THE PENTAGON

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CONTENTS

Loyal Frank Ollmann

In Memoriam

Carl V. Fronabarger, Past President

Members of Kappa Mu Epsilon have been saddened by the knowledge that Dr. Loyal F. Ollmann, National President of Kappa Mu Epsilon, passed from this life on April 8, 1966. Surviving him are his wife, Nila M. (Schwartz) Ollmann, and three children: Naida Jane, Mary Joan, and Loyal Taylor.

Loyal F. Ollmann was born on August 28, 1905. He received an A.B. from Ripon College, 1926; a M.S. from the University of Wisconsin, 1928; and a M.A. and a Ph.D. from the University of Michigan in 1938 and 1939, respectively.

His professional teaching and administrative experiences included serving as: Assistant Instructor of Physics at the University of Wisconsin, 1926-27; Professor of Physics and Mathematics, Elmhurst College, 1929-36; part-time Instructor in Mathematics, University of Michigan, 1936-39; Instructor of Mathematics, Texas Technological College, 1939-40; Assistant Professor of Mathematics, College of Wooster, 1940-41; and he was associated with Hofstra University from 1941 until the time of his death, first as Associate Professor and then as Head of the Mathematics Department; he was Chairman of the Division of Natural Sciences, Mathematics and Engineering, 1957-61.

Dr. Ollmann served as an Associate Editor of the *American Mathematical Monthly*, 1946-51. He was a member of numerous professional organizations including: American Mathematical Society, Mathematical Association of America, American Association of University Professors, Sigma Xi, Sigma Kappa Alpha, Sigma Pi Sigma, and Kappa Mu Epsilon. He is listed in *Who's Who in America, Who's Who in Education,* and in *American Men of Science.*

Kappa Mu Epsilon has benefited greatly by the many contributions that he has made to the organization. He served as treasurer of the organization, 1945-53. He was active in assisting in the initiation of steps to incorporate Kappa Mu Epsilon, and more recently he graciously accepted the responsibility of obtaining legal assistance to incorporate it under the laws of the State of New York and to be declared a non-profit organization by the United States Internal Revenue Service. From 1963 until his death he served Kappa Mu Epsilon as National President during a time of continued expansion and growth.

Dr. Ollmann had some hobbies which gave him a great deal of

(Continued on page 136)

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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Computer Application To Symmetric Double Integration By Hypercubes

JERRY L. LEWIS
Student, Drake University, Des Moines

In this article we shall derive and apply to the digital computer a method for computing the numerical approximations to the double integral of a function of two variables over a two-dimensional hypercube.

BRIEF DERIVATION OF THE GENERAL EQUATION

We shall first turn our attention to the following function table which is represented by the mapping $f{:}x,y \rightarrow f(x,y)$, and let $Z_{mn} = f(x_m, y_n)$:

	x_0	x_1	x_2	$\cdots$	x_m
y_0	Z_{00}	Z_{10}	Z_{20}	$\cdots$	Z_{m0}
y_1	Z_{01}	Z_{11}	Z_{21}	$\cdots$	Z_{m1}
y_2	Z_{02}	Z_{12}	Z_{22}	$\cdots$	Z_{m2}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
y_n	Z_{0n}	Z_{1n}	Z_{2n}	$\cdots$	Z_{mn}

These values are used in forming the following formula for double interpolation, which is derived by O. Biermann's *Mathematische Näherungsmethoden*, pages 138-144, as cited by Scarborough [5]:

$$\begin{aligned} Z = f(x_0 + hs, y_0 + kt) &= Z_{00} + sE_{1+0}Z_{00} + tE_{0+1}Z_{00} \\ &+ \tfrac{1}{2}!\{s(s-1)E_{2+0}Z_{00} + 2stE_{1+1}Z_{00} + t(t-1) \\ &E_{0+2}Z_{00}\} + \tfrac{1}{3}!\,\{s(s-1)(s-2)E_{3+0}Z_{00} + 3s \\ &(s-1)tE_{2+1}Z_{00} + 3st(t-1)E_{1+2}Z_{00} + t(t-1) \\ &(t-2)E_{0+3}Z_{00}\} + \tfrac{1}{4}!\{s(s-1)(s-2)(s-3)E_{4+0} \\ &Z_{00} + 4s(s-1)(s-2)tE_{3+1}Z_{00} + 6s(s-1)t(t-1) \\ &E_{2+2}Z_{00} + 4st(t-1)(t-2)E_{1+3}Z_{00} + t(t-1) \\ &(t-2)(t-3)E_{0+4}Z_{00}\} + R, \qquad (1.1) \end{aligned}$$

where h and k are the intervals between x and y, respectively, and R is the remainder term. The two-way differences $E_{m+n}Z_{00}$ is defined as follows:

$$
\begin{aligned}
E_{1+0}Z_{00} &= Z_{10} - Z_{00},\\
E_{0+1}Z_{00} &= Z_{01} - Z_{00},\\
E_{1+1}Z_{00} &= Z_{11} - Z_{10} - Z_{01} + Z_{00},\\
E_{2+0}Z_{00} &= Z_{20} - 2Z_{10} + Z_{00},\\
E_{0+2}Z_{00} &= Z_{02} - 2Z_{01} + Z_{00},\\
E_{2+1}Z_{00} &= Z_{21} - 2Z_{11} + Z_{01} - Z_{20} + 2Z_{10} - Z_{00},\\
E_{1+2}Z_{00} &= Z_{12} - 2Z_{11} + Z_{10} - Z_{02} + 2Z_{01} - Z_{00},\\
E_{3+0}Z_{00} &= Z_{30} - 3Z_{20} + 3Z_{10} - Z_{00},\\
E_{0+3}Z_{00} &= Z_{03} - 3Z_{02} + 3Z_{01} - Z_{00},\\
E_{3+1}Z_{00} &= Z_{31} - 3Z_{21} + 3Z_{11} - Z_{01} - Z_{30} + 3Z_{20} - 3Z_{10} + Z_{00},\\
E_{1+3}Z_{00} &= Z_{13} - 3Z_{12} + 3Z_{11} - Z_{10} - Z_{03} + 3Z_{02} - 3Z_{01} + Z_{00},\\
E_{4+0}Z_{00} &= Z_{40} - 4Z_{30} + 6Z_{20} - 4Z_{10} + Z_{00},\\
E_{0+4}Z_{00} &= Z_{04} - 4Z_{03} + 6Z_{02} - 4Z_{01} + Z_{00},\\
E_{2+2}Z_{00} &= Z_{22} - 2Z_{21} + Z_{20} - 2Z_{12} + 4Z_{11} - 2Z_{10} +\\
&\qquad Z_{02} - 2Z_{01} + Z_{00}.
\end{aligned}
$$

If we now integrate (1.1) over two intervals in the x and y directions and eliminate all terms involving the two-way differences of E_{3+0}, E_{0+3}, E_{3+1}, E_{1+3}, E_{4+0}, and E_{0+4} because these differences have values of the mapping outside the mechanical cubature we are integrating. Moreover, since $h = dx/du$ and $k = dy/dv$ we have,

$$
\begin{aligned}
I_s = \int_{x_0}^{x_0 + 2h}\int_{y_0}^{y_0 + 2k} Zdydk = hk\int_0^2\int_0^2 \{&Z_{00}\\
+ sE_{1+0}Z_{00} + tE_{0+1}Z_{00} + ½[s(s-1)E_{2+0}Z_{00}&\\
+ stE_{1+1}Z_{00} + t(t-1)E_{0+2}Z_{00}] + 1/6\,[3s(s-1)&\\
tE_{2+1}Z_{00} + 3st(t-1)E_{1+2}Z_{00}] + 1/24[6s(s-1)&\\
t(t-1)E_{2+2}Z_{00}]\}\, dvdu.&
\end{aligned}
$$

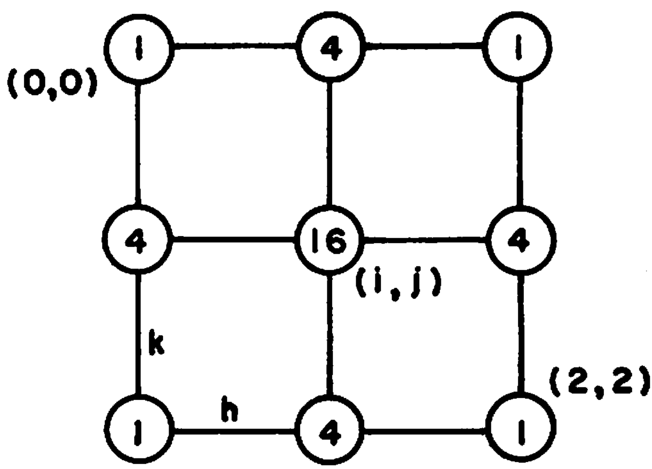
After completing the above integration, we replace the double differences by the values given in the foregoing table, and derive the following general formula:

$$I_s = hk/9\{Z_{00} + Z_{02} + Z_{22} + Z_{20} + 4(Z_{01} + Z_{12} + Z_{21} + Z_{10}) + 16Z_{11}\}, \quad (1.2)$$

which after letting $Z_{11} = f_{i,j}$ can be put into the following form and represented by the "mathematical molecule":

$$I_s = \int_{x_{j-1}}^{x_{j+1}} \int_{y_{j-1}}^{y_{j+1}} f(x,y)\,dxdy$$

$$= kh/9\{(f_{i+1,j+1} + 4f_{i,j+1} + f_{i-1,j+1}) + 4(f_{i+1,j} + 4f_{i,j} + f_{i-1,j}) + (f_{i+1,j-1} + 4f_{i,j-1} + f_{i-k,j-1})\}. \quad (1.3)$$



You will note that a general mapping could be obtained if we added any number of the above molecules, which would correspond to Simpson's Rule for n-intervals in a one-dimensional case.

COMPUTER APPROACH

In applying the above numerical formula (1.3) to the digital computer the scientific programming language called Fortran was used. We shall first discuss briefly the important steps of the program, and then give a listing of the program along with some computed results.

The program was written to utilize the Arithmetic Statement Function defined as follows:

$$\text{KAPPAF}(x,y) = f(x,y).$$

This informs the computer what function is to be evaluated given certain arguments x and y. For example, if $f(x,y) = xy$ and elsewhere in the program the compiler encounters the statement:

AMAP1 = KAPPAF(2,4)

the computer would store the value of 8 in location AMAP1.

The next step is to tell the computer what the limits of integration are to be, and the desired number of intervals. This is accomplished by the following statement:

7 READ 1, XLL, XUL, YLL, YUL, XINC, YINC

where XLL is the x-lower limit of integration, XUL the upper limit, YLL is the y-lower limit, YUL the upper limit, XINC and YINC are the number of intervals.

Now that we have the necessary information into the computer we proceed to evaluate k and h of (1.3), which is derived by the following two statements:

$$h = \text{YY} = (\text{YUL} - \text{YLL})/\text{YINC}$$

$$k = \text{XX} = (\text{XUL} - \text{XLL})/\text{XINC}.$$

We also determine how many hypercubes are to be evaluated in the x-direction (XHCUBE), and the y-direction (YHCUBE). Having determined this we start evaluating the mapping (1.3) with two nested Do loops, which have the following names given to the variables of (1.3):

AMAP1 = $f_{i+1,j+1}$ = KAPPAF(XCHI1, YCHI1)
AMAP2 = $f_{i,j+1}$ = KAPPAF(XCHI2, YCHI1)
AMAP3 = $f_{i-1,j+1}$ = KAPPAF(XCHI3, YCHI1)
AMAP4 = $f_{i+1,j}$ = KAPPAF(XCHI1, YCHI2)
AMAP5 = $f_{i,j}$ = KAPPAF(XCHI2, YCHI2)
AMAP6 = $f_{i-1,j}$ = KAPPAF(XCHI3, YCHI2)
AMAP7 = $f_{i+1,j-1}$ = KAPPAF(XCHI1, YCHI3)
AMAP8 = $f_{i,j-1}$ = KAPPAF(XCHI2, YCHI3)
AMAP9 = $f_{i-1,j-1}$ = KAPPAF(XCHI3, YCHI3)

where XCHI1 = XLL or the x-lower limit of integration, XCHI2 = XLL + XX, XCHI3 = XLL + 2XX, and YCHI1, YCHI2, YCHI3 are defined analogously, using YY.

Having computed the values of AMAP1 through AMAP9 they are then substituted into the mapping EPSILON which is equal to (1.3). It will be noted that the mapping starts out with EPSILON equal to EPSILON; this is because if there are more than one hypercubes to be evaluated, the mapping merely continues to evaluate the nth cube and adds it to the previous result. To help clarify this, in Fortran language the equal sign means "to replace the value with" rather than "is equal to".

After having completed the numerical aspect of the program the computer prints-out the results in the specified Formats given at the bottom of the program, and then either stops or continues with new input values.

PROGRAMMED EXAMPLES

The following pages contain a listing of the problem written in the Fortran computer language along with some computed results. It might be added that SALPHAS defines the mapping to be evaluated in the examples.

```
  SYMMETRIC DOUBLE INTEGRATION BY HYPERCUBES
  HOMOMORPHIC TO SIMPSONS RULE FOR A MAPPING OF ONE VARIABLE

  F(X,Y)=$$XY DXDY
  KAPPAF(X,Y) = X*Y
7 READ 1, XLL,XUL,YLL,YUL,XINC,YINC
  IF(YINC)21,22
  EPSILON = 0
  TEM = YLL
  XHCUBE = XINC/2
  YHCUBE = YINC/2
  YY =(YUL - YLL)/YINC
  XX =(XUL - XLL)/XINC
  PRINT 2,XLL,XUL,YLL,YUL,XX,YY
  DO 10 I = 1,XHCUBE
  XCHI1 = XLL
  XCHI2 = XCHI1 + XX
  XCHI3 = XCHI2 + XX
  DO 11 J = 1,YHCUBE
  YCHI1 = YLL
  YCHI2 = YCHI1 + YY
  YCHI3 = YCHI2 + YY
  AMAP1 = KAPPAF(XCHI1,YCHI1)
  AMAP2 = KAPPAF(XCHI2,YCHI1)
  AMAP3 = KAPPAF(XCHI3,YCHI1)
```

```
   AMAP4 = KAPPAF(XCHI1,YCHI2)
   AMAP5 = KAPPAF(XCHI2,YCHI2)
   AMAP6 = KAPPAF(XCHI3,YCHI2)
   AMAP7 = KAPPAF(XCHI1,YCHI3)
   AMAP8 = KAPPAF(XCHI2,YCHI3)
   AMAP9 = KAPPAF(XCHI3,YCHI3)
   EPSILON = EPSILON + (YY*XX) / 9.*(AMAP1+4*AMAP2 + AMAP3 + 4.*AMAP4
  $          + 16.*AMAP5 + 4.*AMAP6 + AMAP7 + 4.*AMAP8 + AMAP9)
11 YLL = YCHI3
   YLL = TEM
10 XLL = XCHI3
   PRINT 3,EPSILON
   GO TO 7
21 PRINT 4
22 PRINT 5
 1 FORMAT(4F10.0,2F5.0)
 2 FORMAT(1H1/////                                    35X,20HSALPHAS = SS XY
  ADXDY/1H0,31X,27HX-LIMITS OF INTEGRATION ARE/1H0,37X,F6.3,4H TO ,F6
  B.3/1H0,31X,27HY-LIMITS OF INTEGRATION ARE/1H0,37X,F6.3,4H TO ,F6.3
  C/1H0,41X,4HH = ,F5.3/1H0,41X,4HK = ,F5.3/1H0,18X,57HSALPHAS = KH/9
  D(F(I+1,J+1)+4F(I,J+1)+F(I-1,J+1)+4(F(I+1,J)/1H0,26X,51H+4F(I,J)+F(
  EI-1,4))+F(I+1,J-1)+4F(I,J-1)+F(I-1,J-1))/1H0)
 3 FORMAT (35X,10HSALPHAS =  F15.4/1H0,38X,14HEND OF PROBLEM)
 4 FORMAT(///,11HINPUT ERROR)
 5 FORMAT (/1H1/1H0,38X,10HEND OF JOB)
   END
```

