THE PENTAGON

Volume XXIX **Spring, 1970** **Number 2**

CONTENTS

National Officers

Kappa Mu Epsilon, mathematics honor society, was founded in 1931. The object of the fraternity is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; to provide a society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; to disseminate the knowledge of mathematics and to familiarize the members with the advances being made in mathematics. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Finite Metric Spaces*

CHARLES L. BREINDEL
Student, Indiana University of Pennsylvania, Indiana

The undergraduate student, as he begins his studies of mathematics, will become very involved with highly abstract geometry and topology in n-dimensional space. The student, as a result, has to rely upon his "geometric intuition" to aid his understanding of many concepts.

This paper seeks to demonstrate how misleading such an adherence to intuition can be. Through an exposition of metric spaces, the author wishes, not only to explain this basic concept, but also to arrive at a seemingly contradictory property which defies the undergraduate's intuitive reasoning process.

Classical analysis became so complex and varied during the course of its development that it eventually became impossible for the best mathematicians to work in all of its facets. So in the last century mathematicians attempted to discover the fundamental principles on which all analysis rests. Associated with the movement were many outstanding names: Cantor, Hilbert, Lebesque, Riemann, and Weierstrass. The movement had much to do with the rise to prominence of topology, modern algebra, and the theory of measure and integration. Out of the reevaluation and generalizations of classical analysis came modern analysis.

As modern analysis developed many major theorems were given simpler proofs in more general settings. Mathematicians hoped that, with the simplification and clarification of analysis, emphasis would be moved to the underlying theory, rather than the less important, often superfluous, detail of classical analysis.

Since analysis is principally concerned with limit processes and continuity, mathematicians involved in these ideas soon found themselves studying and generalizing two elementary concepts:

(1) convergent sequences of real or complex numbers; and

(2) continuous functions of a real or complex variable. The

*A paper presented at the 1969 National Convention of KME at Cedar Falls, Iowa, May 1-2.

important thing to realize is that each of these concepts is dependent for its meaning on the idea of absolute value of the difference between two real or complex numbers. Note also that the absolute value is the distance between the numbers when they are considered points on the real line, or in the complex plane. Consequently, it has been found very suitable to have available a notion of distance which can be applied to the elements of abstract sets. The metric space is simply that: a non-empty set together with a concept of distance which is applicable to the treatment of convergent sequences in the set and continuous functions on the set.

With this motivation, the metric space is defined.

DEFINITION 1. *A set X, whose elements will be called points, is said to be a metric space, if, with any two distinct points p and q, there is associated a real number $d(p, q)$ called the distance from p to q such that:*

i. $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$

ii. $d(p, q) = d(q, p)$

iii. $d(p, q) \leq d(p, r) + d(r, q)$ *for all r in X.*

A metric space consists of two objects: a non-empty set X and a metric d on X, often symbolically represented (X, d) for convenience. Some of the important examples of metric spaces are the Euclidean spaces E^k, especially E^1 (the real line) and E^2 (the complex plane), where distance is defined by:

$$d(p, q) = |p - q| \quad p, q \text{ in } E^k .$$

The conditions of the above definition are met by this statement.

Every subset X_i of a metric space X is also a metric space, since if conditions *i*, *ii*, and *iii* of the definition are met for p, q, r in X, they also hold if we restrict p, q, and r to lie in some X_i of X. Therefore, every subset of a Euclidean space is a metric space.

There are many kinds of metric spaces, some of which play important roles in geometry and analysis. In this paper, the discussion will be devoted to finite metric spaces, with two examples developed, and ultimately a very novel property illustrated.

The first example, though somewhat trivial, will illustrate very simply the concept of metric spaces.

Example 1. Let X be an arbitrary non-empty set. Define the metric d as follows:

$$d(x, y) = 1 \quad \text{if } x \neq y$$
$$d(x, y) = 0 \quad \text{if } x = y\,.$$

Is this a metric space; that is, are all three parts of the definition satisfied? Clearly *i* is satisfied since $d(x, y) > 0$ when $x \neq y$, and $d(x, y) = 0$ when $x = y$.

Since $d(x, y) = 1 = d(y, x)$ *if* $x \neq y$ and
$d(x, y) = 0 = d(y, x)$ if $x = y$, part *ii*, is true also.

Finally, for x, y, and z in X,
$d(x, y) \leq d(x, z) + d(z, y)$, as stated in *iii* of the definition.

For if $d(x, y)$ equals 0, the condition holds no matter what the value on the right-hand side of *iii*. The right side will either be 0, 1, or 2, and, in any event, will be greater than or equal to 0. If $d(x, y)$ equals 1, then $x \neq y$ and either $x \neq z$ or $z \neq y$. For if

(1) $x = z$, then $y \neq z$ since $x \neq y$ and $d(x, z) = 0$, $d(y, z) = 1$, and *iii* is satisfied.

(2) $y = z$, then $x \neq z$ since $x \neq y$ and $d(x, z) = 1$, $d(y, z) = 0$, and *iii* is satisfied.

(3) $x \neq y \neq z$, then $d(x, z) = 1$, $d(z, y) = 1$, and again *iii* is satisfied.

(X, d) is thus a metric space.

Example 2. Consider a non-empty finite set $X = \{(0, 0)$ $(0, 4)(3, 0)(3, 4)\}$, in E^2 space, where, letting the points be a_1, a_2, a_3, a_4 respectively,

$$d(a_1, a_2) = 4$$
$$d(a_1, a_3) = 3$$
$$d(a_1, a_4) = 5$$
$$d(a_2, a_3) = 5$$
$$d(a_2, a_4) = 3$$
$$d(a_3, a_4) = 4\,.$$

Distance in this metric space is defined in the usual geometric sense, which is, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Now for any two points of X, the distance is equal to 3, 4, or 5. Thus, the distance is always greater than 0. Also,

$$\begin{aligned}
d(a_1, a_2) &= \sqrt{0^2 + 4^2} = \sqrt{4^2 + 0^2} = d(a_2, a_1)\\
d(a_1, a_3) &= \sqrt{3^2 + 0^2} = \sqrt{0^2 + 3^2} = d(a_3, a_1)\\
d(a_1, a_4) &= \sqrt{3^2 + 4^2} = \sqrt{4^2 + 3^2} = d(a_4, a_1)\\
d(a_2, a_3) &= \sqrt{3^2 + 3^2} = \sqrt{3^2 + 3^2} = d(a_3, a_2)\\
d(a_2, a_4) &= \sqrt{3^2 + 0^2} = \sqrt{0^2 + 3^2} = d(a_4, a_2)\\
d(a_3, a_4) &= \sqrt{0^2 + 4^2} = \sqrt{4^2 + 0^2} = d(a_4, a_3) \;.
\end{aligned}$$

And finally, it can be verified easily that for all i, j, k,

$$d(a_i, a_j) \leq d(a_i, a_k) + d(a_k, a_j)\;.$$

Therefore, again this is a metric space, satisfying the definition.

Now before continuing the discussion of metric spaces, there are four terms which need to be defined so that the later statements may be clear and explicit. Though the first two terms, a neighborhood of a point, and the complement of a set, are obviously clear, and are not primary for the development of this paper, it is necessary to understand their meanings so that the precise definitions can be given for open and closed sets.

DEFINITION 2. *With X a metric space, and x a point in X, a neighborhood of x is a set of the form*

$$\{t \mid t \text{ is in } X \text{ and } d(x, t) < r\}$$

for some radius r greater than zero. In other words, a neighborhood around a point x is the set of all points contained in a disc or sphere around x, with center at x, and a radius not equal to zero.

DEFINITION 3. *The complement of a set X, where X is a subset of the universal set, is the set* $x^c = \{x \mid x$ *is in the universal set, and x not in X}: that is, the set of all points not contained in X.*

Now, thus motivated, definitions can be given for open sets and closed sets.

DEFINITION 4. *Let X be a metric space and* X_i *a subset of X. Then* X_i *is called open in X (or simply open, if no confusion is likely) if, for each p in* X_i, X_i *is a neighborhood of p.*

DEFINITION 5. *Let X be a metric space and X_i a subset of X. X_i is called closed in X (or simply closed) if X_i^c is open.*

Although there are those mathematicians who do not accept definitions such as the one just given for closed sets, on the grounds that it is a "negative definition"—it states merely what a closed set is not—this definition is sufficient and adequate for the purposes herein.

These concepts of open and closed sets are considered mutually exclusive by some students of mathematics. A set which is closed cannot be open, nor an open set be simultaneously closed. However, the finite metric space defies this intuitive notion, and such metric spaces are both open and closed. The two examples already given will demonstrate this property of our finite metric spaces. We assume the theorem that the union of a finite number of open sets are open, and the union of a finite number of closed sets are closed.

The first example defines distance as

$$d(x, y) = 1 \quad \text{if } x \neq y,$$
$$d(x, y) = 0 \quad \text{if } x = y.$$

Every subset of (X, d) is both open and closed. For consider any neighborhood around a point with radius $\frac{1}{2}$. For every point p in such a sphere, that sphere is a neighborhood of the point p. Thus the set $\{x\}$ is open since every point in the neighborhood around x with radius $\frac{1}{2}$ is contained in the neighborhood also; and, by the definition, each subset is thus open. In general, any subset A of the finite metric space is open, since A equals the union of a finite number of $\{x\}$, for x in A.

Also each of these subsets A of X is closed, since the complement of A, A^c is an open set of the form given above. The definition of closed sets, therefore, makes each subset A closed, since A^c is open.

Next consider the second example of finite metric spaces, where X is four points in E^2 space and the metric d is defined in the standard geometric manner, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Each point a_i of the space is an open set, for if we take a neighborhood around a_i with radius equal to 1, then every such subset $\{a_i\}$ of X is open, by the definition. For each such neighborhood will contain only the center point a_i and no other, because the radius is less than 3, the distance to the nearest point in the space. And since any finite number of open sets is open, then any subset of the finite metric space is an open set.

However, for any singleton subset $\{a_i\}$ of the spaces, the complement of the set is composed of the union of three other singleton subsets

$$\{a_j\} \cup \{a_k\} \cup \{a_m\} = \{a_j, a_k, a_m\}$$

and it was just shown this set is open. So the complement of any singleton set is open, thus defining each subset $\{a_i\}$ of X to be closed. And again, a finite number of closed sets are closed; therefore, every subset of the finite metric space is closed. Hence, the conclusion is that the metric space is both open and closed.

The conclusion of this discussion points out that finite metric spaces illustrate a very novel property, and one that makes young students of mathematics uneasy with the classical proofs which depend upon definition proofs, and geometric intuition. The terms open and closed sets may be mutually inclusive in finite metric spaces (and other metric spaces), and although this is not the most comfortable idea since so many of the spaces studied by college students are metric spaces, it is fortunate that the primary concern is not with these open and closed finite spaces.

BIBLIOGRAPHY

Randolph, John F. *Basic Real and Abstract Analysis.* Academic Press. New York, 1968.

Rudin, Walter. *Principles of Mathematical Analysis.* McGraw-Hill Book Company, Inc. New York, 1964.

Simmons, George F. *Introduction to Topology and Modern Analysis.* McGraw-Hill Book Company, Inc. New York, 1963.

Wallace, Andrew H. *Differential Topology, First Steps.* W. A. Benjamin, Inc. New York, 1968.

Algebra goes to the root of the matter and ignores the casual nature of particular cases.—E. C. Titchmarsh

Equilateral Triangles and the Parallelogram*

SUSAN M. O'CONNOR
Student, Mount Mary College, Milwaukee

In an article which appeared in the January 1965 issue of *The American Mathematical Monthly,* J. Garfunkel and S. Stahl showed some of the interesting results obtained by constructing equilateral triangles on the sides of a scalene triangle. One of their discoveries was that "if in the middle third of each side of any scalene triangle, equilateral triangles are constructed inwards, then the join of their vertices forms an equilateral triangle." [1, p. 13]

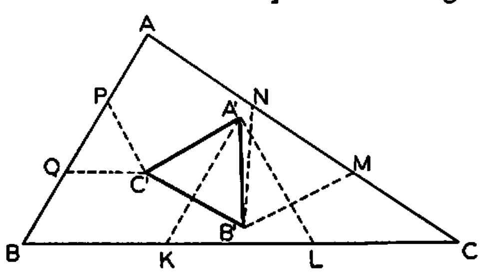


Figure 1

This paper is concerned with the effects of applying the hypothesis of constructing equilateral triangles on various sides of a figure to the parallelogram. The results are interesting and not totally unexpected. Many of the proofs of the theorems in this paper are based on the law of cosines which states that

$$a^2 = b^2 + c^2 - 2bc\,(\cos \alpha)$$

where a, b and c are the lengths of the sides of any triangle and α is the measure of the angle opposite the side whose length is a.

One of the basic theorems of plane geometry states that the perpendicular bisectors of the sides of a triangle meet at a point called the circumcenter. Quite obviously this same conclusion does not hold true for a parallelogram. Rather the intersection of the perpendicular bisectors of the sides of a parallelogram forms another parallelogram.

*A paper presented at the 1969 National Convention of KME at Cedar Falls, Iowa, May 1-2.

THEOREM 1: *ABCD is a parallelogram with W, X, Y and Z the midpoints of AB, BC, CD and DA, respectively. If perpendiculars are constructed at each of the midpoints and extended until they intersect at points A′, B′, C′ and D′, then A′B′C′D′ is a parallelogram.*

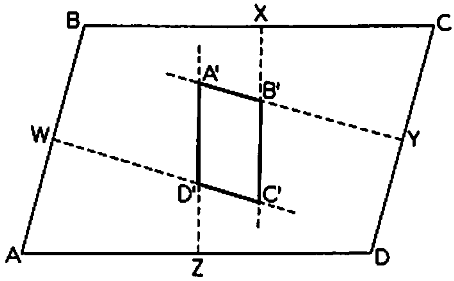


Figure 2

Proof. $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$ because they are opposite sides of a parallelogram. Since lines in a plane perpendicular to parallel lines are themselves parallel, $\overline{C'W} \parallel \overline{A'Y}$ and $\overline{A'Z} \parallel \overline{C'X}$. $\overline{C'D'} \parallel \overline{A'B'}$ and $\overline{A'D'} \parallel \overline{C'B'}$ because they are segments of parallel lines in the same plane. Therefore, *A′B′C′D′* is a parallelogram as it has been shown that both pairs of its opposite sides are parallel.

