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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

An Alternative Method for Obtaining the Equation of the Line at Infinity in the Areal System

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Since the system of areal coordinates for locating points in the plane is not as well known as other coordinate systems we begin with a brief discussion of this type of coordinates.

1. **Areal coordinates.** Areal coordinates, which are only a particular case of a more general class of homogeneous coordinates, are defined as follows: Let ABC be a triangle, hereinafter called the triangle of reference, and P any point in its plane (see Fig. 1). Joining the point P to the vertices we have three triangles and the

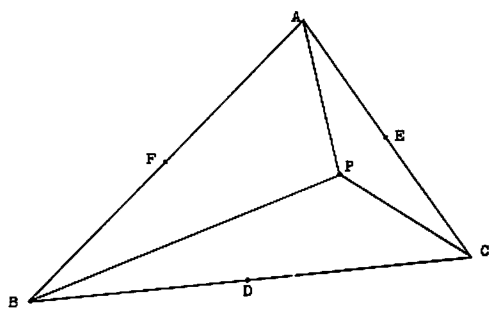

Fig. 1

ratios of the areas

$$X = \frac{\triangle\, PBC}{\triangle\, ABC}, \qquad Y = \frac{\triangle\, PCA}{\triangle\, ABC}, \qquad Z = \frac{\triangle\, PAB}{\triangle\, ABC}$$

are called the areal coordinates of P which are denoted as (X, Y, Z).

The areal coordinates possess sign as well as magnitude based on the fact that the area of a triangle has a sign, being positive when the perimeter is described in the counterclockwise direction and negative when the perimeter is described in the clockwise direction. Thus, wherever the point P may be, inside, outside, or on the triangle, the algebraic sum of the areas will be given by

$$\triangle\, PBC + \triangle\, PCA + \triangle\, PAB = \triangle\, ABC$$

and consequently

$$X + Y + Z = 1.$$

Applying the above definition, it may easily be verified that the areal coordinates of

i) the vertices A, B, and C of the triangle of reference are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ respectively.

ii) the midpoints D, E, and F of the sides BC, CA, and AB are $(0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, 0, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{2}, 0)$ respectively.

iii) the centroid of the triangle of reference are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

2. **Relationship between areal and cartesian coordinates.** Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be the cartesian coordinates of the vertices of the triangle ABC referred to any axes and (x, y) be the cartesian coordinates of any point P in the plane. Let (X, Y, Z) be the areal coordinates of P with respect to the triangle ABC. Then

$$X = \frac{\triangle PBC}{\triangle ABC} = \tfrac{1}{2}\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \div \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Therefore

$$(1) \qquad x(y_2 - y_3) + y(x_3 - x_2) + (x_2y_3 - x_3y_2) = DX$$

where

$$D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Similarly, we obtain

$$(2) \qquad x(y_3 - y_1) + y(x_1 - x_3) + (x_3y_1 - x_1y_3) = DY$$

$$(3) \qquad x(y_1 - y_2) + y(x_2 - x_1) + (x_1y_2 - x_2y_1) = DZ$$

Equations (1), (2), and (3) permit us to find the areal coordinates of P when the rectangular coordinates of P and of the vertices of the triangle of reference are known. To convert areal coordinates of P to rectangular coordinates we solve these equations for x and y by multiplying (1), (2), and (3) by x_1, x_2, and x_3 respectively and adding to obtain

$$xD = D(x_1X + x_2Y + x_3Z)$$

(4) $$x = x_1X + x_2Y + x_3Z$$

Similarly,

(5) $$y = y_1X + y_2Y + y_3Z.$$

3. **Area of a triangle.** Let $P(X_1, Y_1, Z_1)$, $Q(X_2, Y_2, Z_2)$, and $R(X_3, Y_3, Z_3)$ be the areal coordinates of a triangle whose area we are interested in. Let $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$ be the cartesian coordinates of the reference triangle. We know that the area of $\triangle PQR$ is given by

$$\text{Area } (\triangle PQR) = \tfrac{1}{2} \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