$$\int_0^4 \int_0^2 XY \ dx\, dy$$

```
                 SALPHAS = SS XY DXDY

            X-LIMITS OF INTEGRATION ARE

                  0.000 TO  2.000

            Y-LIMITS OF INTEGRATION ARE

                  0.000 TO  4.000

                     H = 1.000

                     K = 1.000

SALPHAS = KH/9(F(I+1,J+1)+4F(I,J+1)+F(I-1,J+1)+4(F(I+1,J)

        +4F(I,J)+F(I-1,4))+F(I+1,J-1)+4F(I,J-1)+F(I-1,J-1))

              SALPHAS =          16.0000

                  END OF PROBLEM
```

$$\int_0^2\int_0^2 XY\ dx\,dy$$

```
$ALPHA$ = $$ XY DXDY
X-LIMITS OF INTEGRATION ARE
0.000 TO 2.000
Y-LIMITS OF INTEGRATION ARE
0.000 TO 2.000
H = 1.000
K = 1.000
$ALPHA$ = KH/9[F[I+1,J+1]+4F[I,J+1]+F[I-1,J+1]+4[F[I+1,J]
+4F[I,J]+F[I-1,4]]+F[I+1,J-1]+4F[I,J-1]+F[I-1,J-1]]
$ALPHA$ =            4.0000
END OF PROBLEM
```

$$\int_0^2\int_0^2 X^2(Y-3)\,1.486\ dx\,dy$$

```
$ALPHA$ = $$(X**2)(Y-3)*1.486 DXDY
X-LIMITS OF INTEGRATION ARE
0.000 TO 2.000
Y-LIMITS OF INTEGRATION ARE
0.000 TO 2.000
H = 0.125
K = 0.125
$ALPHA$ = KH/9[F[I+1,J+1]+4F[I,J+1]+F[I-1,J+1]+4[F[I+1,J]
+4F[I,J]+F[I-1,4]]+F[I+1,J-1]+4F[I,J-1]+F[I-1,J-1]]
$ALPHA$ =          -15.8507
END OF PROBLEM
```

$$\int_0^2 \int_0^4 X^2 (Y-3)\,1.486 \quad dx\,dy$$

```
$ALPHA$ = $$(X**2)(Y-3)*1.486 DXDY

X-LIMITS OF INTEGRATION ARE
      0.000 TO  4.000
Y-LIMITS OF INTEGRATION ARE
      0.000 TO  2.000
         H = 1.000
         K = 1.000

$ALPHA$ = KH/9[F[I+1,J+1]+4F[I,J+1]+F(I-1,J+1)+4[F[I+1,J]
        +4F[I,J]+F[I-1,4]]+F(I+1,J-1]+4F[I,J-1]+F(I-1,J-1]]

      $ALPHA$ =        -126.8053
          END OF PROBLEM
```

$$\int_0^2 \int_0^4 e^Y (1 - X^2)\ dx\,dy$$

```
$ALPHA$ = $$E**Y*(1-X**2) DXDY

X-LIMITS OF INTEGRATION ARE
      0.000 TO  4.000
Y-LIMITS OF INTEGRATION ARE
      0.000 TO  2.000
         H = 1.000
         K = 1.000

$ALPHA$ = KH/9(F[I+1,J+1]+4F[I,J+1]+F[I-1,J+1]+4[F[I+1,J]
        +4F(I,J)+F(I-1,4]]+F[I+1,J-1]+4F(I,J-1)+F(I-1,J-1]]

      $ALPHA$ =        -111.2926
          END OF PROBLEM
```

$$\int_3^{11}\int_2^8 e^Y (1 - X^2)\ dx\,dy$$

```
        $ALPHA$ = $$E**Y*(1-X**2) DXDY
        X-LIMITS OF INTEGRATION ARE
              2.000 TO  8.000
        Y-LIMITS OF INTEGRATION ARE
              3.000 TO 11.000
                 H = 1.000
                 K = 2.000
$ALPHA$ = KH/9(F(I+1,J+1)+4F(I,J+1)+F(I-1,J+1)+4(F(I+1,J)
      +4F(I,J)+F(I-1,4))+F(I+1,J-1)+4F(I,J-1)+F(I-1,J-1))

           $ALPHA$ =  -10270096.1000
              END OF PROBLEM
```

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Conic Sections with Circles as Focal Points

Thomas M. Potts
Student, Kansas State College of Pittsburg

In the high school textbook, *Analytic Geometry,* prepared by the School Mathematics Study Group, each of the conic sections is presented as a locus of points fulfilling specified conditions. This presentation makes understanding of conics relatively easy. However, at one point in the course of study, it is stated that a degenerate circle graphs a point. Then could not the foci of conics be degenerate circles? What would be the results of "regenerating" these focal points?

In order to derive equations representative of such graphs, a few arbitrary definitions are made.

1. *The distance from a point to a circle is measured in the normal manner.*

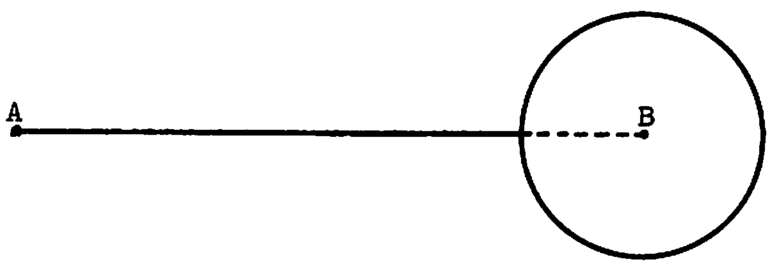

Figure 1

The distance from point A to circle B is equal to the distance from point A to point B minus the radius of circle B.

2. *Distance from a focal circle to a point that includes distance through a second focal circle includes that segment within the second focal circle.*

*A paper presented at the KME Regional Convention at Springfield, Missouri, April 23, 1966.

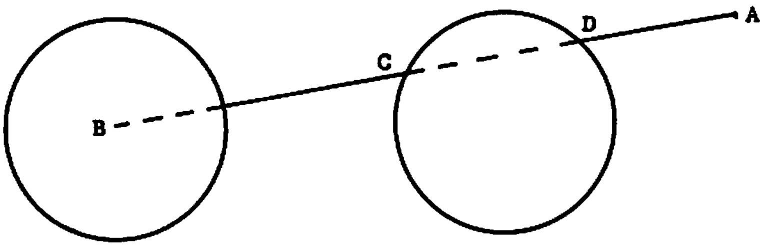

Figure 2

The distance from point *A* to circle *B* includes the segment $\overline{CD}$.

3. *In conics of more than one focal point, the focal circles have the same radius.*
4. *In conics of more than one focal point, the focal circles do not intersect.*
5. *In all conics, the focal circles do not intersect with the directrices.*

These last three definitions are arbitrary rules to simplify this survey. In a more detailed study yet to be made, these three areas will be probed. Further investigations could be made using directrices that are not straight lines with focal points that are either points, circles or ellipses. Other investigations could be made of a conic whose focal "point" is not a closed figure. Still other investigations could be made of the applications of these figures in three-space. At the end of the paper, one such exploration is presented where a "parabola" was graphed with its focal point within a circular directrix. Proof is given that such a graph is an ellipse.

PARABOLA

A parabola is defined as being the locus of all points equidistant from a circle and a straight line.

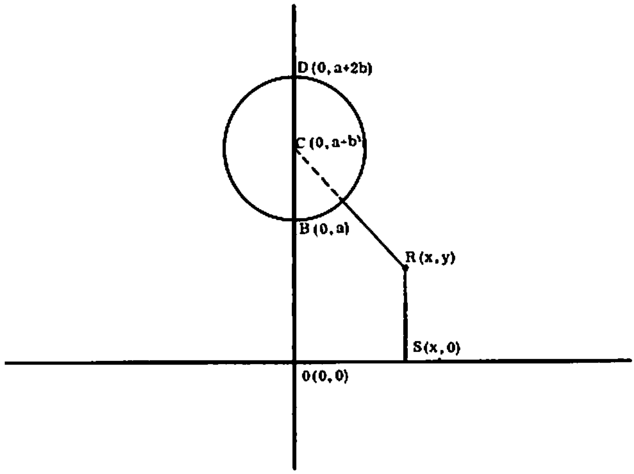

Figure 3

$$
\begin{aligned}
d(R,S) &= d(R,C) - b \\
\sqrt{(x-x)^2 + (y-0)^2} &= \sqrt{(x-0)^2 + [y-(a+b)]^2} - b \\
\sqrt{y^2} + b &= \sqrt{x^2 + [y-(a+b)]^2} \\
y^2 + 2by + b^2 &= x^2 + [y-(a+b)]^2 \\
y^2 + 2by + b^2 &= x^2 + y^2 - 2y(a+b) + (a+b)^2 \\
2by &= x^2 - 2ay - 2by + a^2 + 2ab \\
4by + 2ay &= x^2 + a(a+2b) \\
2y(a+2b) &= x^2 + a(a+2b) \\
2y &= \frac{x^2}{a+2b} + a
\end{aligned}
$$

Vertex: As in the regular parabola, the vertex is the point on the curve closest to the directrix and the focal circle. Setting $x = 0$, we find that the vertex is at $(0, a/2)$.

Latus Rectum: The latus rectum is the distance between the two points on the locus which lie on the line making a right angle with the y-axis at the center of the focal circle. The length of the latus rectum may be determined by setting $y = a + b$ and solving for x. The length of the latus rectum will then be equal to $2|x|$.

$$2y = \frac{x^2}{a + 2b} + a$$

$$/2(a + b) - a = \frac{x^2}{a + 2b}$$

$$(a + 2b)^2 = x^2$$

$$x = \pm(a + 2b)$$

$$2|x| = 2(a + 2b)$$

The length of the latus rectum is then $2(a + 2b)$.

Problem: Sketch the graph of the parabola whose focal circle has a radius of 2 and is 3 units from the directrix, which is the x-axis.

$b = 2 \qquad a = 3$

$$\frac{x^2}{a + 2b} - 2y + a = 0$$

$$\frac{x}{3 + 4} - 2y + 3 = 0$$

$$y = \frac{x^2 + 21}{14}$$

Vertex at $a/2 = (0, 3/2)$

Latus Rectum length $= 2(a + 2b) = 2(3 + 4) = 14$

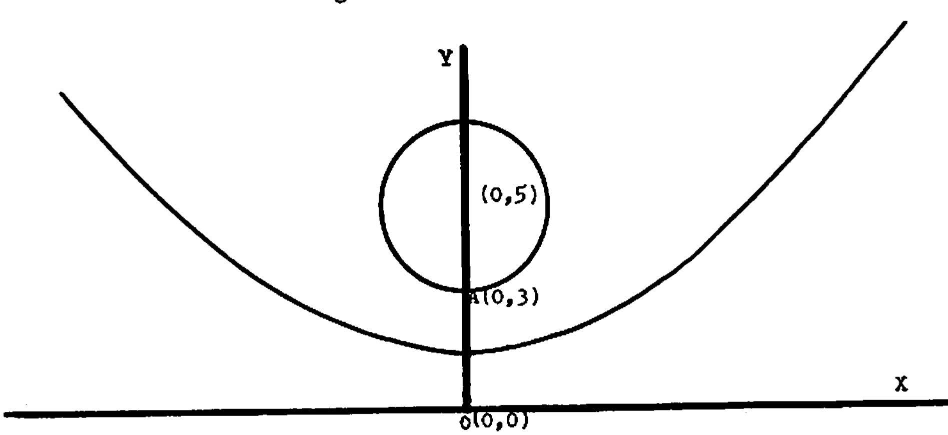

Figure 4

ELLIPSE

An ellipse is defined as the locus of all points such that the distance from any point on the locus to the first focal circle plus the distance from that point to the second focal circle is a constant.

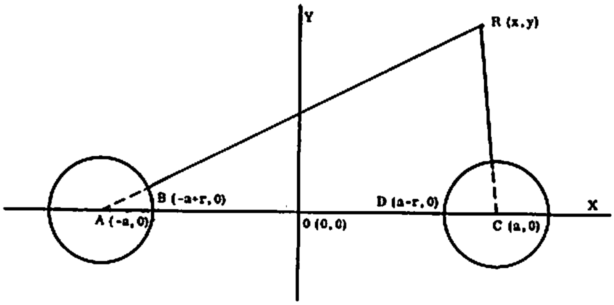

Figure 5

$$d(R,A) - r + d(R,C) - r = K \qquad K = 2p$$

$$\overline{AB} = r$$

$$\sqrt{(x+a)^2 + y^2} - r + \sqrt{(x-a)^2 + y^2} - r = 2p$$

$$\sqrt{(x+a)^2 + y^2} + \sqrt{(x-a)^2 + y^2} = 2(p+r)$$

$$\sqrt{(x+a)^2 + y^2} = 2(p+r) - \sqrt{(x-a)^2 + y^2}$$

$$(x+a)^2 + y^2 = 4(p+r)^2 - 4(p+r)\sqrt{(x-a)^2 + y^2} + (x-a)^2 + y^2$$

$$4xa - 4(p+r)^2 = -4(p+r)\sqrt{(x-a)^2 + y^2}$$

$$(p+r) - \frac{xa}{p+r} = \sqrt{(x-a)^2 + y^2}$$

$$(p+r)^2 - 2(p+r)\frac{xa}{p+r} + \frac{x^2a^2}{(p+r)^2} = (x-a)^2 + y^2$$

$$(p+r)^2 - 2xa + \frac{x^2a^2}{(p+r)^2} = x^2 + a^2 - 2xa + y^2$$

$$(p+r)^2 + \frac{x^2a^2}{(p+r)^2} = x^2 + a^2 + y^2$$

$$-\frac{x^2a^2}{(p+r)^2} + x^2 = -a^2 - y^2 + (p+r)^2$$

$$x^2\left[1 - \frac{a^2}{(p + r)^2}\right] = -y^2 + (p + r)^2 - a^2$$

$$x^2\frac{(p + r)^2 - a^2}{(p + r)^2} = -y^2 + [(p + r)^2 - a^2)]$$

$$\frac{x^2}{(p + r)^2} = -\frac{y^2}{(p + r)^2 - a^2} + 1$$

$$\frac{x^2}{(p + r)^2} + \frac{y^2}{(p + r)^2 - a^2} = 1$$

Problem: Sketch the ellipse whose focal circles have radii of 2 and are 6 units apart when $p = 8$.

Let $x = 0$, then

$$y = \pm\sqrt{(p + r)^2 - a^2}$$

Then $\pm\sqrt{(p + r)^2 - a^2}$ are the y intercepts. If we let $b = \pm\sqrt{(p + r)^2 - a^2}$, then b is the intercept distance and the minor axis is then $2b$.

