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National Officers

Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Gödel's Incompleteness Theorem*

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Many centuries ago the Greeks introduced into the study of mathematics a method called the axiomatic method, in which the truth of a few statements called postulates was assumed, and other statements called theorems were proved from the few. This axiomatic method began to come into its own about two centuries ago when mathematicians began to devise postulate sets and formal developments for many branches of mathematics, old and new.

The axiomatic method was rightfully recognized as one of the most powerful methods and tools ever devised. It possessed beauty, form, and logical rigor. From the time of the primary development of the axiomatic method until 1931, it was commonly felt that, given a branch of mathematics, say geometry or arithmetic or whatever, one could devise eventually a set of postulates from which could be proved all the so-called "true" theorems of that system. In 1931, however, a German mathematician, Kurt Gödel, proved that, for the ordinary arithmetic, if a postulate set was consistent, then there are necessarily true statements which cannot be proved, no matter how complete the postulate set.

There are two ideas basic to this discussion: consistency and completeness.

A system is said to be consistent if it is impossible to deduce a statement A and its negation $\sim A$. This property must be present in any postulate set, for it is easily proven that any statement whatever, true or false, may be derived from an inconsistent system. Of course, a postulate set from which every statement is derivable is useless as a mathematical structure.

A system is said to be (intuitively) complete if a given set of statements, the so called "true" statements, can be derived from it. Gödel proved that if arithmetic is consistent, or free from internal contradiction, then it is necessarily incomplete.

It will be the purpose of this paper to examine the ingenious

*A paper presented at the 1967 National Convention of KME and awarded first place by the Awards Committee.

method which Gödel used to prove an exotic theorem such as this. Indeed, the proof is as interesting as the theorem itself.

Before we examine Gödel's proof, let us examine what is called Richard's paradox, the proof of which is very similar to Gödel's proof.

Consider the set

{sentences | the sentence expresses a property of integers}

restricting ourselves, of course, to sentences which can be written using the undefined terms of the system.

Examples:

x is a prime.

x is greater than 3.

x is a composite number.

x has no divisors besides 3 and 17.

Consider the set of all such statements. Arrange them in numerical order based on the following criteria:

1) the number of letters in the statement.

2) if two statements have the same number of letters, arrange them in alphabetical order.

We now have the set of definitions of properties of integers arranged in numerical order:

$$A_1, A_2, A_3, A_4 \cdots$$

where each A is a definition of properties of integers.

Two things may now happen. Consider the formula A associated with the number n.

Case 1. n may have the property A.

Example: $17 \longleftrightarrow$ "n is a prime"

Case 2. n may not have the property A.

Example: $19 \longleftrightarrow$ "n is not a prime"

Now consider the following statement:

"x does not have the property of the expression associated with the number x."

This statement, being a definition of a property of numbers, is among our list of definitions, and is therefore associated with some number n. Question: Does n have the property of the expression associated with it?

If n has the property of the expression associated with it, then "n does not have the property of the expression associated with the number n." — Contradiction.

If n does not have the property of the expression associated with it, then "n does have the property of the expression associated with n." — Contradiction.

Of course the reasoning is fallacious. A little thought will show that the given "definition" does not belong in the list as we originally described it, and thus the entire argument is fallacious.

However, the basic method:

1) associating a statement of a property with a number,
2) finding a statement which asserted something about the number to which it was associated,

was used by Gödel in his proof.

First of all, Gödel devised a way to associate a unique number with every statement of arithmetic. He did this in the following manner.

All statements are built out of certain formal "pieces." We have

1) logical symbols — $\sim$, $\vee$, $\Rightarrow$, $\exists$, $=$, 0, $'$, $($, $)$, $,$
2) numerical variables — x, y, a, b, $\cdots$
3) sentential variables — p, q, r, $\cdots$
4) predicate variables — P, Q, R, $\cdots$

To each of these Gödel assigned a unique number, in the following manner:

1) logical symbols

$$\begin{array}{cccccccccc} \sim & \vee & \Rightarrow & \exists & = & 0 & ' & (&) & , \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

2) numerical variables

$$\begin{array}{cccc} x & y & z & \cdots \\ \downarrow & \downarrow & \downarrow & \\ 11 & 13 & 17 & \cdots \end{array}$$

3) sentential variables

$$\begin{array}{cccc} p & q & r & \cdots \\ \downarrow & \downarrow & \downarrow & \\ 11^2 & 13^2 & 17^2 & \cdots \end{array}$$

4) predicate variables

$$\begin{array}{cccc} P & Q & R & \cdots \\ \downarrow & \downarrow & \downarrow & \\ 11^3 & 13^3 & 17^3 & \cdots \end{array}$$

We now have a number for every symbol. We assign a number to a formula in the following manner:

$$\begin{array}{cccccccccc} (& \exists & x &) & (& x & = & y & ' &) \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2^8 & 3^4 & 5^{11} & 7^9 & 11^8 & 13^{11} & 17^5 & 19^{13} & 23^7 & 29^9 \end{array}$$

Thus for every formula in arithmetic we have associated a unique number. Moreover, given any number, we can find the statement from which it was derived.

Take, for example, the number 243,000,000.

$$243{,}000{,}000 \text{ equals } \begin{array}{ccc} 2^6 & 3^5 & 5^6 \\ \downarrow & \downarrow & \downarrow \\ 6 & 5 & 6 \\ \downarrow & \downarrow & \downarrow \\ 0 & = & 0 \end{array} .$$

Consider another example:

100 equals $2^2\ 5^2$ is not the Gödel number of any formula whatever, since 2 and 5 are not consecutive primes.

We now have established a 1 - 1 relationship between a subset of the natural numbers and the set of all statements in the arithmetical calculus.

One more item, before we leave Gödel numbering. Consider a statement Q and a series of statements, $P_1, \cdots, P_n$ which constitute a proof of Q. Then if $P_1, \cdots, P_n$ have Gödel numbers $N_1, \cdots, N_n$, then the Gödel number of the proof we will define to be

$$2^{N_1} \cdot 3^{N_2} \cdot \cdots \cdot p_n^{N_n} \text{ where } p_n \text{ is the } n\text{th prime.}$$

Consider this statement:

"The sequence of formulas with Gödel number x is a proof of the formula with Gödel number z." Denote this strictly numerical relationship between x and z by $A(x, z)$.

Consider finally the statement

$$(1) \qquad (\forall x) \sim A(x, z) \quad \text{which says,}$$

"For every x, the sequence of formulas with Gödel number x is not a proof of the formula with Gödel number z."

or in other words,

(2) "The formula with Gödel number z is not demonstrable."

Formula (1) is a legal formula and therefore has a Göbel number, say z'.

Claim: The formula $(\forall x) \sim A(x, z')$ is a true formula which is not derivable from the postulates of the arithmetic.

The formula above is the formal representation of the following sentence:

(3) "The formula with Gödel number z' is not demonstrable."
Call this statement A.

We have constructed a statement A such that

(4) *A means that A is unprovable.*

If arithmetic is consistent, then we must have that

(5) *False formulas are unprovable.*

Now we are in a position to present an intuitive proof of the undecidability of the statement A. There are essentially four possibilities.

1) A is false and provable. This statement contradicts (5).
2) A is false and unprovable. A cannot be false, for then we have that A is not unprovable, or provable, by (4), contradicting (5).
3) A is true and provable. This possibility can not happen since A true means by (4) that A is unprovable, contradicting our assumption.
4) The only possibility left, intuitively, is that A be true and unprovable. Indeed, Gödel proved that this is exactly the case.

This consideration, then, is basically the proof of Gödel's theorem that if arithmetic is consistent, then it is necessarily incomplete. What does this theorem mean? It is full of important meaning.

First of all, it implies that there are some inherent, rather basic limitations to the axiomatic method. It implies that no matter how large or all inclusive our postulate set, there will always be a class of true statements which are not derivable.

It also implies that there will never be a replacement for the thinking ability of man. Suppose that we had a huge computer which was programmed to prove theorems with lightning speed, and suppose we gave this computer a set of postulates and a set of

(continued on page 41)

An Introduction to Geometric Models Based on Axiom Systems*

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Modern mathematics consistently emphasizes the necessity of beginning with a set of undefined terms and relations, then setting up the rules or axioms they are to follow, and finally deducing theorems from the axioms. The possibilities inherent in this method have enabled geometricians to step out of the confined area of Euclidean geometry. We wish to demonstrate how a set of axioms can be interpreted into models when undefined terms are replaced by words with meaning. We shall first introduce the set of axioms comprising Incidence Geometry, an interesting system developed by David Hilbert during the first decade of the twentieth century.

The undefined elements to which no specific content is assigned are point, line, and plane. Intuitively seeing these as dots, pencil streaks, and sheets is helpful in illustrating but should never be used as the basis for a proof of a theorem. The undefined relations are:

1. incidence between a point and a line,
2. incidence between a line and a plane.

The axioms of incidence are as follows

1. Two distinct points are incident with one and only one line.

2. Three distinct points that are not incident with any line are incident with one and only one plane.

3. If distinct points A, B are incident with plane $\mathcal{P}$, then each point incident with line AB is incident with $\mathcal{P}$.

4. If two planes are incident with a point, then they are incident with a second point.

5. Each line is incident with at least two points; and each plane is incident with at least three points that are not all three incident with the same line.

*A paper presented at the 1967 National Convention of KME and awarded second place by the Awards Committee.

Remember that this is an abstract system and the undefined terms point, line, plane could be replaced by the nonsense syllables hoig, boig, loig without changing the logical structure of the theory. For example, one of the theorems derived from these axioms would be translated: If two distinct *loigs* have a *hoig* in common then the set of all their common *hoigs* is a *boig*. What seems like an exercise in nonsense actually helps to point up the advantages of an abstract formulation of a mathematical theory. For if the basic terms are literally meaningless, literally devoid of content, the possibility is opened of assigning them content in new and challenging ways.

These axioms of incidence are not statements in the ordinary sense but abstract propositional functions or open sentences. They become true or false statements when meaningful content is assigned to the basic terms point, line, and plane. A geometric model is just such an interpretation of these undefined terms which makes the axioms of the system true statements. Here then is a simple interpretation.

points:	the numbers 1, 2, 3, 4, 5
lines:	the unordered number pairs (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), and the unordered triple (3, 4, 5)
plane:	the set of five numbers (1, 2, 3, 4, 5)
incidence:	An element is incident with an unordered pair or triple if it is contained in it.

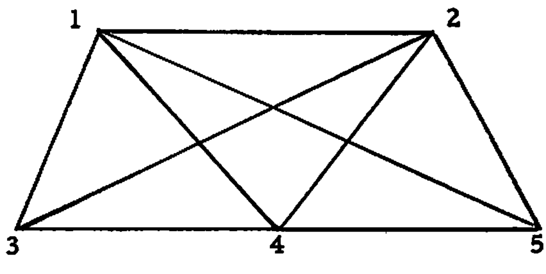

Fig. 1

In order to prove that this interpretation is a model of incidence geometry, it must be verified that it makes the five axioms of incidence geometry true statements. Examining the lines, it is obvious that any two points, such as 2 and 3, are incident with only one of these lines. Axioms 2, 3, and 4 are true only trivially since only one plane is involved. In verifying axiom 5, it is found that each line is in fact incident with at least two points; and there are three points not on the same line which are incident with the plane, such as the line (1, 2) and the point 3, or the line (3, 4, 5) and the point 2.

Now if there is introduced into this set of axioms the additional axiom that corresponding to a line L and a point not incident with it, there is one and only one line incident with the point and parallel to the given line, then a whole new geometry called affine geometry has been defined. There is also the possibility that there may be two lines through a point each parallel to a given line. The type of geometry built upon this axiom is called hyperbolic. Finally, the addition of the axiom that all lines incident with a plane are intersecting, or in other words that there are absolutely no parallel lines, defines projective geometry.

We shall take a look at the following model and try to determine whether it satisfies the axioms of affine, hyperbolic, or projective geometry.

points: the numbers 1, 2, 3, 4, 5

lines: the unordered number pairs (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 5), (3, 4), (3, 5), (4, 5)

plane: the unordered number quintuple (1, 2, 3, 4, 5)

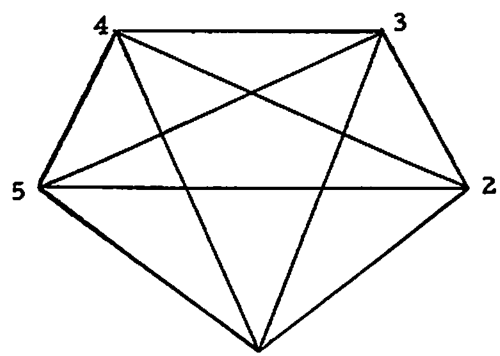

Fig. 2

The verification of the first five axioms is straightforward and in the same manner as was used with the first model. Now, if we choose the line represented by the pair (1, 2) and the point 3, we find that (3, 4) is parallel to (1, 2). But we cannot stop there because so is (3, 5) even though it does not look that way in the diagram. Another instance which is even more startling to our Euclidian-trained minds is the line (3, 5) and the point 1, because both (1, 4) and (1, 2) are parallel to (3, 5). Consequently, it has been verified that this is a model of a hyperbolic geometry.

Since most mathematicians, until they delve into modern geometry, are more familiar with the axiom of unique parallelism which defines affine geometry, it will be more interesting to explore a model of a projective geometry in which there are no parallel lines.

points: the elements A, B, C, D, E, F, G

lines: the unordered triples (A, B, D), (B, C, E), (D, C, F), (D, G, E), (A, E, F), (B, G, F), (A, G, C)

plane: the set (A, B, C, D, E, F, G)

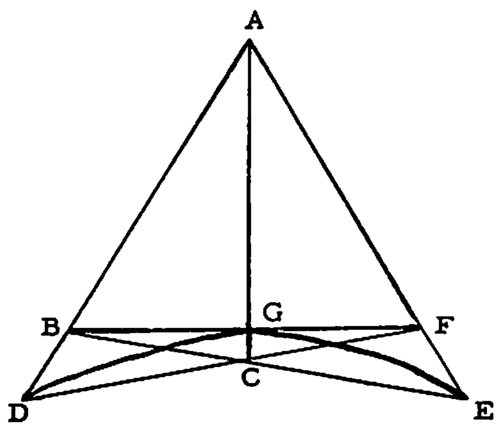

Fig. 3

To verify that all these lines intersect, it is necessary to examine all the triples in the set, and it is found that any two triples have one element in common. Consequently, there are no parallel lines in the projective geometric system of which this model is a representation.

Models can also be very helpful in proving important qualitative properties of a mathematical science. In one such application, models are used to prove that a system is *consistent.* Now a mathematical system, or its set of axioms, is consistent if, within the system, it is impossible to deduce two theorems that contradict each other. In incidence geometry one of the first theorems that can be deduced is that two distinct lines are incident with at most one point. Should it happen that in the course of further development of this geometry it would be found possible to prove that two distinct lines are incident with at least two points, then obviously there would be an inconsistency in our axioms. To prove that a theorem is consistent with a set of axioms one need merely display a model in which the interpretations of both the set of axioms and the theorem are true statements.

