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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Rocketry, Single Stage, Solid Fuel

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The purpose of this paper is to give an analysis of the flight of a single state solid fuel rocket under ideal conditions. These conditions are defined to be as follows: 1) curvature of the earth is nil, 2) air resistance is nil, 3) gravity is homogeneous, and 4) rotation is nil.

As the motion of a rocket is the result of force applied over a length of time, the analysis will begin with the derivation of a thrust formula. The thrust of a rocket engine is due to the change in momentum of the exhaust particles. Momentum is defined to be the mass multiplied by the velocity. If M_1 is the constant fuel burning rate, t is time, and v is the exhaust velocity here also considered to be constant, then the momentum of the exhaust particles is given by $(M_1t)v$.

According to Newton's second law of motion, the time rate of change in momentum is proportional to the force acting on the object. It is conventional to choose units of force so that the constant of proportionality is 1. Thus

$$F = \frac{d(M_1tv)}{dt}$$

$$= M_1v$$

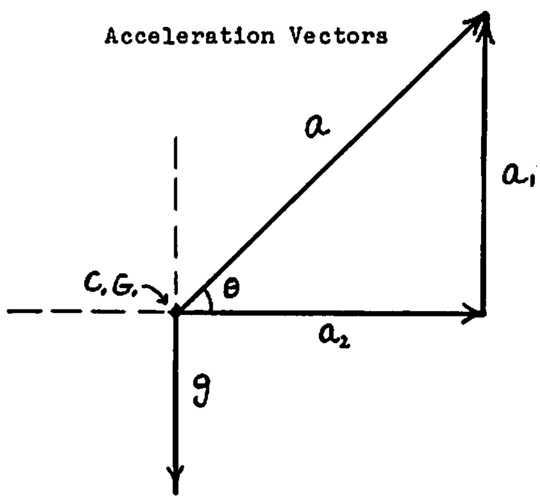
is the force due to the thrust of the rocket engine.

If M_0 is the initial mass of the rocket, then at time t its mass will be $M_0 - M_1t$. Let $V = V(t)$ be the velocity of the rocket relative to the earth, and $a = \frac{dV}{dt}$ the acceleration of the rocket at time t. Also, we have from Newton, the force required to produce an acceleration, a, on a mass, $M_0 - M_1t$, is given by

$$F = (M_0 - M_1t)a.$$

If there are no external forces (we will introduce the effect due to gravity later), these forces are equal and we have

$$(M_0 - M_1t)a = M_1v$$



or

$$a = \frac{M_1 v}{M_0 - M_1 t}.$$

The vertical component of this acceleration for a rocket fired at an angle θ (see the figure) plus the acceleration due to gravity, g, is given by

$$\begin{aligned} a_1 &= a \sin \theta - g \\ &= \frac{M_1 v \sin \theta}{M_0 - M_1 t} - g. \end{aligned}$$

We integrate this with respect to t to find the vertical component of velocity, V_1, at time t:

$$\begin{aligned} V_1(t) &= \int a_1 dt \\ &= \int \left(\frac{M_1 v \sin \theta}{M_0 - M_1 t} - g \right) dt \\ &= -v \sin \theta \ln(M_0 - M_1 t) - gt + C \end{aligned}$$

But $V_1 = 0$ when $t = 0$, hence

$$0 = -v \sin\theta \ln M_0 + C$$

$$C = v \sin\theta \ln M_0$$

and

(1) $V_1(t) = -v \sin\theta \ln(M_0 - M_1 t) - gt + v \sin\theta \ln M_0$

$$= v \sin\theta \ln\left(\frac{M_0}{M_0 - M_1 t}\right) - gt.$$

In a similar fashion, the horizontal component of acceleration is

$$a_2 = \frac{M_1 v \cos\theta}{M_0 - M_1 t}$$

and the horizontal component of velocity is

(2) $$V_2(t) = v \cos\theta \ln\left(\frac{M_0}{M_0 - M_1 t}\right).$$

The distance traveled in the vertical direction in time t, $S_1(t)$, and in the horizontal direction, $S_2(t)$, may be found by integrating the vertical and horizontal components of velocity, V_1 and V_2, respectively. When this is done and the boundary conditions, $S_1 = 0$, $S_2 = 0$ when $t = 0$, are used to evaluate the arbitrary constants, we obtain

(3)

$$S_1(t) = v \sin\theta\left[\left(t - \frac{M_0}{M_1}\right)\ln\left(\frac{M_0}{M_0 - M_1 t}\right) + t\right] - \tfrac{1}{2}gt^2$$

(4)

$$S_2(t) = v \cos\theta\left[\left(t - \frac{M_0}{M_1}\right)\ln\left(\frac{M_0}{M_0 - M_1 t}\right) + t\right]$$

As special cases, suppose $t = t_1$ is the burning time of the fuel. Then substituting t_1 in (3) and (4), we have the burnout altitude, H_b:

(5)

$$H_b = v \sin\theta\left[\left(t_1 - \frac{M_0}{M_1}\right)\ln\left(\frac{M_0}{M_0 - F_1}\right) + t_1\right] - \tfrac{1}{2}t_1^2$$

and the burnout horizontal distance, S_b:

$$(6) \qquad S_b = v \cos \theta \left[\left(t_1 - \frac{M_0}{M_1} \right) \ln \left(\frac{M_0}{M_0 - F_1} \right) + t_1 \right]$$

where $F_1 = M_1 t_1$ is the mass of fuel originally carried by the rocket.

The vertical and horizontal components of velocity, V_{1b} and V_{2b} at the instant of burnout are determined by replacing t with t_1 in equations (1) and (2):

$$(7) \qquad V_{1b} = v \sin \theta \ln \left(\frac{M_0}{M_0 - F_1} \right) - g t_1$$

$$(8) \qquad V_{2b} = v \cos \theta \ln \left(\frac{M_0}{M_0 - F_1} \right)$$

To find the angle the trajectory makes with the surface of the earth, $\theta_1(t)$, at any time t up to the time of burnout, we have

$$\tan \theta_1(t) = V_1(t)/V_2(t)$$

and if θ_b is the angle of the trajectory at the instant of burnout,

$$\tan \theta_b = V_{1b}/V_{2b}$$

where V_{1b} and V_{2b} are given by equations (7) and (8). The velocity at burnout in the direction θ_b is found by adding vectorially the horizontal and vertical components

$$V_b = \sqrt{(V_{1b})^2 + (V_{2b})^2}\,.$$

It would be of interest to know the maximum altitude attained. From free flight ballistics one has

$$h(T) = V_b T \sin \theta_b - \tfrac{1}{2} g T^2$$

for the additional height during free flight in time T seconds after burnout has occurred. Differentiating and setting $h'(T) = 0$, we find that a maximum of $h(T)$ is obtained when

$$T = \frac{V_b \sin \theta_b}{g}$$

and, by direct substitution in $h(T)$,

$$h\left(\frac{V_b \sin \theta_b}{g}\right) = \frac{V_b^2 \sin^2 \theta_b}{2g}\,.$$

Thus, adding the powered altitude, H_b from (5), to the free flight height after burnout, we have for the maximum altitude, H_m,

$$(9) \qquad H_m = H_b + \frac{V_b^2 \sin^2 \theta_b}{2g} .$$

It would also be of interest to know the range of the rocket. Again from free flight ballistics, the horizontal distance covered T seconds after burnout is given by

$$(10) \qquad R_1 = V_b T \cos \theta_b$$

and the height of the rocket above the surface will be

$$H(T) = H_b + h(T)$$
$$= H_b + V_b T \sin \theta_b - \tfrac{1}{2} g T^2.$$

Now set $H(T) = 0$ and solve for T to find

$$T = \frac{V_b \sin \theta_b + \sqrt{V_b^2 \sin^2 \theta_b + 2gH_b}}{g}$$

which is the time after burnout to return the rocket to the surface. But from the equation (9),

$$2gH_m = V_b^2 \sin^2 \theta_b + 2gH_b$$

Hence

$$T = \frac{V_b \sin \theta_b + \sqrt{2gH_m}}{g} .$$

Substitute this in equation (10) to find the horizontal distance traveled in free flight after burnout

$$(11) \qquad R_1 = \frac{V_b \cos \theta_b (V_b \sin \theta_b + \sqrt{2gH_m})}{g} .$$

The total range would be

$$R = S_b + R_1,$$

the sum of the horizontal distance covered during powered flight, S_b from (6), and the additional horizontal distance traveled in free flight, R_1 from (11).

The Use of the Analytical Triangle in Curve Tracing*

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Curves, curves, curves! ! The physicist uses curve traces. By a sketch he can read the resistance of a wire as a function of its temperature, he can read the lag of magnetization behind the magnetizing force. He can read the current through a thermocouple as a function of temperature. The economist uses curve functions to find maximum profits. The engineer uses curve sketches. The plot of one variable against another is an invaluable aid to many professions. The mathematician, too, uses curve sketches. But, how does the mathematician sketch curves?

Take for example $x^2 + y^2 = 4$; it is relatively easy to recognize the equation as that of a circle, and hence make a quick sketch. Or what about $x^2 + xy + y^2 = 6$? Likewise using analytical geometry, one could rotate the axis through an angle of 45°, translate the axis, and again make a quick sketch. Calculating intercepts and points of inflection, finding asymptotes, employing calculus to find maximum and minimum turning points, and recognizing the equation as some standard form are among the many other devices used in curve sketching. But, getting a bit more complicated how would one begin to graph an equation such as $x^4 + x^3y + x^2y^2 + y^2x + xy + x + y = 0$?

By placing the terms of such an equation on DeGua's Analytical Triangle, one can find the relative behavior of a more complex curve near the origin and at points more infinite. This method was republished recently in the book *Curve Tracing* by Percival Frost.[1] To represent DeGua's Analytical Triangle, consider it as being an isoceles right triangle in the first quadrant of a set of rectangular coordinate axis $O\alpha$ and $O\beta$ (Fig. 1).

Upon each axis, $O\alpha$ and $O\beta$, make a number of subdivisions equal to the degree of the equation to be considered. Taking, for

* A paper presented at the KME Regional Convention at Kearney, Nebraska, April 4, 1964.

[1] Frost, Percival, **An Elementary Treatise on Curve Tracing**, Chelsea Publishing Company, New York, New York, 1960, p. 118.

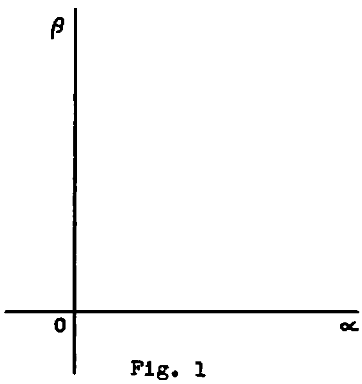

Fig. 1

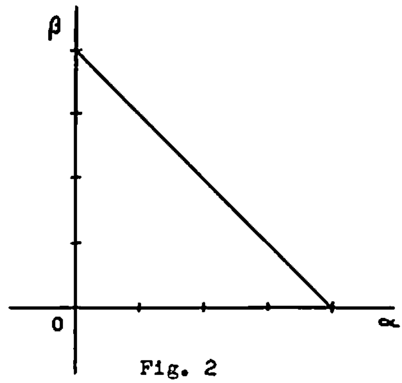

Fig. 2

example,

$$(1) \qquad x^4 + x^3y + x^2y^2 + y^2x + x + xy + y = 0$$

the Analytical Triangle would have four divisions on each axis (Fig. 2).

Each term, $x^\alpha y^\beta$, is placed upon the triangular plane as a point positioned by the coordinates (α, β), exactly as any point (x, y) would be placed on an xy coordinate plane. For example, the term x^3y is positioned at (3, 1) as shown in Fig. 3. An equation is said to be placed upon the Analytical Triangle if there is one dot placed upon the triangle for each term of the equation to be considered.

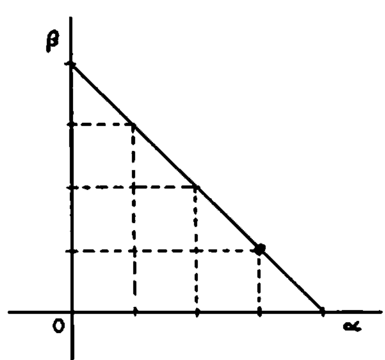

Fig. 3

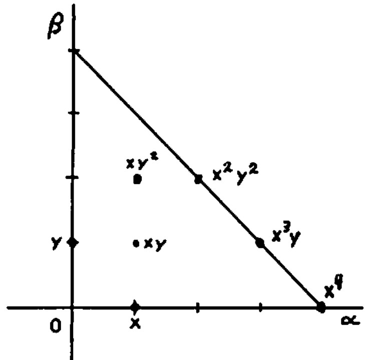

Fig. 4

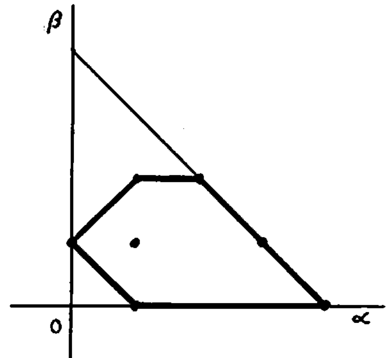

Fig. 5

Equation (1) is placed upon the triangle (Fig. 4). To proceed, connect all points to form a convex polygon, exterior to which no points lie.[2] (Fig. 5).

The benefit of the triangle in finding the relative magnitude and behavior of a curve is gained by considering only those terms lying along a straight line as an approximation to the entire curve.

