these conjectures has been proved although there is considerable evidence to support them. For example, Sierpinski's conjecture has been proved correct for all n in the range $3 \leqq n \leqq 1{,}057{,}438{,}801$. The proof is beyond the scope of this paper.

This problem may be extended by limiting the set from which the denominators may be chosen. In Volume 67 of the *Bulletin of the American Mathematical Society,* Herbert S. Wilf defines an R-basis as a sequence $S = \{n_1, n_2, n_3, \cdots\}$ of distinct integers such that every positive integer is representable as the sum of reciprocals of finitely many integers of S. Hence an immediate consequence of Theorem 2 may be stated as Theorem 2′:

THEOREM 2′: *The set of all positive integers is an R-basis.*

In Volume 61 of the *American Mathematical Monthly,* Robert Breusch proved the fact that the odd positive integers form an R-basis. The generalization of this theorem, that every arithmetic progression is an R-basis is proved by Van Albada and Van Lint in Volume 70 of the same periodical.

Since relatively little work has been done on the problem of Egyptian fractions, this is one of the many fields of number theory which may be extended.

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Square Circles*

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In Euclidean geometry the distance between two points, $P(x_1, y_1)$ and $Q(x_2, y_2)$ is defined to be $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$; and the area of a square is defined to be b^2 where b is the length of a side.

This distance definition is really a "distance" function mapping the set of all ordered pairs of points into the set of non-negative real numbers and satisfying certain other properties; the definition for the area of a square is simply a function mapping the set of all squares into the set of non-negative real numbers and again satisfying other properties.

The purpose of this paper is to define a different distance function and to ascertain if this new definition will necessitate a different function for the assignment of area to a square.

We begin by listing the basic properties of the familiar Euclidean distance function $d(P, Q)$.

For points P, Q and R on the plane

1) $d(P, P) = 0$;
2) $d(P, Q) > 0$ if $P \neq Q$;
3) $d(P, Q) = d(Q, P)$;
4) $d(P, Q) + d(Q, R) \geqq d(P, R)$;
5) if P, Q, R form a right triangle with the right angle at Q, then $d^2(P, Q) + d^2(Q, R) = d^2(P, R)$;
6) $d(P, Q)$ is invariant under a translation of the plane;
7) $d(P, Q)$ is invariant under a rotation of the plane.

Any function which satisfies the first four properties is called a metric. The familiar distance function then is simply a metric which happens to satisfy additional properties essential to Euclidean geometry.

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The basic properties of the familiar Euclidean area function $A(R)$ are the following:

For polygons S and R on the plane,

1) $A(R) = 0$ if and only if R is a single point or a line segment (a degenerate polygon);
2) $A(R) > 0$ if R is not a single point or a line segment;
3) $A(R \cup S) = A(R) + A(S) - A(R \cap S)$;
4) if R is a proper subset of S, then $A(R) < A(S)$;
5) if $R \cong S$, then $A(R) = A(S)$;
6) the area of R is invariant under translations of the plane;
7) the area of R is invariant under rotations of the plane;
8) if R is a square and s is the length of a side, then $A(R) = s^2$.

In place of the original Euclidean metric we define a new function as follows:

DEFINITION 1: *If* $P = (x_1, y_1)$ *and* $Q = (x_2, y_2)$, *then* $m(P, Q) = |x_1 - x_2| + |y_1 - y_2|$.

Then we prove that this function is a metric by showing that it satisfies all four necessary properties.

1) $m(P, P) = |x_1 - x_1| + |y_1 - y_1| = 0 + 0 = 0$
2) If $P \neq Q$, then $x_1 \neq x_2$ or $y_1 \neq y_2$. Thus, $m(P, Q) = |x_1 - x_2| + |y_1 - y_2| > 0$ since at least one of the terms $|x_1 - y_2|$ or $|y_1 - y_2|$ is not zero and neither is negative.
3) $m(P, Q) = |x_1 - x_2| + |y_1 - y_2| = |x_2 - x_1| + |y_2 - y_1| = m(Q, P)$.
4) For the fourth property we will need to examine several cases. For real numbers a and b,

   if $a \geqq 0$ and $b \geqq 0$, then
   $$|a| + |b| = a + b = |a + b|;$$
   if $b \geqq -a \geqq 0$, then
   $$|a| + |b| = -a + b \geqq a + b = |a + b|;$$
   if $-a \geqq b \geqq 0$, then
   $$|a| + |b| = -a + b \geqq -a - b = |a + b|;$$

if $a < b$ and $b < 0$, then

$$|a| + |b| = -a - b = |a + b|;$$

So in all cases $|a| + |b| \geqq |a + b|$. Now we apply this result to our definition. If $R = (x_3, y_3)$ then

$$\begin{aligned} m(P, Q) + m(Q, R) &= |x_1 - x_2| + |y_1 - y_2| \\ &\qquad + |x_2 - x_3| + |y_2 - y_3| \\ &\geqq |x_1 - x_2 + x_2 - x_3| \\ &\qquad + |y_1 - y_2 + y_2 - y_3| \\ &= |x_1 - x_3| + |y_1 - y_3| \\ &= m(P, R)\,. \end{aligned}$$

To see one of the effects of this definition of distance, let us look at the graph of the unit circle with center at the origin. Let $0 = (0, 0)$ and $P = (x, y)$.

$$m(P, 0) = |x - 0| + |y - 0| = |x| + |y| = 1$$

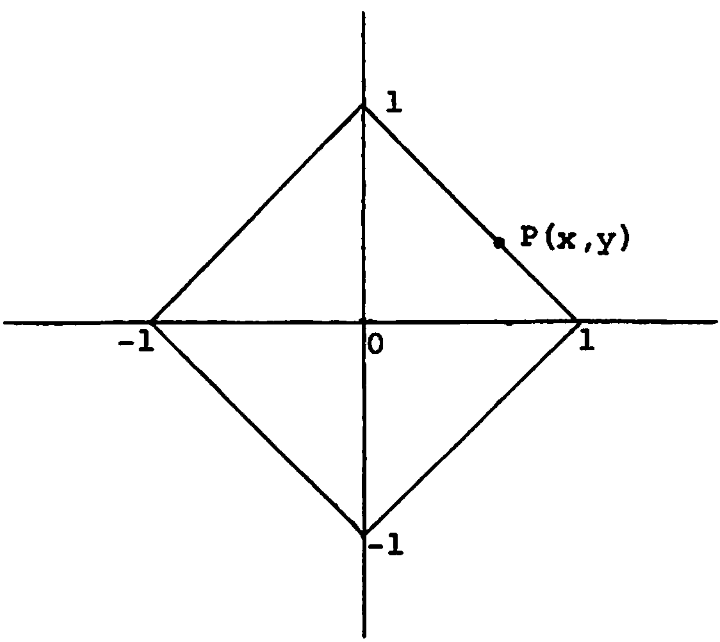


Figure 1

Experimenting further we find that all circles have graphs of this shape. Here is the graph of the circle of radius c with center at the point(a, b).

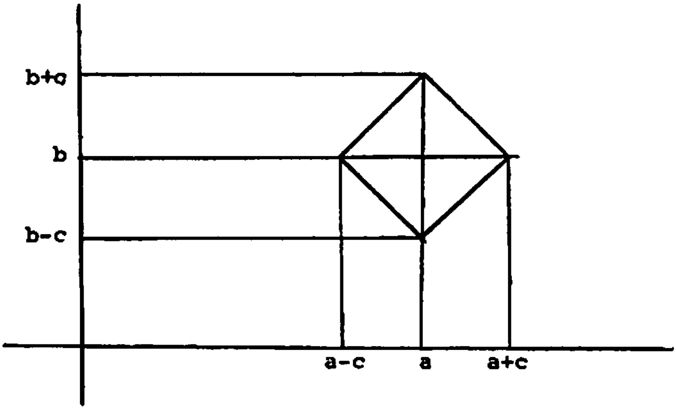


Figure 2

If we define the area of a square of side s to be s^2, we encounter some curious results. Any circle in our new geometry is obviously a square. Let us circumscribe about the unit circle a square with sides parallel to the x- and y-axes.

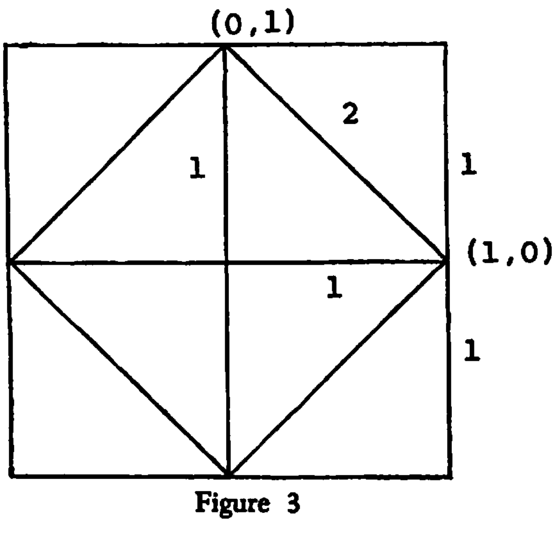
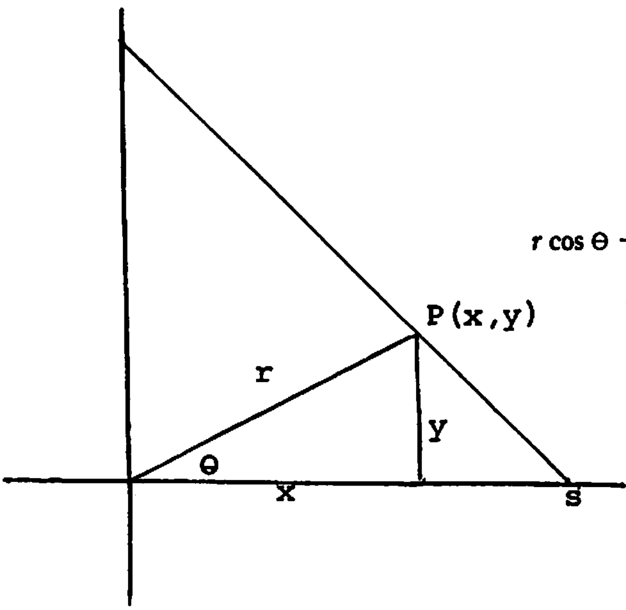


Figure 3

Each side of the circle has length 2, so its area would be 4. But the same thing is true of the "larger" square. This result violates the fourth property of area.

The most obvious definition of area to use in order to obtain as many of the properties of Euclidean area as possible is one which gives Euclidean area. First we find the relationship between the Euclidean metric $d(P, Q)$ and our new metric $m(P, Q)$.

Given a line segment of Euclidean length r and m-length s with non-negative slope and one endpoint at the origin, let Θ be the angle between the positive x-axis and the line segment.



$$x + y = s$$

$$r \cos \Theta + r \sin \Theta = s$$

$$r = \frac{s}{\cos \Theta + \sin \Theta}$$

Figure 4

Definition 2: *The area of a square with side s is:*

$$\frac{s^2}{(\cos \Theta + \sin \Theta)^2} = \frac{s^2}{\cos^2\Theta + \sin^2\Theta + 2 \cos \Theta \sin \Theta} = \frac{s^2}{1 + \sin 2\Theta}$$

where Θ is the angle ($0° \leq \Theta \leq 90°$) between a side of the square with non-negative slope and a line parallel to the x-axis.

Since this definition gives Euclidean area, it is clear that area properties 1 through 4 hold. However, we have lost the validity of property 5 if we use the definition that two squares are congruent if corresponding sides have the same length.

Regarding property 6, lengths are invariant under translations just as in Euclidean geometry. For $m((x_1 + h, y_1 + k), (x_2 + h), (y_2 + k)) = |(x_1 + h) - (x_2 + h)| + |(y_1 + k) - (y_2 + k)| = |x_1 - x_2| + |y_1 - y_2| = m((x_1, y_1), (x_2, y_2))$.