Suppose it is necessary to know if this theorem is consistent with affine geometry.

If each of two intersecting lines incident with one plane is parallel to a second plane, then the two planes are parallel.

The following model of affine geometry can be used and tested to determine whether the theorem is a true statement as interpreted in the model.

Points	*Lines*
a, b, c, d, e, f, g, h	(a, b), (a, c), (a, d), (a, e), (a, f), (a, g), (a, h), (b, c), (b, d), (b, e), (b, f), (b, g), (b, h), (c, d), (c, e), (c, f), (c, g), (c, h), (d, h), (e, f), (e, g), (e, h), (f, g), (f, h), (g, h)

Planes

(a, b, c, d), (e, f, g, h),
(a, b, f, e), (d, c, g, h),
(a, b, g, h), (e, f, c, d),
(a, c, g, e), (b, d, h, f),
(a, d, h, e), (b, c, g, f),
(a, d, g, f), (b, c, h, e),
(a, c, f, h), (b, d, e, g)

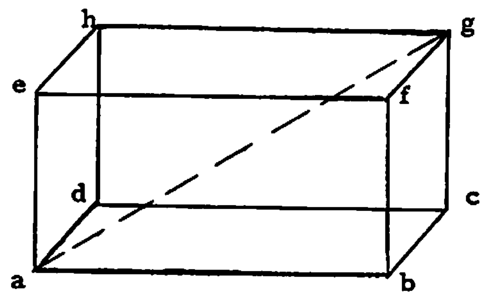

Fig. 4

The intersecting lines (a, b) and (a, c) are incident with the plane (a, b, c, d) and the only plane which is parallel to each of these lines is (e, f, g, h); and (a, b, c, d) is parallel to (e, f, g, h). In the diagram this is obvious. However, if one examines the intersecting lines (a, d) and (a, g), it is found that they are incident with the plane (a, d, g, f) and are each parallel to the plane (b, c, h, e), and these two planes are parallel though it irks ones intuition to think so. Consequently, it has been shown that this theorem is consistent with the axioms of affine geometry.

An interesting but distressing kind of problem often arises in mathematical research when a student conjectures a theorem and has trouble proving it. He may suppose his conjecture wrong and try to prove the opposite theorem. But suppose he does not succeed in this either. He may then wonder whether the theorem cannot be deduced from the postulates of the theory, or whether he has just not been clever enough to find a proof. If the theory is formulated abstractly, as are all the geometric systems presented in this paper, then fortunately there is a procedure for coping with the problem. Consider a specific example. Suppose in the theory of incidence one conjectures the property that every line contains at least three points. If this is a true theorem, that is if it is deducible from the axioms of incidence, then it must be valid for every model of the theory of incidence. But one can easily find a model which falsifies the given property. For example, refer back to the first model of Incidence Geometry. This simple model proves the theorem false because all the lines except one contain only two points. Consequently, the given property cannot be deduced from the axioms of incidence; it is said to be independent of these axioms.

It is also possible to prove that a certain axiom, say axiom A, of any set of axioms is independent of the others, or in other words not deducible from them, if one can display in which axiom A is a false statement and the interpretations of the other axioms are true statements. Yes, models are extremely useful tools for proving the relative consistency and independence of axiom systems not only in geometry but in nearly all fields of the mathematical, physical, and even biological sciences whenever the axiomatic approach is used.

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There is an astonishing imagination, even in the science of mathematics. . . . We repeat, there was far more imagination in the head of Archimedes than in that of Homer. —Voltaire

Adjoining an Idempotent to the Field of Real Numbers

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How can a student of a first course in modern algebra review in a new setting several of the concepts on rings and ideals to which he has just been exposed? It seems likely that an investigation of the nature described below would be of considerable value in clarifying some of the terminology of ring and ideal theory while at the same time providing a deeper feeling for the mathematical ideas involved.

Let R be the field of real numbers and set $K = \{a + bj\colon a, b \in R, j^2 = j, \text{ and } j \notin R\}$. Define equality, addition, and multiplication as follows:

DEFINITION 1. (Equality) $a + bj = c + dj$ if and only if $a = c$ and $b = d$.

DEFINITION 2. (Addition) $(a + bj) + (c + dj) = (a + c) + (b + d)j$.

DEFINITION 3. (Multiplication) $(a + bj)(c + dj) = ac + (ad + bc + bd)j$.

It can easily be verified that $(K, +, \cdot)$ is a commutative ring with unity. Since $j(j - 1) = 0$ but $j \neq 0$ and $j \neq 1$ it is clear that K has proper zero divisors and hence is not an integral domain.

Let $z = a + bj$. We proceed to define the norm of z which we shall denote by $N(z)$.

DEFINITION 4. $N(z) = N(a + bj) = (a + bj)((a + b) - bj) = a^2 + ab$.

THEOREM 1. $N(z_1z_2) = N(z_1)N(z_2)$.

Proof. Let $z_1 = a + bj$ and $z_2 = c + dj$. Then $z_1z_2 = ac + (ad + bc + bd)j$ and $N(z_1z_2) = (ac)^2 + (ac)(ad + bc + bd) = a^2c^2 + a^2cd + abc^2 + abcd$. Also $N(z_1) = a^2 + ab$ and $N(z_2) = c^2 + cd$. Hence $N(z_1)N(z_2) = (a^2 + ab)(c^2 + cd) = a^2c^2 + a^2cd + abc^2 + abcd$.

Therefore $N(z_1z_2) = N(z_1)N(z_2)$.

THEOREM 2. If $z_1z_2 = 0$ then $N(z_1) = 0$ or $N(z_2) = 0$.

Proof. By hypothesis $z_1z_2 = 0$. Hence $0 = N(0) = N(z_1z_2) = N(z_1)N(z_2)$ which implies that $N(z_1) = 0$ or $N(z_2) = 0$.

THEOREM 3. Let $z = a + bj$ be an element of K. Then z has an inverse if and only if $N(z) \neq 0$ and $z^{-1} = ((a + b) - bj)(N(z))^{-1}$

Proof. Let z have an inverse, z^{-1}. Then $zz^{-1} = 1$ and $1 = N(1) = N(zz^{-1}) = N(z)N(z^{-1})$. Hence $N(z) \neq 0$.
Suppose $N(z) \neq 0$. Then $(a + bj)[((a + b) - bj)(N(z)^{-1}] = [(a + bj)((a + b) - bj)] (N(z)^{-1} = N(z)(N(z))^{-1} = 1$.

THEOREM 4. Let $z = a + bj$ be an element of K. If $N(z) = 0$ then $z = bj$ or $z = a - aj$.

Proof. By hypothesis $z = a + bj$ and $N(z) = 0$. Hence $0 = N(z) = a^2 + ab = a(a + b)$ which implies that $a = 0$ or $b = -a$.

Therefore if $N(z) = 0$ then $z = bj$ or $z = a - aj$.

We will denote the set of elements of K whose norm is zero by I; that is, $I = \{z: z \in K \text{ and } N(z) = 0\}$. Perhaps I is an ideal. A little thought, however, quickly indicates that this is not the case because I is not closed under addition since $j \in I$ and $1 - j \in I$ but $1 = (1 - j) + j \notin I$. But, if we consider the two sets $I_1 = \{bj:b \in R\}$ and $I_2 = \{a - aj:a \in R\}$ we find that each is an ideal in K.

THEOREM 5. $I_1 = \{bj:b \in R\}$ and $I_2 = \{a - aj:a \in R\}$ are both ideals in K.

Proof. Let z_1 and z_2 be elements of I_1. Then $z_1 = b_1j$ and $z_2 = b_2j$ which implies that $z_1 - z_2 = b_1j - b_2j = (b_1 - b_2)j \in I_1$.

Let z be an element of K and z_1 be an element of I_1. Then $z = a + bj$ and $z_1 = b_1j$. Hence $zz_1 = (a + bj)(b_1j) = ab_1j + bb_1j = (ab_1 + bb_1)j \in I_1$.

Therefore I_1 is an ideal in K.

In a similar manner it can be shown that I_2 is an ideal in K.

Theorem 6. $I_1 \cup I_2 = I$.

Proof. For each $z \in I_1 \cup I_2$, $z \in I_1$ or $z \in I_2$.

Case 1. $z \in I_1$. Then $z = bj$ and $N(z) = 0$. Hence $z \in I$ and $I_1 \subseteq I$.

Case 2. $z \in I_2$. Then $z = a - aj$ and $N(z) = 0$. Hence $z \in I$ and $I_2 \subseteq I$.

Thus $I_1 \cup I_2 \subseteq I$.

For each $z \in I$, $N(z) = 0$. By Theorem 4, $z = bj$ or $z = a - aj$ which implies that $z \in I_1$ or $z \in I_2$.

Thus $I \subseteq I_1 \cup I_2$.

Therefore $I_1 \cup I_2 = I$.

Theorem 7. $I_1 \cap I_2 = (0)$.

Proof. Clearly $(0) \subseteq I_1 \cap I_2$.

For each $z \in I_1 \cap I_2$, $z \in I_1$ and $z \in I_2$. Hence $z = bj$ and $z = a - aj$ from which it follows that $z = 0$. Thus $I_1 \cap I_2 \subseteq (0)$.

Therefore $I_1 \cap I_2 = (0)$.

Theorem 8. $I_1 \cdot I_2 = (0)$.

Proof. Clearly $(0) \subseteq I_1 \cdot I_2$.

For each $z \in I_1 \cdot I_2$, $z = (bj)(a - aj) = 0$. Thus $I_1 \cdot I_2 \subseteq (0)$.

Therefore $I_1 \cdot I_2 = (0)$.

Theorem 9. I_1 is a principal ideal generated by j and I_2 is a principal ideal generated by $1 - j$.

Proof. Clearly $I_1 \subseteq (j)$.

For each $z \in (j)$, $z = (a + bj)j$ where $a + bj \in K$; that is, $z = (a + b)j \in I_1$. Thus $(j) \subseteq I_1$.

Therefore $I_1 = (j)$.

In a similar manner it can be shown that $I_2 = (1 - j)$.

Theorem 10. Both I_1 and I_2 are prime ideals.

Proof. We begin by showing that I_1 is a prime ideal. Let $z_1z_2 \varepsilon I_1$ and suppose $z_1 \not\varepsilon I_1$. To complete our proof we must show that $z_2 \varepsilon I_1$. Set $z_1 = a + bj$ and $z_2 = c + dj$. $z_1 \not\varepsilon I_1$ implies that $a \neq 0$. $z_1z_2 = (a + bj)(c + dj) = ac + (ad + bc + bd)j$. Thus $ac = 0$ since $z_1z_2 \varepsilon I_1$, and hence $c = 0$ because $a \neq 0$; that is, $z_2 \varepsilon I_1$.

Therefore I_1 is a prime ideal. In a similar manner it can be shown that I_2 is a prime ideal.

Suppose we pause for a moment to look at the residue classes determined by I_1 and I_2. Let $z_1 = a + bj$ and $z_2 = c + dj$. Since $z_1 \equiv z_2 (\text{mod } I_1)$ if and only if $z_1 - z_2 \varepsilon I_1$, $a + bj \equiv c + dj (\text{mod } I_1)$ if and only if $(a - c) + (b - d)j \varepsilon I_1$ which follows if and only if $a = c$. In a similar manner it can be shown that $z_1 \equiv z_2 (\text{mod } I_2)$ if and only if $a - c = -(b - d)$; that is, if and only if $a + b = c + d$.

It is now easy to establish that both K/I_1 and K/I_2 are isomorphic to R.

Theorem 11. $K/I_1 \cong R$.

Proof. Let $\overline{a + bj}$ be the residue class of K/I_1 which contains $a + bj$. Define a mapping ϕ: $K/I_1 \rightarrow R$ by $\phi(\overline{a + bj}) = a$. Clearly ϕ is onto. $\phi(\overline{a + bj}) = \phi(\overline{c + dj}$ implies that $a = c$ and thus that $a + bj \equiv c + dj (\text{mod } I_1)$; that is, that $\overline{a + bj} = \overline{c + dj}$. Hence ϕ is $1 - 1$. $\phi(\overline{a + bj} + \overline{c + dj}) = \phi(\overline{(a + c) + (b + d)j}) = a + c = \phi(\overline{a + bj}) + \phi(\overline{c + dj})$. ϕ $(\overline{a + bj} \cdot \overline{c + dj}) = \phi(\overline{ac + (ad + bc + bd)j}) = ac = \phi(\overline{a + bj})\phi(\overline{c + dj})$.

Therefore $K/I_1 \cong R$.

Theorem 12. $K/I_2 \cong R$.

Proof. Let $\overline{a + bj}$ be the residue class of K/I_2 which contains $a + bj$. Define a mapping ϕ: $K/I_2 \rightarrow R$ by $\phi(\overline{a + bj}) = a + b$. Clearly ϕ is onto. $\phi(\overline{a + bj}) = \phi(\overline{c + dj})$ implies that $a + b = c + d$ and thus that $a + bj \equiv c + dj (\text{mod } I_2)$; that is, that $\overline{a + bj} = \overline{c + dj}$. Hence

ϕ is $1 - 1$. $\phi(\overline{a + bj} + \overline{c + dj}) = \phi(\overline{(a + c) + (b + d)j}) = (a + c) + (b + d) = (a + b) + (c + d) = \phi(\overline{a + bj}) + \phi(\overline{c + dj})$.

$\phi(\overline{a + bj} \cdot \overline{c + dj}) = \phi(\overline{ac + (ad + bc + bd)j}) = ac + (ad + bc + bd) = (a + b)(c + d) = \phi(\overline{a + bj}) \cdot \phi(\overline{c + dj})$.

Therefore $K/I_2 \cong R$.

Theorem 13. I_1 and I_2 are maximal ideals in K.

Proof. We begin by showing that I_1 is a maximal ideal in K. Let I^* be an ideal such that $I_1 \subset I^* \subset K$. If $I_1 \neq I^*$ then there exists a $z \in I^*$ such that $z \notin I_1$. The congruence $zx \equiv b \pmod{I_1}$ is solvable for every b in K since K/I_1 is a field. Hence $zx - b \in I_1$ and $zx - b \in I^*$ from which it follows that $b \in I^*$; that is, $K \subset I^*$. Therefore $I^* = K$ and I_1 is a maximal ideal in K.

In a similar manner it can be shown that I_2 is a maximal ideal in K.

Theorem 14. I_1 and I_2 are the only proper ideals in K.

