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National Officers

Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Paths and Knots as Geometric Groups*

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A group is a useful and versatile part of abstract mathematics which enables classification of objects according to five properties. These are:

1. There must be a binary operation defined on the elements of a set. A *binary* operation is a process which relates in pairs elements of a set.

2. The set must be closed under the binary operation. That is, the element produced from the paired association, called the product, must be an element of the set.

3. The elements must be associative.

4. The set must have an identity element which, paired under the operation with *any* second element, yields the second element.

5. Each of the elements, except the identity, must have an inverse which takes it into the identity element under the operation.

One is guaranteed that the fourth and fifth properties produce the same answer regardless of the order of operation. This property is known as commutativity; however, not all the elements of a group need be commutative. If all elements possess this property, the group is a special type called an Abelian group.

Let us consider the integers modulo 4 under the operation addition $\oplus$. Mod 4 refers to the remainder obtained by dividing an integer by 4. Refer to

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

*A paper presented at the 1968 regional convention of KME at Tahlequah, Oklahoma, April 20 and awarded first place by the Awards Committee.

Observing the inner square, note that the only elements which appear are 0, 1, 2, 3, the elements of the original set. Thus closure holds. Associativity can be tested by exhausting all possible combinations of the four elements three at a time: consider $(2 \oplus 3) \oplus 3 = 0 = 2 \oplus (3 \oplus 3)$.

The identity element is clearly 0 and it is unique. Inverses are for 0, 0; for 1, 3; for 2, 2; for 3, 1.

So we see that the integers mod 4 under addition do form a group. In fact, the group is Abelian since the mod of any two elements is the same regardless of the order in which they are taken: $2 \oplus 3 = 1 = 3 \oplus 2$.

The reason our table contains only the elements 0, 1, 2, 3 is that all other integers can be linked by the same remainder to these four numbers. For example, 6 is in the same class as 2 since 6/4 has a remainder of 2 and 2/4 has a remainder of 2. We call 0, 1, 2, 3 the equivalence classes of mod 4 denoted by [0], [1], [2], [3]. We can manipulate classes just as ordinary numbers:

$[3] \oplus [2] = [1]$. 0, 1, 2, and 3 serve as representatives for the sets $\{0, 4, 8, \cdots\}$, $\{1, 5, 9, \cdots\}$, $\{2, 6, 10, \cdots\}$ and $\{3, 7, 11, \cdots\}$, respectively.

It is fairly obvious that group properties can be tested as long as our set consists of numbers. What would happen if we constructed some type of geometric set, defined an operation on the elements, and tested to see if this set constitutes a group?

Let us define a closed path in space as a directed segment beginning and ending at a fixed point (Fig. 1). The shape of this path is immaterial, but we will be interested in the ability to change

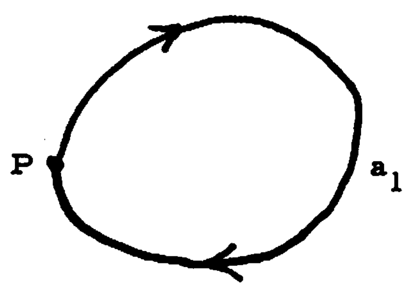


Fig. 1

its shape. We can define an equality relation (known as a homotopy) on our set by saying $a_1 = a_2$ iff a_1 can be continuously deformed into a_2 (Fig. 2). If our space is empty, that is, free from obstacles, then

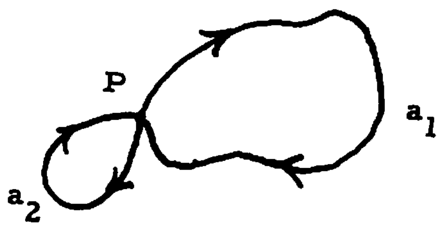


Fig. 2

any closed path a_i is homotopic to any a_j. However, if the Euclidean plane is our space and it contains a fixed disc, then any path enclosing the disc cannot be shrunk continuously into a_2 without passing through the forbidden region R (Fig. 3).

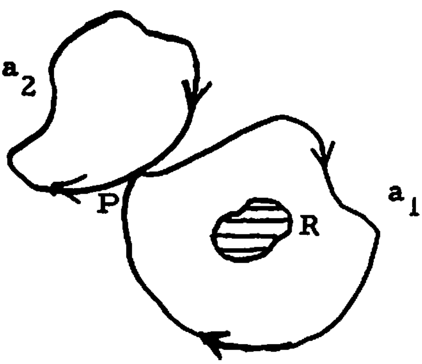


Fig. 3

Now that we have established our set, we can define a binary operation on it. Recall that we call this operation product. Starting at the fixed point P in two-space, detach the terminal side of a_1 at point R (Fig. 4) and the initial point of a_2 at Q (Fig. 5). Then join R to Q resulting in the closed path b (Fig. 6). Thus $a_1 a_2 = b$.

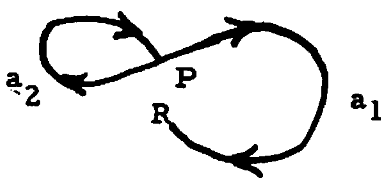


Fig. 4

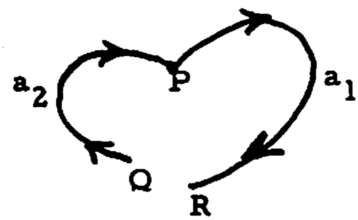


Fig. 5

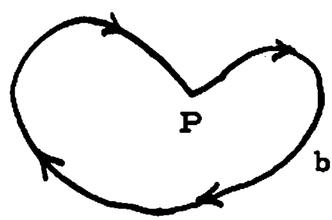


Fig. 6

We construct the equivalence classes of closed paths by requiring that a_i and a_j are in the same class iff they can be continuously deformed into each other. Classes will be denoted by $[a_1]$,$[a_2]$, $\cdots$, $[a_k]$, $\cdots$.

Now let us consider classes of closed curves in three-space. Define A as a closed curve. The set $E_3 \sim A$ is the manifold of A (Fig. 7). Pick P, a point in the manifold, then a_1 can be continuously

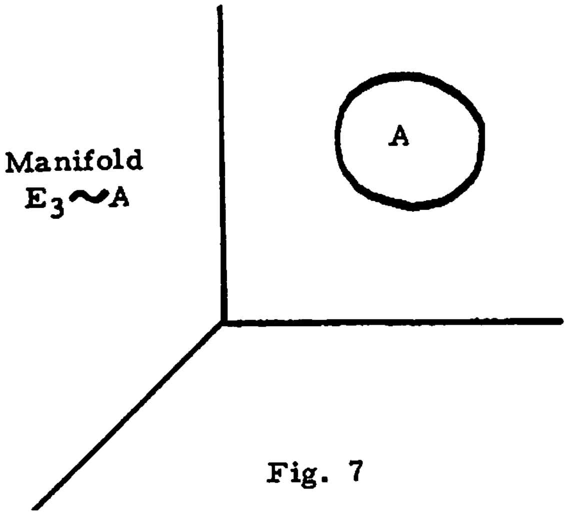


Fig. 7

shrunk into *P* while *b* cannot be without penetrating *A*, just as before in the Euclidean two-space (Fig. 8).

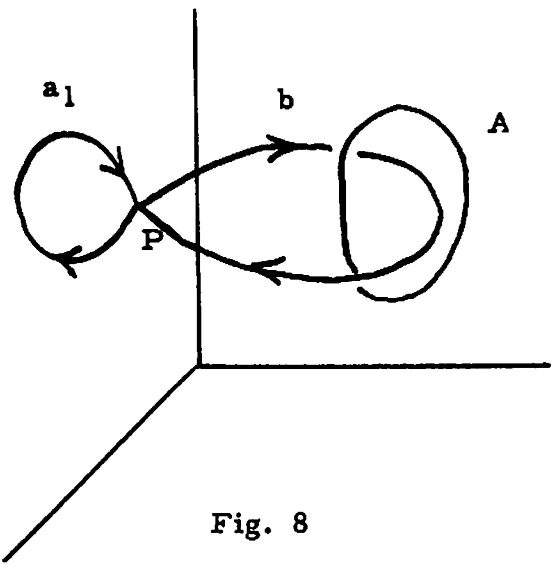


Fig. 8

Thus a_1 falls in one equivalence class $[I]$ and $b \varepsilon [a]$ whose members loop once around A.

Defining an inverse for b, we say that $b^{-1} \varepsilon [b]^{-1}$ is the inverse of b iff $bb^{-1} = b^{-1}b = a \varepsilon [I]$. Let b be an arbitrary closed path. Then b^{-1} is the path obtained by tracing b in the opposite direction

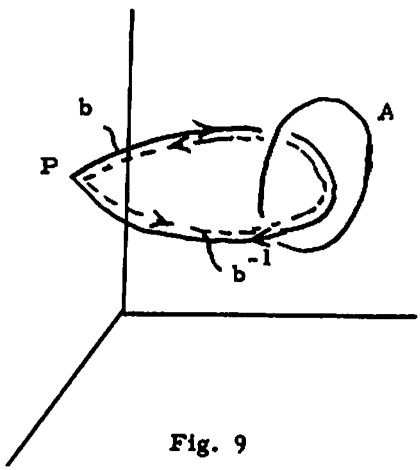


Fig. 9

(Fig. 9). Now form bb^{-1} (Fig. 10). After this, consider $b^{-1}b$ (Fig. 11) using the product as we defined earlier. Continuous deformation yields $P \varepsilon [I]$ in both cases.

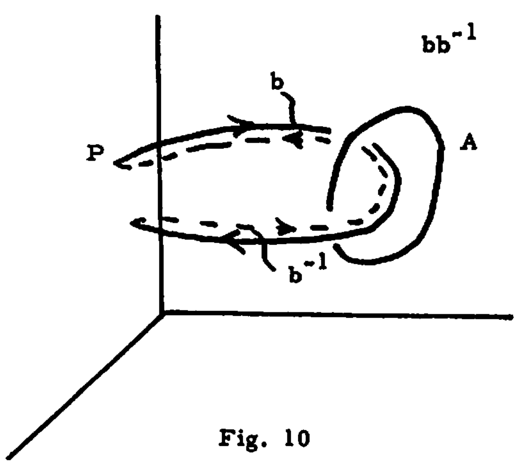


Fig. 10

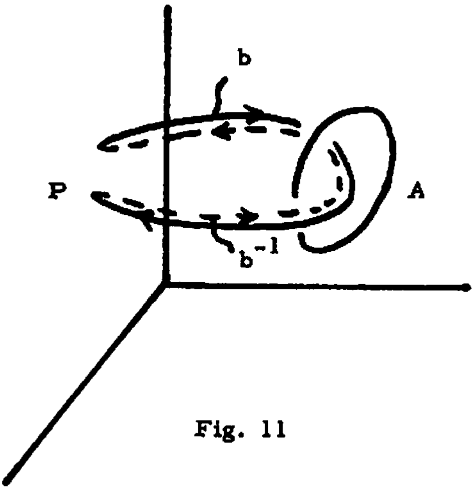


Fig. 11

We must now verify that $[I]\,[a] = [a]\,[I] = [a]$. Recall that $[I]$ is the class of paths which can be continuously deformed to a fixed point. Obviously, the product $a_1 b$ yields a path homotopic to $[b]$ thus $a_1 b = b$ or $[I]\,[a] = [a]$. Similarly, $[a]\,[I] = [a]$ (Figs. 12, 12a, 12b).