Our general equation then becomes

$$\frac{x^2}{(p + r)^2} + \frac{y^2}{b^2} = 1$$

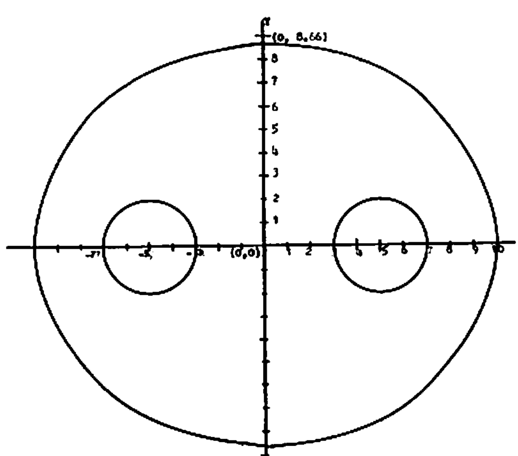

Figure 6

HYPERBOLA

Using the concept that the focal points are circles, a hyperbola is defined as the locus of all points such that the distance from the point to the first focal point minus the distance from the second focal point is equal to a constant.

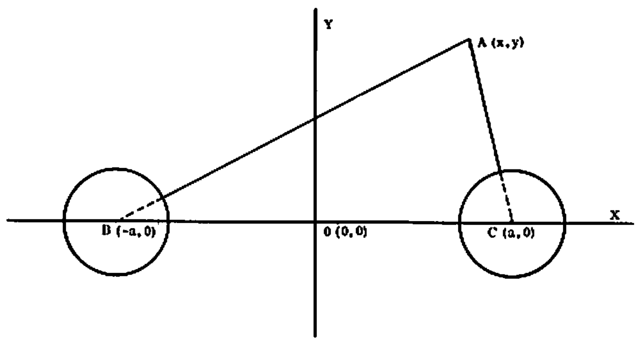

Figure 7

$$(d(A,B) - r) - (d(A,C) - r) = \pm 2a$$

$$d(A,B) - d(A,C) = \pm 2a$$

Therefore, in the hyperbola, focal circles cause the same graph as focal points.

CIRCLE

A circle is defined as the locus of all points the same distance from the focal circle.

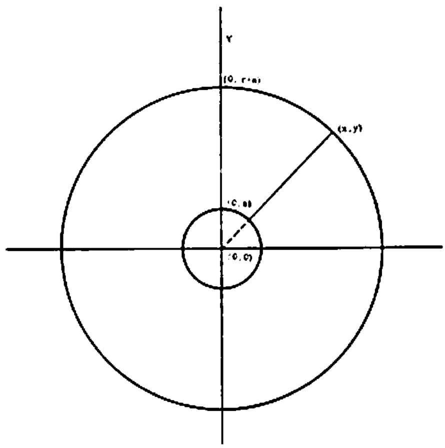

Figure 8

$$\sqrt{(x-0)^2 + (y-0)^2} = r + a$$

$$x^2 + y^2 = (r + a)^2$$

PARABOLA WITH ITS FOCAL POINT IN A CIRCLE

The purpose of this exploration is to derive the equation of a parabola with its focal point in a circular directrix.

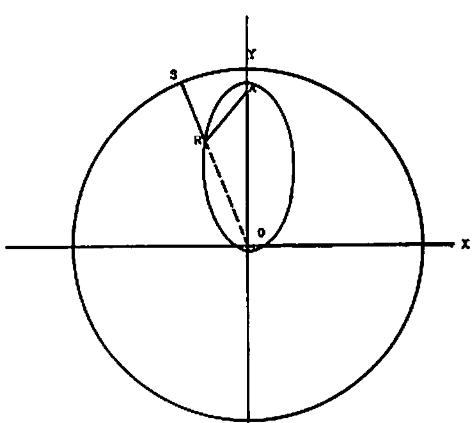

Figure 9

(Continued on page 108)

Concerning Functional Conjugates*

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The problem proposed in this paper is: Given $f(z) = u(x,y) + iv(x,y)$, find necessary and sufficient conditions such that $f(\bar{z}) = \overline{f(z)}$.

By definition: $z = x + iy$, $\bar{z} = x - iy$, $f(z) = u + iv$, and $\overline{f(z)} = u - iv$.

In this proof four lemmas and one theorem are used. The proofs of the lemmas are omitted, but they may be proved by induction. A proof of the theorem may be found in almost any complex variable textbook. The test using the power series is good only for analytic functions; however, the even and odd test is good for both analytic and non-analytic functions.

Lemma 1 — If A_n and B_n are complex functions then $\overline{A_n \pm B_n} = \bar{A}_n \pm \bar{B}_n$.

Lemma 2 — If C_n and D_n are complex functions then $\overline{C_n D_n} = \bar{C}_n \bar{D}_n$.

Lemma 3 — If n is contained in I_+ then $\overline{(z - \beta)^n} = (\bar{z} - \bar{\beta})^n$.

Lemma 4 — $$\overline{\sum_{n=0}^{\infty} \alpha_n (z - \beta)^n} = \sum_{n=0}^{\infty} \bar{\alpha}_n (\bar{z} - \bar{\beta})^n.$$

Theorem 1 — A power series represents an analytic function in its circle of convergence and every analytic function can be uniquely expressed as a power series.[1]

We let $f(z) = \alpha_0 + \alpha_1(z - \beta) + \alpha_2(z - \beta)^2 + \cdots + \alpha_n(z - \beta)^n + \cdots$, where $\alpha_n = a_n + b_n i$ and $\beta = c + di$ for n contained in the non-negative integers.

This implies

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - \beta)^n \text{ and thus } f(\bar{z}) = \sum_{n=0}^{\infty} \alpha_n (\bar{z} - \beta)^n.$$

*A paper presented at the 1965 National Convention of KME.

From the definition of a conjugate and the value of $f(z)$ we see that

$$\overline{f(z)} = \overline{\sum_{n=0}^{\infty} \alpha_n (z - \beta)^n}$$

Thus by Lemma 4:

$$\overline{f(z)} = \sum_{n=0}^{\infty} \overline{\alpha_n} (\bar{z} - \bar{\beta})^n.$$

Equating $f(\bar{z})$ and $\overline{f(z)}$, we see $f(\bar{z}) = \overline{f(z)}$ iff $f(z)$ posses-ses a series expansion on the real axis with real coefficients.

NOTE: For $f(\bar{z}) = \overline{f(z)}$, this does not imply that $f(z)$ cannot be expanded around some point off the real axis.

$f(z) = u(x,y) + iv(x,y)$

$\overline{f(z)} = u(x,y) - iv(x,y)$

$z = x + iy$ $\qquad\qquad \bar{z} = x + i(-y)$

$f(\bar{z}) = u(x,-y) + iv(x,-y)$

Equating $\overline{f(z)}$ and $f(\bar{z})$

$u(x,y) - iv(x,y) = u(x,-y) + iv(x,-y)$

$u(x,y) = u(x,-y)$ implies u even in y.

$v(x,y) = -v(x,-y)$ implies v odd in y.

Therefore: $f(\bar{z}) = \overline{f(z)}$ iff u is even in y and v is odd in y.

In the following examples we are going to use functions expanded at the point $(0,0)$, i.e., $c = 0$ and $d = 0$.

We define $f(z) = u(x,y) + iv(x,y)$. Now $f(z)$ is analytic iff u_x, u_y, v_x, v_y exist, $u_x = v_y$ and $u_y = -v_x$. (Cauchy-Riemann Equations)

Since a constant does not affect the partials, u and v may contain a real constant and $f(z)$ will still be analytic. If $u(x,y)$ has a constant, $\overline{f(z)}$ is not affected; but if $v(x,y)$ has a real constant, $\overline{f(z)}$ is affected.

Let $v(x,y) = g(x,y) + c$ and $u(x,y) = h(x,y) + k$ where c and k are contained in the real numbers.

We will define $P(z) = h(x,y) + ig(x,y)$ such that $P(\bar{z}) = \overline{P(z)}$.

Then letting $m_1(z) = u(x,y) + ig(x,y)$ we see that $m_1(\bar{z}) = u(x,y) - ig(x,y)$ and $\overline{m_1(z)} = u(x,y) - ig(x,y)$.

Therefore $m_1(\bar{z}) = \overline{m_1(z)}$.

Now letting $m_2(z) = h(x,y) + iv(x,y)$ we see that $m_2(z) = h(x,y) + i[g(x,y) + c]$. From the definition of P(z) we see that $m_2(\bar{z}) = h(x,y) - ig(x,y) + c$ and that $\overline{m_2(z)} = h(x,y) - i[g(x,y) + c]$.

Therefore: $m_2(\bar{z}) \neq \overline{m_2(z)}$.

This implies that if v contains a real constant, $f(\bar{z}) \neq \overline{f(z)}$. From the series we can see that this is the case where $b_1 \neq 0$ or $\alpha_1 = a_1 + ci$.

This is a sufficient but not a necessary condition for $f(\bar{z}) \neq \overline{f(z)}$. An example where v does not contain a constant is $f_1(z) = (x + y) + i(y - x)$.

$$f_1(z) = (x + y) + i(y - x).$$

$$u = x + y \qquad v = y - x$$

$$u_x = 1 \qquad v_y = 1$$

$$u_y = 1 \qquad v_x = -1$$

Therefore $f_1(z)$ is analytic.

$$f_1(z) = x + y + iy - ix$$

$$f_1(z) = x + iy - ix + y$$

$$f_1(z) = x + iy + \frac{i}{i}(-ix + y)$$

$$f_1(z) = x + iy + \frac{1}{i}(x + iy)$$

$$f_1(z) = x + iy - i(x + iy)$$

$$f_1(z) = z - iz$$

or

$$f_1(z) = z(1 - i)$$

$$f_1(\bar{z}) = \bar{z}(1-i)$$
$$\overline{f_1(z)} = \overline{z(1-i)} = \bar{z}(1+i) \neq f_1(\bar{z}).$$

In this case $b_2 = -1 \neq 0$.

Examples:

Let $f(z) = e^z$

By Euler's equation $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

$$e^{\bar{z}} = e^{x-iy} = e^x e^{-iy} = e^x(\cos y - i \sin y)$$
$$\overline{e^z} = e^x(\cos y - i \sin y)$$
$$e^{\bar{z}} = \overline{e^z}.$$

Let $f(z) = e^{iz} = e^{-y+ix}$

$$e^{-y}e^{ix} = e^{-y}(\cos x + i \sin x)$$
$$e^{i\bar{z}} = e^{i(x-iy)} = e^y e^{ix} = e^y(\cos x + i \sin x)$$
$$\overline{e^{iz}} = e^{-y}(\cos x - i \sin x)$$
$$e^{i\bar{z}} \neq \overline{e^{iz}}.$$

We could have seen this from the series.

$$e^{iz} = \cos z + i \sin z.$$

Thus the sine series would be multiplied by i. Therefore, since all $b_n \neq 0$, $e^{i\bar{z}} \neq \overline{e^{iz}}$.

Examples of non-analytic functions:

Let $f(z) = x + y$

Then $\overline{f(z)} = x + y$ and $f(\bar{z}) = x - y$

Let $f(z) = x^2 + y^2$

Then $\overline{f(z)} = x^2 + y^2$ and $f(\bar{z}) = x^2 + y^2$

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Incorporation of Some Mathematical Ideas Through Application to an Electrical Circuit*

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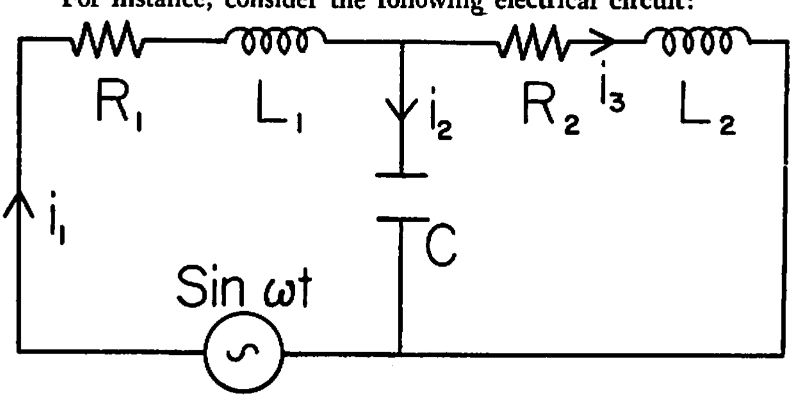
Mathematical ideas and their applications are generally presented to the student as separate entities; that is, the idea is generally presented independently of any application, and then the applications, based upon this particular idea are presented. For example, the calculus student may be subjected first to the idea of integration or the anti-derivative; he then masters the method utilizing the idea; and finally, he meets the application of the idea such as integration to find areas, centroids, moments of inertia, etc. Furthermore, the applications presented during the study of a particular subject are applications emphasizing this idea only. For example, while studying differential equations, applications studied are those to which differential equations are applicable with a minimum of emphasis on other mathematical ideas. Thus, different applications are pertinent while studying in different areas of mathematics; however, it is important to realize that, when working on a practical problem, it is quite probable that many different mathematical ideas might possibly be brought together while solving a single problem. A prime example of this is an application concerning the analysis of electrical circuits.

For instance, consider the following electrical circuit:



*A paper presented at the 1965 National Convention of KME at Fort Collins, Colorado, April 25-26.

This circuit contains linear elements (constant resistances, constant inductances and constant capacitances) and is fed with a variable voltage. Kirchoff's laws are helpful in analyzing such circuits.

(1) His current law states that the excess of current flowing into a given region at any given time over the current flowing out at the same time is the time rate of increase of quantity of electricity within the region at that time. Therefore, if there is no accumulation of electricity within a given region, current flowing into this region equals current flowing out of it.

(2) His electromotive force law states that when several constant elements, resistance R, inductance L, and capacitance C are connected in series, and when an instantaneous current i is flowing in them, there is impressed in the direction of i at the terminals of this series, from a source of power external to the elements, a difference of potential e such that $e = L\frac{di}{dt} + Ri + \int_C idt$ where $\int_C idt$ represents the total quantity of electricity on the capacitor considered.

With the substitution of q for $\int idt$, $\frac{dq}{dt}$ for i and $\frac{d^2q}{dt^2}$ for $\frac{di}{dt}$, this equation can also be written as $e = L\frac{d^2q}{dt^2} + \frac{Rdq}{dt} + \frac{q}{C}$.

Consider the circuit and the assigned direction of the current as shown above. Applying Kirchoff's current law one has the following equations: $i_1 = i_2 + i_3$. Substituting $\frac{dq}{dt}$ for i, one then has $\frac{dq_1}{dt} = \frac{dq_2}{dt} + \frac{dq_3}{dt}$. Applying the second form of Kirchoff's electromotive force law to the loop containing R_1, L_1, C, and the voltage source, one obtains a second order differential equation which is:

$$L_1\frac{d^2q_1}{dt^2} + R_1\frac{dq_1}{dt} + \frac{q_2}{C} = \sin wt$$

Doing the same to the loop containing R_2, L_2, and C one has:

$$L_2\frac{d^2q_3}{dt^2} + R_2\frac{dq_3}{dt} - \frac{q_2}{C} = 0$$

This gives a system of three differential equations in three unknowns. The object, therefore, is to solve for these unknowns.

Laplace Transforms may now be applied to such a system of differential equations. One will remember that the use of the Laplace Transforms is somewhat similar to the use of logarithms. In the use of logarithms, the elements are actually transformed to find their equivalents in a different system, the operations are then performed in this system, and finally this result is transformed back to the original system by taking the antilogarithm. The benefit being that the operations are simpler in the transformed system. The use of Laplace Transforms is analagous to this procedure. A function is transformed to obtain a function in p, a dummy variable, by the relation: $F(p) = \int_0^\infty e^{-pt}f(t)dt$ where $f(t)$ is the function to be transformed. In a system of equations, all variables are transformed and then a solution is found in this simplified system. The inverse transform is then taken giving the solution for each of the variables.