Proof. By Theorem 13 the only possibility is that there exists an ideal I^* such that $(0) \neq I^* \subset I_1$ or $(0) \neq I^* \subset I_2$.
Case 1. $(0) \neq I^* \subseteq I_1$. Then there exists an element

$z \neq 0$ such that $z \in I^*$.

Since $z \in I_1$, $z = bj$. $z \neq 0$ implies that $b \neq 0$. Let $z_1 \in I_1$. Then $z_1 = b_1 j$.

Since R is a field and $b \neq 0$ there exists an $x \in R$ such that $xb = b_1$. Thus $z_1 = b_1 j = (xb)j = x(bj) \in I^*$ and $I_1 \subseteq I^*$.

Hence $I^* = I_1$.

Case 2. $(0) \neq I^* \subset I_2$. Then there exists an element $z \neq 0$ such that $z \in I^*$. Since $z \in I_2$, $z = a - aj$. $z \neq 0$ implies that $a \neq 0$. Let $z_2 \in I_2$. Then $z_2 = a_2 - a_2 j$. Since R is a field and $a \neq 0$ there exists an $x \in R$ such

(continued on page 49)

Harmonic Vibration Figures*

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The harmonograph is a device employing harmonic motion to describe a graphical figure. It was originally designed to have two pendulums swinging at right angles to one another in a back and forth manner only. Such a machine, when properly fitted with a drawing device, produced harmonograms known as Lissajous figures. These figures have found application in the field of electronics as discussed later. The twin elliptical harmonograph, as shown in Figure 1, is used to describe the figures discussed in this paper and is a similar device except that both pendulums are allowed to swing freely in the various phases of the ellipse. Such a machine can produce Lissajous figures as well as a multitude of modified ones.

The various figures are produced by varying the amplitude, the ratio of the pendulums' periods, the respective axes of the ellipses in which the pendulums swing (the shape of the ellipse), and by changing the direction of swing either to concurrent or countercurrent. As is easily ascertained with the number of variables involved, the number of different drawings obtainable is almost endless. However, many of these drawings are similar in many respects, and the purpose of this paper is to show where they are similar and give the mathematical reasons.

Initially a basic understanding of the machine and pendulum motion is necessary. Pendulums, as used in this machine, swing with a relatively small amplitude; and, therefore, the maximum angle the pendulum shaft makes with the vertical axis is small. For small angles, pendulum motion follows a sine curve [1:75]. This knowledge will be used later in deriving equations for the figures. The period, or the time for a pendulum to make one complete swing, is constant for any given pendulum length. This holds true whether or not the pendulum swings back and forth or follows an elliptical path and does not vary even for different amplitudes; therefore, the period relies solely on the length of the simple pendulum [1:80]. This result is usually true for the pendulums used on the machine.

*A paper presented at the 1967 National Convention of KME and awarded third place by the Awards Committee.

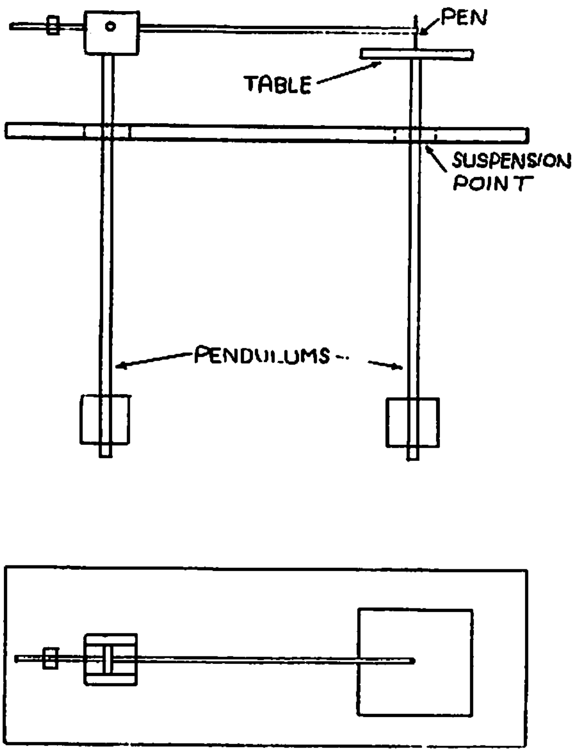


Figure 1

It is complicated only by the fact that there is some top weight on the pendulums. As shown in Figure 1, the pendulum rods extend above the point of suspension. The weight above the suspension point will cause the pendulum to swing slower; the more weight, and the higher up the rod it is placed, the longer the period. However, once this weight is fixed, the period will remain constant no matter how the pendulum is swung. It must be remembered that the amplitudes must be kept relatively small so that the sine curve will be simulated. This is even more important when dealing with top weight.

The machine draws the figures by setting its two pendulums in motion. On top of one pendulum is mounted a table on which the figure is drawn. A long arm with pen attached is mounted on the other pendulum with the pen resting on the table. As the pendulums swing, the pen traces the motion of its pendulum on the table, which is moving in the pattern of the other pendulum. This motion will begin to decrease as friction slows the motion, and the figures will grow smaller. This frictional decrease is of the form e^{-rt} where r is a constant of friction [2:899]. Naturally, the larger r is the faster the system will dampen out, therefore, friction is kept at a minimum. The pendulums are suspended on knife edges, and the pen, where most of the friction is concentrated, is counter-balanced.

With this basic understanding of the machine, we can now move forward into the study of the figures themselves. One of the first things noticed about the figures is in respect to the ratio of the periods of the two pendulums. Good drawings seem to center around simple ratios, *i.e.*, 1:1, 2:3, 3:4, etc. If the ratios are large numbers, 12:17 for example, the figures will lack harmony and symmetry and will seem to be a mere jumble of lines. The figure will have so diminished in size before it has completed so extensive a series of loops that the harmony of the design will not be apparent. These figures are known as discords. Figure 2 is a good example

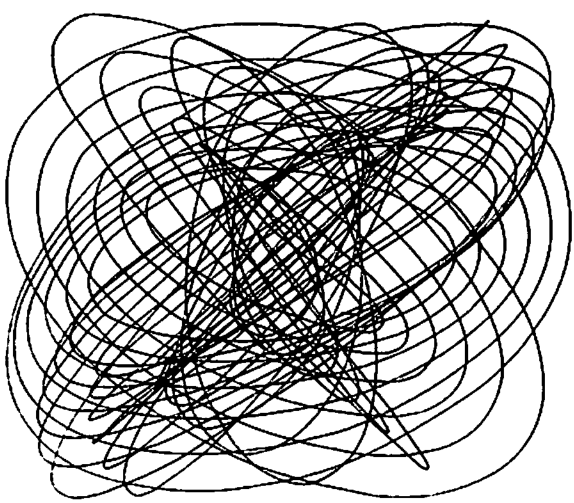


Figure 2

of a discord. The ratio here is approximately 10:13. This figure, as well as all other discords, is actually just as truly the product of harmonic law as the simple ratio figures and holds to the same properties. It is not so pleasing because of its long duration. Compare Figure 2 with the harmony of Figure 3.

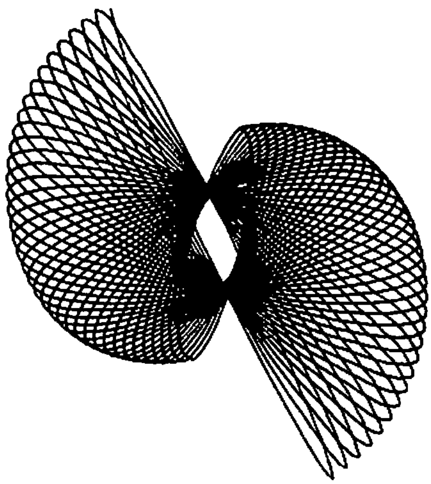


Figure 3

This comparison brings us to an interesting parallel with music in which, as is well known, the harmonious chords are those which combine sound vibrations whose periods are in ratios of small numbers, while the combination of two notes whose ratio of vibrations is represented by high numbers produces an effect of discord on the ear. Table I compares the various ratios with their musical equivalents [1:36].

Another interesting phenomenon takes place when the periods are slightly out of tune, say a ratio of 1:1.05. Here we will get a figure which rotates; each successive loop is slightly rotated — the farther out of tune, the faster the rotation. At the same time the figure is rotating, it will be changing shape. The individual loops which make up the figure will begin to change in shape, broadening or contracting as one pendulum gains on the other.

TABLE I

Comparison of Pendulum Ratios to their Musical Equivalents

Ratio of Period	Musical Chord	Component Notes
1:1	Unison	CC
5:6	Minor 3rd	$CE^{\flat}$
4:5	Major 3rd	CE
3:4	Major 4th	CF
2:3	Major 5th	CG
3:5	Major 6th	CA
1:2	Octave	Cc
1:3	Perfect 12th	Cg
1:4	Double octave	Cc′

Since the ratios are nearly equal, this change is gradual, and the figure remains very pleasing. (See Figure 3.) As the difference in ratio becomes greater, this change will soon become a major factor, and the figure will lose harmony.

For each separate ratio (let us assume we are considering the simple ratios) there is a distinct family of curves. These families are most noticeable when the pendulums are swinging in straight lines perpendicular to each other. Here we will find that the number of loops along each side will correspond to the numerical ratio of the periods. For example, a figure with a 2:3 ratio would have two loops or nodes along one side with three along the perpendicular side (the figures produced being somewhat rectangular in shape). If the pendulums are swinging in ellipses, a different phenomenon will take place. The number of loops in a counter-current figure is the sum of the ratio numbers, while in a concurrent figure, the difference of the ratio numbers gives the number of loops. For the counter-current figure, these loops will point outwards from the center; while when the pendulums are swinging in the same direction, they will point towards the center.

Each family of curves contains almost endless variations, and these are produced by changing the phase, amplitudes, and respective axes of the pendulums. By starting one pendulum ahead of another a phase difference will be effected. A different picture for each phase difference will result, sometimes with so great a variation it is hard for one to distinguish it as a member of its family.

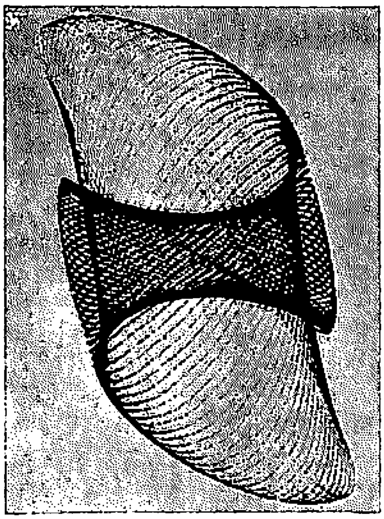


Figure 4

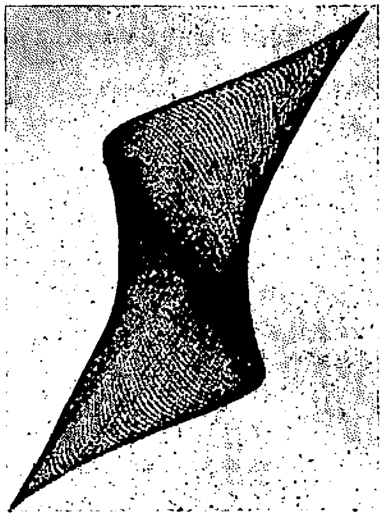


Figure 5

(Compare Figures 4 and 5.) Both figures were drawn identically except for a 45° phase angle difference. Increasing the amplitude of pendulum in any one direction will, of course, stretch the figure in that direction, but since it takes the same time for the pendulum to make its swing, no other changes due to amplitude will result. Changing the various axes of the ellipses in which the pendulums swing is, in effect, varying the amplitude in that particular direction.

Another variable of which we must be concerned is the angle the major axes of the respective ellipses make to one another. The major axis may run anywhere from parallel to perpendicular, and the resulting figure is, of course, changed. This change is often hard to predict. As will be seen later, it is dependent upon how much the rotation of the ellipses change the respective x and y coordinate amplitudes. The last variation is concerned with the direction the pendulums are swinging. If the pendulums are both swinging counter-clockwise, a different figure will result than if they were swinging in opposite directions. As mentioned before, the number of loops will be different as well as the direction in which they point.

Derivation of a General Equation for the Figures Produced

As we know, the figures are produced by combining the movements of two pendulums swinging in an ellipse, or more simply, the combination of two ellipses which are ever decreasing in size. Let

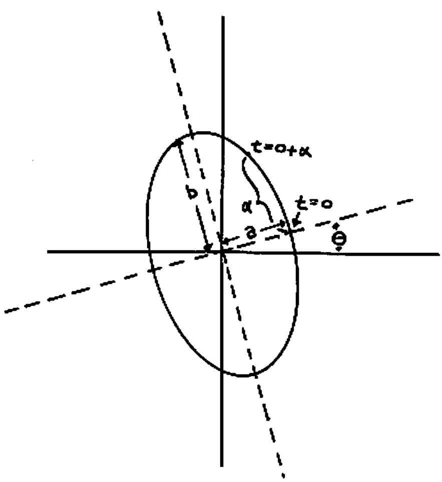


Figure 6

us start with a single ellipse as in Figure 6. The first step is to write the equations for an ellipse in parametric form:

$$\begin{aligned} x &= a \cos \theta \cos t - b \sin \theta \sin t \\ y &= a \sin \theta \cos t + b \cos \theta \sin t \end{aligned} \qquad (1)$$

where t is the phase angle measured in radians [3:203].

Equations (1) would very well describe the pendulum's movement if started at the point $(a, 0)$ and if the angular velocity were $+1$, but in actual practice, the pendulum may be started at any point around the ellipse. We, therefore, must show the initial starting point, or the initial phase angle. This angle will be a constant, α, shown in Figure 6. In equations (1), t will now be replaced by $(t + \alpha)$. The last remaining problem deals with the period of the pendulum. This period can, of course, be changed to almost any desired length. In reality, the shorter the period, the faster the angular velocity. This angular velocity varies directly with the period. We can, therefore, multiply t by ω, the angular velocity, and thereby change the period; ω can be either positive or negative, allowing for either clockwise or counter-clockwise motion. We shall consider counter-clockwise to be the positive direction. Our final equations for x and y are as follows:

$$\begin{aligned} x &= a \cos\theta \cos(\omega t + \alpha) - b \sin\theta \sin(\omega t + \alpha) \\ y &= a \sin\theta \cos(\omega t + \alpha) + b \cos\theta \sin(\omega t + \alpha). \end{aligned} \tag{2}$$

Equations (2) are the undampened description of the pendulum motion. The exponential dampening factor e^{-rt} applies to our conditions of small amplitude; the pendulums will dampen by this factor equally in all directions. Therefore, any point (x^*, y^*) on the curve described by the pendulum at any time t will be given by

$$\begin{aligned} x^* &= e^{-rt}(x) \\ y^* &= e^{-rt}(y) \end{aligned} \tag{3}$$

where r is a constant of friction applying to the considered system. We now have a set of equations (3) which completely describe the motion of one pendulum. Naturally, we have a similar set for the second pendulum, the only difference is that the constants a, b, θ, α, and ω may be different for the second set. It will be of interest to note the relationship between ω_1, and ω_2, the respective angular velocities. Their ratio $\frac{|\omega_1|}{|\omega_2|}$ is exactly the same as the ratio of the two periods.