Also, a line is parallel to its image under a translation, as in Figure 5.

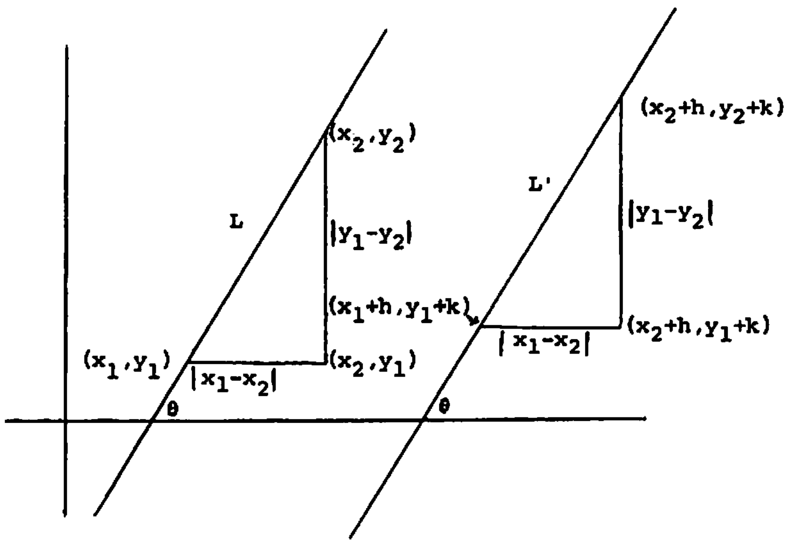


Figure 5

Since lengths are invariant, the slope of line L is the same as the slope of line L'. So angles are also invariant under translation.

Therefore, the area of squares as defined by Definition 2 is invariant under translations.

But with rotation it is a different story. In many cases the image is much different from the original figure. For example, figure 6 shows the result of rotating the points of the line $y = x + 1$ through an angle of $45°$.

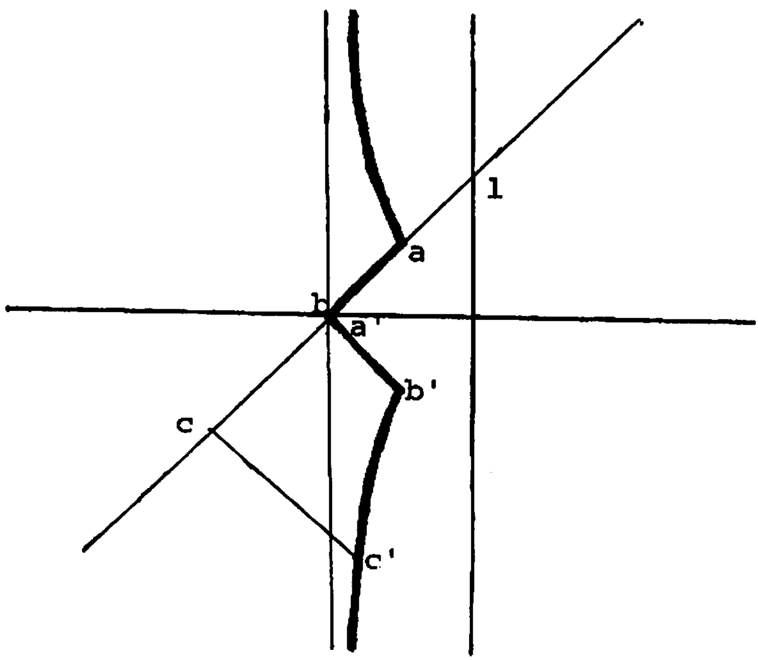


Figure 6

Remember that the *m*-distance of each point on the line from the origin must remain constant. Notice that the segment between $(-1, 0)$ and $(0, 1)$ remains on the unit circle but rotates through an angle of $45°$. Segment *ab* rotates onto segment *a′b′*. But segment *bc* rotates onto the arc *b′c′*. Thus it is seen that a line may not be transformed into a line under rotation.

Finally it is clear that area property 8 no longer applies since we have used a different definition for the area of a square.

Concluding remarks: We know that there exist non-Euclidean geometries; in particular, there exist geometries in which distance is defined and yet in which the Pythagorean Theorem need not hold. In this paper we have introduced one such geometry. The function used to assign distances to pairs of points has been shown to be a metric.

Of the several ways possible to define an area function for this geometry we have selected one and have studied some of its properties. An important property of this new geometry is that length and area are invariant under translation but not under rotation.

The Least Squares Method for the Approximate Solution of Linear Ordinary Differential Equations

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The linear ordinary differential equation of order q

$$(1) \quad L(y) = \frac{d^q y}{dx^q} + A_1(x) \frac{d^{q-1} y}{dx^{q-1}} + \cdots + A_{q-1}(x) \frac{dy}{dx} + A_q(x) y = f(x)$$

often arises in problems in mathematics. Many times it is desirable to have the solution to (1) pass through certain points in that plane.

For the purpose of this article we will assume that (1) has a solution and that this solution is unique when required to pass through q distinct points in the plane.

The case often arises where we must approximate a solution to (1). Then we must find either a numerical approximation to the solution or some analytical expression that approximates the solution.

In this article we want to discuss the method of least squares for finding an approximate solution (in the form of an analytical expression) to (1) along with certain boundary conditions.

To further define the problem, suppose that it is desired to find the solution of (1), defined for the interval $a \leqq x \leqq b$ of the independent variable, and the associated boundary conditions

$$(2) \qquad y(x_j) = y_j, \; j = 1, 2, \cdots, q,$$

which we assume are sufficient to render the solution unique.

Let $u_i(x)$, $i = 0, 1, \cdots, n$, be a set of $n + 1$ independent integrable functions of x, where $u_0(x)$ satisfies the q boundary conditions, and $u_1(x)$, $u_2(x)$, $\cdots$, $u_n(x)$, each satisfy homogeneous boundary conditions at the corresponding q points $x_1, x_2, \cdots, x_q$. In general, the functions $u_1(x)$, $u_2(x)$, $\cdots$, $u_n(x)$, are not completely arbitrary independent functions.

Let

$$(3) \qquad \bar{y} = u_0(x) + \sum_{i=1}^{n} c_i u_i(x) \; .$$

The function $\bar{y}$, where the coefficients c_i, $i = 1, 2, \cdots, n$, are independent of x, also satisfies all the boundary conditions. If the boundary conditions are of the homogeneous type, we omit the function $u_0(x)$.

Thus, the problem is to determine the coefficients in (3) so that $\bar{y}$ is a good approximation to the solution of (1), along with (2), over $a \leqq x \leqq b$.

Let

$$(4) \qquad R(x) = L(\bar{y}) - f(x) \; .$$

If $R(x) \equiv 0$, then $\bar{y} = y$, where y is the exact solution of (1), along with (2). For $R(x) \neq 0$, define

$$(5) \qquad I(c_1, c_2, \cdots, c_n) = \int_a^b [R(x)]^2 dx\,.$$

If we require I to be a minimum, then the coefficients can be found for which $R(x)$ will be the best approximation in the sense of the least squares to zero.

If I is to be a minimum, then it is necessary that $\partial I/\partial c_i = 0$ for each i, $i = 1, 2, \cdots, n$. Now

$$\frac{\partial}{\partial c_i} = \frac{\partial}{\partial c_i}\int_a^b [R(x)]^2 dx$$

$$= \int_a^b 2R(x) \frac{\partial}{\partial c_i} R(x) dx$$

$$= \int_a^b 2R(x) \frac{\partial}{\partial c_i} [Lu_0(x) + \sum_{i=1}^{n} c_i Lu_i(x) - f(x)] dx$$

$$= \int_a^b 2R(x) Lu_i(x) dx\,.$$

Thus, for I to be a minimum, it is necessary that the coefficients be found such that

$$(6) \qquad \int_a^b R(x) Lu_i(x) dx = 0,\ i = 1, 2, \cdots, n\,.$$

We wish to add a note here. There exists a well known method, the method of Galerkin, for the handling of the above outlined problem. In using the method of Galerkin, we pick the coefficients such that

$$(7) \qquad \int_a^b R(x) u_i(x) dx = 0,\ i = 1, 2, \cdots, n\,.$$

Thus, in the least squares method, the c_j's are chosen so that $R(x)$ is orthogonal to the $Lu_i(x)$, $i = 1, 2, \cdots, n$ on $[a, b]$. In the Galerkin method, the c_j's are chosen so that $R(x)$ is orthogonal to the $u_i(x)$, $i = 1, 2, \cdots, n$ on $[a, b]$.

As mentioned earlier, the functions $u_1(x), u_2(x), \cdots, u_n(x)$, are not completely arbitrary independent functions. In general, we also require that the u_j's be also chosen in such a way that the set $Lu_i(x)$, $i = 1, 2, \cdots, n$ is also a set of independent functions. We also want to note that with this added fact, the coefficient matrix of (6) is positive-definite.

For an example, consider the boundary value problem.

$$\begin{aligned} y'' + y' &= 0, \\ y(0) &= 1, \\ y(1) &= e^{-1}. \end{aligned}$$

The exact solution is $y = e^{-x}$. We pick a problem where the solution is known and simple so that we can set forth a comparison.

The range of representation is $[0, 1]$. Let $u_0(x) = (e^{-1}-1)x + 1$ which satisfies both boundary conditions. Let $u_1(x) = x^2 - x$ and let $u_2(x) = x^3 - x^2$. Both $u_1(x)$ and $u_2(x)$ satisfy the homogeneous boundary conditions at $x = 0$ and $x = 1$.

Now

$$\begin{aligned} \bar{y} &= u_0(x) + c_1u_1(x) + c_2u_2(x) \\ &= c_2x^3 + (c_1 - c_2)x^2 + (e^{-1} - 1 - c_1)x + 1, \end{aligned}$$

thus

$$\begin{aligned} R(x) &= L(\bar{y}) - f(x) \\ &= 3c_2x^2 + (4c_2 + 2c_1)x + (e^{-1} - 1 - c_1 - 2c_2). \end{aligned}$$

Now to find the coefficients c_1 and c_2 by the least squares method, we have

$$\int_0^1 R(x)Lu_1(x)dx = \int_0^1 R(x)Lu_2(x)dx = 0$$

which when integrated and cleared of fractions become

$$\begin{cases} 19c_2 + 26c_1 = 12(1 - e^{-1}) \\ 154c_2 + 95c_1 = 30(1 - e^{-1}) \end{cases}$$

from which we have

$$c_1 = \frac{426}{733}(1 - e^{-1}) \text{ and } c_2 = -\frac{120}{733}(1 - e^{-1}) .$$

Thus, $\bar{y}$ is given by

$$\bar{y} = \frac{e^{-1} - 1}{733}[120x^3 - 546x^2 + 1159x] + 1 .$$

If we compute I for the above c_1 and c_2, we find that $I = 0.000545$.

We now present a table to illustrate the above example.

Table 1. Comparison of y and $\bar{y}$.