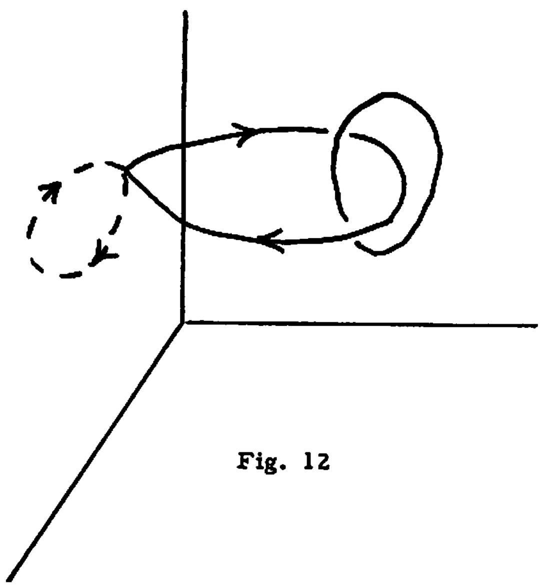
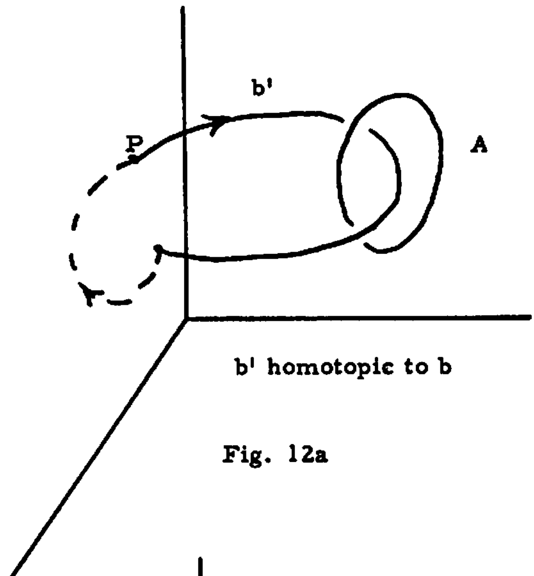
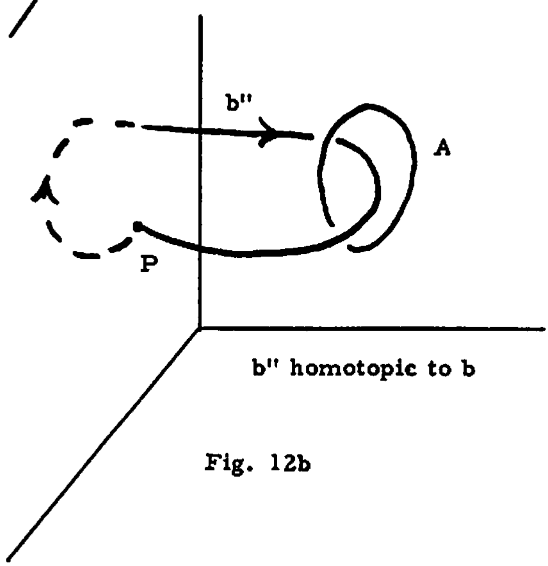


Fig. 12



b' homotopic to b

Fig. 12a



b'' homotopic to b

Fig. 12b

We can form a product using two paths from the same class. Let b and $b' \in [a]$. Then $bb' = b^2$ which loops twice around A

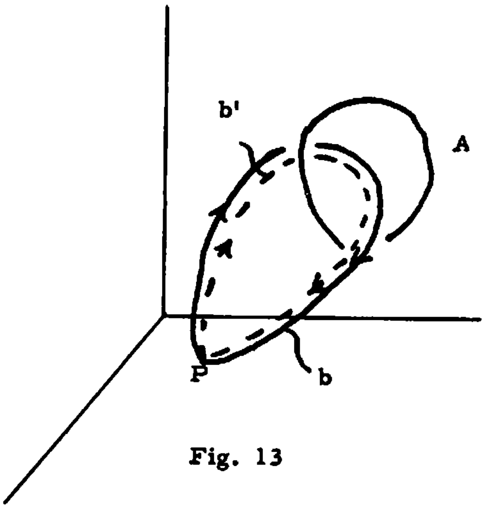


Fig. 13

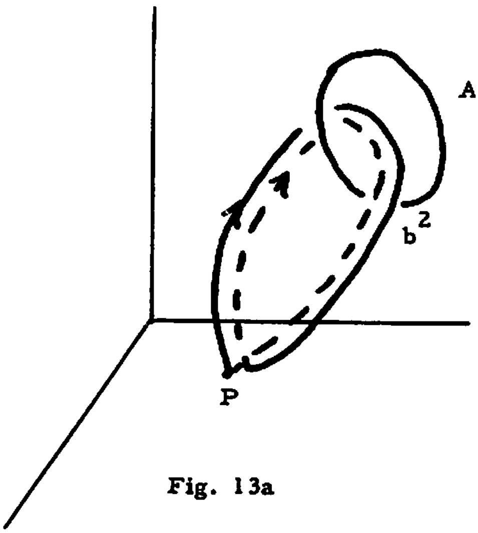


Fig. 13a

(Fig. 13). We can verify the associative law by choosing b, b', and b'' from $[a]$ (Figs. 14, 15, 15a). These result in $b^3 \varepsilon [a]^3$ which

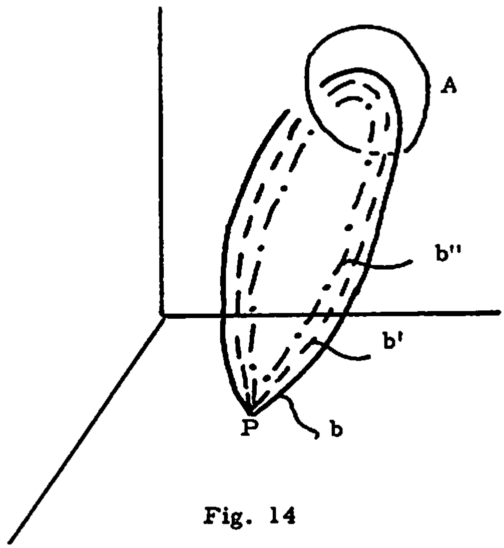


Fig. 14

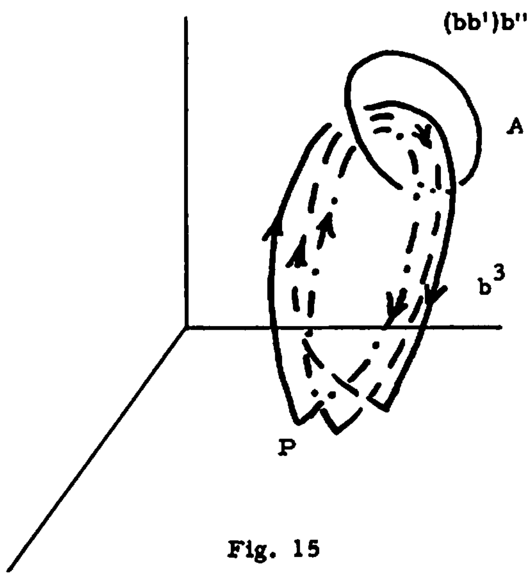


Fig. 15

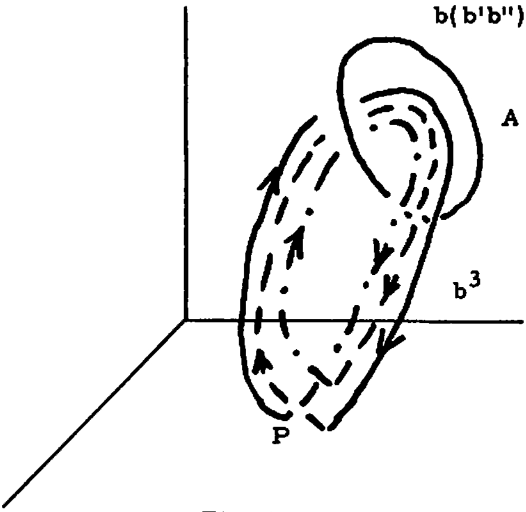


Fig. 15a

loops thrice around *A*.

Since we have satisfied all the necessary conditions, we can conclude that the class of all closed paths homotopic to *a* forms a group. To test the commutative law, look at paths of the form *bb′* and *b′b* (Figs. 16, 16a).

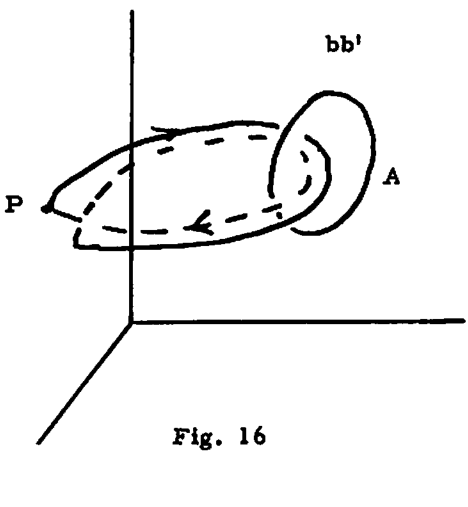


Fig. 16

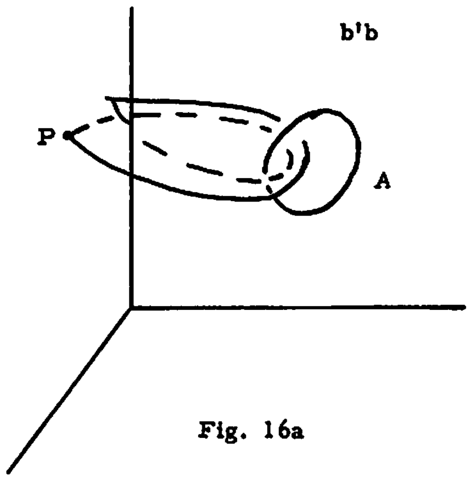
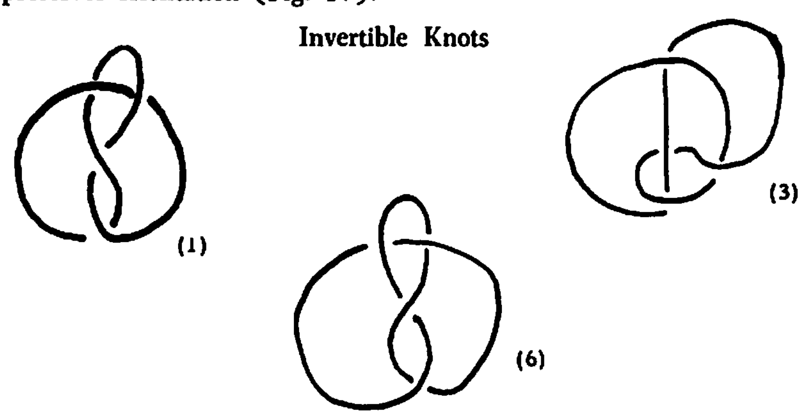


Fig. 16a

This type of work has several applications. One is the use of paths as integration curves in complex variables. A second application is in knot theory of topology. A knot is defined to be a path in three-space which is homomorphic to the unit circle, $x^2 + y^2 = 1$. By working with orientation-preserving operations in different manifolds, one obtains invertible knots of the following types where (1) is a figure-eight knot and (6) is its mirror image. The operation preserves orientation (Fig. 17).

Invertible Knots



(Fig. 17)

Only recently, H. F. Trotter was able to exhibit a class of non-invertible knots. This example is a fascinating one where the number of loops increases by a factor of two moving to the right (Fig. 18).

Non-invertible Knots

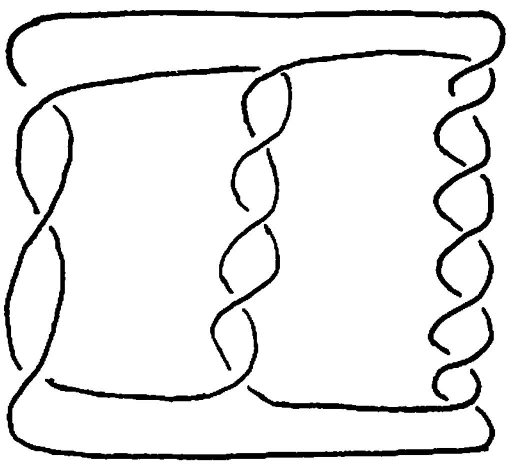


Fig. 18

The books listed in the bibliography give more interesting information about path groups and knots.

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Another Use of the Gradient*

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The gradient has many applications in mathematics and the sciences. In this paper the author will develop another use of this mathematical concept to assist in solving n-degree simultaneous equations.

DEFINITIONS: The gradient is a vector valued function, called the gradient function of F and defined by means of the equation $\nabla F(u) = (F_1\vec{u}, F_2\vec{u})$, where $\vec{u} = (x, y)$ is a vector function (a function with domain of reals and a range of vectors), and F_1 is the first partial derivative with respect to x and F_2 is the first partial derivative with respect to y. The gradient may be extended to R^3 to R functions or F^n to R functions. For R^3 to R functions, the gradient of $F = \nabla F = F_1i + F_2j + F_3k$, where the gradient is written in vector component form, with i, j, and k unit direction vectors.

EXAMPLE: $\phi(x, y, z) = 3x^2y - y^3z^2$. Evaluate the gradient at the point $(-1, -2, -1)$. $\nabla \phi = 6xyi + (3x^2 - 3y^2z^2)j + 2y^3zk = -12i - 9j - 16k$.

APPLICATIONS: The first application of the gradient needed is the concept of directional derivative. The directional derivative is the rate of change of functional values in any direction vector $\vec{u}$. Its value is $D_{\vec{u}}F(\vec{x}) = F(\vec{x}) \cdot u$. Therefore, by the definition of the dot product for vectors, the maximum rate of change of functional values is in the direction of the gradient, as when the definition of dot product is maximum when the cosine is one, which implies that the angle between the vectors is zero. From the above statement the maximum decrease of functional values is in the direction of the negative gradient.

The second concept of the gradient that is needed is the concept of normality to a surface. If a space curve is given by a vector equation, $\vec{R} = F(\vec{t})$, and to say that the curve lies in a given level surface means that $\vec{R}$ satisfies the equation $f(\vec{R}) = c$, for each t, where

*A paper presented at the regional convention of KME at Rosary College, River Forest, Illinois, April 5-6, 1968.

$f(x, y, z) = c$ is the equation of the surface. Taking the derivative of both sides of the equation $f(\vec{R}) = c$, the result is $\nabla f(\vec{R}) \cdot D_t\vec{R} = 0$, which implies that $f(\vec{R})$ is normal to the surface. This result follows because of the definition of the dot product of vectors. (When the dot product is zero the value of the cosine of the angle between the vectors is zero, which implies that the angle is ninety degrees.) Therefore, if P is any point of our level surface, then the gradient of $f(x, y, z) = c$, evaluated at the Point P, is normal to the tangent line $(D_t\vec{R})$ at P, to any curve that lies in the surface and contains the point P. (Recall that a level surface in three dimensions is analogous to that in two dimensions. If c is any number in the range of the function f on R^3 and the graph of the equation $f(x, y, z)$ is a surface in space, then the surface is called a level surface.) There are an infinite number of space curves that pass through the point P and are in the given level surface, and each has a tangent line that is perpendicular to the gradient of f at the point P. These tangent lines all lie in the same plane and thus a tangent plane is formed to the level surface at the point, P.