The last step in deriving the equations for the harmonic figures is relating the two motions. A quick study of the apparatus soon reveals that the pendulums actually subtract from one another. This result is easily demonstrated by having the two pendulums swinging in exactly the same ellipse with equal periods, i.e., $a_1 = a_2$,

$b_1 = b_2$, $\theta_1 = \theta_2$, $\alpha_1 = \alpha_2$ and $\omega_1 = \omega_2$. The resultant figure will be a single point at the origin. Assuming (x_1^*, y_1^*) and (x_2^*, y_2^*) to be any two points of the ellipses described by the respective pendulums at any given time t, the point (X, Y) on the resulting curve will be given by

$$\begin{aligned} X &= x_1^* - x_2^* \\ Y &= y_1^* - y_2^* . \end{aligned} \tag{4}$$

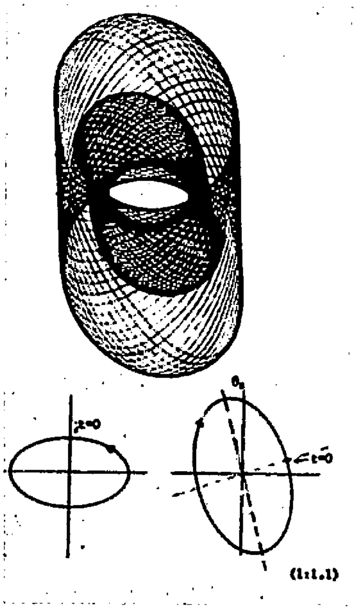
Substituting equations (2) and (3) into (4), we arrive at the final equations for the resultant figure

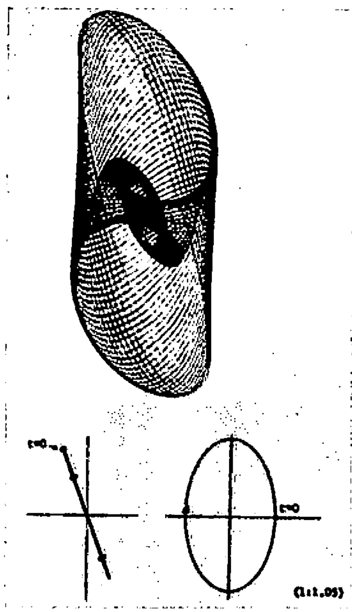
$$\begin{aligned} X = e^{-rt} \{&[a_1 \cos \theta_1 \cos (\omega_1 t + \alpha_1) - b_1 \sin \theta_1 \sin (\omega_1 t + \alpha_1)] \\ &- [a_2 \cos \theta_2 \cos (\omega_2 t + \alpha_2) - b_2 \sin \theta_2 \sin (\omega_2 t + \alpha_2)]\} \\ Y = e^{-rt} \{&[a_1 \sin \theta_1 \cos (\omega_1 t + \alpha_1) + b_1 \cos \theta_1 \sin (\omega_1 t + \alpha_1)] \\ &- [a_2 \sin \theta_2 \cos (\omega_2 t + \alpha_2) + b_2 \cos \theta_2 \sin (\omega_2 t + \alpha_2)]\} . \end{aligned} \tag{5}$$

We have now related all the variables in one expression. With a little careful inspection, one can determine the change that will result if one or more of the conditions are altered. It would now be possible to produce these figures on a computer coupled with one of the new plotting devices by feeding in the equations and the proper data for the constants. By making r small figures with very close lines could be drawn. In effect we could actually make the figures seem as though they were entirely shaded in, or by making r somewhat larger, make them dampen out very swiftly, producing lines far apart.

Naturally, one wonders if these figures have any application. As mentioned earlier, Lissajous' figures are used in electronics. The Lissajous figure is produced by having the pendulums swing back and forth in straight lines perpendicular to one another. The loops along the sides will give the ratio of the periods. By feeding a wave of unknown frequency and a wave of known frequency into an oscilloscope at right angles, a Lissajous figure will be produced on the screen. The number of loops will determine the frequency of the unknown wave. Waves of a desired frequency could be generated by the same method. Even the initial phase angle can be determined. The elliptical figures have found no application to date except for giving a visual account for dampened harmonic motion.

Included are several more examples of the work done by the harmonograph. Below each figure the pattern followed by the pendulums, the initial phase angle, and the ratio of their periods are given. All these figures are near unison, i.e., the 1:1 ratio. The figures here are alike in many ways, yet they differ considerably in appearance.





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On Least Absolute Values

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Least Squares is often used out of convenience. (For a better understanding of this statement, see [2]). Is it not reasonable to suppose that for certain problems, we would be better off with a solution based on least absolute values, least cubes, etc. It is true that it appears that such solutions are, in general, hard to find. It would appear that the "norm" we should use would depend on the problem at hand.

The following theorem illustrates some of the above thoughts. Note especially the wide range of points (x_2, y_2) can be without changing the fit.

THEOREM: Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three points in the real x, y-plane such that $x_1 < x_2 < x_3$. Suppose it is desired to find a and b such that $y = a + bx$ is the best fit to these three points in the sense of least absolute values; i.e., it is desired to find a and b such that

$$(1) \qquad I\,(a, b) = \sum_{i=1}^{3} |\, y_i - a - bx_i \,|$$

is a minimum. Let (a_0, b_0) be the solution of the equations

$$(2) \qquad \begin{aligned} a + bx_1 &= y_1 \\ a + bx_3 &= y_3 \end{aligned}$$

Then, I is a minimum when $I = I(a_0, b_0)$.

LEMMA: If $x_1 < x_2 < x_3$, then $|c + dx_1| + |c + dx_3| > |c + dx_2|$ where c and d are any two real numbers and at least c or d is nonzero.

Proof of Lemma: (i) $d > 0$. If $d > 0$ and c is any real number, then $x_1 < x_2 < x_3$ implies that $c + dx_1 < c + dx_2 < c + dx_3$. If $0 \leqq c + dx_1 < c + dx_2 < c + dx_3$, then $|c + dx_3| > |c + dx_2|$ and therefore $|c + dx_1| + |c + dx_3| > |c + dx_2|$. If $c + dx_1 < 0 \leqq c + dx_2 < c + dx_3$, then $|c + dx_3| > |c + dx_2|$ and therefore $|c + dx_1| + |c + dx_3| > |c + dx_2|$. If $c + dx_1 < c + dx_2 < 0 \leqq c + dx_3$, then $|c + dx_1| > |c + dx_2|$ and therefore $|c + dx_1| + |c + dx_3| > |c + dx_2|$. If $c + dx_1 < c + dx_2 < c + dx_3 < 0$, then

$|c + dx_1| > |c + dx_2|$ and therefore $|c + dx_1| + |c + dx_3| > |c + dx_2|$.

(ii) $d < 0$. If $d < 0$ and c is any real number, then $x_1 < x_2 < x_3$ implies that $c + dx_3 < c + dx_2 < c + dx_1$. If $0 \leqq c + dx_3 < c + dx_2 < c + dx_1$, then $|c + dx_1| > |c + dx_2|$ and therefore $|c + dx_1| + |c + dx_3| > |c + dx_2|$. If $c + dx_3 < 0 \leqq c + dx_2 < c + dx_1$, then $|c + dx_1| > |c + dx_2|$ and therefore $|c + dx_1| + |c + dx_3| > |c + dx_2|$. If $c + dx_3 < c + dx_2 < 0 \leqq c + dx_1$, then $|c + dx_3| > |c + dx_2|$ and therefore $|c + dx_1| + |c + dx_3| > |c + dx_2|$. If $c + dx_3 < c + dx_2 < c + dx_1 < 0$, then $|c + dx_3| > |c + dx_2|$ and therefore $|c + dx_1| + |c + dx_3| > |c + dx_2|$.

(iii) $d = 0$. For $d = 0$ and $c \neq 0$, $|c + dx_1| + |c + dx_3| > |c + dx_2|$ becomes $|c| > 0$ which is clearly true.

Proof of Theorem: Since a_0 and b_0 are solutions of the equations (2), the term $(y_1 - a_0 - b_0x_1)$ and the term $(y_3 - a_0 - b_0x_3)$ are identically zero. Let.

$$I\ (a_0, b_0) = |y_2 - a_0 - b_0x_2| = \varepsilon_1.$$

Assume that there exist some a^* and b^* such that

$$I\ (a^*, b^*) = \sum_{i=1}^{3} |y_i - a^* - b^*x_i| = \varepsilon_2 < \varepsilon_1.$$

Define $a' = a^* - a_0$ and $b' = b^* - b_0$. Now, at least a' or b' is nonzero for otherwise $\varepsilon_1 < \varepsilon_1$. Now,

$$\begin{aligned}
\varepsilon_2 &= \sum_{i=1}^{3} |y_i - a^* - b^*x_i| \\
&= |y_1 - (a' + a_0) - (b' + b_0)x_1| \\
&\qquad + |y_2 - (a' + a_0) - (b' + b_0)x_2| \\
&\qquad + |y_3 - (a' + a_0) - (b' + b_0)x_3| \\
&= |-a' - b'x_1| + |y_2 - a_0 - b_0x_2 - (a' + b'x_2)| \\
&\qquad + |-a' - b'x_3| \\
&\geqq |a' + b'x_1| + |a' + b'x_3| - |a' + b'x_2| + \varepsilon_1
\end{aligned}$$

From the above lemma, $|a' + b'x_1| + |a' + b'x_3| > |a' + b'x_2|$ and therefore it follows that $|a' + b'x_1| + |a' + b'x_3| - |a' + b'x_2| = p$ where p is some positive real number. Thus,

$\varepsilon_2 \geqq p + \varepsilon_1 > \varepsilon_1$. We have a contradiction. Thus, $y = a_0 + b_0 x$ is the best fit in the sense of least absolute values.

Any extension to more points is likely to be very difficult to prove. It is interesting to observe that the choice of the best fit line is independent of y_2. We also observe that it is independent of x_2 as long as x_2 is such that $x_1 < x_2 < x_3$. In certain problems, it might be very desirable to find a fit not subject to some of the ordinates.

Some interesting questions arise. When fitting a line to four points in the sense of least absolute values, would the location of all points be important? What about when five points are used? Would one of them be "free" in the sense that the (x_2, y_2) point in the above theorem was "free"? What about the general case?

The proof of this theorem for the three point case helps to illustrate the fact that working with other "norms" can lead to very difficult problems. Buck [1, page 299] presents a problem concerning least distance. He discusses the solution to the problem and presents the necessary formulas to find a line L which "fits" a set of N points best in the sense that it minimizes $\sum_{j=1}^{N} d_j^2$ where d_j is the distance from the point P_j to the line L. This problem also illustrates that one might, at times, prefer least distance to least squares even considering the fact that it is more difficult to handle.

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Mathematicians are like Frenchmen: whatever you say to them they translate into their own language, and forthwith it is something entirely different. —Goethe

Finite Differences and the Summation of Series*

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The summation of series is a field of mathematics whose basic notions arose through the study of numbers and the attributes of numbers arranged in various patterns. The subject matter of summations is divided into two fields of study: the summation of infinite series and the summation of finite series. Infinite series are relatively familiar to most students of mathematics since the techniques for summing infinite series often utilize calculus. Unfortunately, the sum of a finite series presents a problem which is not adaptable to the methods of integral or differential calculus. For example, without using calculus, find the sum of the first n terms of the finite series whose first four terms are

$$3,\ 9,\ 2,\ 16\ .$$

The method of finite differences offers a relatively simple general approach to the summation of any finite series, such as the series introduced above.

Let a finite series be defined as the sum of a set of numbers determined by some rational integral function of the positive integer n. If each number is represented by u_x, for $x = 1, 2, \cdots, n$, then the series may be written as

$$S = u_1 + u_2 + u_3 + \cdots + u_n \qquad (1)$$

or, in summation notation,

$$S = \sum_{x=1}^{n} u_x .$$

Note that the given series could have been any set of rational numbers. The method is general, but for computational purposes a set of positive integers was selected as an illustrative series.

Finding the general term is the first step in the summation of a finite series. By induction, the general (or xth) term of any

*A paper presented to the 1967 national convention of KME.

finite series may be expressed as

$$u_x = u_1 + \frac{(x-1)}{1\,!}\Delta u_1 + \frac{(x-1)^{(2)}}{2\,!}\Delta^2 u_1 + \cdots + \Delta^{r-1} u_1 \,. \qquad (2)$$

The symbol Δ^r is read as "the rth difference of." A difference is defined by the relation

$$\Delta^r u_k = \Delta^{r-1} u_{k+1} - \Delta^{r-1} u_k \,.$$

From (2), some of the differences may be calculated:

$$\begin{aligned} \Delta u_1 &= u_2 - u_1 \qquad\qquad (\Delta^0 = 1) \\ \Delta^2 u_1 &= \Delta u_2 - \Delta u_1 \\ \Delta^3 u_1 &= \Delta^2 u_2 - \Delta^2 u_1 \,. \end{aligned}$$

The general term is expressed in terms of the first term and its successive differences. The symbol $y^{(n)}$ is read "y to the n factorial." A factorial is defined thus:

$$y^{(n)} = \underbrace{y(y-1)(y-2)\cdots}_{n \text{ factors}}$$

For example, from (2), $(x-1)^{(2)} = (x-1)(x-2)$.

The formula for the general term is most conveniently derived from a table of differences. The table is constructed by separating each term of the series denoted by S, and then subtracting each term from its successor, as shown below.

$$\begin{array}{ccccccccccccc} u_1 & & u_2 & & u_3 & & u_4 & & \cdots & & u_n & & u_{n+1} \\ & \Delta u_1 & & \Delta u_2 & & \Delta u_3 & & & \cdots & & & \Delta u_n & \\ & & \Delta^2 u_1 & & \Delta^2 u_2 & & & & & & \Delta^2 u_{n-1} & & \\ & & & \Delta^3 u_1 & & & & & \cdots & & & & \end{array}$$

The formula for the general term may be read directly from the table of differences by using the first diagonal on the left side of the table. For example, to find the general term of the given series, construct the table as follows:

$$\begin{array}{ccccccccc} 3 & & 9 & & 2 & & 16 & & 85 \\ & 6 & & -7 & & 14 & & 69 & \\ & & -13 & & 21 & & 55 & & \\ & & & 34 & & 34 & & & \\ & & & & 0 & & & & \end{array}$$

Then the general term is read from the first diagonal as:

$$u_x = 3 + 6(x - 1) - \frac{13(x - 1)^{(2)}}{2\,!} + \frac{34(x - 1)^{(3)}}{3\,!}.$$

There are some tacit assumptions inherent in this problem, and in any finite summation problem. First of all, we are assuming that the four given terms determine a unique series, so that the fourth differences are all 0. This implies that all higher differences are also 0. (See the dotted line portion of the table of differences). If the fifth number is projected to be anything but 85, the series is different from the one whose general term was just found. Then the general term is unique for this series determined by the four given terms. The scope of this paper is limited, but for a proof of the above statements for the general finite series, consult Boole's *Calculus of Finite Differences.* [1]

If a relationship may be established between the general term of a series and the sum of the series, we draw a step closer to the final goal of the summation of the finite series. In a manner analogous to the method of integral calculus, the finite integral, v_x, is defined as the function whose first difference is u_x, which says that

$$\Delta v_x = u_x.$$

In a more conventional manner, let the finite integral be defined as the inverse of the first finite difference, so

$$\Delta^{-1} u_x = v_x + C,$$

where C is an arbitrary constant of integration. A definite integral with limits a and b may be evaluated according to the Fundamental Theorem of Finite Integration (analogous to the Fundamental Theorem of Integral Calculus):

$$\Delta^{-1} u_x \Big|_a^b = v_b - v_a ,$$

in which the constant C is evaluated according to the limits.

