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National Officers

Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Geometric Inversion

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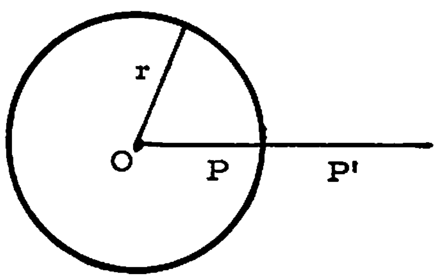
Geometric inversion is a process of setting up a one-to-one correspondence between points and their inverses. To come to a more precise mathematical definition of inverse points it is necessary to define some new terms. These terms are:

1. the circle of inversion—the circle on which we base the correspondence of our points and their inverses,
2. the center of inversion—the center of the circle of inversion, and
3. the radius of inversion—the radius of the circle of inversion.

Now we can define inverse points. Two points are inverse points if they are collinear with the center of the circle of inversion and if the product of their distances from the circle is equal to the square of the radius of the circle. The following illustration may help to indicate more clearly the definition of inverse points.

Consider a circle of inversion. Let the center of inversion be O and the radius of inversion be r. Take a point P. To have the inverse of P, which we can designate by P', P' must be chosen so that it is collinear with O and P and the distances $OP \cdot OP' = r^2$.

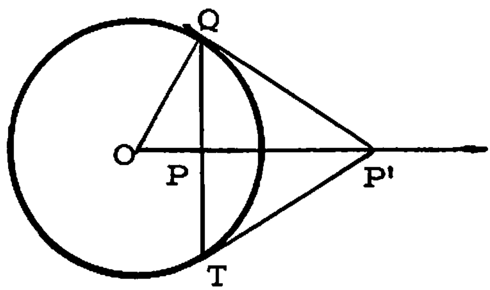


This case will then fulfill our definition for inverse points.

The next thing to consider is how to determine inverse points. We can do this either by construction or analytically. The process of construction would be to consider: a given circle of inversion with center O. If P is our given point and within the circle of inversion, we must find P' to be collinear with O and P. Therefore, extend

*A paper presented at the regional convention of KME at Rosary College, River Forest, Illinois, April 5-6, 1968.

the line segment $\overline{OP}$. At P construct a line which intersects the circle and is perpendicular to $\overline{OP}$. Call the points of intersection of the line and the circle Q and T. At Q or T construct a tangent to the circle. The tangent will intersect $\overline{OP}$ extended at P', the inverse of P.



The proof of this construction is quite simple.

1. $m\angle OPQ = m\angle OQP'$	1. Construction of right angles
2. $m\angle QOP = m\angle QOP'$	2. Identity
3. $m\angle OP'Q = m\angle OQP$	3. If two angles of one triangle are equal respectively to two angles of another triangle, the third angles are equal.
4. $\therefore \triangle OPQ \sim \triangle OQP'$	4. $AAA = AAA$
5. $OP/OQ = OQ/OP'$ or $OP \cdot OP' = (OQ)^2$	5. Corresponding sides of similar triangles are proportional.
6. P and P' are inverse points	6. By definition of inverse points.

If we had used $\overline{OT}$ as the radius, we would have obtained the same result.

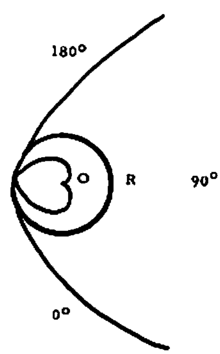
This proof of the construction of the geometric inverse indicates that if P' is the inverse of P, then P is also the inverse of P'. If P' is the given point and is outside the circle of inversion, its inverse P can be found by a reverse process of the previous construction. Using the same figure, construct tangents to the circle from P'. Call the points of tangency Q and T. Draw the line segment $\overline{QT}$. The point where $\overline{QT}$ intersects $\overline{OP'}$ is P.

The other method of determining inverse points is transforming a curve into its inverse curve by using the analytic equation of the curve and its inverse. For convenience in graphing, if one is working with the conic sections (one of the most interesting sets of points and inverses), the polar coordinate system may be used.

For example, take the equation $\rho = \dfrac{2R}{1 - \sin \Theta}$, where R is equal to the radius of inversion. This equation is a parabola with its focus at the center of inversion, its vertex on the circle of inversion, and the latus rectum is the distance $4R$.

For any point P on the parabola, there should be a point P' that is the inverse so that $OP \cdot OP' = R^2$, according to our definition of inverse if O is the center of inversion. To simplify the problem, consider the radius of inversion R to be a unit measure of 1. Then $OP \cdot OP' = 1$ and $OP = 1/OP'$ or the point and its inverse are reciprocals. With $R = 1$, we can go back to the original equation and find the equation of the geometric inverse by finding the reciprocal of the equation.

If $\rho = \dfrac{2R}{1 - \sin \Theta}$ or $\dfrac{\rho}{R} = \dfrac{2}{1 - \sin \Theta}$, then the inverse is $\dfrac{\rho'}{R} = \dfrac{1 - \sin \Theta}{2}$ or $\rho' = \dfrac{R(1 - \sin \Theta)}{2.}$. In graphing this inverted curve, the resulting figure is a cardioid with its cusp at the center of inversion.



Let us investigate some specific points:

Θ	$\sin \Theta$	parabola	cardioid
$0°, 180°$	0	$2R$	$R/2$
$30°$	$\frac{1}{2}$	$4R$	$R/4$
$90°$	1	∞	0
$270°$	-1	R	R

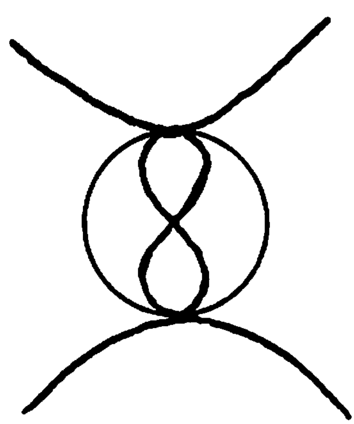
$2R \cdot R/2 = R^2$ and $4R \cdot R/4 = R^2$ which shows that the curves have inverses at these points.

We also notice that at $\Theta = 90°$ the cardioid is at O and the parabola at infinity. This is a general case for any figure. If a point is at the center of inversion its inverse will be in infinity. This seems reasonable if we look for the point by construction; since it takes two points to determine a line and we can find an infinite number of lines that contain the center of inversion and the given point at the center of inversion. Therefore, the inverse is at infinity. Another interesting case is when $\Theta = 270°$. Both the point and its inverse lie on the circle of inversion, which is always the case if the point is on the circle of inversion. This result is logical by our method of construction for if we tried to draw a tangent to the circle from P it would intersect the circle at P. Thus a point on the circle of inversion is its own inverse.

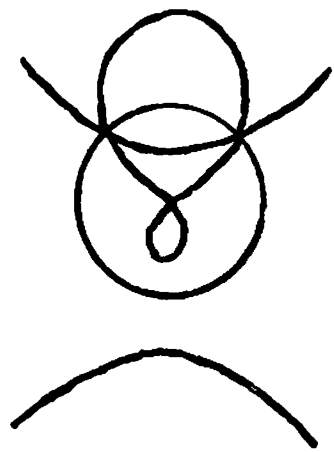
By examining other curves we will be able to obtain more properties of a set of points and their inverses. For a straight line which does not pass through the center of inversion, the inverse is a circle which passes through the center of inversion. The reverse will then also be true. A circle through the center of inversion has a straight line which does not pass through the center of inversion for its inverse. A straight line through the center of inversion will be its own inverse. The circle that does not pass through the center of inversion has another circle as its inverse. A further observation shows that the part of the circle outside the circle of inversion has its corresponding inverse within the circle of inversion and the part inside has its inverse outside.

Just these simple examples are enough to show that in determining an inverse we must not only take into consideration the kind of figure we are working with but its position in regard to the center of inversion. For instance, earlier we considered a parabola with the focus at the center of inversion and found the inverse to be a cardioid. If we took a parabola whose principal vertex was at the center of inversion, we would not have a cardioid but rather a cissoid of Diocles for an inverse.

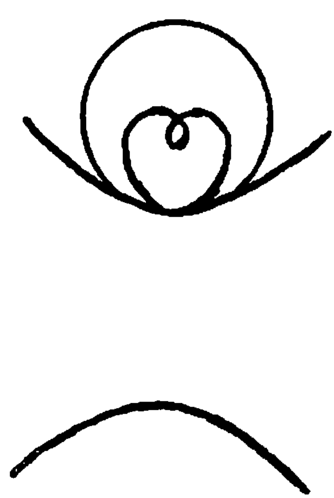
Another example is the hyperbola $r^2 \cos 2\Theta = 1$ and its inverse. If the center of inversion is equidistant from the vertices of the hyperbola and the radius of inversion is one-half the distance between the two branches of the hyperbola, the inverse will be a lemniscate. Each branch of the hyperbola will invert into one of the loops of the lemniscate as illustrated.



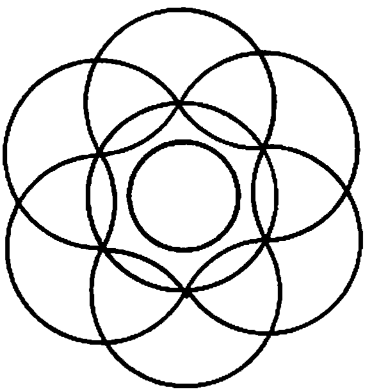
If we move the circle of inversion up or down so that it intersects only one of the branches of the hyperbola, the inversion is still a lemniscate with the larger loop being the inverse of the branch that the circle of inversion intersected, the smaller loop being the inverse of the other branch.



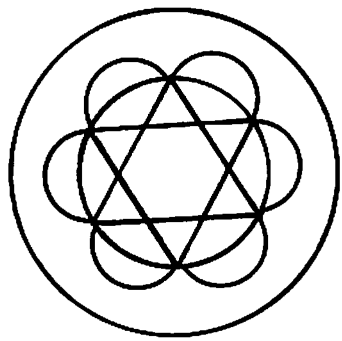
If we move the circle of inversion still further so that it intersects one branch of the hyperbola in only one point the inverse of the hyperbola turns out to be a limacon, the outer loop being the inverse of the branch which the circle intersected and the smaller loop being the inverse of the other branch.



From these examples it can be seen that we can have some very interesting things happen when we find inverses of curves. Not only that but we can build beautiful and symmetrical curves.



figure



inverse

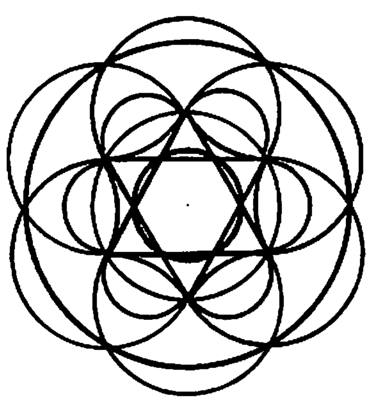


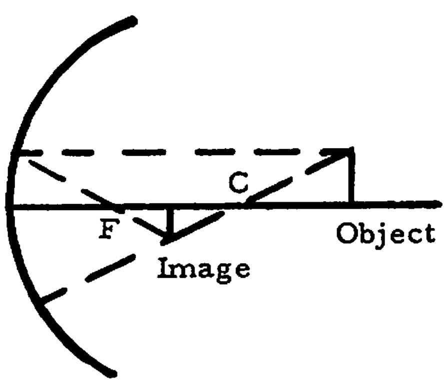
figure and inverse combined

However, aside from the beauty of geometric inversion there are also practical applications. Sir William Thomson used geometric inversion in proving some of the most difficult propositions in the mathematical theory of electricity. John Clark Maxwell applied geometric inversion to his work on electrical images and it can also be used to solve problems in optics.

A simple application from physics is supplied by the relative positions of an object and its image in a mirror or lens. This relationship is expressed by the equation $\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$ where:

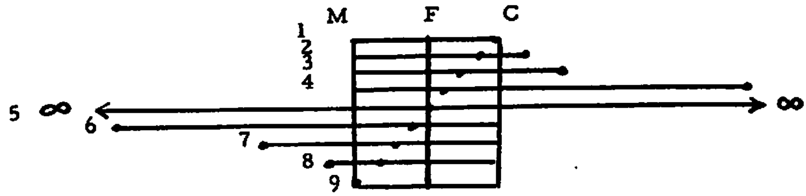
d_o = the distance of the object from the mirror or lens,
d_i = the distance of the image from the mirror or lens,
f = the distance of the principal focus from the mirror or of each of the principal foci from the lens.

For a specific example, consider a concave mirror. Let C equal the center of curvature of the mirror and F the principal focus which lies on the principal axis halfway between C and the mirror.