With these two ideas, the method of approximation can be developed. We can approximate the rate of change of functional values of an R to R function by the change in the tangent line drawn to the curve at a point P. It is assumed that function is continuous and that only very small changes in functional values are taken.

For an R^2 to R function, we can approximate the change in functional values (provided they are small) by a change in the tangent plane, to a given point. Therefore, if we pick any point on our level surface, an approximation can be made of the change in functional values around that point by changes in the tangent plane drawn at that given point.

We will leave our limited three dimensional world behind and travel into fcur-space or n-space. We can discuss an R^s to R function, where some $w = (x, y, z)$. For our function, all the partial derivatives exist and are continuous. It can be proved (by applying the mean value theorem several times) that we can approximate changes in functional values by a new concept of tangency which we call $\triangle w_{app}$. We have no visual idea of what it will be, but one can think of something similar to that in three-space to get an idea.

This $\triangle w_{app}$ is given by the equation $\triangle w_{app} = F_x(x, y, z) \triangle x + F_y(x, y, z) \triangle y + F_z(x, y, z) \triangle z$, where F_x, F_y, and F_z are the partial derivatives of the function with respect to x, y, and z. These partial derivatives are evaluated at some point $P_0 = (x, y, z)$, and $\triangle x$, $\triangle y$, $\triangle z$, are small changes in the function.

With this very general equation we can solve very complex problems. For example let us try to solve the problem below:

PROBLEM: Solve the three equations simultaneously.

1) $2x + 3y + 4z = 5$,
2) $x^2 + y^2 + z^2 = 7$, and
3) $xyz = 4$.

If we write these three equations as a function of x, y, and z as follows we can apply our approximation method. $f(x, y, z) = (2x + 3y + 4z - 5)^2 + (x^2 + y^2 + z^2 - 7)^2 + (xyz - 4)^2$ and solve for $f(x, y, z) = 0$, the problem is solved. The reason for squaring each term is to force each equation to zero.

In any approximation method, we start with a guess. Let $P_0 = (x_0, y_0, z_0)$ be a first guess and suppose the result is not zero, and in this case $F(x_0, y_0, z_0) = f_0$ is positive, which implies that we must decrease the functional value. But in what direction? and how far? From our development of the gradient, we know the maximum rate of change of functional values is in the direction of the gradient, so we will want to decrease the functional values in the direction of the negative gradient, because it will give us the most rapid decrease. Therefore, $\triangle x$, $\triangle y$, $\triangle z$ will be in the direction of the negative gradient, with each change in functional value going in the direction of its counter part of the gradient, evaluated at the point P_0. The next question must also be answered. To insure the right distance let us multiply by a variable h. This implies that $\triangle x = hf_1$, $\triangle y = hf_2$, and $\triangle z = hf_3$. This implies that $\triangle w_{app} = f_1hf_1 + f_2hf_2 + f_3hf_3$, which is our method of approximation. Our next question is what value of h makes $\triangle w_{app} = -f_0$. It stands to reason that if we are five over our goal, we will want to decrease it by five in the next approximation. To solve for this answer, we write the equation $\triangle w_{app} = f_1^2 h + f_2^2 h + f_3^2 h = -f_0$, which implies that $h = \dfrac{-f_0}{f_1^2 + f_2^2 + f_3^2}$. If the point P_0 was not a correct answer, we want to subtract the small change from the point. Therefore, $x = x_0 - \triangle x$, $y = y_0 - \triangle y$, and $z = z_0 - \triangle z$, where x, y, and z is our new guess. Substituting the known values for Δx, Δy, and Δz, the new guess becomes $x = x_0 - hf_1$, $y = y_0 - hf_2$, and $z = z_0 - hf_3$. We can substitute the value of h that we have found and arrive at the values of

$$x = x_0 - \frac{f_0 f_1}{f_1^2 + f_2^2 + f_3^2},\ y = y_0 - \frac{f_0 f_2}{f_1^2 + f_2^2 + f_3^2},\ \text{and } z = z_0 - \frac{f_0 f_3}{f_1^2 + f_2^2 + f_3^2}.$$

For this problem, the first guess was $(-\sqrt{2}, -\sqrt{2}, 2) = P_0$. $f(x, y, z) = f(-\sqrt{2}, -\sqrt{2}, 2) = 18$. To lessen confusion the partial derivatives are listed separately as follows:

$$f_1 = (2x + 3y + 4z - 5) \cdot 4 + 4x(x^2 + y^2 + z^2 - 7) + 2yz(xyz - 4),$$

$$f_2 = (2x + 3y + 4z - 5) \cdot 6 + 4y(x^2 + y^2 + z^2 - 7) + 2xz(xyz - 4),$$

$$f_3 = (2x + 3y + 4z - 5) \cdot 8 + 4z(x^2 + y^2 + z^2 - 7) + 2xy(xyz - 4).$$

For the first guess, $f_1 = -22$, $f_2 = -30$, and $f_3 = -24$, and then $h = .009$. Substituting these values in our approximation method, the new guess is $x = -\sqrt{2} - (.009)(-22) = -1.20$, $y = -\sqrt{2} - (.009)(-30) = -1.14$, and $z = 2 - (.009)(-24) = 2.22$. With these values, the function of $f(x, y, z)$ gives us a value of 5.22, which is greatly reduced from the original guess, which gave us a value of 18. The approximation method can be applied again and the result is a functional value of 3.00. If the method of approximation is repeated enough times the correct answer can be found, provided that one does exist. The beauty of the method is that the number of simultaneous equations that can be solved is unlimited, and their degree can also be unlimited. The computation may look long and hard, but this can be programmed to a computer and it will do all the work in a matter of seconds.

Thus still another use of the gradient and its related concepts has been developed for the use of mathematicians and scientists.

An Interesting Case of "One-Way Only" in Linear Algebra

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Consider the following two theorems, in which the e's are vectors and a, b, c, and d are scalars.

THEOREM 1: If $ae_1 + be_2$ and $ce_1 + de_2$ are linearly independent, then they span the same vector space as e_1 and e_2.

THEOREM 2: If $ae_1 + be_2$ and $ce_1 + de_2$ are linearly independent, then e_1 and e_2 are linearly independent.

The reader should generalize these theorems mentally to the case of n linear combinations of n vectors.

Suppose it is desired to prove these theorems as close to first principles as possible. In particular we will assume no knowledge about solvability of simultaneous linear equations and we assume nothing has been proved yet following the definitions of *vector space, linear dependence*, and *spanning set*. If Theorem 1 is proved first then Theorem 2 follows, as we will show. This writer felt strongly that, vice versa, if Theorem 2 were proved first, then Theorem 1 would follow. But the writer finally convinced himself that Theorem 2 (or the n-vector generalization) cannot be proved first. Here is the case of "one-way-only" that is the subject of this paper—the *way* is from Theorem 1 to Theorem 2, and cannot be from Theorem 2 to Theorem 1. If one sets out to prove Theorem 2 from first principles, he will find that he must prove Theorem 1 first. Those who find this as surprising as the writer found it, may want an "explanation" as much as the writer did, and an explanation will be given.

We will restate the theorems for the case of four linear combinations of four vectors as follows, where the u's and the e's are vectors in the same vector space, Σ.

THEOREM 1. If (u_1, u_2, u_3, u_4) is a linearly independent set and if (e_1, e_2, e_3, e_4) spans Σ, then (u_1, u_2, u_3, u_4) spans Σ.

THEOREM 2. If (u_1, u_2, u_3, u_4) is a linearly independent set and if (e_1, e_2, e_3, e_4) spans Σ, then (e_1, e_2, e_3, e_4) is a linearly independent set.

First we state two lemmas which follow directly from the definitions of dependence and spanning set.

LEMMA 1. If $v = ae_1 + be_2 + ce_3 + de_4$ and if $a \neq 0$, then (v, e_2, e_3, e_4) spans the same vector space as (e_1, e_2, e_3, e_4).

LEMMA 2. If $v = ae_1 + be_2 + ce_3 + de_4$ and if (v, e_2, e_3, e_4) is linearly independent, then (e_1, e_2, e_3, e_4) is linearly independent.

The proofs of these lemmas are good exercises for the beginner in linear algebra.

Note that the hypotheses in Lemma 1 follow as conclusions from the hypotheses of Lemma 2, but not vice versa; in Lemma 1, it is possible for (v, e_2, e_3, e_4) to be linearly dependent (e.g. let $e_1 = e_2 = e_3 = e_4$, and Lemma 1 still holds). Here is the source of the one-way-only phenomenon. We will use Lemma 1 to prove Theorem 1; *then* we will use Lemma 2 to prove Theorem 2. But we can not go the other way and use Lemma 2 to prove Theorem 2 first. We must begin with the lemma that has the weaker hypotheses (Lemma 1). These weaker hypotheses will be satisfied when the lemma is applied, permitting us to draw a conclusion and get started; but the stronger hypotheses of Lemma 2 would not be satisfied if Theorem 1 had not been proved first, as we shall see.

PROOF OF THEOREM 1. The idea of the proof is to begin with (e_1, e_2, e_3, e_4), replace one of the e's by u_1 and observe that the new set is still a spanning set; then replace another e by u_2, and another e by u_3, and the last e by u_4, observing at each step that the new set is still a spanning set.

Since (u_1, u_2, u_3, u_4) is a linearly independent set, no one of these vectors equals zero. Since (e_1, e_2, e_3, e_4) spans Σ, $u_1 = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$. At least one coefficient is not zero since $u_1 \neq 0$. After relabeling the e's if necessary, $a_1 \neq 0$, then, using Lemma 1, (u_1, e_2, e_3, e_4) spans Σ. Since (u_1, e_2, e_3, e_4) spans Σ, $u_2 = b_1u_1 + b_2e_2 + b_3e_3 + b_4e_4$. Now (u_1, u_2) is linearly independent since (u_1, u_2, u_3, u_4) is linearly independent, and so one of the coefficients b_2, b_3, or b_4 is not zero. After relabeling the e's if necessary, $b_2 \neq 0$. Then using Lemma 1, (u_1, u_2, e_3, e_4) spans Σ. Similarly (u_1, u_2, u_3, e_4) spans Σ, and finally (u_1, u_2, u_3, u_4) spans Σ. q.e.d.

Let us put it this way—we have laid a sequence of stepping stones in proving Theorem 1. The stepping stones are the successive spanning sets:

(1) (e_1, e_2, e_3, e_4)
(2) (u_1, e_2, e_3, e_4)
(3) (u_1, u_2, e_3, e_4)
(4) (u_1, u_2, u_3, e_4)
(5) (u_1, u_2, u_3, u_4)

which have been numbered for convenience. In the proof of Theorem 2, we will retrace these stepping stones from (5) back to (1) using Lemma 2. We will see that the stepping stones had to be there before we started; they cannot be laid in the order from (5) to (1).

PROOF OF THEOREM 2. Since (4) is a spanning set (thanks to having proved Theorem 1 first), $u_4 = au_1 + bu_2 + cu_3 + de_4$. Since (5) is a linearly independent set, then (4) is a linearly independent set from Lemma 2. (Note that the hypotheses of Theorem 2 do not insure that $e_4 \neq 0$. If we tried to prove (4) linearly independent without having proved Theorem 1 first, we would first have to strengthen the hypotheses of Theorem 2 with the addition, $e_4 \neq 0$.) Since (3) is a spanning set (from proof of Theorem 1) and (4) is a linearly independent set, then (3) is a linearly independent set from Lemma 2. Similarly (2) is a linearly independent set, and finally (1) is a linearly independent set. q.e.d.

Altogether, if we tried to go from (5) to (1) using Lemma 2 only, without Theorem 1, the hypotheses of Theorem 2 would have to be strengthened by the following additions:

$$u_4 = a_1u_1 + a_2u_2 + a_3u_3 + a_4e_4 \qquad (\text{or } e_4 \neq 0)$$
$$u_3 = b_1u_1 + b_2u_2 + b_3e_3 + b_4e_4$$
$$u_2 = c_1u_1 + c_2e_2 + c_3e_3 + c_4e_4\,.$$

The importance of Theorem 1 (or its generalization to n linear combinations of n vectors) in linear algebra is the almost immediate conclusion that any two bases in a finite dimensional vector space have the same number of elements—which number is then the *dimension* of the space. Suppose, if possible, that Σ contained one basis, S_1, with six elements, and another basis, S_2, with four elements. Let S_3 be any subset of S_1 containing four elements. From Theorem 1, S_3 spans Σ and so is a basis. Then S_1 cannot be a linearly independent set since one of its members equals a linear combination of the others. Then S_1 is not a basis—a contradiction.

Lorentz Geometry

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Introduction. Until the nineteenth century, Euclidean geometry was *the* geometry. As is well known to students of mathematics, Euclidean geometry is an excellent example of the axiomatic method. But by modifying one or more of the axioms of Euclidean geometry, a new and "different" geometry can be developed. In the nineteenth century, the works of Gauss, Bolyai and Lobachevsky resulted in such a new type of geometry, called a non-Euclidean geometry. A little later, at the turn of the century, another "different" type of geometry was developed by Henrik Antoon Lorentz, and this geometry carries his name. Lorentz geometry is related to non-Euclidean geometry but in mathematics it is not as popular as the geometries of Lobachevsky and Riemann. This type of geometry is of more importance in contemporary physics, a topic to be discussed later in this report.