If all terms of the equation are deleted except those corresponding to terms lying along a straight line, then the resulting equation gives terms that can be written $f(xy^{-m}) = 0$. Solving this quadratic for xy^{-m}, one will get xy^{-m} equals one or more constants. $(xy^{-m}) = k$ is a straight line or a simple parabolic curve. To see why this is true, consider $x^{\alpha} y^{\beta}$ as being one of the terms on a straight line. From analytical geometry one knows that the equation of any straight line is $m\alpha + \beta = c$ where m determines the slope and hence is constant for any line (and also for all parallel lines). For any single line, c is constant. Hence, $x^{\alpha} y^{\beta}$ on this line must satisfy the equation $m\alpha + \beta = c$. Therefore, each term on the straight line must be of the form $x^{\alpha}y^{\beta} = x^{\alpha}y^{-m\alpha+c} = (xy^{-m})^{\alpha} y^{c}$, α being the only variable. Hence, if all terms on the straight line are set equal to zero, instead of the entire equation, the result gives $f(xy^{-m}) = 0$. Solving for xy^{-m}, one gets one or more constants.[3]

2 Ibid., p. 119.
3 Ibid., pp. 120-121.

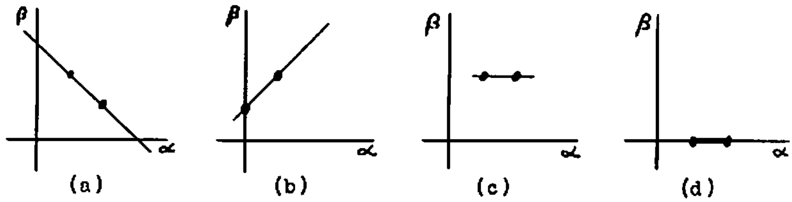

Fig. 6

There are one of four alternatives for any side of the polygon:[1]

1. L intersects both $O\alpha$ and $O\beta$, $0 < \alpha$ and $0 < \beta$ (Fig. 6a).
2. L intersects $O\alpha$ or $O\beta$, $\alpha < 0$ or $\beta < 0$ (Fig. 6b).
3. L is parallel to $O\alpha$ or $O\beta$ (Fig. 6c).
4. L is coincident with $O\alpha$ or $O\beta$ (Fig. 6d).

In the first case, if L intersects both legs of the triangle, one can use those terms on the line as an approximation to the entire equation: 1) if one considers x and y infinitely large when rejected terms lie on the same side of L as O (Fig. 7a), or 2) if one considers x and y infinitely small when rejected terms lie on the opposite side of L as O (Fig. 7b).

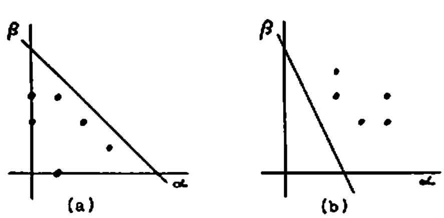

Fig. 7

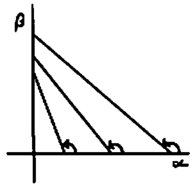

Fig. 8

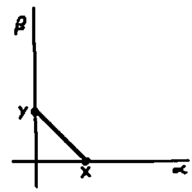

Fig. 9

One can prove this as follows. If the equation of the line on the Analytical Triangle intersects both legs of the triangle, its slope must be negative because the angle of inclination will be greater than 90° (See Fig. 8). But α and β must satisfy the equation $m\alpha + \beta = c$ and the slope of $m\alpha + \beta = c$ is, by analytical geometry, $m/-1$, and in the case being considered one desires a negative slope. Hence, one is requiring that m and c both be positive. Hence, letting m and c be positive, construct a line parallel to L, called L_1, through the point $x^{\alpha'} y^{\beta'}$, and substitute c' for c. The roots of the equation determining the constant (xy^{-m}) will not be identical in the cases of L and L_1 but they will be independent of the values of x and y.[4] Hence, the difference in value must depend on the order of the two equations, determined by y^c and $y^{c'}$. The rejected terms disappear with respect to those remaining if the ratio $y^{c'}/y^c$ disappears. However, $y^{c'}/y^c$ vanishes if y is infinitely great and $c' < c$, or when y is infinitely small and $c' > c$. If $c' < c$, one is saying that the constructed line, L' must have been on the same side of L as the origin, and hence the rejected terms must be on the same side as the origin. Likewise, the opposite is true if $c' > c$. Furthermore, if $(xy^{-m}) = k$, x varies directly as y^m and is correspondingly great when y is, or near 0 when y is. Hence, we have shown that if a line (L) drawn through terms intersects both sides of the Analytical Triangle, and 1) if rejected terms are on the same side of L as O, one can consider values of y and x

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[4] Ibid., p. 121.

infinitely great, or 2) if rejected terms lie on the opposite side of L as O, one can consider values of y and x near 0 and all terms on L give a reasonable approximation to the entire equation.[5]

For example, in the equation (1), the term x and the term y lie on a line which intersects both legs of the triangle, $O\alpha$ and $O\beta$ (Fig. 9). If only these terms are considered, that is $x + y = 0$, they will give a reasonable approximation to the curve at points near the origin since the rejected points lie on the opposite of line L as the origin. We can see why this is true, because if x is infinitely small, the value of x^4 would be extremely small and for all practical purposes negligible. Likewise, x^3y would be small, etc.

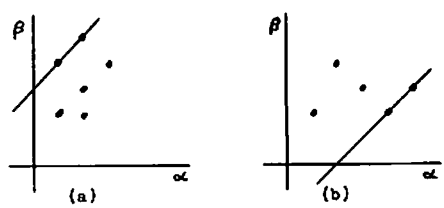

Fig. 10

In the second case, if L intersects either $O\alpha$ or $O\beta$ at a point less than 0, and all terms are rejected except those which lie on the line, then the resulting equation gives approximate values to the total equation if we consider x as being infinitely great and y infinitely small or conversely. All the terms of the equation vanish with respect to those retained if: 1) we consider y infinitely great and x near 0 when the line intersects $O\alpha$, or 2) if we consider x infinitely great and y is near 0 when the line intersects $O\beta$. (See Fig. 10). To prove this we proceed as before. However, if the line L intersects an axis produced backwards through O, then the slope must be positive, hence m must be negative, and $x^{\alpha}y^{\beta} = (xy^{-m})^{\alpha}y^{c}$ becomes $(xy^{k})^{\alpha}y^{c}$, $k > 0$, $k = |m|$. But if $(xy^{k}) = a$, then x varies inversely as y^{k}. Therefore if y is great, x is small; and if x is great, y is small. Thus, case (2) is proved.[6]

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5 Ibid., pp. 120-121.
6 Ibid., pp. 120-121.

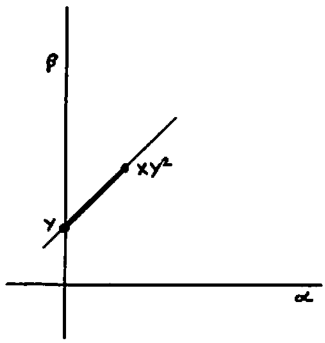

Fig. 11

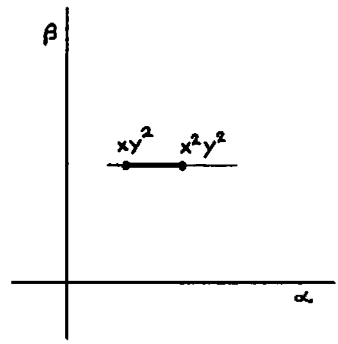

Fig. 12

Again considering equation (1), one sees that L through xy^2 and y intersects $O\alpha$ at $\alpha < 0$ (Fig. 11). Since all rejected points lie on the same side of L as O, we consider y great and x small. Therefore, we can see that

$$y + xy^2 = 0,$$
$$y(1 + xy) = 0,$$
$$1 + xy = 0,$$
$$xy = -1$$

gives a reasonable approximation to the entire equation for infinite y. For example, if x is small, x^4 would be negligible; likewise x^3y, etc.

A third case occurs if the line connecting the terms is parallel to $O\alpha$ or $O\beta$. If a line, for example, is parallel to $O\alpha$ then the result gives one or more straight lines parallel to the y axis. (Conversely, a line parallel to $O\beta$ produces one or more straight lines parallel to the x axis).[7]

The deleted terms on the triangle will be relatively meaningless if one considers infinite values of y when the line is parallel to $O\beta$ or if one considers large values of x if the line is parallel to $O\alpha$. In equation (1), xy^2 and x^2y^2 lie on a straight line parallel to $O\beta$ (Fig. 12). If $xy^2 + x^2y^2 = 0$, $xy^2(1 + x) = 0$, then $x = -1$,

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[7] Ibid., pp. 120-121.

so when y is infinitely large, $x = -1$ gives the straight line parallel to the y axis.

In a fourth case, if a line L coincides with a leg of the triangle, say $O\alpha$, then the corresponding equation gives the points of intersection with, in this case, the x axis.[8]

For example, in the equation previously considered, equation (1), there is a line lying along $O\alpha$ connecting x^4 and x (Fig. 13). Setting only $x^4 + x$ equal to zero one has:

$$x^4 + x = 0$$

$$x(x^3 + 1) = 0$$

$$x = 0$$

$$x = -1$$

Fig. 13

Obviously these are the x intercepts since direct substitution shows that the points $(-1, 0)$ and $(0, 0)$ satisfy the equation.

Looking at the line lying along the hypotenuse one sees that it is a line of the first case. Setting the sum of the appropriate terms equal to zero should allow us to consider infinite values of x and y

$$x^4 + x^3y + x^2y^2 = 0$$

$$x^2(x^2 + xy + y^2) = 0$$

$$x = \frac{-y \pm \sqrt{y^2 - 4y^2}}{2}$$

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8 Ibid., pp. 120-121.

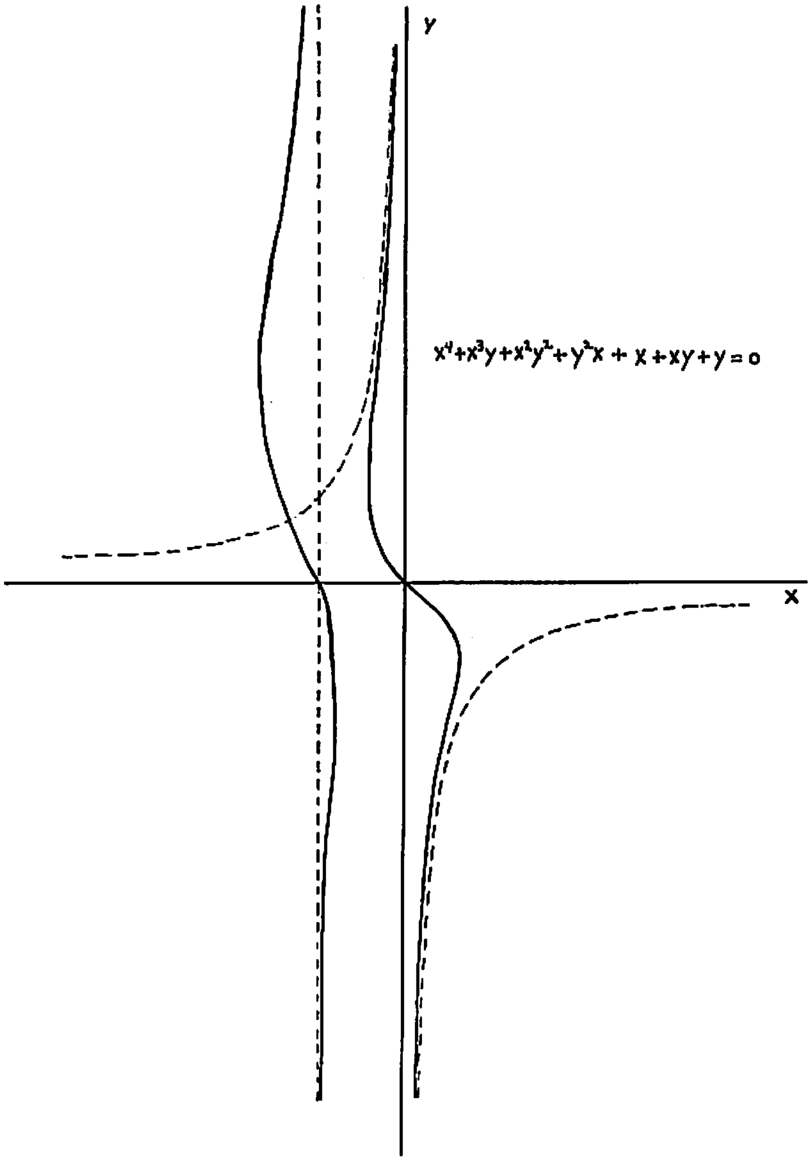

Fig. 14

which simply says that x and y are not simultaneously infinite.

To sketch the entire curve, one knows that:

(1) the x-intercepts are $(0, 0)$ and $(-1, 0)$

(2) near the origin, the equation acts nearly as $x + y = 0$

(3) at points where y is large and x is small, the equation acts as $xy = -1$

(4) at points where y is large, the curve approaches the line $x = -1$.

The graph is shown in Fig. 14.

The Analytical Triangle is not, in itself, a complete curve sketching device, but it can be of considerable help in sketching curves of high degree equations. The use of the triangle gives approximations which are near enough, for all practical purposes, in the vicinity of the origin and at points where either x or y is large. However, the inverse process — finding the equation of a curve by studying the curve properties — is also greatly assisted by considering the Analytical Triangle. Hence, again one must say, the Analytical Triangle is a very valuable aid in the study of complex curves or in the study of higher order equations.

Mathematics—in a strict sense—is the abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations.

—J. A. H. Murray

The Decimal Place Accuracy of Newton's Method[1]

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Ball State Teachers College
and
F. Max Stein
Colorado State University

1. In his initial introduction to the study of approximation using Newton's method, the student of calculus often inquires as to the accuracy of the successive approximations that are obtained. We recall that Newton's method, or the Newton-Raphson method, is used to approximate a real root of the equation $f(x) = 0$. It is based on the principle that the tangent line is a reasonable approximation to the curve $y = f(x)$ in a small enough neighborhood of the root $x = r$ of $f(x) = 0$.