If the summation problem can be reduced to the problem of evaluating a finite integral, the work is reduced considerably: instead of the addition of n terms, the summation may be expressed as a single subtraction. It is necessary to prove that the summation of any series may be expressed as a finite integral.

Let v_x be a function whose first difference is u_x, so

$$\Delta v_x = u_x \ .$$

Then, from the definition of a first difference:

$$\begin{aligned}
\Delta v_1 &= v_2 - v_1 = u_1 \\
\Delta v_2 &= v_3 - v_2 = u_2 \\
\Delta v_3 &= v_4 - v_3 = u_3 \\
&\cdots \\
\Delta v_{n-1} &= v_n - v_{n-1} = u_{n-1} \\
\Delta v_n &= v_{n+1} - v_n = u_n
\end{aligned}$$

If the middle terms are added, the equality of the sum of the right side of the expression is unchanged, so

$$v_{n+1} - v_1 = u_1 + u_2 + u_3 + \cdots + u_n \, .$$

Then, the right side is just the earlier definition of S, and the left side may be written in limit notation, so

$$v_x \Big|_1^{n+1} = S \, .$$

But u_x is the first difference of v_x, which implies that

$$v_x = \Delta^{-1} u_x + C \, .$$

Then if we substitute for v_x its equivalent, the summation of a series becomes

$$S = \Delta^{-1} u_x \Big|_1^{n+1} \quad . \qquad (3)$$

Thus, to sum any series, first determine the general term. Then evaluate the definite integral of the general term between the limits 1 and $n + 1$.

Note that any "polynomial" in finite calculus must be expressed in terms of factorials. The factorial in finite calculus plays the same role as does the power in integral calculus. For example, the student of mathematics knows that, in general,

$$\int x^n dx = \frac{x^{n+1}}{n + 1} + C \quad .$$

If we set up a mapping between integral calculus and finite calculus, the finite integral would be

$$\Delta^{-1}x^{(n)} = \frac{x^{(n+1)}}{n+1} + C\ .$$

Then, to find the sum of a finite series, the general term is expressed in terms of factorials (as in (2)), then the finite integral is set up and evaluated. In terms of the specific example, the sum is

$$S = \Delta^{-1}u_x = \Delta^{-1}\left(3 + 6(x-1) - \frac{13(x-1)^{(2)}}{2\,!} + \frac{34(x-1)^{(3)}}{3\,!}\right)\Bigg|_{1}^{n+1}$$

Then the sum is

$$S = 3n^{(1)} + 3n^{(2)} - \frac{13}{3\,!}\,n^{(3)} + \frac{34}{4\,!}\,n^{(4)}\ .$$

Now the sum of the first n terms of the finite series has been determined. However, there still remains one problem: the sum is in factorial form, which is not convenient for calculations from a given n. Then we wish to transform this factorial form to a more familiar polynomial form.

The transformation from factorials to polynomials could be effected by multiplying the factors and grouping like terms. However, imagine the work involved in finding $x^{(10)}$, which is not an unusual factorial to encounter in sums. Then, instead of using multiplication, define a new function which will effect the transformation directly. That is, let

$$x^{(n)} = S_1^n x^1 + S_2^n x^2 + S_3^n x^3 + \cdots + S_n^n x^n\ ,$$

where S_i^n are constants known as Stirling Numbers. A recursion formula is readily developed (see Richardson's text) which generates a table of Stirling Numbers, part of which is shown below. For example, $n^{(4)} = -6n + 11n^2 - 6n^3 + n^4$. Each term of the sum may be evaluated in similar manner to yield:

$$S = 3n + 3(n^2 - n) - \frac{13}{3\,!}(n^3 - 3n^2 + 2n) + \frac{34}{4\,!}(n^4 - 6n^3 + 11n^2 - 6n)$$

so that

$$S = \frac{1}{12}(17n^4 - 128n^3 + 301n^2 - 154n).$$

S_1^n / n	S	S	S	S
1	1			
2	−1	1		
3	2	3	1	
4	−6	11	−6	1

Now that the sum of the series is in polynomial form, exactly what does it mean? If, for example, we evaluate the expression for $n = 4$, we find that the sum of the first four terms is 30. If we evaluate the expression for $n = 20$, we find the sum of the first twenty terms. This particular problem has illustrated the goal of the numerous statements and equations found in the earlier sections: the summation of a finite series.

For a long time finite series have been important to men such as actuaries, who must determine a trend for a short period of time. It seems that now, in the era of rapid change, finite series will demonstrate once again the ever-growing role of mathematics in industry and business.

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Installation of New Chapters

EDITED BY SISTER HELEN SULLIVAN

ILLINOIS ZETA CHAPTER

Rosary College, River Falls, Illinois

On Sunday, February 26, 1967, the Illinois Zeta Chapter was installed by Dr. Jerome Sachs, president of Illinois Teachers' College, Chicago North. Dr. Sachs gave a short talk on "Euclid, Mersenne, Perfect Numbers and the Binary Notation." After the ceremony a tea was held for the initiates, their parents, and guests.

The following are charter members: Carol Anderson, Elaine Bardick, Elizabeth Brennan, Thomasyne Campbell, Joanne Capito, Karen Charvat, Victoria Davis, Verona Fischer, Joan Gengler, Mary Jo Guzzardo, Mary Pat Hawley, Marie Hill, Patricia Husson, Kathleen Hytry, Patricia Johnston, Katherine Kahler, Jean Karasch, Judy Kaiser, Carol Kenealy, Christine Krol, Alice Kuehne, Margaret McShane, Gayle Madonna, Janet Plaza, Patricia Pung, Gail Rihacek, Sister John Grace, Sister Susan Mary, Diane Shields, Celina Tannura, Joan Weiss. The faculty members are: Dr. John M. Mihaljan, Mrs. Richard Schooley, Sister M. Colum, Sister M. Raimonda, Sister M. Philip.

The officers of the chapter are:

President	Gayle Madonna
Vice-President	Jean Karasch
Secretary	Janet Plaza
Treasurer	Marie Hill
Faculty Sponsor	Sister M. Philip
Corresponding Secretary	Mrs. Patricia Schooley

SOUTH CAROLINA BETA CHAPTER

South Carolina State College, Orangeburg, South Carolina

South Carolina Beta Chapter was installed on May 6, 1967, by Alabama Beta Chapter of Florence State College. Dr. Elizabeth T. Wooldridge, corresponding secretary of Alabama Beta Chapter, was the installing officer, and Miss Pamela Sams and Miss Mary Virginia Darby assisted with the ritual. Miss Barbara Wright, Benjamin Fouts, and Ronald Williams also represented Alabama Beta at the installation. Mrs. Geraldyne P. Zimmerman of South Carolina State

College was the conductor. A luncheon was held in the Walnut Room of the College Dining Hall following the ceremony.

Charter members are: Gary E. Bell, John T. Bowen, Daniel M. Ferguson, Jr., Lutricia W. Gaillard, William Gilyard, III, Robert Gyles, Titus J. Hastie, Jr., Eugene Lomax, Binah R. Miller, Mary A. Nash, Alexander Nichols, Jr., Paul E. Perry, and Harold D. Thompson. The faculty members are: E. Melvin Adams, Randall R. Harris, Dr. George W. Hunter, Mrs. C. Allen Jones, Frank M. Staley, Jr., and Mrs. Geraldyne P. Zimmerman.

The chapter officers are:

President	Gary E. Bell
Vice-President	Lutricia W. Gaillard
Secretary	Binah R. Miller
Treasurer	Alexander Nichols, Jr.
Corresponding Secretary	Frank M. Staley, Jr.
Sponsor	Mrs. C. Allen Jones

TEXAS ZETA CHAPTER

Tarleton State College, Stephenville, Texas

Texas Zeta Chapter was installed on May 14, 1967, by Dr. David Cecil, sponsor of Texas Epsilon Chapter at North Texas State College, Denton. Following the ceremony, a banquet was held and Dr. Cecil spoke on "Frieze Groups."

Charter members are: John Ammons, Kenneth Carrol, Sam Daniel, Jr., William Daniel, Dennis Dillin, Floyd R. Hamiter, M. I. Knudson, Jr., Suzanne Marx, Carroll Wayne Pilgrim, Mary Jo Stewart, Paul W. Todd, and Marty Wrinkle.

The officers are:

President	Dennis Dillin
Vice-President	John Ammons
Secretary	Mary Jo Stewart
Treasurer	William Daniel
Corresponding Secretary	Suzanne Marx
Publicity Chairman	Paul W. Todd

CONNECTICUT ALPHA CHAPTER

Southern Connecticut State College, New Haven, Connecticut

On May 29, 1967, the Connecticut Alpha Chapter was installed by Professor J. D. Daugherty of Kutztown State College, Kutztown, Pennsylvania. A banquet was held in the Faculty Dining Hall for the twenty-three initiates and Professor and Mrs. Daugherty. The installation followed the dinner, and Professor Daugherty spoke on "The History of Kappa Mu Epsilon."

The charter members include five faculty members and eighteen students: Chester F. Bass, Helen Bass, Francis J. Degnan, Henry P. Gates, J. Philip Smith, Lynne A. Alexander, Robert G. Bishop, Jr., Joyce A. Cromie, Marjorie A. Ewer, Clayton R. Hall, Arthur H. Hourwitz, Michele A. Joyce, Rae E. Lawson, Richard A. Loris, Karen M. McDermott, Ronald P. Mileski, Joan A. Moeckel, Rita C. O'Brien, Janice E. Olcsvary, Rita Renee Parks, Patricia B. Parsons, Nicholas J. Rinaldi, and Dominic L. Santossio.

The officers are:

President	Janice Olcsvary
Vice-President	Dominic Santossio
Secretary	Karen McDermott
Treasurer	Rae E. Lawson
Corresponding Secretary	Loretta K. Smith
Faculty Sponsor	Chester F. Bass

Mrs. Smith is a member of Virginia Beta Chapter.

(continued from page 7)

rules for derivation of theorems. If we let it run forever, proving theorems by these rules, there would always be some theorems which a man, by his ability to think and reason, could prove which a machine could not. There will never be a replacement for the mind of man.

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Kleene, S. C., *Introduction to Metamathematics*. New York: D. Van Nostrand Company, Inc., 1952.

The Problem Corner

EDITED BY H. HOWARD FRISINGER

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before March 1, 1968. The best solutions submitted by studnts will be published in the Spring 1968 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor H. Howard Frisinger, Department of Mathematics and Statistics, Colorado State University, Fort Collins, Colorado 80521.

PROPOSED PROBLEMS

206. *Proposed by Raymond Huck, Marietta College, Marietta, Ohio.*

Show that $\tan^2 18° + \tan^2 36° + \tan^2 54° + \tan^2 72° = 12$.

207. *Proposed by Charles W. Trugg, San Diego, California.*

There is only one three-digit number which is equal to twice the sum of the squares of its digits. Find this number.

208. *Proposed by Steven R. Conrad, Flushing, New York.*

Find all values of x for which the expression $4^x + 4^8 + 4^{11}$ is a perfect square.

209. *Proposed by Thomas F. Cleary III, State University of New York at Albany, Albany, New York.*

Given: An arbitrary triangle $\triangle ABC$ and an arbitrary point P in the interior of the triangle.

Prove: The sum of the lengths of the perpendiculars from point P to each of the sides of $\triangle ABC$ equals the length of an altitude of $\triangle ABC$.

210. *Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.*

The Fibonacci sequence $\{F_n\}$ is defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geqq 2$.

Similarly, the Lucas sequence $\{L_n\}$ is defined by $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$ for $n \geqq 3$.

Then for a given positive integer k, find

$$\lim_{n \to \infty} F_{n+k} / L_n$$

SOLUTIONS

201. *Proposed by William Mikesell, Indiana University of Pennsylvania, Indiana, Pennsylvania.*

Prove the following statement: In the set of regular polygons only three, the triangle, square, and the hexagon are such that they can fit together exactly without any gap or overlap.

Solution by Myron J. Fouratt, Montclair State College, Upper Montclair, New Jersey.

Suppose P is a regular polygon of k sides with vertex angle a. By drawing a line from each vertex to the center of the polygon, we construct k isosceles triangles in P. (Each side of P acts as a base of a triangle). Since there are 180° in each triangle, the

$$\text{Sum of all angles in } P = 180k.$$

Of this, a certain amount occurs at the vertices but the amount at the center is a constant 360°. Thus, the

$$\begin{aligned}\text{Sum of the angles at the vertices of } P &= 180k - 360\\ &= 180(k-2).\end{aligned}$$

Since there are k vertices in P, each vertex angle is

$$a = \frac{180(k-2)}{k}. \qquad (1)$$

Since we do not know how many polygons P are needed to fit together about a common point, let n denote the requisite number. Now, each of these n polygons contributes its vertex angle to the common point and, in order that there are no gaps or overlaps, the number of vertices times the size of each vertex angle must equal 360°:

$$na = 360° \text{ or } a = \frac{360°}{n}. \qquad (2)$$

Equating (1) and (2), and simplifying,

$$n = \frac{2k}{k-2}.$$

Adding and subtracting 4 from the numerator, factoring the numerator, and then dividing, we obtain

$$n = 2 + \frac{4}{k - 2} .$$

We recall, at this point, the restriction on n:

a) n must be a positive integer, and

b) n must be greater than or equal to 3.

Since k must also be a positive integer greater than or equal to 3, we try $k = 3$ in the last equation and obtain $n = 6$, a valid solution. If $k = 4$, we have $n = 4$, another valid solution. However, if $k = 5$, $n = 3 + 1/3$, which fails to satisfy the first condition. Thus, the regular pentagon will not work. If $k = 6$, $n = 3$, and a third valid solution is possible. If $k = 7$, $n = 2 + 4/5$, which fails to satisfy both conditions. In fact, all numbers for k greater than 6 afford no other valid solutions since the corresponding n will be a fraction between 2 and 3.