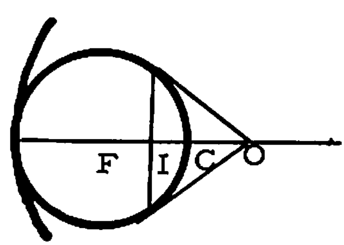
The dotted lines on the illustration are the constructions of the image. It operates with two rays from the object, one through the center of curvature which is reflected in itself and the other in a direction parallel to the principal axis which is reflected through the principal focus.

The next illustration is a diagram of the relative positions of the object and image as their positions change in relation to the same mirror, focus, and center of curvature.



In the first case the object and image are both at C. In the second, third, and fourth cases, as the object is moved toward *F*, the image is moving farther from C. In the fifth case, the object is at *F* and the image at infinity. In the sixth, seventh, and eighth cases the object approaches the mirror and the image is beyond the mirror and imaginary. The positions of the object and the image are also interchangeable.

Now we must correlate this example with geometric inversion. Consider a circle of inversion with the principal focus as the center and with a radius equal to the distance from the focus to the mirror. The circle of inversion also passes through C.



By definition $FI \cdot FO = (FC)^2$ or

$$(d_i - f)(d_o - f) = f^2$$

$$d_i d_o - f d_o - f d_i + f^2 = f^2$$

$$d_i d_o - f d_o - f d_i = 0$$

$$f d_o + f d_i = d_i d_o$$

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$

We have just derived the formula for the relationship of the object and its image. Therefore, we can consider the object to be a point and the image to be its inverse or vice versa. Going back to our previous illustration, we can think of the diagram as a relation of a point to its inverse. The results would be similar to what we have previously discovered about points and their inverses. For instance the circle of inversion passes through C and we said that a point on the circle of inversion is its own inverse. From this it can be seen that if the point is within the circle of inversion its inverse is outside the circle of inversion. And the point at *F*, which is the center of inversion, does have its inverse at infinity. This shows that there is a definite relation between objects and images to points and their inverses. It also shows that the possibilities for using geometric inversion are numerous, whether for the practical, theoretical, or aesthetic aspects.

REFERENCE

Scripta Mathematica, Vol. 14, pp. 113-125 and pp. 266-272.

To Thales . . . the primary question was not What do we know but How do we know it?

—Aristotle

Cubic Quadruples from Pythagorean Triples*

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When Fermat was studying the Pythagorean equation, $a^2 + b^2 = c^2$, he considered the general equation, $x^n + y^n = z^n$, which resulted in his "last" theorem. In this brief discussion the study of a slightly different analogue of the equation of Pythagoreas will be suggested.

In general it would seem to be desirable to know something of the integral solutions of:

$$(1) \quad x_1^n + x_2^n + \cdots + x_n^n = x_{n+1}^n .$$

By way of approaching that goal, the present consideration will deal with the cubic case of (1):

$$(2) \quad x^3 + y^3 + z^3 = w^3.$$

It goes without saying that, in addition to the restriction to integral solutions, one need not consider the trivial combinations of 0, -1, and 1 which offer solutions. Furthermore, it follows that should one find a quadruple of positive solutions, the corresponding four negative solutions will also be obtained. Dealing with the cubic form, however, causes one to exercise more care with the signs than in the Pythagorean case. For the present the discussion will be restricted to all positive solutions.

By trial, it is easily verified that the arithmetic quadruple $\{3, 4, 5, 6\}$ is one solution set of the equation. This is quite interesting when one recalls that the most famous of the Pythagorean triples is $\{3, 4, 5\}$. Perhaps, then, the solutions can be related to these triples.

Consider the parametric equation which generates all Pythagorean triples:

$$(3) \quad (2uv)^2 + (u^2 - v^2)^2 = (u^2 + v^2)^2,\ u > v$$

and both integral.

This equation yields primitive Pythagorean triples whenever $(u, v) = 1$. Hence, the simplest triple $\{3, 4, 5\}$ comes when $u = 2$, $v = 1$. Not only is this choice of u and v special in that $u > v$ and

*A paper presented at the regional convention of KME at Northeastern State College, Tahlequah, Oklahoma, April 20, 1968.

$(u, v) = 1$, but also in that $u = 2v$. So consider the sum $3^3 + 4^3 + 5^3$ using the parameters:

$$(u^2 - v^2)^3 + (2uv)^3 + (u^2 + v^2)^3 =$$

$$[(2v)^2 - v^2]^3 + (4v^2)^3 + [(2v)^2 + v^2]^3 =$$

$$4^3v^6 + 3^3v^6 + 5^3v^6 = 6^3v^6,$$

and $$6^3v^6 = (6v^2)^3 = (3uv)^3$$

also, $$6^3v^6 = (6v^2)^3 = [2(3v^2)]^3 = [2(u^2 - v^2)]^3.$$

Thus, one concludes that using the same parameters as for Pythagorean triples, and if $u = 2v$, one has a parametrization which yields "cubic quadruples" where one chooses the fourth element of the quadruple, say d, as $d = 3uv$, or $d = 2(u^2 - v^2) = 2b$.

Now consider the sum of three cubes for the more general case where $u = kv$, k a positive integer:

$$(2uv)^3 + (u^2 - v^2)^3 + (u^2 + v^2)^3 =$$

$$8k^3v^6 + (k^2 - 1)^3v^6 + (k^2 + 1)^3v^6 =$$

$$[8k^3 + (k^2 - 1)^3 + (k^2 + 1)^3]v^6 =$$

$$(2k^6 + 8k^3 + 6k^2)v^6.$$

Hence, one sees that the parameters $u = kv$ yield another cubic quadruple whenever

(4) $2k^6 + 8k^3 + 6k^2 = I^3$ has integral solution for any positive integer, I.

For instance, let $k = 2$, as discussed, then

$$\sqrt[3]{2k^6 + 8k^3 + 6k^2} = \sqrt[3]{216} = 6.$$

Checking further, one sees that $k = 3$ gives

$$\sqrt[3]{2k^6 + 8k^3 + 6k^2} = \sqrt[3]{2 \cdot 729 + 8 \cdot 27 + 6 \cdot 9}$$

$$= \sqrt[3]{1458 + 216 + 54} = \sqrt[3]{1728} = 12, \text{ integral.}$$

Thus $u = 3v$ is a parametric solution. When $v = 1$ the triple $\{6, 8, 10\}$ is obtained. And, from statement (4) above, the fourth member of the quadruple is 12 (since $v = 1$). Hence, the second solution set is also arithmetic: $\{6, 8, 10, 12\}$. This fact is quite interesting and deserves further attention.

In general, when $u = 3v$, one sees that:

$$12^3v^6 = (12v^2)^3 = (4 \cdot 3v \cdot v)^3 = (4uv)^3$$

implies the fourth member of the quadruple of this type is determined by the parametrization $d = 4uv$. Recalling that in the case where $k = 2$, the parametrization was $d = 3uv$ it seems that if there is a general formula for the fourth member, it must be $d = (k + 1)uv$.

At this point consider the quadruple generating triples of the parametric form $u = 2v$:

$v = 1$ yields $a = 4$; $b = 3$; $c = 5$; $d = 6$, and

$v = 2$ yields $a = 16$; $b = 12$; $c = 20$; $d = 24$, etc.

And with regards to the parametric form $u = 3v$ one has:

$v = 1$ yields $a = 6$; $b = 8$; $c = 10$; $d = 12$, and

$v = 2$ yields $a = 24$; $b = 32$; $c = 40$; $d = 48$, etc.

In spite of the fact that we are dealing with a cubic equation, a given family of solutions is related by a factor of v^2. This, of course, is a result of the dependence on the Pythagorean parameters for the solutions:

$$a = 2uv = 4kv;\ b = u^2 - v^2 = (k^2 - 1)^2v;$$

$$c = u^2 + v^2 = (k^2 + 1)v^2;\ d = (k + 1)uv = (k^2 + k)v^2$$

Thus, there are at least two non-trivial families of cubic quadruples. A formula has not been derived which can be proved to give sufficiently all cubic quadruples, as does the discussed parametrization give all Pythagorean triples.

In closing consider the solutions to the equation, $a^4 + b^4 + c^4 + d^4 = e^4$. In considering the cubic form, all the work here has been based on the nicety of the triple $\{3, 4, 5\}$ which yields the quadruple $\{3, 4, 5, 6\}$. However, as $(3^4 + 4^4 + 5^4 + 6^4)$ is not a perfect quadruple, such a treatment will not suffice. How shall the search be conducted? Further, how shall equation (1) be generalized?

Addendum:

Since this paper was first prepared, the author has had the opportunity to have access to an IBM system 1130 computer. By application of some elementary programming and techniques from the theory of numbers, he has successfully demonstrated that the above mentioned solutions for equation (4) are the only such solutions for $k \leqq 66$. The computation of solutions for (4) quickly becomes exceedingly complex. The value of the left-hand side of (4) for $k = 66$ is 165,310,226,136. The size of this number develops into a problem with memory space for the equipment available. Furthermore, the role of k in this discussion is such that to know that there are only two solutions for $k \leqq 66$ is somewhat meaningful. For instance, it may lead one to suspect that the number of solutions is very sparse, if not finite.

Concerning any of the problems herein proposed or discussed, the author would be indebted to any person who could extend this study in any way, and he will strive to be of assistance in discussing the many ramifications of this topic and in sharing his results.

Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world.

—Roger Bacon

Irrational Roots of Complex Numbers*

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Using DeMoivre's Theorem a simple method of finding integral roots of complex numbers can be developed. By extending this method, a method of finding irrational roots of complex numbers can be found and this can be used to prove some interesting properties of these irrational roots.

The method of finding integral roots of complex numbers can be obtained in the following manner (see [1] pp. 214-216).

DEFINITION 1:

P cis Θ is an n^{th} root of Q cis ϕ if $(P \text{ cis } \Theta)^n = Q \text{ cis } \phi$.

DeMoivre's Theorem:

$$(P \text{ cis } \Theta)^n = P^n \text{ cis } (n\Theta) .$$

DeMoivre's Theorem and Definition 1 can be used to prove the following theorem:

THEOREM 1:

The n^{th} roots of Q cis ϕ are of the form

$r_k = Q^{1/n} \text{ cis } \Theta_k$ where

$$\Theta_k = (\phi + 2k\pi)/n \quad , \quad k = 0, 1, 2, \cdots, (n-1) .$$

It can be seen that by letting $k = n$

$$\Theta_n = (\phi + 2n\pi)/n = \phi/n + 2\pi = \Theta_o + 2\pi .$$

Since cis Θ_o = cis$(\Theta_o + 2\pi)$, $r_o = r_n$. Therefore, for values of $k \geqq n$, $\Theta_k = \Theta_{k'}$ where $0 \leqq k' < n$ and there are only n distinct n^{th} roots. Also $| \Theta_{k_1} - \Theta_{k_2} | \geqq 2\pi/n$ for $k_1 \neq k_2$.

A method will now be developed to find irrational roots of complex numbers. To begin with, the following definition should be made:

DEFINITION 2:

Let a be an irrational number. Then P cis Θ is an a^{th} root of Q cis ϕ if $(P \text{ cis } \Theta)^a = Q \text{ cis } \phi$.

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*A paper presented at the regional convention of KME at Rosary College, River Forest, Illinois, April 5-6, 1968.

The following assumptions will be made:

1. $P \text{ cis } \Theta = Pe^{i\Theta}$.
2. Complex exponents obey the same laws as real exponents.
3. For P and a, positive real numbers, P^a is a positive real number.

THEOREM 2:

If a is an irrational number, $(P \text{ cis } \Theta)^a = P^a \text{ cis } (a\Theta)$.

Proof:

$$(P \text{ cis } \Theta)^a = (Pe^{i\Theta})^a$$
$$= P^a e^{ia\Theta}$$
$$(P \text{ cis } \Theta)^a = P^a \text{ cis } (a\Theta).$$

THEOREM 3:

If a is an irrational number, the a^{th} roots of Q cis ϕ are of the form:

$$R_k = Q^{1/a} \text{ cis } \Theta_k \quad \text{where}$$
$$\Theta_k = (\phi + 2k\pi)/a, \quad k \text{ an integer.}$$

Proof:

If $P \text{ cis } \Theta$ is an a^{th} root of Q cis ϕ then

$$(P \text{ cis } \Theta)^a = Q \text{ cis } \phi$$
$$P^a \text{ cis } (a\Theta) = Q \text{ cis } \phi \quad \text{or}$$
$$P^a(\cos(a\Theta) + i \sin(a\Theta)) = Q(\cos \phi + i \sin \phi) .$$

This equality will hold only if

$$(1) \quad P^a \cos(a\Theta) = Q \cos \phi \quad \text{and}$$
$$(2) \quad P^a \sin(a\Theta) = Q \sin \phi .$$

Squaring (1) and (2) and adding we get

$$P^{2a}(\cos^2(a\Theta) + \sin^2(a\Theta)) = Q^2(\cos^2 \phi + \sin^2 \phi)$$
$$\therefore \quad P^{2a} = Q^2 \quad \text{or} \quad P = Q^{1/a} .$$

Then from (1) and (2)

$$(Q^{1/a})^a \cos(a\Theta) = Q \cos \phi$$

$(Q^{1/a})^a \sin(a\Theta) = Q \sin\phi$ or

$\cos(a\Theta) = \cos\phi$ and $\sin(a\Theta) = \sin\phi$

$\therefore$ $a\Theta = \phi + 2k\pi$ or

$\Theta = (\phi + 2k\pi)/a$, k an integer.