Value of x	$y = e^{-x}$	$\bar{y}$
0.00	1.000000	1.000000
0.05	0.951229	0.951190
0.10	0.904837	0.904656
0.15	0.860708	0.860321
0.20	0.818731	0.818108
0.25	0.778801	0.777938
0.30	0.740818	0.739736
0.35	0.704688	0.703421
0.40	0.670320	0.668917
0.45	0.637628	0.636147
0.50	0.606531	0.605032
0.55	0.576950	0.575496
0.60	0.548812	0.547461
0.65	0.522046	0.520848
0.70	0.496585	0.495580
0.75	0.472367	0.471580
0.80	0.449329	0.448770
0.85	0.427415	0.427073
0.90	0.406570	0.406410
0.95	0.386741	0.386705
1.00	0.367879	0.367879

Different Proofs of Cauchy-Schwarz Inequality

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STATEMENT: *If $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are any real numbers, then the following inequality holds true:*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geqq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$$

Proof No. 1 Observe that

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) = \left(\sum_{i=1}^{n} a_i b_i\right)^2 +$$

$$\sum_{i=2}^{n} (a_1 b_i - a_i b_1)^2 + \sum_{i=3}^{n} (a_2 b_i - a_i b_2)^2 + \cdots +$$

$$(a_{n-1} b_n - a_n b_{n-1})^2 .$$

Clearly, therefore, $\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geqq \left(\sum_{i=1}^{n} a_i b_i\right)^2.$

Proof No. 2 Let $x_i = \dfrac{a_i}{\sqrt{\sum_{i=1}^{n} a_i^2}}, \; y_i = \dfrac{b_i}{\sqrt{\sum_{i=1}^{n} b_i^2}}$.

Since $x_i^2 + y_i^2 > 2x_i y_i$,

we have

$$\text{(A)} \qquad \frac{a_i^2}{\sum_{i=1}^{n} a_i^2} + \frac{b_i^2}{\sum_{i=1}^{n} b_i^2} \geqq \frac{2a_i b_i}{\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}}$$

for $i = 1, 2, \cdots, n$.

The relations at (A) on addition yield

$$\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{\sum_{i=1}^{n} a_i^2} + \frac{b_1^2 + b_2^2 + \cdots + b_n^2}{\sum_{i=1}^{n} b_i^2} \geq$$

$$2 \cdot \frac{a_1b_1 + a_2b_2 + \cdots + a_nb_n}{\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}}$$

$$\Rightarrow 1 + 1 \geq 2 \frac{\sum_{i=1}^{n} a_ib_i}{\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}}$$

$$\Rightarrow \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} \geq \sum_{i=1}^{n} a_ib_i$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_ib_i\right)^2.$$

Proof No. 3 We shall use the following elementary result, the proof of which is left for the reader:

(B) If $a > 0$, then $ax^2 + 2bx + c \geq 0$ for all values of x if $ac \geq b^2$. Observe now that

$$(a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2 \geq 0$$

$$\Rightarrow (a_1^2x^2 + 2a_1b_1x + b_1^2) + (a_2^2x^2 + 2a_2b_2x + b_2^2) + \cdots +$$

$$(a_n^2x^2 + 2a_nb_nx + b_n^2) \geq 0$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i^2\right)x^2 + 2\left(\sum_{i=1}^{n} a_ib_i\right)x + \left(\sum_{i=1}^{n} b_i^2\right) \geq 0.$$

Applying the result stated at (B) to the above inequality we get

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_ib_i\right)^2.$$

Proof No. 4 The proof is by induction.

When $n = 1$, the inequality is trivially true. Assume that the inequality is true for $n = k$, that is,

$$\left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right) \geqq \left(\sum_{i=1}^{k} a_ib_i\right)^2 .$$

For $n = k + 1$, the inequality would read as

(C) $\left(\sum_{i=1}^{k+1} a_i^2\right)\left(\sum_{i=1}^{k+1} b_i^2\right) \geqq \left(\sum_{i=1}^{k+1} a_ib_i\right)^2$ and we want to ascertain if this is true. Observe that the left hand side of (C) equals

$$\left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right) + \left(\sum_{i=1}^{k} a_i^2\right) b_{k+1}^2 + \left(\sum_{i=1}^{k} b_i^2\right) a_{k+1}^2 + a_{k+1}^2 b_{k+1}^2$$

$$\geqq \left(\sum_{i=1}^{k} a_ib_i\right)^2 + \sum_{i=1}^{k} (a_i^2b_{k+1}^2 + b_i^2a_{k+1}^2 - 2a_ib_ia_{k+1}b_{k+1})$$

$$+ 2\left(\sum_{i=1}^{k} a_ib_i\right) a_{k+1}b_{k+1} + a_{k+1}^2b_{k+1}^2$$

$$= \left(\sum_{i=1}^{k+1} a_ib_i\right)^2 + \left(\sum_{i=1}^{k} a_ib_{k+1} - b_ia_{k+1}\right)^2$$

$$\geqq \left(\sum_{i=1}^{k+1} a_ib_i\right)^2 = \text{right hand side of (C)}.$$

Thus we see that the inequality in question does hold for $n = k + 1$. Therefore, by mathematical induction we see that the Cauchy-Schwarz inequality holds in general.

Proof No. 5 Let vector $A = (a_1, a_2, \cdots, a_n)$ and vector $B = (b_1, b_2, \cdots, b_n)$.

Then $\sum_{i=1}^{n} a_i b_i = A \cdot B = |A||B| \sin \Theta$, Θ being the angle between vectors A and B,

$$= \left(\sqrt{\sum_{i=1}^{n} a_i^2}\right)\left(\sqrt{\sum_{i=1}^{n} b_i^2}\right) \sin \Theta .$$

Therefore, $\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \sin^2\Theta = \left(\sum_{i=1}^{n} a_i b_i\right)^2$.

Hence $\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geqq \left(\sum_{i=1}^{n} a_i b_i\right)^2$, since $0 \leqq \sin^2\Theta \leqq 1$.

On Various Many-Valued Logics

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"It is raining."

"You mean it is raining in Ithaca, New York, at 2 p.m., July 14, 1950, for you do not know whether or not it is now raining in El Paso, Texas."

"Would you agree then that my statement is neither true nor false?" [5; p.3]

This short dialogue illustrates the ambiguity which might arise if one were to accept a two-valued logic as the absolute truth. From the time there was first a clear enunciation of the principle 'Every proposition is either true or false,' there have been those who questioned it. Modern developments in physics indicate that the two-valued logic is not always adequate to explain physical behavior. C. G. Darwin said, ". . . the old logic was devised for a world that was thought to have hard outlines, and now that the new mechanics has shown that the outlines are not hard, the method of reasoning must be changed." [4; p. 207].

In a two-valued logic we have a group of propositions, p, q, etc., each of which has a truth value of 0 or 1. When interpreted in the usual way, $p = 1$ means "p is true," and $p = 0$ means "p is false." A unique feature of this two-valued system is that when the property, 0 or 1, of the propositions p, q, etc., is given, then any other proposition derived from p, q, etc., is automatically determined to have either the property 0 or 1.

Although the two-valued system is currently the most widely used, there are other truth-value systems which are being developed. These alternative systems do not necessarily contradict the principle that 'Every proposition is either true or false, and none is both,' which might seem to determine the character of the two-valued logic.

The simplest many-valued logic is a three-valued logic. Lukasiewicz and Tarski were among the first to develop a logic with three possible truth values: $p = 1$, which will mean "p is certainly true;" $p = 0$, which will mean "p is certainly false;" and $p = \frac{1}{2}$, which will mean "p is doubtful." It must be remembered that p and $p = 1$ may not be equivalent in this system.

In order to distinguish from the two-valued calculus, we will introduce the symbols Lukasiewicz and Tarski used for their logical connectives:

pOq for $p \vee q$ read "p or q;"
pAq for $p \wedge q$ read "p and q;"
Np for $\sim p$ read "not p;"
pCq for $p \rightarrow q$ read "p implies q;"
pEq for $p \leftrightarrow q$ read "p is equivalent to q" [1; p. 214].

Negation and implication are taken as the fundamental operations, and the other operations are defined in terms of these in the following way:

$$pOq = {}_{D}{:}pCq.Cq;$$
$$pAq = {}_{D}N{:}Np.O.Nq;$$
$$pEq = {}_{D}{:}pCq.A.qCp.$$

Two propositions with the same truth values are considered logically equivalent. That is, if $\tau(a) = \tau(b)$, then $a = b$. All logical operations are defined in terms of truth values alone. Thus a proposition is true if it is a tautology, i.e., if it has all 1's down its major connective.

The following truth-table analyzes a set of truth values which can be assigned to the various propositions formed using the connectives defined above:

p	q	Np	pOq	pAq	pEq	pCq
1	1	0	1	1	1	1
1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	0	1	0	0	0
$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1	0	0	1
0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1
0	0	1	0	0	1	1

We should examine carefully the truth-values of these different relationships for the corresponding values of p and q. Most of them follow if we remember the assumed meaning of the connectives, except perhaps pCq. Here our logical intuition does not seem to be sufficient.

Lukasiewicz in "Philosophische Bemerkungen zu mehwertigen Systemen des Aussagen-kalkuls," explains a general method which helps clarify this connective. If p and q designate certain numbers of the interval $0 - 1$, then

Cpq (p implies q) $= 1$ for $p \leq q$;
$Cpq = 1 - p + q$ for $p > q$;
Np (p is false) $= 1 - p$. [1; p. 213.]

At this point we should make it clear that the values assigned in the above table are completely arbitrary. Other three-valued logics, different from Lukasiewicz and Tarski, can be easily developed using other truth values.

In terms of Np and Cp a further function of one element, Mp, is defined as follows: $Mp = {}_D{:}Np.Cp$, read 'p is possible.' The truth-table for Mp is

p	Mp
1	1
$\frac{1}{2}$	1
0	0

It is important that we clarify the distinction between the truth-values of 1, $\frac{1}{2}$, and 0, and the truth-functions of p, Mp, and Np. The first are labels for truth values, the latter are propositions.

In comparing the various many-valued logics it is interesting to note that for every principle which holds in the three-valued calculus, its analogue also holds in two-valued calculus, but the converse is not true. In fact, in general, the more truth values there are, the fewer provable laws there are. We can easily see that every law in three-valued calculus will be valid in two-valued calculus if we look at the defining truth-table on page 25. If we strike out each line which contains a $\frac{1}{2}$, we obtain the following truth table:

p	q	Np	pOq	pAq	pEq	pCq
1	1	0	1	1	1	1
1	0	0	1	0	0	0
0	1	1	1	0	0	1
0	0	1	0	0	1	1

If we compare this table to our two-valued truth table, we see that $pCq \equiv p \rightarrow q$; $pOq = p \vee q$; $pAq = p \wedge q$; $pEq \equiv p \leftrightarrow q$; and $Np \equiv \sim p$.

It is important for us not to conclude that since every law of the three-valued calculus holds in the two-valued system, and since there are some propositions of the two-valued system which do not hold in our system, that the three-valued calculus is a weaker system. There are many distinctions which can be made in three-valued calculus which cannot be made in the two-valued calculus due to the elimination of the value $\frac{1}{2}$, namely—the distinction of Mp from p, $NMNp$ from p, NMp from Np, and MNp from Np. [1; p. 218]

An important issue arises about the falsity of the law of the excluded middle in our system. Shall we say that our system is false because it denies this law? Or must we conclude that the law is false because in our system it is false? The latter case can be contradicted since in the two-valued system it is necessarily true. Lewis and Langford discuss this issue:

> The way out of this dilemma lies in the reflection that the tautological laws of any truth-value system are necessarily true; but that the symbolic system itself does not tell us what it is true *of*. It is true for whatever interpretation of its truth-values will make them exhaustive of the relations and other truth-functions which will then be consonnant with their matrix-properties. But such an interpretation—for this or any other symbolic system—is something which has to be found; and something concerning which it is easily possible to make a mistake. To suppose that because of the rigorous character of its method, it must therefore represent 'the truth of logic' would be excessively naive. [1; p. 222]

There are many applications for a three-valued logic, especially in modern physics. One possible use is in Dirac's interpretation of his 'principle of superposition.' Here he affirms that between the states of an atomic system, there exists a peculiar relationship such that "whenever the system is definitely in one state we can equally well consider it as being in each of two or more states. The original state must be regarded as the result of a kind of superposition of the two or more new states in a way that cannot be conceived on classical ideas." [4; p. 208] In a two-valued logic, a system is either in a state or not in it. In three-valued logic a system can fail to be either in a state or not in it, which situation can be described by saying that the system is partly in the state. In this application $\frac{1}{2}$ would probably be interpreted differently from the interpretation in our development, but this illustration serves as an example of the applicability of a three-valued logic to physical reality.