Consider now the system of equations obtained by applying Kirchoff's Laws to the electrical circuit. The system is:

$$(1)\ \frac{dq_1}{dt} - \frac{dq_2}{dt} - \frac{dq_3}{dt} = 0$$

$$(2)\ L_1\frac{d^2q_1}{dt^2} + R_1\frac{dq_1}{dt} + \frac{q_2}{C} = \sin wt$$

$$(3)\ L_2\frac{d^2q_3}{dt^2} + R_2\frac{dq_3}{dt} - \frac{q_2}{C} = 0$$

One can transform each of the terms of the equations and use the relation that the sum of the transforms equals the transform of their sum to hold the equality relation. In order to transform the system of equations one needs the transform of the derivative. This relation may be derived as follows:

Let D denote $\frac{d}{dt}$ and $T\{f(t)\} = F(P)$. Then by definition

$$T\{Df(t)\} = \int_0^\infty e^{-pt} Df(t)dt.$$

Integrating by parts where

$$u = e^{-pt} \qquad dv = Df(t)dt$$
$$du = -pe^{-pt}dt \qquad v = f(t)$$

one has

$$\int_0^\infty e^{-pt}Df(t)dt = \Big[e^{-pt}f(t)\Big]_0^\infty + p\int_0^\infty e^{-pt}f(t)dt$$

But

$$\Big[e^{-pt}f(t)\Big]_0^\infty = -f(0)$$

And

$$p\int_0^\infty e^{-pt}f(t)dt = pF(p)$$

Hence

$$T\{Df(t)\} = pF(p) - f(0)$$

All transforms used in this paper may be derived in a similar manner. In using this transform it is necessary that initial conditions be given. The circuit was analyzed under the following initial conditions:

$$q_1 = 0,\ F(q_1) = 0;\ \text{when } t = 0$$
$$q_2 = 0;\ \text{when } t = 0$$
$$q_3 = 0,\ F(q_3) = 0;\ \text{when } t = 0$$

Denote the transform of a variable by a capital letter in a manner such that the transforms of q_1, q_2, and q_3 are Q_1, Q_2 Q_3, respectively. The inverse transforms of Q_1, Q_2, and Q_3 will then, of course, be q_1, q_2, and q_3, respectively. Transforming equation (1), we then have $Q_1 - Q_2 - Q_3 = 0$.

In order to transform equation (2) we also need the relation that $T\{\sin at\} = \frac{a}{p^2 + a^2}$. The transformed equation is then:

$$L_1p^2Q_1 + R_1pQ_1 + \frac{1}{C}Q_2 = \frac{w}{p^2 + w^2} \qquad (12)$$

Transforming equation (3) one has $L_2p^2Q_3 + R_2pQ_3 - \frac{1}{C}Q_2 = 0$ (13)

The problem now becomes one of solving for Q_1, Q_2, and Q_3 in terms of p.

The solution of this system could have been accomplished by several methods, however the method utilized is a variation of the single division scheme, which is essentially a systematic method for the elimination of variables from a system of simultaneous equations. First construct a table consisting of four columns and three rows. The first column will be used for the coefficients of the first unknown; the second for the coefficients of the second unknown; the third for the coefficients of the third unknown and the fourth for the constant term. Note that three rows are needed to contain the coefficients from each of the three equations.

1	-1	-1	0
$L_1p^2 + R_1p$	$\frac{1}{C}$	0	$\frac{w}{p^2 + w^2}$
0	$-\frac{1}{C}$	$L_2p^2 + R_2p$	0

The table is extended by dividing the first row by the element appearing in the first row and first column. Then by taking combinations of the rows one gets the elements (except in Row 1) in the first column to be zero, which amounts to the elimination of Q_1 from two of the equations.

Applying these ideas to the table one sees the first element in the first row is already one, therefore it is not necessary to divide the first row by the first element.

Row one is now multiplied by $L_1p^2 + R_1p$ and subtracted from row two giving a zero in the first column second row and the other elements are as shown in row four of the figure below. The first element in the third row is already zero therefore a similar operation need not be applied. Using this single division scheme, it is possible to interchange rows whenever convenient without affecting the solution of the system. Since it was convenient row five was multiplied by $(-C)$ and interchanged with row four giving rows six and seven. Row six is now multiplied by $(\frac{1}{C} + L_1p^2 + R_1p)$ and subtracted from row seven giving row eight. Row eight actually represents the equation,

$$[L_1p^2 + R_1p + (CL_2p^2 + CR_2p)(\frac{1}{C} + L_1p^2 + R_1p)]Q_3 = \frac{w}{p^2 + w^2}$$

Q_1	Q_2	Q_3	C
1	-1	-1	0
$L_1p^2 + R_1p$	$\frac{1}{C}$	0	$\frac{w}{p^2 + w^2}$
0	$-\frac{1}{C}$	$L_2p^2 + R_2p$	0
	$\frac{1}{C} + L_1p^2 + R_1p$	$L_1p^2 + R_1p$	$\frac{w}{p^2 + w^2}$
	$-\frac{1}{C}$	$L_2p^2 + R_2p$	0
	1	$-CL_2p^2 - CR_2p$	0
	$\frac{1}{C} + L_1p^2 + R_1p$	$L_1p^2 + R_1p$	$\frac{w}{p^2 + w^2}$
		$(L_1p^2 + R_1p) + (CL_2p^2 + CR_2p) \cdot \left(\frac{1}{C} + L_1p^2 + R_1p\right)$	$\frac{w}{p^2 + w^2}$

Dividing through by the coefficient of Q_3 gives the solution for Q_3

which is:

$$Q_3 = \frac{w}{(p^2 + w^2)(p)(L_2p + R_2 + CL_1L_2p^3 + CL_1R_2p^2 + CL_2R_1p^2 + CR_1R_2p + L_1p + R_1)}$$

The equation obtained from row six is:

$Q_2 - (CL_2p^2 + CR_2p)\, Q_3 = 0$ or $Q_2 = (CL_2p^2 + CR_2p)Q_3$.

Now substituting for Q_3 gives:

$$Q_2 = \frac{Cw(L_2p + R_2)}{(p^2 + w^2)(L_2p + R_2 + CL_1L_2p^3 + CL_1R_2p^2 + CL_2R_1p^2 + CR_1R_2p + L_1p + R_1)}$$

Finally the equation obtained from row one is: $Q_1 - Q_2 - Q_3 = 0$ or $Q_1 = Q_2 + Q_3$. It is convenient to leave the solution for Q_1 in this form.

After the solution of the system for Q_2 and Q_3, all that remains is to take the inverse transform of these two expressions. One can reduce these expressions to partial fractions. The computer proves helpful in this phase of the solution as well as in taking the inverse transform. Consequently, the general solution was dropped at this point. The data and numerical results obtained are not included in the paper as it does not further develop any mathematical ideas and the central theme revolves about the means of obtaining rather than actual results. The computer solution is developed for specific values of R_1, R_2, L_1, L_2, C, and w. The computer program is developed such that R_1, R_2, L_1, L_2, C, and w may be varied giving a solution for each set of data. This enables one to analyze the circuit and study the effects produced by different variations of the data.

In order to take the inverse transforms for q_2 and q_3, it would be helpful to express Q_2 and Q_3 in the following form:

$$Q_2 = \frac{Ap + B}{p^2 + w^2} + \frac{C}{p + r} + \frac{Dp + E}{p^2 + bp + c}$$

$$Q_3 = \frac{A_1p + B_1}{p^2 + w^2} + \frac{C_1}{p} + \frac{D_1}{p + r} + \frac{E_1p + F_1}{p^2 + bp + c}$$

Recall that Q_2 and Q_3 are in the form

$$Q_2 = \frac{Cw(L_2p + R_2)}{(p^2 + w^2)(a_1p^3 + b_1p^2 + c_1p + d_1)}$$

$$Q_3 = \frac{w}{(p^2 + w^2)(p)(a_1p^3 + b_1p^2 + c_1p + d_1)}$$

where a_1, b_1, c_1, and d_1 are constants to be evaluated. The values of R_1, R_2, L_1, L_2, C, and w were read into the computer. The computer then evaluated the coefficients of p^3, p^2, and p and the constant terms.

Consider the factor of the denominator of Q_3 which can now be written (with evaluated coefficients) in the form of $(a_1p^3 + b_1p^2 + c_1p + d_1)$. Both the numerator and denominator of Q_3 may then be divided by a_1 giving the third degree polynomial the form of $(p^3 + b_2p^2 + c_2p + d_2)$ where $b_2 = \frac{b_1}{a_1}$, $c_2 = \frac{c_1}{a_1}$ and $d_2 = \frac{d_1}{a_1}$. In order to break this third degree polynomial into factors, one can set it equal to zero, solve for the roots of the resulting cubic equation, and then put it into the desired form. The computer evaluated b_2, c_2, and d_2.

One has a cubic equation in the form of $p^3 + b_2p^2 + c_2p + d_2 = 0$. To solve this cubic equation, it is helpful to make the substitution $x = p - \frac{b_2}{3}$. This then gives a cubic in the form of $x^3 + mx + n = 0$ where $m = \frac{1}{3}(3c_2 - b_2^2)$ and $n = \frac{1}{27}(2b_2^3 - 9b_2c_2 + 27d_2)$. The computer evaluated m and n.

The discriminant of this equation is given by $\frac{n^2}{4} + \frac{m^3}{27}$. If this discriminant is greater than zero, there will be one real root and two conjugate imaginary roots; if equal to zero, there will be three real roots of which at least two are equal; if less than zero, there will be three real and unequal roots.

If the discriminant was positive or zero, a form of Cardan's formulas was used to effect the solution. For such a solution, let

$$A = \sqrt[3]{\frac{-n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} \quad \text{and} \quad B = \sqrt[3]{\frac{-n}{2} - \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}.$$

Then the solutions for x are given by:

$$x = A + B, x = -\frac{A + B}{2} + \frac{A - B}{2}\sqrt{3}i,$$

and

$$x = -\frac{A+B}{2} - \frac{A-B}{2}\sqrt{3}i.$$

These roots of x are now transformed by the relation $p = x + \frac{b_2}{3}$ to give roots of the original cubic in p. These roots may now be written in the following form:

$$p = r_1, \quad p = r_2 + r_3 i, \quad p = r_2 - r_3 i$$

where

$$r_1 = \frac{b_2}{3} + A + B, \quad r_2 = \frac{b_2}{3} - \frac{A+B}{2}, \quad \text{and } r_3 = \frac{A-B}{2}.\sqrt{3}$$

The factor of the denominator originally given by $(p^3 + b_2p^2 + c_2p + d_2)$ can now be given as $(p - r_1)(p - r_2 + r_3i)(p - r_2 - r_3i)$. Multiplying out the factors containing imaginaries, one obtains $(p - r_1)(p^2 - 2r_2p + r_2^2 + r_3^2)$. Letting $r = -r_1$, $b = -2r_2$, and $c = r_2^2 + r_3^2$, this becomes $(p + r)(p^2 + bp + c)$. The computer evaluated the discriminant. If the discriminant was greater than zero, A and B were evaluated. Then, r_1, r_2, and r_3 were evaluated. From these values of r_1, r_2, and r_3; r, b, and c were evaluated. The computer punched out a data card which contained r, b, and c. The relations for all these evaluations are given above.

If the discriminant was negative the trigonometric solution of the cubic was used. In this solution the following formulas were utilized:

$$\cos o = \frac{-n}{2} \div \sqrt{\frac{-m^3}{27}}, \quad \text{then } x = 2\sqrt{\frac{-m}{3}}\cos o,$$

$$x = 2\sqrt{\frac{-m}{3}}\cos\left(\frac{o}{3} + \frac{2\pi}{3}\right), \text{ and } x = 2\sqrt{\frac{-m}{3}}\cos\left(\frac{o}{3} + \frac{4\pi}{3}\right)$$

If the discriminant was negative the computer evaluated $(\cos o)$ from the above relation. Then $(\tan o)$ was evaluated by the relation

$$\tan o = \sqrt{\frac{1}{(\cos o)^2} - 1}. \qquad \text{Let } k = \sqrt{\frac{1}{(\cos o)^2} - 1},$$

then $o = \tan^{-1}k$. The computer evaluated $\cos o$, k, and then

ϕ in that order. Next the computer evaluated x from the three appropriate formulas shown above. The transformation $p = x + \frac{b_2}{3}$ was then made to obtain the roots of the original equation. The factor $(p^3 + b_2p^2 + c_2p + d_2)$ of the denominators can now be written as $(p + r)(p + r_1)(p + r_2)$ where r, r_1, and r_2 are the roots of the above equation multiplied by -1. Then $(p + r_1)(p + r_2)$ can be written as $p_2 + bp + c$ where $b = r_1 + r_2$ and $c = r_1r_2$. After evaluating the three values of x, the computer transformed to get roots of p by adding $\frac{b_2}{3}$ to the roots of x. It then multiplied these roots by -1 to give r, r_1, and r_2. Next it computed b and c by the relations $b = r_1 + r_2$ and $c = r_1r_2$. It then punched out a data card containing r, b, and c in that order.

Q_2 and Q_3 are now in the following form:

$$Q_2 = \frac{(CwL_2p + CwR_2)/a_1}{(p^2 + w^2)(p + r)(p^2 + bp + c)}$$

$$Q_3 = \frac{w/a_1}{(p^2 + w^2)(p)(p + r)(p^2 + bp + c)}$$

where all constants have already been evaluated. Note that both numerators are divided by a_1. This is a result of dividing both the numerator and denominator by a_1 during the solution of the cubic equation.

The computer now evaluated k, G, and H where $k = w/a_1$, $G = \frac{CwL_2}{a_1}$, and $H = \frac{CwR_2}{a_1}$. It then punched out a data card containing these three values in the order given above.

The computer work is divided into three different programs due to the fact that there was not enough core storage to execute the entire program at once. The work preceding this point was all included in the first program. The first program punches out data cards which are used in the second and third programs. These data cards contain the values of k, G, H, r, b, and c as determined above.

Q_2 and Q_3 are now in the following form:

$$Q_2 = \frac{Gp + H}{(p^2 + w^2)(p + r)(p^2 + bp + c)}$$

$$Q_3 = \frac{k}{(p^2 + w^2)(p)(p + r)(p^2 + bp + c)}$$

Breaking these expressions into partial fractions of the form

$$Q_2 = \frac{Ap + B}{p^2 + w^2} + \frac{C}{p + r} + \frac{Dp + E}{p^2 + bp + c}$$

$$Q_3 = \frac{A_1p + B_1}{p^2 + w^2} + \frac{C_1}{p} + \frac{D_1}{p + r} + \frac{E_1p + F_1}{p^2 + bp + c}$$

greatly facilitates inverse transformation. In order to accomplish this, the following relations are used:

$$(1)\ \frac{Ap + B}{p^2 + w^2} + \frac{C}{p + r} + \frac{Dp + E}{p^2 + bp + c} = \frac{Gp + H}{(p^2 + w^2)(p + r)(p^2 + bp + c)}$$

$$(2)\ \frac{A_1p + B_1}{p^2 + w^2} + \frac{C_1}{p} + \frac{D_1}{p + r} + \frac{E_1p + F_1}{p^2 + bp + c} = \frac{k}{(p^2 + w^2)(p)(p + r)(p^2 + bp + c)}$$

The object of the second program is to solve for A_1, B_1, C_1, D_1, E_1, and F_1, as shown in relation (2). To begin, both sides were multiplied by $(p^2 + w^2)(p)(p + r)(p^2 + bp + c)$ giving in factored form: $(A_1p + B_1)(p)(p + r)(p^2 + bp + c) + C_1(p^2 + w^2)(p + r)(p^2 + bp + c) + D_1(p^2 + w^2)(p)(p^2 + bp + c) + (E_1p + F_1)(p^2 + w^2)(p)(p + r) = k$.