Definition of Lorentz Geometry. Lorentz geometry is concerned with a special set of transformations. These transformations "take the point (x, y) into the point (x', y') where

$$x' = kx + c$$

$$y' = \frac{y}{k} + d$$

with c and d any real numbers and k any positive real number" [5; p. 292]. It can easily be shown that this set of transformations constitutes a group of transformations. And ". . . the study of the properties of the points of a plane which remain invarient under this group of transformations is known as Lorentz geometry" [3; p. 149].

Lorotations. When the above transformation equations are reduced to the case where $c = d = 0$, a special type of Lorentz geometry results. This more limited set of transformations ($x' = kx$; $y' = y/k$) is similar to Euclidean rotations in that it leaves the point $(0, 0)$ fixed. Therefore, this special case of Lorentz geometry is sometimes called lorotations. The remainder of this report will be concerned primarily with lorotations.

Effects of Lorotating. As was stated earlier in this report, k is restricted to being a positive real number. If $0 \leq k < 1$, then under a lorotation the x-axis will contract while the y-axis will stretch. As an example let $k = \frac{1}{2}$. The "new" x coordinates will be determined by $x' = \frac{1}{2}x$, which implies a contraction. And similarly the "new" y coordinates will be determined by $y' = y/\frac{1}{2}$ or $y' = 2y$, which indicates an expansion. If $k = 1$, it is readily seen that all points remain fixed under this lorotation, and finally the case where $k > 1$ would result in an expansion of the x-axis and a contraction of the y-axis.

The effects of lorotating suggest that under most circumstances Euclidean distance and angle measure will not remain the same. This result can be verified by any simple numerical example where $k \neq 1$; therefore, the Euclidean concepts of distance and angle measure are not invarient properties of Lorentz geometry.

The Lorcle. What happens to a circle under a lorotation? A simple computation will show that it is transformed into an ellipse.

Let the circle be given by

$$x^2 + y^2 = r^2, \qquad r > 0$$

Under a lorotation

$$\begin{aligned} x' &= kx \\ y' &= y/k \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= x'/k \\ y &= ky' \end{aligned}$$

Substituting in the equation for the circle gives

$$(x'/k)^2 + (ky')^2 = r^2$$

or

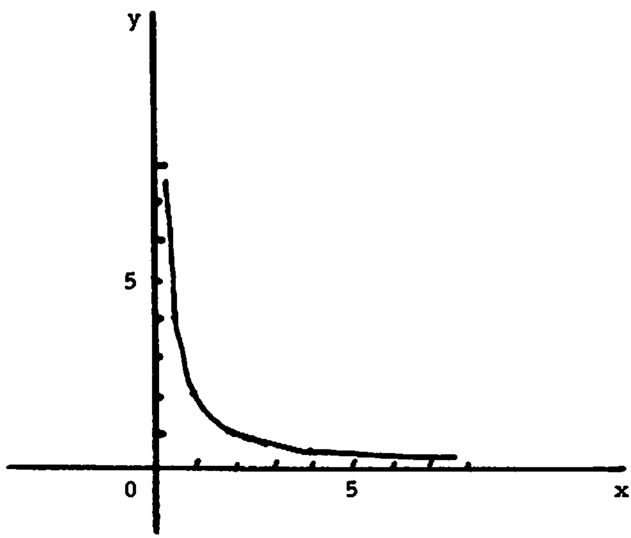
$$(x')^2 + k^4(y')^2 = k^2r^2$$

which is the equation of an ellipse.

In Euclidean geometry, a circle has a special property: ". . . we can get as many points of it as we please by rotating about the center through different angles any point on the circle" [5; p. 297]. Since a lorotation is in many ways analogous to an Euclidean rotation, the next logical step would be to find a figure derived from lorotations (instead of Euclidean rotations) which has the same fundamental property in Lorentz geometry that the circle has in Euclidean geometry. This new figure will be called a lorcle and is defined as ". . . the locus of all points obtainable from a single point by lorotations about a fixed point called the center of the lorcle"

[5; p. 298]. The following example demonstrates how a lorcle can be obtained.

> Let the center of the lorcle be at the origin and let (1, 2) be a point on the lorcle. In order to find more points on the lorcle, lorotations must be applied to the point (1, 2). A few examples of such lorotations are:
>
> $k = \frac{1}{3}$: transforms (1, 2) into $(\frac{1}{3}, 6)$
> since $x' = \frac{1}{3}x = \frac{1}{3} \cdot 1 = \frac{1}{3}$
> $y' = 3y = 3 \cdot 2 = 6$
>
> $k = \frac{1}{2}$: transforms (1, 2) into $(\frac{1}{2}, 4)$
>
> $k = 1$: transforms (1, 2) into (1, 2)
>
> $k = 2$: transforms (1, 2) into (2, 1)
>
> $k = 3$: transforms (1, 2) into $(3, \frac{2}{3})$
>
> Or in general any lorotation transforms (1, 2) into $(k, 2/k)$. Graphing these points results in the graph of a portion of the rectangular hyperbola $xy = 2$.



The graph of this lorcle is restricted to the first quadrant since the point (1, 2) of the lorcle is in the first quadrant and all other points of this lorcle are of the form $(k, 2/k)$ where k is positive.

Therefore, the quadrant in which a lorcle centered at the origin lies can be determined by the location of any point on the lorcle. If the center of the lorcle is not at (0, 0), the resulting lorcle is a regular hyperbola of the form $x^2 - y^2 = a^2$. As with the straight line in Euclidean geometry, a lorcle in Lorentz geometry cannot be drawn in its entirety.

Tangents to a Lorcle. The definition of a tangent to a lorcle in Lorentz geometry is analogous to the definition of a tangent to a circle in Euclidean geometry. That is, there is only one line through a point on a lorcle which touches the lorcle in only one point and this line is called the tangent to the lorcle at that point. Since a lorcle is a hyperbola, we know that the tangent exists and there is only one tangent through a given point on the lorcle, and the equation of the tangent to a lorcle can be found using elementary calculus. In order to more fully justify this statement, it is necessary to recognize that in Lorentz geometry, as in Euclidean geometry, two points determine one and only one straight line.

The general equation of a lorcle centered at the origin is $xy = c$.

Taking the derivative of this equation gives the slope of the tangent

$$\frac{dy}{dx} = -\frac{y}{x}$$

And putting this into the point slope equation of a straight line, the tangent to a lorcle at a point (x_0, y_0) will be

$$(y - y_0) = -\frac{y}{x}(x - x_0)$$

As an example, using the previously derived lorcle, $xy = 2$, the equation of the tangent to this lorcle at the point (1, 2) is

$$(y - 2) = -\frac{2}{1}(x - 1)$$

or

$$y = -2x + 1$$

Lopendicular Lines. As was seen earlier, Euclidean angle measure is not an invariant property in Lorentz geometry. This presents a problem when defining "perpendicular" or lopendicular lines in this new geometry. Here is where the lorcle is very useful. ". . . The tangent to a lorcle is [said to be lopendicular] to the line connecting the center [of the lorcle] with the point of contact" [5; p. 300]. This definition is analogous to the theorem in Euclidean geometry which states that "a line perpendicular to a radius at its extremity is tangent to the circle" [1; p. 283].

Lortance. Since Euclidean distance is not an invariant property in Lorentz geometry, a new definition of "distance" is necessary. This new definition of "distance" must have the property that the "distance" between two points is left unchanged by lorotating. This new "distance" is called the lortance between two points (x_1, y_1) and (x_2, y_2) and is defined to be $(x_1 - x_2)(y_1 - y_2)$. The following statements provide a proof that the lortance between two points is an invariant property of Lorentz geometry.

Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be any two given points. The lortance between P and Q by definition is

$$(x_1 - x_2)(y_1 - y_2).$$

Lorotating P and Q will give

$P' = (kx_1, y_1/k)$ and $Q' = (kx_2, y_2/k)$ respectively.

The lortance between P' and Q' by definition is

$$(kx_1 - kx_2)(y_1/k - y_2/k)$$

which is equal to

$$k(x_1 - x_2)\frac{(y_1 - y_2)}{k} = (x_1 - x_2)(y_1 - y_2)$$

which is the lortance between P and Q.

Rotations and Lorotations: An Analogy. In Euclidean geometry, all points on the circumference of a circle can be obtained by rotating a line of fixed length (the radius) about a fixed point. For the sake of simplicity, let the fixed point be at the origin and let (x, y) be a point on the circle. In Euclidean geometry the length of this radius by definition is $\sqrt{(x - 0)^2 + (y - 0)^2}$ or $\sqrt{x^2 + y^2}$. In order for the length of the radius to remain invariant, the distance of any other point (x', y') on the circle to the center of the circle must be equal to $\sqrt{x^2 + y^2}$. That is,

$$\sqrt{(x')^2 + (y')^2} = \sqrt{x^2 + y^2}$$

or by squaring both sides

$$(x')^2 + (y')^2 = x^2 + y^2.$$

In Euclidean geometry, this type of rotation which transforms a point on a circle into another point on the circle can be represented by the following transformation:

$$\begin{aligned} x' &= rx + sy \\ y' &= -sx + ry \qquad \text{where } r^2 + s^2 = 1. \end{aligned}$$

In their present form, this transformation looks unfamiliar. But by consulting any analytic geometry book, it can be seen that these equations are just a restatement of the equations used to rotate the axes of a coordinate system through an angle θ about the origin.

$$\begin{bmatrix} x' = x \cos \theta + y \sin \theta \\ y' = -x \sin \theta + y \cos \theta \\ \text{where } \cos^2\theta + \sin^2\theta = 1 \end{bmatrix} [4;\ \text{p. } 124].$$

The following gives a proof that this transformation actually does take any point of a circle into another point of the circle.

> Let (x, y) be a given point on a circle centered at the origin. By definition its distance from the center of the circle is the square root of $x^2 + y^2$.
>
> Using the above equations, the point (x, y) can be transformed into the point (x', y') whose distance from the origin is the square root of $(x')^2 + (y')^2$. In order to show that (x', y') is also on the circle, it must be proved that:
>
> $$(x')^2 + (y')^2 = x^2 + y^2.$$
>
> The proof is as follows:

$$(x')^2 + (y')^2 = (rx + sy)^2 + (-sx + ry)^2$$
$$= r^2x^2 + 2rsxy + s^2y^2 + s^2x^2 - 2rsxy + r^2y^2$$
$$= (r^2 + s^2)x^2 + (r^2 + s^2)y^2$$
$$= x^2 + y^2 \qquad \text{since } r^2 + s^2 = 1\ .$$

[5; p. 302].

Therefore, every circle in Euclidean geometry is left unchanged by this transformation.

Since lorcles in Lorentz geometry are analogous to circles in Euclidean geometry, the next logical step is to find a similar transformation in Lorentz geometry which takes any point on a lorcle into another point on the lorcle. Again for the sake of simplicity, let the lorcle (hyperbola) be represented by $x^2 - y^2 = a^2$ ($xy = b$ is a special case of this more general form of an hyperbola) where (x, y) is a point on the lorcle. Let (x, y) be transformed into (x', y') by the following equations:

$$x' = rx - sy$$
$$y' = -sx + ry \qquad \text{where } r^2 - s^2 = 1\ .$$

In order to prove that the point (x', y') is also on the lorcle, it must be shown that:

$$(x')^2 - (y')^2 = a^2$$

or

$$(x')^2 - (y')^2 = x^2 - y^2.$$

The proof follows:

$$(x')^2 - (y')^2 = (rx - sy)^2 - (-sx + ry)^2$$
$$= (r^2x^2 - 2rsxy + s^2y^2) - (s^2x^2 - 2rsxy + r^2y^2)$$
$$= (r^2 - s^2)x^2 - (r^2 - s^2)y^2$$
$$= x^2 - y^2 \qquad \text{since } r^2 - s^2 = 1\ .$$

[5; p. 302].

Therefore, in Lorentz geometry, the lorcle (hyperbola) is invariant under this transformation.

H. A. Lorentz and the Theory of Relativity. This simplified version of the sort of geometry developed by Henrik Antoon Lorentz plays a very important role in the theory of relativity.

"At the turn of the century, H. A. Lorentz was regarded by theoretical physicists of all nations as the leading spirit . . ." [7; p. 5]. To the people of the Netherlands, Lorentz was a national hero. As a student, he was always precocious. He excelled in all subjects but his favorites were mathematics and physics. After graduating summa cum laude from Leyden University in Holland, Lorentz decided ". . . that he could study the subjects necessary for his Doctor's examination just as well by himself . . ."[7; p. 27]. Three years later he passed his exam, summa cum laude.

During the years that followed, Lorentz became acquainted with the works of the great English physicist James Clark Maxwell, the originator of the electromagnetic theory of light. Lorentz devoured everything that Maxwell published. This feat was remarkable since Maxwell's writings ". . . were sometimes referred to as an 'impenetrable intellectual jungle-forest' "[7; p. 32].

From Maxwell's works, Lorentz went on to develop ". . . a complete theory covering all electromagnetic phenomena known at the time, including the electromagnetics of moving bodies" [7; p. 7]. According to Einstein, this was a work of rare clarity, logical consistency and beauty. In this work, Lorentz presented his famous transformations:

$$x' = k(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = k(t - vx/c^2)$$

where $k = \sqrt{1 - v^2/c^2}$, c being the speed of light.

[2; p. 303].