In this paper we shall discuss Newton's method and investigate the accuracy obtained by its repeated use. We shall find that, in general, the decimal place accuracy of successive approximations in-

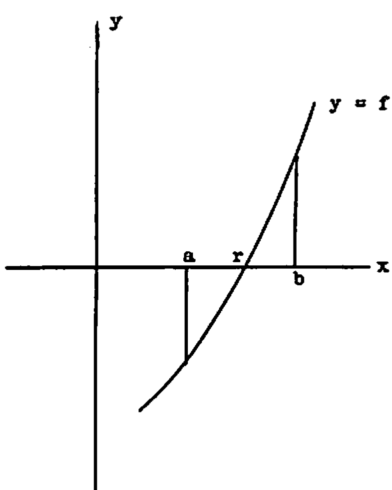

Fig. 1

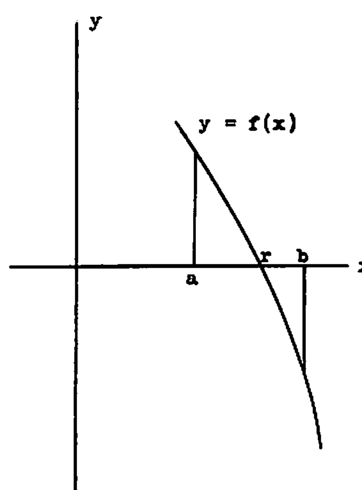

Fig. 2

1 Prepared in a National Science Foundation Undergraduate Science Education Program at Colorado State University by Mr. Allen under the direction of Professor F. Max Stein.

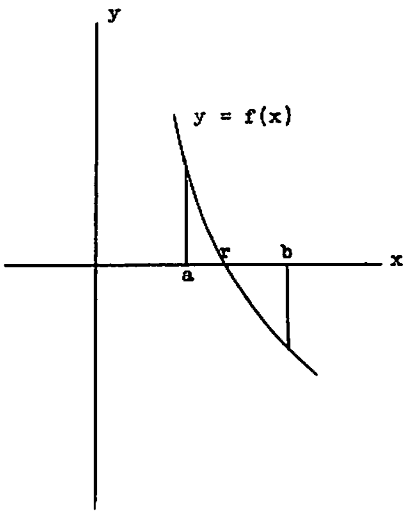

Fig. 3

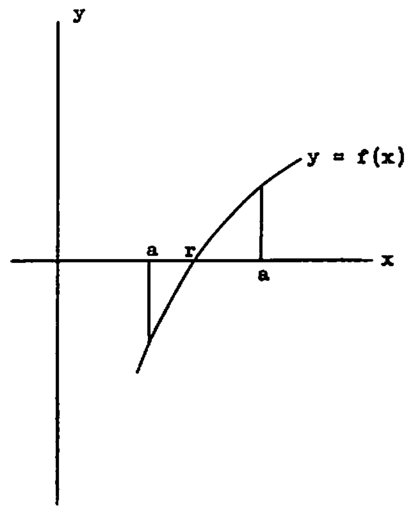

Fig. 4

creases approximately as the geometric progression 1, 2, 4, 8, 16, 32, ...

2. In the interval $[a, b]$, a and b integers, containing r as the only root of $f(x) = 0$, we consider the curve $y = f(x)$ and assume that $f'(x)$ and $f''(x)$ are continuous and non-zero. Since r is the only root of $f(x) = 0$ in the interval and $f''(x) \neq 0$, it follows that $f(a)$ and $f(b)$ will necessarily be of opposite sign. We consider four possible cases, see Figures 1 - 4.

From the figures we note that if $f(b)$ is positive in cases 1 and 4 and negative in the other two, while $f''(x)$ is positive over $[a, b]$ in cases 1 and 3 and negative in the other two. We list these four cases in the following table:

Case	$f(b)$	$f''(x)$ over $[a, b]$
1	+	+
2	−	−
3	−	+
4	+	−

As can be seen by examining the possible cases in (5) below, the facts

(a) that $f''(x)$ does not change sign throughout $[a, b]$ and the sign is known, and

(b) that the sign of $f(b)$ is known

are sufficient for us to make a first approximation $x = x_1$ so that

(A) the tangent line to $y = f(x)$ at $x = x_1$ intersects the x-axis in the interval $[a, b]$ nearer to r than x_1, and

(B) successive approximations to the root of $f(x) = 0$ approach r from the same side.

In view of the preceding remarks, the following rule gives us a method for choosing our first approximation x_1 to the root r of $f(x) = 0$ in applying Newton's method.

RULE. If, on the interval $[a, b]$, $f'(x) \neq 0$, $f(x) = 0$ has $x = r$ as its only root, and the product $f(b)\, f''(x) > 0$, choose $x_1 = b$ as a first approximation. If $f(b)\, f''(x) < 0$, choose $x_1 = a$ as a first approximation.

3. We shall discuss only case 1; the remaining three cases can be developed in a similar manner. Thus, upon making our choice of x_1 according to the rule, we have that $x_1 = b$. In Fig. 5 (an enlargement of Fig. 1) the equation of the tangent line to the curve $y = f(x)$ at $x = x_1$ is

$$y - f(x_1) = f'(x_1)(x - x_1). \tag{1}$$

If we let $x = x_2$ be the point at which the tangent line crosses the x-axis, we have

$$x_2 = x_1 - f(x_1)/f'(x_1), \tag{2}$$

a second approximation x_2 in terms of x_1. By going through the above process again with x_2 in (2) replacing x_1, we obtain a third approximation x_3. Upon repeating this procedure we obtain the recurrence relation for any approximation

$$x_{i+1} = x_i - f(x_i)/f'(x_i); \tag{3}$$

this step-by-step approximation is known as *Newton's method.*

The interative procedure described above gives us a decreasing sequence $x_1, x_2, x_3, \ldots$ that is bounded below by r. That the sequence is decreasing can be seen from the fact that $f'(x)$ and $f(x)$ are positive in the interval $[r, b]$ thus making x_{i+1} less than

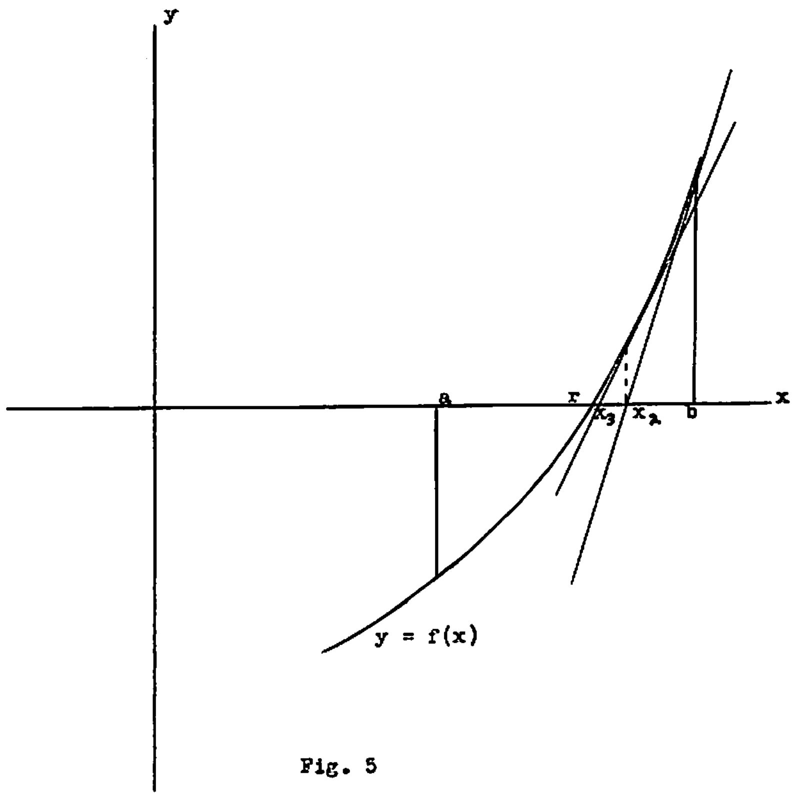

Fig. 5

x_i in (3). A proof of the convergence of this sequence to the limit r can be found in [2].

4. We now investigate the accuracy of Newton's method. For a function $f(s)$ satisfying Taylor's Theorem, we have in the neighborhood of $s = h$ that

$$(4) \qquad f(s) = f(h) + f'(h)(s - h) + \frac{f''(c)}{2}(s - h)^2,$$

where c is some number between s and h. If we let $s = r$ and $h = x_i$, we can write (4) as

$$(5) \qquad r = x_i - \frac{f(x_i)}{f'(x_i)} - \frac{f''(c)}{2f'(x_i)}(r - x_i)^2.$$

In passing we note in (5) that if the rule is applied for each of the cases 1 - 4, the conditions (A) and (B) given above are satisfied.

By (3) we know that

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)},$$

therefore we have from (5) that

$$(6) \qquad |x_{i+1} - r| = |f''(c)/2f'(x_i)|(x_i - r)^2.$$

By taking the absolute value we include the possibilities of approaching the root from the right or the left.

We see that since $|x_i - r|$ is the error in the i-th iteration, that the error in the $(i + 1)$-th iteration equals the error in the i-th iteration squared and multiplied by the factor $|f''(c)/2f'(x_i)|$, where c is some number between x and r. If, in the interval containing x_{i+1}, x_i, and the root r, a maximum absolute value M can be established for $f''(c)/2f'(x_i)$, we have for (6)

$$(7) \qquad |x_{i+1} - r| < M(x_i - r)^2.$$

By taking the common logarithm of (7) and multiplying the resulting inequality by -1, we obtain

$$(8) \qquad -\log|x_{i+1} - r| > -2\log|x_i - r| - \log M.$$

In (8) we take the negative of the logarithm to avoid dealing with negative logarithms under the assumption that $|x_i - r| < 1$. If we let n be the largest integer in $-\log| x_i - r|$, then (8) can be written as

$$(9) \qquad -\log| x_{i+1} - r| > 2n - \log M.$$

From (9) we see that if the i-th iteration is correct to n decimal places, then the $(i + 1)$-th iteration will be correct to at least m decimal places, where m is the largest integer in $2n - \log M$. If $M < 10$, we can say that if x_i approximates r to one decimal place, then x_{i+1} approximates r to at least two decimal places, x_{i+2} approximates r to at least four decimal places, etc.

Thus, in general, for $M < 10$, Newton's method will produce successive approximations in which the decimal place accuracy increases by at least the geometric progression 1, 2, 4, 8, 16, . . . , when the first approximation is correct to one decimal place.

5. In numerical work a decision must be made regarding the rounding off of the approximations at each step. This method we shall use, assuming $|x_i - r| < 1$, is as follows. If the first approximation is the first integer greater than r, we round off to one decimal place in our second approximation by taking the next larger digit in the tenths place as our value for this place. If the approximation turns out to be a digit in the tenths place followed by an infinite number of zeros, we take this digit as our value for this place. For example, if the second approximation is 1.5321, we round off to 1.6.

On the other hand if the first approximation is the greatest integer less than r, we round off by truncating after the first decimal place in our second approximation. For example, if our second approximation is 3.2563, we round off to 3.2.

For successive approximations we round off by continuing this procedure. Thus, having chosen an integer as our first approximation, we round off (using the above procedure) starting with our second approximation, to the number of decimal places given by the progression 1, 2, 3, 5, 9, 17, 33, ... We note that each term in this progression is one greater than the corresponding term in the progression 0, 1, 2, 4, 8, 16, 32, ... By following the procedure given for rounding off, we can insure that successive iterations will approach a root from the same side.

6. Example. A single root of

$$f(x) = 3x^3 - 5x^2 + 12x - 20 = 0$$

is known to lie between 1 and 2. Use Newton's method to approximate this root correct to at least four decimal places.

Here we have that

$$f'(x) = 9x^2 - 10x + 12$$

and

$$f''(x) = 18x - 10.$$

Taking $a = 1$ and $b = 2$ we that that $f''(x) > 0$ throughout the interval and that $f(b) = f(2) = 8$; thus $f''(x)$ and $f(b)$ are of the same sign. Therefore we choose $x_1 = b$ as our first approximation. We then have, by rounding off each x_i according to the given progression, that

$$x_2 = 1.8,$$

$$x_3 = 1.68,$$

$$x_4 = 1.667,$$

and

$$x_5 = 1.66667.$$

By the progression for determining the accuracy of successive approximations we have that x_5 is correct to at least 4 decimal places. It is readily verified for this example that $r = 5/3$.

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It is remarkable that a science (probabilities) which began with the consideration of games of chance, should have become the most important object of human knowledge.

—LAPLACE

On Maxima and Minima*

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THEOREM: Given a circle m, center at C, P an interior point of m, $P \neq C$, there is no point on m nearest P.

Proof: Let the circle m with center at C have a radius r. For any interior point P, P lies on a diameter of m. Without loss of generality the diameter could be considered in the horizontal position so that the configuration would appear as in Figure 1 with A and B as the end points of the diameter.

Let a line L be perpendicular to AB thru P. A rectangular coordinate system is now established by letting AB be the x-axis

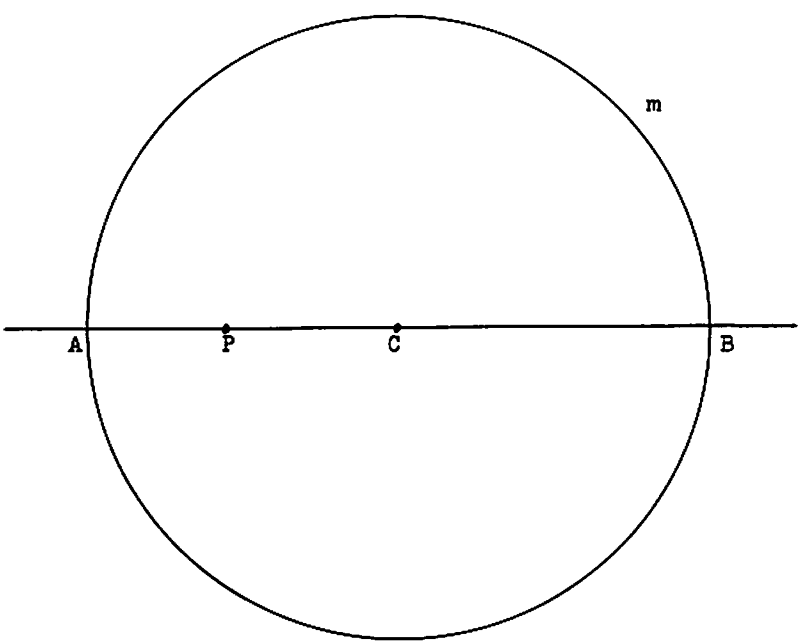

Fig. 1

* A paper presented at the KME Regional Convention at Kearney, Nebraska, April 4, 1964.