Suppose now that one retains the midpoints of the sides of the parallelogram and constructs equilateral triangles on alternate segments of the perimeter so that the triangles fall on the outside of the parallelogram. The join of the vertices of these equilateral triangles forms another parallelogram.

THEOREM 2: *ABCD is a parallelogram with W, X, Y and Z the midpoints of* $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ *and* $\overline{DA}$, *respectively. If equilateral triangles are constructed on* $\overline{AW}$, $\overline{BX}$, $\overline{CY}$ *and* $\overline{DZ}$ *so that they fall on the outside of* ▱ *ABCD, then A′B′C′D′ is a parallelogram.*

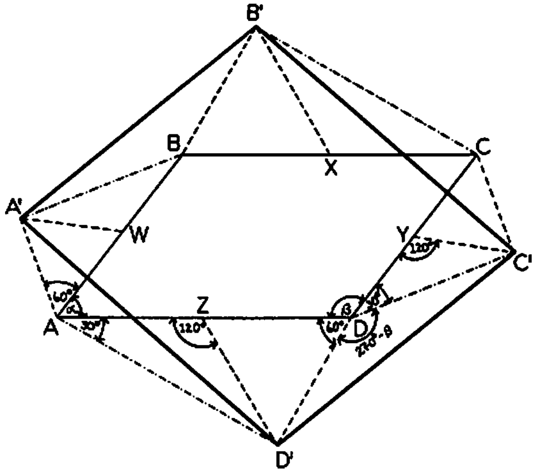


Figure 3

Proof. Let the lengths of $\overline{AD}$ and $\overline{BC}$ be equal to a and those of $\overline{AB}$ and $\overline{CD}$ to b. By the law of cosines, $(AD')^2 = \frac{3a^2}{4}$ and $(A'D')^2 = \frac{3a^2 + b^2 - 2\sqrt{3}ab \sin \alpha}{4}$. By the same process $(B'C')^2 = \frac{3a^2 + b^2 - 2\sqrt{3}ab \sin \alpha}{4}$. Thus it can be shown that $\overline{A'D'} \cong \overline{B'C'}$. Again using the law of cosines, $(DC')^2 = \frac{3b^2}{4}$, $(C'D')^2 = \frac{a^2 + 3b^2 + 2\sqrt{3}ab \sin \beta}{4}$, and $(A'B')^2 = \frac{a^2 + 3b^2 + 2\sqrt{3}ab \sin \beta}{4}$. As before $\overline{C'D'}$ can be proven congruent to $\overline{A'B'}$. Therefore, $A'B'C'D'$ is a parallelogram since both pairs of its opposite sides are congruent.

If the triangles are constructed inwardly, the join of their vertices again forms a parallelogram. The theorem and its proof are similar to that of the preceding theorem and for that reason have been omitted.

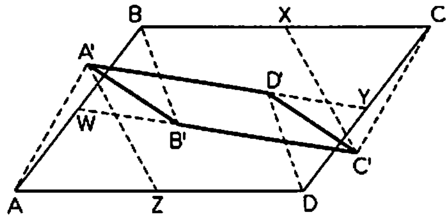


Figure 4

By joining the midpoints of the sides of a parallelogram, a new parallelogram is formed and the join of the vertices of the equilateral triangles constructed on these segments forms a parallelogram.

THEOREM 3. *ABCD is a parallelogram with W, X, Y and Z the midpoints of* $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ *and* $\overline{AD}$, *respectively. If line segments are drawn connecting the midpoints, then WXYZ is a parallelogram.*

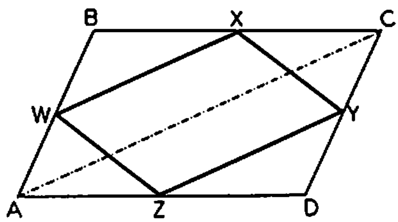


Figure 5

Proof. This proof necessitates the construction of the diagonal $\overline{AC}$. $\overline{WX}$ is a line segment connecting the midpoints of two sides of a triangle. Therefore $\overline{WX} \parallel \overline{AC}$ and $WX = \frac{AC}{2}$. Similarly, $\overline{YZ} \parallel \overline{AC}$ and $YZ = \frac{AC}{2}$. Thus $\overline{WX} \parallel \overline{YZ}$ and $\overline{WX} \cong \overline{YZ}$. Since one pair of opposite sides of the quadrilateral are both parallel and congruent to each other, $WXYZ$ is a parallelogram.

THEOREM 4. *ABCD is a parallelogram with W, X, Y and Z the midpoints of* $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ *and* $\overline{AD}$, *respectively. If A'WX, B'XY, C'YZ and D'WZ are equilateral triangles, then A'B'C'D' is a parallelogram.*

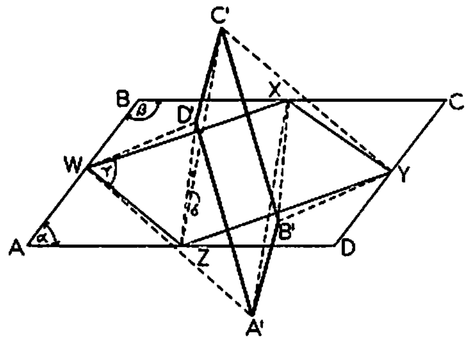


Figure 6

Proof. Let the lengths of $\overline{AD}$ and $\overline{BC}$ be equal to a and those of $\overline{AB}$ and $\overline{CD}$ to b. By the law of cosines, $(WZ)^2 = \dfrac{x^2}{4}$, where $x^2 = a^2 + b^2 - 2ab \cos \alpha$ and $\alpha = m\angle WAZ$. Again by the law of cosines, $(WX)^2 = \dfrac{y^2}{4}$, where $y^2 = a^2 + b^2 - 2ab \cos \beta$ and $\beta = m\angle WBX$. Because $\triangle D'WZ$ and $\triangle A'WX$ are equilateral triangles, $\overline{WZ} \cong \overline{WD'}$ and $\overline{WX} \cong \overline{WA'}$. Therefore, $(A'D')^2 = \dfrac{x^2 + y^2 - 2xy \cos \gamma}{4}$, where $\gamma = m\angle A'WD'$. Using the theorems concerned with angle addition and the transitive property of real numbers one can show that $\angle A'WD' \cong \angle B'YC'$. Thus it follows that $(B'C')^2 = \dfrac{x^2 + y^2 - 2xy \cos \gamma}{4}$ and that $\overline{A'D'} \cong \overline{B'C'}$. Similarly $\overline{A'B'} \cong \overline{C'D'}$. Since both pairs of opposite sides of the quadrilateral are congruent, $A'B'C'D'$ is a parallelogram.

Now consider a parallelogram whose sides have been divided into three congruent parts—trisected. (The endpoints of these three congruent segments are called "points of trisection.") If perpendiculars are constructed at the points of trisection of any two adjacent sides of a parallelogram, the intersection of these lines forms a parallelogram.

THEOREM 5. *ABCD is a parallelogram with S, T, U, V, W, X, Y and Z the points of trisection of its sides. If perpendiculars* *are constructed at points W, X, Y and Z and extended until they intersect at points A′, B′, C′ and D′, then A′B′C′D′ is a parallelogram.*

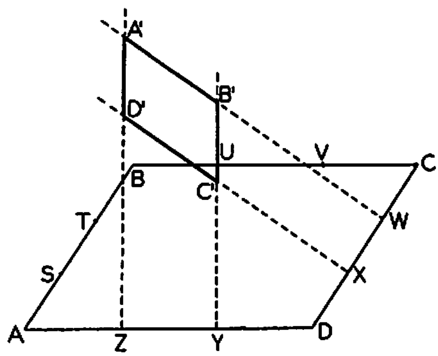


Figure 7

Proof. Since $\overline{A'Z}$ and $\overline{B'Y}$ are both perpendicular to $\overline{AD}$, $\overline{A'Z} \parallel \overline{B'Y}$. $\overline{A'D'} \parallel \overline{B'C'}$ as they are segments in parallel lines. Similarly it can be shown that $\overline{A'B'} \parallel \overline{C'D'}$. Therefore, because both pairs of its opposite sides are parallel, $A'B'C'D'$ is a parallelogram.

In all of the theorems of the following portion of this paper (Theorems 6 through 9), the sides of the parallelogram have been trisected. The theorems state the conclusions reached when equilateral triangles are constructed on various segments of the parallelogram. Each of these proofs hinges on the law of cosines.

First consider the case where equilateral triangles are constructed in the middle third of each side of a parallelogram. If the triangles fall either on the outside of or towards the interior of the parallelogram, then the join of their vertices forms a new parallelogram.

THEOREM 6. *ABCD is a parallelogram with S, T, U, V, W, X, Y and Z the points of trisection of its sides. If A′ST, D′UV, C′WX and B′ZY are equilateral triangles constructed inwardly, then A′B′C′D′ is a parallelogram.*

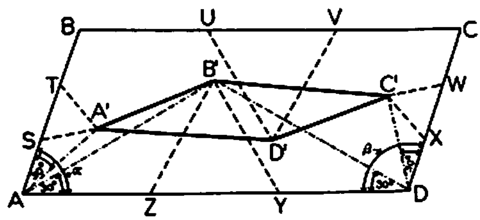


Figure 8

Proof. Let the lengths of $\overline{AD}$ and $\overline{BC}$ be equal to a and those of $\overline{AB}$ and $\overline{CD}$ to b. Using the law of cosines, $(AB')^2 = \frac{a^2}{3}$, $(AA')^2 = \frac{b^2}{3}$, and $(A'B')^2 = \frac{a^2+b^2-ab(\cos\alpha+\sqrt{3}\sin\alpha)}{3}$, where $\alpha = m\angle BAD$. $\angle BAD \cong \angle BCD$ which implies that $\alpha = m\angle BCD$. Again, by the law of cosines, $(C'D')^2 = \frac{a^2+b^2-ab(\cos\alpha+\sqrt{3}\sin\alpha)}{3}$. Thus $\overline{A'B'} \cong \overline{C'D'}$. Similarly $(B'C')^2 = A'D')^2 = \frac{a^2+b^2-ab(\cos\beta+\sqrt{3}\sin\beta)}{3}$ and $\overline{B'C'} \cong \overline{A'D'}$. Therefore, since both pairs of opposite sides of the quadrilateral are congruent, $A'B'C'D'$ is a parallelogram.

THEOREM 7. *ABCD is a parallelogram with S, T, U, V, W, X, Y and Z the points of trisection of its sides. If A'UV, B'WX, C'YZ and D'ST are equilateral triangles constructed in the exterior of ▱ ABCD, then A'B'C'D' is a parallelogram.*

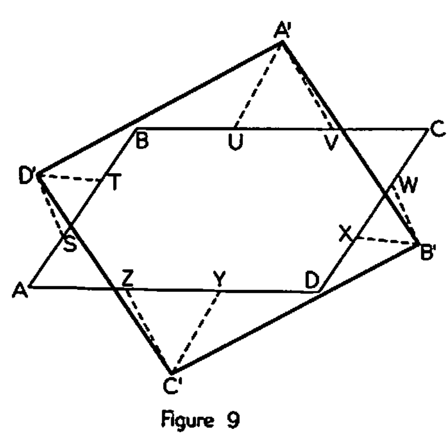


Figure 9

The proof of Theorem 7 is analagous to that of Theorem 6 and for this reason has not been included in this paper.

If instead of on the middle third, equilateral triangles are constructed on any corresponding third of the sides of a parallelogram, then the join of their vertices forms a parallelogram.

THEOREM 8. *ABCD is a parallelogram with S, T, U, V, W, X, Y and Z the points of trisection of its sides. If A'TB, B'VC, C'XD and D'ZA are equilateral triangles, then A'B'C'D' is a parallelogram.*

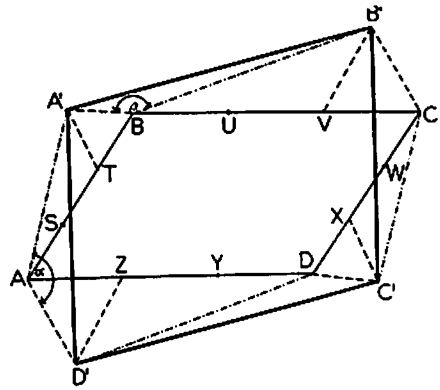


Figure 10

Proof. Let the lengths of $\overline{AD}$ and $\overline{BC}$ be equal to a and those of $\overline{AB}$ and $\overline{CD}$ to b. By the law of cosines $(A'D')^2 =$

$\dfrac{a^2 + 7b^2 - 2\sqrt{7}ab\cos\alpha}{9}$ where $m\angle A'AD' = \alpha$. It is possible to show that $\angle A'AD' \cong \angle B'CC'$ and thus that $(B'C')^2 =$

$\dfrac{a^2 + 7b^2 - 2\sqrt{7}ab\cos\alpha}{9}$. Therefore $\overline{A'D'} \cong \overline{B'C'}$. Similarly

$(A'B')^2 = (D'C')^2 = \dfrac{7a^2 + b^2 - 2\sqrt{7}ab\cos\beta}{9}$ and $\overline{A'B'} \cong \overline{D'C'}$.

Since both pairs of opposite sides of the quadrilateral are congruent, $A'B'C'D'$ is a parallelogram.

This same conclusion holds true if the triangles are constructed interiorly. Because this theorem and its proof follow so closely to that of Theorem 8, they have been omitted.

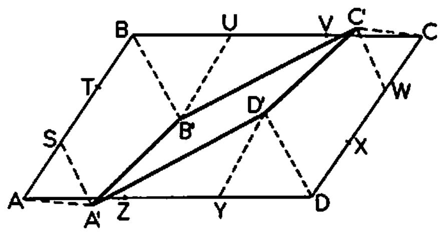


Figure 11

Lastly, suppose that equilateral triangles are constructed on corresponding segments of the sides of the parallelogram with the length of the sides of the triangles equal to ⅔ the length of the side of the original figure. Again, the join of the vertices of the triangles forms a parallelogram.