In nature as we know it, the measure of length is a simple thing. "Suppose that a passenger walks from one position to another along the deck of a moving ship. What is the distance from his initial to his final position?" [6; p. 433]. This distance can easily be found using the Galileo transformation:

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

[2; p. 301].

where v is the velocity of the ship. In the seventeenth century, Newton and other physicists believed that this small scale representation was sufficient to represent the mechanics of the whole universe. Newton believed in absolute space and absolute time.

But in 1905, the Michelson-Morley experiment on the velocity of light showed that the velocity of the earth does not affect the velocity of light relative to the earth. This result was a contradiction of the Newtonian belief that the velocity of the earth would either add to or decrease the velocity of light, just as the velocity of the current in a river would add to or decrease the velocity of a row boat. Therefore in order for physical laws to conform to experimental fact, ". . . the basic assumption that the velocity of light is the same for all observers in the universe regardless of how they may be moving relative to each other . . . [and the assumption] . . . that no physical body has a velocity which exceeds that of light [had to be adopted]" [6; p. 439].

"The concepts of absolute space and time, which Newton needed to frame the true laws of the universe, [were discarded by the theory of relativity]. Accepting the fact that two observers moving relative to each other will disagree on the measurements of space and time, [was introduced as] the motions of 'local length' and 'local time' "[6; p. 439]. Two observers who are in motion relative to each other will obtain "different" measurements of distance and time between the same two events. And what each observer sees is, in the theory of relativity, governed by the previously stated Lorentz transformation:

$$x' = k(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = k(t - vx/c^2)$$

where $k = \sqrt{1 - v^2/c^2}$, c being the speed of light

In this transformation, (x, y, z, t) represents the coordinate system of one observer and (x', y', z', t') represents that of the other. The velocity of their motion relative to each other is represented by v.

"H. A. Lorentz had shown that in the same manner in which

Newton's equation $F = ma$ [force = mass times acceleration] is invariant against the Galileo transformation $x' = x - vt$, Maxwell's equations [on electromagnetism] are invariant against the transformations [shown above]. Although Lorentz did not realize the full significance of this fact, these transformations carry his name" [2; p. 303]. The most important consequence of the Lorentz transformations is that space and time are no longer considered to be two continua existing independently beside each other.

So what connection does this have with Lorentz geometry? As was previously stated in this report, the transformations in Lorentz geometry which leave a lorcle invariant under a lorotation are given by:

$$x' = rx - sy.$$

$$y' = -sx + ry.$$

With a little manipulation it can be shown that these equations are a form of the Lorentz transformations used in the theory of relativity.

To make things easier, let $ct = y$ and $ct' = y'$ (since c, the speed of light is the same in any system). Substituting in the above equations gives:

$$x' = rx - sct$$

$$ct' = -sx + rct$$

or

$$t' = -sx/c + rt.$$

Letting $r = k$ and $s = kv/c$, results in:

$$x' = (k)x - (kv/c)ct \qquad x' = k(x - vt)$$
$$\text{or}$$
$$t' = -(kv/c)(x/c) + kt \qquad t' = k(t - vx/c^2)$$

which are the Lorentz transformations so famous in the theory of relativity.

(Continued on page 39)

A Simple and Interesting Topological Space

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When devising examples to illustrate properties of topological spaces it is natural to examine the real line and subspaces on the real line This space is the one with which students are most familiar and it illustrates nicely most of the elementary concepts of topology. The example illustrated below demonstrates that, with a slight alteration in the topology, several of the properties of the subspace may be altered. The example was originally composed to study certain types of invertible spaces, but its usefulness in illustrating more ordinary properties was soon realized.

EXAMPLE: Consider first the subset $S = [0, 1) \cup (2, 3) \cup (4, 5]$ of the real line. Let $\sigma = \{(a, b) \cap S \mid a, b$ real numbers and $a < b\}$ be a collection of open sets in S. This basis produces the relative topology, τ, of S induced by the usual topology for the real line. Now S as a subspace of the real line has the following properties:

a) T_i for $i = 0, 1, 2, 3, 4, 5$. (A T_3-space is regular and T_1; a T_4-space is normal and T_1; a T_5-space is completely normal and T_1.)

b) First countable.

c) Second countable.

d) Separable.

e) Perfect ($S \subset S'$, where S' is the derived set of S).

f) Regular. (S is regular if F is a closed subset of S and $x \in S$, but $x \notin F$, then there exist two disjoint open subsets of S, one containing F and the other containing x.)

g) Lindelöf. (Every open covering of S is reducible to a countable subcovering.) Every second countable space is Lindelöf.

h) Normal. (S is normal if F_1 and F_2 are two disjoint open subsets of S, one containing F_1 and the other containing F_2.) Every regular Lindelöf space is normal.

i) Metrizable. Every regular second countable T_1-space is metrizable.

j) Completely normal. (S is completely normal if A and B are two separated subsets of S, then there exist two disjoint open subsets of S, one containing A and the other containing B.) Every regular second countable space is completely normal.

k) Completely regular. (S is completely regular if F is a closed subset of S, $x \varepsilon S$ and $x \notin F$, then there exists a continuous mapping $f:S \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.) Any normal, regular space is completely regular.

l) Tychonoff. (S is a Tychonoff space if S is completely regular and T_1.)

m) Locally connected.

n) Locally compact.

o) Paracompact. (S is paracompact if for every open covering of S there is a locally finite open cover which refines it. When A and B are two families of subsets of S, A is a refinement of B, or A refines B, if each member of A is a subset of some member of B. E, a family of subsets of S is locally finite (descrete) iff every point of S has a neighborhood which has a nonempty intersection with at most a finite number of the members of E.) Every regular Lindelöff space is paracompact.

On the other hand, S is neither connected, compact, nor countably compact.

Now let σ' be the collect of open sets constructed as follows: $\{1/2\} \cup (2, d)$ or $\{9/2\} \cup (2, d)$ where $2 < d \leqq 3$. $(a, b) \cap S$ where $1/2 \notin (a, b)$, $9/2 \notin (a, b)$, a and b are real numbers where $a < b$. σ' is a base for a topology of S since $S = \bigcup_{A_\alpha \varepsilon \sigma'} A_\alpha$ and for each $x \varepsilon S$ and each pair $U, V \varepsilon \sigma'$ for which $x \varepsilon U$ and $x \varepsilon V$, there exists $W \varepsilon \sigma'$ such that $x \varepsilon W \subset (U \cap V)$. A topology, τ', formed from arbitrary unions of sets of σ' alters S in the following ways:

1) S is not T_2 since there do not exist disjoint, open subsets of S such that one contains $1/2$ and the other contains $9/2$.

2) S is not regular since there do not exist disjoint, open subsets of S such that one contains $[1/4, 3/4]$ and the other contains $9/2$.

3) S is not normal since there do not exist disjoint, open subsets of S such that one contains $[1/4, 3/4]$ and the other contains $[17/4, 19/4]$.

4) Since every completely regular space is regular, S cannot be completely regular as it is not regular. It is a simple exercise to

show that there does not exist a continuous mapping f which takes S into $[0, 1]$ such that $f(1/2) = 0$ and $f\{(4, 5]\} = 1$.

5) Since every completely normal space is normal, S cannot be completely normal as it is not normal. The sets of 3) above also demonstrate that S is not completely normal.

6) S is not T_3.

7) S is not T_4.

8) S is not T_5.

9) S is not Tychonoff.

10) S is not metrizable.

11) Since a paracompact space is normal (a regular, Lindelöf Space), S cannot be paracompact as it is not normal.

We leave it to the reader to show that (S, τ') is T_0, T_1, first countable, second countable, separable, perfect, and Lindelöf. It should be noted that $(S, \tau'$ is locally connected. In fact the sets, $(2, 3) \cup \{1/2, 9/2\}$, $(2, d) \cup \{1/2\}$, and $(2, d) \cup \{9/2\}$ are all connected. Also, (S, τ') is locally compact. Showing local compactness at the points $1/2$ and $9/2$ is all that is of interest. This exercise is left for the reader.

The space (S, τ) is the union of the three connected separated sets, $[0, 1)$, $(2, 3)$, and $(4, 5]$ each of which is both open and closed in S. The space (S, τ') is the union of five connected, separated sets, $[0, 1/2)$, $(1/2, 1)$, $(4, 9/2)$, $(9/2, 5]$, and $(2, 3) \cup \{1/2, 9/2\}$ each of which is both open and closed in S. Also, the sequence $\{2 + 1/n\}$ converges to each of the points $1/2$ and $9/2$ in (S, τ') but does not have a limit in the space (S, τ).

The interested reader should be able to find other properties of both (S, τ) and (S, τ').

Thus all human cognition begins with intuitions, proceeds from thence to conceptions, and end with ideas.

—I. Kant

In Memoriam

Dr. Harold E. Tinnapel, National Vice-President of KME from 1961 to 1965, died last spring at his home in Pemberville, Ohio. He had been a member of the Bowling Green State University faculty since 1949. He was graduated from Ohio State University with B.A., M.A. and Ph.D. degrees. He taught at Indiana Technical College in Ft. Wayne from 1940 to 1943. He was a member of Phi Delta Kappa, Sigma Xi, American Mathematical Society, Mathematics Association of America, Ohio Council of Teachers of Mathematics and the Greater Toledo Council of Teachers. He was editor of the book review section of the *Mathematics Teacher.*

In tribute to Dr. Tinnapel, Past President Carl V. Fronabarger said, "Those who knew Harold E. Tinnapel have been saddened to learn of his untimely death at the age of forty-nine. He served Kappa Mu Epsilon in many ways—as one of those on the Bowling Green Faculty who planned so well for the National Convention which was held on the Bowling Green State University campus and as editor of the Book Shelf section of *The Pentagon.* As National Vice-President from 1961-1965, he capably assumed the responsibility of giving directions for the preparation and the selection of those papers to be presented at two biennial conventions. As a person, he had a warm personality, a fine sense of humor, and a sense of responsibility that enabled him to effectively carry out any obligations that he assumed. I am glad to have had an opportunity to know him as a friend and as a colleague in the activities of KME."

Installation of New Chapters

EDITED BY SISTER HELEN SULLIVAN

NEW YORK ETA CHAPTER

Niagara University, Niagara, New York

The New York Eta Chapter of Kappa Mu Epsilon was installed at Niagara University, Niagara, New York, on Saturday, May

18, 1968. The ceremonies were held in the presidential suite of O'Shea Hall with Dr. Wilbur J. Waggoner of Central Michigan University serving as the installing officer. Twenty-seven charter members were inducted among whom were seven faculty members.

New officers are:

President	Robert W. Pierce
Vice-President	Gregory Sliwa
Secretary	Mary Burgess
Treasurer	Peter Milonni

The faculty sponsor and corresponding secretary is Robert L. Bailey.

Seventeenth Biennial Convention

May 2-3, 1969

The seventeenth biennial convention of Kappa Mu Epsilon will be held on the campus of University of Northern Iowa, Cedar Falls, Iowa, on May 2-3, 1969. Students are urged to prepare papers to be considered for presentation at the convention. Papers must be submitted to Professor George R. Mach, National Vice-President, California State Polytechnic College, San Luis Obispo, California, before February 3, 1969. For complete directions with respect to the preparation of such papers, see page 38 of the Fall 1968 issue of *The Pentagon*.

I hope that every chapter will be well represented at the convention.

Fred W. Lott
National President

Directions for Papers to be Presented at the Seventeenth Biennial Kappa Mu Epsilon Convention

CEDAR FALLS, IOWA
May 2-3, 1969

A significant feature of this convention will be the presentation of papers by student members of KME. The mathematics topic which the student selects should be in his area of interest, and of such scope that he can give it adequate treatment within the time allotted.

WHO MAY SUBMIT PAPERS: Any student KME member may submit a paper for presentation at the convention. Papers may be submitted by undergraduates and graduates; however, undergraduates will not compete with graduates.

SUBJECT: The material should be within the scope of the understanding of undergraduates, preferably the undergraduate who has completed differential and integral calculus. The Selection Committee will naturally favor papers that are within this limitation, and which can be presented with reasonable completeness within the time limit prescribed.

TIME LIMIT: The usual time limit is 20 minutes, but this may be changed on the recommendation of the Selection Committee if requested by the student.

PAPER: The paper to be presented, together with a description of charts, models or other visual aids that are to be used in the presentation should be presented to the Selection Committee. A bibliography of source materials, together with a statement that the author of the paper is a member of KME, and his official classification in school, undergraduate or graduate (at the time of the convention), should accompany the paper.

DATE AND PLACE DUE: The papers must be received in the office of the national vice-president no later than February 3, 1969.

SELECTION: The Selection Committee will choose about ten to twelve papers for presentation at the convention. All other papers will be listed by title and student's name on the convention program, and will be available as alternates. The authors of all papers submitted will be notified as soon as possible after the selection is made.

PRIZES: The author of each paper presented will be given a two-year extension of his subscription to THE PENTAGON. Authors of the two or three best papers presented by undergraduates, according to the judgment of the Awards Committee composed of faculty and students, will be awarded copies of suitable mathematics books. If enough papers are presented by graduate students then one or more similar prizes will be awarded to this group.

George R. Mach
National Vice-President, Kappa Mu Epsilon
Department of Mathematical Sciences
California State Polytechnic College
San Luis Obispo, California 93401

(Continued from page 32)

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The Problem Corner

EDITED BY ROBERT L. POE

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before March 1, 1969. The best solutions submitted by students will be published in the Spring 1969 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor Robert L. Poe, Department of Mathematics, Texas Technological College, Lubbock, Texas 79409.