In the set of regular polygons, then, the only ones that fit together exactly without gap or overlap are the equilateral triangle ($k = 3$), the square ($k = 4$), and the regular hexagon ($k = 6$).

Also solved by John C. Kieffer, University of Missouri at Rolla, Rolla, Missouri; Layne Watson, University of Evansville, Evansville, Indiana.

202. *Proposed by R. S. Luthar, Colby College, Waterville, Maine.*

Show that there are infinitely many primes of the form:

$$x^3 + y^3 + z^3 + u^3 + t^3.$$

Solution by Layne Watson, University of Evansville, Evansville, Indiana.

$$\begin{aligned} \text{Let } x &= k + 1 \\ y &= k - 1 \\ z &= -k \\ u &= -k \\ t &= 1 \end{aligned}$$

Then $x^3 + y^3 + z^3 + u^3 + t^3 = 6k - 1$, and there are infinitely many primes of the form $6k - 1$.

203. *Proposed by Layne Watson, University of Evansville, Evansville, Indiana.*

Prove that the sum of N vectors of equal length radiating from a point P is zero, where the angle between a vector and the preceding one is $\frac{2\pi}{N}$. Use this result to prove that $\sum_{n=0}^{N-1} \cos \frac{2\pi n}{N} = 0$ and $\sum_{n=0}^{N-1} \sin \frac{2\pi n}{N} = 0$, where N is an integer > 1.

Solution by Calvert A. Jared II, Butler University, Indianapolis, Indiana.

We will look at two cases.

Case I. Let $N = 2$. Then the angle between the two vectors is π (they both lie on the same line). So if $\mathbf{w}_1$ and $\mathbf{w}_2$ are the two vectors, $\mathbf{w}_2$ can be expressed as $-\mathbf{w}_1$. Hence

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 - \mathbf{w}_1 = \mathbf{0}.$$

Case II. Let $N > 2$. Pick any of the vectors as a stationary one. Then if we take the vector immediately following it (as we go from right to left), and place its origin at the endpoint of the stationary one, the angle between the two vectors will be $(\pi - 2\pi/N)$. If we do this for all N vectors, i.e., keep putting the next vector at the endpoint of the preceding one, we will have a figure of N equal sides and all angles having size $(\pi - 2\pi/N)$. Hence the figure will be a regular polygon of N sides. Therefore it will be closed; the last vector will have its endpoint at P. Hence the vector sum will be zero.

Let $\sum_{n=0}^{N-1} \mathbf{w}_n = \mathbf{0}$ denote the sum of the N vectors. Since each vector is of the same length we can express $\mathbf{w}_n$ as $w\boldsymbol{\alpha}_n$ where $\boldsymbol{\alpha}_n$ is a unit vector in the direction of $\mathbf{w}_n$.

So $\sum_{n=0}^{N-1} \mathbf{w}_n = \sum_{n=0}^{N-1} w\boldsymbol{\alpha}_n = w \sum_{n=0}^{N-1} \boldsymbol{\alpha}_n$. But each vector $\boldsymbol{\alpha}_n$ can be expressed by a component in the x direction $(x_n\mathbf{i})$ and a component in the y direction $(y_n\mathbf{j})$. Hence

$$\sum_{n=0}^{N-1}\mathbf{w}_n = w\sum_{n=0}^{N-1}(x_n\mathbf{i} + y_n\mathbf{j})$$
$$= w\sum_{n=0}^{N-1}x_n\mathbf{i} + w\sum_{n=0}^{N-1}y_n\mathbf{j}\,.$$

But $x_n = r\cos\theta_n$

$y_n = r\sin\theta_n$ where $r = 1$.

So $x_n = \cos\theta_n$

$y_n = \sin\theta_n$

But if $n = 0$ $\theta_0 = 0$

$n = 1$ $\theta_1 = 2\pi/N$

$n = 2$ $\theta_2 = 2\pi(2)/N$

$\vdots$

$n = n$ $\theta_n = 2\pi n/N$.

So $x_n = \cos\dfrac{2\pi n}{N}$

$y_n = \sin\dfrac{2\pi n}{N}$.

So $w\sum_{n=0}^{N-1}\cos\dfrac{2\pi n}{N}\mathbf{i} + w\sum_{n=0}^{N-1}\sin\dfrac{2\pi n}{N}\mathbf{j} = \mathbf{0}$.

Hence $w\sum_{n=0}^{N-1}\cos\dfrac{2\pi n}{N}\mathbf{i} = \mathbf{0}$ and $w\sum_{n=0}^{N-1}\sin\dfrac{2\pi n}{N}\mathbf{j} = \mathbf{0}$ or

$\left(w\sum_{n=0}^{N-1}\cos\dfrac{2\pi n}{N}\right)\mathbf{i} = \mathbf{0}$ and $\left(w\sum_{n=0}^{N-1}\sin\dfrac{2\pi n}{N}\right)\mathbf{j} = \mathbf{0}$.

But w is non-zero and $\mathbf{i}$ and $\mathbf{j}$ are both unit vectors by definition, so

$$\sum_{n=0}^{N-1}\cos\frac{2\pi n}{N} = 0 \text{ and } \sum_{n=0}^{N-1}\sin\frac{2\pi n}{N} = 0\,.$$

204. *Proposed by the Editor.*

Show that among any ten consecutive positive integers, at most five can be primes, and that five actually occur in only one case. Must at least one of any ten consecutive positive integers be prime?

Solution by Thomas F. Cleary III, State University of New York at Albany, Albany, New York.

Among any ten consecutive positive integers five are odd and five are even. With the exception of the integer 2, which is both even and a prime, the five even integers cannot be prime since they are divisible by 2. The five odd integers can all be prime, thus there can be at most five primes among the ten consecutive integers.

Consider the exception described above where the integer 2 is among the ten consecutive positive integers. There are two possibilities:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10 or 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

The first contains the four primes 2, 3, 5, and 7. The second contains the *five* primes 2, 3, 5, 7, and 11. To show that this second possibility is the only one that contains five primes, consider *all* remaining sets of consecutive positive integers whose first integer is greater than 2. From before, every such set contains a subset of five consecutive odd integers all of which may be prime. Denote the first three of these integers A, $A + 2$, and $A + 4$ where A is odd. If A is divisible by 3 then A is composite, therefore there can be at most *four* primes among the five odd integers. This proves the uniqueness of the above set of five primes. However, if A is not divisible by 3, then either $A = 3k + 1$ or $A = 3k + 2$, where k is a positive integer.

If $A = 3k + 1$ then $A + 2$ is divisible by 3. If $A = 3k + 2$ then $A + 4$ is divisible by 3. Thus in any case one of the three odd integers A, $A + 2$, and $A + 4$ is divisible by 3. Therefore among the ten consecutive integers at most *four* are prime. This proves the uniqueness of the above set of five primes.

It is possible that among the ten consecutive positive integers there are no primes as is the case in the following examples:
114, 115, $\cdots$, 123 or 200, 201, $\cdots$, 209 or 212, 213, $\cdots$, 221

Also solved by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio; Calvert A. Jared II, Butler University, Indianapolis, Indiana; John C. Kieffer, University of Missouri at Rolla, Rolla, Missouri; James Lander, Illinois State University, Normal, Illinois; R. S. Luthar, Colby College, Waterville, Maine; Layne Watson, University of Evansville, Evansville, Indiana.

205. *Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.*

The Fibonacci sequence $\{F_n\}$ is defined as

$$F_0 = 0,\ F_1 = 1,\ F_k = F_{k-1} + F_{k-2} \text{ for } k \geq 2.$$

Now let $f(n)$ represent the continued fraction

$$f(n) = F_0 + \cfrac{1}{F_1 + \cfrac{1}{F_2 + \cfrac{1}{F_3 + \cfrac{1}{F_4 + \ddots + \cfrac{1}{F_n}}}}}$$

and let $g(n)$ represent the continued fraction

$$g(n) = F_0 + \cfrac{F_1}{F_2 + \cfrac{F_3}{F_4 + \cfrac{F_5}{F_6 + \ddots + \cfrac{F_{n-1}}{F_n}}}}$$

Determine, whether possible or not, and if so, an exact value for $\lim\limits_{n \to \infty} f(n)$ and $\lim\limits_{n \to \infty} g(n)$.

Solution by Layne Watson, University of Evansville, Evansville, Indiana.

The continued fractions for $f(n)$ and $g(n)$ are infinite, so they are not rational numbers. Further, since the continued fractions are not periodic, $f(n)$ and $g(n)$ are not quadratic surds. Because of the nature of continued fractions, $f(n)$ is greater than each odd convergent and less than each even convergent, and each convergent lies between the preceding two. Each convergent is a rational number, and $f(n)$ is the least upper bound of the (infinite) set of odd convergents. Therefore $f(n)$ is a real number. Similarly, since $g(n)$ is the least upper bound of an infinite set of rationals (its odd convergents), it is a real number. $f(n)$ and $g(n)$ can be thought of as infinite nonrepeating decimals, and therefore no exact values can be given. Numerically,

$$.5888739525489335 < f(n) < .5888739525489339$$

and

$$.6416600010214519 < g(n) < .6416600010214521\ .$$

(continued from page 19)

that $xa = a_2$. Thus $z_2 = a_2 - a_2j = xa - xaj = x(a - aj) \varepsilon I^*$ and $I_2 \subseteq I^*$.

Hence $I^* = I_1$.

Therefore I_1 and I_2 are the only proper ideals in K.

Corollary. Every ideal in K is principal.

It can also be shown that K is isomorphic to a particular set of 2×2 matrices under the correspondence

$$a + bj \longleftrightarrow \begin{bmatrix} a & b \\ 0 & a + b \end{bmatrix} .$$

The Book Shelf

EDITED BY JOHN C. BIDDLE

This department of *The Pentagon* brings to the attention of its readers published books (both old and new) which are of a common nature to all students of mathematics. Preference will be given to those books writen in English or to English translations. Books to be reviewed should be sent to Dr. John C. Biddle, Mathematics Department, Central Michigan University, Mt. Pleasant, Michigan 48858.

Modern Mathematics — An Elementary Approach, Ruric E. Wheeler, Brooks/Cole Publishing Co., Belmont, California, 1966. 438 pp., $8.95.

This text is one of many recent publications designed to provide the elementary education major with a sufficient mathematical background to teach the modern elementary arithmetic curriculum. The preface states that the text essentially fulfills CUPM recommendations. The preface also notes that the book grew out of training of elementary school teachers. The coverage of topics is adequate and the presentation is good; hence it can be safely recommended as an addition to the growing collection of similar adequate texts.

Specifically the text covers arithmetic and number systems through the real numbers in 268 pages. Geometry is allotted eighty-three pages and algebra (including complex numbers) is given only thirty-two pages. Within the arithmetic coverage, fifty pages is devoted to number theory. The text is clearly sufficient for a course in arithmetic. However, coverage is too light to provide sufficient background in geometry or algebra, particularly algebra, for the teaching of modern elementary programs.

A brief introductory chapter discusses mathematical "reasoning"; an equally brief development of sets follows in the second chapter. The development of equivalence relation and one-one correspondence is not adequate as a foundation for the rest of the material of the text. The third chapter discusses the concepts of natural number, order, cardinal number, ordinal number, infinite, whole number, and zero in the restricted space of less than two pages. The rest of the chapter adequately develops the ring properties of the whole numbers as well as the inequality laws.

In chapter four on numeration systems (covered in less than thirty pages of exposition) topics are handled too lightly. Four pages are used to cover the standard historical systems. Another four pages are allocated to our decimal place value system together with the properties of exponents. The rest of the chapter concerns other base systems.

The next three chapters cover integers, rational numbers, and real numbers. This development is good and uses sufficient exposition. Proofs of properties are adequate. The author weakens the text when he discusses (on page 131) the "likeness" between natural numbers and positive integers; he also mentions the idea of "embedding." However, he has defined the integers to be an extension of set of natural numbers, so there is indeed identity not isomorphism between the sets. The rationals are defined as usual as equivalence classes of ordered pairs of integers. Here only isomorphism can exist between the integers and the appropriate subset of rationals, yet on page 159, he says "the integers are 'imbedded in' or 'contained as a subset of' the set of rational numbers." Clearly, the careful reader will be confused. The author fails to mention the term "isomorphism," but stresses the idea of extension.

The treatment of real numbers is very good. A careful development, of course, is not proper for this type of text. A real number is defined to be an infinite decimal number. Addition is considered (informally) as the limit of the partial sum; multiplication is similarly defined. This work is preceded by the usual topics of repeating and non-repeating decimals.

The geometry chapters are good but brief. They include a good development of measure of a segment and an angle, but a weak treatment of area. A good beginning is made for congruence and similarity proportion of triangles. The treatment of algebra is quite inadequate and deals primarily with quadratic relations.

In summary, the text is a good one and can be handled by the elementary major. Most of the flaws mentioned above can be easily handled and overcome by the instructor. The level and tone of the text is moderate and reasonable. Problem sections are adequate and a complete answer key is provided as an appendix. Chapter summaries are provided and additional readings are suggested.

J. K. Bidwell
Central Michigan University

Mathematical Quickies, Charles W. Trigg, McGraw-Hill, New York, 1967, Cloth, 210 pages, $7.95.

To anyone familiar with the problem sections of *Mathematics*

Magazine, The Pentagon, School Science and Mathematics, the name Charles W. Trigg is a well-known one. Mr. Trigg, now Dean Emeritus of Los Angeles City College, was for forty years a teacher of science and mathematics. He has published over 600 articles and problem solutions, has a collection of over 16,000 intriguing problems, and from these he has selected 270 that he considers most stimulating and worthy of the label "Quickies". The reader is presented with a double challenge — to solve the problems and to devise more elegant solutions than those provided by Mr. Trigg and 102 of his fellow problemists (including such names as N. Anning, M. Beberman, M. Klamkin, L. Moser, C. Read). An elegant solution is defined as "one characterized by clarity, conciseness, logic, and surprise."

Some of Mr. Trigg's "Quickies" proved to be "Longies" for me (e.g. "Solve $x^3 + 1 = y^2$ as a Diophantine Cubic.") I did indeed find solutions of $(-1, 0)$ and $(0, \pm 1)$ quickly. $(2, \pm 3)$ took a bit longer, but then to prove there were no more — well, if it's a "Quicky" for you, congratulations! There are many problems elementary school children will enjoy, and some problems involving algebra, geometry, trigonometry, or number theory. Consider these intriguing problem names: "The End of the World," "The Dozing Student," "The Beauty Contest," "The Hula Hoop." For the purists, we find the Steiner-Lehman Theorem and the Fibonacci Series.