THEOREM 9. *ABCD is a parallelogram with S, T, U, V, W, X, Y and Z the points of trisection of the sides. If SBA′, UCD′, WDC′ and YAB′ are equilateral triangles constructed so that they fall on the outside of ▱ ABCD, then A′B′C′D′ is a parallelogram.*

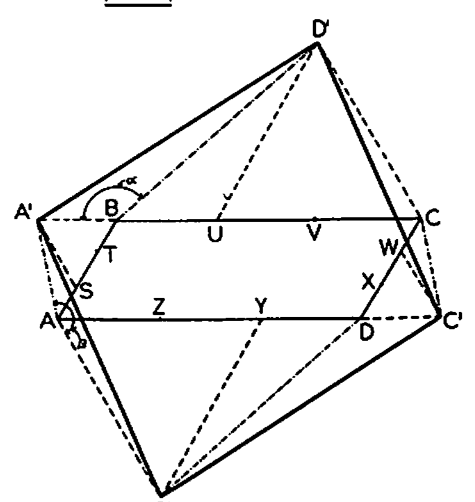


Figure 12

Proof. Let the lengths of $\overline{AD}$ and $\overline{BC}$ be equal to a and those of $\overline{AB}$ and $\overline{CD}$ to b. Using the law of cosines, $(BD')^2 =$

$\frac{7a^2}{9}$ and $(A'D')^2 = \frac{7a^2 + 4b^2 - 4\sqrt{7}ab \cos \alpha}{9}$ where $\alpha = m\angle A'BD'$. It can be shown that $\angle B'DC' \cong \angle A'BD'$ and thus $m\angle B'DC' = \alpha$. By the law of cosines, $(B'C')^2 = \frac{7a^2 + 4b^2 - 4\sqrt{7}ab \cos \alpha}{9}$. Therefore $\overline{B'C'} \cong \overline{A'D'}$. Similarly, $(A'B')^2 = (D'C')^2 = \frac{4a^2 + 7b^2 - 4\sqrt{7}ab \cos \beta}{9}$, where $\beta = m\angle A'AB' = m\angle D'CC'$. Thus $\overline{A'B'} \cong \overline{D'C'}$. Since both pairs of opposite sides of the quadrilateral are congruent, $A'B'C'D'$ is a parallelogram.

And as before, this same conclusion would hold if the triangles fell towards the interior of $\square$ $ABCD$. No theorem or proof accompanies this figure.

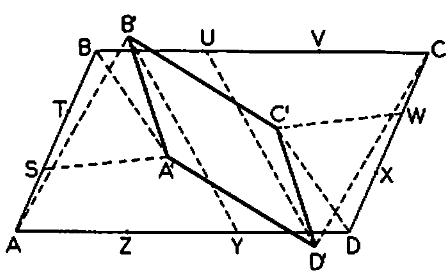


Figure 13

Ending the paper at this point is by no means an indication that the subject has been exhausted. Rather the closing has been necessitated due to a time factor. Many parallelograms appear in the figures of which no mention has been made. Many questions still remain unanswered. For example, suppose the parallelogram were restricted to a rhombus or rectangle. Would the figure formed by the join of the vertices of the equilateral triangles continue to have the same properties as the original figure? Or, suppose that instead of a parallelogram, a trapezoid or merely a quadrilateral had been the original figure. Would the figures produced by joining the vertices of the equilateral triangles have had any special characteristics of their own or in relation to the original figure? These and many other questions have been investigated in a much more extensive honors paper which the author wrote as part of the requirements for graduation from Mount Mary College.

Some might wonder whether or not these conclusions are of any useful value. To a pure mathematician such a question would be of little importance. But to placate any of a more practical bent, there is one such use for them. If the parallelogram is restricted to a rectangle or even more so to a square, then several interesting designs result from constructing equilateral triangles on the sides of the figures. No proofs accompany this figure.

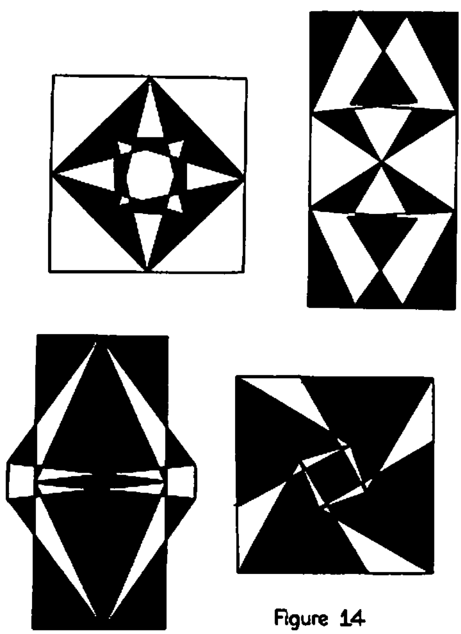


Figure 14

BIBLIOGRAPHY

J. Garfunkel and S. Stahl. "The Triangle Reinvestigated," *The American Mathematical Monthly,* LXXII (January 1965), 12-20.

The Theory of Positional Numeration

JOHN FLAIG
Student, California State Polytechnic College, Pomona

INTRODUCTION

In a positional system, after the base (say x) is adopted, symbols for 0, 1, 2, $\cdots$, $(x - 1)$ must be selected. In these systems, therefore, there are only x basic symbols of which all numbers are formed. Now any integer number, N, in this system (base x) may be written uniquely in the following form:

$$\begin{aligned} N_x &= U_n U_{n-1} \cdots U_1 U_0 \\ &= U_n 10_x^n + U_{n-1} 10_x^{n-1} + \cdots + U_1 10_x + U_0 \end{aligned}$$

or

$$N_x = \sum_{i=0}^{n} U_i (10_x)^i ,$$

(read 10_x, one - zero, base x). This expression is called the numeric polynomial of degree n, where $U_i \varepsilon I$, $0 < U_n < | 10_x |$, $0 \leqq U_i < | 10_x |$ for $i = 0, 1, \cdots, n - 1$ and $| x | > 1$ where $x \varepsilon R$, (I = integers, Q = rationals, R^* = irrationals, R = reals).

Note: Normally when we talk about different bases, say $x = 12$, the duodecimal scale, we are thinking in base ten. Thus $x = 12_{10}$ but $12_{10} = 10_{12}$, therefore $x = 10_x$, and this relation is the reason 10_x is employed in the definition. Consider the Division Algorithm; we may replace the condition $0 \leqq U < x$ with $0 \leqq U < 10_x$, since they mean the same thing. One should keep this in mind when reading Theorem (1,1).

SECTION 1

DEFINITION 1:

$$\textit{Base } x = 10_x = x_{10} .$$

It should be noted that the first equality is the precise notation, however, the last statement means the same thing and is intuitively clearer since most of our numerical thinking is done in base ten.

DEFINITION 2 (The Fundamental Set):

The set of numerals in the base set is called the Fundamental Set, i.e., Fundamental Set $= \{0, 1, \cdots, x - 1\}$, *in base* x.

THEOREM 1 (Division Algorithm for Positive Integers): *If* $N, x \varepsilon I$ *with* $x > 0 \Rightarrow \exists$ *unique* $q, U \varepsilon I$ *such that*

$$N = qx + U, \qquad 0 \leqq U < x.$$

Proof:

(1) Existence. Consider the set $S = \{qx \mid q \varepsilon I\}$; this set is unbounded and N is bounded so there is a $q_0 \varepsilon I$ such that

$$q_0x \leqq N < (q_0 + 1)x$$

adding a negative q_0x to each part

$$q_0x - q_0x \leqq N - q_0x < q_0x + x - q_0x,$$

then

$$0 \leqq N - q_0x < x.$$

Let $U = N - q_0x$ and $q_0 = q$, then $N = qx + U$ where $0 \leqq U < x$.

(2) Uniqueness will be proved in Theorem 2.

THEOREM (1, 1): *Let* x *be a positive integer* $x > 1$. *If* N *is any positive integer, then there exists a non-negative integer* n, *such that* N *can be expressed uniquely in the form:*

$$N = U_nx^n + U_{n-1}x^{n-1} + \cdots + U_1x^1 + U_0, \qquad \text{(I)}$$

where $0 < U_n < x$ *and* $0 \leqq U_i < x$ *for* $i = 0, 1, \cdots, n - 1$.

Proof. By repeated use of the Division Algorithm (Theorem 1).

If $N < x$, we have the desired form at once with $U_0 = N$ and $n = 0$. If $N \geqq x$, we may write

$$N = q_1x + U_0, \qquad 0 \leqq U_0 < x, \qquad (1)$$

by the Division Algorithm, and clearly $q_1 > 0$.

If $q_1 < x$, we set $U_1 = q_1$ and have equation (I) with $n = 1$. On the other hand, if $q_1 \geqq x$, we apply the Division Algorithm to q_1 and x, to obtain

$$q_1 = q_2x + U_1, \quad q_2 > 0, \quad 0 \leqq U_1 < x. \qquad (2)$$

Now if $q_2 < x$, we set $U_2 = q_2$ and have the desired expression (I) with $n = 2$. If $q_2 \geqq x$, we write

$$q_2 = q_3x + U_2\,, \qquad q_3 > 0\,, \qquad 0 \leqq U_2 < x. \qquad (3)$$

If $q_3 < x$, we set $U_3 = q_3$ and are finished. If not, we continue as before.

We continue this process as long as possible, getting an ordered set of q_i's,

$$N > q_1 > q_2 > q_3 > \cdots.$$

This set of q_i's is strictly decreasing with $q_i \in$ I. The sequence must stop after at most n steps. Failure to terminate would contradict the Archimedean property of real numbers.

So suppose $q_k > x$ and $0 < q_{k+1} < x$, then there is a last q_i, say q_n ($q_n = q_{k+1}$). We set $U_n = q_n$ and observe the following set of equations:

$$\begin{array}{rlc}
N &= q_1x + U_0 & (1)\\
q_1 &= q_2x + U_1 & (2)\\
q_2 &= q_3x + U_2 & 3\\
\vdots & & \vdots\\
q_{n-1} &= q_nx + U_{n-1} & (n-1)\\
q_n &= U_n & (n)
\end{array}$$

Now substitute the second equation in the first one, the third in the first, the fourth in the first, etc., finally the n^{th} in the first, The result is that N is expressed in the form of (I), i.e.,

$$N = U_nx^n + U_{n-1}x^{n-1} + \cdots + U_1x + U_0\,,$$

$0 < U_n < x$ and $0 \leqq U_i < x$ for $i = 0, 1, \cdots, n-1$ (recall Note in Introduction), or

$$N = U_nU_{n-1}\cdots U_1U_0\,.$$

With N expressed in this form we say N equals $U_nU_{n-1}\cdots U_1U_0$ in the base x,

$$\text{symbolically: } N_e = (U_nU_{n-1}\cdots U_1U_0)_x\,.$$

The uniqueness of (I) follows immediately from the uniqueness of the various remainders when the Division Algorithm is applied (this fact is proved in Theorem 2).

Remark: We must make a restriction on our notation to guarantee unique representation. Consider the problem in base ten, .9999 ··· = 1.000 ···, usually all numbers which are from some point on always 9 are ruled out. We make the same restriction in each base x, i.e., rule out all numbers that are $(x - 1)_x$ from some point on.

SECTION 2

DEFINITION 3 (Basis for a Positional System):

A basis for a system of numeration is a set of unique elements of the fundamental set, constants called U_i's, and powers of the base x, $(10_x)^i$, such that the $U_i(10_x)^i$ are linearly independent, and

$$N_x = c \sum_{i=-\infty}^{\infty} U_i(10_x)^i .$$

Every $N \in R$ is uniquely expressible in the above form, where $c = +1$, $N > 0$ and $c = -1$, $N < 0$.

THEOREM (1, 2): *The infinite limits on Σ are only the most general conditions.* However, any finitely expressible number in base x is represented:

$$N_x = c \sum_{i=n_0}^{n_1} U_i(10_x)^i .$$

We should note though that a number which is finitely expressible in base x may not be in base y. This will be shown in Section 3.

THEOREM (2, 2): *If $N \in R$, then*

$$N_x = c \sum_{-\infty}^{\infty} U_i(10_x)^i ,$$

for some base x. The next portion of Section 2 deals with the possible values of x. Also we say N_x spans R and mean that the set of all N_x elements is just R expressed in base x.

Proof. Since $R = Q \cup R^*$ and $N_x \in Q$ or $N_x \in R^*$ are in any case expressed in the above form, we have the required result.

In this section we shall use $\sum_{0}^{n}$ and $\sum_{n_0}^{n_1}$ both to mean that the number N is finitely expressible and with a change of variable, i, these can be made equal.

THEOREM (3, 2): *The "positive integers" form a basis for positional number systems.*

Proof. By Theorem (1, 1) we have that if $N, x \in I, x > 1 \Rightarrow \exists$ unique $n \in I, n \geqq 0 \ni N$ $(N > 0)$ can be expressed uniquely in the form:

$$N_x = U_n U_{n-1} \cdots U_1 U_0 = U_n 10_x^n + U_{n-1} 10_x^{n-1} + \cdots + U_1 10_x + U_0$$

or

$$N_x = \sum_{i=0}^{n} U_i (10_x)^i$$

where $U_i \in I$ and $0 < U_n < 10_x$, $0 \leqq U_i < 10_x$ for $i = 0, 1, \cdots, n-1$. If $N \in R$, this result plus Theorem (2, 2) yields

$$N_x = c \sum_{i=-\infty}^{\infty} U_i (10_x)^i .$$

THEOREM (4, 2): *The "integers" form a basis for positional number systems.*

Proof. In order to prove this theorem we must strengthen the Division Algorithm and from this proceed, as before, to generate a theorem similar to Theorem (1, 1). However, this theorem will be much stronger than (1, 1). With these factors in mind we present the following.