Multiplying this expression out and grouping coefficients, one obtains: $(A_1 + C_1 + D_1 + E_1)p^5 + (A_1r + B_1 + A_1b + C_1r + C_1b + D_1b + F_1 + E_1r)p^4 + (B_1r + A_1rb + B_1b + A_1c + C_1w^2 + C_1br + C_1c + D_1w^2 + D_1c + F_1r + E_1w^2)p^3 + (B_1br + A_1cr + B_1c + C_1w^2r + C_1bw^2 + C_1cr + D_1bw^2 + F_1w^2 + E_1rw^2)p^2 + (B_1rc + C_1bw^2r + C_1cw^2 + D_1cw^2 + F_1rw^2)p + C_1cw^2r = k$. Since this is an identity, coefficients of equivalent powers of p on each side of the equality may be equated. This gives a linear system of six simultaneous equations in six unknowns. The resulting system is:

(1) $A_1 + C_1 + D_1 + E_1 = 0$

(2) $(b + r)A_1 + B_1 + (b + r)C_1 + bD_1 + rE_1 + F_1 = 0$

(3) $(c + br)A_1 + (b + r)B_1 + (w^2 + br + c)C_1$
$+ (w^2 + c)D_1 + w^2E_1 + rF_1 = 0$

(4) $crA_1 + (br + c)B_1 + (w^2r + bw^2 + cr)C_1$
$+ bw^2D_1 + rw^2E_1 + w^2F_1 = 0$

(5) $rcB_1 + (bw^2r + cw^2)C_1 + cw^2D_1 + rw^2F_1 = 0$

(6) $cw^2rC_1 = k$

A working knowledge of simple matrix methods and matrix terminology shall be assumed at this point. The system of equations may therefore be written in the following matrix form:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ b + r & 1 & b + r & b & r & 1 \\ c + rb & b + r & w^2 + br + c & w^2 + c & w^2 & r \\ cr & br + c & w^2r + bw^2 + cr & bw^2 & rw^2 & w^2 \\ 0 & rc & bw^2r + cw^2 & cw^2 & 0 & rw^2 \\ 0 & 0 & cw^2r & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ E_1 \\ F_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ k \end{bmatrix}$$

Let M represent the matrix of coefficients (on the left), let X represent the matrix of unknowns (in the middle), and let C represent the matrix of constants (on the right). This system can now be written as: $MX = C$. The problem now becomes one of finding M^{-1}. If such a matrix exists it may be multiplied by both sides of the equation in the following manner.

$$M^{-1}MX = M^{-1}C$$

$$\text{Since } M^{-1}M = U$$

$$UX = M^{-1}C$$

$$X = M^{-1}C$$

Denote the matrix $M^{-1}C$ by N. It is readily seen that both X and N will be of order (6, 1). The object now becomes one of finding M^{-1} and multiplying it by C in the above manner. This equation may now be written as follows:

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \\ E_1 \\ F_1 \end{bmatrix} = \begin{bmatrix} n_{1,1} \\ n_{2,1} \\ n_{3,1} \\ n_{4,1} \\ n_{5,1} \\ n_{6,1} \end{bmatrix}$$

Then, from matrix equality, the solution for A_1 will be the element $n_{1,1}$ of the product, the solution for B_1 will be the element $n_{2,1}$ of the product, etc.

The matrix M, was inverted to find M^{-1} by the use of the Cayley-Hamilton Theorem. That is, M^{-1} is given by the following equation:

$$M^{-1} = -\frac{1}{a_n}\left(M^{n-1} + a_1M^{n-2} + a_2M^{n-3} + \cdots + a_{n-1}U\right)$$

The coefficients a_1 through a_n are given by Bôcher's formulas in the following manner:

$$a_1 = -s_1$$

$$a_2 = -\tfrac{1}{2}(a_1s_1 + s_2)$$

$$a_3 = -\tfrac{1}{3}(a_2s_1 + a_1s_2 + s_3)$$

$$\text{- - -} \qquad \text{- - -} \qquad \text{- - -}$$

$$a_n = -\frac{1}{n}\left(a_{n-1}s_1 + a_{n-2}s_2 + \cdots + a_1s_{n-1} + s_n\right)$$

where s_n is the trace of the n^{th} power of the matrix in question. After the matrix inversion and multiplication of $M^{-1}C$, one now has the roots of the six simultaneous equations; that is, one now has A_1, B_1, C_1, D_1, E_1, F_1. The next thing to do is to take the inverse transform of the equation.

$$Q_3 = \frac{A_1p + B_1}{p^2 + w^2} + \frac{C_1}{p} + \frac{D_1}{p + r} + \frac{E_1p + F_1}{p^2 + bp + c}$$

It was observed at this time that, for the range of data to be run through the computer, c would always be greater than $\left(\frac{b}{2}\right)^2$ in the expression $p^2 + bp + c$; therefore, the computer was programmed to read out the inverse transformation for this situation only. After the solution for A_1, B_1, C_1, D_1, E_1, and F_1, it is possible to read the inverse transforms directly from a table. Thus, using the following relations,

$$T^{-1}\left\{\frac{p}{p^2 + a^2}\right\} = \cos at,\ T^{-1}\left\{\frac{a}{p^2 + a^2}\right\} = \sin at$$

$$T^{-1}\left\{\frac{a}{p}\right\} = a,\ T^{-1}\left\{\frac{1}{p + a}\right\} = e^{-at}$$

$$T^{-1}\left\{\frac{p}{p^2 + bp + c}\right\} = \frac{-\sqrt{c}}{d} e^{-\frac{b}{2}t} \sin(dt - \theta) \quad \left\{ \begin{array}{l} \text{if } c > \left(\frac{b}{2}\right)^2 \\ d = \sqrt{c - \left(\frac{b}{2}\right)^2} \\ \tan \theta = \frac{2d}{b} \end{array} \right.$$

$$T^{-1}\left\{\frac{1}{p^2 + bp + c}\right\} = \frac{e^{-\frac{b}{2}t}}{d} \sin dt$$

the inverse transform for q_3 gives:

$$q_3 = A_1 \cos wt + \frac{B_1}{w} \sin wt + C_1 + D_1 e^{-rt} + \frac{-E_1\sqrt{c}}{d} +$$

$$e^{-\frac{b}{2}t} \sin(dt - \theta) + \frac{F_1 e^{-\frac{b}{2}t}}{d} \sin dt$$

In essence, the second computer program did this. It first read in the values of k, G, H, r, b, and c as determined by the first computer program. After reading in these values, it then computed the coefficients of the 6 x 6 matrix storing these values in matrix form. It then determined the inverse matrix. This involved raising the matrix to the 2nd through 6th powers and storing each of these. The trace of each was then determined and a_1 through a_6 were computed. The inverse matrix was then determined in accord with the Cayley-Hamilton Theorem. Next, this inverse coefficient matrix was multiplied by the matrix of constant terms and A_1, B_1, C_1, D_1, E_1, and F_1 were set equal to the proper elements of this product matrix. It was then simply a matter of writing out the correct values in the correct position and the solution for q_3 was complete.

The third computer program was built along exactly the same lines as the second computer program. The reduction to partial fraction involved only five unknowns: therefore, only a 5 x 5 matrix was generated. This matrix solution was done in exactly the same manner as the other. The equation involving the partial fraction is as follows:

$$\frac{Ap + B}{p^2 + w^2} + \frac{C}{p + r} + \frac{Dp + E}{p^2 + bp + c} = \frac{Gp + H}{(p^2 + w^2)(p + r)(p^2 + bp + c)}$$

Note that the factor of p does not appear in the denominator making the reduction to partial fractions a little simpler. Upon multiplying through by the denominator and equating coefficients of like powers of p as done before, one has a system of five simultaneous equations in five unknowns. This equation in matrix form is given by:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ b + r & 1 & b & r & 1 \\ br + c & b + r & c + w^2 & w^2 & r \\ cr & br + c & bw^2 & w^2r & w^2 \\ 0 & cr & cw^2 & 0 & w^2r \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ G \\ H \end{bmatrix}$$

The further solution of q_2 shall not be discussed as it was done in the same manner as that for q_3.

The inverse transformation for q_2 gives:

$$q_2 = A \cos wt + \frac{B}{w} \sin wt + Ce^{-rt} +$$

$$D\frac{\sqrt{c}}{d}e^{-\frac{b}{2}t} \sin (dt - \emptyset) + E \frac{e^{-\frac{b}{2}t}}{d} \sin dt$$

After the solution for q_3 and q_2, q_1 can easily be found by the relation $q_1 = q_2 + q_3$. Once q_1, q_2, and q_3 are known it is easy to find i_1, i_2, and i_3 by the relation $i = \frac{dq}{dt}$. At this point the mathematical solution is complete. The reader may note that there were many approaches that could possibly have been taken in solving this electrical circuit problem. Probably, some would have been somewhat simpler and quicker, however, this approach is interesting in that the solution incorporates many mathematical ideas.

Recall that to solve this electrical circuit, various mathematical ideas were employed. First, Kirchoff's laws were applied to the circuit to obtain a system of second order differential equations. The Laplace Transform was applied to this system of differential equations reducing it to a system of algebraic equations. Methods of matrix algebra were applied to obtain the solution of this system. The IBM 1620 computer was employed to affect the inverse transformation.

Writing the computer program involved such ideas from the theory of equations as the use of Cardan's formulas or the trigonometric method to solve the cubic equation, and such ideas from matrix algebra as solving a system of six equations in six unknowns by use of a matrix equation. The solution of this electrical circuit has aptly demonstrated how many mathematical ideas can be used in the solution of a single problem.

"A great Discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery."

—GEORGE POLYA

Factoring a Polynomial of the Fourth Degree

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Faculty, Colby College

We shall show that a polynomial of the fourth degree with integral coefficients is factorable if certain relations exist in its coefficients.

Theorem: Given a polynomial

$$Ax^4 + Bx^3 + Cx^2 + Dx + E.$$

Let

$$B = p + q, D = p' + q', C = l + m + k \qquad \text{(i)}$$

such that

$$\frac{pq}{A} = \frac{p'q'}{E} = k \qquad \text{(ii)}$$

and

$$\frac{pp'}{k} = l, \ \frac{qq'}{k} = m. \qquad \text{(iii)}$$

Show that the polynomial is factorable.

Proof:

$$Ax^4 + Bx^3 + Cx^2 + Dx + E$$

$$= Ax^4 + (p + q)x^3 + (l + m + k)x^2 + (p' + q')x + E$$

from (i)

$$= Ax^4 + px^3 + lx^2 + qx^3 + kx^2 + p'x + mx^2 + q'x + E$$

$$= \left(\frac{pq}{k}x^4 + px^3 + \frac{pp'}{k}x^2\right) + \left(qx^3 + kx^2 + p'x\right) + \left(\frac{qq'}{k}x^2 + q'x + \frac{p'q'}{k}\right)$$

from (ii) and (iii)

$$= \frac{p}{k}x^2\left(qx^2 + kx + p'\right) + x\left(qx^2 + kx + p'\right) + \left(qx^2 + kx + p'\right)\frac{q'}{k}$$

$$= \left(qx^2 + kx + p'\right)\left(\frac{p}{k}x^2 + x + \frac{q'}{k}\right).$$

Hence the theorem.

We shall now illustrate the usefulness of this method by solving a couple of biquadratic equations.

Example I. Solve the equation

$$x^4 + 10x^3 + 35x^2 + 50x + 24 = 0.$$

Solution: Let $10 = 5 + 5$ and $50 = 30 + 20$.

Since

$$\frac{5 \times 5}{1} = 25 = \frac{30 \times 20}{24}$$

we can hopefully move forward and split 35 into three parts, one of which is 25. The sum of the other two terms is therefore 10. Now 10 is to be broken into two parts whose product is $24 \times 1 = 24$ (as conditions (ii) and (iii) together imply $AE = lm$). The two parts are obviously 6 and 4. Thus the given equation

$$x^4 + 10x^3 + 35x^2 + 50x + 24 = 0$$

can be written as

$$x^4 + 5x^3 + 5x^3 + 4x^2 + 6x^2 + 25x^2 + 30x + 20x + 24 = 0$$

$$x^2(x^2 + 5x + 6) + 5x(x^2 + 5x + 6) + 4(x^2 + 5x + 6) = 0$$

$$(x^2 + 5x + 6)(x^2 + 5x + 4) = 0$$

$$(x + 2)(x + 3)(x + 4)(x + 1) = 0$$

$$x = -1, -2, -3, -4.$$

Example II. Solve the equation

$$x^4 - 8x^2 - 24x + 7 = 0.$$

Solution: First of all we shall write the equation without any of its terms missing as follows:

$$x^4 + 0x^3 - 8x^2 - 24x + 7 = 0$$

$$0 = 4 + (-4), -24 = -28 + 4.$$

Since

$$\frac{4 \times (-4)}{1} = -16 = \frac{(-28)(4)}{7},$$

we can hopefully move forward. Now -16 must be one of the terms in which -8 is to be broken. The sum of the remaining two terms will therefore be 8. Split 8 into two parts whose product is $7 \times 1 = 7$. The two parts obviously are 7 and 1.

Hence the given equation

$$x^4 + 0x^3 - 8x^2 - 24x + 7 = 0$$

can now be written as

$$x^4 + 4x^3 - 4x^3 + 7x^2 + x^2 - 16x^2 - 28x + 4x + 7 = 0$$

$$x^2(x^2 + 4x + 7) - 4x(x^2 + 4x + 7) + (x^2 + 4x + 7) = 0$$

$$(x^2 + 4x + 7)(x^2 - 4x + 1) = 0$$

$$x = -2 \pm \sqrt{-3},\ 2 \pm \sqrt{3}.$$

REFERENCE

Luthar, R. S. "Solution of the Cubic Equation by Inspection," *Mathematics Student,* Vol. XIV, Nos. 1, 2, March, June, 1946.

(Continued from page 85)

Given the directrix circle with center at 0 and any point S on the circle and any point A within the circle.

$$d(S,R) = d(A,R)$$

$$d(S,R) = r - d(R,0) \qquad r = 0S$$

$$d(A,R) = r - d(R,0)$$

$$(A,R) + d(R,0) = r$$

Therefore, a "parabola" within a circular directrix determines an ellipse with focal points at A and 0, and with a major axis of r.

The Problem Corner

EDITED BY H. HOWARD FRISINGER

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before October 1, 1966. The best solutions submitted by students will be published in the Fall 1966 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor H. Howard Frisinger, Colorado State University, Fort Collins, Colorado.

PROPOSED PROBLEMS

191. *Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.*

All irrational numbers can be expressed in terms of an infinite continued fraction. For example,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

Show how to express the square root of all positive integers in a general continued fraction formula, in which the formula consists entirely of integers. In other words, express $\sqrt{N}$ as an infinite continued fraction, where N is a positive integer.

192. *Proposed by LeRoy Simmons, Washburn University, Topeka, Kansas.*

Find a positive integer x such that $ax + b(x + 1)$ will be equal to *all* integers greater than or equal to 110, but will not equal 109; where $a,b \epsilon [0,1,2,3, \ldots\ldots, \infty]$.

193. *Proposed by Patricia Robaugh, Duquesne University, Pittsburgh, Pennsylvania.*

To find the product of 13 and 35, first list the powers of 2 which are less than or equal to 13. In a second column, successively double 35. Choose the numbers in the left column whose sum is 13. Add the corresponding number in the right column for the product.

*1	35*
2	70
*4	140*
*8	280*

$$35 + 140 + 280 = 455$$

Prove that this method of multiplication gives correct results in all cases.