Then $P \operatorname{cis} \Theta = Q^{1/a} \operatorname{cis}((\phi + 2k\pi)/a)$.

As an example, Theorem 3 will be used to find the π^{th} roots of 1:

$$1 = (1) \operatorname{cis}(0)$$

$$R_k = (1)^{1/\pi} \operatorname{cis}((0 + 2k\pi)/\pi)$$

$$R_k = \operatorname{cis}(2k) \quad , \quad k \text{ an integer.}$$

Theorem 3 can be used to prove some interesting properties of irrational roots of complex numbers. The first is stated in the following theorem.

Theorem 4:

If a is an irrational number, there is an infinite number of a^{th} roots of $Q \operatorname{cis} \phi$.

Proof:

If the theorem is false, then there would exist Θ_1 and Θ_2 such that

(1) $Q^{1/a} \operatorname{cis} \Theta_1 = Q^{1/a} \operatorname{cis} \Theta_2$ where

$\Theta_1 = (\phi + 2k_1\pi)/a$ and $\Theta_2 = (\phi + 2k_2\pi)/a$,

k_1 and k_2 integers.

From (1), $\operatorname{cis} \Theta_1 = \operatorname{cis} \Theta_2$ or

$$\cos\Theta_1 + i \sin\Theta_1 = \cos\Theta_2 + i \sin\Theta_2 \Rightarrow$$

$$\cos\Theta_1 = \cos\Theta_2 \quad \text{and} \quad \sin\Theta_1 = \sin\Theta_2 \Rightarrow$$

$$\Theta_1 = \Theta_2 + 2m\pi, \quad m \text{ an integer}$$

or $(\phi + 2k_1\pi)/a = (\phi + 2k_2\pi)/a + 2m\pi$.

Simplifying:

$$k_1 = k_2 + ma .$$

Since k_1 and k_2 are integers, ma must be an integer. This is a contradiction, since a is irrational.

Two sets will now be defined in order to demonstrate the last property of irrational roots which will be discussed. Let a be an irrational number.

$$R = \{r_n \mid r_n = Q^{1/a} \operatorname{cis}((\phi + 2n\pi)/a))\}$$

R is merely the set of a^{th} roots of Q cis ϕ.

$$A = \{\Theta_n \mid \Theta_n = (\phi + 2n\pi)/a - 2k_n\pi\,, \quad k_n \text{ an integer chosen so that } 0 \leqq \Theta_n < 2\}\,.$$

Each Θ_n is found by taking the angle in r_n and subtracting a sufficient number of even multiples of 2π so that $0 \leqq \Theta_n < 2\pi$. The Θ_n may be located on a circle in the following manner:

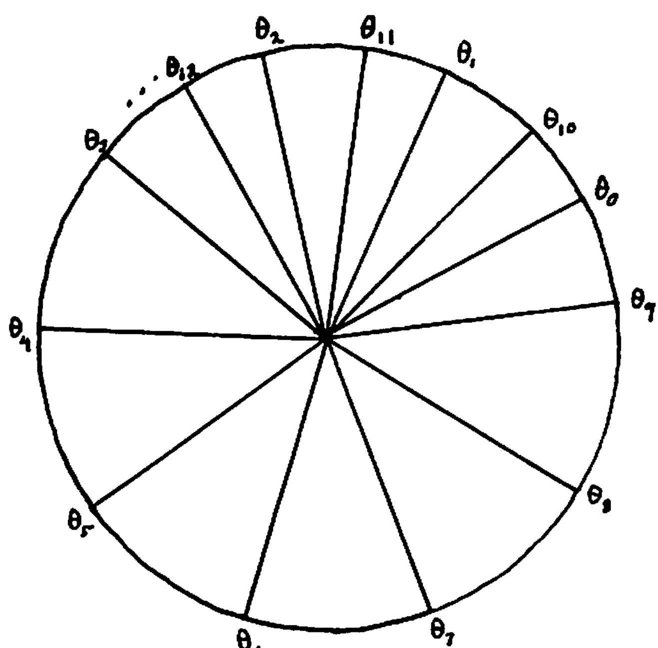

Figure 1

The question arises: How are these Θ_n distributed on the circle? The lemma and theorem following definition 3 will provide an answer to this.

DEFINITION 3:

$$D(\alpha, \beta) = \min \left(| \alpha - \beta |, \Big| | \alpha - \beta | - 2\pi \Big| \right).$$

$D(\alpha, \beta)$ is merely the shortest distance between two points on the circle. For example, if $\alpha = \pi/4$ and $\beta = 7\pi/4$ then

$$| \pi/4 - 7\pi/4 | = 3\pi/2 \quad \text{and}$$

$$\Big| | \pi/4 - 7\pi/4 | - 2\pi \Big| = | 3\pi/2 - 2\pi | = \pi/2.$$

Therefore $D(\pi/4, 7\pi/4) = \pi/2$.

LEMMA 1:

For every $\Theta_n \in A$, given any $\varepsilon > 0$, $\exists\ \Theta_m \in A$ such that $D(\Theta_m, \Theta_n) < \varepsilon$.

Proof:

If the conclusion is false, then for some Θ_n, $\exists\ \varepsilon_n > 0$ such that for all m, $D(\Theta_m, \Theta_n) \geqq \varepsilon_n \Rightarrow$ for all m, $| \Theta_m - \Theta_n | \geqq \varepsilon_n$ Let $\varepsilon = \min(\varepsilon_n, \Theta_n, 2\pi - \Theta_n)$ if $\Theta_n \neq 0$. If $\Theta_n = 0$, let $\varepsilon = \varepsilon_n$. There are two possibilities:

I. For all p and q, $| \Theta_p - \Theta_q | \geqq \varepsilon$.

II. $\exists\ \Theta_r$ and Θ_s such that $| \Theta_r - \Theta_s | < \varepsilon$.

I cannot be true since there is an infinite number of distinct Θ_n. Therefore Θ_r and Θ_s exist such that $| \Theta_r - \Theta_s | < \varepsilon$. Now add $T = (2\pi n - 2\pi s)/a - 2\pi k_n + 2\pi k_s$ to Θ_s.

$$\begin{aligned}\Theta_s + T &= (\phi + 2\pi s)/a - 2\pi k_s + (2\pi n - 2\pi s)/a \\ &\quad - 2\pi k_n + 2\pi k_s = (\phi + 2\pi n)/a - 2\pi k_n = \Theta_n.\end{aligned}$$

$$\begin{aligned}\text{Now } | \Theta_r - \Theta_s | &= | (\Theta_r + T) - (\Theta_s + T) | \\ &= | (\Theta_r + T) - \Theta_n | < \varepsilon.\end{aligned}$$

$$\begin{aligned}\Theta_r + T &= (\phi + 2\pi r)/a - 2\pi k_r + (2\pi n - 2\pi s)/a \\ &\quad - 2\pi k_n + 2\pi k_s = (\phi + 2\pi(r + n - s))/a \\ &\quad - 2\pi(k_r + k_n - k_s).\end{aligned}$$

Let $r + n - s = m$ and $k_r + k_n - k_s = k_m$.

Then $\Theta_r + T = (\phi + 2\pi m)/a - 2\pi k_m$ and

$$|\Theta_r - \Theta_s| = |\Theta_m - \Theta_n| < \varepsilon \Rightarrow$$

$$-\varepsilon < \Theta_m - \Theta_n < \varepsilon \quad \text{or}$$

(1) $\Theta_n - \varepsilon < \Theta_m < \Theta_n + \varepsilon.$

Now assume that $\Theta_n \neq 0$.

Then since $\varepsilon \leqq \Theta_n$ and $\varepsilon \leqq 2\pi - \Theta_n$.

$$\Theta_n - \varepsilon \geqq \Theta_n - \Theta_n = 0 \text{ and } \Theta_n + \varepsilon \leqq \Theta_n + 2\pi - \Theta_n = 2\pi$$

$$\Rightarrow 0 < \Theta_n < 2\pi.$$

This means that $\Theta_m \in A$ and $|\Theta_m - \Theta_n| < \varepsilon$, which contradicts our assumption that no such Θ_m existed.

If $\Theta_n = 0$, then from (1)

$$-\varepsilon < \Theta_m < \varepsilon.$$

If $0 \leqq \Theta_m < \varepsilon$, then $\Theta_m \in A$ and we have a contradiction as before.

If $-\varepsilon < \Theta_m < 0$, let $\Theta'_m = \Theta_m + 2\pi = (\phi + 2m\pi)/a - 2\pi(k_m - 1)$.

Then

(2) $2\pi - \varepsilon < \Theta'_m < 2\pi$ and $\Theta'_m \in A$.

From (2)

$$-\varepsilon < \Theta'_m - 2\pi < 0 < \varepsilon \Rightarrow$$

$$-\varepsilon < |\Theta'_m - 0| - 2\pi < \varepsilon \Rightarrow$$

$$\Big|\,|\Theta'_m - \Theta_n| - 2\pi\Big| < \varepsilon \Rightarrow$$

$$D(\Theta'_m, \Theta_n) < \varepsilon.$$

Again we have a contradiction and the theorem is proved.

THEOREM 4:

For any $\beta \in [0, 2\pi)$, given any $\varepsilon > 0$, $\exists\ \Theta_n \in A$ such that $D(\beta, \Theta_n) < \varepsilon$.

Proof:

If the theorem is false then $\exists\ (b, c) \subset [0, 2\pi)$ such that $\Theta_n \notin (b, c)$ for all n, and

I. $b = \Theta_r$ and $c = \Theta_s$ or

II. $b = 0$ and $c = \Theta_s$.

Let $\varepsilon = \min(D(\Theta_s, 0), (c - b))$.

From Lemma 1 there is a Θ_m such that

(1) $D(\Theta_m, \Theta_s) < \varepsilon$.

From the conditions on ε it must be that

(2) $D(\Theta_m, \Theta_s) = \Theta_m - \Theta_s$.

Assume *I* is true.

Now $\Theta_s + (2\pi r - 2\pi s)/a - 2\pi k_r + 2\pi k_s = \Theta_r$

$$\Theta_m + (2\pi r - 2\pi s)/a - 2\pi k_r + 2\pi k_s =$$

$$(\phi + 2\pi(m + r - s))/a - 2\pi(k_m + k_r - k_s) = \Theta_n$$

where $m + r - s = n$ and $k_m + k_r - k_s = k_n$.

From (1) and(2) we have

$$0 < \Theta_m - \Theta_s < \varepsilon \Rightarrow 0 < \Theta_n - \Theta_r < \varepsilon \leqq c - b = \Theta_s - \Theta_r.$$

$\therefore \quad \Theta_r < \Theta_n < \Theta_s \quad$ and $\quad \Theta_n \epsilon (b, c)$, a contradiction.

Assume *II* is true.

Then $\varepsilon \leqq c - b = \Theta_s - 0 = \Theta_s$.

Now $\Theta_m + (2\pi s - 2\pi m)/a - 2\pi k_s + 2\pi k_m = \Theta_s$

$$\Theta_s + (2\pi s - 2\pi m)/a - 2\pi k_s + 2\pi k_m = \Theta_n \quad \text{where}$$

$$n = 2s - m \text{ and } k_n = 2k_s - k_m.$$

Since $0 < \Theta_m - \Theta_s < \varepsilon \leqq \Theta_s$,

$$0 < \Theta_s - \Theta_n < \Theta_s \quad \text{or}$$

$$0 < \Theta_n < \Theta_s.$$

Therefore $\Theta_n \epsilon (b, c)$ and again a contradiction.

In conclusion it is interesting to compare the properties of integral roots to those of irrational roots. As noted before there is only a finite number of integral roots of a complex number; however, there is an infinite number of irrational roots. Also when the integral roots are located on a circle, each root is at least $2\pi/n$ radians from the other roots. In contrast to this the irrational roots are densely distributed on the circle.

REFERENCE

[1] Robison, J. Vincent. *Modern Algebra and Trigonometry.* New York: McGraw-Hill, 1966.

Exponential Decay, An Easy Way

THOMAS L. VICKREY
Faculty, Central Missouri State College

Early in the study of differential equations, one encounters the first order linear equation

$$k\,\frac{dy}{dt} + y = q, \quad k > 0, \tag{1}$$

which is useful in a wide range of applications. For example, certain problems in heat conduction, motion against friction, electric circuits, etc. can be analyzed by means of (1). The purpose of this discussion is most appropriately demonstrated by restricting the input $q = q(t)$ to be a constant.