THEOREM 2 (Division Algorithm for Integers):
If $N, x \in I$ and $x \neq 0 \Rightarrow \exists$ unique $q, U \in I$ such that $N = qx + U$ and $0 \leqq U < |x|$.

Proof:

(1) Existence. The existence proof is exactly the same as Theorem 1 except that we consider both $x > 0$ and $x < 0$ (for $x < 0$, let $x = -x$ in Theorem 1) the result is:

$$N = qx + U \qquad \text{where } 0 \leqq U < |x|.$$

(2) Uniqueness. Assume

$$\begin{cases} N = qx + U \\ N = q_1 x + U_1 . \end{cases} \tag{1}$$

We must show that $q = q_1$ and $U = U_1$. Then from (1),

$$qx + U = q_1x + U_1,$$

but this implies

$$qx - q_1x = U_1 - U,$$

which implies

$$(q - q_1)x = (U_1 - U),$$

and

$$| q - q_1 | \, | x | = | U_1 - U | . \tag{2}$$

Now assume $U_1 \neq U$, then the right side of (2) is $| U_1 - U | \geqq 1$ and

$$0 < | U_1 - U | < | x | , \tag{a}$$

but on the left side of (2) then $| q - q_1 | \geqq 1$ implies

$$| q - q_1 | \, | x | \geqq | x | . \tag{b}$$

But since $| q_1 - q | \, | x | = | U_1 - U |$, inequalities (a) and (b) contradict each other, i.e., $| U_1 - U | < | x |$ and $| U_1 - U | \geqq | x |$, therefore $U = U_1$ and then $q = q_1$.

Now, with this stronger form of Theorem 1 we repeat the process of Theorem (1, 1) using the new conditions and restraints. From this we arrive at the result:

If $N, x \in I$ and $| x | > 1 \Rightarrow \exists$ unique $n \in I$, $n \geqq 0 \ni N$ can be expressed uniquely in the form:

$$N_x = c \sum_{i=0}^{n} U_i(10_x)^i$$

where $U_i \in I$ and $0 < U_n < | 10_x |$, $0 \leqq U_i < | 10_x |$ for $i = 0, 1, \cdots, n - 1$. ①

If $N \in R$, this result plus Theorem (2, 2) yields

$$N_x = c \sum_{i=-\infty}^{\infty} U_i(10_x)^i .$$

However, this is the result we sought and Theorem (4, 2) is proved.

We arrive now at a critical point, the theoretical structure of most common methods of numeration has been rigorously developed in Theorem (3, 2), and Theorem (4, 2), where x is an "integer". Now suppose x is rational or irrational. Can x now function as a basis for a system of numeration? We assert that such a construction is possible.

We know if $x \in I$ $(x \neq 0, \pm 1)$ and $N \in R$, then

$$N_x = c \sum_{-\infty}^{\infty} U_i(10_x)^i ,$$

(thus the "integer bases" span R). If we could show that the same thing was true for $x \in Q$ or $x \in R$, we would be finished. However, we can do this if we can show that any real number in any integer base can be converted into a number in a rational or irrational base (with uniqueness preserved).

An example of our problem is $7_{10} = (?)_\pi$. Does this equation have meaning? Recall that to convert numbers in integer bases one merely used a division process.

Example: $25_{10} = (?)_2$

$$\begin{array}{r|r|l} 2 & 25 & \\ & 12 & 1 \\ & 6 & 0 \\ & 3 & 0 \\ & 1 & 1 \end{array}$$

Therefore,

$$25_{10} = (1\ 1\ 0\ 0\ 1)_2 .$$

Conversion properties between various bases will be discussed more fully in Section 3. Let us now construct a method for converting to rational and irrational bases.

THEOREM (5, 2) (the method of reduced remainders):

A real number in an integer base can be converted to a rational (or irrational) base number.

Proof. Let a be a real number in base x $(x \in I, |x| > 1)$; without loss of generality, we consider only those numbers such that

$$\begin{aligned} a_x &= .U_1U_2U_3 \cdots U_n \cdots \\ &= \frac{U_1}{10_x} + \frac{U_2}{(10_x)^2} + \cdots . \end{aligned}$$

Let y be a rational or irrational number. We are going to convert a_x to a'_y (denoted $a_x \longleftrightarrow a'_y$). Now there is only one more stipulation before we proceed; the U_i's must always be non-negative integers.②

Then to establish the theorem we proceed as follows: Convert a_x to a (base ten), hence $a = .a_1a_2a_3 \cdots a_m \cdots$, or approximately, $\hat{a} = .a_1a_2a_3 \cdots a_m$. Now let us approximate $\hat{a}$ in base y,

$$\hat{a} \cong \sum_{i=-\infty}^{\infty} U_i(10_y)^i = U_{n_1}U_{n_1-1} \cdots U_0.U_{-1} \cdots U_{n_0} \cdots, \qquad (1)$$

where the $0 \leqq U_i < |y|$. Then we can further approximate $\hat{a}$ in finite form as

$$\hat{a} \cong \sum_{i=n_0}^{n_1} U_i(10_y)^i .$$

This last expression, however, is just defined as $\hat{a}'$ in base ten. Therefore, $\hat{a} \cong \hat{a}'$ and this implies $a_x \cong a'_y$.

Now the only question remaining is: Can we find the U_i which satisfy (1)? To do this, consider

$$\hat{a} \div y^{n_1} \Rightarrow \hat{a} = q_1y^{n_1} + U_1 .$$

Therefore $a'_y \cong q_1 \cdots$, where q_1 is in the $(10_y)^{n_1}$ place. Then

$$U_1 \div y^{n_1-1} \Rightarrow U_1 = q_2y^{n_1-1} + U_2 .$$

Therefore, $a'_y \cong q_1q_2 \cdots$. Then,

$$(U_1 - U_2) \div y^{n_1-2} \Rightarrow (U_1 - U_2) = q_3y^{n_1-2} + U_3.$$

Therefore, $a'_y \cong q_1q_2q_3 \cdots$. We continue in this manner until

$$(U_1 - U_2 - U_3 - \cdots - U_{n_0}) \div y^{n_1-n_0}$$

$$\Rightarrow (U_1 - U_2 - \cdots - U_{n_0}) = q_{n_0+1}y^{n_1-n_0} + U_{n_0+1} .$$

Therefore, $a'_y \cong q_1q_2q_3 \cdots q_{n_1}.q_{-1} \cdots q_{n_0}$.

This process can be carried out to any degree of accuracy desired depending on $\hat{a}$ and the number of divisions.

Example: $7_{10} = (?)_\pi$.

Assume $7_{10} \cong a_1\pi^2 + a_2\pi + a_3\pi^0 + a_4 \left(\frac{1}{\pi}\right) + a_5 \left(\frac{1}{\pi}\right)^2.$

$$\begin{array}{r|l} & 2 \\ \pi & \overline{7.00000} \\ & \underline{6.28318} \\ & .71681 \end{array}$$

i.e.,

$$2 + \text{remainder } (.7168).$$

Therefore $a_1 = 0$, $a_2 = 2$ and then since $.7168 < 1 \Rightarrow a_3 = 0$. We continue,

$$.7168 \div \left(\frac{1}{\pi}\right) \Rightarrow a_4 = 2.$$

Then subtract the remainders and continue,

$$.080 \div \left(\frac{1}{\pi}\right)^2 \Rightarrow a_5 = 0,$$

and

$$.080 \div \left(\frac{1}{\pi}\right)^3 \Rightarrow a_6 = 2.$$

Therefore $7_{10} \cong (20.202)_\pi$. Notice that $7_{10} = 21_3$ and $7_{10} = 13_4$ thereby giving us an idea that our approximation is in the proper range.

THEOREM (6, 2): *The "rational numbers" form a basis for positional number systems.*

Proof. By Theorem 2, Theorem (5, 2) and repeating the process of Theorem (1, 1), we have that if $N, x \in Q$ (Q = rationals) and $|x| > 1 \Rightarrow \exists$ unique $n \in I$, such that N can be expressed uniquely to any desired degree of accuracy in the form

$$N_x = \sum_{i=0}^{n} U_i(10_x)^i$$

where $U_i \in I$ ②, and $0 < U_n < |10_x|$, $0 \leqq U_i < |10_x|$ for $i = 0, 1, \cdots, n-1$. If $N \in R$, this result plus Theorem (2, 2) yields

$$N_x = \sum_{i=-\infty}^{\infty} U_i(10_x)^i.$$

THEOREM (7, 2): *The "irrational numbers" form a basis for positional number systems.*

Proof. The proof remains the same as that of rational numbers, only the condition $x \in Q$ is replaced with $x \in R^*$ (R^* = irrationals).

We may sum up the previous theorems in one general theorem.

THEOREM (8, 2): *The "real numbers" form a basis for positional number systems.*

Proof. From Theorems (1, 2) through (7, 2) we have the desired result. If $|x| > 1$, then N can be expressed to any desired degree of accuracy as

$$N_x = c \sum_{i=0}^{n} U_i(10_x)^i$$

where $U_i \in I$, $0 < U_n < |10_x|$ and $0 \leq U_i < |10_x|$ for $i = 0, 1, \cdots, n-1$ and $x \in R$. If $N \in R$, this result plus Theorem (2, 2) yields

$$N_x = c \sum_{i=-\infty}^{\infty} U_i(10_i)^i .$$

The next logical question and an interesting result is the following theorem.

THEOREM (9, 2): *The "complex numbers" do not form a basis for positional number notation.*

Proof. In order that x form a basis for a system of numeration there must exist an order relation on the q_i when the Division Algorithm is applied (see the proof of Theorem (1, 1)). But division of complex numbers with imaginary parts in general gives rise to complex numbers with imaginary parts. Now since the complex numbers are not an ordered field the q_i are not orderable and therefore the theorem is proved.

THEOREM (10, 2): *Any real number N can be expressed uniquely in any real base x, $|x| > 1$, to any desired degree of accuracy in the following manner:*

$$N_x = c \sum_{i=-\infty}^{+\infty} U_i(10_x)^i$$

where $U_i \in I$, $0 < U_n < |10_x|$ *and* $0 \leq U_i < |10_x|$ *for* $i = -\infty, \cdots, 0, 1, \cdots, +\infty$.

Proof. Follows from Theorems (1, 2) through (8, 2).

SECTION 3

An excellent discussion of elementary convertability theory is presented in Fehr and Hill *Contemporary Mathematics.* These ideas plus Theorem (5, 2) form the underlying structure of the material presented below.

THEOREM 3: *We have shown in Theorem (10, 2) that any real number N may be expressed uniquely in the base x. However, the number N itself must be expressed in some base, say y. Therefore, we have the immediate result that if*

$$N_x = \sum_{i=-\infty}^{\infty} U_i(10_x)^i$$

and if N can be expressed in scale y notation also, i.e., say

$$N'_y = \sum_{i=-\infty}^{\infty} U'_i(10_y)^i$$

then $N_x = N'_y$ *and* N_x *can be converted into* N'_y *and vice versa.*

We say $N_x = \Sigma\, U_i(10_x)^i = \Sigma\, U'_i(10_y)^i = N'_y$ and employ the primes to denote the fact that in general the U_i will be different in different bases, and therefore, so will the representation of the numeric polynomial.

Example: Convert a number in base seven to base ten.

$N_7 = \Sigma\, U_i(10_7)^i$. Now to change N_7 to N'_{10} we must change 10_7 to t_{10}. The object being to find t. Here the solution is simple, $t = 7$.

Example: Convert 24_7 to base ten.

$24_7 = 2(10_7)^1 + 4(10_7)^0 = 2(7_{10})^1 + 4(7_{10})^0 = 14_{10} + 4_{10} = 18_{10}$.

Now just as we could extend the Division Algorithm and Theorem (1, 1) to integers, rational and irrational number bases, we can also extend the notion of convertability.

Examples:

(1)	$12.25_{10} = 121_{5/2}$	(2)	$5_{10} = 10001_{\sqrt{2}}$
(3)	$(3 + \sqrt{2})_{10} = 111_{\sqrt{2}}$	(4)	$7_{10} \cong 20.202_{\pi}$
(5)	$15_{10} = 195_{-10}$	(6)	$-20_{10} = 20_{-10}$
(7)	$-75_{10} = 85_{-10}$	(8)	$10_{-2} = -2_{10}$
(9)	$10_{12} = 12_{10}$	(10)	$1_e = 1_{10} = 1_{\pi}$
(11)	$0_{\pi} = 0_{1000} = 0_{-50}$	(12)	$10_e \cong 2.718_{10}$.

Application:

The reader might find the following example both interesting and enlightening.

Consider the negative binary system. The fundamental set is $\{0, 1\}$ with operations defined by the following table:

+	0	1
0	0	1
1	1	110

$\cdot$	0	1
0	0	0
1	0	1

Then:

(1) $110_{-2} = 1(-2)^2 + 1(-2)^1 + 0 = 2_{10} = 10_2$;

(2) Compute: $111_{-2} + 1_{-2} = (?)$

$$\begin{array}{r} \cdot \\ \cdot \\ \cdot\ 11 \\ 11 \\ 11 \\ 11 \\ 111 \\ +1 \\ \hline \cdots\ 000100_{-2} \end{array}$$

Answer $= 100_{-2}$

(3) In negative base systems we can express negative numbers without using a minus sign (any number with an even number of digits is negative). Now when coding numbers for computers in the positive binary scale, we must use sign

bits; using negative binary coding eliminates this and increases the storage capacity of the machine.

The reader is urged to consider this method of coding and its consequences.

BIBLIOGRAPHY

Bower, Henery and Joan. *Arithmetical Excursions.* New York: Dover Publications, 1961.

Dantzig, Tobias. *Number, the Language of Science.* New York: The Macmillan Company, 1954, 4th ed.

Eves, Howard. *An Introduction to the History of Mathematics.* New York: Holt, Rinehart and Winston, 1963.

Fehr, Howard and Thomas Hill. *Contemporary Mathematics.* Boston: D. C. Heath and Company, 1966.