194. *Proposed by E. R. Deal, Colorado State University, Fort Collins, Colorado.*

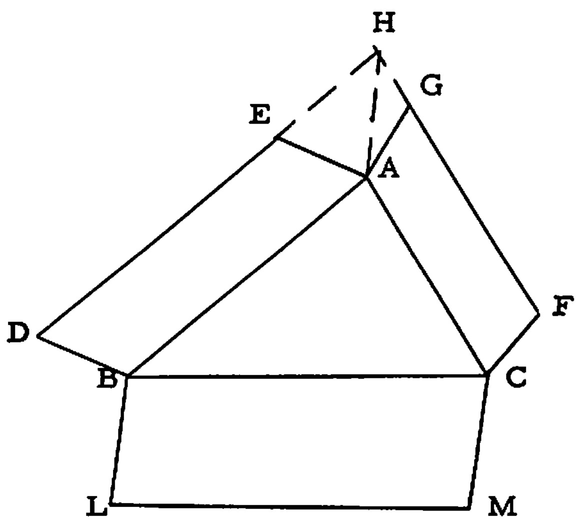
Fill in the missing digits

```
               x x 8
       ________________
 x x x | x x x x x x x
         x x x 5
         -------
          x x x x
            9 x x
           ------
            x x x
            x x x
           ------
```

195. *Proposed by the Editor.*

Given any $\triangle$ *ABC*, any two parallelograms *DBAE* on *AB* and *ACFG* on *AC*. Let *DE* and *FG* meet in *H* and draw *BL* and *CM* equal and parallel to *AH*.

Prove area *ABDE* + area *ACFG* = area *BCML*.



SOLUTIONS

186. *Proposed by Fred W. Lott, State College of Iowa, Cedar Falls, Iowa.*

Prove that the square of an integer ends in 6 if and only if the ten's digit of the square is odd.

Solution by G. Nicholas Lauer, Central Michigan University, Mount Pleasant, Michigan.

Any integer may be broken up into the following form:

$$(\cdots d(1000) + c(100) + b(10) + a(1))$$

The square of an integer is just

$$(\cdots d(1000) + c(100) + b(10) + a)^2$$

a,b,c,d are all < 10 and $\geqq 0$ and integers. In this expression, the only terms affecting the ten's and unit's place will be.

$$2(ab)(10) + a^2$$

If the expression $2(ab)(10) + a^2$ ends in 6, $a = 4$ or 6 and $a^2 = 16$ or 36. Now $2(ab)$ is even. Since $2(ab)$ is even and the ten's place of a^2 is odd (1 or 3) when the expression ends in 6, the ten's place is always odd.

Next, assume that the ten's place is odd. Again we use the expression: $2ab(10) + a^2$. As the ten's place is odd and since $2ab$ is even, a^2 must have a ten's place and it must be odd. This means $a = 4, 5, 6, 7, 8$, or 9 and a^2 is odd. The squares of all these have an even ten's place except the 4 and 6. Therefore, $a = 4$ or 6, and the squares of these two numbers end in 6.

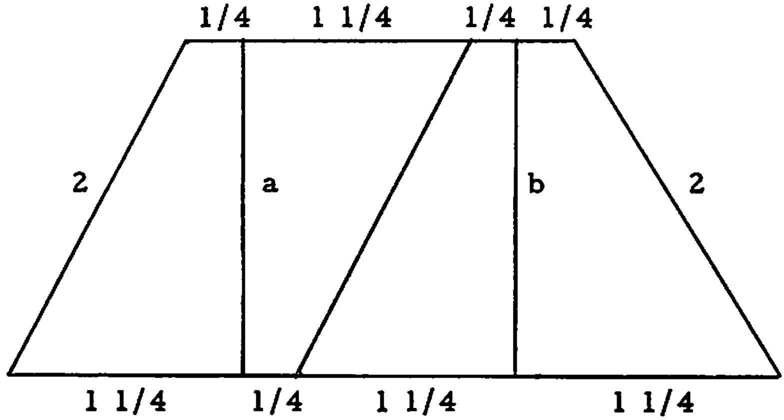
Also solved by David C. Lantz, Kutztown State College, Kutztown, Pennsylvania: Charles Parry, State University College, Oswego, New York.

187. *Proposed by E. R. Deal, Colorado State University, Fort Collins, Colorado.*

Break the isosceles trapezoid up into four congruent parts.

Solution by Stephanie Nagurny, Immaculate College, Pennsylvania.

Let the top line of a trapezoid be divided into a ratio of $\frac{3}{4}:1\frac{3}{4}:\frac{1}{4}:\frac{1}{4}$. Let the base line be divided into $1\frac{1}{4}:\frac{3}{4}:1\frac{1}{4}:1\frac{1}{4}$. The diagonal line in the center is parallel to the left side of the trapezoid and of the same length. Lines *a* and *b* are perpendicular to the top and bottom lines.



Also solved by David Lantz, Kutztown State College, Kutztown, Pennsylvania; G. Nicholas Lauer, Central Michigan University, Mount Pleasant, Michigan; William H. Mikesell, Indiana University, Indiana, Pennsylvania; Philip Naverstick, William Jewell College, Liberty, Missouri; Larry Prilliman, Emporia State Teachers College, Emporia, Kansas; Martin M. Simons, Washburn University, Topeka, Kansas; Charles Parry, State University College, Oswego, New York; Patricia Robaugh, Duquesne University, Pittsburgh, Pennsylvania.

188. *Proposed by T. L. Zimmerman, Kansas State Teachers College, Emporia, Kansas.*

Prove that $\sqrt{p}$ is an irrational number if p is a prime number.

Solution by Charles Parry, University College, Oswego, New York.

If $\sqrt{p}$ is rational then there exists integers m and n such that $(m,n) = 1$ and $m/n = \sqrt{p}$ or $m^2/n^2 = p$. Thus $pn^2 = m^2$ so p divides m^2 so p must also divide m. (If p does not divide m, then by the fundamental theorem p does not divide m^2.) Thus $m = kp$ for some integer k. From the above equation we get $pn^2 = (kp)^2 = k^2p^2$, dividing by p, $n^2 = k^2p$ so p divides n also, contradicting the assumption $(m,n) = 1$. Thus m and n can not exist so $\sqrt{p}$ is irrational.

Also solved by David C. Lantz, Kutztown State College, Kutztown, Pennsylvania; Anne Marie Victor, Marywood College, Scranton, Pennsylvania.

189. *Proposed by Howard Frisinger, Colorado State University, Fort Collins, Colorado.*

Consider the table:

1	0 + 1
2 + 3 + 4	1 + 8
5 + 6 + 7 + 8 + 9	8 + 27
10 + 11 + 12 + 13 + 14 + 15 + 16	27 + 64

Express the general law suggested by this table and prove it.

Solution by T. E. Berg, Illinois State University, Normal, Illinois.

A general law generating respective rows of the above table can be expressed as follows:

$$(n^2 - 2n + 2) + (n^2 - 2n + 3) + (n^2 - 2n + 4) \cdots + [(n^2 - 2n + 2) + (2n - 2)] = (n - 1)^3 + n^3$$

For $n = 1, 2, 3, \cdots$ we will get the respective rows of the table. Now we must show that the above equality holds: There are $2n - 1$ elements on the left hand side of the equality. We can add the elements on the left hand side of the proposed equality by regrouping them, i.e., we want to add:

$$[n^2 + n^2 + n^2 + \cdots + n^2]^* + [-2n - 2n - 2n - \cdots - 2n]^* + (2 + 3 + 4 + \cdots + 2n) \overset{?}{=} (n - 1)^3 + n^3$$

*There are two $2n - 1$ terms in the expression.

$$(2n - 1)(n^2) + (-2n)(2n - 1) + \frac{(2n)(2n + 1)}{2} - 1 =$$

$$2n^3 - n^2 - 4n^2 + 2n + 2n^2 + n - 1 =$$

$$2n^3 - 3n^2 + 3n - 1 = (n - 1)^3 + n^3$$

Hence, the equality holds.

Also solved by David C. Lantz, Kutztown State College, Kutztown, Pennsylvania; G. Nicholas Lauer, Central Michigan University, Mount Pleasant, Michigan.

190. *Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.*

Find three right triangles such that the lengths of all sides are integral and the lengths of the legs are consecutive integers.

Solution by David C. Lantz, Kutztown State College, Kutztown, Pennsylvania.

All integral-sided right triangles can be found by the formula

$$** \quad (2uv)^2 + (u^2 - v^2) = (u^2 + v^2)^2$$

where each quantity represents a side of the triangle, i.e.,

$$a = 2uv, b = u^2 - v^2 \text{ and } c = u^2 + v^2$$

To find legs which were consecutive integers, I set

$$2uv = u^2 - v^2 + 1$$

and by the quadratic formula solved for u.

$$u = v \pm \sqrt{2v^2 - 1}$$

For u to be an integer, the quantity $2v^2 - 1$ had to be a square integer. This is true when v is 1 (so that u is 2) or 5 (so that u is 2* or 12). *[This should be -2, to get -20, -21, but as all terms are squared in **, we can take the absolute values.] The resulting triangles are (3, 4, 5), (20, 21, 29), and (119, 120, 169).

Also solved by Charles Parry, University College, Oswego, New York; Martin M. Simons, Washburn University, Topeka, Kansas.

Installation of New Chapters

EDITED BY SISTER HELEN SULLIVAN

PENNSYLVANIA EPSILON CHAPTER

Kutztown State College, Kutztown, Pennsylvania

Pennsylvania Epsilon Chapter was installed on April 3, 1965, by Dr. Loyal F. Ollmann, National President and Head of the Department of Mathematics at Hofstra University, Hempstead, New York. The initiation was held prior to the banquet attended by the charter members, members of the mathematics staff, their spouses, and invited guests. President and Mrs. I. L. deFrancesco were honored guests. Dr. deFrancesco extended greetings of the college to the group and commended the Mathematics Society on its achievement. Dr. Ollmann gave an interesting talk on the history and purposes of Kappa Mu Epsilon.

Five faculty members and twenty-nine students are charter members. They are: Ellen Blose, Beverly Bouchat, William Csencsits, J. Dwight Daugherty, Caroline Deisher, Concetta DiLazzaro, Edward W. Evans, Joseph Fassman, William Feichtl, James Gibson, Thomas Gitch, Floyd Godshalk, Jr., John D. Gorman, Terrance Grady, John Herbine, Edward Hetrick, June Hower, Caroline Kehs, Paul A. Knedler, Carl Kostival, Jean Mozeko, Nancy Pankowski, S. Jane Portz, Linda Rothenberger, Melvin Rose, Edward Schnable, Donald Schneider, Earl Swartz, Ivy Silfies, Donald Smith, Robert Sterner, Eleanor Trout, Beverly Wetmore, and Jason W. White.

The officers of the chapter are:

President	Carl Kostival
Vice-President	Floyd Godshalk, Jr.
Recording Secretary	S. Jane Portz
Treasurer	Beverly Wetmore
Faculty Sponsor	Edward W. Evans
Corresponding Secretary	J. Dwight Daugherty

The Mathematics Society of Kutztown State College was organized in 1961 and has held monthly meetings regularly. Some of the topics discussed were: various aspects of infinity, linear programming, mathematical induction, topology, inversive geometry, mathematical computers, and the dynamic beauty of geometric forms. Kutztown State College is the first of the fourteen Pennsylvania State Colleges to install a chapter of Kappa Mu Epsilon.

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of *The Pentagon*. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor James P. Burling, State University College, Oswego, New York.

Differential Equations, Max Morris and Orley E. Brown. Fourth Edition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964, VI + 366 pp., $8.50.

Written in a careful, lucid style, this fourth edition of a well-known textbook continues to offer a noteworthy development of differential equations. In addition to customary materials there are new topics dealing with the Laplace transform, Fourier series, and the solution of partial differential equations . . . "a novel and effective treatment for regions with irregular boundaries," according to the authors. They use nodal points and Lagrangian approximations in this treatment; in the numerical solution of ordinary differential equations they use the methods of Milne and Runge-Kutta.

A characteristic of the present edition is the extensive space allotted to computational methods—sixty-five pages, almost a fifth of the book—whereas the Laplace transform occupies but twelve pages (excluding exercises).

Another characteristic is the abundance of well-chosen examples—many of them physical applications—worked out so that the student may reread the corresponding theory with increased understanding.

In many cases the authors have not merely provided the differential equation to which a problem gives rise, but have attempted to give the background of the problem as well.

Unlike some writers they have seen fit not to curtail or eliminate the presentation of differential operators and their inverses.

The reviewer observed one misprint: on page 123 an equality sign should replace the plus sign between the last two limiting forms in illustration 2 of L'Hospital's rule. The format and typography of the book are excellent.

Appended to the volume is a table of some hundred integrals as well as other useful tables of exponential, logarithmic, hyperbolic, and trigonometric functions. Answers are supplied for all exercises.

As a textbook this edition can be adapted to students with traditional mathematical background as well as to those with more sophisticated training and experience. All in all, at its level of rigor, the book maintains a commendable degree of merit.

—Raymond Huck
Marietta College

Ordinary Differential Equations, Philip Hartman, John Wiley & Sons, Inc., New York, 1964, 612 pp., $20.00.

Both the quantity and quality of the topics treated make this an outstanding textbook on the theory of differential equations. The book contains more material than could be taught in a one-year course and is written for advanced undergraduates and graduate students in mathematics, engineering, and the physical sciences who have a sound background in matrix theory, functions of real variables, vector spaces, point set topology, and abstract algebra.

As the author indicates in the preface, a good "basic course" in ordinary differential equations could be taught from portions of Chapters 1, 2, 3, 4, 5, 7, 8, 10, and 12. In addition to the sections of these chapters not included in the "basic course" the book contains chapters that deal with total and partial differential equations, invariant manifolds and linearizations, linear second order equations, dichotomies for solutions of linear equations monotone solutions, boundary layer theory, and global asymptotic stability. The four appendices, which are placed after Chapters 4, 7, 9, and 11, discuss analytic linear equations, Poincare-Bendixson theory on 2-manifolds, smooth equivalence maps, and disconjugate systems, respectively. In most cases the subject matter is arranged so that the material non-essential to the theoretical development of differential equations complements the "basic course" content. In general, the more difficult and less basic topics appear at the end of chapters.

A noteworthy feature of the "basic course" is the treatment given integration of differential inequalities. The pertinent statements inserted within the discussions, labeled as remarks, enhance the quality of the book. A list of notes which cite the sources of theorems, definitions, solutions, and procedures, and point out references for further reading is the last section of each chapter.

Exercises are more or less grouped into three categories: routine, difficult, and most difficult. Routine problems are designed to test the students' comprehension of the material presented. Hints are provided at the back of the book to aid the student in solving the difficult exercises. The most difficult exercises, which usually lead to more detailed developments of theory and applications, require further reading in order to obtain solutions. References are given for this reading in the list of notes found at the end of each chapter. The serious mathematics student should indeed appreciate these useful notes. The book is well-written and completely rigorous, but perhaps a bit short on illustrations and examples. However, illustrations and examples are not necessarily advantageous for study at the level for which the book is written and would serve to make it more voluminous.

This book is a must for the list of references furnished the students taking a course in ordinary differential equations if it is not to be used as the textbook.

—Robert L. Poe
Kansas State Teachers College

Elements of Numerical Analysis, Peter Henrici, John Wiley and Sons, New York, 1964, 328 pp., $8.00.

This book joins the family of excellent books on numerical analysis which is so important in applied mathematics. It is an introductory book based on the author's lecture notes from a course he has taught repeatedly. The book is intended for undergraduate students and is divided into three parts. The first part deals with the solution of equations using various iteration methods and the quotient-difference algorithm. The second part discusses interpolation and approximation using all of the classical methods and applies the concepts to numerical differentiation, numerical integration, and numerical solution of differential equations. The third part is concerned with computation and has an excellent chapter on roundoff errors. There is also a three chapter introduction which discusses the history and definition of numerical analysis, complex numbers and polynomials, and difference equations.