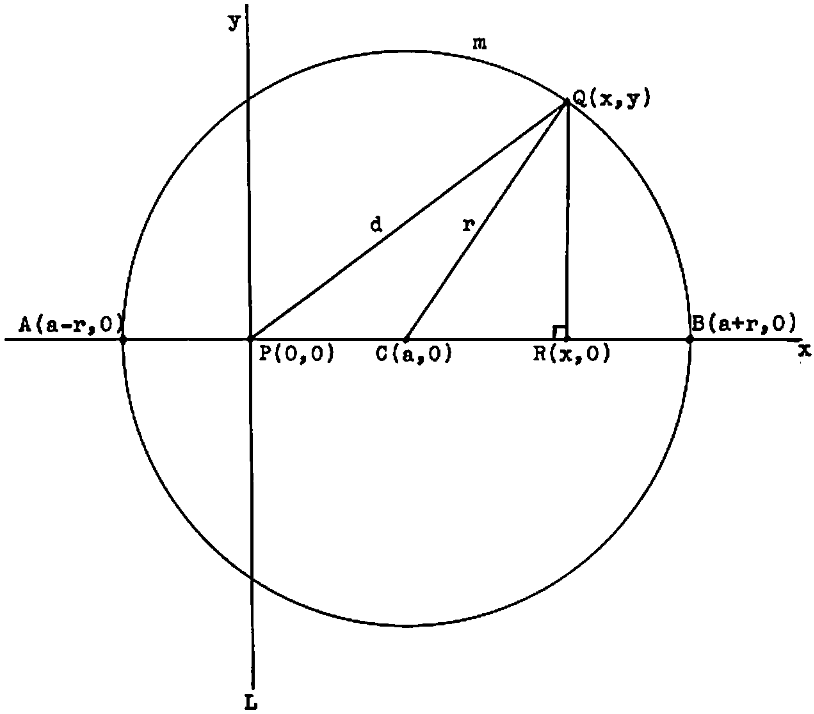

Fig. 2

and the line L be the y-axis with the origin at P. The coordinates of point P, C, A, and B are $(0, 0)$, $(a, 0)$, $(a - r, 0)$, and $(a + r, 0)$, respectively. Let Q with coordinates (x, y) be a point on the circle, and the line segment CQ join it with the center of the circle. Let the line segment PQ be the distance d from P to Q. Let the projection of Q on the x-axis be the point R with coordinates $(x, 0)$. The completed construction appears as Figure 2.

The equation of the circle m is

$$(x - a)^2 + y^2 = r^2. \tag{1}$$

From the Pythagorean Theorem, it can be seen that

$$d^2 = x^2 + y^2. \tag{2}$$

Solving for y^2.

$$y^2 = d^2 - x^2. \tag{3}$$

Substituting this expression for y^2 in equation (1), the following equation is obtained:

(4) $$(x - a)^2 + (d^2 - x^2) = r^2.$$

Solving equation (4) for d:

$$d^2 = r^2 + x^2 - x^2 + 2ax - a^2$$

(5) $$d^2 = 2ax - a^2 + r^2$$

(6a) $$d = \sqrt{2ax - a^2 + r^2}$$

or

(6b) $$d = -\sqrt{2ax - a^2 + r^2}.$$

It can be seen that equation (6a) defines a function d, where $d(x) = \sqrt{2ax - a^2 + r^2}$ is the distance from P to Q, Q any point on m. The domain of d, as is evident from Figure 2 is

$$\{x \mid a - r \leqq x \leqq a + r, x \in \text{Real Numbers}\}.$$

Since d is continuous in $[a - r, a + r]$, d has a maximum and a minimum value in this interval. In order to find the critical numbers where relative maximum or minimum values of a function occur, one must set the derivative of the function equal to zero. The derivative of d with respect to x is

$$d'(x) = \tfrac{1}{2}(2ax - a^2 + r^2)^{-\frac{1}{2}}(2a)$$

which simplifies to

(7) $$d'(x) = \frac{a}{\sqrt{2ax - a^2 + r^2}} = \frac{a}{d}.$$

Distance d is never zero since P was chosen as an interior point of m. Hence $d'(x) = 0$ if and only if $a = 0$. But if $a = 0$, P = C as can be seen in Figure 2. Since C is equidistant from all points on m and $P = C$, then P is equidistant from all points on m. Since P is equidistant from all points on m, it must follow that there is no point on m nearest to P.

At this point, the reader may want to pause briefly to reflect on the preceding argument, for there is obviously a fallacy in the reasoning. It is apparent from Figure 2 that not all points on m are equidistant from P.

As often happens when dealing with problems involving maxima and minima, one jumps to the conclusion that if an extrema exists, it must be a relative one. Setting the derivative of a function equal to zero and then finding the critical numbers work very nicely in discovering relative extrema. However, this method does not apply if the extremum is an end point minimum or an end point maximum. This is exactly the case for the function d.

If one were to plot the graph of equation (5) it would be a parabola with the vertex at $(\frac{a^2 - r^2}{2a}, 0)$, opening to the right with y intercepts at $(0, \sqrt{r^2 - a^2})$ and $(0, -\sqrt{r^2 - a^2})$.

Since

$$r > a,$$

$$a + r > 2a$$

$$(a + r)(a - r) < 2a(a - r)$$

$$a^2 - r^2 < 2a(a - r).$$

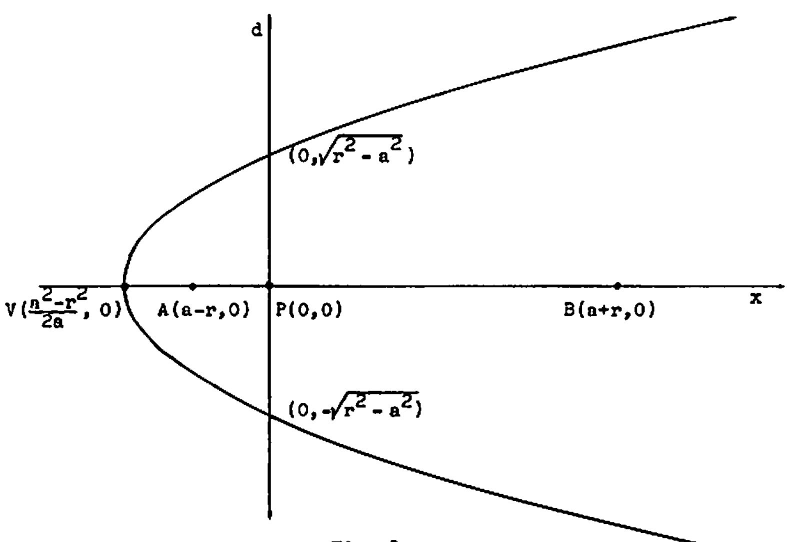

Fig. 3

Therefore

$$\frac{a^2 - r^2}{2a} < (a - r).$$

It is obvious that the x coordinate of the vertex is to the left of point $A(a - r, 0)$, and hence the graph of the equation (5) would appear as Figure 3.

Recalling the distance function d, one may remember that the domain of d is the closed interval $[a - r, a + r]$. The function d then would graph as a subset of the points on the parabola of Figure 3. The graph of d is shown as Figure 4.

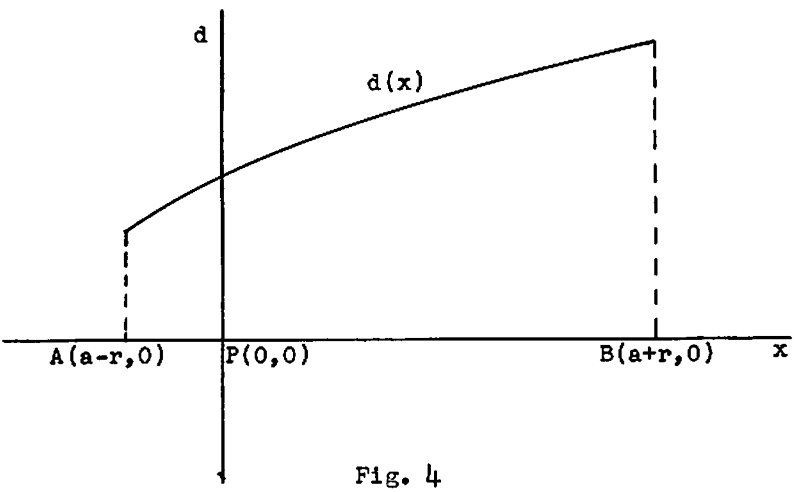

Fig. 4

It is obvious that d is increasing throughout the interval $[a - r, a + r]$; and consequently, no relative minimum can exist in this interval. However, it can be seen that an end point minimum value of the function does occur at the number $(a - r)$ and that this minimum is

$$\begin{aligned} d(a - r) &= \sqrt{2a(a - r) - a^2 + r^2} \\ &= \sqrt{a^2 - 2ar + r^2} \\ &= | a - r |. \end{aligned}$$

This conclusion is, of course, consistent with our intuitive notions gained simply by looking at Figure 2.

In short, the theory of relative extrema did not apply to the problem posed; and applying it led to the wrong conclusion. Instead, the theory of end point extrema is the real key to the understanding of this problem.

This problem together with a solution different from the solution presented in this paper was printed in a book called *Fallacies in Mathematics* by E. A. Maxwell (see Reference [3] below).

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Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the natural world.

—B. RUSSELL

Orthogonality of Vectors With Respect to a Weight Vector[1]

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and
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1. **Introduction.** Many authors attempt to lead the student into the concept of the orthogonality of two functions over a certain closed interval as an extension of the orthogonality of two finite dimensional vectors. In such approaches certain functions are presented as infinite dimensional vectors. The set of real valued functions which are square integrable on a particular closed interval can in fact be shown to constitute a vector space over the reals, see [2] for instance. However, in the midst of the analogy, the student is then introduced to the concept of the orthogonality of two square integrable functions over a closed interval with respect to a weight function or weighting function. This concept has no commonly presented analog in the work previously done in the finite dimensional vector spaces.

It is the purpose of this paper to exhibit the existence of a property in the finite dimensional vector spaces which could be considered the analog to this orthogonality of functions with respect to a weight function. The analog will be arrived at quite naturally as a result of a new multiplication of two finite dimensional vectors which will be introduced. The properties of this multiplication will then be examined with respect to the usual operations with vectors including addition, multiplication by a scalar, scalar product, and, in three dimensions, the vector product. Finally the results will be given a physical application and an example will be shown.

2. **Notation and well-known vector properties.** Let all n-tuples of the form $(u_1, u_2, \cdots, u_n)$ denote vectors in V_n, the vector space of n-tuples of real scalars over the real numbers. Vectors

[1] Prepared in a National Science Foundation Undergraduate Science Education Program at Colorado State University by Mr. Ward under the direction of Professor F. Max Stein.

will also be denoted by bold face capital letters; e.g., $\mathbf{U} = (u_1, u_2, \cdots, u_n)$ will represent a vector in V_n. We shall represent scalars by lower case letters e.g., b will denote a real scalar.

We first define the following properties and state some theorems for vectors in V_n in addition to to the usual properties assumed for a vector space over the real numbers.

(I) If $\mathbf{U} = (u_1, u_2, \cdots, u_n)$, $\mathbf{V} = (v_1, v_2, \cdots, v_n)$ are any two vectors in V_n, then

$$(\mathbf{U}, \mathbf{V}) = \sum_{i=1}^{n} u_i v_i$$

is the *scalar product* of $\mathbf{U}$ and $\mathbf{V}$, which in V_3 is customarily denoted by $\mathbf{U} \cdot \mathbf{V}$ and is called the *dot product.*

(II) $$(\mathbf{U}, \mathbf{V}) = (\mathbf{V}, \mathbf{U});$$

i.e., the scalar product is commutative.

(III) If b is any scalar, then

$$b(\mathbf{U}, \mathbf{V}) = (b\mathbf{U}, \mathbf{V}) = (\mathbf{U}, b\mathbf{V}).$$

(IV) If $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ are any three vectors in V_n, then

$$(\mathbf{U} + \mathbf{W}, \mathbf{V}) = (\mathbf{U}, \mathbf{V}) + (\mathbf{W}, \mathbf{V});$$

i.e., the scalar product distributes with respect to addition.

(V) $(\mathbf{U}, \mathbf{U}) \geqq 0$, and $(\mathbf{U}, \mathbf{U}) = 0$, if and only if $\mathbf{U} = \mathbf{O}$, where $\mathbf{O}$ is the *zero vector.* The zero vector is an n-tuple all of whose components are zero.

We also have the following properties in V_3. For any two vectors $\mathbf{U}$ and $\mathbf{V}$ in V_3.

(VI) $\mathbf{U} \times \mathbf{V} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$, is called the *vector product* or the *cross product* of U and V.

(VII) $$\mathbf{U} \times \mathbf{V} = -\mathbf{V} \times \mathbf{U};$$

i.e., the vector product is not commutative.

(VIII) $$(\mathbf{U} \times \mathbf{V}) \cdot \mathbf{W} = \mathbf{U} \cdot (\mathbf{V} \times \mathbf{W});$$

i.e., the dot and cross products commute for the *triple scalar product.*

(IX) $(\mathbf{U} \times \mathbf{V}) \times \mathbf{W} \neq \mathbf{U} \times (\mathbf{V} \times \mathbf{W})$ in general;

i.e., the *triple vector product* is not associative.

We also have the following general concepts for vectors in V_n.

(X) A set of vectors $\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_m$ is said to be *linearly independent* if and only if, for constants $c_1, c_2, \cdots, c_m$, $\sum_{i=1}^{m} c_i\mathbf{U}_i = \mathbf{O}$ implies $c_i = 0$ $(i = 1, 2, \cdots, m)$.

(XI) A set of vectors Q is said to *span* the vector space V_n if and only if every vector in V_n can be expressed as a linear combination of the vectors in Q.

3. **The vector-component product ¢.** We have thus far defined two types of vector multiplication. The first (I) produces a scalar and the second (VI) produces a vector. We now define a third type of multiplication for vectors which will produce a vector. It is applicable in V_n, whereas we recall that the vector product was defined for V_3.