Manual of the Duodecimal Society, The Duodecimal Society of America, Staten Island, New York.

McCoy, Neal H. *Introduction to Modern Algebra.* Boston: Allyn and Bacon, 1960.

Thompson, R. C. and Adil Yaqub. *Introduction to Abstract Algebra.* Scott, Foresman and Company, to appear in 1970.

______________, *Introduction to Linear Algebra.* Scott, Foresman and Company, to appear in 1969.

Well, Chauncey. "Using a Negative Base for Number Notation." *The Mathematics Teacher,* LVI, 1963.

FOOTNOTES

① The condition $|x| > 1$ is necessary, since the U_i are non-negative integers and $0 \leq U_i < |x|$. Hence, if $|x| \leq 1$, then the U_i must all be zero, which is quite trivial.

② The U_i must remain non-negative integers, since letting them be rational or irrational at once destroys the uniqueness of representation and hence destroys the entire value of systemized numeration. For a proof, see "Systems of Numerations" by John Flaig, available through California State Polytechnic Library, Pomona, California (page 44).

Directions for Papers to Be Presented At the Eighteenth Biennial Kappa Mu Epsilon Convention

INDIANA, PENNSYLVANIA

April 2-3, 1971

A significant feature of this convention will be the presentation of papers by student members of KME. The mathematics topic which the student selects should be in his area of interest and of such a scope that he can give it adequate treatment within the time allotted.

WHO MAY SUBMIT PAPERS: Any student KME member may submit a paper for use on the convention program. Papers may be submitted by graduates and undergraduates; however, graduates will not compete with undergraduates.

SUBJECT: The material should be within the scope of the understanding of undergraduates, preferably those who have completed differential and integral calculus. The Selection Committee will naturally favor papers within this limitation and which can be presented with reasonable completeness within the time limit prescribed.

TIME LIMIT: The usual time limit is twenty minutes, but this may be changed on the recommendation of the Selection Committee if requested by the student.

PAPER: The paper to be presented, together with a description of the charts, models, or other visual aids that are to be used in the presentation, should be presented to the Selection Committee. A bibliography of source materials, together with a statement that the author of the paper is a member of KME, and his official classification in school, undergraduate or graduate, should accompany his paper.

DATE AND PLACE DUE: The papers must be submitted no later than January 9, 1971, to the office of the National Vice-President.

(Continued on p. 99)

Installation of New Chapters

EDITED BY SISTER HELEN SULLIVAN

NEW YORK THETA CHAPTER

St. Francis College, Brooklyn, New York

On Friday, October 24, 1969, the installation of New York Theta Chapter of Kappa Mu Epsilon was held at St. Francis College. Robert Vincent, chapter president, welcomed the initiates and guests and introduced the installing officer, Professor James Lightner, Chairman of the Mathematics Department at Western Maryland College, Westminster, Maryland.

Those persons initiated included A. Amodeo, L. Backes, V. Carriero, A. Chupa, R. Higgins, W. Imbriale, A. Rotolo, P. St. John, G. J. Towusma, R. Vincent, A. Voltz, R. Wendt, K. Westly, Sister Mary Lois, J. Andres, J. Burke, D. Coscia, J. Lazzara, T. O'Hara, J. Tremmel, and Bro. L. Quinn, O.S.F.

Following the installation, Professor Lightner installed the officers of New York Theta. These include:

President—Robert Vincent
Vice-President—Robert Higgins
Secretary—William Imbriale
Treasurer—Vincent Carriero
Corresponding Secretary—Donald Coscia

After the installation ceremony, the members and guests attended a dinner-reception at the Hamilton Restaurant. Professor Lightner presented the topic, "The Role of Honor Societies—In Particular: Mathematics Honor Societies."

PENNSYLVANIA IOTA CHAPTER

Shippensburg State College, Shippensburg, Pennsylvania

On November 1, 1969, at 7:00 p.m. the members of the Shippensburg State College Honorary Mathematics Fraternity were initiated as charter members of the Pennsylvania Iota Chapter of Kappa Mu Epsilon.

Thomas R. Cook, a charter member and past president of the fraternity, gave a brief history of the group from its origin to its installation into KME. Professor John S. Mowbray then introduced the installing officer, Dr. J. Dwight Daugherty from Kutztown State College. Forty-two members including twelve faculty members and thirty students of Shippensburg State College were presented for membership by the president. After being initiated, each member signed the constitution and received his membership card. The purpose of KME, the description and meaning of the badge, crest, and seal and the translation of the motto were presented. Dr. Daugherty then presented the charter of the Chapter to its president, Miss Fortna.

Following the initiation of the members was the installation of the new officers. They include:

President—Sharon L. Fortna
Vice-President—Nancy J. White
Recording Secretary—M. June Gutshall
Corresponding Secretary—John S. Mowbray
Treasurer—William A. Gould
Historian—Carol R. Kreider
Faculty Sponsor—James L. Sieber.

Dr. Daugherty gave a brief talk on the origin of KME and the benefits of belonging to this mathematics honor society. To close the program William A. Gould read some of the letters of welcome from other chapters of KME. Among those read were the letters from George R. Mach, the National President, and Fred W. Lott, the Past-President.

— Δ —

(Continued from p. 97)

SELECTION: The Selection Committee will choose ten to twelve papers for presentation at the convention. All other papers will be listed by title and student's name on the convention program and will be available as alternates.

William R. Smith
National Vice-President, Kappa Mu Epsilon
Department of Mathematics
Indiana University of Pennsylvania
Indiana, Pennsylvania 15701

The Problem Corner

EDITED BY ROBERT L. POE

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before October 1, 1970. The best solutions submitted by students will be published in the Fall 1970 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor Robert L. Poe, Department of Mathematics, Berry College, Mount Berry, Georgia 30149.

PROPOSED PROBLEMS

231. *Proposed by Pat LaFratta, Waukesha, Wisconsin.*

Find all the integral values of a, b, and c, if any exist, such that $x/a + y/b = 1$ is tangent to the graph of $x^{3/4} + y^{3/4} = c^{3/4}$.

232. *Proposed by R. S. Luthar, Waukesha, Wisconsin.*

Construct a function that is continuous at one point but discontinuous at every other point of its domain.

233. *Proposed by Leigh Janes, Pleasantville, New Jersey.*

If $x + y = k$, k a constant, and $z = x^p y^q$ maximi z e z.

234. *Proposed by Pat LaFratta, Waukesha, Wisconsin.*

Prove that $[(n + 1)(2n + 1)]^n \geqq 6^n(n\,!)^2$ for any positive integer n.

235. *Proposed by R. S. Luthar, Waukesha, Wisconsin.*

For any positive reals x and y prove that the following inequality holds:

$xy(1/x + 1/y + 1)^5 \geqq 108(1/x + 1/y)$.

SOLUTIONS

226. *Proposed by John Caffrey, Washington, D.C.*

A, B, C, and D are arranged in a Greek square. The sum of the four columns is given using letters of the same digit value.

Obviously, the simple sum would be the same if the square were rotated. What are the four digits?

$$\begin{array}{r} BCDA \\ CABD \\ DBAC \\ \underline{ADCB} \\ ABBBA \end{array}$$

Solution by Michael J. Handley, Eastern Illinois University, Charleston, Illinois.

Assume $0 < A < 10$
$0 < B < 10$
$0 < C < 10$
$0 < D < 10$
and $A \neq B \neq C \neq D$
and are integers.

If A, B, C, and D have values of 1, 2, 3, and 4 (not necessarily in that order) their sum = 10 which indicates that for any values of the four parmeters their sum $\geqq 10$. Thus a digit will be carried after column addition.

From the ones column (1) $A + D + C + B = 10X + A$ (where X is carried to the tens column).

From the tens column (2) $X + D + B + A + C = 10Y + B$ (Y carried to the hundreds column).

From the hundreds column (3) $Y + C + A + B + D = 10Y + B$ (Y carried to the thousands column).

From the thousands column (4) $Y + B + C + D + A = 10A + B$.

By # (2) and # (3) $Y = X$, and from # (3) and # (4) $Y = A$.

therefore $Y = X = A$.

By substitution into # (1) $A + D + C + B = 11A$.

By substitution into # (2) $A + D + B + A + C = 10A + B$.

Combining terms and solving #'s (1) and (2) for B in terms of A

$-2A + B = 0$ implies $2A = B$
and for C and D in terms of A or B we get
(5) $C + D = 8A = 4B$.

Substituting the maximum values for C and D into the above equation we have

$$9 + 8 = 8A = 4B \; .$$

It is seen that A must be less than 3. In fact the sum of C and D must be an even integer. Therefore, C and D are either both odd or both even. Letting $A = 1$, then $B = 2$, and from # (5) $C = 3$ or 5 and $D = 5$ or 3 respectively. Letting $A = 2$, then $B = 4$, and from # (5) $C = 7$ or 9 and $D = 9$ or 7. Hence there are four different solutions:

(a) $A=1, B=2, C=3, D=5$, (b) $A=1, B=2, C=5, D=3$,

$$\begin{array}{r} 2\,3\,5\,1 \\ 3\,1\,2\,5 \\ 5\,2\,1\,3 \\ 1\,5\,3\,2 \\ \hline 1\,2\,2\,2\,1 \end{array}\;; \qquad \begin{array}{r} 2\,5\,3\,1 \\ 5\,1\,2\,3 \\ 3\,2\,1\,5 \\ 1\,3\,5\,2 \\ \hline 1\,2\,2\,2\,1 \end{array}\;;$$

(c) $A=2, B=4, C=7, D=9$, (d) $A=2, B=4, C=9, D=7$,

$$\begin{array}{r} 4\,7\,9\,2 \\ 7\,2\,4\,9 \\ 9\,4\,2\,7 \\ 2\,9\,7\,4 \\ \hline 2\,4\,4\,4\,2 \end{array} \qquad \begin{array}{r} 4\,9\,7\,2 \\ 9\,2\,4\,7 \\ 7\,4\,2\,9 \\ 2\,7\,9\,4 \\ \hline 2\,4\,4\,4\,2 \end{array}\,.$$

Also solved by Karen Dowdy, Southern Methodist University, Dallas, Texas; Don Ehman, Bowling Green State University, Bowling Green, Ohio; Vance L. Johnson, Western Illinois University, Macomb, Illinois; Don N. Page, William Jewell College, Liberty, Missouri; Alana Rohr, Kansas State Teachers College, Emporia, Kansas; Kenneth M. Wilke, Topeka, Kansas.

227. *Proposed by Leigh Janes, Pleasantville, New Jersey.*

The exact value of $N!$, N a positive integer greater than or equal to 5, is an even positive integer which terminates in one or more zeros; that is, the exact value of $N!$, $N \geqq 5$, has one or more trailing zeros. Are there any factorials with exactly 5 trailing zeros? If we say that $N!$ has k trailing zeros is it possible to determine which values of k have no corresponding values of N?

Solution by Don N. Page, William Jewell College, Liberty, Missouri.

Each trailing zero indicates 10 as a factor of $N!$, so k trailing zeros indicates $10^k = 2^k 5^k$ as a factor. Each even number $2n \leqq N$ contributes one or more factors of 2 for $N!$, but the factors of 5 are contributed only by every fifth number $\leq N$ and thus determine the number of trailing zeros. For $N < 25$, every fifth number $5n \leq N$ gives one factor of 5, so $N! = 24!$ has $[N/5] = 4$ trailing zeros, where $[x]$ denotes the largest integer $I \leq x$. But the next factorial, $N! = 25!$, has the additional factor $25 = 5^2$ which gives two factors of 5, so $25!$ has $[N/5] = [N/5] = [25/5] + [25/25] = 5 + 1 = 6$ trailing zeros. Thus there are no factorials with exactly 5 trailing zeros.

In general, the number of trailing zeros of $N!$ is

$k = [N/5] + [N/5] + [N/5] + \cdots = \sum_{j=1}^{\infty} [N/5^j]$, with each term after $[N/5^j] < 5$ or with $j > \log_5 N$ being zero. k increases by one, since $[N/5]$ also increases by one at that time. This occurs when $N = 5^2 n$, where n is some integer. Thus the values of k which have no corresponding values for N are those between $\sum_{j=1}^{\infty} \left[\frac{5^2 n - 1}{5^j}\right]$ and $\sum_{j=1}^{\infty} \left[\frac{5^2 n}{5^j}\right]$, where n is any integer greater than zero. The first few such values are $k = 5, 11, 17, 23, 29, 30, 36, 42, \cdots$

Also solved by Kenneth M. Wilke, Topeka, Kansas.

228. *Proposed by Leigh Janes, Pleasantville, New Jersey.*

If a polynomial over the field of complex numbers containing k distinct terms is raised to the nth power what is the least possible number of distinct terms in the result and what is the greatest possible number of terms in the result?

Solution by Leigh Janes (proposer of the problem), Pleasantville, New Jersey.

A term of the result is determined only by the powers of its factors. Distributing N powers among k distinct terms is equivalent to distributing N identical marbles among k distinct boxes. Let $n(k, N)$ be the number of ways of distributing N identical marbles among k distinct boxes. Arrange the boxes in some sort of order; each box may have from zero to N marbles inclusive. If all N marbles go into the first box, there are $n(k - 1, 0)$ ways of distributing the remaining marbles in the remaining boxes. If $N - 1$ marbles go into the first box, there are $n(k - 1, 1)$ ways of distributing the remaining marbles into the remaining boxes. If x marbles go into the first box, there are $n(k - 1, N - x)$ ways of distributing the remaining marbles to the remaining boxes. Therefore it can be seen that

$$
\begin{aligned}
n(k, N) &= \sum_{j=0}^{N} n(k-1, j) \\
&= \sum_{j=0}^{N-1} n(k-1, j) + n(k-1, N) \\
&= n(k, N-1) + n(k-1, N)\,.
\end{aligned}
$$

To make a table of $n(k, N)$, note that $n(k, 0) = 1$, $n(k, 1) = k$, $n(1, N) = 1$, $n(2, N) = N + 1$, and the remainder of the table may be filled in by using $n(N, k) = n(k, N-1) + n(k-1, N)$.