The author gives extensive treatment to difference equations and "in fact, difference equations form one of the unifying themes of the book". There is no treatment of numerical methods in algebra and matrix theory nor the fitting of numerical data by means other than by interpolating polynomials. The book also omits material on programming and programming languages, which any book on numerical analysis should since these topics are computer oriented and belong in a separate book. There are no tables in the book.

The author's background in the teaching of the material in this book is evident throughout and his clear discussions make for delightful reading. A clear cut distinction is made between algorithms and theorems. "An algorithm is a computational procedure; a theorem is a statement about what an algorithm does." In mixing the practical with the theoretical, the author succeeds admirably in proving his point "that numerical analysis is primarily a *mathematical* discipline."

Since the author was concerned about the students' knowledge of the basic properties of the complex field; perhaps, he should also have been concerned about the students' ability to handle vector notation.

The inclusion of "Recommended Reading" and "Research Problems" is a welcomed sight in any book. There are also many examples and some 300 problems but no answers to the problems are given.

—Kaj L. Nielson
Battelle Memorial Institute

A Guide to Algol Programming, Daniel D. McCracken, Wiley & Sons, Inc., New York, 1962, 106 pp., $3.95.

This paperback book does considerably more than explain the computer language, ALGOL. It develops the fundamental idea of an Algorithm for the beginning programmer and is considered a good textbook for any programming course which includes ALGOL. The presentation of how to use the ALGOL language is good. However, in some cases, the author tends to define by an example rather than a clear definition. The examples, case studied, and exercises show how computers are applied in science and engineering and include some techniques of numerical analysis. The explanations might be considered a little wordy for a reference manual, but are usually excellent when the book is used as a textbook. No errors were found by the reviewer or reported to the reviewer by other users.

—Ralph E. Lee
The University of Missouri at Rolla

Statistics and Experimental Design in Engineering and the Physical Sciences, Volumes I and II, Norman L. Johnson and Fred C.

Leone, John Wiley and Sons, Inc., New York; Volume I, 523 pp., $10.95; Volume II, 399 pp., $11.50.

This two-volume set is an excellent text for a two-or three-semester course in statistics for engineering and for science majors at the undergraduate level, assuming "engineering mathematics" as a prerequisite.

It combines an extraordinary amount of material, including much from recent journal articles, in the way of methods, with enough theory to justify them, of statistics applied to the analysis of experimental results in science and engineering.

Motivation and explanation are given in very great detail, and topics of interest to engineers are especially well represented. For example (after a comprehensive introduction to the basic theory of statistics), an entire chapter is devoted to the various aspects of control charts as used in modern quality control.

The basic introduction to probability and statistical viewpoints has the advantage that it mentions a large number of special distributions and situations which engineers encounter, but which are not covered in the usual introductory course. This advantage may lead to a disadvantage in that the beginning student may find it difficult to see the general structure of statistical methods amid the multitude of special cases. However, if the instructor is to be effective, perhaps this is the best place for it, for then the book may be retained as a helpful reference, rather than having to be replaced or supplemented with material which does contain actual cases arising frequently in practice. A long and much-needed discussion of the uses and misuses of regression analysis is given.

From this standpoint Johnson and Leone seems to be a very good textbook indeed. The second volume is devoted to analysis of variance models for the design of experiments, a chapter each on construction of models, modified designs and factorial designs, a chapter on sequential analysis, one on multivariate methods (including response surfaces), and one on sampling.

—Paul D. Minton
Southern Methodist University

The Fibonacci Quarterly—The official journal of THE FIBONACCI ASSOCIATION, Editor: V. E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California.

No doubt it is safe to say that every one who has ever had anything to do with mathematics has at some time or other come across something about the Fibonacci numbers. But to most people the mention of the Fibonacci numbers brings to mind some patterns in nature, such as the arrangement of sunflower seeds. Probably few are aware of the widespread interest in these numbers that exists today and the extent to which these numbers, or others like them, impinge on many mathematical topics.

A group of mathematicians in Northern California became interested in these numbers toward the end of 1962 and soon this interest resulted in their forming The Fibonacci Association, in order "to exchange ideas and stimulate research in Fibonacci numbers and related topics." Plans were soon afoot, according to an article in the fourth number of the first volume, for "a regular type of organization and activity." By that time two conferences had already been held at San Jose College, and these may now have become a regular feature of the organization.

This group soon realized that it would be highly desirable to have a journal for the rapid dissemination of their research. But they, as any one naturally would, wondered whether there are enough people interested in such a specialized subject and even whether there is enough in it to justify the publication of such a journal. They nevertheless went ahead with this project, and began publication in February of 1963. By the end of its first year the subscribers numbered one thousand, so it would seem that there are more people interested in these numbers than one would have first supposed.

An article in Volume I, Number 4, entitled "History of the Fibonacci Quarterly," answers several questions concerning the possibility of there being enough ideas to justify this publication. By the end of the first year, according to this article, some three hundred and twenty pages of mathematics were produced without too much strain, but to obviate the danger of narrowness, the *Quarterly* is officially described as "A Journal Devoted to the Study of Integers with Special Properties." Most of the papers in the first three volumes, however, concern the Fibonacci and Lucas numbers. The Lucas numbers are formed in the same way as the Fibonacci numbers, but they begin a bit differently.

The Fibonacci Quarterly is now in its fourth year and the wide range of interest in this specialized field is indicated by the fact that the contributors to the first three volumes represent the North

American continent from the Atlantic to the Pacific, from Canada to Texas, and there are even a few from abroad.

The *Quarterly* also caters to a wide range of people. It is divided into two parts in about the ratio of five to three. Part I is for the advanced readers and Part II for beginners. Both parts end with a problem and a solution section.

Part I presupposes a good amount of background in the study of recurrence relations and indeed of mathematics in general. A few titles, picked at random, will indicate the degree to which these numbers impinge on various phases of mathematics: "Gaussian Fibonacci and Lucas Numbers," "Fibonacci Matrix Modulo m," "The Golden Ratio and the Fibonacci Geometry," "Fibonacci Numbers from a Differential Equation," "Continued Fractions of Fibonacci and Lucas Ratios".

Part II, for beginners, has a "Beginners' Corner" which has continued through these first three volumes. This presents "background material for the beginning Fibonacci explorer." Another section, entitled "A Primer on the Fibonacci Sequence," ran through five parts. These articles lay the foundation for the theory of these and similar numbers. The topics treated are mathematical induction, Cramer's Rule, the algebra of 2 x 2 matrices, two-dimensional vectors, with applications to the Fibonacci and Lucas identities, some twenty-four of which are listed. Groundwork is also done in the general field of sequences and infinite series. An article on "Linear Recurrence Relations" further prepares the beginner for his investigation of these numbers.

There is an elementary research department titled "Exploring Recurrent Sequences," initiated by Brother U. Alfred, which is an invitation to newcomers in the field to take part in some interesting research. Each issue contains a number of suggested problems for the beginner.

Of particular interest and a source of encouragement to young people who want to do some independent studying is the article "Fibonacci — Tribonacci", by Mark Feinberg, a fourteen-year-old ninth grader from a junior high school. It is a summary of a winning project in a junior high school science fair.

In conclusion, *The Fibonacci Quarterly* is a journal of wide appeal and of special interest for undergraduates, but it gives evidence that even the seasoned mathematician can find it worth reading. The managing editor of the journal is Brother U. Alfred, St. Mary's College, St. Mary's California, 94575, and the subscription price is $4.00 per year.

—Sister Mary Felice
Mount Mary College

Theory of Functions of a Complex Variable, Volume I, A. I. Markushevich, Translated by R. A. Silverman, Prentice-Hall, Englewood Cliffs, New Jersey, 1965, xiv + 459 pp., $12.00.

The translator of this work written by a professor of mathematics at Moscow State University has made some format changes, breaking the original text into 3 volumes, and adding over 300 problems and new references.

The book covers many of the usual topics of a first course in Complex Variables with a high level of rigor and detail and with numerous extensions to more advanced topics. There are many remarks and notes to amplify points, and a generous collection of examples.

The book is divided into three parts. Part 1, Basic Concepts, includes topological concepts used in the text. Part 2, Differentiation and Elementary Functions, includes careful descriptions of the mappings by the elementary functions, and a thorough discussion of the linear fractional transformation, including an introduction to "Lobachevskian Geometry." Part 3, Integration and Power Series, contains a thorough proof of Cauchy's Integral Theorem and its consequences. The chapters on series include numerous extensions of the usual convergence theorems, especially regarding the behavior of a power series on its circle of convergence.

The book is a well-written, thorough first course in Complex Variables, and makes full use of modern concepts. It has more in it than most first courses would require but would be well worth considering for highly qualified students, or in any case as a rigorous, yet readable reference.

—R. N. Townsend
Bowling Green State University

BOOKS RECEIVED

Cybernetics, Second Edition, Norbert Wiener, The Massachusetts Institute of Technology Press and John Wiley & Sons, Inc.

(440 Park Avenue South), New York, xiv + 212 pp., 1961, $6.50.

Diophantine Geometry, Serge Lang, Interscience Publishers, New York, 1962, 170 pp., $7.45.

Elementary Vector Geometry, Seymour Schuster, Wiley & Sons, Inc., New York, 1962, 213 pp., $4.95.

The Mathematical Principles of Natural Philosophy, Isaac Newton, Philosophical Library, New York, 1964, 447 pp., $10.00.

Partial Differential Equations, Lipman Bers, Fritz John and Martin Schechter, Wiley & Sons, New York, 1964, 343 pp., $10.70.

Problems in the Sense of Riemann and Klein, Josip Plemelj, Wiley & Sons, New York, 1964, 173 pp., $8.00.

The Theory of Graphs, Claude Berge, Wiley & Sons, 1962, 246 pp., $6.50.

Unified Algebra and Trigonometry, Dick Wick Hall and Louis O. Kattsoff, Wiley & Sons, New York, 1962, 455 pp., $6.75.

NOTICE

The National Council of Kappa Mu Epsilon announces the appointment of Fred W. Lott, Jr., to fill the vacancy created by the death of the National President, Loyal F. Ollmann. For six years Dr. Lott served very effectively as Editor of THE PENTAGON, and at the 1965 National Convention he was elected to serve as National Vice-President.

The Council also announces the appointment of George R. Mach to fill the vacancy of National Vice-President. Professor Mach is the Scrapbook Editor of THE PENTAGON.

The Mathematical Scrapbook

EDITED BY GEORGE R. MACH

Readers are encouraged to submit scrapbook material to the editor. Material will be used where possible and acknowledgment will be made in *The Pentagon*. The first submissions appear in this issue.

Consider a few numbers, their factors, and the number of factors each has.

Number	Factors	Number of Factors
1	1	1
2	1, 2	2
3	1, 3	2
4	1, 2, 4	3
5	1, 5	2
6	1, 2, 3, 6	4
7	1, 7	2
8	1, 2, 4, 8	4
9	1, 3, 9	3
10	1, 2, 5, 10	4

Notice that only three of the above numbers have an odd number of factors. Try the numbers from 11 to 20 and you will find only one more. The next ten numbers again produce only one. Isn't it strange that so few have an odd number of factors? Do you notice anything interesting about those that do? What do you conjecture? Can you prove it? Here's how. Group the factors of 30 by complementary pairs: (1,30), (2,15), (3,10), (5,6). Now try the same procedure for 36. There's your proof.

=Δ=

Note that $\sqrt{5\frac{5}{24}} = 5\sqrt{\frac{5}{24}}$, and $\sqrt[3]{2\frac{2}{7}} = 2\sqrt[3]{\frac{2}{7}}$. The 5 and the 2 can apparently be "removed" from the radicals. Can you find other similar examples? Can you discover some rules or principles which will help you?

A person would be very fortunate to find a place to invest his money with an annual interest rate of 100%. In a year, one dollar so invested would return two. Many savings institutions now compute and pay interest biannually or quarterly and we call this compounding the interest. Suppose that we could invest our dollar with an annual rate of 100% compounded not biannually, quarterly, monthly, or even daily, but *compounded instantaneously!* What a bonanza! We could surely retire in a year. Or could we?

Let's calculate the annual return of a dollar. The old formula, $P = A(1 + i)^n$, must be modified as follows for compounding the interest.

Compounding Periods	Formula	Return
One	$1(1 + 1.00)^1$	\$2.00
Two	$1(1 + 0.50)^2$	2.25
Four	$1(1 + 0.25)^4$	2.44+
n	$1(1 + \frac{1}{n})^n$	

To compound instantaneously, we take the $\lim_{n \to \infty} (1 + \frac{1}{n})^n$. This limit should be an old friend of yours. If you don't recognize it, look in your calculus book. You might find it as $\lim_{h \to 0} (1 + h)^{1/h}$, which is the same thing. Well, it would be a good return on your investment, but hardly a bonanza.

=Δ=

Editor's Note: The following material was submitted by J. M. Brooke, Sweany, Texas.

DEFINITION: A circle has no corners. An oval has no corners, but not nearly so no corners as a circle has.

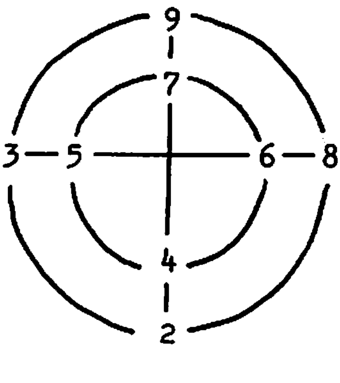
=Δ=

STRICTLY FOR SQUARES: 49 is a square. If we put 48 in the center of 49 we get 4489, which is 67^2. Another 48 gives 444889, which is a square. This can continue indefinitely;

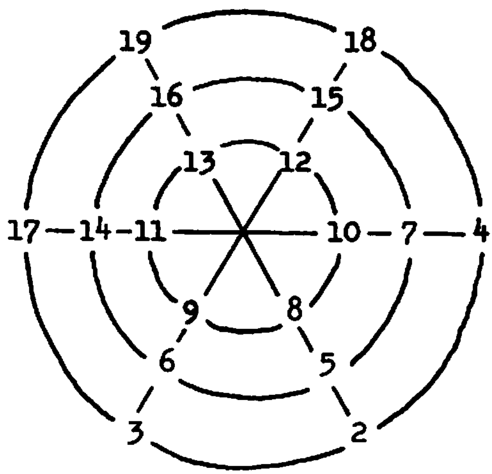
44448889, 4444488889, etc. Can you find another pair of consecutive numbers that have this same property?

=△=

Magic squares receive lots of attention. A magic square has the sum of the numbers in each row, each column, and each diagonal the same. The fascinating subject of magic circles gets much less attention. Here are two Japanese examples:



n = 22



n = 63

A number of concentric circles are divided by an equal number of diagonals. The problem is to put consecutive numbers at the intersections so that the sum of the numbers around each circle and across each diagonal will be the same. Can you find other second and third order magic circles or magic circles of higher orders?

Kappa Mu Epsilon News

EDITED BY J. D. HAGGARD, HISTORIAN

Alabama Beta, Florence State College, Florence

At an initiation banquet in April, 1965, fourteen new members were initiated into Alabama Beta Chapter. Other programs in the spring of 1965 included student papers by Bettye Bergin and Frances Haney.

A Coffee Hour at homecoming was attended by sixty-five alumni members and guests from eighteen different years.

Programs this year have featured Marjory Johnson, who described her summer work with I.B.M. in Huntsville, Alabama; Frances Haney, who did summer missionary work in the mountains of Virginia; Mr. William Scott, who was sent by T.V.A. to Nigeria to evaluate the fertilizer potential of the country; Dr. Charles Robinson, of the University of Mississippi, who gave a paper on "The Square Circle" and discussed graduate school possibilities with individual members.