We have looked extensively at a three-valued logic, but this is only the simplest of the many-valued logics. The many-valued logic proposed by Lukasiewicz and Tarski is not functionally complete, so we will look at the system of E. L. Post. In the development of our three-valued logic we said that a statement was valid if and only if it was a tautology when tested by a truth-table. A truth-table development has the merit of being simple and easily applicable. It also clearly shows us that we are dealing with a many-valued logic. However, there is an alternative procedure for accepting statements, that of the axiomatic method, which is used so extensively in modern mathematics.

Post generalized a system of logic with m distinct truth-values, $t_1, t_2, \cdots, t_m$, where m is any positive integer. A function of order n will have m^n configurations in its truth-table, and there will be m^{m^n} truth-tables of order n. Post says that a system is complete if it has all possible tables. The two following tables generate a complete system:

p	$N_m p$
t_1	t_2
t_2	t_3
$\cdots$	$\cdots$
t_m	t_1

p	q	$pO_m q$	
t_1	t_1	t_1	
$\cdots$	$\cdots$	$\cdots$	
t_{i_1}	t_{j_1}	t_{i_1}	
$\cdots$	$\cdots$	$\cdots$	
t_{i_2}	t_{j_2}	t_{j_2}	$i_1 \leqq j_1$
$\cdots$	$\cdots$	$\cdots$	$i_2 \geqq j_2$
t_m	t_n	t_m	

If we take a close look at these tables we see that $N_m p$ permutes the truth-values cyclically, while $pO_m q$ has the higher of the truth-values. (The higher truth-value has the smaller subscript.) [2; p. 180]

From this we can generalize a set of postulates by first assuming arbitrary primitive functions, then selecting a set of postulates that will give us the desired system. From these postulates theorems can be derived.

At the beginning it was mentioned that if our logic could be applied it would be a more successful development. We have already looked at a possible application to physics. Now, let us look at an application to other fields of mathematics.

Alan Rose suggests an application of an eight-valued logic to geometry. It has been found that Euclid's fifth postulate "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles," is independent of his other four. Two other consistent forms of geometry have been developed in which this postulate has been rejected. These geometries are known as hyperbolic and elliptic geometry. Euclidean geometry is known as parabolic geometry. If l is a line, and P is a point not on the line, in parabolic geometry there is one line through P parallel to the given line; in hyperbolic, there are two lines through P parallel to l; and in elliptic there are no lines through P parallel to l. We can thus regard geometry as being a system whose truth-values form the eight-element lattice as shown in the figure. When a proposition is true in all three geometries, it has the value I. When a proposition

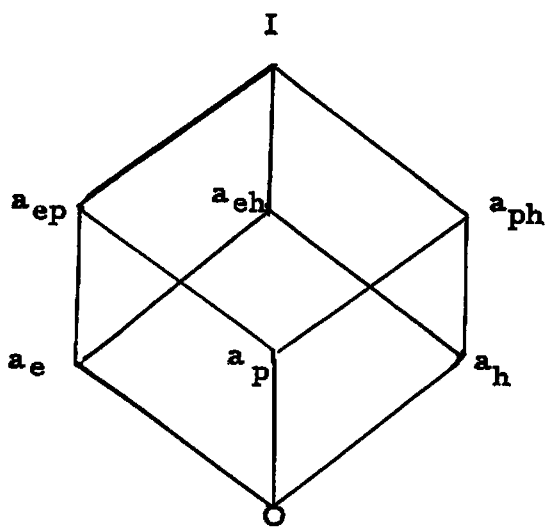

Figure 1

is true in elliptic and parabolic, it has the truth-value a_{ep}. The truth-values of a_{eh} and a_{ph} can be interpreted similarly. When a proposition is true in elliptic, but not in parabolic or hyperbolic, it has the truth-value a_e. The truth values of a_p and a_h can be interpreted similarly. When a proposition is false in all three, it has the truth value 0. This idea could be developed much more extensively, but that is not the purpose of this paper. [3; p. 42]

We see that various many-valued logics can be developed, and many of these have important applications. Perhaps some day in mathematics many-valued logics may be used as extensively as the two-valued logic is today.

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People used to think that when a thing changes, it must be in a state of change, and that when a thing moves, it is in a state of motion. This is now known to be a mistake. —Bertrand Russell

Some Sums of Fibonacci Numbers and P* Numbers

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An interesting sequence of positive integers arises from an exercise suggested by the thirteenth century mathematician Leonardo of Pisa (about 1170-1250), who is called Fibonacci since he was the son (figlio) of Bonaccio. In his famous book *Liber Abaci* the following problem appears: How many pairs of rabbits can be produced from a single pair, if it is supposed that every month each pair begets a new pair, which from the second month on becomes productive?

We are led to the following sequence of positive integers:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \cdots,$$

where each new term (beginning with the third term) is found by adding the last term to its predecessor. Thus $2 = 1 + 1$, $3 = 2 + 1$, $5 = 3 + 2$, $8 = 5 + 3$, $13 = 8 + 5, \cdots$. In general, f_1 (the first term in the Fibonacci sequence) $= 1$, $f_2 = 1$, and $f_{n+2} = f_n + f_{n+1}$, where n is any positive integer. Examples of the Fibonacci sequence occur in plant growth and in art, as well as in geometry. The positive integers $1, 2, 3, 5, 8, 13, 21, \cdots$ are called the Fibonacci numbers. Over the years numerous properties of these numbers have been discovered by mathematicians; however, there are still some open questions concerning them. For example, to this date we do not know if there exist infinitely many prime numbers among the Fibonacci numbers.

Clearly the set of Fibonacci numbers is a proper subset of the set of positive integers. It is also evident that some Fibonacci numbers are even numbers and some are odd numbers. We observe that 1, 2, and 3 are all Fibonacci numbers, $4 = 3 + 1$, 5 is a Fibonacci number, $6 = 5 + 1$, $7 = 5 + 2$, 8 is a Fibonacci number, $9 = 8 + 1$, $10 = 8 + 2$, $11 = 8 + 3$, $12 = 8 + 3 + 1$, 13 is a Fibonacci number, $14 = 13 + 1$, $15 = 13 + 2$, $16 = 13 + 3$, $17 = 13 + 3 + 1$, $18 = 13 + 3 + 2$, $19 = 13 + 3 + 2 + 1$, and $20 = 13 + 5 + 2$. After considering several additional examples, we are led to conjecture that every positive integer is either a Fibonacci number or a sum of distinct Fibonacci numbers.

In establishing this result (and later in proving our second major result), we shall use strong (sometimes also called course-of-values) induction. Let $p(x)$ be a statement form over the positive integers. (That is, for any given positive integer n, $p(n)$ is a statement and is therefore either true or false.) Then the strong form of the principle of mathematical induction allows us to derive the conclusion $p(n)$ for all positive integers n from the hypotheses (1) $p(1)$ is true and (2) for each positive integer m, if $p(m)$ is true for all $m < n$ then $p(n)$ is true. Readers who are interested in investigating strong induction and ordinary induction in greater depth should consult Shepherdson, "Weak and Strong Induction," *The American Mathematical Monthly,* Volume 76, Number 9 (November, 1969), pp. 989-1004. We now prove the following theorem.

THEOREM. *Every positive integer is either a Fibonacci number or a sum of distinct Fibonacci numbers.*

Proof. Let n be a positive integer. If n is a Fibonacci number, the desired result is immediate. Hence assume n is not a Fibonacci number (thus $n \neq 1, 2, 3$ in particular). Assume that for every positive integer $m < n$ that m is either a Fibonacci number or a sum of distinct Fibonacci numbers. Now there is a Fibonacci number f_k (the kth term in the Fibonacci sequence) such that $f_k < n < f_{k+1}$ where $k \geqq 3$. Let $m = n - f_k$. By the assumption, $m = m_1 + m_2 + \cdots + m_q$, $q \geqq 1$, where $m_1, m_2, \cdots, m_q$ are distinct Fibonacci numbers. Since $n < f_{k+1}$, $m = n - f_k < f_{k+1} - f_k = f_{k-1} < f_k$ whence $f_k \neq m_i$ for $i = 1, 2, \cdots, q$. Therefore $n = f_k + m_1 + \cdots + m_q$ as required. The result follows by strong induction.

Let P^* be the set of non-composite positive integers; that is, x is an element of P^* if and only if x is a positive prime number or $x = 1$. The elements of P^* will be called P^* numbers. We observe that 1, 2, and 3 are all P^* numbers, $4 = 3 + 1$, 5 is a P^* number, $6 = 5 + 1$, 7 is a P^* number, $8 = 5 + 3$, $9 = 7 + 2$, $10 = 7 + 3$, 11 is a P^* number, $12 = 7 + 5$, 13 is a P^* number, $14 = 11 + 3$, $15 = 13 + 2$, $16 = 13 + 3$, 17 is a P^* number, $18 = 13 + 5$, 19 is a P^* number, and $20 = 17 + 3$. Perhaps every positive integer is either a P^* number or a sum of distinct P^* numbers. To assist us in establishing this result, we will need some additional information. We begin by citing a theorem that was conjectured by Bertrand in 1845 and first proved by Tchebycheff in 1850, and then obtain two corollaries of this theorem.

THEOREM *(Tchebycheff). If n is a positive integer > 3, then between n and $2n - 2$ there is at least one prime number.* [For a proof of this result see Sierpinski, *Elementary Theory of Numbers*, New York: Hafner Publishing Company, 1964, p. 137.]

COROLLARY 1. *If n is a positive integer $\geqq 2$, then between n and $2n$ there is at least one prime number.*

Proof. By virtue of Tchebycheff's Theorem the corollary is true for positive integers > 3. To verify it for $n = 2$ and $n = 3$ we note that the prime number 3 is between the positive integers 2 and 4 and the prime number 5 is between the positive integers 3 and 6.

COROLLARY 2. *If n is a positive integer $\geqq 4$, then between $n/2$ and n there is at least one prime number.*

Proof. Case 1. n is an even positive integer. Then $n/2$ is a positive integer $\geqq 2$ and by Corollary 1 there is at least one prime number p between $n/2$ and n.

Case 2. n is an odd positive integer. Then $n - 1$ is an even positive integer $\geqq 4$ because $n \geqq 5$ and $(n - 1)/2$ is a positive integer $\geqq 2$. Thus there is a prime number p such that $(n - 1)/2 < p < n - 1$. Since $(n - 1)/2 < p$ and both p and $(n - 1)/2$ are positive integers, $(n + 1)/2 = (n - 1)/2 + 1 \leqq p$. Hence $n/2 < (n + 1)/2 \leqq p < n - 1 < n$. Therefore there is at least one prime number p between $n/2$ and n.

THEOREM. *Every positive integer is either a P^* number or a sum of distinct P^* numbers.*

Proof. Let n be a positive integer. If n is a P^* number, the desired result is immediate. Hence assume n is not a P^* number (thus $n \neq 1, 2, 3$ in particular). Assume that for every positive integer $m < n$ that m is either a P^* number or a sum of distinct P^* numbers. By Corollary 2 of Tchebycheff's Theorem there is a prime number p such that $n/2 < p < n$ for $n \geqq 4$. Let $m = n - p$. By the assumption, $m = m_1 + m_2 + \cdots + m_q$, $q \geqq 1$, where $m_1, m_2, \cdots, m_q$ are distinct P^* numbers. Since $n/2 < p$, $n < 2p$ and $m = n - p < p$ whence $p \neq m_i$ for $i = 1, 2, \cdots, q$. Therefore $n = p + m_1 + \cdots + m_q$ as required. The result follows by strong induction.