The general solution of (1), namely

$$y = q + C_1\exp(-\,t/k) \tag{2}$$

may be obtained in a variety of ways using techniques developed in the standard course in elementary differential equations. After imposing a prescribed condition relative to a given problem, a particular solution can be gotten from (2). If $y = y_o$ when $t = 0$, then it follows from (2), with q transposed, that

$$y - q = (y_o - q)\exp(-\,t/k). \tag{3}$$

The value of k is a consequence of the physical nature of the problem under investigation but can be determined through additional observations or conditions imposed on (3). Hence, for a fixed q, y may be computed for any given value of t.

However, it is worth noting that if the observations to be used in (2) are arranged properly, an advantage is gained. In particular, if the values of the independent variable t fall in an arithmetic progression, say $t = 0, mk, 2mk, 3mk, \cdots$, where m is some positive real number, then the corresponding values of $y - q$ obtained from (3) form a geometric progression

$$(y_o - q), (y_o - q)e^{-m}, (y_o - q)e^{-2m}, (y_o - q)e^{-3m}, \cdots,$$

with common ratio $e^{-m} < 1$. This property can be useful and makes some calculations relatively trivial.

By way of example, one may consider an application involving Newton's Law of Cooling: The temperature of a body changes at a rate which is proportional to the difference in temperature between the surrounding medium and the body itself. Expressing this in symbols, where t is time, y is the temperature of the body, and q is the temperature of the surrounding medium, yields

$$\frac{dy}{dt} = -\frac{1}{k}\left(y - q\right),$$

with proportionality constant $-1/k$, $k > 0$. The minus sign indicates that the temperature of the body decreases if $y > q$ and increases if $y < q$. This equation, after rearrangement, is the same as (1).

A thermometer reading 30°C is taken outdoors. Two minutes later it reads 15°C, and another two minutes later it reads 10°C. With no further waiting or observation, what is the outside temperature? The solution could of course be gotten by determining C_1 and k (or $e^{-1/k}$) in (2). However, since the observations are made at evenly spaced intervals of time, the differences in the thermometer readings and the outside temperature form a geometric progression, namely

$$30 - q,\ 15 - q,\ 10 - q, \cdots,$$

with common ratio

$$\text{c.r.} = \frac{15 - q}{30 - q} = \frac{10 - q}{15 - q}.$$

Solving the equation yields $q = 7.5$°C.

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1. Kaplan, Wilfred, *Ordinary Differential Equations,* Massachusetts, Addison-Wesley Publishing Company, Inc., 1958.

2. Rainville, Earl D., *Elementary Differential Equations,* New York, The Macmillan Company, 1952.

The Mathematical Scrapbook

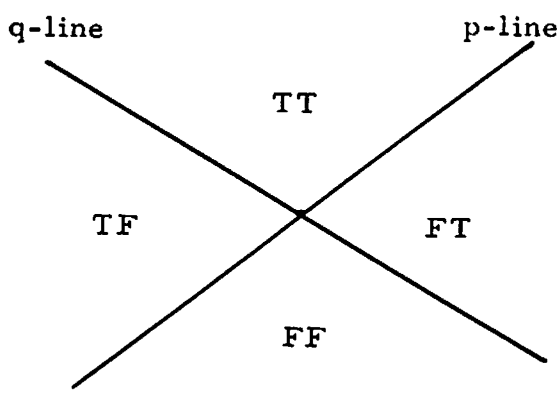
Edited by George R. Mach

Readers are encouraged to submit Scrapbook material to the editor. Material will be used where possible and acknowledgement will be made in THE PENTAGON. All of the Scrapbook material for this issue was submitted by members of the Nebraska Beta Chapter.

Editor's note: The following was submitted by Larry Babcock.

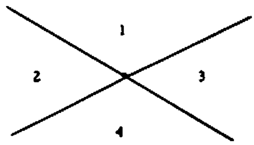
An alternative to the common truth table is the method of using the St. Andrews cross $\times$. This method is quite unusual and offers interesting applications.

The first case considered will be the case having two simple statements p and q. The ordinary truth table has four possible combinations of p and q: TT, TF, FT, FF. These four cases can be represented by the St. Andrews cross as:



where T is always above the line and F below the line for the respective statements p and q.

For simplification purposes, we shall denote the corresponding cases as follows:



Case No.	p	q
1	T	T
2	T	F
3	F	T
4	F	F

A dot in one quadrant of the St. Andrews cross shall represent a compound statement whose entry is T in the truth table and for all F entries the quadrant is left blank. Therefore, $\bullet\times = p \wedge \sim q$ since $p \wedge \sim q$ is true only in case 2.

The common statements and connectives can thus be represented as:

$p \quad \overset{\bullet}{\bullet\times}$ $\qquad$ $q \quad \overset{\bullet}{\times\bullet}$

$\sim p \quad \underset{\bullet}{\times\bullet}$ $\qquad$ $\sim q \quad \underset{\bullet}{\bullet\times}$

$p \wedge q \quad \overset{\bullet}{\times}$ $\qquad$ $p \vee q \quad \overset{\bullet}{\bullet\times\bullet}$

$p \veebar q \quad \bullet\times\bullet$ $\qquad$ $p \rightarrow q \quad \overset{\bullet}{\underset{\bullet}{\times\bullet}}$

$p \longleftrightarrow q \quad \overset{\bullet}{\underset{\bullet}{\times}}$

Since $\overset{\bullet}{\underset{\bullet}{\bullet\times\bullet}}$ is logically true, it denotes a tautology and $\times$ is logically false; it denotes a contradiction.

Three rules are necessary: (1) the conjunction of two crosses is another cross having a dot iff both of the original crosses had a dot. (2) The disjunction of two crosses is another cross having a dot iff at least one of the two original crosses had a dot. (3) The negation of a cross is a cross which has a dot only where blanks were present in the original cross.

As examples,

$$(p \wedge \sim q) \vee (p \longleftrightarrow q) = \bullet\times \vee \overset{\bullet}{\underset{\bullet}{\times}} = \overset{\bullet}{\underset{\bullet}{\bullet\times}} = q \rightarrow p,$$

whereas

$$(p \wedge \sim q) \wedge (p \longleftrightarrow q) = \bullet\times \wedge \overset{\bullet}{\underset{\bullet}{\times}} = \times \text{ which is a contradiction.}$$

$$\sim \overset{\bullet}{\underset{\bullet}{\bullet\times}} = \times\bullet .$$

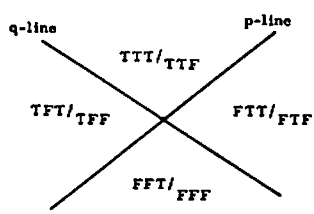
The St. Andrews cross can be efficiently used to verify the validity of arguments:

$$\begin{array}{l} \text{Test } p \wedge q \\ \underline{\sim p \rightarrow q} \\ \therefore \sim q\,. \end{array}$$

Test $[(p \wedge q) \wedge (\sim p \rightarrow q)] \rightarrow \sim q$ which should be a tautology if the argument is valid.

$[(p \wedge q) \wedge (\sim p \rightarrow q)] = \dot{\times}\ \overline{\wedge}\ \cdot\dot{\times}\cdot = \dot{\times}$ which will not yield our desired conclusion $\sim q$ since $\sim q = \cdot\underset{\bullet}{\times}$ and a true statement cannot imply a false statement. Hence the argument is invalid.

The St. Andrews cross can be extended to the case of three statements p, q, and r.



corresponds to the usual eight cases of the truth table for p, q and r. In each quadrant an additional r-line is drawn.

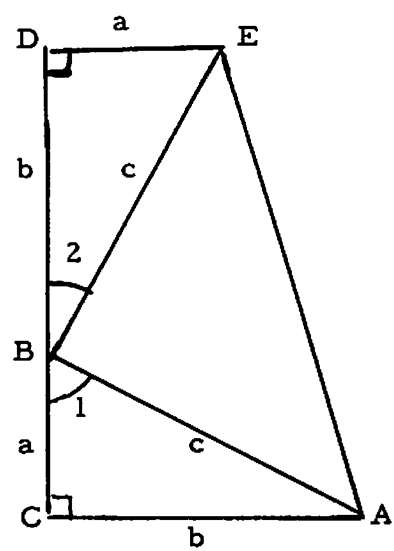
Using this cross, devise a proof of the argument

$$\begin{array}{l} q \rightarrow p \\ q \rightarrow r \\ \underline{\sim r} \\ \therefore \sim p\quad. \end{array}$$

Editor's note: The following was submitted by Greg Gass.

Quite often a student after studying the many proofs in the beginning mathematics courses will begin to feel that only a well versed mathematician could begin to derive a "new" proof. An interesting counterexample to this belief is the following geometric proof of the Pythagorean theorem which was discovered by General James A. Garfield several years before he became the twentieth President of the United States. (There are some 250 additional geometric proofs of the Pythagorean theorem.)

Consider the figure given, with right $\triangle ABC \cong \triangle BED$. Note that $\triangle ABE$ is a right triangle since $\angle 1 + \angle 2 = 90°$.



Hence, area trapezoid $ACDE$ = sum of areas of $\triangle BED$, $\triangle ABC$, and $\triangle ABE$.

$$1/2(a + b)(a + b) = 1/2ab + 1/2ab + 1/2c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$a^2 + b^2 = c^2$$

Editor's note: The following was submitted by Gerald Esch.

A different twist on the use of Pascal's triangle was shown by D. C. Duncan in *The American Mathematical Monthly* of December, 1965. Everyone knows that the n^{th} row of Pascal's triangle gives the coefficients of the expansion of $(x + y)^n$. However, much less known is the fact that the n^{th} diagonal of the triangle gives the coefficients in the Maclaurin series:

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k, \text{ for } (1-x)^{-n}.$$

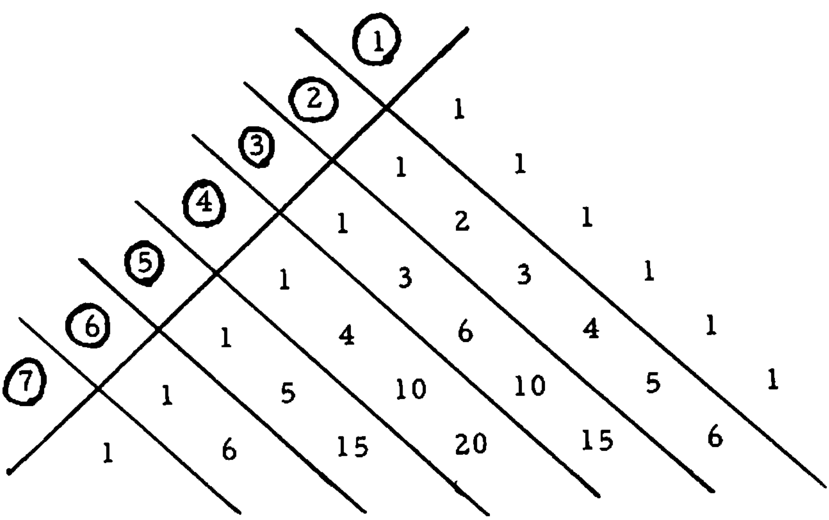
Maclaurin's formula states:

$$F(x) = F(0) +$$

$$\frac{F'(0)}{1!}x + \frac{F''(0)}{2!}x^2 + \frac{F'''(0)}{3!}x^3 + \cdots + \frac{F^{(n)}(0)}{n!}x^n + \cdots.$$

As an example, let us take $F(x) = (1-x)^{-n}$ with $n = 3$. Then the Maclaurin series is:

$F(x) = 1 - 3x + 6x^2 - 10x^3 + \cdots$, and you will note that the coefficients are equal to the third diagonal of Pascal's triangle.



Editor's note: The following was submitted by Pam Herman Poppe and Betty Kruse.

In the Mathematical Scrapbook section of the fall 1968 issue of *The Pentagon* there was a discussion on the number of years necessary for September 18 to fall on every day of the week. This discussion reminds one of the formula derived in some number theory texts.

It was in the year 1582 that the Gregorian revision of the Julian Calendar was promulgated. As a consequence of this revision, a formula has been derived in which the day of the week is expressed as a function of the calendar date.

It is familiar to everyone that in the revised calendar, a common year consists of 365 days and each leap year of 366 days. (Leap years are the years for which the number is divisible by 4, the exception being centurial years which are leap years divisible by 400.)

With this information we can demonstrate the formula. It was developed for 1600 and the subsequent years. To simplify the formula further, the start of the new year will be March 1, making it easier to add the extra day for leap years at the end of the year.