$k \backslash N$	0	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	10	15	21	28	36	45
4	1	4	10	20	35	56	84	120	165
5	1	5	15	35	70	126	210	330	495
6	1	6	21	56	126	252	462	792	1287
7	1	7	28	84	210	462	924	1716	3003

Observe the indicated diagonals. If the array were tilted 45° clockwise, it would appear as Pascal's triangle whose elements are of the form

$$ {}_R C_E = \frac{R!}{E!(R-E)!} $$

where R is the row number and E is the number of the element from either end of the row (starting from zero at one end). The relation between R and E and $k + N$ is: $R = N + K - 1$ and $E = N$. Therefore, $n(k, N) = N + k - 1\ C_N = \dfrac{(N + k - 1)!}{N!(k-1)!}$.

The least possible number of terms (degree of polynomial)^n + 1, where the polynomial contains all powers $\leq n$. The greatest number of terms is $n(k, N)$.

229. *Proposed by the Editor.*

Today students who either take drugs, or belong to student activist organizations which oppose the Establishment, or are in opposition to the Vietnam War give college and university administrators many uneasy moments. It is not unreasonable to believe that many of these administrators would like to enroll in their schools as few students as possible who may be classified as belonging to one or more of the above categories. Certainly then, such schools must attempt to enroll the minimum number of students which may be identified with all three of these groups. However, the government and other organizations that provide institutions with lucrative research and student aid grants insist that such colleges and universities maintain fair and unbiased admission policies. Recently a study conducted by Trivia Researchers, Inc. and sponsored by the Senate revealed that 5% or less of the student body in the average university was composed of students who could be characterized as drug takers who oppose the Establishment and the Vietnam War. Now Dr. Grant Ghettar, the president of BIG University (Bountiful Institutional Grants University), through subjective questions asked on admission applications has been able to determine that a fraction, p, of the total applicants for admission to BIGU next fall are potential drug users, a fraction, q, of them will belong to activist organizations which oppose the Establishment, and a fraction, r, of them oppose the Vietnam War. Since Dr. Grant Ghettar is more of a politician than an academician, will you determine for him the least number of students who may be members of the drug-taking opposition to the establishment and the Vietnam War that should be admitted to BIG University next fall? Keep in mind that some must be admitted in order to obtain grants.

Solution by the Editor (proposer of the problem)

Let P mean "take drugs", Q mean "oppose the Establishment", and R mean "oppose the Vietnam War". Now the least possible number of those who, being P and Q, are also R, may be found by arranging the applications in a row, so that the PQ-category may begin from one end of the row, and the R-category from the other end, and counting the part where they overlap; and, the smaller the PQ-category, the smaller the common part. Thus the PQ-category must be made a minimum.

This may be accomplished by re-arranging the application forms, so that the P-category begins at one end of the row, and the

Q-category from the other; and the least possible number for the PQ-category is the common part of this row, i.e.,

$[p - (1 - q)] = [p + q - 1]$. This may be pictured as

$$\overbrace{\underbrace{}_{(1-q)}\underbrace{}_{q}}^{p} .$$

Now the least possible number for the PQR-category is the common part, $[p + q - 1 - (1 - r)] = [p + q + r - 2]$. This may be represented as

$$\overbrace{\underbrace{}_{(1-r)}\underbrace{}_{r}}^{(p+q-1)} .$$

Therefore, Dr. Grant Ghettar should select $p + q + r - 2$ applicants who are members of the drug-taking opposition to the Establishment and the Vietnam War for entrance at BIGU next fall. Note that $p + q + r \geqq 3$ and that the maximum of $(1, p + q + r - 2)$ would be the best policy.

230. *Proposed by the Editor.*

In the long division problem below each of the odd digits has been replaced by the letter O and each of the even digits has been replaced by the letter E. Can you state the problem in digits?

$$\begin{array}{r} OOE \\ OOE\overline{\big)EEOOE} \\ \underline{EOE} \\ OOO \\ \underline{OEE} \\ EOE \\ \underline{EOE} \end{array}$$

Let the quotient be $OOE = \Theta_1\Theta_2E_1$ so as to distinguish it from the divisor. Let $OOE \leqq \Theta_1\Theta_2E_1$. Since $EEOOE \leqq 88998$, we have $OOE < 199 < 298 < \sqrt{88998}$. Hence the first digit of the divisor is 1. Furthermore, $\Theta_1\Theta_2E_1 \leqq 798$. An examination of the products EOE, OEE, EOE reveals $7 \geqq \Theta_1 \geqq 3$, $\Theta_2 > 1$ and $\Theta_2 \neq \Theta_1$.

Case I. $\Theta_1 = 3$.	Then $OO =$ 11	$3 \times OO =$ 33	
	13	39	
	15	45	
	17	51	
	19	57.	

Only $OO = 13$ offers any chance that $OOE \times 3 = EOE$. This is possible only if $OOE = 138$. Now $\Theta_2 = 5$ or 7 and only $\Theta_2 = 7$ yields a product of the form $138 \times \Theta_2 = OEE = 966$. But an examination of the possible values of *EEOOE* shows this is impossible. Hence $\Theta_1 \neq 3$.

Case II. $\Theta_1 = 5$.	Then $OO =$ 11	$5 \times OO =$ 55	
	13	65	
	15	75	
	17	85.	

Reasoning as in case I, $OO = 13$ or $OO = 17$. If $OO = 17$, then $\Theta_2 = 3$ which is impossible since it leads to the same kind of contradiction as in case I. If $OO = 13$, then $\Theta_2 = 3$ or 7. Reasoning as above $OOE = 138$, $\Theta_2 = 7$ leads to a contradiction $\Theta_2 = 3$ implies quotient $\Theta_1\Theta_2E = 532$ but 138×532 contradicts *EEOOE*. Hence $\Theta_1 \neq 5$.

Case III. $\Theta_1 = 7$. Then $OO = \begin{matrix} 11 \\ 13 \end{matrix}$ $\quad 7 \times OO = \begin{matrix} 77 \\ 91 \end{matrix}$

Clearly only $OO = 11$ is acceptable and $7 \times OOE = EOE$ implies $OOE = 116$. Then $\Theta_2 = 3$ or 5. Also $E_1 \times 116 = EOE$ implies $E_1 = 2$ or 6. Hence the quotient is among the numbers 732, 736, 752, or 756. Of these only $\Theta_1\Theta_2E_1 = 732$ gives a product of the form *EEOOE*. Hence the unique solution is:

$$
\begin{array}{r|r}
 & 732 \\ \hline
116 & 84912 \\
 & 812 \\ \hline
 & 371 \\
 & 348 \\ \hline
 & 232 \\
 & 232 \\ \hline
\end{array}
\quad .
$$

The Book Shelf

EDITED BY JAMES BIDWELL

This department of *The Pentagon* brings to the attention of its readers published books (both old and new) which are of a common nature to all students of mathematics. Preference will be given to those books written in English or to English translations. Books to be reviewed should be sent to Dr. James Bidwell, Central Michigan University, Mount Pleasant, Michigan 48858.

Advanced Calculus, Stephen Hoffman, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1970, 383 pp., $10.50.

The number of different courses in advanced calculus being offered today is a sizable percentage of the number of colleges which offer the subject. It is, therefore, impossible for an author of such a text to satisfy everyone. Mr. Hoffman has made no attempt to write an all-purpose text but has concentrated on topics which he considers important for his course in advanced calculus. This procedure has contributed to a continuity and clarity of ideas not always found in such texts. The text is for the traditional advanced calculus course, but has fewer topics than are sometimes found and a greater continuity of material with more proof of theory. The level of material has as prerequisite one year of calculus, and a student at this level should be able to read this text with understanding. Careful motivation and good explanations are given to the many theorems which are carefully stated and proved.

The development is continuous and must be pursued in the order given. The first one-third of the text is devoted to the development of vectors and vector operations, leading up to the classical Divergence Theorem and Stokes' Theorem in the real space of three dimensions. The development of vectors is on n-dimensional vector spaces satisfying the usual vector space axioms. Good proofs are given to major theorems, and an adequate selection of exercises is available with each section.

The middle third of the text is concerned with integration, continuity, and differentiation. A theoretical treatment of integration is developed for functions on "quadrable" n-dimensional sets into the reals, making use of concepts of upper and lower Riemann integrals. Continuity of functions on n-dimensional sets is done carefully with good balance between explanation and proof. The

Heine-Borel Theorem is proved and applied to continuous functions to obtain criteria for uniform continuity.

The last third of the text is devoted to improper integrals, including the Divergence Theorem and Stokes' Theorem with improper integrals, infinite series, and power series representations. This section includes a *minimal* treatment of the Laplace Transform and Gamma Function, Fourier Series, and Orthogonal Functions. For self-help, a short appendix on series solutions of differential equations is adequate, but appendices on matrices, and partial differential equations provide little more than definitions.

In general the book is well written with good explanations and problem examples. If the series of topics listed above is desired in an advanced calculus course, this text is worthy of examination.

C. J. Pipes
Southern Methodist University

Analytic Geometry and the Calculus, Second Edition, A. M. Goodman, The Macmillan Company, New York, 819 pp., $12.95.

A. W. Goodman's second edition of his *Analytic Geometry and the Calculus* is a rewrite of his 1963 textbook that includes several additions in content, portions of which are adaptations of material from his two volume series *Modern Calculus with Analytic Geometry*. Chapters I and II are new material for the second edition as is Chapter XXII. Chapter I is devoted to an explanation of terms, notation, and the background mathematics assumed to be known by the reader. Chapter II is a brief and quite concise discussion of real inequalities, absolute value of real numbers, and directed distances. The other new material, Chapter XXII, also the last chapter of the text, presents basic linear algebra beginning with the concept of n-dimensional vector spaces and then develops the ideas of matrix algebra, linear transformations and eigen-vectors in quick succession.

In this new edition most of the illustrations have been redrawn and a number of exercises have been added. The sections on integration, primarily the definite integral, conics and series have been revised. These revisions improve the content of the textbook and enhance its value for classroom use. The material of the book is organized so that it may be taught as a four-hour per semester, three-semester sequence course, or as a three-hour per semester, four-semester sequence course, or as a five-hour per quarter, four-quarter sequence course. That is, the organization and presentation of calculus with analytic geometry in this textbook is standard and consequently quite similar to the majority of the books available for such courses. It is written for average science and engineering majors. Mathematics majors and above average students would need to supplement the material as presented by extra work taken from the plentiful supply of references cited and the three appendices in the book.

Outstanding features of the book are (1) it is readable and well written, (2) it is clean of errors, (3) important equations and formulas are enclosed in boxes, (4) it has ample exercises, (5) many of the chapters are prefaced by a section entitled "Objective" which outlines the intent of the presentation in the chapter and also includes some appropriate historical notes, (6) answers are given to the majority of the problems, (7) useful indices and references are included, and (8) it has many very good illustrations.

Items which might not appeal to some are (1) it makes use of the delta-increment notation in the presentation of limits, (2) a number of important theorems are not proved and often when proofs are given they are more intuitive than rigorous, (3) much of the development of theory is intuitive in nature, (4) the material on linear algebra stands by itself and for all practical purposes appears to have no connection with calculus and analytic geometry, and (5) the concepts of line integrals and Jacobians have been omitted.

It can be stated that this is a good, standard calculus with analytic geometry textbook which may help illustrate the observation that no new calculus and analytic geometry problems have been created in the past one hundred years.

Robert L. Poe
Berry College

Linear Analysis and Differential Equations, Richard C. MacCamy and Victor J. Mizel, The Macmillan Company, Toronto, Ontario, 1969, xiii, 561 pp.

This book is highly recommended for anyone having a background in elementary calculus that includes an introduction to functions of several variables. The reader whose main interest is applied mathematics *or* pure mathematics can profit from reading this book.

The authors manage to integrate linear algebra and differential equations while presenting physical motivation for most of the important topics. The book is not complete in the sense that not every theorem is proved. Indeed, the authors deliberately omit proofs that involve technical details. On the other hand, these theorems are illustrated through applications and heuristic arguments are never passed off as proofs.

The authors cover a wide range of material, including such topics as infinite dimensional spaces, partial differential equations and Stokes' Theorem. The presentation is modern. For example, many concepts are formulated by means of approximation theory. Indeed, Fourier series are presented as an extension of least square approximation.

Richard Dowds
State University College, Fredonia, N.Y.

Calculus with Analytic Geometry, Volume I, Angus Taylor &. C. J. Halberg, Prentice-Hall, Englewood Cliffs, N.J., 1969, 950 pp., $12.50.

This book has a balance of theory and applications approached from both the intuitive and the theoretical. Perhaps the intuitive approach to limits is carried to the extreme; that is, the student could fail to have any appreciation for a more formal approach after such a lengthy intuitive introduction.

An important asset of this text is the exercises—well chosen and arranged so as to lead the student to the more difficult ideas. On each topic there seems to be an ample supply of problems so that the student needing additional practice has a ready supply. The standard topics of functions of one variable are covered thoroughly with numerous examples. The concept of a function is developed after a review of the essential ideas of pre-calculus mathematics. The distinction between a function and a relation is made quite clear. The length of discussion on various topics could be too much. The above average student would perhaps become bored with the details. For the average or below average student (if you can get this student to read) this text would be excellent.

In the mass of calculus texts that have for years flooded the market, this text seemingly could be rated in between the average to above average category. An instructor would have to leave much of the material as reading assignments, obviously not the basic ideas but the discussion of these ideas. Any difficulties encountered from using this book as a text could, as in most cases, be overcome by a competent instructor.

Lloyd Koontz
Eastern Illinois University

MINIREVIEWS

Elements of Algebra, Francis J. Mueller, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1969, 356 pp., $7.95.

This book is a college text for students who have had little or no algebra in secondary school. The content is standard secondary school algebra with little modern terminology or use of sets in the body of the book. Many examples are worked out. The appendices include intuitive probability, progressions, set theory, and statistics. Answers to odd-numbered exercises are included.

Modern Algebra with Trigonometry, John T. Moore, The Macmillan Company, Toronto, Ontario, 1969, 423 pp.