It is also of interest to observe that it can be shown that if n is a positive integer > 6 then n is a prime number or a sum of distinct prime numbers [See Sierpinski, *Elementary Theory of Numbers*, p. 144].

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4. Henry L. Alder, "The Number System in More General Scales", *Mathematics Magazine,* May 1962, pp. 145-151.
5. J. L. Brown, Jr., "On the Equivalence of Completeness and Semi-Completeness for Integer Sequences", *Mathematics Magazine,* September 1963, pp. 224-226.
6. G. M. Benson, "Solution to Problem E 1492", *The American Mathematical Monthly,* Vol. 69, 1962, p. 567.
7. Problem 9, p. 102 and Problem 4, p. 191 as found in Ivan Niven and Herbert S. Zuckerman, *An Introduction to the Theory of Numbers* (Second Edition), New York: John Wiley and Sons, Inc., 1966.

To the eyes of the man of imagination, nature is imagination itself. —William Blake

Directions for Papers to Be Presented At the Eighteenth Biennial Kappa Mu Epsilon Convention

INDIANA, PENNSYLVANIA

April 2-3, 1971

A significant feature of this convention will be the presentation of papers by student members of KME. The mathematics topic which the student selects should be in his area of interest and of such a scope that he can give it adequate treatment within the time allotted.

WHO MAY SUBMIT PAPERS: Any student KME member may submit a paper for use on the convention program. Papers may be submitted by graduates and undergraduates; however, graduates will not compete with undergraduates.

SUBJECT: The material should be within the scope of the understanding of undergraduates, preferably those who have completed differential and integral calculus. The Selection Committee will naturally favor papers within this limitation and which can be presented with reasonable completeness within the time limit prescribed.

TIME LIMIT: The usual time limit is twenty minutes, but this may be changed on the recommendation of the Selection Committee if requested by the student.

PAPER: The paper to be presented, together with a description of the charts, models, or other visual aids that are to be used in the presentation, should be presented to the Selection Committee. A bibliography of source materials, together with a statement that the author of the paper is a member of KME, and his official classification in school, undergraduate or graduate, should accompany his paper.

DATE AND PLACE DUE: The papers must be submitted no later than January 9, 1971, to the office of the National Vice-President.

SELECTION: The Selection Committee will choose ten to twelve papers for presentation at the convention. All other papers will be listed by title and student's name on the convention program and will be available as alternates.

William R. Smith
National Vice-President, Kappa Mu Epsilon
Department of Mathematics
Indiana University of Pennsylvania
Indiana, Pennsylvania 15701

(Continued from p. 7)

Van Albada, P. J., and J. H. van Lint, "Reciprocal Bases for the Integers", *American Mathematical Monthly,* 70(1963) pp. 170-173.

Wilf, H. S., "Reciprocal Bases for the Integers", *Bulletin of the American Mathematical Society,* 67(1961) p. 456.

The Problem Corner

EDITED BY ROBERT L. POE

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before February 15, 1971. The best solutions submitted by students will be published in the Spring 1971 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor Robert L. Poe, Department of Mathematics, Berry College, Mount Berry, Georgia 30149.

PROPOSED PROBLEMS

236. *Proposed by John Caffrey, American Council on Education, Washington, D.C.*

Beginning in the upper left corner of the table below consider the inverse of any square matrix whose elements are listed. Prove that the inverse matrix has elements all of which are integers and define a function which generates the elements of the inverse.

	0	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	10	15	21	28	36	45
4	1	4	10	20	35	56	84	120	165
5	1	5	15	35	70	126	210	330	495
6	1	6	21	56	126	252	462	792	1287
7	1	7	28	84	210	462	924	1716	3003

If the array were tilted 45° clockwise, it would appear as Pascal's triangle.

Problems 237, 238, 239, and 240 are considered to be problems *of antiquity whose proposers are unknown to the Editor. However, sincere appreciation is expressed to those individuals who suggested the problems to the Editor.*

237. Consider three noncollinear points taken at random on an infinite plane. Determine the probability of these points being the vertices of an obtuse-angled triangle.

238. Consider the angle determined by two rays with a common initial point as the vertex and a given interior point of the angle. Construct the line through the given point which with the two rays forms a triangle with the least area.

239. Solve the system $x/y = x - z$; $x/z = x - y$; and determine the limiting values of all real solutions.

240. A bag contains two marbles of which nothing is known except that each is either black or white. Determine their colors without taking them out of the bag or looking into the bag.

SOLUTIONS

231. *Proposed by Pat LaFratta, Waukesha, Wisconsin.*

Find all the integral values of a, b, and c, if any exist, such that $x/a + y/b = 1$ is tangent to the graph of $x^{3/4} + y^{3/4} = c^{3/4}$.

Solution by Pat LaFratta (proposer of the problem), University of Wisconsin, Waukesha, Wisconsin.

A general point satisfying (1) can be taken as $x = c \sin^{8/3} \Theta$, $y = c \cos^{8/3} \Theta$. By elementary calculus, the equation of the tangent to the graph of (1) at the aforesaid point is given by

$y - c \cos^{8/3} \Theta = - \dfrac{\cos^{2/3} \Theta}{\sin^{2/3} \Theta} (x - c \sin^{8/3} \Theta)$ which can be

written in the form $\dfrac{x}{c \sin^{2/3} \Theta} + \dfrac{y}{c \cos^{2/3} \Theta} = 1$

implying that $a = c \sin^{2/3} \Theta$, $b = c \cos^{2/3} \Theta$ implies that $a^3 + b^3 = c^3$ which is known to be impossible in integers. Hence, there is no solution to our problem.

232. *Proposed by R. S. Luthar, Waukesha, Wisconsin.*

Construct a function that is continuous at one point but discontinuous at every other point of its domain.

Solution by Kenneth Rosen, University of Michigan, Ann Arbor, Michigan.

Consider $f(x) = \begin{cases} 0 \text{ for } x \text{ rational} \\ x \text{ for } x \text{ irrational} \end{cases}$.

Since the rationals and the irrationals are both dense in the reals, for every point a there exist sequences $\{x_i\}$ (all rationals) and $\{y_i\}$ (all irrational) such that $\lim\limits_{\to\infty} x_n = a = \lim\limits_{n\to\infty} y_n$. A necessary condition if f is continuous at a is that $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} f(y_n)$. But, $\lim\limits_{n\to\infty} f(x_n) = 0$ while $\lim\limits_{n\to\infty} f(y_n) = a$. Hence, the only possible point of continuity is $a = 0$. But given any $\varepsilon > 0$ take $S = \varepsilon$, $|x| < S \Rightarrow |f(x)| < \varepsilon$ regardless of whether x is rational or irrational. Hence $\lim\limits_{x\to 0} f(x) = f(0) = 0$. Hence f is continuous at 0, but nowhere else.

Also solved by Leslie Paul Jones, Marietta College, Marietta, Ohio.

233. *Proposed by Leigh Janes, Pleasantville, New Jersey.*

If $x + y = k$, k a constant, and $z = x^p y^q$ maximi z e z.

Solution by Leslie Paul Jones, Marietta College, Marietta, Ohio.

Case 1: Either p or q, or both, equal 1. Without loss of generality it may be assumed that $p = 1$. Then $z = xy^q$. Since $x + y = k$, then $y = k - x$ and $z = x(k - x)^q$. It follows that $dz/dx = (k - x)^q - qx(k - x)^{(q-1)}$. If $dz/dx = 0$, then $(k - x)^q = qx(k - x)^{(q-1)}$. Obviously $x = k$ is a solution. By dividing both sides of the equation by $(k - x)^{(q-1)}$ one obtains $qx = (k - x)$ and $x = k/(q + 1)$ follows as a solution. Then $x = k$ or $x = k/(q + 1)$ will maximize z.

Similarly, if $q = 1$, then $y = k$ or $y = k/(p + 1)$ will maximize z.

Case 2: Neither p nor q equals 1.

Obviously $z = x^p(k - x)^q$. Again applying elementary calculus $dz/dx = px^{(p-1)} (k - x)^q - qx^p(k - x)^{(q-1)}$. If $dz/dx = 0$, then $px^{(p-1)} (k - x)^q = qx^p(k - x)^{(q-1)}$. Immediately x

$= 0$ and $x = k$ are solutions (having assumed $p \neq 1$ and $q \neq 1$). Dividing both sides of the equation by $x^{(p-1)}(k - x)^{(q-1)}$ yields $p(k - x) = qx$. Then $pk - px = qx$ so $x = pk/(p + q)$, the generalized form of the second solution in Case 1. Then z will be maximized by one of these values.

Either by repeating the process with $x = k - y$ or by substituting the solutions for x above directly into $x + y = k$, the solutions for y, $y = 0$, $y = k$, and $y = qk/(p + q)$ are easily obtained.

Also solved by Charles Traine, St. Francis College, Brooklyn, New York; Kenneth M. Wilke, Topeka, Kansas.

234. *Proposed by Pat LaFratta, Waukesha, Wisconsin.*

Prove that $[(n + 1)(2n + 1)]^n \geqq 6^n(n\,!)^2$ for any positive integer n.

Solution by Kenneth M. Wilke, Topeka, Kansas.

The desired result can be rewritten equivalently as follows:

$$\frac{[n(n + 1)(2n + 1)]^n}{(6n)^n} \geqq (n!)^2$$

or upon taking nth roots since n is a positive integer,

$$\frac{n(n + 1)(2n + 1)}{6n} \geqq \sqrt[n]{(n!)^2}$$

or

$$\frac{1}{n} \sum_{i=1}^{n} i^2 \geqq \left(\prod_{i=1}^{n} i^2 \right)^{\frac{1}{n}}.$$

But this last inequality states that the arithmetic mean of the first n squares is greater than or equal to the geometric mean of the first n squares. The truth of this statement is a direct result of the well known inequality: The arithmetic mean of n numbers is greater than or equal to the geometric mean of the same n numbers. Hence the original inequality is true for all positive integers n with equality only if $n = 1$.

Also solved by Kenneth Rosen, University of Michigan, Ann Arbor, Michigan.

235. *Proposed by R. S. Luthar, Waukesha, Wisconsin.*

For any positive reals x and y prove that the following inequality holds:

$xy(1/x + 1/y + 1)^5 \geqq 108(1/x + 1/y)$.

Solution by Kenneth Rosen, University of Michigan, Ann Arbor, Michigan.

Since the geometric mean of a set of positive reals is greater than or equal to their harmonic mean

$$\sqrt[3]{1 \cdot x \cdot y} \geqq \frac{3}{1/x + 1/y + 1}$$

This implies that $xy(1/x + 1/y + 1)^3 \geqq 27$. Equality holds if and only if $1/x = 1/y = 1$ or equivalently when $x = y = 1$.

Define $z = 1/x + 1/y$.

Now $(1 - z)^2 \geqq 0$,
$1 - 2z + z^2 \geqq 0$,
$1 + 2z + z^2 \geqq 4z$,
$(1 + z)^2 \geqq 4z$, and $(1 + 1/x + 1/y)^2$

$\geqq 4(1/x + 1/y)$, with equality if and only if $z = 1$ or when $1/x + 1/y = 1$. Hence, $xy(1/x + 1/y + 1)^5 \geqq 27 \cdot 4(1/x + 1/y) = 108(1/x + 1/y)$.

Note: In the 1970 Spring issue of THE PENTAGON the Editor failed to indicate that the solution to Problem 230 was by Kenneth M. Wilke, Topeka, Kansas. The problem was also solved by Karen Dowdy, Southern Methodist University, Dallas, Texas; Russell C. Mills, State University College, Oswego, New York; Alana Rohr, Kansas State Teachers College, Emporia, Kansas.