DEFINITION 1. Let $\mathbf{U} = (u_1, u_2, \cdots, u_n)$, $\mathbf{V} = (v_1, v_2, \cdots, v_n)$ be any two vectors in V_n. Then the *vector-component product* of $\mathbf{U}$ and $\mathbf{V}$ is defined to be the vector $(u_1v_1, u_2v_2, \cdots, u_nv_n)$. This product will be denoted by $\mathbf{U} ¢ \mathbf{V}$.

4. **Properties of the vector-component product.** We have, directly for Definition 1, the following properties of the vector-component product.

(1) If b is any scalar, then

$$b(\mathbf{U} ¢ \mathbf{V}) = (b\mathbf{U}) ¢ \mathbf{V} = \mathbf{U} ¢ (b\mathbf{V}).$$

(2) If $\mathbf{W}$ is any other vector in V_n, then

$$(\mathbf{U} + \mathbf{V}) ¢ \mathbf{W} = (\mathbf{U} ¢ \mathbf{W}) + (\mathbf{V} ¢ \mathbf{W}),$$

i.e., the vector-component product distributes with respect to addition.

(3) $$\mathbf{U} ¢ \mathbf{V} = \mathbf{V} ¢ \mathbf{U},$$

i.e., the vector-component product is commutative. Thus from (2) and (3)

$$\mathbf{U} ¢ (\mathbf{V} + \mathbf{W}) = (\mathbf{U} ¢ \mathbf{V}) + (\mathbf{U} ¢ \mathbf{W}).$$

(4) $$(\mathbf{U} ¢ \mathbf{V}) ¢ \mathbf{W} = \mathbf{U} ¢ (\mathbf{V} ¢ \mathbf{W});$$

i.e., the vector-component product is associative.

DEFINITION 2. The *scalar product* of two vectors **V** and **W** in $\mathbf{V}_n$ *with respect to the weight vector* **U** in V_n is

$$\mathbf{U} ¢ \mathbf{V} \cdot \mathbf{W} = \sum_{i=1}^{n} u_i v_i w_i.$$

Here we must recognize that the vector-component product must be performed before the scalar product.

THEOREM 1. (**U** ¢ **V**, **W**) = (**U** ¢ **W**, **V**) = (**V** ¢ **U**, **W**) = (**V** ¢ **W**, **U**) = (**W** ¢ **U**, **V**) = (**W** ¢ **V**, **U**).

Proof: The proof here follows immediately from the associativity and commutivity of the factors in each term of the summation in Definition 2.

5. Further immediate properties of the scalar product of two vectors with respect to a weight vector.

(5) b(**U** ¢ **V**, **W**) = (*b***U** ¢ **V**, **W**) = (**U** ¢ *b***V**, **W**) = (**U** ¢ **V**, *b***W**).

If **R** is an other vector in V_n, then

(6) ([**U** + **V**] ¢ **W**, **R**) = (**U** ¢ **W**, **R**) + (**V** ¢ **W**, **R**).

Thus from Theorem 1 we have results similar to that of (6) for (**U** ¢ [**V** + **W**], **R**) and (**U** ¢ **V**, [**W** ¢ **R**]).

If $u_i > 0$, (i = 1, 2, $\cdots$, n), then

(7) (**U** ¢ **V**, **V**) $\geqq$ 0, for arbitrary **V** in V_n,

and

(8) (**U** ¢ **V**, **V**) = 0, if and only if **V** = **O**.

We note here that unless **U** is restricted to have positive components, a vector can be orthogonal to itself with respect to an appropriate **U**.

6. Additional properties of the vector-component product in V_3.

(9) (**U** ¢ **V**) × **W** $\neq$ **U** ¢ (**V** × **W**)

in general, for in fact

(A) (**U** ¢ **V**) × **W**

$$= (u_2v_2w_3 - u_3v_3w_2,\ u_3v_3w_1 - u_1v_1w_3,\ u_1v_1w_2 - u_2v_2w_1)$$

whereas

$\mathbf{U} \cent (\mathbf{V} \times \mathbf{W})$

$$= (u_1 v_2 w_3 - u_1 v_3 w_2,\ u_2 v_3 w_1 - u_2 v_1 w_3,\ u_3 v_1 w_2 - u_3 v_2 w_1).$$

We can see that we have equality in (9), for instance, if $u_1 = u_2 = u_3$, or if one of the vectors is the zero vector.

From (VI) and Definition 1 we have

(10) $\quad \mathbf{U} \cent (\mathbf{V} \times \mathbf{W}) = - \mathbf{U} \cent (\mathbf{W} \times \mathbf{V}).$

(11) In general $(\mathbf{U} \cent \mathbf{V}) \times \mathbf{W} \neq - (\mathbf{U} \cent \mathbf{W}) \times \mathbf{V}$, since
(B) $(\mathbf{U} \cent \mathbf{W}) \times \mathbf{V}$

$$= (u_2 v_3 w_2 - u_3 v_2 w_3,\ u_3 v_1 w_3 - u_1 v_3 w_1,\ u_1 v_2 w_1 - u_2 v_1 w_2).$$

A comparison of (A) and (B) reveals that equality holds in (11) when $u_i v_i w_j = - u_i v_j w_i$; $i \neq j$; $(i, j = 1, 2, 3)$.

7. **Orthogonality of vectors with respect to a weight vector.**

DEFINITION 3. The m vectors $\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_m$ in V_n are said to be *mutually orthogonal with respect to the weight vector* $\mathbf{P}$ in V_n if and only if

(C) $\quad (\mathbf{P} \cent \mathbf{U}_i, \mathbf{U}_j) = 0;\ i \neq j;\ i, j = 1, 2, \cdots, m.$

DEFINITION 4. The m vectors $\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_m$ in V_n are said to be *mutually orthonormal with respect to the weight vector* $\mathbf{P}$ in V_n if and only if condition (C) holds and

(D) $\quad (\mathbf{P} \cent \mathbf{U}_i, \mathbf{U}_i) = 1,\ (i = 1, 2, \cdots, m)$

Note: If $\mathbf{P} = (1, 1, \cdots, 1)$ we have orthogonality in the usual sense if (C) holds and orthonormality in the usual sense if (C) and (D) hold.

8. **Existence theorems.** Before stating theorems regarding the existence of weight vectors with respect to which certain sets of n-tuples are orthogonal, let us consider some facts that readily follow from what has been developed thus far. If we are given m linearly independent vectors in V_n; say $\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_m$, $m \leqq n$; then we can produce at most $m \frac{(m-1)}{2}$ different vectors by forming all possible different vector-component products of the $\mathbf{U}_i$; i.e., all possible combinations of m vectors taken two at a time. Since each of the vectors thus formed is n-dimensional, at most n of the $m \frac{(m-1)}{2}$ vectors can be linearly independent. We denote the

highest power set of linearly independent vectors which can be formed from all possible vector-component products of the $\mathbf{U}_i$ by the symbol L_{max}. The set L_{max} may or may not span V_n; i.e., L_{max} may or may not consist of n vectors.

System (C) may be written in the form

(E)

$$\sum_{i=1}^{n} p_i u_{ji} u_{ki} = 0; (j = 1, 2, \cdots, m - 1), (k = j + 1, \cdots, m),$$

where $\mathbf{U}_i = (u_{i1}, u_{i2}, \cdots, u_{in})$, $(i = 1, 2, \cdots, n)$. System (E) is then a system of $m \frac{(m-1)}{2}$ equations in n unknowns. But we have already seen that at most n of these equations can be linearly independent. Thus we can reduce system (E) to

$$\text{(F)} \qquad \sum_{i=1}^{n} p_i h_{ji} = 0; j = i, 2, \cdots, k \leqq n,$$

where $\mathbf{L}_j = (h_{j1}, h_{j2}, \cdots, h_{jn})$, $(j = 1, 2, \cdots, k)$ are the elements of L_{max} for the $\mathbf{U}_i$.

With these observations and the notation developed within them we state the following existence theorems.

THEOREM 2. If, for the set whose elements are the vectors $\mathbf{U}_i$, L_{max} has m elements, $m < n$, then there exists at least one non-zero weight vector $\mathbf{P}' = p'_1, p'_2, \cdots, p'_n$ with respect to which the vectors $\mathbf{U}_i$ are mutually orthogonal.

Proof: Since system (E) can be reduced to a system of the form (F) with m homogeneous equations in n unknowns, $m < n$, there will always exist a non-trivial solution for the system. We denote such a non-trivial solution in the form of a vector $\mathbf{P}'$ to obtain the desired result.

THEOREM 3. If, for the set whose elements are the vectors $\mathbf{U}_i$, the elements of L_{max} span V_n, then there exists a non-zero weight vector $\mathbf{P}'$ with respect to which the vectors $\mathbf{U}_i$ are mutually orthogonal if and only if the n-th order determinant $|h_{ij}|$ is equal to zero, where for vectors $\mathbf{L}_i$ in L_{max}, $\mathbf{L}_i = (h_{i1}, h_{i2}, \cdots, h_{in})$, and h_{ij} is an n by n matrix whose row vectors are the vectors in L_{max}.

Proof: If we reduce system (E) to the form of system (F) with $m = n$ we then have a system of n homogeneous equations in

n unknowns. Such a system is known to have a non-trivial solution if and only if its coefficient matrix has a determinant which is zero. We may interpret the determinant $|h_{ij}|$ as a coefficient matrix for the system which we have expressed in the form of (F). We then have the desired result that $|h_{ij}|$ must be zero, if a non-zero weight vector **P'** is to exist as the theorem stipulates.

9. **An example of the use of the vector-component product.** If we restrict the weight vectors to that set of vectors all of whose members have only positive components we have a readily available physical application of the vector-component product. Vectors in a non-cartesian coordinate system N can be transformed into vectors in a rectangular cartesian coordinate system by use of an appropriate vector. If we let **U** be a vector whose components are given in N and **R** be a vector such that its components have the same respective ratios as the scales on the axes of N, then **R** ¢ **U** will represent a vector in a rectangular cartesian coordinate system. It is this usage which gives partial rise to the name weight vector which has previously been used.

If we have two vectors **U** and **V** in a rectangular non-cartesian coordinate system in three-space, we may want to determine the scalar product of **U** and **V** in a rectangular cartesian system. We could then take

$$([\mathbf{R} \cent \mathbf{U}], [\mathbf{R} \cent \mathbf{V}]) = ([\mathbf{R} \cent \mathbf{R}] \cent \mathbf{U}, \mathbf{V}) = ([\mathbf{P} \cent \mathbf{U}], \mathbf{V})$$

where $\mathbf{R} = (r_1, r_2, r_3)$, $\mathbf{P} = (r_1^2, r_2^2, r_3^2)$.

Example: Let the velocities of particles have their x component given in millimeters-per-second, y component given in centimeters-per-second, and z component given in decimeters-per-second. Is the path of a particle moving in this system outward along the vector $(-8, 4, .2)$ perpendicular to the path of another particle moving outward along the vector $(5, 5.1, -1)$?

Solution: Under the conditions of the problem, take

$$\mathbf{R} = (1, 10, 100), \mathbf{P} = (1, 100, 10{,}000),$$

then

$$(\mathbf{P} \cent \mathbf{U}, \mathbf{V}) = (-8)(5) + (4)(5.1)(100) + (.2)(-1)(10{,}000)$$
$$= 0.$$

(Continued on page 64)

The Problem Corner

EDITED BY F. MAX STEIN

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on seperate sheets before March 1, 1965. The best solutions submitted by students will be published in the Spring 1965 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor F. Max Stein, Colorado State University, Fort Collins, Colorado.

PROPOSED PROBLEMS

176. *Proposed by Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.*

Discuss the sequence of integers $[n + \sqrt{n} + \frac{1}{2}]$, $n = 1, 2, \cdots$, where $[x]$ is the greatest integer not exceeding x.

177. *Proposed by Howard Frisinger, Assistant Professor Mathematics, Colorado State University, Fort Collins, Colorado.*

Given a rectangle of length d_1 and width d_2, $d_1 > d_2$. If a square of side d_2 is removed from the rectangle, the remaining rectangle has length d_2 and width d_3, $d_2 > d_3$. If this process is continued, find the number r where $r = d_{i+1}/d_i$, $i = 1, 2, 3, \cdots$.

178. *Proposed by Douglas A. Engel, Hays, Kansas.*

Prove that the following formula is true:

$$n! = (n-1)(n-1)! + (n-2)(n-2)! + \cdots + 2(2!) + 1(1!) + 1(0!).$$

179. *Proposed by Leigh R. Janes, Houston, Texas.*

Using the base eight or nine, determine mappings from digits into letters that will make the following addition correct:

$$\begin{array}{r} W\ R\ O\ N\ G \\ W\ R\ O\ N\ G \\ \hline R\ I\ G\ H\ T \end{array}$$

180. *Proposed by Fred W. Lott, Jr., State College of Iowa, Cedar Falls, Iowa.*

My house is on a road where the numbers run 1, 2, 3, $\cdots$ consecutively. My number is a three digit one, and, by a curious coincidence, the sum of all house numbers less than mine is the same as the sum of all house numbers greater than mine. What is my number, and how many houses are there on my road?

SOLUTIONS

171. *Proposed by Robert A. Bruce, Colorado State University, Fort Collins, Colorado.*

Find x such that $x^{x^{\cdot^{\cdot^{\cdot}}}} = 2$. Similarly solve the equation $x^{x^{\cdot^{\cdot^{\cdot}}}} = 4$ for x and compare the results.

Partial solution by John L. Lebbert, Washburn University, Topeka, Kansas.

The numbers $x, x^x, x^{x^x}, \cdots, x^{x^{\cdot^{\cdot^{\cdot}}}}$ form an infinite sequence; this sequence converges to two or to four as indicated in the problem. Denote x by a_0. Then $a_{n+1} = a_0^{a_n}$. Assume this series converges to some number p.