PROPOSED PROBLEMS

216. *Proposed by Thomas P. Dence, University of Colorado, Boulder, Colorado.*

Show that given a natural number, n, we have

$$n = \log_{(0+1+2+3+4)/5}\left[\log_{\underbrace{\sqrt{\sqrt{\sqrt{\vdots\sqrt{-6+7+8}}}}}_{n \text{ radicals}}} 9\right].$$

217. *Proposed by Charles W. Trigg, San Diego, California.*

(a) Using the nine positive digits just once each form two positive integers, A and B, such that $A = 7B$. Find all possibilities.

(b) Find the unique solution to part (a) above when all ten digits are used.

218. *Proposed by Ali R. Amir-Moez, Texas Technological College, Lubbock, Texas.*

Prove the following known theorem of plane geometry (a) directly, (b) indirectly, (c) algebraically, and (d) analytically. *Theorem.* If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

219. *Proposed by Rosser J. Smith III, Texas Technological College, Lubbock, Texas.*

Show that Q, the number of positive integers no greater than the positive integer M with initial digit no greater than n $(n = 1, 2, 3, \cdots, 9)$, is

$$Q = \begin{cases} M, \text{ if } 1 \leqq M \leqq n \\ M - \dfrac{(9-n)(10^k - 1)}{9}, \text{ if } T \leqq M < (n+1)10^k \\ \dfrac{n(10^{k+1} - 1)}{9}, \text{ if } (n+1)10^k \leqq M < 10^{k+1} \end{cases}$$

where $k = 0, 1, 2, 3, \cdots$, and $T = \max\{n + 1, 10^k\}$ for $10^k \leqq M < 10^{k+1}$.

220. *Proposed by Charles W. Trigg, San Diego, California.*
In the following cryptarithm each letter represents a distinct digit in the decimal scale.

$$6\ (HITFLY) = FLYHIT$$

Identify the digits.

SOLUTIONS

211. *Proposed by J. F. Leetch, Bowling Green State University, Bowling Green, Ohio.*

Prove that in the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$, every fifth term is divisible by 5 and that these are the only terms having this property.

Solution by John David Nichols, Union University, Jackson, Tennessee.

Expanded the sequence is 1, 1, 2, 3, 5, $\cdots$, F_{n-2}, F_{n-1}, F_n, $F_n + F_{n-1}$, $2F_n + F_{n-1}$, $3F_n + 2F_{n-1}$, $5F_n + 3F_{n-1}$, $8F_n + 5F_{n-1}$, $\cdots$. Let I_{n_5} be the statement: F_{n_5}, the nth-fifth term of the Fibonacci sequence 1, 1, 2, 3, 5, $\cdots$, is divisible by 5. I_{1_5} is true. If I_{k_5} is true, then $I_{(k+1)_5}$ is true since if $5 \mid F_{k_5}$ then $5 \mid (8F_{k_5} + 5F_{k_5-1})$. Hence by induction every fifth term is divisible by 5.

Now if $5 \nmid F_n$, then $5 \nmid (F_{n+5} = 8F_n + 5F_{n-1})$. Since 5 does not divide F_1, F_2, F_3, and F_4, 5 cannot divide F_{in+5} for $i = 1, 2, 3, 4$ and n a positive integer not divisible by 5. Therefore, only fifth terms are divisible by 5.

Also solved by Dana Mabbott, Marietta College, Marietta, Ohio; William R. MacHose, Grove City College, Grove City, Pennsylvania; Don N. Page, William Jewell College, Liberty, Missouri.

212. *Proposed by Charles W. Trigg, San Diego, California.*

There is only one three-digit number which is six times the sum of the fourth powers of its digits. Find this number.

Solution by Edgar C. Torbert III, Alabama College, Montevalle, Alabama.

Let x, y, z be the three digits of the number. The number is then represented by the expression: $100x + 10y + z$. The sum of the fourth powers of the digits is represented algebraically as: $x^4 + y^4 + z^4$. It is given that: $100x + 10y + z = 6(x^4 + y^4 + z^4)$. If the number has three digits, then $1 \leqq x \leqq 9$, $0 \leqq y \leqq 9$, and $0 \leqq z \leqq 9$. Expansion of the right side of the equation yields the following: $100x + 10y + z = 6x^4 + 6y^4 + 6z^4$. $6x^4 + 6y^4 + 6z^4$ must be less than $100(x + 1)$. Furthermore, acceptable values for x must satisfy the following condition: $6x^4 < 100(x + 1)$, and these values must be in the range stated above. Reference to the table below reveals that one and two are the only values which satisfy the two conditions stated for x, since larger values yield progressively greater values for $6x^4 - 100(x + 1)$.

x	x^4	$6x^4$	$100(x + 1)$
1	1	6	200
2	16	96	300
3	81	486	400

Letting $x = 1$, we can derive the following equation:

$$6(1)^4 + 6y^4 + 6z^4 = 100(1) + 10y + z$$
$$6 + 6y^4 + 6z^4 = 100 + 10y + z$$
$$6y^4 + 6z^4 = 94 + 10y + z.$$

Now a condition may be set forth for the possible values for y when $x = 1$. We may state that: $6y^4 + 6z^4 < 94 + 10(y + 1)$. Then, $6y^4 < 94 + 10(y + 1)$. From the following table we find three suitable values: 0, 1, 2.

y	y^4	$6y^4$	$10(y+1)$	$94 + 10(y+1)$
0	0	0	10	104
1	1	6	20	114
2	16	96	30	124
3	81	486	40	134

Letting $x = 1$ and $y = 0$, we derive the following equations:

$$6(1)^4 + 6(0)^4 + 6z^4 = 100 + 0 + z$$
$$6 + 0 + 6z^4 = 100 + z$$
$$6z^4 - z - 94 = 0\,.$$

Solving this fourth degree equation we find one acceptable root. That value for z is 2. Therefore, the three-digit number 102 is a solution satisfying all conditions established for the three digits. A quick check reveals that 102 also satisfies the condition stipulated in the proposed problem.

The uniqueness of the answer is affirmed by the following checks. Letting $x = 1$ and $y = 1$, we derive this fourth degree equation: $6z^4 - z - 98 = 0$. This equation has no roots in the given range for z. Letting $x = 1$ and $y = 2$, we derive the following: $6z^4 - z - 8 = 0$. Again no acceptable roots are found. This verifies that there are not other three-digit numbers with $x = 1$ that are satisfactory.

Letting $x = 2$, we can make tables showing that 0, 1, and 2 are values for y satisfying the condition that: $6y^4 < 104 + 10(y + 1)$. No acceptable solutions are found for the resulting equations: $6z^4 - z - 104 = 0$, $6z^4 - z - 108 = 0$, and $6z^4 - z - 28 = 0$. It may be concluded that there are no solutions for this problem with $x = 2$. Since x cannot have values other than one or two, we can say that the uniqueness of our solution, 102, is affirmed.

Also solved by Alvin M. Black, North Texas State University, Denton, Texas; Anthony D. Girolama Jr., Texas Technological College, Lubbock, Texas; Roy J. Holt, Southern Methodist University, Dallas, Texas; Dana Mabbott, Marietta College, Marietta, Ohio; Frank Mathis, Southern Methodist University, Dallas, Texas; Ronald Mileski, Southern Connecticut State College, New Haven, Connecticut; Peter W. Milonni, Niagara University, Niagara University, New York; William R. MacHose, Grove City College, Grove City, Pennsylvania; Don N. Page, William Jewell College, Liberty, Missouri.

213. *Proposed by R. S. Luthar, The University of Wisconsin, Waukesha, Wisconsin.*

If n is an odd integer with at least two distinct factors, prove that:

$$\log n \geqq (k - 1) \log 3 + \log 5$$

where k is the number of distinct prime factors of n.

Solution by Dana Mabbott, Marietta College, Marietta, Ohio.

We assume that n is positive; otherwise $\log n$ is undefined.

Let $a_1, a_2, \cdots, a_k$ be the k distinct prime factors of n so that $n = a_1^{z_1} \cdot a_2^{z_2} \cdot \cdots \cdot a_k^{z_k}$, where the z's are all positive integers. We are given that $k \geqq 2$. Since the a's are prime and odd, no a is 1 or 2. Hence $a_i \geqq 3$ for all $i = 1, 2, \cdots, k$. At least one of the a_i's is greater than or equal to 5, since at least two distinct a_i's exist. We will not sacrifice generality if we assume that $a_1 \geqq 5$, so that $a_2 \geqq 3, a_3 \geqq 3, \cdots, a_k \geqq 3$.

Hence $n \geqq 5^{z_1} \cdot 3^{z_2} \cdot 3^{z_3} \cdot \cdots \cdot 3^{z_k}$. We have $z_1 \geqq 1$, $z_2 \geqq 1, z_3 \geqq 1, \cdots, z_k \geqq 1$, so that $5^{z_1} \cdot 3^{z_2} \cdot 3^{z_3} \cdot \cdots \cdot 3^{z_k} \geqq 5 \cdot \underbrace{3 \cdot 3 \cdot \cdots \cdot 3}_{k\,-\,1 \text{ terms}}$.

Therefore $n \geqq 5 \cdot 3^{(k-1)}$. Taking logarithms of both sides, we obtain $\log n \geqq \log[5 \cdot 3^{k-1}]$, or $\underline{\log n \geqq \log 5 + (k - 1) \log 3}$.

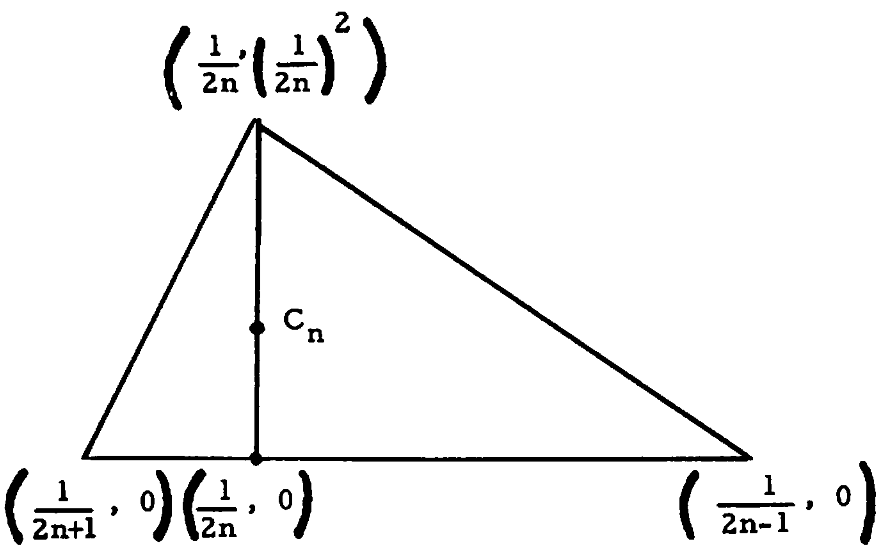
Also solved by Don N. Page, Williams Jewell College, Liberty, Missouri.

214. *Proposed by J. F. Leetch, Bowling Green State University, Bowling Green, Ohio.*

Join consecutively the points $(1,0)$, $(\frac{1}{2},(\frac{1}{2})^2)$, $(\frac{1}{3},0)$, $(\frac{1}{4},(\frac{1}{4})^2)$, $\cdots$, $\left(\frac{1}{2n},\left(\frac{1}{2n}\right)^2\right)$, $\left(\frac{1}{2n+1},0\right)$, $\cdots$ with line segments, and include $(0,0)$ in the resulting graph. Does this graph have length?

Solved by Peter Milonni, Niagara University, Niagara University, New York.

The graph forms an infinite number of triangles with base on the x-axis. Each triangle is of the form



Call L_n the perimeter of each triangle minus the segment of the triangle along the x-axis. Then if $\sum_{n=1}^{\infty} L_n$ is a convergent series we can say that the graph "has length." From the diagram we see that

$$Ln = \sqrt{\left(\frac{1}{2n-1} - \frac{1}{2n}\right)^2 + \left(\frac{1}{2n}\right)^4} + \sqrt{\left(\frac{1}{2n} - \frac{1}{2n+1}\right)^2 + \left(\frac{1}{2n}\right)^4}$$

$$= \left(\frac{1}{2n}\right)^2 \left[\sqrt{\left(\frac{2n}{2n-1}\right)^2 + 1} + \sqrt{\left(\frac{2n}{2n+1}\right)^2 + 1}\right]$$

Now let the height of the n^{th} triangle—$(1/2n)^2$—define the diameter of a circle C_n. The circumference of circle C_n is

$$S_n = \left(\frac{1}{2n}\right)^2 \pi = 2\pi\left(\frac{1}{2n}\right)^2$$

Form the ratio $L_n/S_n =$

$$\frac{\sqrt{\left(\frac{2n}{2n-1}\right)^2 + 1} + \sqrt{\left(\frac{2n}{2n+1}\right)^2 + 1}}{\pi}.$$

It is easily seen that

$$\frac{L_n}{S_n} < 1 \text{ (the largest value } \sqrt{\left(\frac{2n}{2n-1}\right)^2 + 1}$$

takes is $\sqrt{5}$ and the largest value

$$\sqrt{\left(\frac{2n}{2n+1}\right)^2 + 1}$$

takes is $\sqrt{2}$, and $\sqrt{5} + \sqrt{2}$ is greater than π.) Also note that

$$\lim_{n\to\infty}\left(\sqrt{\left(\frac{2n}{2n-1}\right)^2 + 1} + \sqrt{\left(\frac{2n}{2n+1}\right)^2 + 1}\right) = 2\sqrt{2} < \pi.$$

Now consider the infinite series $\sum_{n=1}^{\infty} C_n, \sum_{n=1}^{\infty} 2\pi(1/2)\left(\frac{1}{2n}\right)^2 = \frac{\pi}{4}$

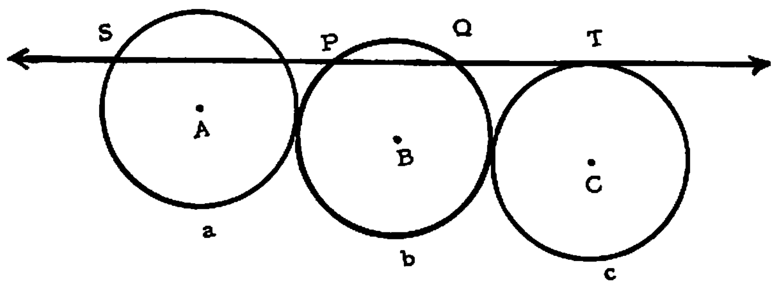
$\sum_{n=1}^{\infty} L_n$ is term by term less than the convergent series $\sum_{n=1}^{\infty} C_n$, $\sum_{n=1}^{\infty} L_n$ must be a convergent series so that our graph does indeed have length.