This book is the second edition of a text designed as a precalculus text without analytic geometry. Emphasis is on systems of equations and matrices, exponential, logarithmic, and circular functions. Also included are chapters on complex numbers, theory of equations, and the binomial theorem. Properties of real numbers are developed and concept of function integrates the book. The approach used appears quite good for prospective majors. It could be used either as a high school or college text although written for the college level. Answers for odd-numbered exercises are included.

Management Decision Making under Uncertainty: An Introduction to Probability and Statistical Decision Theory, T. R. Dyckman, S. Smidt, and A. K. McAdams, The Macmillan Company, Toronto, Ontaria, 1969, 662 pp., $10.95.

"This book is designed for use in an introductory course in probability and statistics for students interested in the solution of managerial problems." Calculus is not required, but it is recommended as a prerequisite. Problems are oriented for business applications. Contents include, besides normal introductory material, chapters on random variables, two-person games, utility theory, sampling, and decision theory with and without sampling. Proofs are often given for the basic theory involved.

Go with the Odds, Charles Goren, The Macmillan Company, New York, 1969, 308 pp., $6.95.

This delightful book on gambling and bridge combines experience and mathematics to give a practical guide for players. Although little real mathematics is involved, many probabilities and expectations are discussed. Casino games, lotteries, bridge, and other card games are emphasized. The teacher may find the examples useful in the classroom or in his private interests!

Calculus, Arthur B. Simon, The Macmillan Company, New York, 1970, 626 pp., $11.95.

This text on calculus is designed to cover three semesters. It contains no development of analytic geometry. The book begins with a chapter on sets and functions. Chapter II is on computational calculus *without* proof. Succeeding chapters develop the theory underlying derivative and integral computation. This approach is novel. The later chapters include an introduction to linear algebra, multiple integrals and differential equations. A complete answer key is appended.

Elementary Concepts of Mathematics, Third Edition, Burton W. Jones, The Macmillan Company, New York, 1970, 400 pp.

The new edition of this text, originally published in 1940, differs little from the second edition. Chapters include sets, logic, number systems, topics in elementary algebra, probability, minor geometry, Lorentz geometry, and topology. It would be of interest to liberal arts students as well as prospective teachers of elementary or junior high school. An answer key for odd-numbered problems is included.

I do not know what I may appear to the world, but to myself I seem to have been only a boy playing on the seashore, and diverting myself in now-and-then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.—I. NEWTON

The Mathematical Scrapbook

EDITED BY RICHARD LEE BARLOW

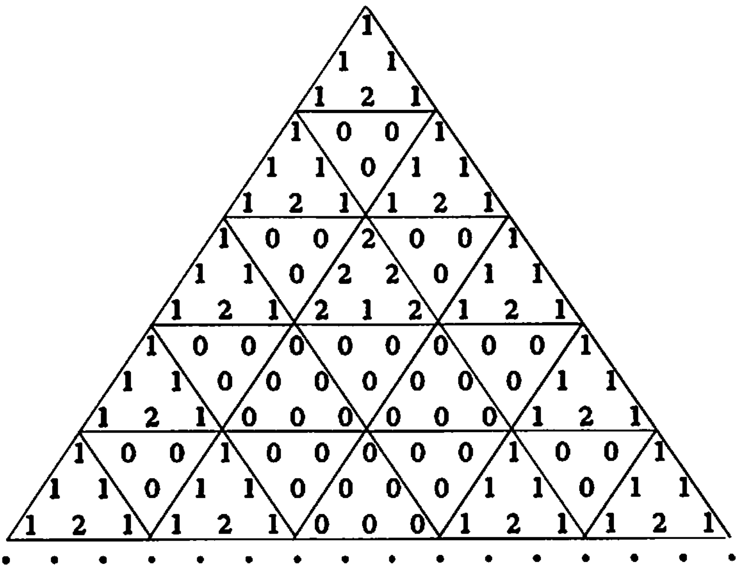
Readers are encouraged to submit Scrapbook material to the Scrapbook editor. Material will be used where possible and acknowledgment will be made in THE PENTAGON. If your chapter of Kappa Mu Epsilon would like to contribute the entire Scrapbook section as a chapter project, please contact the Scrapbook editor, Professor Richard L. Barlow, Kearney State College, Kearney, Nebraska.

In practically every elementary algebra course, Pascal's triangle is derived by arranging in an infinite triangular table the coefficients of the terms resulting from the expansion of the binomial $(a + b)^n$, where $n = 0, 1, 2, 3 \cdots$. Thus, Pascal's triangle becomes:

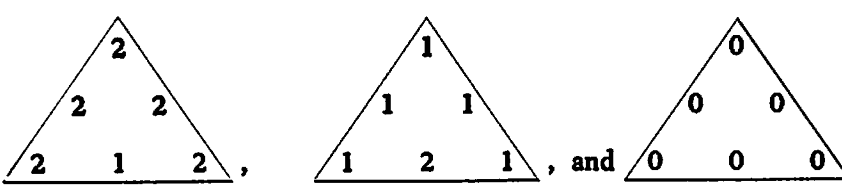
$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1&&&&&&&& \\
&&&&&&&1&&1&&&&&&& \\
&&&&&&1&&2&&1&&&&&& \\
&&&&&1&&3&&3&&1&&&&& \\
&&&&1&&4&&6&&4&&1&&&& \\
&&&1&&5&&10&&10&&5&&1&&& \\
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1 \\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&&&&&&&
\end{array}$$

After carefully observing the pattern of the numbers in each row, the student soon notes that every number (except the first and the last) in each row is equal to the sum of the two numbers which lie above it to the right and left.

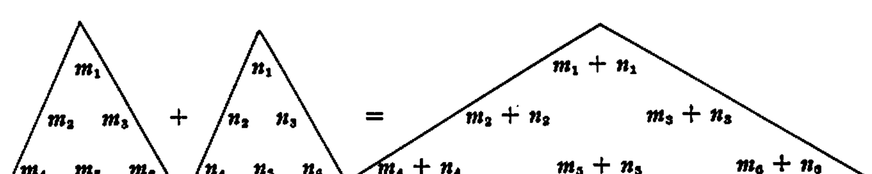
An interesting result occurs when Pascal's triangle is written in terms of a prime modulus. For example, Pascal's triangle modulo 3 is:



By forming the smaller upward pointed triangles of three rows each as indicated above, it is noted that the resulting downward pointed triangles contain only zero entries. The upward pointed triangles, which we shall call basic triangles, consist of triangles of three types:



Now define the addition of two basic triangles to be:



Observing the triangular patterns, one will note that the basic triangles forming the ends of rows of triangles are all the same type, namely

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

The other basic triangles, however, are formed by taking the sum of the two basic triangles lying above it to the right and to the left, as was the case with the numbers in the usual Pascal's triangle. This concept, which we developed for modulo three, also holds for Pascal's triangle in any prime modulus.

Can you show this is true modulo five?

— Δ —

In the Scrapbook section of the Spring 1969 issue of *The Pentagon,* a formula was presented for determining the day of the week upon which the rth day of the mth month of the year N would fall. At that time, it was noted that the formula presented was valid only for the years $N \geqq 1600$, which were years after the 1582 Gregorian revision of the Julian calendar. One might ask why was the calendar revision necessary in 1582 and is our present calendar completely accurate?

One of the earliest known calendars was developed in primitive Egypt as an aid to the farmers plagued by the floods of the Nile. The Nile floods occurred quite regularly and approximately coincided with the heliacal rising of the star Sirius. Hence, in the year (approximately 2773 B.C.) in which the Egyptian calendar was adopted, the first day of the first month began with the observation of the heliacal rising of Sirius, which also approximately coincided with the beginning of the summer season. Their year consisted of three seasons of four months each, with each of the twelve months having exactly thirty days followed by an intercalary period of five days. This resulted in each year having a uniform length of 365 days. Since the constant 365 day calendar year was approximately one-fourth of a day shorter than the actual solar year, the calendar advanced until, at the end of 1456 solar years, it had gained one complete solar year and arrived back at its starting point. Even with the Egyptian calendar's inaccuracies, it served them well and it was retained for almost 3000 years.

In 46 B.C., Julius Caesar introduced the Julian calendar which was created by the Alexandrian astronomer Sosigenes. This calendar was to correct the 0.25 day per year error of the Egyptian calendar by regularly adding one day every fourth year (the years divisible by four), called leap years.

However, the estimate of 0.25 day rather than the actual 0.2422 day resulted in an error of the Julian calendar. This slight but ever increasing error in the Julian calendar resulted in an accumulated ten day error by the 16th century. In 1582, the Gregorian revision of the Julian calendar was introduced by Pope Gregory XIII to correct this error. This correction was made by omitting ten days of that year (the day following October 4 was October 15) to bring the calendar and the sun back into correspondence. To correct the leap year problem, leap years were to occur every year divisible by four except in the centurial year. A centurial year is a leap year only if it is divisible by 400. (Thus the years 1700, 1800, and 1900 would not be centurial leap years but the year 2000 would be a centurial leap year.) While the Gregorian calendar was adopted by many countries around 1582, England and her colonies did not adopt the Gregorian calendar until 1752 and Russia not until 1918. Is our calendar completely accurate? Not quite. There is a slight error in the fourth decimal place, which will amount to a full day in about 3,300 years.

— Δ —

The classical algebraic proof that $2 = 1$ is known by almost every mathematics student. The following proof is one which involves a method which can be adapted to show the equivalence of any natural number and 1 by properly choosing the algebraic function.

Suppose $y = \sqrt{x}$. Then by solving for x we have $x = y^2$.

Then,

$$\iint dy\,dx = \int (y + k_1)dx$$

$$= \int (x^{1/2} + k_1)dx$$

$$= \tfrac{2}{3}x^{3/2} + k_1x + k_2\,,$$

where the k_i's represent some constant.

Also,

$$\iint dx\,dy = \int (x + k_3)dy$$

$$= \int (y^2 + k_3)dy$$

$$= \tfrac{1}{3}y^3 + k_3y + k_4$$

$$= \tfrac{1}{3}x^{3/2} + k_3x^{1/2} + k_4\,,$$

where the k_i's again represent some constant.

But since,

$$\iint dy\,dx = \iint dx\,dy, \text{ we have}$$

$$\tfrac{2}{3}x^{3/2} + k_1x + k_2 = \tfrac{1}{3}x^{3/2} + k_3x^{1/2} + k_4\,.$$

Since these polynomials are equal, their respective coefficients of like terms must be equal.

Hence,

$$\tfrac{2}{3} = \tfrac{1}{3}\,,$$
$$k_1 = 0\,,$$
$$0 = k_3\,,$$
$$k_2 = k_4\,.$$

But if $\frac{2}{3} = \frac{1}{3}$, then $2 = 1$. Can you find an error?

Kappa Mu Epsilon News

EDITED BY EDDIE W. ROBINSON, *Historian*

Albama Beta, Florence State University, Florence

25 members, Charles E. Wilson, President; Dr. Elizabeth T. Woolridge, Corresponding Secretary.

Fourteen new members were initiated at a banquet in April, 1969. The speaker was William Green, a 1935 charter member of the chapter. Activities include a tutoring service for elementary and secondary students in the local area as well as university students. A coffee hour was held at Homecoming with twenty-five alumni members from twenty-one different years attending.

Mrs. Mary R. Hudson, sponsor, retired at the end of the 1968-69 school year. She received the University's annual "Faculty Member of the Year" award and was honored by a "This is Your Life" program at a KME meeting.

A 1966 initiate, Eddy Joe Brackin, has returned to the teaching faculty.

Alabama Gamma, University of Montevallo

21 members, 8 pledges, Edgar C. Torbett, III, President; Ned A. Lowrey, Corresponding Secretary.

Eight members were initiated in December. Two programs presented were "Kinds of Infinity" and "An Introduction to Non-Euclidean Geometry." The chapter sponsored a get-acquainted party for freshman mathematics majors and minors.

Alabama Epsilon, Huntingdon College, Montgomery

8 actives, 3 pledges, Gloria Spikes, President; Dr. Rex Jones, Corresponding Secretary.

The chapter sponsored a trip to Oakridge National Laboratories and is planning a trip to Houston's Manned Space Flight Center.

California Gamma, California State Polytechnic College, San Luis Obispo

46 student members, 23 faculty, 24 pledges, Dick Bradshaw, President; Dr. George Mach, Corresponding Secretary.

Dr. Leonard Tornheim, Chenson Research Company, spoke at the January initiation banquet honoring twenty-four new members. The chapter started a free tutorial service, available to all students.

California Delta, California State Polytechnic College, Kellogg Voorhies Campus, Pomona

20 members, 11 pledges, Jack Parker, President; Professor A. Konigsberg, Corresponding Secretary.

The chapter has a five-day tutoring program each week and a lecture series, "Careers in Mathematics." Activities include a display at the annual California Polytechnic Open House, "Poly Vue," and a year-end picnic for graduating members.

Connecticut Alpha, Southern Connecticut State College, New Haven

67 members, Donald Browning, President; Mrs. Loretta Smith, Corresponding Secretary.

Two programs were "The Use of Computers in the Classroom" by Dr. Washburn and "Inconsistencies in Euclidean Geometry" by Dr. Grant. Chapter members helped to revise the mathematics curriculum.

Illinois Beta, Eastern Illinois University, Charleston

Denny Han, President; Ruth Queary, Corresponding Secretary.

The highlight of the year is a banquet which follows the spring quarter initiation. The O'Brien Scholarship, the Van Deventer Calculus Prize, the Taylor Award, and the Freshman Award are to be presented at this banquet. The chapter and the Mathematics Club is sponsoring a trip to the Museum of Science and Industry in Chicago.

Illinois Epsilon, North Park College, Chicago

14 members, Roberta Nuckals, President; Alice Iverson, Corresponding Secretary.

Programs have included the movie, "The Search for Solid Ground," and three lectures: "Color Plus Math Equals Insight" by P. McCray, "Geometric Solitaire" by C. A. Jacakes, and "Continued Fractions" by Roger Griffith. The Visiting Lecturer for 1970 will be Professor Wade Ellis.