The following numbers were assigned to the m months:

1	March	5	July	9	November
2	April	6	August	10	December
3	May	7	September	11	January
4	June	8	October	12	February

The following numbers were assigned to d, days of the week:

0	Sunday	4	Thursday
1	Monday	5	Friday
2	Tuesday	6	Saturday
3	Wednesday		

The r^{th} day of the m^{th} month of the year N, which is $100C + D$ where $0 \leqq D < 100$, falls on the day number:
$d = r + [2.6m - 0.2] - 2C + D + [C/4] + [D/4] \pmod 7$,
where [ ] represents the greatest integer function.

This year Richard M. Nixon was inaugurated the thirty-seventh U.S. President on Monday, January 20. George Washington was inaugurated as the first U.S. President on April 30, 1789. Using the above formula show that this latter date was a Thursday.

The Problem Corner

EDITED BY ROBERT L. POE

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before October 1, 1969. The best solutions submitted by students will be published in the Fall 1969 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor Robert L. Poe, Department of Mathematics, Texas Technological College, Lubbock, Texas 79409.

PROPOSED PROBLEMS

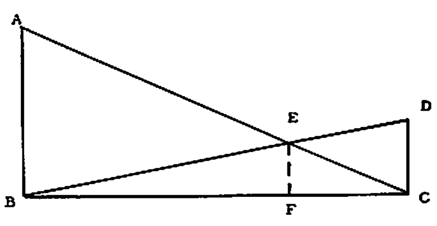
221. *Proposed by R. S. Luthar, University of Wisconsin, Wankesha, Wisconsin.*

Show that for any positive integer n,

$2^n + 3^n + 4^n \equiv (-1) \pmod{10}$ iff 4 does not divide n.

222. *Proposed by Rosser J. Smith, III, Sun Oil Company, Dallas, Texas.*

Given the two right triangles, $\triangle\ ABC$ and $\triangle\ DCB$, with $AC = 70$, $DB = 60$, and $EF = 10$, find BC.



223. *Proposed by Charles W. Trigg, San Diego, California.*

Show that no pentagonal number can be partitioned into three three-digit primes which together contain the nine positive digits.

224. *Proposed by R. S. Luthar, University of Wisconsin, Wankesha, Wisconsin.*

$$\text{Evaluate } \lim_{x \to 0} \left(\frac{x + \log \sec x}{x} \right)^{\frac{x}{\log \cos x}}.$$

225. *Proposed by Thomas P. Dence, University of Colorado, Boulder, Colorado.*

Consider six different points in the plane. Draw all line segments connecting one point with another. Now color each line segment with one of two different colors. Prove that there will always exist a triangle of the resulting colored drawing whose sides have the same color.

SOLUTIONS

216. *Proposed by Thomas P. Dence, University of Colorado, Boulder, Colorado.*

Show that given a natural number, n, we have

$$n = \log_{(0+1+2+3+4)/5} \Big[\log_{\underbrace{\sqrt{\sqrt{\sqrt{\vdots\sqrt{-6+7+8}}}}}_{n \text{ radicals}}} 9 \Big].$$

Solution by Don N. Page, William Jewell College, Liberty, Missouri.

By combining terms and expressing the repeated radical as a fractional power of the radicand, we have

$$\log_{(0+1+2+3+4)/5} \Big[\log_{\underbrace{\sqrt{\sqrt{\sqrt{\vdots\sqrt{-6+7+8}}}}}_{n \text{ radicals}}} 9 \Big] = \log_2 [\log_{9^{(1/2)^n}} 9].$$

But $\log_a x = \dfrac{\log_b x}{\log_b a}$, so for $a = 9^{(1/2)^n}$, $x = 9$, $b = 9$, we have $\log_2[\log_{9^{(1/2)^n}} 9] = \log_2 \left[\dfrac{\log_9 9}{\log_9 9^{(1/2)^n}}\right] = \log_2 \left[\dfrac{1}{(1/2)^n}\right]$

$= \log_2 2^n = n.$

217. *Proposed by Charles W. Trigg, San Diego, California.*

(a) Using the nine positive digits just once each form two positive integers, A and B, such that $A = 7B$. Find all possibilities.

(b) Find the unique solution to part (a) above when all ten digits are used.

Solution by Kenneth M. Wilke, Attorney at Law, Topeka, Kansas.

(a) Since the two numbers A and B use all of the nine positive digits without repetition, their sum $A + B = 8B \equiv 0 \pmod 9$. Hence both A and B are divisible by 9. Furthermore, B contains four digits whose sum is 18 or 27 for if a is a digit of B, then $9 - a$ is also a digit of B unless B is composed of the following digits: (2, 3, 5, 8), (2, 3, 4, 9), (1, 3, 5, 9),(1, 2, 6, 9), (1, 4, 6, 7), (3, 7, 8, 9), (4, 6, 8, 9) or (5, 6, 7, 9).

Combining this with the possible combinations of units place digits in the numbers A and B, the following solutions were found: (A, B) = (18459, 2637), (16758, 2394), (31689, 4527), (36918, 5274), (37926, 5418), (41832, 5976), and (53298, 7614).

(b) Clearly A and B contain five digits each. Furthermore A contains the digit 9 while B contains the digit 0 and begins with the digit 1 followed by the digit 2, 3, or 4. By applying considerations similar to those used in part (a) above, the following values were found for A and B:

$$A = 98532 \text{ and } B = 14076 .$$

Also solved by Mary Brousil, Rosary College, River Forest, Illinois.

218. *Proposed by Ali R. Amir-Moez, Texas Technological College, Lubbock, Texas.*

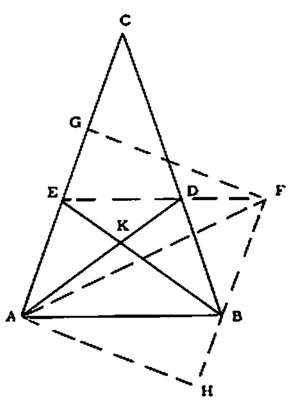
Prove the following known theorem of plane geometry (a) directly, (b) indirectly, (c) algebraically, and (d) analytically. *Theorem*. If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

Solution by Elisha S. Loomis, Ph.D. (1852-1940) printed in ORIGINAL INVESTIGATION or HOW TO ATTACK AN EXERCISE IN GEOMETRY by Elisha S. Loomis published as a public service by The Bonded Scale and Machine Company of Columbus, Ohio. (This interesting and enlightening monograph may be obtained free by writing Arthur Gluck, President, The Bonded Scale and Machine Company, Columbus, Ohio.)

(a) Given: The triangle ABC, in which the bisectors AD and BE are equal.

To prove: That the triangle is isosceles.

Proof:



1. Make angle BEF = angle BAD and angle FBE = angle ADB.

2. Extend FB to H and draw the perpendiculars FG and AH.

3. Join AF.

4. Since angles BAD and ADB are two angles of a triangle they are together less than two right angles, and hence EF and BF will meet.

5. Therefore, triangle ABD = triangle EFB.

6. Therefore, $BD = BF$ and $AB = EF$.

7. Angle AKB = angle KAE + angle AEK = angle BEF + angle AEK = angle AEF.

8. But angle AKB = angle KDB + angle DBK = angle FBK + angle KBA = angle FBA.

9. Therefore, angle AEF = angle FBA.

10. Therefore, angle FEG = angle ABH.

11. Therefore, triangle FGE = triangle AHB.

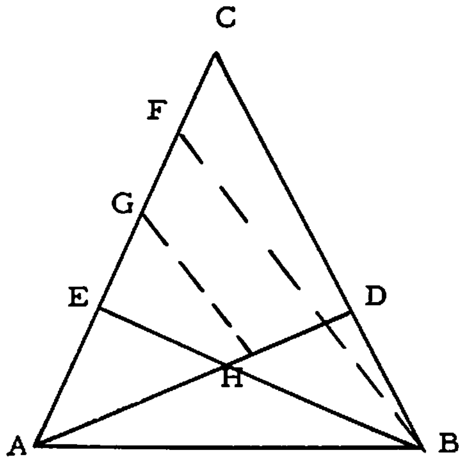
12. Therefore $GE = HB$ and $GF = HA$.

13. Therefore triangle FGA = triangle AHF.

14. Therefore $GA = HF$.

15. Therefore $AE = FB = BD$.
16. Therefore triangle ABE = triangle BAD.
17. Therefore angle EBA = angle BAD.
18. Therefore angle CBA = angle BAC.
19. Therefore $CA = CB$.

(b) Symbolization



Given: The triangle ABC in which the bisector AD equals the bisector BE.

To prove: That the triangle ABC is isosceles.

Proof: Analysis.

1. There are three possible suppositions: first, angle A is greater than angle B; second, angle A is less than angle B; third, angle A is equal to angle B.

2. First, suppose angle A less than angle B; therefore, one-half of angle A is less than one-half of angle B.

3. Construct angle FBE = angle DAF.

4. Now in the triangle FBA, FA is greater than FB; the greater side is opposite the greater angle.

5. Take $AG = BF$ and draw GH parallel to FB.

6. The triangles AHG and BEF are equal, having a side and two adjacent angles of the one equal, respectively, to a side and two adjacent angles of the other.

7. Therefore $AH = BE$, being homologous sides of equal triangles. But this is absurd, since $AD = BE$, by hypothesis, and AH is only a part of AD.

8. Second, in like manner it can be shown that angle A cannot be greater than angle B.

9. Third, as angle A is neither less than nor greater than angle B, it must be equal to angle B.

10. Therefore triangle ABC is isosceles.

Synthesis.

1. As consequences in steps 7 and 8 above contradict known truths, then assumptions first and second above are false, and their contradictory, angle A = angle B, is the basis for our synthetic proof.

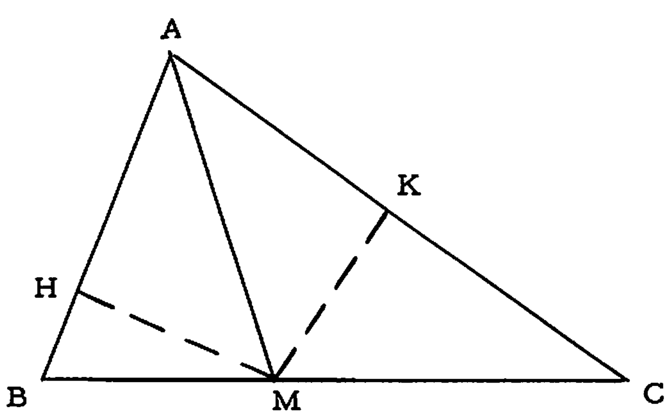
2. Since angle A = angle B, $CA = CB$, being sides opposite equal angles.

3. Therefore the triangle ABC is an isosceles triangle. But this synthesis is unnecessary, since step 10 above is obvious from the analysis which led up to it.

Solution by Ali R. Amir-Moez (proposer of the problem), Texas Technological College, Lubbock, Texas.

(c) In a triangle ABC for h_a, the altitude through A, we have

$$h_a^2 = (a + b + c)(a + b - c)(a + c - b)(b + c - a)/4a^2$$



Let $AM = v = v_a$, the angle bisector of the angle A. Let $BM = x$ and $MC = y$. Then

$$x = ac/(b + c),\ y = ab/(b + c)$$

Let MH and MK be respectively perpendicular to AB and AC. Then MH is the altitude through M of the triangle MAB. Therefore $MH^2 = (v + c + x)(v + c - x)(v + x - c)(x + c - v)/4c^2$. This gives

$$-4c^2MH^2 = [v + (x + c)][v - (x + c)][v + (x - c)][v - (x - c)]$$
$$= [v^2 - (x + c)^2][v^2 - (x - c)^2].$$

Simplifying we get

(1) $-4c^2MH^2 = v^4 - 2(x^2 + c^2)\,v^2 + (x^4 - 2c^2x^2 + c^4).$

Similarly for MK we get

(2) $-4b^2MK^2 = v^4 - 2(y^2 + b^2)\,v^2 + (y^4 - 2b^2y^2 + b^4).$

If we multiply both sides of (1) by b^2 and both sides of (2) by $-c^2$ and add, considering $MH = MK$, we get

(3) $$0 = (b^2 - c^2)\,v^4 + 2[c^2(y^2 + b^2) - (b^2x^2 + c^2)]v^2 + b^2(x^4 - 2c^2x^2 + c^4) - c^2(y^4 - 2b^2y^2 + b^4)\,,$$

or

$$(b^2 - c^2)\,v^4 + 2(c^2y^2 - b^2x^2)\,v^2 + b^2(x^4 - 2c^2x^2 + c^4) - c^2(y^4 - 2b^2y^2 + b^4) = 0\,.$$

Now we see that

$$b^2x^2 - c^2y^2 = [a^2b^2c^2/(b + c)^2] - [a^2b^2c^2/(b + c)^2] = 0\,.$$

Also

$$b^2(x^4 - 2c^2x^2 + c^4) - c^2(y^4 - 2b^2y^2 + b^4)$$
$$= b^2[a^4c^4/(b + c)^4 - 2a^2c^4/(b + c)^2 + c^4] - c^2[a^4b^4/(b + c)^4 - 2a^2b^4/(b + c)^2 + b^4]$$
$$= a^4b^2c^2(c^2 - b^2)/(b + c)^4 + 2a^2b^2c^2(b^2 - c^2)/(b + c)^2 + b^2c^2(c^2 - b^2)\,.$$