Also solved by Thomas P. Dence, University of Colorado, Boulder, Colorado; Roy J. Holt, Southern Methodist University, Dallas, Texas; Dana Mabbott, Marietta College, Marietta, Ohio; William R. MacHose, Grove City College, Grove City, Pennsylvania; Don N. Page, William Jewell College, Liberty, Missouri.

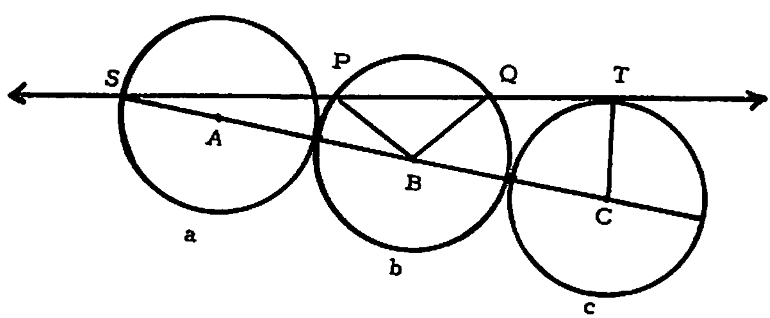
215. *Proposed by Leigh Jones, State University of New York at Albany, Albany, New York*

Circles a, b, c with respective centers A, B, C and radius one are such that a and c are tangent to b. Points S, A, B, C are

collinear. Line $\overleftrightarrow{ST}$ is tangent to c at T, and intersects circle b at P and Q. How long is PQ?



Solution by Frank Mathis, Southern Methodist University, Dallas, Texas.



Recalling from geometry that a tangent to a circle is perpendicular to the radius drawn to the point of contact and that the line of centers of tangent circles passes through the point of contact, we know that STC forms a right triangle with $SC = 5$ and $CT = 1$. Then $\sin \angle CST = 1/5$. Also $SB = 3$ and $PB = QB = 1$. Using the law of sines in triangle SBQ:

$$\frac{\sin \angle BSP}{1} = \frac{\sin \angle SQB}{1}$$

or $\sin \angle SQB = 3/5$ and $\cos \angle SQB = 4/5$.

Using the law of cosines in triangle PBQ:

$$PB^2 = BQ^2 + PQ^2 - 2PQ \cdot BQ \cdot \cos \angle PQB$$

or, $1 = 1 + PQ^2 - (8/5)\ PQ$.

Then, $PQ = 0$ or $PQ = 8/5$.

If $PQ = 0$ then $\overleftrightarrow{PT}$ would be tangent to b and $\overleftrightarrow{SA}$ perpendicular to $\overleftrightarrow{AC}$ which contradicts the given information that S, A, B, C are collinear.

Therefore, $PQ = 8/5$.

Trigonometric solutions also given by Alvin Black, North Texas State University, Dalton, Texas; Gregory Holdan, Indiana University of Pennsylvania, Indiana, Pennsylvania; Mickey Kerr, William Jewell College, Liberty, Missouri.

Geometric solutions were given by Dana Mabbott, Marietta College, Marietta, Ohio; William R. MacHose, Grove City College, Grove City, Pennsylvania; Don N. Page, William Jewell College, Liberty, Missouri; Edgar C. Torbet III, Alabama College, Montevallo, Alabama.

An analytical solution was presented by Jerry K. Stonewater, Drake University, Des Moines, Iowa.

EDITORIAL NOTE:

It has been pointed out by Don N. Page, William Jewell College, Liberty, Missouri, and Kenneth M. Wilke, Topeka, Kansas, that the solution printed for problem 208 in the Spring, 1968 issue of *The Pentagon* is incomplete. They both observed that $x = 4$ also makes $4^x + 4^8 + 4^{11}$ a perfect square, i.e., $4^4 + 4^8 + 4^{11} = 2^8 + 2^{16} + 2^{22} = 2^8 + 2 \cdot 2^{15} + 2^{22} = (2^4 + 2^{11})^2 = (16 + 2048)^2 = (2064)^2$.

The Mathematical Scrapbook

EDITED BY GEORGE R. MACH

Readers are encouraged to submit Scrapbook material to the editor. Material will be used where possible and acknowledgement will be made in THE PENTAGON. All of the Scrapbook material for this issue was submitted by members of the Iowa Gamma Chapter.

Editor's note: The following was submitted by S. Ron Oliver.

Today is Wednesday, September 18, 1968. Did you ever wonder how long it will be before September 18 has fallen on every day of the week? By an application of modular arithmetic this question is rather easily answered. Since 365 is congruent to 1, modulo 7, the answer would obviously be 7, if it were not for leap year. Ah! But we do have a leap year in our wonderfully accurate system of measuring time.

Now four times three hundred and sixty-five plus one is congruent to 5, modulo 7. Hence, every four years the 18th of September falls over a span of five days of the week. Obviously, then, one of those five days is skipped as a result of leap year. Thus, in a seven-year period, one day is skipped and one day is hit twice in a row as our four-year spans of five pass through the seven-day week twice.

Since we obviously skip one day the first time through the week, we must go through at least twice. But since we are working with a cyclic group, twice will be sufficient. To span two weeks we need to go through three, four-year cycles (a span of five days per four-year cycle). Since this will span fifteen days, it will actually be more than enough. Hence, over a twelve-year period, the 18th of September will have hit every day of the week at least once. And we can see that there will have been five duplications (12-7). Since we needed to go through only two complete four-year cycles to span a two-week period, we necessarily skipped only two days. Hence, only two of the five duplications were necessary. Thus, a nine-year period will insure us that the 18th of September has fallen on every day of the week. This nine-year period is both sufficient and necessary.

Another interesting question that arises is how many years must pass before September 18 has fallen on every day of the week an equal number of times. To answer this question we simply need to determine how long it will take before our four-year period ends on the same day of the week it began. In other words, what is the order of 5, modulo 7, under addition? Obviously, since we are working with a cyclic group, the order of 5 is 7. Hence, seven four-year periods or twenty-eight years are required for the 18th of September (or any other day of the year, February 29 included) to fall on every day of the week an equal number of times.

Editor's note: 1968 is a leap year. The seven-year, twelve-year, and nine-year periods mentioned above have patterns if started in 1968 that they might not have if started in other years. Also, periods containing a century year would be interesting to contemplate because the year 2000 will be a leap year but the year 2100 will not.

Editor's note: The following was submitted by Mrs. Pamela Fehr.

Some squares involving only the digit "1" have interesting patterns when written as fractions in just the right way. Consider $(11)^2$ written as follows:

$$(11)^2 = 121 = \frac{484}{4} = \frac{22 \cdot 22}{1+2+1}.$$

Notice the numerator uses all "2's" and the denominator has a "2" in the middle. Also, the denominator has the same digits as the beginning number.

Another similar number is 12,321.

$$(111)^2 = 12{,}321 = \frac{333 \cdot 333}{1+2+3+2+1}.$$

Each factor in the numerator is made up of three "3's" and again "3" is the middle digit of 12,321. The denominator has a "3" in the middle and also is made up of the same digits as 12,321.

Try to write similar fractions for the following numbers:

123,454,321
12,345,654,321
1,234,567,654,321
123,456,787,654,321.

Editor's note: The following was submitted by Craig Bainbridge.

In the Mathematical Scrapbook section of the spring 1968 issue of THE PENTAGON there was a discussion of a quick method of squaring numbers that end in 5. This presentation brings to mind a "quick" system of computation developed by Jakow Trachtenberg, the founder of the Mathematical Institute of Zurich. To give an example of his method of multiplication, several definitions are in order.

A *neighbor* of a digit X is the digit immediately to the right of X. "Half" of a digit is the greatest integer less than or equal to the algebraic half of the digit.

With these terms in mind, to multiply a number by 7, the rule is: double the digit on the extreme right and add "half" the neighbor; further, if the digit is odd add 5. An example follows:

7×4631

7	1 doubled is 2, but 1 is odd; therefore, add 5. Result, 7.
17	3 doubled is 6, but 3 is odd; therefore, add 5. Add "half" of neighbor 1 which is 0. Result, 11. Record 1 ten.
417	6 doubled is 12. Add "half" of neighbor 3 as well as the 1 hundred. Result, 14 hundreds. Hence, write 4 hundred.
2417	4 doubled plus "half" of 6 gives 11 and the 1 thousand makes the total 12 thousands. Write 2 thousand.
32417	0 doubled is 0. Add "half" of 4 and the 1 ten thousand to yield 3 ten thousands.

A glance tells us that continuing the process results in successive zeros on the left and the product is completed. Therefore, $7 \times 4631 = 32{,}417$.

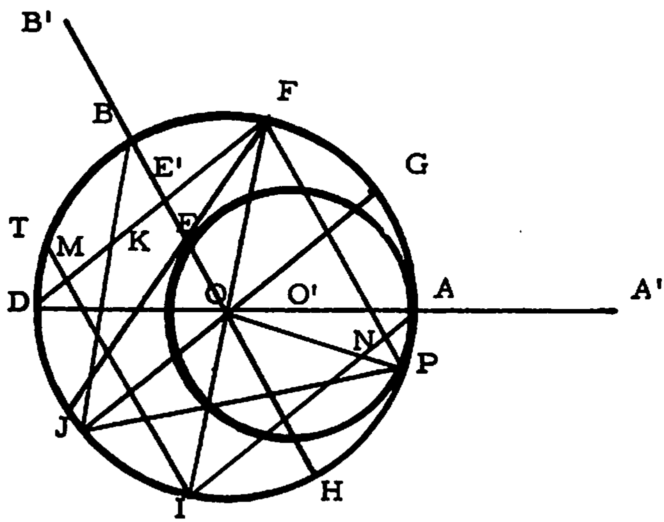
Can you prove this multiplication is valid in general?

Editor's note: The following was submitted by Tom Cooper.

In the December, 1916, issue of THE AMERICAN MATHEMATICAL MONTHLY there appears a proposed solution by Aron Ingvale to an old problem, the trisection of an angle.

Given: $\angle A'OB'$.

Construct circle C with radius, R, and center, O, and call the intersection with $\overleftrightarrow{OA'}$ and $\overleftrightarrow{OB'}$, A and B, respectively. Construct circle, C', tangent to C at A with center on $\overline{OA}$ and radius ¾ R. Call the intersection of $\overline{OB}$ and C', E, and construct a tangent to C' at E and call the intersection with C, F. Bisect $\angle AOF$ and call the intersection with C, G. Extend $\overline{BO}$, $\overline{FO}$, $\overline{GO}$, and $\overline{AO}$ to C and call the intersection points H, I, J, and D, respectively. Now construct lines through I and F parallel to $\overline{BH}$ and call the intersection with C, T and P, respectively.



What's wrong with the following proof?

$\angle\ FOG = \angle\ GOA$, by construction. $\overline{FD}$ is parallel to $\overline{GJ}$ is parallel to $\overline{AI}$ since they subtend equal arcs. $\overline{TI}$ is parallel to $\overline{BH}$ is parallel to $\overline{FP}$ by construction. Therefore, triangle BOJ is isosceles.

K is the intersection of $\overline{BJ}$ and $\overline{DF}$, E' is the intersection of $\overline{BH}$ and $\overline{DF}$. $\overline{DF}$ parallel to $\overline{JG}$ implies that triangle BKE' is isosceles and triangle BKE' is similar to triangle POJ. N is the intersection of $\overline{FP}$ and $\overline{AI}$. $\overline{DF}$ is parallel to $\overline{NI}$, $\overline{BH}$ is parallel to $\overline{FN}$, and $\overline{BJ}$ is parallel to $\overline{FI}$. Triangle FNI is isosceles and is similar to triangle BKE'. M is the intersection of $\overline{DF}$ and $\overline{TI}$. Triangle FNI is congruent

(Continued on page 55)

The Book Shelf

EDITED BY JOHN C. BIDDLE

This department of *The Pentagon* brings to the attention of its readers published books (both old and new) which are of a common nature to all students of mathematics. Preference will be given to those books written in English or to English translations. Books to be reviewed should be sent to Dr. James Bidwell, Central Michigan University, Mount Pleasant, Michigan 48858.