Augustinus (354-430 A.A.) made the first recorded statement that creation took six days because both God's creation and 6 are perfect numbers. —*Excursions into Mathematics*

The Mathematical Scrapbook

EDITED BY RICHARD LEE BARLOW

Readers are encouraged to submit Scrapbook material to the Scrapbook editor. Material will be used where possible and acknowledgment will be made in THE PENTAGON. If your chapter of Kappa Mu Epsilon would like to contribute the entire Scrapbook section as a chapter project, please contact the Scrapbook editor, Professor Richard L. Barlow, Kearney State College, Kearney, Nebraska.

Practically every mathematics student at one time or another has studied numbers written in a numeration system other than the usual Hindu-Arabic system of base ten. At that time it is noted that the position of the digit is important; that is, the system has place-value. For example,

$$3629_{10} = 3 \cdot 10^3 + 6 \cdot 10^2 + 2 \cdot 10^1 + 9 \cdot 10^0 .$$

It is noted that in base ten notation, one uses the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and only these ten symbols in writing any number. The ideas of zero and place-value are the main differences between the Hindu-Arabic system and the various numeration systems which preceded it. In addition, the Hindu-Arabic numeration system allows somewhat easier computation than do its predecessors. (Have you ever tried multiplying in Roman numerals?)

In working with bases other than ten, one notes that many of the properties of the base ten system also hold. For example, in base six one uses only the six symbols 0, 1, 2, 3, 4, 5 to write any number. The position of digits is also important. Consider

$$4203_6 = 4 \cdot 6^3 + 2 \cdot 6^2 + 0 \cdot 6^1 + 3 \cdot 6^0 = 939_{10} .$$

One will observe that the positions here all represent powers of the base six. A similar result is true in any base. The base may be larger than ten, say thirteen, where one will use thirteen symbols, possibly 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e, w where $t_{12} = 10_{10}$, $e_{12} = 11_{10}$, and $w_{12} = 12_{10}$. Hence, $2wt_{13} = 2 \cdot 13^2 + 12 \cdot 13^1 + 10 = 504_{10}$.

Nothing is usually said about bases which are fractional or negative. We shall consider these cases here with their properties. The place-value for a base b system, $b \neq 0$ are:

$$\cdots \quad \overline{}_{b^3} \quad \overline{}_{b^2} \quad \overline{}_{b^1} \quad \overline{}_{b^0} \quad \cdot \quad \overline{}_{b^{-1}} \quad \overline{}_{b^{-2}} \quad \overline{}_{b^{-3}} \quad \cdots.$$

For any non-zero number b, $b^0 = 1$ and hence each numeration system will have a units or ones position. We shall use the usual decimal point " . " to indicate the position of the units digit of a numeral.

The following examples illustrate this notation:

$$621.25_{10} = 6 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0 + 2 \cdot 10^{-1} + 5 \cdot 10^{-2},$$

and

$$\begin{aligned} 53.026_7 &= 5 \cdot 7^1 + 3 \cdot 7^0 + 0 \cdot 7^{-1} + 2 \cdot 7^{-2} + 6 \cdot 7^{-3} \\ &= 5 \cdot 7 + 3 \cdot 1 + 0 \cdot \frac{1}{7} + 2 \cdot \frac{1}{7^2} + 6 \cdot \frac{1}{7^3} \\ &= 35 + 3 + \frac{2}{49} + \frac{6}{343} \\ &= 38\ \frac{20}{343}\ . \end{aligned}$$

An unusual situation occurs when the base itself is fractional. Consider a base of one-fourth whose place values are indicated below:

$$\cdots \quad \overline{}_{(1/4)^2} \quad \overline{}_{(1/4)^1} \quad \overline{}_{(1/4)^0} \quad \cdot \quad \overline{}_{(1/4)^{-1}} \quad \overline{}_{(1/4)^{-2}} \quad \overline{}_{(1/4)^{-3}} \quad \cdots$$

or

$$\cdots \quad \overline{}_{1/16} \quad \overline{}_{1/4} \quad \overline{}_{1} \quad \cdot \quad \overline{}_{4} \quad \overline{}_{16} \quad \overline{}_{64} \quad \cdots.$$

It first appears strange that the place values of the digits to the right of the decimal point could be larger than those to the left, but this must be the case to keep this system consistent with the cases previously considered. We will also note that this numeration system in base 1/4 requires only the four symbols 0, 1, 2, 3 to write any number.

Now consider the base 4/3 numeration system. The place values are indicated below:

$$\cdots \quad \overline{}_{(4/3)^2} \quad \overline{}_{(4/3)^1} \quad \overline{}_{(4/3)^0} \quad \cdot \quad \overline{}_{(4/3)^{-1}} \quad \overline{}_{(4/3)^{-2}} \quad \overline{}_{(4/3)^{-3}} \quad \cdots$$

or

$$\cdots \underline{}_{16/9}\ \underline{}_{4/3}\ \underline{}_{1}\ .\ \underline{}_{3/4}\ \underline{}_{9/16}\ \underline{}_{27/64} \cdots.$$

This system appears to meet more of our expectations of a numeration system since the digits to the left of the decimal point have a greater place value than those to the right. But this numeration system will require four symbols (not three as one might first suspect), namely 0, 1, 2, 3. As a general rule, the number of symbols required to represent any number in the fractional base a/b, $(a, b) = 1$, $b \neq 0$ is the maximum of a and b. For our base 4/3, the maximum of 4 and 3 is 4, so we must use four symbols. This is consistent with the previous work since 1/4 required four symbols (the maximum of 1 and 4), base 13/1 requires 13 symbols (the maximum of 13 and 1), etc. In base 4/3, if we did not use all four symbols 0, 1, 2, 3 then we could not write the number 4_{10} in base 4/3 notation since $4_{10} = 30_{4/3}$. Can you verify this?

If one wishes a negative base, somewhat unusual results occur. The number of symbols necessary to write any number in a negative base system is the same as it would be if the base were a positive number, and so we shall ignore the sign of the base. For example, $-4/3$ requires four symbols as did the base 4/3.

Consider the place-value for a base -5 numeration system:

$$\cdots \underline{}_{(-5)^2}\ \underline{}_{(-5)^1}\ \underline{}_{(-5)^0}\ .\ \underline{}_{(-5)^{-1}}\ \underline{}_{(-5)^{-2}}\ \underline{}_{(-5)^{-3}} \cdots$$

or

$$\cdots \underline{}_{25}\ \underline{}_{-5}\ \underline{}_{1}\ .\ \underline{}_{-1/5}\ \underline{}_{1/25}\ \underline{}_{-1/25} \cdots.$$

One will note that the positions alternate in sign.

Hence,

$$\begin{aligned} 321.413_{-5} &= 3\cdot 25 + 2\cdot(-5) + 1\cdot 1 + 4\cdot(-1/5) \\ &\qquad + 1\cdot(1/25) + 3(-1/125) \\ &= 66 - 98/125 \\ &= 65\ 27/125 \end{aligned}$$

The basic operations may be easily performed in bases other than ten. For example, in addition one finds

$$\begin{array}{r} 621.23_7 \\ +\ \ 50.65_7 \\ \hline 1002.21_7 \end{array}$$

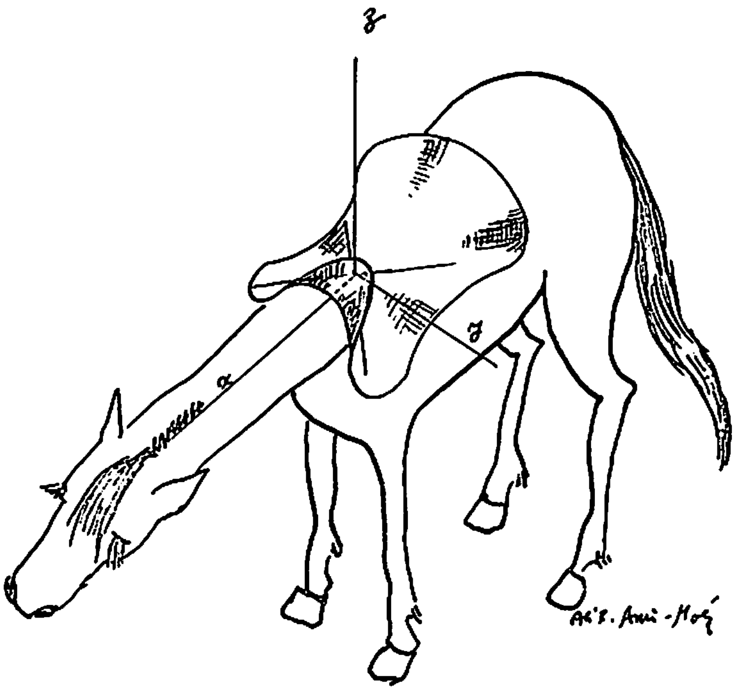
by recalling that each digit represents 7 of the groups preceding it on the right.

Similarly,

$$\begin{array}{r} 321.22_{-5} \\ +\ 240.34_{-5} \\ \hline 411.41_{-5} \end{array}$$

Can you prove this?

Editor's note: The following was submitted by Professor Moez of Texas Technological College.



A portion of $z = ax^2 - by^2$,

$a > 0$, $b > 0$.

The Book Shelf

Edited by James Bidwell

This department of *The Pentagon* brings to the attention of its readers published books (both old and new) which are of a common nature to all students of mathematics. Preference will be given to those books written in English or to English translations. Books to be reviewed should be sent to Dr. James Bidwell, Central Michigan University, Mount Pleasant, Michigan 48858.

Introduction to Mathematical Statistics, 3rd edition, Robert V. Hogg and Allen T. Craig, MacMillan, New York, 1970. X + 415 pp., $20.

If you liked the first two editions, you will love the third. It is the best available (undergraduate-level) textbook for preparing students for graduate study in mathematical statistics and for mathematics majors who wish to study theoretical statistics. This edition differs from its predecessors mainly in that it contains an expanded (and excellent) section on random variables and a chapter on nonparametric methods.

On the other hand, this book might not be suitable for students interested primarily in the applications of statistics. As a student, I studied "statistics" from the first edition, but I did not gain an appreciation for how statistics is used. Perhaps a separate laboratory course in which "real-life" problems are discussed would serve to give students a feeling for the applications of statistics.

Edward M. Bolger
Miami University

Modern Elementary Mathematics, Anne E. Kenyon, Prentice-Hall, Englewood Cliffs, N.J., 1969, 365 pp., $8.95.

The author states that "this textbook is intended for the elementary credential candidate, to give him some of the background necessary to understand and teach from the new elementary school mathematics textbooks. It is also considered appropriate for the liberal arts student . . ." The reviewer approached the text from the viewpoint of the former rather than the latter. The scope is not restricted to arithmetic, geometry, or algebra *a la* CUPM, but rather includes both algebra and geometry.

The first three chapters deal with elementary set theory and logic. Set theory is considered first and the algebra of propositions follows. Many texts consider these topics in reverse order. The logic material is well written, the exercises are interesting; however, it is this reviewer's opinion that this section has little relevance to the work of the elementary teacher unless truth tables are tied to inference patterns. The remainder of the text is independent of this section.

Chapters 5, 6, and 7, which treat respectively whole numbers, integers, and rational numbers, are very complete. The approach is to define the operations for the system and show that the usual properties follow. In some texts the properties are assumed and the "how-to" of performing the operations is proved. Either approach is mathematically sound, nevertheless, it is the reviewer's opinion that the author's approach is better for the intended audience.

Particular attention is given to the repeated additions interpretation of multiplication. This interpretation is most useful in elementary school mathematics but is often ignored by writers of texts for these teachers. Multiplication is also defined in terms of the Cartesian product. An excellent job is done in relating the operations to the number line.