As

$n \rightarrow \infty : a_n \rightarrow p,\ a_{n+1} \rightarrow p$, and we have $p = a_0^p$ or $p = x^p$ or

$$x = (p)^{1/p}.$$

In the two above equations we have $p = 2$ and $p = 4$.

$$(1)\quad p = 2:\ x = (2)^{1/2} = \sqrt{2}.$$

$$(2)\quad p = 4:\ x = (4)^{1/4} = (2)^{1/2} = \sqrt{2}.$$

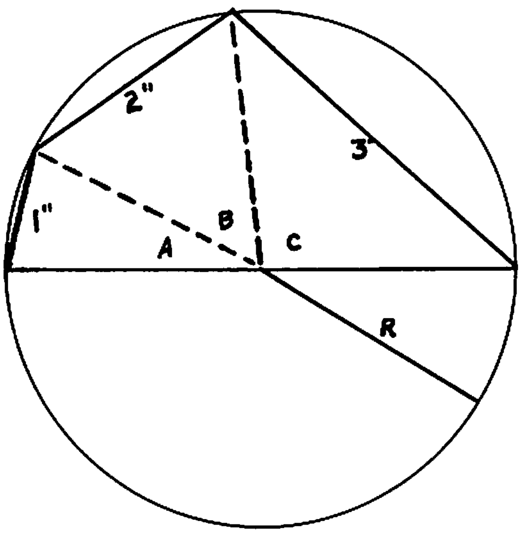
The results are the same in both instances: $x = \sqrt{2}$.

Editor's Note: Although the sequence apparently converges to two numbers, 2 and 4, it can be shown that 2 is the correct limit.

Solved completely by the proposer; also solved partially by Yeuk-Laan Chui, Anderson College, Anderson, Indiana, and Thomas A. Jones, Colorado State University, Fort Collins, Colorado.

172. *Proposed by James F. Rasmussen, Wayne State College, Wayne, Nebraska.*

Determine the radius of the circle shown below.



Solution by George W. Norton, Marietta College, Marietta, Ohio.

By adding the dotted lines and letters $A,B,C,$ in the figure and using the law of cosines we can write

$\cos A = 1 - 1/2R^2$, $\cos B = 1 - 4/2R^2$, $\cos C = 1 - 9/2R^2$.

Since $A + B + C = 180°$,

$\cos(A + B) = -\cos C = \cos A \cos B - \sin A \sin B,$

and from $\sin^2\theta + \cos^2\theta = 1,$

$\sin A = \sqrt{4R^2 - 1} / 2R^2$ and $\sin B = 2\sqrt{R^2 - 1} / R^2$.

$$\text{Then } \left(1 - \frac{1}{2R^2}\right)\left(1 - \frac{2}{R^2}\right) - \left(\frac{\sqrt{4R^2 - 1}}{2R^2}\right)\left(\frac{2\sqrt{R^2 - 1}}{R^2}\right)$$

$$= -\left(1 - \frac{9}{2R^2}\right),$$

which simplifies to

$$(R^2)^3 - 7(R^2)^2 + \frac{49}{4}(R^2) - \frac{9}{4} = 0.$$

This can be solved for R^2 by a method for cubic equations found in the Handbook of Chemistry and Physics, p. 318. Then $R = \pm\ 2.056, \pm\ 0.452, \pm\ 1.6$, of which only 2.056 is pertinent here; the others are not reasonable answers. Therefore $R = 2.056''$, approximately.

Also solved by José A. Boix, University of Southwestern Louisiana, Lafayette, Louisiana; Yeuk-Laan Chui, Anderson College, Anderson, Indiana; Thomas P. Deuce, Bowling Green State University, Bowling Green, Ohio; John L. Lebbert, Washburn University, Topeka, Kansas; and the proposer.

173. *Proposed by J. Frederick Leetch, Asst. Professor of Mathematics, Bowling Green State University, Bowling Green, Ohio.*

If G is a finite cyclic group of order n, generated by a, then the product of the n distinct elements is either a^n (if n is odd), or $a^{n/2}$ (if n is even).

Solution by Yeuk-Laan Chui, Anderson College, Anderson, Indiana.

Let $a, a^2, a^3, \ldots, a^n$ be elements of the group. Then $a^m = a^{m+nl}$ (where $m = 1,2,3,\ldots, n$ and $l = 0,1,2,\ldots$)

Case 1: n is odd $\rightarrow n = 2k + 1$ (where $k = 0,1,2,\ldots$)

Product of elements:

$$\begin{aligned} a \cdot a^2 \cdot a^3 \cdot \cdots \cdot a^{2k+1} &= a^{\frac{1}{2}(2k+1)(2k+2)} \\ &= a^{(k+1)(2k+1)}. \\ &= a^{2k(k+1)} \cdot a^{k+1} \\ &= [a^{2k+2}]^k \cdot a^{k+1} \\ &= [a^{(2k+1)+1}]^k \cdot a^{k+1} \\ &= a^k \cdot a^{k+1} \\ &= a^{2k+1} = a^n \end{aligned}$$

Case 2: n is even $\rightarrow n = 2k$ (where $k = 1,2,3, \ldots$)

Product of elements:

$$a \cdot a^2 \cdot a^3 \cdot \cdots \cdot a^{2k} = a^{\frac{1}{2}(2k)(2k+1)}$$
$$= a^{(1+2k)k}$$
$$= [a^{1+2k}]^k$$
$$= a^k = a^{\frac{1}{2}n}.$$

Also solved by John L. Lebbert, Washburn University, Topeka, Kansas.

174. *Proposed by George Tzelepis, Ulster County Community College, Kingston, New York.*

Consider the equations

$$\text{(A)} \qquad x^3 + 3mx + 2k = 0$$
$$\text{(B)} \qquad x^2 + 2mx + k = 0$$

Suppose that $k \neq 0$.

a. Find a relation between m and k, such that equations (A) and (B) have a common root.

b. Express k as a function of m.

c. Find the least positive integer m, which is even, such that k is rational.

d. Solve equation (A) for x and solve equation (B) for x completely, using the values that you found for m and k.

Solution by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.

Eliminating k from the two equations gives the equation

$$x(x^2 - 2x - m) = 0$$

with roots $x = 0,\ 1 \pm \sqrt{1+m}$. The root $x = 0$ can be neglected since this would force k to be zero.

Solving equation (B) for x we get

$$x = -m \pm \sqrt{m^2 - k}.$$

Therefore, if equations (A) and (B) are to have a common root we may write

$$1 \pm \sqrt{1+m} = -m \pm \sqrt{m^2 - k}.$$

Transposing $-m$, squaring, and simplifying results in the following expression for k in terms of m:

$$k = -2 - 3m \mp (2 + 2m)\sqrt{1 + m}.$$

The least positive even integer m, which makes k rational is clearly $m = 8$; this makes $k = -80, 28$.

Using $k = -80$ we obtain the two equations.

$$(A') \quad x^3 + 24x - 160 = 0$$

$$(B') \quad x^2 + 16x - 80 = 0.$$

The first has roots of $x = 4, -2 \pm 6i$, and the second has roots of $x = 4, -20$.

Using $k = 28$ we obtain the two equations

$$(A'') \quad x^3 + 24x + 56 = 0$$

$$(B'') \quad x^2 + 16x + 28 = 0.$$

The first has roots of $x = -2, 1 \pm 3i\sqrt{3}$, and the second has roots of $x = -2, -14$.

Also solved by Yeuk-Laan Chui, Anderson College, Anderson, Indiana; John L. Lebbert, Washburn University, Topeka, Kansas; George W. Norton, Marietta College, Marietta, Ohio; and the proposer.

175. *Proposed by the Editor.*

It is known that integers 0 through 112 can be expressed by using exactly four 4's and the operation of addition, subtraction, multiplication, division, extracting the square root, factorial, decimal and powers. Show how to write integers 0 through 20 in this manner.

Solution by William Bramley, State College of Iowa, Cedar Falls, Iowa.

$$4 + 4 - 4 - 4 = 0$$

$$4 - 4 + \frac{4}{4} = 1$$

$$\frac{4}{4} + \frac{4}{4} = 2$$

$$\sqrt{4} + \sqrt{4} - \frac{4}{4} = 3$$

$$4 + 4 - \sqrt{4} - \sqrt{4} = 4$$

$$\frac{4!}{4} - \frac{4}{4} = 5$$

$$\frac{4!}{4} + 4 - 4 = 6$$

$$\frac{4!}{4} + \frac{4}{4} = 7$$

$$4 + 4 + 4 - 4 = 8$$

$$4 + 4 + \frac{4}{4} = 9$$

$$(4)(4) - 4 - \sqrt{4} = 10$$

$$\frac{4!}{\sqrt{4}} - \frac{4}{4} = 11$$

$$\frac{4!}{4} + 4 + \sqrt{4} = 12$$

$$\frac{4!}{\sqrt{4}} + \frac{4}{4} = 13$$

$$(4)(4) - \frac{4}{\sqrt{4}} = 14$$

$$(4)(4) - \frac{4}{4} = 15$$

$$\frac{4^4}{(4)(4)} = 16$$

$$(4)(4) + \frac{4}{4} = 17$$

$$4! - \frac{4!}{(\sqrt{4})(\sqrt{4})} = 18$$

$$4! - 4 - \frac{4}{4} = 19$$

$$4! + \sqrt{4} - \frac{4!}{4} = 20$$

Also solved by Yeuk-Laan Chui, Anderson College, Anderson, Indiana; Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio; Thomas A. Jones, Colorado State University, Fort Collins, Colorado; John L. Lebbert, Washburn University, Topeka, Kansas; George W. Norton, Marietta College, Marietta, Ohio; and Jerry L. Roger, University of Tulsa, Tulsa, Oklahoma.

Directions for Papers to be Presented at the Fifteenth Biennial Kappa Mu Epsilon Convention

FORT COLLINS, COLORADO
April 25-27, 1965

A significant feature of this convention will be the presentation of papers by student members of KME. The topic on mathematics which the student selects should be in his area of interest and of such a scope that he can give it adequate treatment within the time allotted. By this time the preparation of his paper should be well under way, and he should take advantage of all opportunities available to present his paper before groups interested in mathematics.

Who may submit papers: Any member may submit a paper for use on the convention program. Papers may be submitted by graduates and undergraduates; however, undergraduates will not compete against graduates. Awards will be granted for the best papers presented by undergraduates. Special awards may be given for the best papers presented by graduates, if a sufficient number are presented.

Subject: The material should be within the scope of the understanding of undergraduates, preferably the undergraduate who has completed differential and integral calculus. The Selection Committee will naturally favor papers that are within this limitation and which can be presented with reasonable completeness within the time limit prescribed.

Time Limit: The usual time limit is twenty minutes but this may be changed on the recommendation of the Selection Committee.

Paper: The paper to be presented together with a description of charts, models, or other visual aids that are to be used in the presentation of the paper should be submitted to the Selection Committee. A carbon copy of the complete paper may be submitted, and in lieu of the complete paper an outline (sufficient in detail to give the committee a clear idea of the content, methods, and scope of the paper) may be submitted before the February 1st deadline to be followed by the complete paper before March 1, 1965. A bibliography of source materials together with the statement that the author of the paper is a member of KME and his official classification in school, undergraduate or graduate, should accompany his paper.

Date and Place Due: The papers must be submitted no later than February 1, 1965, to the office of the National Vice-President.

Selection: The Selection Committee will choose about eight papers for presentation at the convention. All other papers will be listed by title on the convention program. The authors of these papers selected for presentation will be notified as soon as possible after the selection is made.

Prizes:

1. The author of each paper presented will be given a two-year extension of his subscription of *The Pentagon.*
2. Authors of the two or three best papers presented by undergraduates, according to the judgment of a committee composed of faculty and students will be awarded copies of suitable mathematics books.
3. If a sufficient number of papers submitted by graduate students are chosen for presentation, then one or more similar prizes will be awarded for the best paper or papers from this group.

Harold Tinnappel
National Vice-President
Bowling Green State University
Bowling Green, Ohio

The Mathematical Scrapbook

EDITED BY J. M. SACHS

The typical mathematician feels great confidence in a conclusion reached by careful reasoning. He is not convinced to the same degree by experimental evidence. For the typical engineer these statements may be reversed. Confronted by a carefully thought-out theory which predicts a certain result, and a carefully performed experiment which fails to produce it, the typical mathematician asks first, "What is wrong with the experiment?" And the typical engineer, "What is wrong with the argument?"

—T. C. FRY

=Δ=

We can examine a few examples of products of three consecutive integers such as $1\cdot2\cdot3 = 6$, $2\cdot3\cdot4 = 24$, $3\cdot4\cdot5 = 60$, $4\cdot5\cdot6 = 120$. None of the products is a square of an integer. Is it true that $n(n + 1)(n + 2)$ cannot be the square of an integer?

Suppose $n(n + 1)(n + 2) = p^2$, p an integer. Then either p is divisible by n or p is not divisible by n.

Case 1: p is divisible by n, that is $p = qn$, where q is an integer. Then

$$n(n + 1)(n + 2) = q^2n^2$$

$$(n + 1)(n + 2) = q^2n.$$

Thus $n^2 + 3n + 2$ is divisible by n which is possible only if $n = 1$ or $n = 2$.

Case 2: p is not divisible by n. If p^2 is divisible by n, can you complete an argument that this is true only for limited values of n? Can you construct an argument that $n(n + 1)(n + 2)$ is not a cube or fourth power of a positive integer?

=Δ=

The familiar definition: An axiom is a self-evident truth means, if it means anything, that the proposition which we call an axiom has been approved by us in the light of our experience and intuition. In this sense mathematics has no axioms, for mathematics is a formal subject over which formal and not material implication reigns.

—E. B. WILSON

There are nine candidates for mayor of our city in this election year. A total of 321,823 votes are cast for this office. The winner, according to the returns, has only 35 votes more than his nearest rival, 65 votes more than the next, 110 votes mores than the next, 1321 votes more than the next, 2721 votes more than the next, 5630 votes more than the next, 12135 votes more than the next and 21,627 votes more than the last. When the astute Professor Noitall sees these results, he immediately tells the authorities that there has been a mistake made in the count. (Can you tell why without any further hints.) The puzzled officials demand an explanation. Professor N. irritated by this request to belabor the obvious gives this answer, "The sum of 35, 65, 110, 1321, 2721, 5630, 12135, 21627, and 321832 is 365476. The sum of the digits in the last total is 31. Three plus one is four. Now please go away and let me work!" Can you now make a convincing argument that there has been a mistake?

=Δ=

The nineteenth century, which prided itself upon the invention of steam and evolution might have derived a more legitimate title to fame from the discovery of pure mathematics.