A Nonparametric Introduction to Statistics, Charles H. Kraft and Constance van Eeden, The MacMillan Co., New York, 1968, 342 pp., $9.95.

To the reader who is familiar with the usual texts for an introductory, non-calculus prerequisite, course in elementary statistics, this book will be a distinctly different text designed for the same type of course. For as the author states in his preface "Part I of this text is designed as a one-term introduction to statistics. We chose nonparametric methods as the principal vehicle for this introduction because of the simplicity of their basic probability theory. This simplicity permits an introduction to inference, that is, to the establishment of a relationship between observations and a family of models, to precede a discussion of probability."

Part II of this text contains descriptions of nonparametric tests such as the Wilcoxon, Mann-Whitney test, the sign and median tests, and so on. The author suggests that "With supplementation, by examples from experimental sciences, Part II could serve as a basis for a second course."

For the instructor who is searching for an unusual introductory text in statistics, this book would fit that description. This reviewer, however, is of the opinion that this text will be less readable for the student than the introductory texts using a parametric approach. On the other hand the student would be made more aware of the relationship between statistical inference and mathematical models. Most of the first part of this text is concerned with building a model for treatment effect.

The problem sections are not long and after chapter twelve are non-existent which probably will detract from its desirability as a textbook. The table section at the end of the book is excellent and presents the distributions of several test statistics.

For the student who is interested in nonparametric statistics, this book will serve as an admirable introduction to this topic, but for the student who is interested in an introduction to statistics, this reviewer feels that the present texts in introductory statistics are more suitable.

Wilbur Waggoner
Central Michigan University

Mathematics for Applied Engineering, Edward J. Cairns, Prentice-Hall, Englewood Cliffs, New Jersey, 1966, $10.50.

The text covers a wide range of mathematical topics presently covered in high school/college freshman level courses: algebra, including complex numbers, trigonometry, analytic geometry, and differential and integral calculus.

The format of the book is to provide rules and formulas followed by worked examples and a collection of routine exercises with both of these areas involving engineering applications whenever appropriate. Thus, the mathematical level is below that necessary for a student of science or modern engineering, although as the cover advertises, it could be regarded as a "manual for use by design engineers, draftsmen, and technicians as well as readers preparing to enter these vocations."

T. Robertson
Occidental College

Calculus of Vector Functions, Second Edition, R. E. Williamson, R. H. Crowell, and H. F. Trotter, Prentice-Hall, Englewood Cliffs, New Jersey, 1968, $10.50.

The book under review is the second edition of a book which first appeared in 1962.

The book is a modern course on functions of several variables. The authors develop the necessary linear algebra in the first chapter, so that the only prerequisite for reading this book is the customary year of elementary (one variable) calculus.

The amount of revision that went into this second edition is substantial and the reviewer feels that the current edition is much superior to the first edition. In particular, the chapter on linear algebra has been completely rewritten and expanded. This chapter is very well written and students should find it quite readable. In addition, new material on the various forms of Stokes' theorem (including a brief introduction to differential forms) has been included. Now the student is made aware of the elegant fact that

Stokes' theorem is a generalization of the fundamental theorem of calculus!

R. E. Dowds
State University College, Fredonia, New York

College Algebra and Trigonometry, Daniel E. Dupree and Frank L. Harmon, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1968, $7.95.

The authors have developed a book which should be a great help to those students who have a rather meager background in high school algebra and trigonometry, especially those students who wish to continue their study of mathematics.

The subject is approached from a deductive line of reasoning, beginning with an excellent chapter dealing with logic, sets, and the real number system. The students are then led through a series of chapters involving functions, inverse functions, equations, inequalities, identities, applications of algebraic and trigonometric functions. The text concludes with chapters on determinants and matrices and exponential and hyperbolic functions.

The format of this book presents a very good background for the study of the calculus. The book is well suited to a basic course for college freshmen.

Sister Edmund Marie
St. Bonaventure School

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to triangle FMI. Therefore, triangle FMI is isosceles. $\angle DFI = \angle IFP = \angle BJG = \angle FIA$. But, $\angle IFP = \angle IOH = \angle FOB$. $\angle IFP = \angle BJG = \angle FOG$. Therefore, $\angle BOF = \angle FOG$, and $\angle FOG = \angle GOA$ by construction. $\angle BOF = \angle FOG = \angle GOA$. Therefore, $\angle BOA$ has been trisected.

Prove that the above is correct if $\angle AOB = 90°$ or $180°$, but fails in general.

Kappa Mu Epsilon News

EDITED BY EDDIE W. ROBINSON, *Historian*

Twenty years ago: Colorado Alpha, Missouri Delta, and California Alpha were installed, making a total of forty chapters on the roll.

Ten years ago: L. P. Woods, one of the founders of KME, died on February 26. Members began preparing papers for the 1959 convention at Bowling Green, Ohio. California Beta sponsored a Mathematics Field Day for five hundred students from southern California high schools. Kansas Beta hosted the 1958 regional KME convention.

Five years ago: The Fourteenth Biennial KME Convention was held at Illinois State Normal University. Forty-five chapters were represented by a total of 307 registrants.

Indiana Alpha, Manchester College, North Manchester

Meetings were held every other week with some outside speakers and some from the local chapter. Emphasis was on problem solving and prizes were awarded to those who had solved the most problems. A field trip was taken to Chicago to attend an area meeting of the National Council of Teachers of Mathematics.

Indiana Gamma, Anderson College, Anderson

Last year's President Larry McFarling is now a graduate assistant at Purdue University and KME alumnus Gary Wood is now a graduate assistant at Miami University. Chapter members attended the North Central Regional Convention at Rosary College.

Kansas Gamma, Mount St. Scholastica College, Atchison

Activities for the spring semester, 1967-1968, included the presentation of expository papers by students, a pledge party for actives, an evaluation meeting and a farewell banquet. Special activities were attending the Regional Convention at Tahlequah, Oklahoma, and hosting the High School Invitational Mathematics Contest. The guest lecturer for the semester was Dr. Fred Van Vlick, who spoke on "Applications of Matrix Theory in Economics and Social Sciences."

Missouri Alpha, Southwest Missouri State College, Springfield

Chapter activities for the spring semester, 1967-1968, included the presentation of expository papers by students and faculty, a banquet at which new members were initiated and attendance at the Regional Convention at Tahlequah, Oklahoma. The KME Chapter Merit Award was presented to Harold Weatherwax, chapter vice-president.

Missouri Epsilon, Central Methodist College, Fayette

Ten members were initiated on March 5, 1968, and new officers were installed on May 7, 1968. The programs for the chapter meetings consisted of papers on number theory, statistics, election predictions, teaching projections and other selected topics. Each new member made an abacus and presented it to his sponsor.

New York Gamma, State University College, Oswego

Dr. John Walcott is the corresponding secretary and Dr. James Burling and Dr. Frederic Fischer are faculty sponsors.

Ohio Epsilon, Marietta College, Marietta

Chapter members held tutorial sessions for beginning mathematics students before mid-term and final exams. New members were inducted at the end of February.

Texas Alpha, Texas Technological College, Lubbock

On November 16, 1967, a meeting was held by interested faculty and students who qualified for membership in Kappa Mu Epsilon and it was decided to reactivate the Texas Alpha Chapter. Forty new members were initiated on December 7, 1967, at a banquet which featured Dr. Patrick Odell, Chairman of the Department of Mathematics, as the speaker. Three important chapter meetings had the following programs:

January 4, Mrs. Harmon Jenkins, who spoke on the function and services of the placement office.

February 8, Dr. Thomas Baullion presented a talk on "Techniques for Summing a Series."

March 14, Dr. George Innis, who spoke on "Computer Models in Agriculture."

Thirty-four new members were initiated in April, 1968, bringing the total membership to 728. The chapter officers for 1968-1969 are: Wayne Woodward, President; David Henneki, Vice-President; Judy Murrah, Secretary; Judy Forsman, Treasurer; and Derald Walling, Faculty Sponsor.

Wisconsin Beta, Wisconsin State University, River Falls

Twenty-six new members were initiated bringing the total membership to fifty-six. Dr. Warren Land, University of Minnesota, was the guest speaker. His topic was "The Place of Abstraction in Applied Mathematics." Other speakers for the year were Mr. Douglas Mountain, speaking on "The Portrait of Pi," and Mr. Bruce Williamson, speaking on "The Golden Section, Nature's Divine Proportion."

Report on the 1968 North Central Regional Convention

Ninety Kappa Mu Epsilon members from the North Central area attended the 1968 Regional Convention held at Rosary College on April 5-6. Colleges and universities participating included Anderson College, Central Michigan University, Drake University, Illinois State University, Mt. Mary College, North Park College, Rosary College, and the University of Northern Iowa at Cedar Falls.

Friday evening registration and dinner opened the convention, followed by a lecture given by the guest speaker for the weekend, Dr. Fred C. Leone, presently in the program of visiting professors and from the Department of Statistics and the Department of Industrial Management and Engineering at the University of Northern Iowa at Iowa City. His very interesting talk accompanied by amusing and realistic examples concerned "Statistics—Its Use and Abuse." Later, students and faculty members and guests exchanged ideas and became acquainted over coffee, punch, and cookies at an informal gathering in the college student lounge.

Following an evening at the Oak Park Arms Hotel near Rosary, out-of-town members attended breakfast and further registration at the college. The presentation of student papers began shortly afterwards. The papers and students participating were:

"The Nine-Point Circle" by Mary Brousil
Illinois Zeta Chapter, Rosary College,

"Irrational Roots of Complex Numbers" by Dennis McGavran,
Iowa Beta Chapter, Drake University,

"Geometric Inversion" by Marilyn Lalich,
Wisconsin Alpha Chapter, Mount Mary College,

"Konigsberg Bridge Problem" by Judy Kaiser,
Illinois Zeta Chapter, Rosary College,

"Approximation Method Using the Gradient" by Glenn Grove,
Iowa Alpha Chapter, University of Northern Iowa,

"Properties Common to Fields and Groups"
by Susan O'Connor,
Wisconsin Alpha Chapter, Mount Mary College,
"Superellipses" by Robert Otto,
Michigan Beta, Central Michigan University.

Students and faculty members then voted on the three papers they felt were the most interesting and worthy of special merit. Robert Otto and his "Superellipses" ranked first, followed by Glenn Grove's "Approximation Method Using the Gradient," and Marilyn Lalich's "Geometric Inversion." Special notice was also given to Dennis McGavran's paper on the "Irrational Roots of Complex Numbers."

The convention was also honored by the attendance of Kappa Mu Epsilon's National President, Dr. Fred Lott, from the University of Northern Iowa at Cedar Falls. He presented the welcome and delivered a brief address to those attending the Saturday luncheon that succeeded the presentation of the student papers. Judy Kaiser, President of the Rosary College Chapter, then introduced the officers of the Illinois Zeta Chapter: Sr. M. Philip, Faculty Moderator; Mrs. Richard Schooley, Corresponding Secretary; Joan Weiss, Vice-President; Joanne Capito, Secretary; Pat Husson, Treasurer; and Sr. Marie de Ricci, Dean of Studies at Rosary. Attending faculty members from each attending chapter were then introduced.

Dr. Fred Leone closed the convention with an excellent presentation on the topic, "Why Design Experiments Statistically," emphasizing the accuracy and decision making involved and resulting from this method.

Report on the South Central Regional Convention

Kappa Mu Epsilon chapters in Arkansas, Iowa, Kansas, Missouri, Nebraska, New Mexico, Oklahoma and Texas attended the Regional Convention held at Northeastern State College, Tahlequah, Oklahoma, on April 19-20, 1968. Oklahoma Alpha, the first chapter of KME, was the host chapter.

Many chapters had large delegations including students and faculty members. Three national officers attended: Laura Z. Greene, Secretary, Eddie W. Robinson, Historian, and Dr. Carl V. Fronabarger, Past President. The luncheon speaker was Dr. Emmit R. Wheat, Northeastern State College, who spoke on "Math and Music."

Papers presented were the following:

"Identification of Geometries through Transformation"
Norma Henkenius, Kansas Gamma Chapter

"Magic Squares"
Mrs. Doris Standley, Missouri Beta Chapter

"Cubic Quadruples from Pythagorean Triples"
S. Ron Oliver, Iowa Gamma Chapter

"Relation Method'
Ross Roye, Oklahoma Alpha Chapter

"A Comparative Study of Some Special Methods of Approximate Integration" Judy Graney,
Kansas Gamma Chapter

"Paths and Knots as Geometric Groups"
Barbara Elder, Kansas Delta Chapter

"The Use of Matrices in the Classification of Conics"
Mary Peterson, Iowa Gamma Chapter

"Contributions by Dedekind"
Carolyn Wyatt, Iowa Gamma Chapter

Pat Hossman, President, Oklahoma Alpha Chapter presided at the presentation of the papers and introdouced the following officers of Oklahoma Alpha Chapter:

James Fisher, Vice-President; Patti Compton, Secretary; Dale Moris, Treasurer; Dr. Raymond Carpenter, Corresponding Secretary; Mike Reagan, Sponsor.

It is difficult to estimate the probability of the results of induction.

—Laplace