Chapter 8 briefly considers the real numbers. Chapters 9, 10, 11, 12, and 13 cover most of the content of a contemporary high school geometry course. The material in Chapter 9 is informal; the remaining material is formal. The geometry terminology, symbolism and content are similar to that of SMSG Geometry. Congruence for segments and angles is defined in terms of the ruler and protractor postulates, respectively. Transformational geometry (reflections, translations and rotations) which is beginning to appear informally in elementary school materials is not mentioned.

The selection of exercises is good. Provision is made in the exercises for adequate practice of concepts and skills. In addition, most exercises contain problems which are interesting and thought-provoking. Answers are provided for the odd numbered exercises.

All things considered, this is a "readable" and "teachable" text for use in Level I courses. The development of the rational numbers is especially good. The book is worthy of consideration for a course for future elementary teachers.

William A. Miller
Central Michigan University

Elementary Linear Algebra, Bernard Kolman, The Macmillan Company, New York, 1970. 255 pages. $8.95.

In this text the author has included more than the usual introduction to linear algebra. His purpose is to provide sophomores who have had a year of calculus with an opportunity to develop a facility with abstract ideas. The computational aspects of the subject are accompanied by a gradual introduction into the postulational character of mathematics.

After a preliminary chapter, which contains a clear and precise treatment of the basic language of sets and the mathematics of functions as used throughout the text, matrices and matrix operations are associated with the solution of linear equations. In the chapter treating real vector spaces effective use is made of the matrix material. The theory of linear transformations flows naturally from the elementary geometric one of rotations, projections and reflections. Determinants and their properties receive attention mainly for the role they play in the "study of the properties of a linear transformation mapping of vector space V into itself." Included in the chapter on eigenvalues and eigenvectors are Euclidean spaces, the Gram-Schmidt process, real quadratic forms and the diagonalization problem for symmetric matrices.

An unusual feature is the inclusion of a chapter dealing with the usefulness of linear algebra in the solution of differential equations. There are, also, frequent references to computer implementation of the techniques developed.

Sufficient illustrative examples are given. Throughout the text the proofs are understandable and are constructed in such a way as to help the student formulate his own. The exercises are ample, providing opportunity for application of concepts and for the development of ease with independent thinking. Answers to selected exercises are included; a separate solutions manual accompanies the text.

This text is recommended for its sound and complete introduction to the concepts of linear algebra.

Sister Marie Augustine
College of Notre Dame of Maryland

Modern Elementary Algebra for College Students, Vivian Shaw Groza and Susanne Shelley, Holt, Rinehart and Winston, New York, 1969, xi + 418 pp.

The goal of the authors was to provide a text for a "modern" course covering the standard topics of elementary algebra and thereby providing the background necessary fot studying intermediate algebra and the subsequent courses of trigonometry and calculus. They have assumed as prerequisite a background in set concepts and elementary mathematical logic, although introductions to sets and logic are placed in the Appendix so that any instructor who feels a need for the above concepts may introduce them at his discretion.

The text covers the standard topics of elementary algebra including a treatment of the real number system; linear equations and their solutions; relations, functions and graphing; solving a system of two linear equations by various methods; polynomials (including multiplying and factoring); fractions; radicals, exponents, and complex numbers; and quadratic equations.

The book is well designed and very readable. Good use is made of a second color (blue), especially to set off items of importance. In general, the numerous exercises appear reasonable, allowing for a selection of problems appropriate to a range of abilities and applicable to the material which preceded the exercises. Answers are provided to almost all of the problems and a check of a small sample revealed no incorrect answers. Theorems and definitions are set off in bold type and the tables and graphs are easy to read. Each chapter ends with a summary of concepts and a list of properties and/or theorems plus a chapter review. Beginning with Chapter 2 there is also a set of exercises for a comprehensive review of all previous material studied.

The most impressive features of the book were the use of interesting historical notes throughout the text to supply additional information and interest to the material under consideration, the many verbal problems in almost every chapter with diagrams and tables as aids in setting up equations, and the introduction of partial fractions in the unit on fractions as preparation for the subsequent calculus courses.

One criticism might be offered. On pages 62 and 239 a statement of the Axiom of Completeness is made as follows, "Each point on the number line corresponds to exactly one real number and each real number corresponds to exactly one point on the number line". In the opinion of the reviewer, this is not the Axiom of Completeness and cannot be shown equivalent to it.

The overall impression was that the text should be considered seriously for a course in basic algebra from the modern viewpoint for colleges, junior colleges and for the more mathematically talented high school students.

Ronald D. Dettmers
Wisconsin State University
Whitewater, Wisconsin

College Algebra and Trigonometry by Steven J. Bryant, Jack Karush, Leon Mower, and Daniel Saltz, Goodyear Publishing Company, Inc., Pacific Palisades, California, 1970, 477 pp., $10.95.

The book does a fine job of covering topics traditional to college algebra and/or trigonometry courses. The problem lists are adequate and offer ample opportunity to gain algebraic and trigonometric skills needed in the calculus. Also included are challenging problems for the good student. Answers to the odd problems are included in the back of the book.

The material is introduced in the postulate, definition, and theorem manner, with proofs supplied for most theorems. Although set terminology is sprinkled throughout the basic material, the subject of sets is not formally examined until Part Five of the book.

Part Four, which covers vectors, analytic geometry, and matrices, offers only a brief treatment of these topics and seems to present only a brief overview to material better left to separate courses.

Ramon L. Avila
Ball State University

Elements of Number Theory, Anthony J. Pettofrezzo and Donald R. Byrkit, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1970, 256 pp., $7.95.

This book has been criticized by one instructor of a number theory course as incomplete, wordy, and given to assuming results without proof. All of which is true, but unjust. The authors obviously intended to write an introduction to the basic facts and methods of number theory which could be read by an interested and capable high school student. Putting it positively, this reviewer feels that this book is as complete, as direct, and as full of proofs as is consistent with its emphasis on the elements of number theory. The subjects treated include: 1. the divisibility properties of the integers,

2. the theory of congruences, and 3. continued fractions. Several topics such as quadratic reciprocity and algebraic integers are not discussed. In addition the authors have devoted a few pages to well chosen examples which suggest the method of proof for the theorem which follows. This feature alone increases the usefulness of the book many fold, because it should permit many students to read the book on their own without the assistance of an instructor. It is also true that certain results are assumed without proof. On the whole, though, this is the sort of thing that will be obvious to the student of mathematics in a few years anyway. For example, the authors accept without proof the fact that a polynomial congruence of degree n in one variable with a prime modulus has at most n incongruent solutions. Another feature that deserves favorable comment is the introductory first chapter which presents the basic properties of the integers and gives the reader some practice with induction and summation and product notation.

Here, then, is a readable book that says some interesting things about numbers and shows the student how to prove some of these things. As such, it should be a good textbook for a freshman or sophomore course in number theory or perhaps for a special course for teachers. Failing this, it would simply be a good book for a student to read on his own.

Charles Holmes
Miami University

Intermediate Analysis, M. S. Ramanujan and Edward S. Thomas, The Macmillan Co., New York, 1970, 165 pp., $7.95.

This book is designed to serve as an introduction to post-calculus analysis, where the emphasis is to be on proofs and rigor, rather than on formulas. The first half of the book treats sets, functions, relations, and cardinality carefully and slowly. The reader is introduced to the basic ideas and is made to use them in the many exercises.

Next there is a chapter on the real numbers, including least upper bounds, convergence, Cauchy sequences, etc. (The construction of the reals by Dedekind cuts is in an appendix.) The last two chapters, on infinite series and power series, are the type found in a standard calculus text. These are the only chapters where any real use is made of a calculus prerequisite.

The first five chapters would make an excellent one quarter course for students going on to Advanced Calculus. The chapters on series seem to have been borrowed from a calculus course in order to fill out a semester's work.

E. R. Deal
Colorado State University

MINIREVIEWS

Calculus and Analytic Geometry, Douglas F. Riddle, Wadsworth Publishing Co., Belmont, California, 1970, 748 pp., $13.

An attempt to avoid the usual rigorous epsilon-delta approach to limits. The book uses geometric definition for limit and continuity. More analytic geometry and curve sketching than usual. Rigor comes late in book. Last chapters are on partial derivatives and multiple integrals.

Fields and Functions. A Course in Precalculus Mathematics, C. W. Bedford, E. E. Hammond, Jr., G. W. Best, J. R. Lux, The Macmillan Co., Collier-Macmillan Ltd., London, 1970, 639 pp., $9.

This text was developed at Phillips Academy. It covers the usual material, including a chapter on complex numbers. The book relies on the rigor of logic, definition, and structure. Some intuitive examples are included. Could be used by superior secondary students. Selected answers and tables are included.

Applications of College Mathematics, A. William Gray and Otis M. Ulm, Glencoe Press, Beverly Hills, California, 1970, 360 pp.

This is a nontechnical text for students planning careers in business, education, or the social sciences. Material on probability, statistics, computers, matrices, and linear programming are featured. Interest tables and selected answers to exercises are included. A standard high school program is a required prerequisite.

Basic Mathematics Review, James A. Cooley and Ralph Mansfield, The Macmillan Co., New York, 1970, 414 pp., $6.95.

This is a consumable workbook for reviewing arithmetic and algebra in college. Each lesson has exposition, examples, and problems. Space for student work included. The material is also available in separate volumes for arithmetic and algebra. The level is suitable for high school evening classes.

Fundamentals of Modern Mathematics, A. J. Jackowski and J. B. Sbrega, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1970, 420 pp., $9.50.

This book is designed for prospective elementary and junior high school teachers. The authors suggest two years of high school mathematics as a prerequisite. The usual material is covered, including four chapters on geometry and measuration. The chapter on real numbers is brief. The chapter on rational numbers is quite structure oriented. This is a content book with no methodology included. Some abstract algebra is interspersed in the book.

Fundamentals of Algebra, Dale W. Lick, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1970, 533 pp., $9.95.

This is a precalculus text without analytic geometry. The stress is on the usual topics. Two chapters review elementary algebra. There is a chapter on linear algebra and probability. There is a college algebra flavor to the book. Half of the exercises have answers in the appendix.

Mathematics, rightly viewed, possesses not only truth but supreme beauty—a beauty cold and austere . . . yet sublimely pure. . . . Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home. . . .

—Bertrand Russell

Kappa Mu Epsilon News

EDITED BY EDDIE W. ROBINSON, *Historian*

REGIONAL CONVENTION

April 25, 1970

Missouri Beta, Central Missouri State College

ARKANSAS - COLORADO - IOWA - KANSAS - MISSOURI - NEBRASKA - OKLAHOMA

Program: First General Session, Mary Lou Russell, Presiding

Welcome, Burnell Haldiman

"Egyptian Fractions," by Barbara Shappard, Washburn University.

"On Generating Functions in Pascal's Triangle," by Susan Jarchow, Mount St. Scholastica College.

"Trisection of an Angle," by Kathy Peterson, Kansas State College of Pittsburg.

"For All Rational Numbers," by Melvin Watson, Kansas State College of Pittsburg.

"Square Circles," by Michael Brandley, Kansas State Teachers College.

Luncheon Address by Dr. L. T. Sheflett, Southwest Missouri State College.

Second General Session, Mary Lou Russell, Presiding.

"A Statistical Study of the Draft Lottery," by Don Page, William Jewell College.

"Use of the LaPlace Transformation in Characterization of a Continuous Random Variable," Ozdagan Yilmoz, University of Missouri at Rolla.

"Is There A Difference between Algebraic and Geometric Groups," by Mary Graney, Mount St. Scholastica College.

Presentation of Awards—Dr. H. Keith Stumpff, Central Missouri State College.

Other Papers Submitted:

"Some Properties of Door Spaces," by Craig Bainbridge, Morningside College.