Five of our graduating seniors, Bettye Bergin, Norman Cooper, Harold Darby, Cecilia Holt, and Marjory Johnson, have applied for admission to graduate college.

Alabama Epsilon, Huntingdon College, Montgomery

Dr. Robert L. Plunkett, acting head of the Mathematics Department of the University of Alabama, was guest lecturer at a joint meeting of Kappa Mu Epsilon and Math Club on February 24, 1966. He spoke on the function concept. His visit was sponsored by the Mathematics Association of America.

Alabama Epsilon has initiated a tutorial clearinghouse service for high school students in the city which enables members of Alabama Epsilon to obtain practice in teaching and also helps the local students. We hope to see this service expanded to all departments in the college.

California Delta, California State Polytechnic College, Pomona

Dr. Robert James from the Claremont College was a guest speaker of this chapter on Thursday, February 24, and spoke on "Area, Measure, and Integration—a Geometric Viewpoint." He remained after his talk for an informal discussion of the "New Mathematics." Dr. James is sponsored by the National Science

Foundation and the Visiting Lectureship Program administered by the Mathematical Association of America.

Florida Alpha, Stetson University, De Land

We have six new actives and seven pledges. Another initiation will be held in April.

Illinois Beta, Eastern Illinois University, Charleston

Plans are being made for the initiation ceremony and banquet to be held April 21, 1966. Last year we initiated twenty-four new members and we anticipate a similar number this year.

Plans are under way to attend the regional convention of Kappa Mu Epsilon at Mount Mary College, Milwaukee, Wisconsin.

Illinois Gamma, Illinois Teachers College, Chicago

During the year 1965, we initiated twenty-eight new members into Kappa Mu Epsilon. March 2, 1966, we initiated another sixteen new members.

Indiana Gamma, Anderson College, Anderson

Two of last year's KME graduates received graduate assistantships in mathematics—President Yeuk-Laan Chui Chien from Purdue University and Stanley Stephens from Lehigh University.

A group of ten KME members visited the Indiana sectional meeting of the Mathematical Association of America in November to hear Professor Eilenberg speak on "Categories."

Iowa Beta, Drake University, Des Moines

President Jerry L. Lewis held a series of meetings for members of the Chapter to learn basic programming by using the scientific computer language, FORTRAN. Mr. Lewis is an experienced programmer and had access to General Electric's internal processing center which the class utilized throughout the series of lessons.

Iowa Gamma, Morningside College, Sioux City

Eight new members were initiated on December 9, 1965. They were Daryl Arnold, Gerald Larson, Gary Pfeister, James McDonald, Joseph Hilber, John Verzani, Carolyn Wyatt, and Robert Green.

Three members, Carl Bylin, Richard Cloud, and Charlene Schnepf, participated in the William Lowell Putnam competition.

Kansas Alpha, Kansas State College, Pittsburg

On October 28, 1965, Kansas Alpha became the first chapter to initiate one thousand members. Six charter members, including Jessie Bailey, R. W. Hart, May Kriegsman Stange, Ruth Kriegsman Way, Pansy Lewis Sale, Margaret Parker, and R. G. Smith, attended the initiation and banquet and messages were received from two other charter members, Violet Lewis Covell and Esther Pease Seaman. The program for the evening was presented by Dennis Ferman, who spoke on the topic "Bertrand Russell's Definition of Number."

Another interesting program was given by two Chinese graduate students, Peter Jack Chi and Tang-Yung Lo, who described the mathematics curriculum of the schools in Formosa.

Kansas Gamma, Mount St. Scholastica College, Atchison

Visiting lecturer, Professor William Stamey from Kansas State University, spoke to the Chapter on October 26. His topic was "Phases of Modern Geometry." A tea for Dr. and Mrs. Stamey followed the lecture.

Three students applied for grants in the Undergraduate Research Program being sponsored by the Mathematics Department with support from N.S.F. They are David Pyne, Kathleen Mudd, and Jo Ingle.

Kansas Gamma pledged twenty-three members on October 4. There are twenty-six active members. Meetings were held bi-weekly and student papers presented at each.

The second annual Mathematics Tournament for neighboring high schools was held on February 12. Nine schools participated by sending sixty-five representatives. The school having the highest over-all average in the competition receives the trophy. For the second time this honor went to Bishop Miege High School, Mission, Kansas.

Professor R. V. Andree from the University of Oklahoma visited area colleges on February 2. The Kansas Gamma Chapter heard him speak on "Unusual Uses of the Computer" at St. Benedict's College Science Hall.

Social events for the year included: Initiation banquet, September 11; Chili supper, October 12; Christmas Wassail Bowl Ceremony, December 12; Pledges' Taffy Pull, January 30; Spring Banquet, April 30.

Maryland Alpha, College of Notre Dame of Maryland, Baltimore

1965-1966 activities include:

Student papers: Bernadette Wegemer, "The Mathematics of the Dark Ages;" Barbara Tipton, "Operations in the Negative."

Field Trip to the Baltimore IBM center.

Dr. Joan Rosenblatt, Statistician from National Bureau of Standards, spoke on "Applications of Statistics to Colorimetry."

May Initiation meeting: Five new members to be initiated; Speaker: Professor Noel Balthasar, "Number Pairs."

Initial meeting of the year was centered about a panel on "Op Goes the Easel," in which Sister Marie Augustine discussed the mathematics of Op Art; and Sister John de Matha (Art Dept.) discussed the artist's view of Op Art.

Maryland Beta, Western Maryland College, Westminster

Maryland Beta Chapter was installed May 30, 1965. The October meeting was devoted to organization and planning for the year which followed an informal dinner at the home of the corresponding secretary. The November meeting was highlighted by guest speaker Dr. Miriam Whaples of the music faculty who discussed the mathematics of music including the group structure of the tonal system. The December-January meeting featured Mr. Perry L. McDonnell who discussed the basic elements and problems of topology. Mr. James E. Lightner of the mathematics faculty spoke at the February meeting on the impact of "modern mathematics" on the secondary and college curricula, and projected the future of the mathematics curriculum at Western Maryland College. The March meeting was devoted to a talk by another faculty member, Mr. Raymond Albert, Director of the Data Processing Center, who presented some ideas on the computer in society and on elementary programming. The highlight of the year was the April banquet meeting at which time five new members were initiated and several members (seniors) presented their seminar papers before the assembled group. During the month of May, the chapter was sponsor and host to the departmental seminars where other seniors presented their papers in public.

Plans are under way for a chapter-sponsored series of seminars and talks by several mathematicians in industry and in education during the Science Convocation in the fall, during which time the addition to our science building will be dedicated in formal ceremonies.

Mississippi Alpha, Mississippi State College for Women, Columbus

The annual initiation ceremony and banquet were held October 27, 1965. There were thirteen pledges initiated at that time. Throughout the academic year the local KME chapter meets monthly in conjunction with the Mathematics Club. The program this month was presented by the local representatives of the IBM Corporation.

Missouri Alpha, Southwest Missouri State College, Springfield

Some of the activities of Missouri Alpha during this year included: initiated twenty members in the fall; sponsored candidate for Yearbook Queen; KME team entered preliminary contest of SMS College Bowl; sponsored KME Regional Convention on April 23, 1966.

Dates, topics, and speakers at meetings this year:

October 12, 1965	The Tetrahedron Group	Eddie W. Robinson
November 9, 1965	Extension Fields	John W. Bridges
December 14, 1965	A Special Vector Space	David McNeill
January 11, 1966	Computer Programs	Jackie Barker
February 8, 1966	Astronomy	Donald H. McInnis

Missouri Beta, Central Missouri State College, Warrensburg

Missouri Beta increased its membership by ten at its fall initiation. We now have an active membership of twenty-five and hope to increase the membership by at least an additional ten members when the spring initiation is held in April.

Several of our programs were provided by staff members and members of the club this past year. We are concluding the academic year with a banquet held jointly with the honorary physics organization, Sigma Zeta.

The club took part for the first time in the local campus quiz program called "Toss-Up," which is a variation of G. E. College Bowl. Much valuable experience was gained from this venture and we are looking forward to next year's contest.

In addition, the organization plans to submit three papers to the regional convention to be held at Southwest Missouri State College this spring.

Missouri Gamma, William Jewell College, Liberty

Our faculty sponsor, Mr. Darrel Thoman, left the college this second semester for Rolla, Missouri, to teach and to work toward his

Ph.D. He received a National Science Foundation Faculty Fellowship. Dr. Elman Morrow is serving as our faculty sponsor the remainder of this year.

We held an initiation on February 3, 1966, for fifteen new members, including two faculty members, Mr. Ekblaw in the Mathematics Department and Dr. Watkins, Head of the Economics Department.

Missouri Zeta, University of Missouri at Rolla, Rolla

Missouri Zeta presented an award to the outstanding freshman and senior in mathematics at the spring K.M.E. banquet. The group also sponsored an award for the oustanding achievement in mathematics at the annual Phelps County Science Fair. Last year's award was on the use of computers. The group is looking forward to grading the coming science exhibits. The organization is presently preparing to present film programs on mathematics to the student body. Missouri Zeta initiated six new members last semester and is again surveying for prospective initiates this semester. The group plans to present individual papers at the regional convention this year. The Chapter has obtained Professor John M. H. Olmsted, a Mathematical Association of America Lecturer, for a mathematics seminar open to students and faculty, and as the K.M.E. banquet speaker. Professor Olmsted's talks, on May 5 and 6, will climax the programs and talks K.M.E. Missouri Zeta has sponsored this year.

New York Alpha, Hofstra University, Hempstead

Nine new members were initiated on January 7, 1966.

Plans are underway for the development of a mathematics library to be operated by and for the students.

North Carolina Alpha, Wake Forest College, Winston-Salem

Six new members were initiated at our Christmas banquet.

Recent programs have included talks on "The Perils of Installment Buying," "The Mathematician in Medical Research," and "Jobs for Mathematicians in Communications and Defense."

Ohio Gamma, Baldwin-Wallace College, Berea

In February, Dick Bohrer presented a paper on "Mathematics in Warfare" and Marie Haushalter presented a film on the weather, Terry Furman presented a paper on the "Moebius Strip."

The election of officers was held at the meeting in May. Also, Nancy Plumb presented a paper on the "History of Probability," and Elaine Westervelt gave a paper on the "History of Numbers."

Initiation was held at the first meeting of the 1965-66 school year. The following members were initiated: Sue Benein, William Chen, Charlotte Clark, Tom Day, Dave Dudik, Camille Falcone, Dave Fortier, Eric Hansen, Hugh Harris, Dan Harry, Joyce Little, Lois Manahan, Marilyn Mills, Kent Maffett, Karen Palmer, Neil Podolnick, Rose Marie Randall, Charles Ryan, and Jim Whipple.

In November, Charlotte Clark, Jim Whipple and Dan Harry presented papers on "The Birth of Numbers," "One, Two, Three . . . Infinity," and "Computers," respectively.

Professor Martin Schulte from Case Institute of Technology was the guest speaker in December. His topic was the "Gerschgorin Methods of Matrix Theory." In January, a field trip to the Cleveland NASA Lewis Research Center was taken. The computing facilities were particularly interesting.

Student papers were also presented in February: Hugh Harris on "Ballistics," Lois Manahan on "Stonehenge, Megalith to the Heavens," and Rose Randall on "The History of the Slide Rule."

Ohio Epsilon, Marietta College, Marietta

The chapter now has twenty-seven members and is expected to bring the membership up to forty with initiation in the spring.

Pennsylvania Beta, LaSalle College, Philadelphia

Some of the papers presented at Kappa Mu Epsilon meetings this year include: Idempotents in the Group Algebra of the Symmetric Group of Order Six; Convergence of the Fourier series for $f(x) = x, -\pi < x < \pi$; Semi-direct Products of Groups; Calculus of Variations; Stereographic Projection and Linear Fractional Transformations.

Pennsylvania Gamma, Waynesburg College, Waynesburg

Some of the programs thus far this year have included: The Analog Computer—a talk and demonstration given by Mr. Harold Hartley, instructor in Physics; Mendel's Law—a talk given by Dr. Charles Bryner, professor of Biology.

Pennsylvania Delta, Marywood College, Scranton

On May 11, 1965, initiation for new members of Kappa Mu Epsilon was held. Those accepted were: Elaine Biglin, Sandra Drula, Michele Fordiani, Phyllis Franks, Mary Ann Gibbons, Lois

Lane, Anyarita Martyak, Helen Mokriski, Mary Ann Tabeski, Diana Warner, and Dorothy Zientak.

Highlights of the 1965-66 term have been:

(1) Presentation of a seminar paper on Orthonormal Polynomials by Helen Mokriski. The paper was the result of her summer's research at Oklahoma University sponsored by the National Science Foundation.

(2) I.B.M. films on the relation of man to machine. Mr. Squish, representative of the local I.B.M. office, answered questions following the movie.

(3) Demonstration of a model computer by Sr. M. Coleman.

(4) Lecture on new trends for teaching mathematics by Miss Marjorie O'Neill, Supervisor of Mathematics in the Scranton Public Schools.

Tennessee Beta, East Tennessee State University, Johnson City

The speaker at the February meeting was Joe Gueron, a member of Tennessee Beta Chapter and a graduate assistant in the Mathematics Department. Mr. Gueron spoke on The University Systems in South America and especially in his own state of Venezuela.

The Chapter pledged support to the Heart Fund Drive and voted to contribute money and effort to this drive.

New members were voted upon and plans were made for the initiation ceremony on March 1.

Tennessee Gamma, Union Univrsity, Jackson

Tennessee Gamma was organized on Union University Campus in the Spring of 1965, and received its national charter at the installation banquet May 24, 1965. Our guest speaker was Mrs. Laura McCormick from East Tennessee State College in Johnson City, Tennessee. We have had excellent member participation this first year and our programs have been presented by individual members. Dr. Wesson from Vanderbilt is to speak at our March meeting. We are pleased that our sponsor has been honored with a National Science Foundation grant.

Texas Epsilon, North Texas State University, Denton

Ten new members were initiated on December 14, 1965. We are currently engaged in preparing a freshman mathematics test to be given March 1, 1966. Spring pledgeship is in progress with pledges required to obtain signatures of all current members of the chapter.

Virginia Alpha, Virginia State College, Petersburg

Virginia Alpha Chapter is preparing for its Fourth Annual Spring Banquet. All members who have graduated are invited back for this annual event which includes a banquet speaker invited from among the graduate members of Virginia Alpha Chapter or a visiting scholar.

Wisconsin Alpha, Mount Mary College, Milwaukee

Some of the programs and activities of the year include:

October 13 — Talk given on "Partial Fractions" by Sue Kasseckert.

November 10 — Guest speaker, Mr. Vern Peterka, who talked on "Math and Magic."

February 9 — Initiation of twelve new members into K.M.E.

We are hosting the Regional K.M.E. Convention the weekend of March 18-19.

We are making preparations for our annual mathematics contest to be held on April 2, 1966.

(Continued from page 67)

pleasure and satisfaction. He loved to go deep-sea fishing, and a visit to his home on Long Island in the summer would reveal a lawn that was meticulously trimmed and bordered with a variety of flowers and shrubs. Under a shading canopy you would have observed thousands of Africian violets (he was an authority on these plants) and in a basement room you would have found a multitude of aquariums with fish ranging from guppies to miniature sharks, each tank labeled with the name of the species and the value.

Dr. Ollmann will no longer lead us as our National President. Yet, in a sense, he is still with us who have known his friendly manner, his courteous deeds, and his example of leadership in the important job of making Kappa Mu Epsilon a significant organization in the life of the undergraduate student of mathematics.