From a teacher's standpoint, when using the book as a source of problems to spice a particular unit, I would have preferred the problems to be grouped according to branches of mathematics. The heterogeneous mixing of types by the author is intentional, however, as he feels an important part of the challenge of any problem is to settle upon the particular branch of mathematics to be used. The teacher will therefore have to meet the challenge before he can challenge his students. If you're not always able to find the time for all the challenges a teacher is faced with these days, the detailed solutions of each problem provide quick and welcome help. I predict the reader of this book will have fun and also learn many helpful problem-solving techniques — whether his solutions be "Quickies" or "Longies."

DONALD F. MARSHALL
Harvard Graduate School of Education
Cambridge, Mass.

The Mathematical Scrapbook

EDITED BY GEORGE R. MACH

Readers are encouraged to submit Scrapbook material to the editor. Material will be used where possible and acknowledgement will be made in THE PENTAGON.

The Fall 1965 MATHEMATICAL SCRAPBOOK had a note about the reciprocals of primes as repeating decimals. Let's take another look at some of them.

When we divide any number by 13 there are just twelve possible non-zero remainders. As we saw before, $\frac{1}{13} = \overline{0.076923}$, where the bar indicates the repeating digits. Since this decimal repeats six digits, just six of the possible remainders appear (10, 9, 12, 3, 4, 1 in that order) and then they repeat if the division is continued.

It is interesting to note that $\frac{2}{13} = 0.\overline{153846}$ and that this division yields the other six possible remainders (7, 5, 11, 6, 8, 2 in that order) and then they repeat. Of course, zero is not a possible remainder and no remainder can belong to both sets. Why? Note that the sum of each set of remainders is 39. Why 39?

However, more interesting properties of these repeating decimals are yet to be seen. As expected, $\frac{2}{13} = 2\left(\frac{1}{13}\right) =$ $2(0.\overline{076923}) = 0.\overline{153846}$. The repeating decimals for all proper fractions with denominator 13 exhibit only these two cyclic sequences of digits, the different ones simply starting with a different digit. The patterns are easily seen when they are arranged as follows

$$\frac{1}{13} = \overline{.076923} \qquad \frac{2}{13} = \overline{.153846}$$

$$\frac{10}{13} = \overline{.769230} \qquad \frac{7}{13} = \overline{.538461}$$

$$\frac{9}{13} = \overline{.692307} \qquad \frac{5}{13} = \overline{.384615}$$

$\frac{12}{13} = .\overline{923076}$ $\frac{11}{13} = .\overline{846153}$

$\frac{3}{13} = .\overline{230769}$ $\frac{6}{13} = .\overline{461538}$

$\frac{4}{13} = .\overline{307692}$ $\frac{8}{13} = .\overline{615384}$

Can you find a pattern which determines or limits how the fractions are segregated into these two sets? What happens in the case of a reciprocal like $\frac{1}{7}$, where all six of the possible non-zero remainders appear in the division process? Well, $\frac{1}{7} = 0.\overline{142857}$ and $\frac{2}{7} = 0.\overline{285714}$ looks like an interesting start. What happens in the case of a reciprocal like $\frac{1}{41}$, where just four of the possible forty non-zero remainders (18, 16, 37, 1 in that order) appear and then they repeat if the division is continued? A few of the repeating decimals follow:

$\frac{1}{41} = .\overline{02439}$ $\frac{5}{41} = .\overline{12195}$ $\frac{9}{41} = .\overline{21951}$

$\frac{2}{41} = .\overline{04878}$ $\frac{6}{41} = .\overline{14634}$ $\frac{10}{41} = .\overline{24390}$

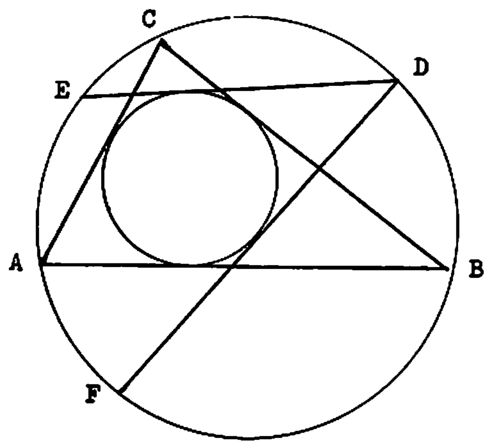
$\frac{3}{41} = .\overline{07317}$ $\frac{7}{41} = .\overline{17073}$ $\frac{11}{41} = .\overline{26829}$

$\frac{4}{41} = .\overline{09756}$ $\frac{8}{41} = .\overline{19512}$ $\frac{12}{41} = .\overline{29268}$

Note the patterns appearing already. Into how many cyclic sets do you expect all forty of them to fall? Is it important to note that there are five digits in the repeating portion of the decimals? Think of a good question yourself and then try to answer it.

=Δ=

Consider a triangle *ABC*, its circumscribed circle, and its inscribed circle. (The sides of the triangle are chords of the circumscribed circle and they are tangent to the inscribed circle.) Select any point D other than A, B, or C on the circumscribed circle and lay out from it two chords DE and DF of that circle, each tangent to the inscribed circle as indicated.

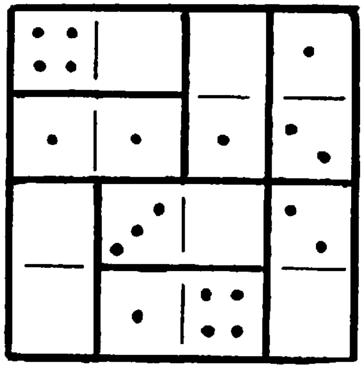


Is it possible that a third chord EF, which completes the triangle DEF, might also be tangent to the original inscribed circle? What happens when ABC is equilateral? Are there any other interesting special cases? Can any generalizations be made?

=Δ=

Magic squares and magic circles have had recent mention in the MATHEMATICAL SCRAPBOOK. Domino magic squares may be formed in much the same manner as magic squares. Using all members of an ordinary double six set of dominoes, a 7 x 7 square (a column of blank sides of the dominoes on one edge completes it) can be formed with each row, column, and diagonal summing to 24.

Using just a part of the dominoes, many even order squares can be formed. An example of a 4 x 4, whose sum is five, is given below.



If we use the complement of each number with respect to six, we get a square whose sum is nineteen. If we complement each number with respect to four, we get a square whose sum is eleven, etc. However, many essentially different squares of order four (as well as other orders) exist. Get out some dominoes and see what you can do.

=△=

The late Professor W. F. White reported a student's question and his answer regarding fourth dimension by analogy as follows:

> Q. If the path of a moving point (no dimension) is a line (one dimension), and the path of a moving line is a surface (two dimensions), and the path of a moving surface is a solid (three dimensions), why isn't the path of a moving solid a four-dimensional magnitude?
>
> A. If your hypotheses were correct, your conclusion should follow by analogy. The path of a moving point is, indeed, always a line. The path of a moving line is a surface *except* when the line moves in its own dimension, "slides in its trace." The path of a moving surface is a solid only when the motion is in a third dimension. The generation of a four-dimensional magnitude by the motion of a solid presupposes that the solid is to be moved in a fourth dimension.

Kappa Mu Epsilon News

EDITED BY EDDIE W. ROBINSON, HISTORIAN

The Sixteenth Biennial Convention of Kappa Mu Epsilon was held April 6, 7, and 8, 1967, with Kansas Gamma at Mount St. Scholastica College, Atchison, as host chapter.

THURSDAY, APRIL 6, 1967

Following the registration, a Relaxer was held in the Riccardi Center Game Room. Chapter news and ideas were exchanged, chapter songs were sung and all delegates to the convention got acquainted. The National Council met in the Library Seminar Room.

FRIDAY, APRIL 7, 1967

The meetings were held in the Administration Building Auditorium. National President, Fred W. Lott, of Iowa Alpha presided. President Gustane C. Zader of Mount St. Scholastica College gave the address of welcome and National Vice-President George R. Mach responded for the society. The roll call of the chapters was made by Laura Z. Greene, National Secretary. The following chapters, approved for membership since the last national convention, were welcomed:

Iowa Gamma, Morningside College, Sioux City,

Maryland Beta, Western Maryland College, Westminster,

New York Zeta, Colgate University, Hamilton,

Illinois Zeta, Rosary College, River Forest.

Petitions for new chapters at Grove City College, Grove City, Pennsylvania, South Carolina State College, Orangeburg, and Tarleton State College, Stephenville, Texas, were presented and approved.

Dr. George R. Mach presided during the presentation of the following papers:

1. *A Platonic Philosophy of Zero,* Joseph E. Hilber, Iowa Gamma, Morningside College.
2. *Finite Differences and the Summation of Series,* Joyce R. Curry, California Gamma, California State Polytechnic College.
3. *The Movable Figures,* Andrea Lee Meyer, Kansas Gamma, Mount St. Scholastica College.

4. *Communications Networks Using Matrices,* Judy Kaldenberg, Iowa Alpha, State College of Iowa.

After lunch in Riccardi Center and the taking of the group picture, the faculty members and students met separately in two "Let's Exchange Ideas" discussion sections.

The convention reconvened at 2:30 p.m. and after reports from the two sections, the following student papers were presented:

5. *Gödel's Incompleteness Theorem,* John W. Bridges, Missouri Alpha, Southwest Missouri State College.
6. *An Introduction to Topological Groups,* Jo Ingle, Kansas Gamma, Mount St. Scholastica College.
7. *Harmonic Vibration Figures,* Bradley J. Beitel, California Delta, California State Polytechnic College.
8. *Fibonacci Numbers,* Karen Johnson, Wisconsin Alpha, Mount Mary College.

A banquet was served in Riccardi Center with Sister Malachy Kennedy, Kansas Gamma, as Mistress of Ceremonies. Sister Helen Sullivan, Chairman, Department of Mathematics, Mount St. Scholastica College, gave the invocation. After some vocal selections from the Mount Ensemble, the guest speaker, Professor Seymour Schuster, University of Minnesota, spoke on "Mathematicians that Work on Films." He showed two films: "Geometry" and "Curves of Constant Width."

SATURDAY, APRIL 8, 1967

The program began at 8:30 a.m. with the following student papers:

9. *An Introduction to Geometric Models Based on Axiom Systems,* Leora Ernst, Kansas Gamma, Mount St. Scholastica College.
10. *Symbolic Logic and its Relation to Computers,* Donald Marks, Michigan Beta, Central Michigan University.
11. *A Method Beyond the Taylor Series for Computing Values of an Integral for Large X,* Martha Robinette, Missouri Alpha, Southwest Missouri State College.
12. *Significant Theories and Subsequent Observations on the Subject of Twin Primes,* Elizabeth Murphy, Kansas Gamma, Mount St. Scholastica College.

The following papers were listed by title:

1. *The Axiom of Choice,* Lawrence D. Tomlin, Kansas Beta, Kansas State Teachers College.
2. *On the Nature of Mathematics,* Sheila R. Predmore, New York Beta, State University of New York at Albany.
3. *An Introduction to Fibonacci Numbers,* Bernita Meyer, Kansas Gamma, Mount St. Scholastica College.

Professor Charles B. Tucker, Kansas Beta reported for the nominating committee. The following list of national officers was elected for 1967-1969.

President	Dr. Fred W. Lott, Jr. University of Northern Iowa
Vice-President	Dr. George R. Mach California State Polytechnic College
Secretary	Professor Laura Greene Washburn University of Topeka
Treasurer	Professor Walter C. Butler Colorado State University
Historian	Professor Eddie W. Robinson Southwest Missouri State College

Professor Ronald G. Smith, Kansas Alpha, Chairman of the Awards Committee made the following awards to the students listed below for papers presented during the convention.

First Place	John W. Bridges, Missouri Alpha
Second Place	Leora Ernst, Kansas Gamma
Third Place	Bradley J. Beitel, California Delta

Professor D. V. LaFrenz, Missouri Gamma, reported for the resolution committee. The following resolution was adopted:

Whereas the sixteenth biennial convention on this beautiful college campus has been a very enjoyable and profitable conference, be it resolved that we express our appreciation to:

1. The host chapter, Kansas Gamma, and its moderator Sister Helen Sullivan, to Mount St. Scholastica College of Atchison, Kansas, for their hospitality, and efficient organization of all major and minor details that contributed so well to the success of the convention, and to the Mount Ensemble for their musical entertainment.

2. Each national officer: President Professor Lott, Vice-President Professor Mach, Secretary Miss Laura Greene, Treasurer Professor Butler, and Historian Professor Haggard, and all their efficient assistance.

3. Professor Kriegsman, the editor of THE PENTAGON, and to Professor Waggoner, the business manager of THE PENTAGON, who have so satisfactorily maintained the quality of our magazine.

4. Professor Seymour Schuster, who provided the educational inspirational program at the convention banquet.

5. The fifteen students who have prepared and presented the excellent papers which formed an integral part of the convention program.

REPORT OF THE NATIONAL PRESIDENT

These are days of rapid change in colleges and universities which is reflected in the continued growth of Kappa Mu Epsilon. There were 2,836 new members initiated this biennium as compared with 2,364 in the previous biennium, an increase of twenty per cent. During the two years of this biennium we have installed eight new chapters. These are: Pennsylvania Beta at Indiana University of Pennsylvania; Arkansas Alpha, Arkansas State College; Tennessee Gamma, Union University; Wisconsin Beta, Wisconsin State University at River Falls; Illinois Zeta at Rosary College; Iowa Gamma, Morningside College; Maryland Beta, Western Maryland College; and New York Zeta at Colgate University. A new chapter was recently approved for Southern Connecticut State College by vote of the Chapters and arrangements are now being made for its installation. By your vote yesterday three more chapters will soon be added. We now have seventy-six active chapters and the four additional chapters to be installed in the near future will make a total of eighty.

There were two regional conventions held last year. Mount Mary College in Wisconsin was the host chapter for the North Central Region. There were four chapters represented with sixty KME members attending. The Middle West Regional Convention was hosted by Southwest Missouri State College with an attendance of 101 from fifteen chapters. We should continue to support these regional conventions in the even-numbered years and encourage other regions to plan such meetings. For this purpose the national organization provides up to $100 to help defray the expenses of the host chapter of each regional conference.

The action of this convention in strengthening the membership requirements and authorizing the National Council to seek membership in the Association of College Honor Societies is an important step forward and will have significant implications for the future development of Kappa Mu Epsilon.

As you noted from the Treasurer's Report, our operations for the last biennium resulted in a small deficit for the first time. Part of this is due to the greater distances for most chapters to the previous national convention resulting in somewhat larger travel allowances. Another reason of course is the increase in general costs of operation in the last fourteen years since the present dues were established in 1953. Perhaps an increase in initiation fees is inevitable sometime in the future; however, after careful consideration, your National Council has decided against making such a recommendation at this time. We are a non-profit organization and there is no reason to make substantial increases in the treasury each biennium. This situation will be under continual study to make certain that the assets of Kappa Mu Epsilon remain at a level that will assure sound financing.