Since in general $b \neq c$, (3) can be written as

$$v^4 - a^4b^2c^2/(b+c)^4 + 2a^2b^2c^2/(b+c)^2 - b^2c^2 = 0, \text{ or}$$

$$v^4 = b^2c^2[1 + a^4/(b+c)^4 - 2a^2/(b+c)^2], \text{ and}$$

$$v^4 = [b^2c^2/(b+c)^4][a^2 - (b+c)^2]^2.$$

Therefore

$$v_a^2 = bc[a^2 - (b+c)^2]/(b+c)^2;$$

$$v_b^2 = ca[b^2 - (c+a)^2]/(c+a)^2;$$

$$v_c^2 = ab[c^2 - (a+b)^2]/(a+b)^2.$$

Now if $v_b = v_c$, then

$$ca[b^2 - (c+a)^2]/(c+a)^2 = ab[c^2 - (a+b)^2]/(a+b)^2.$$

This can be simplified to

$$cb^2/(c+a)^2 - c = bc^2/(a+b)^2 - b, \text{ or}$$

$$bc[b/(c+a)^2 - c/(a+b)^2] + b - c = 0, \text{ or}$$

$$bc[a^2(b-c) + 2a(b^2-c^2) + b^3 - c^3]/(c+a)^2(a+b)^2 + (b-c) = 0;$$

$$(b-c)\{[bc/(c+a)^2(a+b)^2][a^2 + 2a(b+c) + (b^2+bc+c^2)] + 1\} = 0.$$

Since the interior of braces is positive, we have

$$b - c = 0, \text{ or } b = c.$$

219. *Proposed by Rosser J. Smith III, Texas Technological College, Lubbock, Texas.*

Show that Q, the number of positive integers no greater than the positive integer M with initial digit no greater than n ($n = 1, 2, 3, \cdots, 9$), is

$$Q = \begin{cases} M, \text{ if } 1 \leq M \leq n \\ M - \dfrac{(9-n)(10^k - 1)}{9}, \text{ if } T \leq M < (n+1)10^k \\ \dfrac{n(10^{k+1} - 1)}{9}, \text{ if } (n+1)10^k \leq M < 10^{k+1} \end{cases}$$

where $k = 0, 1, 2, 3, \cdots$, and $T = \max\{n + 1, 10^k\}$ for $10^k \leq M < 10^{k+1}$.

A sample case and a partial solution is presented by Rosser J. Smith III, now of the Sun Oil Company, Dallas, Texas, in hopes that interested readers will extend these hints and submit completed proofs. Such proofs will be printed in the Fall 1969 issue of THE PENTAGON.

To illustrate a systematic way to find Q, consider the special case for Q_4. Let $M = 1, 2, 3, \cdots$, and as this simple counting progresses, note the successive values of Q_4. The following table is a partial listing of what one would observe.

M	Q_4
1	1
4	4
5	4
9	4
10	5
49	44
50	44
99	44
100	45
499	444
500	444
999	444
1000	445
4999	4444
5000	4444
9999	4444
10,000	4445
49,999	44,444
50,000	44,444
100,000	44,445
.	.
.	.
.	.
M	Q_4

A study of this pattern will show that Q_t is given by the following expression:

$$Q_t = \begin{cases} M, \text{ if } 1 \leqq M \leqq 4 \\ M - 5 \displaystyle\sum_{j=1}^{k} 10^{k-j}, \text{ if } T \leqq M < 5 \times 10^k \\ 4 \displaystyle\sum_{j=0}^{k} 10^{k-j}, \text{ if } 5 \times 10^k \leqq M < 10^{k+1} \end{cases}$$

where $k = 0, 1, 2, \cdots$, and

$$T = \max\{n + 1, 10^k \mid 10^k \leqq M < 10^{k+1}\}.$$

The summations are geometric series and can be summed to obtain

$$\sum_{j=1}^{k} 10^{k-j} = (10^k - 1)/9 \text{ and}$$

$$\sum_{j=0}^{k} 10^{k-j} = (10^{k+1} - 1)/9$$

so that $\left| 1/9 - \sum_{j=1}^{k} 10^{-j} \right| = \left| 10/9 - \sum_{j=0}^{k} 10^{-j} \right| = \dfrac{1}{9 \times 10^k}$.

220. *Proposed by Charles W. Trigg, San Diego, California.*

In the following cryptarithm each letter represents a distinct digit in the decimal scale.

$$6\ (HITFLY) = FLYHIT$$

Identify the digits.

Solution by Margaret M. Conway, Marywood College, Scranton, Pennsylvania.

$6\ H$ + some remainder r must be a one digit number. The only integer which could satisfy this condition is 1. Then $H = 1$.

If $H = 1$, then F must be greater than or equal to 6. Also $6F$ + some $r < 6$ gives a number whose last digit equals 1.

F must equal either 8 or 6 to satisfy these two conditions.

Let $F = 6$. Then L must equal 8 or 9 to permit $6L$ + *a* remainder to have 5 for an initial digit $(6 \cdot 6 + 5 = 41)$.

L cannot $= 8$, since $L = 8$ in the answer would imply that $6\ I$ + no remainder has a final digit of 8. This is impossible.

L cannot $= 9$, since $L = 9$ in the answer implies that $6\ I + 0$ has 9 for its last digit. This is also impossible. Then $F = 8$.

We know that ($6\ F$ + some remainder < 6) has 1 for its final digit. If $6\ F = 48$, then the remainder must be 3.

Therefore 3 must be the first digit of $6\ L + r$. $L = 6$ or 5.

Let $L = 6$. Then $6\ I + n$ (another remainder < 6) $= 26$. Then $I = 4$, and $20 \leqq 6\ T + 5 \leqq 29$.

$T = 3$ or 4. $T \neq 4$, since T and I are distinct.

$T \neq 3$, since $6\ Y$ must then end in 3, and this is impossible. Then $L = 5$.

If so, $6\ I + n = 25$. This makes $I = 4$.

$6\ L + 5$ ends in 4. Since $L = 5$, the initial digit of $6\ Y = 4$. Then $Y = 7$ or 8.

$Y \neq 8$ since Y and F are distinct. Therefore $Y = 7$.

If $Y = 7, I = 2$.

Proof:
$$\begin{array}{r} 142857 \\ \times\ 6 \\ \hline 857142 \end{array}$$

Also solved by Mary Brousil, Rosary College, River Forest, Illinois; Judith Brunori, Marywood College, Scranton, Pennsylvania; Rosemary Krummenoehl, Kutztown State College, Kutztown, Pennsylvania; Don N. Page, William Jewell College, Liberty, Missouri; Kenneth M. Wilke, Topeka, Kansas.

There is no branch of mathematics however abstract which may not some day be applied to phenomena of the real world.

—Lobachevsky

The Book Shelf

EDITED BY JAMES BIDWELL

This department of *The Pentagon* brings to the attention of its readers published books (both old and new) which are of a common nature to all students of mathematics. Preference will be given to those books written in English or to English translations. Books to be reviewed should be sent to Dr. James Bidwell, Central Michigan University, Mount Pleasant, Michigan 48858.

Elementary Differential Equations, Earl D. Rainville and Phillip E. Bedient, fourth edition, The Macmillan Company, New York, 1969, xiv + 466 pp., $8.95.

A Short Course in Differential Equations, Earl D. Rainville and Phillip E. Bedient, fourth edition, The Macmillan Company, New York, 1969, xi + 281 pp., $7.50.

It would be difficult to find an experienced differential equations teacher who has not used a Rainville book as a text at one time or another. Generally, the Rainville books have been readily adaptable for use by all groups of students, whether they be engineers, mathematics majors, or those having other reasons for taking the course, and this latest edition, bearing also the name of a new author, is no exception.

At first glance it appears that few changes have been made by the new co-author, but as one progresses in his reading he finds a number of desirable alterations. Several chapters have been redesigned, their number being reduced from thirty to twenty-six in the larger book, with no loss of important course material. There are 460 pages in the new edition compared to 511 in the previous one, a fact welcomed by those who prefer a more concise textbook.

The smaller book is simply the first sixteen chapters of the larger. The introductory first and second chapters, acquainting a student with differential equations of first order and first degree, are followed by an excellent chapter on applications. The next six chapters deal with more advanced equations and techniques. These are followed by two chapters on Laplace transforms. Next is a short chapter of applications of higher order equations and boundary value problems. Chapters 13, 14, 15, and 16 are called, respectively, "Systems of Equations," "Electric Circuits and Networks," "Existence and Uniqueness of Solutions," and "Nonlinear Equations."

The last ten chapters, appearing only in the larger book, take up power series methods, hypergeometric equations, Bessel's and Legendre functions, numerical methods, partial differential equations and applications, orthogonality, Fourier series and applications, more on Laplace transforms.

Any criticism by this reviewer is purely personal. It is felt that the book could be made even more concise by minor rearrangements of material, perhaps a desirable reversal of the apparent present-day trend toward "wordy" books whose authors attempt to teach the course in their texts. A less awkward approach to the exponential shift and a more complete treatment of operator methods of solving linear equations $f(D)y = g(x)$ would be worthwhile considerations. There is some question about the choice of topics that make up the *Short Course;* for example, engineering classes might wish to use the smaller book, but they would find nothing in it on infinite series solutions, while a ten page chapter is devoted to existence theorems and uniqueness.

Whatever bad points the books may have, they are far outweighed by good ones. The lists of exercises are lengthy, plentiful and usually well graded in difficulty. One is not apt to find, anywhere, a better elementary treatment of the Laplace transform, a topic presented early enough for use in solving practical problems. Almost every conceivable topic covered by a first, three semester-hour course in differential equations is included in the larger book with outstanding explanatory material for each. There is more than ample coverage for a full year course for engineers and science students, if judiciously augmented here and there by the teacher. Of course classes other than engineering and science may use the book advantageously, but it is slanted somewhat toward "applied" mathematics.

The physical makeup of each book is very good. Errors and misprints are few, a result of three highly successful previous editions and obvious care in preparation on the part of the new co-author and others. This reviewer considers the Rainville-Bedient books a welcome addition to the storehouse of many fine differential equation texts now in print. Wherever it is not used as a text, it should be available for reference on the library shelves.

D. H. Erkiletian, Jr.
University of Missouri, Rolla

Topics in Regression Analysis, Arthur S. Goldberger, The Macmillan Co., New York, 1968, 144 pp., $6.95.

This book is not a systematic development of the theory and practice of regression analysis, but rather a discussion of topics from the point of view of a researcher who is interested in using this tool in empirical investigations. Some familiarity of the reader with multiple regression analysis and matrix algebra is presumed. Also a knowledge of probability and statistical theory would be desirable. The plan of the book as given by the author is:

> In Chapter 2 the concept of the population regression function as a key feature of a stochastic relationship is developed and the sample regression function is proposed as its estimator. Chapter 3 considers the interpretation of the regression coefficients obtained when a linear regression function is fit to a body of data. A related interpretation of the coefficient of determination is attempted in Chapter 4. Some bases for judging the reliability of regression coefficients are presented in Chapter 5, and Chapter 6 is devoted to further analysis of this problem and to the related question of multicollinearity. The treatment up to this point proceeds with a minimum of formal statistical theory, a limitation which is removed in Chapter 7. Chapter 8 serves to establish that linear regression analysis is in fact applicable in a wide variety of nonlinear problems. Finally, Chapter 9 offers some suggestions for the selection of appropriate functional forms.

The reviewer feels the usefulness of this book will be as a reference book rather than as a textbook. There are no problem sections in the book. Although the author is a professor of economics and the references are in general to economic data, the material covered by this book would be of interest to any statistician concerned with regression analysis. The book is an excellent reference for serious students of regression analysis.

Wilbur Waggoner
Central Michigan University

Applied Mathematics for Engineering and Science, Wares Shere and Gordon Love, Prentice-Hall, Englewood Cliffs, N.Y., 1969, 672 pp., $10.95.

Some mathematics teachers, who have accepted the trend towards "modern math" with great reluctance, may welcome this book as a sign that the pendulum is swinging back to a new emphasis on the "practical" and on manipulative skills. This swing could be a good thing for some teachers and students—provided the authors take advantage of the basic goals of "modern math"—especially a development based on logic, reason, and correctness. However, this book is "modern" in the sense that it is involved with day-to-day engineering problems.

Although the reviewer hasn't counted, she believes the authors' claims of "over 3,000 exercises" and "approximately 1,400 solved examples". Thus if used properly the student should be able to acquire a good deal of manipulative skill.