Indiana Alpha, Manchester College, North Manchester

14 members, Phil Bantz, President; David Neuhouser, Corresponding Secretary.

The chapter meets every other week with programs given by faculty, students and outside speakers. Speakers have been Dr. Zimmerman, Goshen College, and Dr. Bittinger from Indiana University.

Iowa Gamma, Morningside College, Sioux City

36 members, Craig Bainbridge, President; Elsie Muller, Corresponding Secretary.

The chapter has sponsored visits by lecturers from MAA, SIAM, and the Statistics Association and co-sponsors a colloquium with two neighboring liberal arts colleges. Four members participated in the William Lowell Putnam Competition.

Kansas Alpha, Kansas State College of Pittsburg

55 members, James Harlin, President; Dr. Harold L. Thomas, Corresponding Secretary.

Curtis Woodhead discussed the problem of the probability that Friday falls on the 13th day of the month at the October meeting. Kathy Peterson presented the November program with a paper on the trisection of an angle. Additional programs were the "Projective Plane" by Dr. Elwyn Davis and a discussion of student teaching experiences. Recipients of the annual Robert Miller Mendenhall Award for scholastic achievement were Mary Blood and Helen Gardner. Steven Armstrong was recognized as the Outstanding Senior in Secondary Education.

Kansas Gamma, Mount St. Scholastica, Atchison

8 members, 7 pledges, Judy Graney, President; Sister Helen Sullivan, Corresponding Secretary.

Semimonthly meetings are held, featuring student papers. A program about computer mathematics was presented by Sr. DeMontfort Knightly of Lillis High School, Kansas City. Other programs included a pledge presentation, "The History of Mathematics."

Maryland Alpha, College of Notre Dame of Maryland, Baltimore

7 members, 3 pledges, Susan Landgraf, President; Sister Marie Augustine, Corresponding Secretary.

Programs have included "The Magic of Magic Squares" by Susan Landgraf, "Odd Numbers" by Jeanette Gilmore, and a joint meeting with Maryland Beta when Dr. A. I. Thaler spoke on "Braids."

Maryland Beta, Western Maryland College, Westminster

20 members, David K. Baugh, President; James E. Lightner, Corresponding Secretary.

The planning meeting for the year was held at the home of the faculty adviser. Two programs are geared for mathematics education and mathematical applications in industry and government. The chapter established the Clyde A. Spicer Award to be given to the sophomore who shows the most potential as a mathematics major. The first recipient was Miss Gloria Phillips.

Michigan Alpha, Albion College, Albion

Cathy Amos, President; W. K. Moore, Corresponding Secretary.

Programs and meetings: Movie, "Let's Teach Guessing," "Sequences" by John Wenzel, Albion College; "Graph Theory" by Professor Arthur White, Michigan State; "Regular Polyhedra with Holes," Professor B. M. Stewart; "Topology," Cathy Wassick; "Space Filling Curves," Professor Donald Malm.

Michigan Beta, Central Michigan University, Mount Pleasant

45 members, Marie Barns, President; Dean Hinshaw, Corresponding Secretary.

Activities included the fall picnic, tutoring, high school visitation program, initiation, showcase, and planning for the Regional Convention on April 17-18.

Mississippi Alpha, Mississippi State College for Women, Columbus

12 members, 14 pledges, Susan Vaughan, President; Dr. Donald King, Corresponding Secretary.

Activities include a Christmas party and a picnic. Program speakers included Dr. Noel Childrens and Dr. Stephen Puckette.

Mississippi Gamma, University of Southern Mississippi, Hattiesburg

20 members, William Marshall, President; Jack V. Munn, Corresponding Secretary.

The chapter plans to undertake a membership drive. Plans are being made to send members to speak at surrounding high schools and to sponsor a tutoring service.

Missouri Beta, Central Missouri State College, Warrensburg

21 members, 19 pledges, Mary Lou Russell, President; Velma S. Birkhead, Corresponding Secretary.

Programs have included a film, "Mathematical Peep Shows," and guest speakers: Mr. Wallace Griffith, "Concepts in Computer Programming;" Dr. Norman Royal, "Facts and Fantasies of the Space Age;" Mr. John Myrick, "Applications of Mathematics at Western Electric;" and Dr. Henry Polowy, "How to Count Fish in a Pond, Lake or Ocean." Activities have included: Hub participation, selling CRC handbooks, sponsoring mathematics mixer for all students interested in mathematics, a car in the homecoming parade, College Bowl, registration at $(MAT)^2$ Conference, Christmas party for members and their dates and the Spring Banquet for KME and Sigma Zeta.

The 1970 Regional Conference is to be held on CMSC campus, April 25.

Missouri Gamma, William Jewell College, Liberty

8 members, Don Page, President; Sherman Sherrick, Corresponding Secretary.

Programs and activities: Mr. Truett Mathis, "An Enumeration of Logical Functions;" movie, "Donald in Mathemagicland;" Wai Mui Lau, "Some Nonparametric Methods;" Judy Wyss, "Mathematics in the Secondary Schools;" Dr. Darrell Thomas, "Special Topics in Probability and Statistics;" banquet and initiation, Seymour Schuster, speaker.

Missouri Epsilon, Central Methodist College, Fayette

12 members, 7 pledges, Barbara Richardson, President; W. H. Enrich, Corresponding Secretary.

At least one paper is presented at each meeting. Activities included a display and entertainment for Parent's Day, tutoring sessions for freshmen and publication of a brochure on graduates and their activities for promotional activities.

Missouri Zeta, University of Missouri, Rolla

17 members, 16 pledges, Quince Hadley, President; Lyman T. Smith, Corresponding Secretary.

Activities include two meetings a month, banquet and initiation, and various speakers at the different meetings.

Missouri Eta, Northeast Missouri State College, Kirksville

24 members, 3 pledges, Patrick O'Rourke, President; William Weber, Corresponding Secretary.

Activities have included: annual spring picnic, attending National Convention, tutoring program, guest lecture series, attending professional meetings. Programs have been: "Fermat's Last Theorem," "Pythagorean Triples," "A Study of N-Dimensional Equations," "The Area of a Pythagorean Triangle and the Number Six," "Inversions of Conic Sections through a Unit Circle," and "Fixed Points."

Nebraska Alpha, Wayne State College, Wayne

30 members, Doris Haltorf, President; Maurice Anderson, Corresponding Secretary.

Meetings are held once a month with initiates presenting papers. Thirteen new members were initiated this fall. A banquet is planned for April in conjunction with Lambda Delta Lambda. Eleven students attended the National Convention at Cedar Falls, Iowa, and the chapter plans to send a delegation to Warrensburg.

Nebraska Beta, Kearney State College, Kearney

46 members, 19 pledges, Larry Babcock, President; Richard Barlow, Corresponding Secretary.

Activities were: Christmas party, spring banquet with a visiting lecturer, distribution of 1500 copies of the chapter newsletter to the area high schools. A $50 scholarship will be awarded to a deserving KME member at the spring banquet. Fourteen members and two faculty sponsors are going to attend the Regional Convention in Warrensburg.

New York Zeta, Colgate University, Hamilton

20 members, Bruce Schwaidelson, President; T. K. Frutiger, Corresponding Secretary.

Meetings have included two lectures: "Buffor's Needle Problem" by Professor A. R. Strand and "Godel's Theorem" by John T. Koranda. The initiation meeting will have Professor Wade Ellis as speaker.

New York Theta, St. Francis College, Brooklyn

18 members, 8 pledges, Robert Vincent, President; Donald Cascia, Corresponding Secretary.

This chapter was organized and installed during the past year. Programs have included a reception, linear algebra seminar, and two films, "Limits of Sequences" and "Definite Integral."

Ohio Alpha, Bowling Green State University

30 members, James Eiting, President; Harry Mathias, Corresponding Secretary.

The Chapter visited Marathon Oil's Computer Center and sponsored an open forum for the mathematics curriculum. The primary event was publication of the magazine *The Origin* with biographies on all faculty and KME members.

Oklahoma Alpha, Northeastern State College, Tahlequah

43 members, Robert Hughes, President; Dr. Raymond Carpenter, Corresponding Secretary.

Meetings are held on the first and third Thursday of each month with programs given by chapter members. Fall initiation occurs at the Christmas party and spring initiation at the Founders' Day Banquet in April.

Oklahoma Beta, University of Tulsa

31 members, Gary Miessler, President; Dr. Thomas Cairns, Corresponding Secretary.

During the month of October, the officers, assisted by a number of other members, assisted in the arrangement of a new faculty evaluation system, particularly as it concerned the mathematics department. This evaluation system involved the filling out of appropriate forms by students in all classes; subsequent evaluation of the faculty was accomplished by punching the responses on computer cards which were then analyzed by use of a Control Data computer.

The current officers were fortunate in having a considerable sum of money in the treasury from previous years. The society members are now in the process of arranging a special "Kappa Mu Epsilon Award" to be given to some deserving undergraduate mathematics student. It is hoped that the local finances can be restructured so that this award may be made an annual one.

Pennsylvania Alpha, Westminster College, New Wilmington

15 members, 19 pledges, Brian Pontuis, President; Dr. Thomas Nealigh, Corresponding Secretary.

Thirty members toured the Gulf Petroleum Research Facilities and one program was Mr. Alan Sternberg discussing employment opportunities for mathematics majors.

Pennsylvania Epsilon, Kutztown State College, Kutztown

21 members, 15 faculty, 10 pledges, David P. Zerbe, President; Dr. Dwight Daugherty, Corresponding Secretary.

Programs consist of KME students presenting papers. Two annual initiation dinners are held with outstanding speakers. Freshmen are invited to attend and participate in meetings.

Pennsylvania Zeta, Indiana University of Pennsylvania

47 members, Charles Breindel, President; Ida Arms, Corresponding Secretary.

Nineteen members were initiated on October 21, 1969. Mr. Joseph Angelo, a member of the mathematics faculty, presented a talk on "Continued Fractions." At the regular meeting in November, Mr. William R. Smith, Faculty Adviser, presented "The Dance Problem." He also presented this topic later at a regional meeting of NCTM in Cleveland. At the regular meeting in December, Mr. John Busovicki, a new member of the mathematics faculty this year and a graduate of IUP, talked about graduate study and opportunity for assistantships. Mr. Busovicki is a charter member of Pennsylvania Zeta Chapter. At the meeting in December officers for the next two semesters were elected. They are: President, Donald Laughery; Vice-President, Elaine Eichorn; Secretary, Bonita Miller; Treasurer, Frances Coledo.

During the spring semester we are having as a guest lecturer Dr. Haskell Cohen from the University of Massachusetts, and a banquet is planned for May.

The members and faculty advisers of Pennsylvania Zeta Chapter are pleased that the National Council has decided that the Biennial Convention will be held on the campus of Indiana University of Pennsylvania during April of 1971. We are looking forward to serving as your hosts for this important event.

Pennsylvania Eta, Grove City College

30 members, Richard Gies, President; Cameron Barr, Jr., Corresponding Secretary.

Initiates are required to prepare a ten-minute paper as requirement for admission into KME and these papers are used for programs. One recent program was Cindy Sexton's paper, "Computers in Education."

Pennsylvania Theta, Susquehanna University, Selinsgrove

15 members, 15 pledges, Margaret Harris, President; Carol Jensen, Corresponding Secretary.

Activities included a mathematics faculty reception, a tutoring program for students, and attendance at a mathematics convention at Swarthmore College. Two recent papers presented were "The Unique Colorability of Maps" and "The Value of an Infinite Series."

Pennsylvania Iota, Shippensburg State College

42 members, Sharon L. Fortna, President; John S. Mowbray, Corresponding Secretary.

This chapter was installed November 1, 1969, and since that time their activities have included a motion picture and discussion and a visiting lecturer, Dr. Preston Hammer of Pennsylvania State University, who spoke on "Beware of Axiom Mongers" and "Continuity—What It Means."

Texas Alpha, Texas Technological College, Lubbock

53 members, John Harris, President; Dr. Derald Walling, Corresponding Secretary.

Seven new members were initiated at the fall banquet and a picnic was held in September. Programs included Dr. Russel Seacat, "Math and Engineering;" Dr. George Innis, "Computers and Mathematics;" and an IBM representative who spoke on "Activities in a Computer Center."

Texas Zeta, Tarleton State College, Stephenville

11 members, 2 pledges, German Daniel, President; Timothy Flinn, Corresponding Secretary.

Programs have been three guest lecturers and one film. Activities included a pizza party for new mathematics majors and the Mathematics Club.

Virginia Alpha, Virginia State College, Petersburg

26 members, 3 pledges, Linda Bailey, President; Emma B. Smith, Corresponding Secretary.

The Spring 1969 Banquet was in honor of the sponsor, Dr. R. R. McDaniel, who retired as head of the Mathematics Department.

Wisconsin Alpha, Mount Mary College, Milwaukee

20 members, Mary Ellen Naber, President; Sister Mary Petronia, Corresponding Secretary.

Plans are being made for the annual mathematics contest. Programs have been the following: "Friday the Thirteenth," Jeanette Hoene; "Some Ideas in Topology," Mary Schwamb; "Solutions to Mr. and Mrs. Adams' Problem in Moving Their Furniture—Permutations," Gloria Saffald; "Introductory Ideas about Vectors," Susan Gesell; "Tricks in Multiplication," Barbara Kasseckert; "Linear Programming," Cathy Palzin; "Permutations and the Insanity Cubes," Linda Schneider; "Paper Folding," Barbara Pozorski; "Gonks," Betty Witt; and "Computing Devices," Sister Catherine Ann.

Eighteenth Biennial Convention

April 2-3, 1971

The eighteenth biennial convention of Kappa Mu Epsilon will be hosted by the Pennsylvania Zeta chapter and will be held on the campus of Indiana University of Pennsylvania on April 2-3, 1971. Students are encouraged to prepare and submit papers for presentation at the convention. Complete directions for the submission of papers are found on page 97 of this issue of *The Pentagon*.

All chapters are encouraged to plan early for as large a delegation of students and faculty as possible.

George R. Mach
National President