The method of basing travel allowances of convention delegates on railroad costs has become quite awkward. In recent years it has become more difficult to travel by train or even to get a quotation of what rail travel would cost. Last night your council approved a change to a simple five cents per mile allowance. Since this requires a change in the constitution, this matter will be submitted to the chapters for their vote during the fall term.

One of the heartwarming aspects of being a part of Kappa Mu Epsilon is the way people respond to help with the on-going program of the organization. A society such as Kappa Mu Epsilon could not exist without the voluntary efforts of many people. The national officers whose reports you have just heard spend many long hours carrying out their responsibilities. We are indebted to the Editor and Business Manager of THE PENTAGON for the excellent journal they produce. This magazine is among the most important projects of Kappa Mu Epsilon. These last two days one is immediately aware of gracious services of our host chapter in planning and providing for the needs of this convention. Then there are those who serve on committees and other convention responsibilities, those who have prepared and presented papers, those who have made the regional conventions possible, the student officers, faculty sponsors and corresponding secretaries at local chapters throughout the country. The list is extensive. To all those persons who have had a part in carrying forward the work of Kappa Mu Epsilon I would like to express both my personal appreciation and that of the entire organization.

FRED W. LOTT, JR.

REPORT OF THE NATIONAL VICE PRESIDENT

On April 23, 1966, the National Council appointed me vice-president to fill an existing vacancy. I appreciated the confidence placed in me by the National Council. Since then, I have conferred personally with the National President and the National Secretary and by mail with the other officers. I have enjoyed working with the National Council under President Lott's direction.

My major effort this past year has been toward arranging and conducting the student presentations for this convention. An announcement including instructions for submitting papers was sent to all chapters in October and a similar announcement appeared in the Fall 1966 issue of THE PENTAGON. I appointed and acted as chairman of the Student Paper Selection Committee, which included as members: Professor Raymond Carpenter, Oklahoma Alpha; Professor Z. T. Gallion, Mississippi Alpha; and Professor Margaret Martinson, Kansas Delta. Fifteen papers were submitted from ten chapters. Twelve were selected for presentation and three as alternates. All instructions and correspondence with the students was done by me.

Kappa Mu Epsilon is a student society and it exists to honor and serve its members. It is recommended that we seek to attain the widest possible representation at the biennial conventions so that the mathematical experiences of the delegates and their chapters may be enriched. It is also recommended that sincere encouragement be given to students to prepare papers for presentation at our conventions and publication in THE PENTAGON. Kappa Mu Epsilon offers to its student members these unique and valuable opportunities which they might not get in any other way.

GEORGE R. MACH

REPORT OF THE NATIONAL SECRETARY

Kappa Mu Epsilon has more than twenty-three thousand members with seventy-six chapters in twenty-seven states.

All routine correspondence with the national organization comes to the secretary's office. We process membership reports, orders for invitations, jewelry, and supplies. We maintain a file of the original membership cards, initiation reports and chapter by-laws.

I appreciate the cooperation of the corresponding secretaries of each chapter in carrying out the many details of the office.

LAURA Z. GREENE

REPORT OF THE EDITOR OF THE PENTAGON

During the past biennium there have been several changes in the editorial staff of THE PENTAGON. At the last convention Fred W. Lott requested that a new editor be appointed as he assumed other responsibilities in Kappa Mu Epsilon. At the same time Jerome Sachs asked to be relieved of his position, and George R. Mach, California State Polytechnic College, became the new associate editor in charge of the Mathematical Scrapbook. Beginning with the Spring, 1966, issue H. Howard Frisinger replaced F. Max Stein as Problem Corner editor — both of these men are from Colorado State University. Recently, James P. Burling, State University College of Oswego, has replaced Harold Tinnappel of Bowling Green State University, who served as Book Shelf Editor for the first three issues. J. D. Haggard, Kansas State College of Pittsburg, the National Historian, has continued to keep the fraternity informed of the activities of the various chapters by editing the Kappa Mu Epsilon News. The reports of Installation of Chapters have been handled by Sister Helen Sullivan of Mount St. Scholastica College. Each of these people has worked diligently, and I do wish to express appreciation to each of the former editors who gave valuable assistance to his successor.

Two other individuals who deserve much recognition for their untiring efforts and patience are the business manager, Wilbur J. Waggoner, and the manager of the University Press, Irwin Campbell, both of Central Michigan University. They have been especially helpful to the new editor through their valuable suggestions. We would also be remiss if we did not recognize the assistance of the sponsors, corresponding secretaries, and other faculty members of the chapters who have supplied information and encouraged students to submit papers and who have refereed papers.

There have been three issues of THE PENTAGON published since the last convention and the fourth issue is now in the process of being printed. In addition to the regularly published sections there have been twenty-one articles, including fifteen student papers and six written by faculty members and others.

THE PENTAGON, as the official journal of Kappa Mu Epsilon, attempts to reflect the objectives of the organization, and, therefore, stresses the role of the undergraduate student of mathematics. I would like to urge each of the students to become an active participant in this publication through writing papers, proposing problems, or submitting solutions to ones already posed, and I hope that each faculty member will encourage the students in their endeavors. This work will require time and effort on the part of both the student who prepares the manuscript and the faculty member who guides him, but I am sure that the rewards of publication will be gratifying. Of course, we are always ready to consider any article which is interesting and stimulating to the student so papers from faculty members and others are also welcome.

A few of you have made suggestions and comments relative to THE PENTAGON and these have been given the careful attention of the editor involved. We welcome these ideas and invite your criticisms, but above all contribute your articles, problems, solutions, and other appropriate items.

HELEN KRIEGSMAN

REPORT OF THE BUSINESS MANAGER OF THE PENTAGON

I feel somewhat like an elder citizen or patriarch in making this report as Business Manager of THE PENTAGON. This is the fifth biennial conference of Kappa Mu Epsilon to which I have reported. During the ten-year period covered by these reports, every editor of THE PENTAGON and officer of Kappa Mu Epsilon has changed except for our distinguished National Secretary, Miss Greene. Also during this ten-year period, I have seen many changes in the circulation of our national magazine.

Every comparative statistic reported concerning higher education in the United States is one of marked increase.. The number of PENTAGONS printed also shows this marked increase. Last fall we printed three thousand fifty copies. This represents more than a fifty percent increase in ten years. The Fall, 1966, PENTAGON went to every state except Idaho and Montana and also was mailed to many foreign countries.

The circulation pattern of THE PENTAGON has changed due to new chapters being added and also to varying enrollment expansion among our chapters. In the first two reports I made to this convention concerning circulation, more PENTAGONS were mailed to Kansas than any other state and Pennsylvania was not listed among the six states receiving the most magazines. For each of the last two issues, the greatest number of copies was mailed to Pennsylvania. Over one-half of all copies went to just nine states: California, Illinois, Indiana, Kansas, Missouri, New York, Ohio, Pennsylvania, and Tennessee.

Since THE PENTAGON is published twice yearly, in May and December, some subscribers might justifiably question receiving their first journal in September or March. THE PENTAGON is not a magazine that goes out of date. Therefore, rather than having an initiate wait as long as six months before receiving his first issue, I try to stock enough magazines so that I may mail PENTAGONS to initiates as soon as their subscription card is received from the national secretary. This procedure is followed until a reserve of fifty copies is reached. This reserve is needed to fill requests for back issues.

The inside front cover of THE PENTAGON carries a statement that copies lost because of failure to notify the business manager of change of address cannot be replaced. I would urge each of the delegates present to apprise their chapter members of this fact. Many subscribers do not receive one or more of the journals to which they are entitled because of failure to keep a current address on file. Once a magazine is returned to my office because the post office could not deliver it, the subscription card for this subscriber is pulled and no more PENTAGONS are mailed to him.

To those of you who presented papers to this biennial convention, your subscription will be extended two years. Authors of articles printed in THE PENTAGON receive five additional copies of that issue.

I would like to express my appreciation to our editor, Helen Kriegsman and her associate editors for their efforts in publishing a journal of which this society can be proud. As circulation manager, I have an unequaled opportunity to see the interest that libraries and individuals have in THE PENTAGON. It has again been my privilege to serve Kappa Mu Epsilon by directing the distribution of our journal.

WILBUR J. WAGGONER

REPORT OF THE NATIONAL HISTORIAN

The Office of National Historian is primarily that of a depository of records and documents relative to events and activities occurring in the several chapters across the country. We have followed the practice, over the last four years, of soliciting from each active chapter the names of the local officers and an account of new items they consider newsworthy. In this connection, we would urge each corresponding secretary to respond to this annual inquiry concerning the activities of the local club. By this means, we can maintain a permanent record of many worthwhile activities that would otherwise be forgotten.

The cooperation over the past years of the many corresponding secretaries, the national officers, and THE PENTAGON editor has been outstanding.

We are now able to deposit with our successor, a fairly complete file on each active chapter and a complete collection of all back issues of THE PENTAGON.

J. D. HAGGARD

KAPPA MU EPSILON

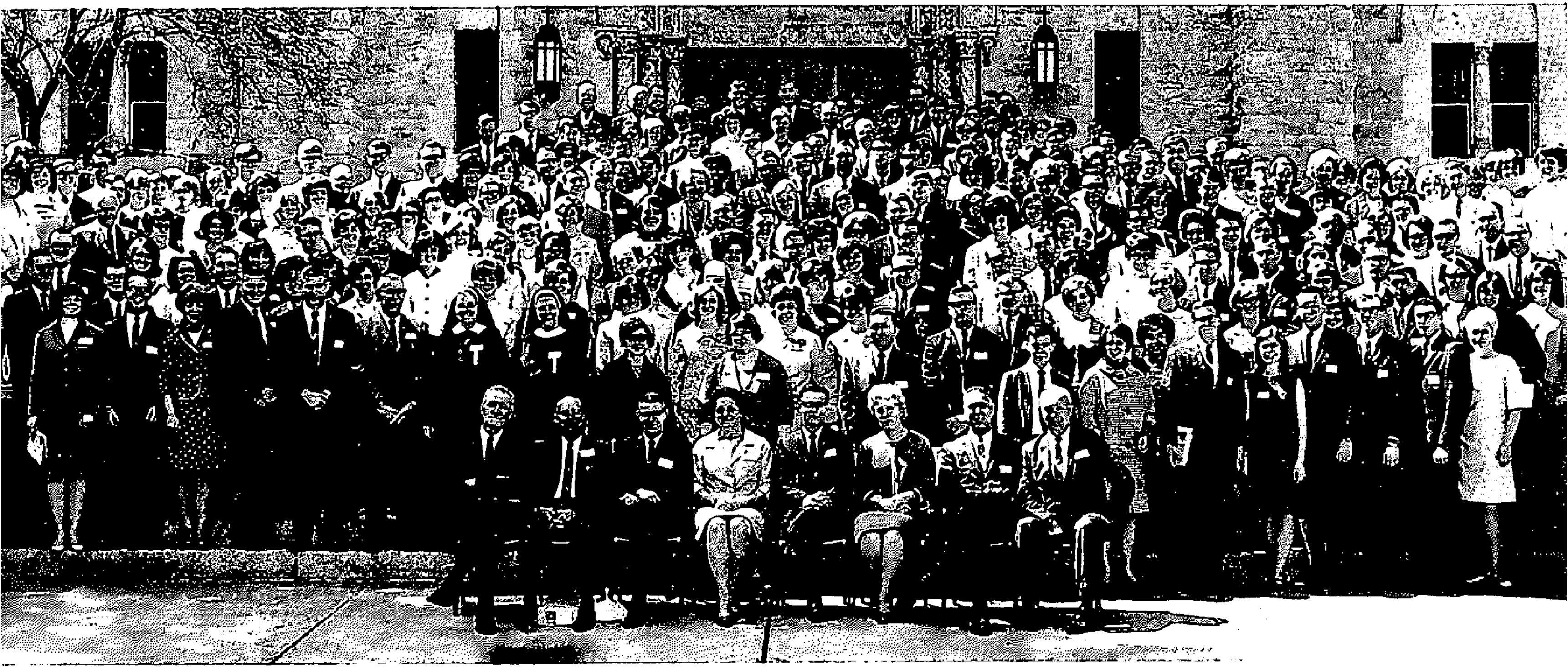
FINANCIAL REPORT OF THE NATIONAL TREASURER

For the period April 21, 1965 to April 1, 1967

1. CASH ON HAND APRIL 21, 1965			$ 9,079.37	
RECEIPTS				
2. RECEIPTS FROM CHAPTERS (See Accompanying Sheet)				
Initiates (2836 at $5.00)	$14,180.00			
Miscellaneous (Supplies, Jewelry, Installations, etc.	3,650.98			
Total Receipts from Chapters		$17,830.98		
3. MISCELLANEOUS RECEIPTS				
Interest on Bonds	187.92			
Pentagon (Surplus)	44.94			
So. Carolina State (Escrow)	40.00			
Over Payments	66.00			
Short Checks	20.00			
Total Miscellaneous Receipts		358.86		
4. TOTAL RECEIPTS			$18,189.84	
5. TOTAL RECEIPTS PLUS CASH ON HAND				$27,269.21
EXPENDITURES				
6. NATIONAL CONVENTION, 1965				
Paid to Chapter Delegates	$ 4,050.43			
Officers Expenses	1,148.50			
Miscellaneous (Speaker, Prizes, etc.)	161.10			
Host Chapter	156.48			
Total National Convention		$ 5,516.51		
7. BALFOUR COMPANY (Memberships, Certificates, Stationery, etc.)		5,342.65		
8. PENTAGON (Printing, Mailing of 4 Issues)		6,307.44		
9. INSTALLATION EXPENSE		179.75		
10. NATIONAL OFFICE EXPENSE		169.04		
2 Regional Conventions		116.58		
11. MISCELLANEOUS EXPENDITURES				
Refunds		69.64		
Treasurer's Bond		62.50		
Short Checks		20.00		
Secretarial Expense		1,093.68		
12. TOTAL EXPENSE			$18,877.79	
13. CASH BALANCE ON HAND APRIL 1, 1967			8,391.42	
14. TOTAL EXPENDITURES PLUS CASH ON HAND				$27,269.21
15. BONDS ON HAND APRIL 1, 1967	$ 3,000.00			
16. SAVINGS ACCOUNT + 268.91 INT.	3,512.85			
		$ 6,512.85		
17. TOTAL ASSETS AS OF APRIL 1, 1967			$14,904.27	
18. TOTAL ASSETS 1965			15,323.31	
19. NET LOSS FOR PERIOD				$ 419.04

Respectfully submitted,

Walter C. Butler, Nat'l Treas.
April 1, 1967



Kappa Mu Epsilon Convention, April 6-8, 1967.
Mount St. Scholastica College, Atchison, Kansas