"Your Answer is 1099," by James Harlin, Kansas State College of Pittsburg.

"Modular Pseudo-Valuations," by Ron Oliver, Morningside College.

"That's Life," by Tony Soukup, Kearney State College.

"N-Dimensional Coordinate Systems," by Jerel Williams, Kansas State Teachers College.

CHAPTER NEWS

Illinois Beta, Eastern Illinois University

Eight new members were initiated on November 6, 1969, and twenty-six members were initiated on May 17, 1970. This makes a total of 641 members initiated during the thirty-six years of the chapter's existence. The formal initiation ceremony and the business meeting were followed by a banquet in the University Union honoring all those initiated during the past year. The speaker was Larry Johansen, who is presently doing graduate work and teaching in the Mathematics Department at Eastern. He plans to start teaching in Kishwaukee Junior College at Malta, Illinois, in the fall of 1970.

In honor of scholarship and potential in mathematics the following awards were presented at the banquet: Mr. Claire Krukenberg presented the Freshman Award to Marcia Meers and Keith Lyons; Mrs. Helen Van Deventer presented the K.M.E. Calculus Award to Leonard Storm; Mr. Charles Pettypool presented the O'Brian Scholarship to Sandy Roediger and Peggy Ping; and Dean Lawrence Ringenberg presented the Taylor Award to John McJunkin. The recipients and their parents were guests of the chapter and the Mathematics Department.

New officers of the chapter for 1970-71 are:

Roy McKittrick—President
Tim Burke—Vice President
Bev Dilliner—Secretary-Treasurer
Mr. Larry Williams—Adviser
Mrs. Ruth Queary—Corresponding Secretary.

Indiana Delta, University of Evansville

New officers for Indiana Delta Chapter are:

Wayne Roell—President
Paula Perlitz—Vice President
Lana Turner—Secretary.

Missouri Alpha, Southwest Missouri State College

The annual picnic was held in May where David Tartar was presented the Kappa Mu Epsilon Merit Award. New officers for 1970-71 are:

Peggy Turnbough—President
Michael Ridlen—Vice President
Peggy Stuckmeyer—Secretary
Linda Allgeier—Treasurer
John Kubicek—Corresponding Secretary.

Ten members of the chapter attended the Regional Convention in Warrensburg.

Pennsylvania Epsilon, Kutztown State College

One hundred members and friends of the Pennsylvania Epsilon Chapter of Kappa Mu Epsilon were present at the Spring initiation and dinner held on Friday, April 24, 1970, in the College Red Dining Hall. After the initiation of eleven members, the program was suddenly changed into a surprise program honoring the sponsor and corresponding secretary, Dr. J. Dwight Daugherty.

Among the distinguished people present who paid tribute to Dr. Daugherty was the President of the College, Dr. Lawrence M. Stratton. He spoke glowingly about the honoree's outstanding work and contributions during the past ten and one-half years. Also present and honoring the guest of honor was Mr. Walter Hollenbach, a former president of the New York Schoolmasters Club, of which Dr. Daugherty is a member of Board of Governors. The department chairman and each student officer of the chapter expressed his appreciation to Dr. and Mrs. Daugherty for their many contributions to the success of Pennsylvania Epsilon. Several letters and telegrams from nationally known professional friends of Dr. Daugherty were read.

Dr. Daugherty accepted the gifts, which consisted of a large plaque appropriately engraved and a fine poem suitably framed and written by the vice president, Miss Rosemarie DiSante. He expressed his thanks for the surprises and most of all for the many courtesies, kindnesses, and enthusiastic support of the Pennsylvania Epsilon

Chapter. He pleaded for greater student effort in attempting to get their names in print by carefully writing and submitting to the editors of magazines, including *The Pentagon,* the fine talks that they will give during the coming year. President David Zerbe presided at the meeting.

The Kutztown State College Mathematics Society was established by Dr. Daugherty in December, 1960, to meet the need of greater student participation in talking and writing mathematics. The Society prospered and became the Pennsylvania Epsilon Chapter of Kappa Mu Epsilon on April 3, 1965. Meetings are held monthly and the programs consist of the presentation of student prepared talks on mathematical topics. In addition two initiation dinners are held annually, one in the fall and the other in the spring. At these meetings outstanding mathematicians are invited to speak. Pennsylvania Epsilon membership is 136, including one honorary member.

The Pennsylvania Epsilon Chapter presents an annual Award in Mathematics at the Spring Commencement. This award consists of an engraved certificate with the name of the winner embossed and twenty dollars in cash. The winner's name and the Award Announcement appear on the commencement program. It is presented to the senior graduating in the current academic year who has attained the highest honors during the study of four years of mathematics at Kutztown State College. It is properly signed by the Adviser and Corresponding Secretary and is one of the chapter's outstanding contributions to the college. It has been presented annually during the last four academic years.

Texas Epsilon, North Texas State University

Fifteen active members, two pledges.

Stuart Anderson—President
Joyce McFarland—Vice President
Laura Fisher—Secretary
Carol Congleton—Treasurer
Dr. Melvin Hagan—Corresponding Secretary
Dr. John Allen—Faculty Sponsor.

The February meeting featured Dr. Parrish who spoke on "Infinite Mathematics." "Number Theory" was the topic of the program in March and Robert Manning and Jerry Walden presented the April program on computers. Other meetings were a Christmas Party and the annual picnic.

LIST OF CORRESPONDING SECRETARIES

During the past year a number of chapters have selected new corresponding secretaries. Since some members may wish to contact these individuals for approval for official jewelry or other society matters, a current list is presented.

Active Chapter	Institution	Corresponding Secretary
Alabama Beta	Florence State University	Elizabeth Woolridge
Alabama Gamma	Univ. of Montevallo	Dr. Angela Hernandez
Alabama Epsilon	Huntingdon College	Rex C. Jones
Arkansas Alpha	Arkansas State University	J. L. Linnsteadter
California Gamma	Calif. State Polytechnic College	Dr. George R. Mach
California Delta	Calif. State Polytechnic College, Kellogg Voorhis Campus	Albert Konigsberg
Colorado Alpha	Colorado State Univ.	Dr. H. Howard Frisinger
Connecticut Alpha	Southern Connecticut State College	Ray Erwin Sparks
Florida Alpha	Stetson University	Emmett S. Ashcraft
Illinois Alpha	Illinois State University	Clyde T. McCormick
Illinois Beta	Eastern Illinois Univ.	Ruth Queary
Illinois Gamma	Chicago State College	Thomas P. Roelle
Illinois Delta	College of St. Francis	Sister Loretta Tures
Illinois Epsilon	North Park College	Alice Iverson
Illinois Zeta	Rosary College	Sister Nona Mary Allard
Illinois Eta	Western Illinois Univ.	Professor Kent Harris
Indiana Alpha	Manchester College	David L. Neuhouser
Indiana Beta	Butler University	Dr. Barry Lobb
Indiana Gamma	Anderson College	Paul Saltzman
Indiana Delta	University of Evansville	Gene Bennett
Iowa Alpha	University of Northern Iowa	John S. Cross
Iowa Beta	Drake University	Joseph Hoffert
Iowa Gamma	Morningside College	Elsie Muller
Kansas Alpha	Kansas State College of Pittsburg	Dr. Harold L. Thomas
Kansas Beta	Kansas State Teachers College	Charles B. Tucker
Kansas Gamma	Mount St. Scholastica College	Sister Jo Ann Fellin
Kansas Delta	Washburn University of Topeka	Margaret Martinson
Kansas Epsilon	Fort Hays Kansas State College	Eugene Etter
Maryland Alpha	College of Notre Dame of Maryland	Sister Marie Augustine
Maryland Beta	Western Maryland College	James E. Lightner

Massachusetts Alpha	Assumption College	Richard Houde
Michigan Alpha	Albion College	W. Keith Moore
Michigan Beta	Central Michigan Univ.	Dean Hinshaw
Mississippi Alpha	Mississippi State College for Women	Donald King
Mississippi Beta	Mississippi State University	Claris Marie Armstrong
Mississippi Gamma	University of Southern Mississippi	Jack V. Munn
Missouri Alpha	Southwest Missouri State College	John Kubicek
Missouri Beta	Central Missouri State College	Dr. Homer Hampton
Missouri Gamma	William Jewell College	Sherman Sherrick
Missouri Epsilon	Central Methodist College	R. C. Carnett
Missouri Zeta	The University of Missouri at Rolla	Dr. Roy Rakestraw
Missouri Eta	Northeast Missouri State College	Joe Flowers
Nebraska Alpha	Wayne State College	Fred A. Webber
Nebraska Beta	Kearney State College	Richard Barlow
Nebraska Gamma	Chadron State College	Lenora D. Briggs
New Jersey Alpha	Upsala College	Don Lintvedt
New Jersey Beta	Montclair State College	Dr. Evan Maletsky
New Mexico Alpha	University of New Mexico	Merle Mitchell
New York Alpha	Hofstra University	Alexander Weiner
New York Gamma	State University College	John W. Walcott
New York Delta	Utica College of Syracuse University	Thomas J. Burke
New York Epsilon	Ladycliff College	Sister Clare Bernadette
New York Zeta	Colgate University	Theodore Frutiger
New York Eta	Niagara University	Robert L. Bailey
New York Theta	St. Francis College	Donald R. Coscia
North Carolina Alpha	Wake Forest University	Dr. John V. Baxley
Ohio Alpha	Bowling Greene State University	Harry Mathias
Ohio Gamma	Baldwin-Wallace College	Robert E. Schlea
Ohio Epsilon	Marietta College	George W. Trickey
Ohio Zeta	Muskingum College	James L. Smith
Oklahoma Alpha	Northeastern State College	Raymond Carpenter
Oklahoma Beta	University of Tulsa	Dr. T. W. Cairns
Pennsylvania Alpha	Westminster College	J. Miller Peck
Pennsylvania Beta	LaSalle College	Brother Damian Connelly
Pennsylvania Gamma	Waynesburg College	Lester T. Moston
Pennsylvania Delta	Marywood College	Marie Loftus
Pennsylvania Epsilon	Kutztown State College	Irving Hollingshead
Pennsylvania Zeta	Indiana University of Pennsylvania	Ida Z. Arms
Pennsylvania Eta	Grove City College	Marvin C. Henry
Pennsylvania Theta	Susquehanna University	Carol N. Jensen
Pennsylvania Iota	Shippensburg State College	John S. Mowbray, Jr.

South Carolina Alpha	Coker College	G. C. Metz
South Carolina Beta	South Carolina State Col.	Frank Staley, Jr.
Tennessee Alpha	Tennessee Technological University	Evelyn Brown
Tennessee Beta	East Tennessee State University	Lora D. McCormick
Tennessee Gamma	Union University	Richard Dehn
Texas Alpha	Texas Tech University	Robert Moreland
Texas Beta	Southern Methodist Univ.	C. J. Pipes
Texas Gamma	Texas Women's University	Ronald V. McPherson
Texas Epsilon	North Texas State Univ.	Melvin R. Hagan
Texas Zeta	Tarleton State College	Timothy Lee Flinn
Virginia Alpha	Virginia State College	Emma B. Smith
Virginia Beta	Radford College	Ruth Ann Poe
Wisconsin Alpha	Mount Mary College	Sister Mary Petronia
Wisconsin Beta	Wisconsin State University	Lyle Oleson

Eighteenth Biennial Convention

April 2-3, 1971

The eighteenth biennial convention of Kappa Mu Epsilon will be hosted by the Pennsylvania Zeta chapter and will be held on the campus of Indiana University of Pennsylvania on April 2-3, 1971. Students are encouraged to prepare and submit papers for presentation at the convention. Complete directions for the submission of papers are found on page 35 of this issue of *The Pentagon.*

All chapters are encouraged to plan early for as large a delegation of students and faculty as possible.

George R. Mach
National President