The book begins with the following quote from Lord Kelvin: "I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it: but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of Science, whatever the matter may be."—and this sets the stage. The first chapter starts with a study of operations with numbers and algebraic operations, then on to logarithms, trigonometry, linear equations and determinants, vector algebra, analytical geometry, the oblique triangle, complex numbers, theory of equations, probability and statistics, differentiation and integration. There are three appendices which cover over 100 pages. Appendix A is on the use of the sliderule, Appendix B contains various formulas and mathematical constants (e.g., sin 1" is given to 12 significant figures), and Appendix C consists of various tables.

Although the authors' aim is "to present mathematics as a useful tool" and they try to implement this by using simple "easy to understand" language, their presentation lacks clarity, unity, and correctness at various points. For example: (1) On page 4 "We say that the number $\frac{p}{q}$ is a *rational number* if and only if both p and q are integers" (no mention made that $q \neq 0$). Further, "If either p or q or both are not integers, then the number $\frac{p}{q}$ is called an *irrational number*" (Thus $\frac{.3}{7}$ and $\frac{\sqrt{-1}}{23}$ are irrational!) They then proceed to give examples without any mention of proof. (2) On page 13, "Rule 1-15: Division by zero is not permitted because the quotient $\frac{a}{0}$ is not defined." There is no other explanation. (3) On page 69, "Definition 2-19: Whenever the relationship between two variables is such that if the value of one is given, one or more values of the other variable can be determined; the latter variable is called a function of the first variable." One advantage of having this definition in the book is that it gives the instructor an opportunity to emphasize the correct definition—as well as the fact that the students shouldn't believe everything they read. (4) Although the introduction to trigonometry looks more like that of the good old days (than the more sophisticated "modern" approach dealing with the unit circle and wrapping function), the reviewer believes that it would be more consistent with the objectives and with the rest of the book if it started with the right triangle—and developed it more completely before discussing trigonometric functions of any angle." (5) On page 235 the graph of $y = \sin x$ reflects an inaccurate use of units (e.g., $\frac{\pi}{2}$ appears to be less than 1). (6) It would seem that the discussion of "Oblique Triangles" was postponed until after the chapter on Analytical Geometry so that the distance formula could be employed in deriving the law of cosines. But it was not.

Well, those who have been hoping that the "old math" would soon return on a respectable basis, will have to continue to hope.

Gloria Olive
Wisconsin State University—Superior

MINIREVIEWS

(The editor is introducing one paragraph reviews as a new feature of the *Book Shelf*. This procedure is being used to accommodate the large number of books being sent for review and the lack of space to devote to complete reviews. Some of the minireviews will later be reviewed in full.)

Elementary Mathematics—A Logical Approach, by Paul Sanders and Arnold McEntire, second edition, International Textbook Company, Scranton, Pa., 1969, 351 pp.

This is a survey type book designed for both liberal arts students and mathematics majors. Chapters include topics in logic and proof, number systems, abstract systems, polynomial algebra, analytic geometry, vector and matrices, probability and statistics. Modern terminology is used; the depth of material is designed for college freshmen.

Geometry, A Perspective View, by M. F. Rosskopf, J. S. Levine, and B. R. Vogeli, McGraw-Hill Book Company, New York, 1969, 306 pp.

The text is designed to meet CUPM recommendations for a course in geometry for elementary school teachers. The four parts of the book are titled: Mathematics Creativity and Formality; Measurement of Sets of Points; Congruence Parallelism, and Similarity of Sets of Points; Geometry of Transformations. There are many excellent line drawings. Explanations are divided into brief attacks at the material. The book would provide elementary teachers with an excellent background.

Basic Algebraic Systems, An Introduction to Abstract Algebra, by Richard Laatsch, McGraw-Hill Book Company, New York, 1968, 224 pp.

This book is a standard elementary introduction to abstract structures. After a chapter of fundamental concepts, the group concept is developed through normal subgroup and the fundamental homomorphism theorem. Rings, fields, polynomials introduced. More group theory and introduction to vector spaces (via groups with operators) complete the major portion of this text. A brief, well-written text for undergraduates.

College Mathematics With Business Applications by John E. Freund, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1969, 625 pp., $9.95.

This text is a pre-calculus text and introduction to calculus combined with side issues that include stress on mathematical models, probability, matrices and strategy games. Business-oriented problem work is integrated throughout. Most mathematics of finance material is covered.

The golden age of mathematics — that was not the age of Euclid, it is ours.

—C. J. Keyser

Kappa Mu Epsilon News

EDITED BY EDDIE W. ROBINSON, *Historian*

Twenty-five years ago: War demands cause most chapters to drastically curtail their activities. The National Council met in Cedar Falls, Iowa.

Twenty years ago: The Seventh Biennial Convention was held in Topeka, Kansas, at Washburn University. Dr. H. Van Engen was elected president. Miss Franklee Gilbert, Alabama Gamma, was awarded the prize for the best paper published in *The Pentagon.*

Ten years ago: New York Gamma, Tennessee Beta, Pennsylvania Gamma, Virginia Beta, and Nebraska Beta Chapters were installed into the society, making a total of fifty-four chapters.

Five years ago: Oklahoma Beta, California Delta, and Pennsylvania Delta Chapters were installed. Plans were made for the Fifteenth Biennial Convention at Bowling Green State University.

Alabama Epsilon, Huntingdon College, Montgomery

Pledges at Alabama Epsilon are provisional members who have the required grade point average, but are deficient in course requirements. Members must have three courses in mathematics, including calculus, have completed thirty-six semester hours, have a 3.35 grade point average in mathematics and rank in the upper thirty-five per cent of their class. One member will be attending the next convention.

Connecticut Alpha, Southern Connecticut State College, New Haven

Meetings are held twice a month and the chapter has seven active members and twenty-three pledges. Chapter officers are: Roberta Page, president; Joseph Yearsley, vice-president; Susan Ragozzino, secretary; Brian McNamara, treasurer; Mrs. L. Smith, corresponding secretary; Kevin Lonergan, historian; Mrs. Loretta Smith and Mrs. Helen Bass, faculty sponsors.

Speakers at recent meetings were Dr. A. J. Pettofrezza, speaking on the teaching of mathematics and new curriculum; Mr. Ronald P. Mileski, speaking on number theory; Mr. Harvey Blau, "Banach Tarski Paradox;" and Miss Roberta Page, "Boolean Algebra."

Illinois Alpha, Illinois State University, Normal

Pledges of this chapter are required to donate time to the chapter in addition to writing an original research paper. Pledges participate in skits for social meetings and chapter dances. Four new members were initiated and twenty-five students pledged during the fall semester. A homecoming breakfast, an all school dance, and a Christmas party were all fall activities. The homecoming float entry, entitled "The Clock Socked Won" received first place in the competition of organization floats. The homecoming theme was "Mad Mother Goose."

Chapter officers are: Larry Dennis, president; Richard Ryder, vice-president; Jean Linenweber, secretary; Mike Weimer, treasurer; Dr. Clyde T. McCormack, corresponding secretary; and Dr. Orlyn Edge, faculty sponsor.

NSF Pre-service Institute for Prospective Teachers of High School Mathematics was the topic of discussion by Dr. Charles Morris and a panel of students at a recent meeting. Other programs included Miss Janet Cook, speaking on "Simulation and Dissimulation," and Mr. William Black, speaking on "Computerized Communications." Expenses for the National Convention were earned by sponsoring an all-school dance.

Illinois Zeta, Rosary College, River Forest

Four students and two faculty members are planning to attend the National Convention at Cedar Falls, Iowa.

Programs for the semester included: Initiation ceremony; Film showing—"Infinite Acres" and "Limit." Reports by members on the history and theory of probability and game problems were also given. Officers are: Christine Krol, president; Joanne Capito, vice-president; Mary Brousil, secretary; Vicki Davis, treasurer; Sister Nona Mary Allord, corresponding secretary; and Sister Mary Philip Steele, O. P., faculty sponsor. The chapter has nineteen members.

Iowa Alpha, University of Northern Iowa, Cedar Falls

New members are required to present a paper at one of the monthly meetings. This chapter is hosting the National Convention on May 2-3, 1969. Chapter officers are: Jerry Jurschak, president; Anina Christensen, vice-president; Janet Proescholdt, secretary-treasurer; John S. Cross, corresponding secretary.

Iowa Gamma, Morningside College, Sioux City

In the fall the chapter held a computer dance to raise money for expenses connected with attending the national convention. The chapter is sponsoring a tutoring service for all mathematics students.

Three members participated in the William Lowell Putnam Competition. A colloquium was held with Briar Cliff and Westmore Colleges. Student papers were presented and Professor Walter Mientka of the University of Nebraska spoke. Officers are: S. Ron Oliver, president; Craig Bainkredge, vice-president; Marjorie Kaye, secretary; Allen Widrowicz, treasurer; and Elsie Muller, corresponding secretary.

Indiana Alpha, Manchester College, North Manchester

New members are required to present a paper on some aspect of mathematics. Programs of the semester included a skit on the history of mathematics. Officers are: Ted Noffsinger, president; Phil Bantz, vice-president; Marcia Kump, secretary; Jeff Williams, treasurer; Dr. David Neuhouser, corresponding secretary; and George Adams, faculty sponsor.

Kansas Alpha, Kansas State College of Pittsburg, Pittsburg

Officers are: Brian VanLaningham, president; James Harlin, vice-president; Helen Blood, secretary; James Ciardullo, treasurer; Dr. Harold L. Thomas, corresponding secretary; J. Bryan Sperry, faculty sponsor.

The October meeting was primarily a business meeting to formulate programs for the coming year. In November, Ken Lauburg presented "A Proof of Barbier's Theorem." The December program included initiation of new members. The chapter president gave a program on curve fitting using computer techniques. The January program was given by a member of the music department faculty, Dr. Donald Key. His talk discussed the role of mathematics in music.

The annual fall picnic was held in October. This event sponsored by the KME chapter traditionally brings together all students and faculty of the mathematics and physics departments who wish to participate.

Recipients of the Robert Miller Mendenhall Award for scholastic achievement were Homer J. Watson and Curtis Woodhead. Each received a KME pin.

Kansas Gamma, Mount St. Scholastica, Atchison

To be invited to pledgeship, a student must give promise as a mathematician, plan to be a mathematics major or minor, have a B average and be in the upper one-third of her class. To pledge, she must have the recommendation of two members in addition to that of a faculty member. She must then receive a two-thirds vote of the chapter to be admitted to pledgeship. At the time of pledgeship, the pledge promises to attend all meetings of the chapter regularly, to do all in her power to further the interest of the society, and to render service whenever called upon. The pledges are asked to present a program in the spring as evidence of their interest in KME and their ability to work together as a group. Ordinarily, a student pledges for two semesters, and is initiated at the beginning of the second semester of her sophomore year.

Officers are: Norma Henkenius, president; Betty Struckhoff, vice-president; Judy Graney, secretary; Alice Hunninghake, treasurer; Sister Helen Sullivan, O.S.B., corresponding secretary.

Kansas Gamma holds semi-monthly meetings which are highlighted by the presentation of student papers. Mr. Robert Curry, new member of the mathematics faculty, lectured on "Distribution of Prime Numbers" at a November meeting. Activities for the fall semester included the initiation ceremony and banquet for two new members, Mary Jane Turner and Alyne Robino. Twelve members journeyed to Midwest Research in Kansas City for the annual field trip. The traditional Wassail Bowl ceremony in December was the highlight of the chapter's activities. A freshman seminar was conducted semi-monthly for students capable and willing to extend their search for knowledge of mathematics outside the classroom. Three members, Alice Henninghake, Alyne Robino, and Judy Graney, participated in the William Lowell Putnam Competition.

Kansas Epsilon, Fort Hays Kansas State College, Hays

Kansas Epsilon has twenty-six members among whom are the following officers: Leo K. Weigel, president; Dean Meenen, vice-president; Sharon Ruder, secretary; Eugene Etter, corresponding secretary; Mrs. Marilyn Wilson, faculty sponsor.

Maryland, Alpha, College of Notre Dame of Maryland, Baltimore

Pledgeship is a temporary membership which implies a continued interest in mathematics and a continuation of one's achievement in the study of mathematics. These members may serve as minor officers, committee members and may prepare programs. Membership requirements include completion of twelve semester hours of calculus and analytic geometry and a semester of modern algebra. Each member is required to prepare an independent study and present it at one of the monthly meetings. The chapter has four active members and five pledges. Officers are: Jean Decker, president; Kathleen Tipton, vice-president; Joan Weazka, secretary; Kathleen Tipton, treasurer; Sister Marie Augustine, corresponding secretary; Jeannette Gilmore, faculty sponsor.

The past semester Gerri Smith and Kathleen Pleines gave papers demonstrating how physicists and chemists use mathematics, Mrs. Sylvia Sorkin spoke on "Continuity and Topology," and at a joint meeting with the Biology Club Sister Mary Paula spoke on computers.