—Bertrand Russell

=Δ=

Plutarch mentions that there are two rectangles for which the measure of area in square units is the same as the measure of perimeter in linear units. The conditions of the problem indicate that integral solutions are being sought. If we use L and W to represent the length and width respectively we can state Plutarch's solution in our symbolism. Let $L \geqq W$.

$$A = L \cdot W \qquad p = 2L + 2W$$

$$LW = 2L + 2W$$

$$W = 2 + \frac{4}{L-2}.$$

If the solutions are to be integral $L - 2$ is a divisor of 4. Thus $L - 2$ is equal to 1, 2, or 4. For $L - 2 = 1$ we get a contradiction of $L \geqq W$. For $L - 2 = 2$ we get one solution, $L = 4$, $W = 4$. For $L - 2 = 4$ we get the other solution $L = 6$, $W = 3$. Suppose we remove the restriction that the solutions must be integral. If both L and W are to be positive rationals what can you say about the set of solutions? Suppose we permit L to be irrational, what can you say about the set of solutions?

=△=

Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable. Solving problems is the specific achievement of intelligence and intelligence is the specific gift of mankind. Solving problems can be regarded as the most characteristically human activity.

—G. POLYA

=△=

The perfect number, one equal to the sum of all of its proper divisors, has intrigued mathematicians for thousands of years. Some perfect numbers can be generated in the following way. Let $p = 2^0 + 2^1 + 2^2 + \cdots + 2^n$. If p is a prime then $2^n p$ *is* perfect. This is proved in the writing of Euclid. Let us look at a few applications. For $n = 1$, $p = 3$ which is a prime so $2^1 \cdot 3$ or 6 is perfect. For $n = 2$, $p = 7$ which is a prime so $2^2 \cdot 7$ or 28 is perfect. For $n = 3$, $p = 15$ which is not a prime. For $n = 4$, $p = 31$ which is a prime so $2^4 \cdot 31$ or 496 is perfect. Among the many conjectures about perfect numbers and this method of generating them are the following:

i. This method generates all perfect numbers.

ii. Perfect numbers end alternately in the digits 6 and 8.

iii. There is no more than a single perfect number between successive powers of 10.

iv. All perfect numbers are even.

v. Every perfect number is triangular.

vi. Odd perfect numbers, if they exist, must be of the form $(4n + 1)c^2$ where $(4n + 1)$ is a prime.

Some of these conjectures contradict others. How many do you believe to be true?

=△=

Explicit definitions describe the thing being defined in terms of things whose meaning is already assumed to be understood. This is different from the definitions as given in dictionaries where a "mesh-type" of definition is given and there is no clear understanding as to which term is defined first, but enough synonyms are included so that a person who nows the meaning of many other words is likely to have enough information to decipher the meaning of the word under consideration. In mathematics, definitions are given in a "tree-like" fashion where a term is defined by using previously accepted terms. We try to avoid the "circular" lines of definitions found in dictionaries.

—R. H. Bing

=△=

For the non-Euclidean geometries created independently by the Hungarian, John Bolyai, and the Russian, Lobachevski, were so shocking to the regnant philosophy of the time as not only to challenge attention but to compel it and they were at the same time so invulnerable that logicians, mathematicians, physicists and philosophers were obliged to scutinize as never before the method by which the strange new doctrines had been constructed. There ensued a revolution in the theory of knowledge, the philosophy of science was greatly advanced, and postulational thinking was henceforth to play increasingly a double role, that of builder and that of critic or judge.

—C. J. Keyser

=△=

The problems suggested by Fermat include one which can be solved by proving that successive powers of 5 exhibit a pattern of remainders upon division by 7.

5/7	Remainder 5
25/7	Remainder 4
125/7	Remainder 6
625/7	Remainder 2
3125/7	Remainder 3
15625/7	Remainder 1
78125/7	Remainder 5

What kind of a proof can you construct that this is a repeating pattern? Does a pattern emerge when the powers of 3 are divided by 5? Does the emergence of a pattern depend upon both or one of the numerator and denominator being primes? Are there any conditions on numerator or denominator?

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of *The Pentagon*. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Harold E. Tinnappel, Bowling Green State University, Bowling Green, Ohio.

Elements of General Topology, D. Bushaw, John Wiley and Sons, New York, 1963, 166 pp., $6.95.

This book is written for readers who have a background of about three years of sound undergraduate mathematics. After a chapter devoted to a brief sketch of the history of topology, including mention of the original Hausdorff axioms, the book provides a discussion of general topological spaces and continues with chapters on "Continuity and Homeomorphism," "The Construction of Topologies," "Separation, Compactness, and Connectedness," "Uniform Spaces," and "Completeness." The discussion of the idea of uniform spaces, and the use of this concept, together with that of filter, in the discussion of completeness is rather unusual in a book at this level. Two appendices conclude the book. One of these is devoted to a summary of the needed results from set theory, the other to a very helpful list of suggestions for further reading. The author presents numerous exercises, some of which extend the theory and some of which illustrate its application to specific examples. Hints are provided for most of these exercises at the end of the book.

One unfortunate aspect of the style of presentation used is the fact that definitions and important remarks are often hidden in conversational paragraphs. This fact, together with the assumption of a relatively large amount of set theory, makes the reading rather hazardous for the beginner, who in a number of instances might miss a number of finer details unless carefully guided in his study. The book's usefulness as a reference suffers from an incomplete index of special symbols which make it difficult to determine whether a given notation is presumed familiar, is to be found in the appendix on set theory, or introduced in the body of the text. The book is relatively free from typographical errors, but in one or two cases errors appear in examples or statements of exercises.

The book includes a treatment of the topics of product and quotient spaces. The important hierarchy of separation axioms, giving rise to the notion of T_0, T_1, T_2 spaces, etc., is given its proper place in the development. The discussion of compactness relates this concept to the ideas of sequential and countable compactness.

In spite of the difficulties mentioned, this book is a respectable addition to the growing library of books on topology for undergraduates.

—EDWARD Z. ANDALAFTE
Southwest Missouri State College

Elements of Point Set Topology, John D. Baum, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964, 150 pp., $5.95.

The author has written this book mindful of the present and hopeful of the not too distant future training of most undergraduate majors. It is felt that the book contains material which "every future undergraduate mathematics major should know." The author is offering an introductory text which in spirit parallels many of the present texts in modern algebra. As such the book lays (in one semester) a foundation that will enable the student to pursue a study of analysis, advanced point set topology, or algebraic topology.

Several mathematical concepts are introduced in a manner somewhat different from the so-called accepted practice. However, in so doing, the author is extremely careful to point up such deviations from the usual. In most cases he substantiates why he has so chosen to depart from existing nomenclature.

Chapter 0 contains a study of the algebra of sets. In Chapter 1 we find a systematic presentation of neighborhood system, topology, open set, limited point, derived set, closure, closed set, subspace, limit of sequence, Hausdorff Space, bases, countability axioms, separability, sub-bases, product space, and the product topology. Continuous functions and homeomorphisms are discussed in Chapter 2. T_i, $i = 0, 1, 2, 3, 4$, normal and regular spaces and varieties of compactness are the main concern in Chapter 3. Chapter 4 concerns itself primarily with varieties of connectedness. Since geometric notions are used in motivating the presentation of the previous chapters, the metric topology is presented in Chapter 5 as a rather special sort of topological space in which there is defined a distance concept. Many and meaningful exercises are included for the student. Even though much material is presented the author has kept it from being a collection of definitions and theorems.

The book is written in "lecture form." The reviewer appreciated greatly the expository portions of the book and the diction used. Misprints are minimal. This excellent book will be welcomed by the instructors of undergraduate courses in topology.

—S. Elwood Bohn
Miami University

Tables of Series, Products, and Integrals, I. M. Ryshik and I. S. Gradstein, New York: Plenum Press, 1963, 198 pp., $7.95.

This is a copious and skillful compilation of formulas designed principally for physicists and engineers. Little space is given to explanations and there are no derivations or proofs of theorems. All definitions and commentary appear in parallel columns in both German and English translations from the Russian.

An introductory chapter lists various finite series, infinite series, and infinite products of numbers. Chapter I consists of summation formulas and series representations of the simple elementary functions and various functions of them.

Chapters II, III, IV, present an extensive collection of indefinite and definite integrals. There are over nine hundred indefinite integrals and nearly a thousand definite integrals of elementary functions as well as 225 definite integrals of special functions. There is a more comprehensive selection of functions than has heretofore been available.

When looking up an integral, the user of these tables must first simplify his integral by suitable substitutions in such a way that the arguments of the external function take a simple form. The required substitution in a given instance may not be evident.

The order adopted for the individual elementary functions is: rational, algebraic, exponential hyperbolic, trigonometric, logarithmic, inverse hyperbolic, and inverse trigonometric. For definite integrals of special functions the order is: elliptic integrals and functions, integral, integral sine and cosine, error integral, Fresnel integrals, Gamma functions and related functions, Bessel functions, Legendre functions, orthogonal functions (Tschebyscheff, Hermite, Jacobi, Laguerre), hypergeometric and confluent hypergometric functions, and Riemann zeta functions.

In determining order, the external function is dominating; expressions built with the same dominating function are ordered according to the rank of the next inner function with the operators ranked in the order: polynomial, rational, algebraic, and power. If the integrand contains two external functions $\phi_1 (x)$ and $\phi_2 (x)$, the integral is placed after all integrals containing only ϕ_1 if ϕ_1 is the function with the higher rank. Integrals with the same ϕ_1 and different ϕ_2 are arranged according to the rank of ϕ_2.

Chapter V is concerned with integral transforms and their inverses. The Fourier transformation is defined, followed by a table of 56 Fourier transforms. Next, the Laplace transformation together with a table of 158 transforms, and finally, the Hankel transformation with a table of 12 transforms.

Chapters VI and VII are devoted to special functions: definitions, properties, representations by integrals, representations by series of various types, representations by products, functional equations relating special functions, transformations of special functions, and asymptotic series. There is an index of special functions and their notations.

The volume concludes with a bibliography of 42 principal references, one-third of them in Russian, which form the basis for constructing most of the tables. There is an extensive list of supplementary references on Fourier and Laplace transformations and on special functions.

An errata sheet listing 75 errors accompanies the book. It is not difficult to detect additional errors and obvious misprints.

This book fills a real need in that it collects in well-organized fashion, the series, products, and integrals that are most important for present-day applications.

—David Krabill
Bowling Green State University

Topics in Modern Mathematics, Edited by Ralph G. Stanton and Kenneth D. Fryer, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963, 187 pp., $5.95.

The book under review (hereafter referred to as TMM) is a collection of ten chapters, the subjct of each chapter being some topic in "modern mathematics." In addition, there is an Epilogue entitled "Some Questions about Modern Mathematics in the Secondary School Curriculum." In all, there are six authors represented, each of whom is a member of the Department of Mathematics at the University of Waterloo. TMM is a revision and amplification of a set of lecture notes used in a series of seminars (on modern mathematics) sponsored by the Department of Waterloo in 1959. The purpose of the seminars was to acquaint secondary school teachers, presumably with a "traditional" background in mathematics, with some topics now finding their way into the secondary school due to various curriculum changes. TMM is published with this purpose in mind.

The chapter titles are as follows: 1. Groups and Fields (15 pages); 2. Set Theory (21 pages); 3. Boolean Algebra (12 pages); 4. Logic and Computing (16 pages); 5. Vector Spaces and Matrices (19 pages); 6. Numerical Analysis (13 pages); 7. Functions of a Single Variable (31 pages); 8. Fundamental Concepts of Calculus (17 pages); 9. Probability Theory and Statistics (19 pages); 10. Some Types of Geometry (16 pages).

TMM is an unusual book in the following sense: It is a book about modern mathematics, yet it is written by a set of men who are openly critical of the modern methods of presenting mathematics. Perhaps a more suitable title for TMM would be "Topics in Modern Mathematics, Written in Traditional Language". It should be remarked that many of the criticisms of modern mathematics brought out by the authors are well taken and are, indeed, valid. For example, it is pointed out that a premature emphasis on the axiomatic method can very well deaden interest in mathematics. Most everyone would agree with this. In view of this, the reviewer was amused to discover that the authors themselves are apparently afflicted with the "axiomatic disease". In particular, reference is made to the introduction of complex numbers in TMM. There one finds that i is introduced axiomatically as an abstract symbol satisfying the axiom $i^2 + 1 = 0$. On the other hand, most modern books will define a complex number as an ordered pair of real numbers, in which case no new axioms are needed!

Little technical knowledge of mathematics is needed to read TMM, and thus it should be accessible to a large circle of readers. Moreover, each chapter contains an ample supply of exercises to test the reader's understanding of the material presented. From this point of view, TMM is a good book. On the other hand, the reviewer feels that TMM has a number of shortcomings.

Very often the authors did not make a sharp distinction between informal or intuitive discussions and the more formal mathematics. For example, on pages 60 and 61, a heuristic argument is given for the principle of mathematical induction. However, the inexperienced reader (for whom the book in intended) will very likely get the idea that he has just read a rigorous argument establishing the truth of the principle of mathematical induction.

In a number of places, the material needs quite a bit more "polishing". As an example of what is meant, consider the following: In chapter 1, the first section is devoted to the concept of an equivalence relation. It is stated that an equivalence relation is wanted in any algebraic structure. Yet, after this first section, the reviewer was unable to find a single mention of the term "equivalence relation", even though there were a number of situations in which the concept could be used very effectively.

Finally, a number of inaccuracies were discovered. (Even the mistake $\sqrt{\lambda^2} = \lambda$, so common in older books, was found.) A serious misprint occurs on page 14, where an inexperienced reader will conclude that $\log_{10}2 = .30103$.

—R. E. Dowds
Butler University

Calculus for Students of Engineering and the Exact Sciences, Vol. I, Hugh A. Thurston, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963, 193 pp., $5.95.

Calculus for Students of Engineering and the Exact Sciences, Vol. II, Hugh A. Thurston, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963, 208 pp., $5.95.

These two little volumes, covering the calculus usually required for beginning engineering students, represent an interesting addition to the large family of such textbooks now available. The books are written primarily for students of the applied phases of mathematics.

The chapters are indicated by letter rather than by number, Volume I containing Chapters A through E and Volume II covering Chapters F through L. Solutions to the exercises appear at the end of each volume.