Michigan Alpha, Albion College, Albion

The chapter has twenty-four members and fourteen pledges. Chapter officers are: James Rauff, president; Margaret Lamb, vice-president; Cathy Amos, secretary-treasurer; Dr. Keith Moore, corresponding secretary; Charles Stocking, faculty sponsor.

Programs included: October, Mrs. Joan Van Deventer spoke on "Instant Insanity;" November, a speaker from Dow Chemical Company; December, Dr. R. C. Fryxell spoke on "Areas Without Limits."

Michigan Beta, Central Michigan, University, Mt. Pleasant

The chapter has sixty-four members. Monthly meetings consist of a business meeting followed by papers presented by student members. Officers are: Robert Otto, president; Betty Baissonneault, vice-president; Joan Siderman, secretary; Jim Osborn, treasurer; Arnold Hammel, corresponding secretary and faculty sponsor.

Twenty Michigan Beta members split into teams of two or three students each and visited area high schools. The chapter has been in charge of a display case in the Mathematics Building. One display was on symbols used in various areas of mathematics, and another was a Christmas greeting from Kappa Mu Epsilon using an operation table for a group.

Mississippi Gamma, University of Southern Mississippi, Hattiesburg

The chapter has twenty-four members including the following officers: Emily Vornsdo, president; Pamela Brown, vice-president; Jack Vice, secretary; Jack Munn, treasurer, corresponding secretary, and faculty sponsor.

Meetings usually consist of a talk given by a faculty member or visiting lecturer with refreshments following. The fall activities included the initiation of fourteen new members, a Christmas party where a tree was decorated with mathematics symbols.

Missouri Alpha, Southwest Missouri State College, Springfield

The chapter has forty-six members including these officers: Timothy Murphy, president; Michael Truskowski, vice-president; Phyllis Payton, secretary; Nichol Dryton, treasurer; Eddie W. Robinson, corresponding secretary; Dr. L. T. Shiflett, faculty sponsor.

Meetings are held monthly with presentations by students and faculty. Dr. William Brown spoke on "Philosophy and Mathematics." Frank Gillespie spoke on the history of mathematics and Patrick Clayton discussed his discovery of the comet Bully-Clayton (1968d). The chapter has instituted a tutoring service for mathematics students. Each semester, new members are initiated at a banquet. Five members will be attending the national convention.

Missouri Gamma, William Jewell College, Liberty

The chapter presents an award to the member who solves the most problems in *The Pentagon*. Monthly meetings are held and programs are presented by students. These include mathematical problems which members present on historical and biographical aspects of mathematics. The mathematics department gave a large supply of mathematics books to KME to sell, thus allowing members to acquire good books at a very reasonable cost.

Missouri Zeta, University of Missouri at Rolla, Rolla

The chapter has twenty-five members. Campus tours were provided for visiting high school mathematics clubs, and the chapter presented awards to outstanding exhibits at the South Central Missouri Science Fair. The members have been experimenting with a mathematics help program. Officers are: Quince E. Hedley, president; James Byer, vice-president; Sandra Hartmann, secretary; Stanley Webb, treasurer; Lyman Smith, corresponding secretary; Dr. Troy Hicks, faculty sponsor.

Nebraska Alpha, Wayne State College, Wayne

The chapter has thirty-three members, thirteen of whom will be attending the national convention. Officers are: Dorren Steffen, president; John Martens, vice-president; Gayle Kloppel, secretary; Arlo Pinkerman, treasurer; Fred A. Webber, corresponding secretary; Maurice Anderson, faculty sponsor.

Activities have included the building of a homecoming float and sponsoring a dance to raise funds for the convention. New members present short talks to the chapter.

Nebraska Beta, Kearney State College, Kearney

Membership includes thirty-nine active members and fourteen pledges. Officers are: Pam Herman Poppe, president; Al Neis, vice-president; Marilyn Koch, secretary; George Weaver, treasurer; Richard Lee Barlow, corresponding secretary; Calvin Nelson, faculty sponsor.

Meetings are held twice monthly. Four new members were initiated during the fall semester 1968-69. Programs have included talks on the mathematician in industry, mathematics in Israel by a foreign exchange student, using the K.S.C. Placement Bureau in securing a job, and student papers.

Judy Maul received our chapter's K.M.E. scholarship for this year. The "Mathematics Booster Hour" is being held twice each week in the evenings for helping students in our freshman mathematics courses. The annual KME Christmas party was held, organized by the first semester pledge class. Work was done by our members on our chapter's "Newsletter" which is given out in the junior high mathematics classes in area schools. Our chapter has also been working on the "Scrapbook" section of the Spring 1969 issue of *The Pentagon.*

New York Epsilon, Ladycliff College, Highland Falls

The chapter has eight members. Officers are: Marilyn Jugan, president; Martha Martorano, vice-president; Carol Evans, secretary-treasurer; Sister Clare Bernadette, corresponding secretary.

Programs have consisted of guest speakers and lecturers at Mount St. Mary College, Morist College, and West Point. The chapter started a tutorial service for freshmen and the local high school students.

New York Zeta, Colgate University, Hamilton

The chapter has twenty members including the officers: Kenneth Reynolds, president; Edwin Morris, vice-president; Leonard Wilson, secretary-treasurer; Theodore Frutiger, corresponding secretary; James Wardwell, faculty sponsor.

The chapter has joint meetings with mathematics clubs at Wells College and Skidmore College, usually with student presentations. Professor Preston Hammer, Chairman of the Department of Computer Science at Pennsylvania State University, spoke to the chapter in a two-day visitation. Seventeen new members were added during the fall semester.

Oklahoma Alpha, Northeastern State College, Tahlequah

The chapter has forty active members including the following officers: Thad Langford, president; Charles Gambill, vice-president; Charles Masters, secretary; Linda Graham, treasurer; Dr. Raymond Carpenter, corresponding secretary; Mike Reagan, faculty sponsor.

The chapter meets twice each month with attendance averaging twenty members. Students and faculty present papers at the regular meetings. Students who have met the chapter requirements for membership are initiated at the Christmas meeting and at the Founders' Day banquet in the spring.

Pennsylvania Beta, LaSalle College, Philadelphia

Membership in the chapter is composed of eight members and twenty-five pledges. Officers are: Richard Mitchell, president; Kenneth Kryszczun, vice-president; Joseph Struh, secretary; Michael Marinelli, treasurer; Jack Fitzsimmons, corresponding secretary; Brother Hugh Albright, faculty sponsor.

The co-developer of the computer at the University of Pennsylvania, Dr. Mauchly, was a guest lecturer to the chapter. The chapter maintains a club room for mathematics majors, enabling continued communication between students and faculty. Activities include the LaSalle Open House and intramural football, basketball, and baseball games. These activities help to create unity within the chapter.

Pennsylvania Gamma, Waynesburg College, Waynesburg

The chapter has twenty members, including the following officers: Sharon Mattie, president; Deminick Oliverie, vice-president; Becky Sweeting, secretary-treasurer; Lester T. Moston, corresponding secretary.

Members are required to present a problem at the initiation meeting.

Pennsylvania Delta, Marywood College, Scranton

Officers of the chapter are: Linda M. Balimon, president; Judith Brunori, vice-president; Margaret Mary Conway, secretary; Patricia Cusick, treasurer; Miss Marie Loftus, faculty sponsor.

The chapter has twenty-two members and has had two fall meetings, including a speech by Mr. Norman Mair, who spoke on "The Motivation for the Definition of Topological Spaces."

Pennsylvania Epsilon, Kutztown State College, Kutztown

The chapter has 106 members. Officers are: Martin Johnson, president; Ronald Long, vice-president; Robin Fies, secretary; Carolyn Caton, treasurer; Dr. J. D. Daugherty, corresponding secretary; Dr. E. W. Evans, faculty sponsor.

Dr. Cyrus E. Belkey, President of Kutztown State College, was initiated as an honorary member. He was an undergraduate mathematics major at Albright College. Activities of the semester have been a field trip to the N.C.T.M. meeting in Philadelphia and a reception for freshman mathematics majors. Two initiations are held annually. Two outstanding speakers have been Dr. Herman Von Baravalle and Sister Helen Sullivan. The chapter awards an honor certificate at commencement to the student who majors in mathematics and has the highest grade point average.

Pennsylvania Zeta, Indiana University of Pennsylvania, Indiana

There are sixty-one members in the chapter. Officers are: Mark Smith, president; Randall Drake, vice-president; Donna Carlson, secretary; Patricia Phenicie, treasurer; Ida Z. Arms, corresponding secretary; William R. Smith, faculty sponsor.

At the first meeting of Pennsylvania Zeta Chapter, Dr. Maher Shawer, a new member of the mathematics staff and a native of Cairo, Egypt, spoke on the topic "Arab Contributions to Mathematics." This meeting was held in October and at this time twenty-five eligible students were pledged to become members of the chapter. One week later these students were initiated into the organization. At the regular monthly meetings held in November and December Mr. Edwin Bailey and Dr. Dal Shafer, both members of the mathematics staff presented talks. A major activity of the members is to conduct HELP sessions for those students taking mathematics courses who are having some difficulty. The members of the chapter give freely of their time and this has become a real asset in assisting students.

Tennessee Alpha, Tennessee Technological University, Cookeville

There are seventy-five members in the chapter. Officers are: Carol Ramsey, president; Larry Gregory, vice-president; Rita Duff, secretary; Felix Hoots, treasurer; Evelyn Brown, corresponding secretary; Ronald Sircy and Herbert Willcox, faculty sponsors.

Membership requirements are a quality point average of 3.20 in mathematics and a 3.00 overall quality point average.

Tennessee Beta, East Tennessee State University, Johnson City

Meetings consist of a business session and a snack period. Yolanda Grindstaff spoke on "Light" at the November meeting. Chapter members have been guests of the Mathematics Department at their regular monthly colloquium. Mike Dzvonik, president of the chapter last year, was the recipient of the Faculty Award at the June Commencement. Officers are: David M. Combs, president; Thomas E. LaGuardis, Jr., vice-president; Nancy Rawls and Yolanda Grindstaff, secretaries; George L. Lee, treasurer; Mrs. Lora McCormick, corresponding secretary; Miss Sallie Pat Carson, and T. Henry Jablonski, Jr., faculty sponsors; Richard Counts, historian; Carmen E. Stallard, representative.

Texas Gamma, Texas Women's University, Denton

The chapter has eight members including the following officers: Linda Lee Jones, president; Marilinda DeLeon, vice-president; Anna Gorzalez, secretary-treasurer; Ron McPherson, corresponding secretary.

The Mathematics Club and the chapter met jointly for the annual departmental picnic. Spring activities include a field trip and an initiation banquet.

Texas Epsilon, North Texas State University, Denton

There are twenty-eight members and eight pledges. Officers are: Kenneth Foster, president; Johnny Simpson, vice-president; Judy Moore, secretary; Linda Ball, treasurer; Dr. Melvin R. Hagan, corresponding secretary; Dr. David R. Cecil, faculty sponsor; Sarah Taylor, historian.

At various meetings in the year local professors give short lectures on topics of their own field. North Texas State University has made an annual event of a North Texas State Mathematics Conference at which 1,500 local high school mathematics teachers attended. Only in its second year, it has grown to such proportions that speakers from as far away as Washington, D.C. brought lectures. KME members plan to continue helping in the organization and promotion of this worthwhile event which is sponsored by the teacher education professors of the Mathematics Department.

Dr. John J. Kamerick, the new president of North Texas State, will be formally introduced to the mathematics faculty and to KME members at a banquet which will be sponsored by the chapter. Dr. Kamerick will bring the banquet address.

The chapter is striving for renewed cooperation and participation with the mathematics faculty's activities for the purpose of promoting interest in mathematics. Experience has shown that participation in the mathematics conference, the lecture series, and social events have caused a closer relationship between teacher and student.

Wisconsin Alpha, Mount Mary College, Milwaukee

There are eight members and seven pledges. Officers are: Maureen O'Donnell, president; Judy Pokrop, vice-president; Mary Ellen Naber, secretary; Shirley Bruder, treasurer; Sister Mary Petronia, corresponding secretary and faculty sponsor.

Chapter meetings were as follows: October—a socratic Dialogue on Mathematics (from the book, *Dialogues on Mathematics* by A. Renyi) delivered by Sister Mary Cheryl and Kathy Greif; November—Talk entitled, "Progressions—Arithmetical, Geometric, and Harmonic" by Mary Mower, Marsha Allgeyer, and Pat Paul; talk entitled "Ancient Number Systems" by Sister Kathleen Marie, Christine Kummer, and Cathy Durasch; December—Panel of student teachers, Ellen Levine, Grace Makarewicz, Tiia Ostrovski, and Susan O'Connor. These girls told about their teaching experiences which was followed by a question and answer period.

The chapter will hold its annual mathematics contest for high school students on March 22. A number of the members are making plans to attend the National Convention at the University of Northern Iowa on May 2-3.

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