After a preliminary review of some of the more useful concepts of analytic geometry, the differential calculus is introduced and continued, with few applications, through the study of the fundamentals of functions of several valuables and partial differentiation. Integration, with some applications, is considered in the third chapter. The fourth chapter continues to deal with integration, but of more difficult expressions. As a final chapter in the first volume, the usual applications of the derivative are taken up.

The second volume is stated to be a full year course designed to follow the first volume. It covers, in the first two chapters, the study of functions of several variables and multiple integrations, with applications of these. An unusual feature is the inclusion of short treatments of such topics as La Grange Multipliers, Green's Theorem and Stokes' Theorem, with applications of these. Following these is a chapter on differential equations through linear differential equations with constant coefficients. Finally, the remaining chapters, in order, are called Sequences and Series, Mathematical Rigor, Double Limits and Differentials.

The books contain many interesting problems, but for some topics applications are perhaps too few. The chapter on sequences and series seems thin in this respect, for example; and there are no applications, either geometric or physical, of differential equations. Thus, it would be necessary to draw such problems from other sources. The chapter called "Mathematical Rigor", appearing late in the second volume, develops rigor without the use of "ε - δ" language. It seems difficult to state whether this is less confusing to beginning students or whether it is of any value at all, since the ideas involved cannot be called upon until so late in the course.

The books are generally well written and easy to read. The author, in an obvious attempt to be concise, has done so without complete omission of topics of definite value to engineering and science students. It would seem that a very strong course in analytic geometry would be prerequisite to the use of these books, and that at least two, possible three, semesters be required to complete them. One wonders why the author and publisher chose to use two volumes rather than just one, for all students of science and engineering need the material contained in both. Thus, in order to take full advantage of the breakdown, it should be used as a text in a college where instructional timing and context are compatible with the author's choice of distribution of text material.

—D. H. Erkiletian, Jr.
University of Missouri at Rolla

Installation of New Chapters

EDITED BY SISTER HELEN SULLIVAN

CALIFORNIA DELTA CHAPTER

California State Polytechnic College, Pomona, California

California Delta Chapter of Kappa Mu Epsilon was installed on Thursday, November 5, 1964 at California State Polytechnic College, Kellog-Voorhis Campus, Pomona, California.

Nineteen members, nine undergraduate students and ten faculty members, were initiated by Dr. Chester G. Jaeger, Claremont Mens' College, assisted by Dr. Hugh J. Hamilton, Pomona College. The installation ceremonies were witnessed by a delegation from the California Gamma Chapter located on the San Luis Obispo campus of California State Polytechnic College. The visitors were Dr. George R. Mach (Faculty Advisor), Lawson Maddox (President, Gamma Chapter), Steve Corlett (Vice-President), Deanna Wilber (Treasurer), and Nora Smith (member). The presence of the San Luis Obispo contingent was much appreciated by the members of the new Delta Chapter.

Officers of Delta Chapter installed at the ceremonies are:

Donald N. Holbrook	President
David M. Almos	Vice-President
Fredric O. Hardy	Secretary-Treasurer
Albert Konigsberg	Corresponding Secretary
Eldon Vought	Faculty Advisor

PENNSYLVANIA DELTA CHAPTER

Marywood College, Scranton, Pennsylvania

Pennsylvania Delta Chapter was installed on November 8, 1964, by Dr. Loyal F. Ollman, National President of Kappa Mu Epsilon. The ceremony took place in Room 127 of the Liberal Arts building following a dinner for members and guests in the dining room of Nazareth Hall, the Student Union. Dr. Ollman gave an interesting talk on the history of Kappa Mu Epsilon and of *The Pentagon.*

Charter members are the faculty members Sister M. Coleman, I.H.M., Sister M. Cormac, I.H.M., Miss Elizabeth Blewitt, Miss Marie Loftus and the students:

Baker, Barbara Ann	Hovan, Rosemarie	Purcell, Jane
Banick, Joan	Hudak, Elaine	Quinn, Winifred
Bruno, Mary D.	Kaiser, Jane	Raisch, Judith
Burns, Elizabeth	Kaiser, Nancy	Rokita, Barbara
Burns, Mary Elizabeth	Kessler, Barbara	Simpson, Bonnie
Clark, Marilyn	McGinnis, Anne	Simpson, Mary Jane
Corgan, Patricia	Loftus, Kathleen	Tierney, Mary Ann
Coyne, Claire	Luddy, Margaret	Vanderhoven, Audrey
Diehl, Mary	Meyers, Joan	Victor, Ann
Farrell, Mary Lynn	Mulligan, Linda	Waering, Marianne
Forquer, Jeannine	Nealon, Joanne	Wagner, Perpetua
Gimber, Marilyn	Pretko, Rita	Waltz, Sharon
Hinz, Carol		

The new chapter's officers are Elaine Hudak, President; Barbara Kessler, Vice-President; Mary Jane Simpson, Recording Secretary; Joanne Nealon, Treasurer; Marie Lotfus, Corresponding Secretary; Sister M. Coleman, I.H.M., is Faculty Sponsor.

In 1962 the mathematics club, known on campus as the SEMI-group (the society for the enrichment of the mathematically interested), was formed and some of its first members set membership in Kappa Mu Epsilon as one of their primary goals. The club has sponsored an annual dinner, a lecture on "Boolean Rings and their Algebra" by Mr. Norman Muir, a trip to visit I.B.M. and Metropolitan Life Insurance Co. in New York. Several interesting topics have been presented by the members of the club.

Marywood College, Scranton, Pennsylvania, was founded in 1915 by the Sisters, Servants of the Immaculate Heart of Mary. Marywood now has approximately 1000 students in the undergraduate program.

Kappa Mu Epsilon News

EDITED BY J. D. HAGGARD, HISTORIAN

REGIONAL CONFERENCE, Kearney State College, Kearney, Nebraska.

The Colorado-Missouri-Oklahoma-Iowa-Nebraska-Kansas Regional Conference of Kappa Mu Epsilon was held at Kearney State College on April 4, 1964. There were 13 chapters represented, with 105 students and faculty sponsors registered for the conference.

Dr. Hubert L. Hunzeker, Chairman of the Mathematics Department at Omaha University, Omaha, Nebraska, was guest speaker at the luncheon. His topic was, "Ramifications of Linear Vector Spaces". Student papers presented at the conference were as follows:

"An Effective Decision Procedure for the Elementary Theory of Ordering Reals", Jerry Atwood, Missouri Alpha.

"A Nonarchimedian Ordered Ring", George Poole, Kansas Beta.

"The Derivation and Application of Equations to Chrystal Deformations in a Magnetic Field", Neal E. Busch, Iowa Beta.

"Mathematical Logic", William F. Fulkerson, Missouri Beta.

"Development and Applications of the Theory of Complex Variables in the Solution of Real Integrals", Mary Huff, Missouri Gamma.

"Set Theoretic Matrices", Robert H. Lohman, Kansas Alpha.

"Using the Analytical Triangle in Curve Tracing", Dale Schoenefeld, Nebraska Alpha.

"On Maxima and Minima", Carol Stimpert, Kansas Beta.

The awards for presentation of papers were: Dale Schoenefeld, Nebraska Alpha, first; George Poole, Kansas Beta, second; Carol Stimpert, Kansas Beta, third. Robert H. Lohman, Kansas Alpha, being a graduate student was in a separate category and was given an award for his presentation.

Faculty and student discussion groups were held during the afternoon followed by a final business meeting.

REGIONAL CONFERENCE, Bowling Green State University, Bowling Green, Ohio.

The Third Regional Conference of Kappa Mu Epsilon chapters in Illinois, Indiana, Michigan, Ohio, and Wisconsin was held at Bowling Green State University, Bowling Green, Ohio, on April 10-11, 1964. Nine chapters from the five state region were represented at the conference with a total of 70 students and faculty sponsors attending.

Professor Robert L. Wilson, Chairman of the Mathematics Department of Ohio Wesleyan University, was the guest speaker at the banquet meeting on Friday evening. Papers presented during the two days were:

"The Theorems of Miquel", Catherine Anne Modjeska, Wisconsin Alpha.

"The Derivative of $y = x^{nx}$", Bernard Taheny, Illinois Gamma.

"The Star Product", Yeuk-Laan Chui, Indiana Gamma.

"A Conjecture about Prime Numbers", Andrew J. Samide, Illinois Alpha.

"The Meaning of Finite Markov Chains", S. Eric Steg, Michigan Beta.

"Valtran", Arthur Valiant, III, Indiana Delta.

Colorado Alpha, Colorado State University, Ft. Collins.

Professor Walter C. Butler, national treasurer of Kappa Mu Epsilon, received the 1964 Harris T. Guard Distinguished Service Award at Colorado State University. The selection is made annually by the Faculty and is based upon outstanding teaching and contribution to the University. The award carries with it a $1000 cash gift to the recipient. Kappa Mu Epsilon is honored to have one of its members receive such a distinction.

Indiana Alpha, Manchester College, North Manchester.

Monthly meetings, preceded by a dinner, were held during the year. Professor Ralph Shively of Swarthmore College spoke at one of the banquet meetings on the topic "Topological Spaces".

Professor Phillip S. Jones of the University of Michigan spoke at the initiation banquet on May 12. His topic was "History of Complex Numbers". Professor Jones is a past president of the National Council of Teachers of Mathematics.

Kansas Alpha, Kansas State College, Pittsburg.

We had ten students and faculty in attendance at the Regional Conference at Kearney, Nebraska, April 4, where Mr. Robert Lohman gave a paper entitled, "Set Theoretic Matrics."

Kansas Alpha initiated 23 members during 1963-64. This brings our total membership to 953.

Kansas Gamma, Mount St. Scholastica College, Atchison.

The chapter has had another very active year, pledging 18 new members to add to the 20 actives. Papers presented at local meetings during last year include: "Factorial Analysis," by Sheila M. Catrambone; "Latin Squares," by Mary Ann Sieckhaus; "Mathematics in Aesthetics," by Frances J. Barry; "The Constant *e*," by Susan Voigt; "Introduction to the Theory of Games," by Dolores Stiefermann; "Nomographs," by Linda Coughlin. Martha Heidlage presented a paper "Coordinatization of 25 Point Geometry," to the Kansas Section of the Mathematical Association of America.

Two panel discussions were conducted during the year. One on "Modern Mathematics" was held primarily to interest high school teachers from the area. Panelists were: Carol Crnic, Mary Ann Montgomery, Mary Westrup. The other topic was "Vectors," and the panelists were: Marcia Crawford, Linda DeJonghe, Dolores Meyer.

The chapter had 13 members and the Faculty Sponsor in attendance at the Regional Conference held April 4, on the campus of the Nebraska Beta Chapter at Kearney.

The chapter sponsored a high school mathematics tournament on May 2, 1964. Ten schools in the area participated in the meetings with a total of 79 students and a number of faculty members in attendance.

Dr. Mario Juncosa of the Rand Corporation spoke to the chapter meeting on April 27.

Sister Helen Sullivan, Chairman of the Mathematics Department, will be on sabbatical leave for the year 1964-65. Sister will be engaged in mathematical research at the University of Minnesota.

Michigan Beta, Central Michigan University, Mount Pleasant.

The outstanding news item from the Michigan Beta Chapter during the 1963-64 school year was the awarding of a Woodrow Wilson Fellowship to senior Eric Steg. The award is for graduate study at the school of his choice for the academic year 1964-65. Eric has been very active in the local Kappa Mu Epsilon chapter for the past three years. He was secretary his junior year, president during his senior year, attended two regional conferences where he gave a paper, "Finite Markov Chains" at the most recent one, and attended one national meeting.

Eric has maintained a straight A average in all his university work and has been very active in several honors activities at his school.

Missouri Zeta, Missouri School of Mines, Rolla.

Dr. J. H. Senne was the guest speaker at the March 18 meeting, where his topic was, "Earth Satellites and Space Travel."

Dr. J. H. Bramble spoke at the initiation banquet, April 15, on "Finite Difference Methods in Boundary Value Problems,"

Nebraska Alpha, Wayne State College, Wayne, Nebraska.

During the school year we initiated a total of thirteen new members into Kappa Mu Epsilon. This represents one of the largest groups of initiates in the history of the local chapter. We had nine members attend the regional conference at Kearney, Nebraska, in April. We sponsored a dance as a money making project for the club.

New Mexico Alpha, University of New Mexico, Albuquerque.

New Mexico Alpha initiated several new projects this past year.

One of these was the presentation of Certificates of Achievement to approximately 30 graduating high school seniors. Candidates were selected by their high school principals for outstanding excellence in mathematics. This will be a continuing project, designed to create greater interest in mathematics.

A college student project called "Problems of the Fortnight" is underway. A mathematical problem is to be presented each two weeks and a small cash award goes to the student with the best solution at the end of each period. This, hopefully, will generate added interest in mathematics at the college level.

Other efforts along this same line are the contests conducted to select the best paper on some phase of mathematics, the compilation of a list of problems for initiates to solve, and the display of geometrical models in a showcase in the Student Union Building.

Guest speakers, both professors and students, have been on the programs at our regular meetings. Speech subjects have included "Mathematics Education in Europe," "An Infinitude of Primes," "Paradoxes," and "What is a Law?"

We initiated 23 new members in January, and concluded our year by initiating 14 more in May.

North Carolina Alpha, Wake Forest College, Winston-Salem.

There are now 47 active members of the chapter. During this last school year we have held monthly meetings with a wide variety of topics being presented by the members. We have also had speakers representing computer centers, insurance companies, teachers, professors, and mathematics. We had a Christmas banquet at which all the mathematics professors were in attendance, and also a spring picnic.

Our by-laws have recently been revised and rewritten to bring them up to date. In the process, our requirements for membership have been substantially raised.

Texas Epsilon, North Texas State University, Denton.

We had an inspiring freshman initiation in conjunction with a banquet. Seven new members were initiated into the chapter. Dr. W. David L. Appling, of the North Texas State University Mathematics Department, was the guest speaker.

(Continued from page 37)

Thus the particles are actually moving at right angles to one another.

Here we can also see that the vector **P** becomes a weight vector with respect to which **U** and **V** are orthogonal.

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[1] Churchill, R. V., *Fourier Series and Boundary Value Problems*, 2nd ed., McGraw-Hill Book Company, Inc., New York, 1963.

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