where (a_1, b_1), (a_2, b_2), and (a_3, b_3) are the rectangular coordinates of P, Q, and R. Using equations (4) and (5) we obtain

$$\text{Area } (\triangle PQR) =$$

$$\tfrac{1}{2} \begin{vmatrix} x_1X_1 + x_2Y_1 + x_3Z_1 & y_1X_1 + y_2Y_1 + y_3Z_1 & X_1 + Y_1 + Z_1 \\ x_1X_2 + x_2Y_2 + x_3Z_2 & y_1X_2 + y_2Y_2 + y_3Z_2 & X_2 + Y_2 + Z_2 \\ x_1X_3 + x_2Y_3 + x_3Z_3 & y_1X_3 + y_2Y_3 + y_3Z_3 & X_3 + Y_3 + Z_3 \end{vmatrix}$$

From the property that the determinant of the product of two square matrices is the product of the determinants of the matrices, this can be written as

$$\text{Area } (\triangle PQR) = \tfrac{1}{2} \begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} \cdot \text{Area } (\triangle ABC)$$

The above result shows that the condition for the collinearity of three points whose areal coordinates are (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), and (X_3, Y_3, Z_3) is given by the equation

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = 0.$$

4. **Equation of a line.** To find the equation of the line joining two points whose areal coordinates are (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2), we let (X, Y, Z) be the areal coordinates of any point on the line joining the two given points. Since the three points are collinear,

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = 0$$

which is the required equation.

We now show that the equation $AX + BY + CZ = 0$ represents a straight line. Suppose (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), and (X_3, Y_3, Z_3) are any three points on the locus of the given equation. Then

$$AX_1 + BY_1 + CZ_1 = 0$$
$$AX_2 + BY_2 + CZ_2 = 0$$
$$AX_3 + BY_3 + CZ_3 = 0$$

This is a system of homogeneous linear equations in A, B, and C, and in order to have a non-trivial solution we must have

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = 0.$$

Thus the area of the triangle formed by any three points taken on the locus of $AX + BY + CZ = 0$ is zero. Clearly then the locus of this equation represents a straight line.

5. **Polar of a point with respect to the triangle of reference.** Let ABC be the triangle and $P(f, g, h)$ be any point in the plane (see Fig. 2). Let AP, BP, and CP meet BC, CA, and AB at D, E,

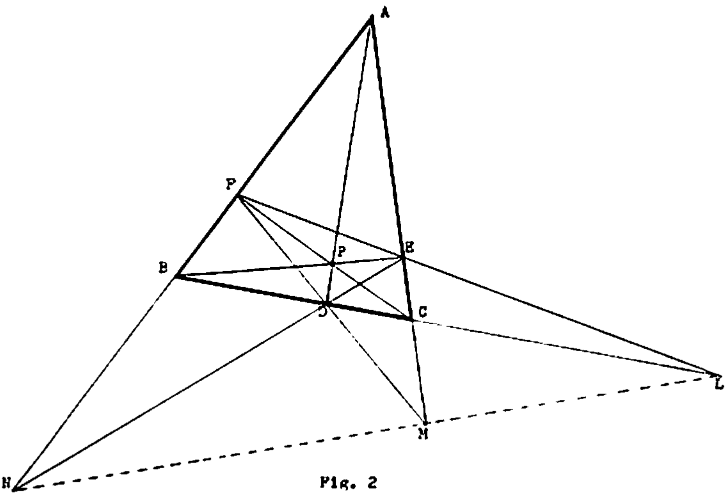

Fig. 2

and F respectively. Let FE, FD, and ED meet BC, AC, and AB at L, M, and N respectively. We shall show that L, M, and N are collinear. The straight LMN is called the polar of the point P with respect to the triangle ABC.

The equation of BC is given by

$$\begin{vmatrix} X & Y & Z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0, \text{ or } X = 0.$$

Similarly, the equations of CA and AB are $Y = 0$ and $Z = 0$ respectively. The equation of BP is given by

$$\begin{vmatrix} X & Y & Z \\ 0 & 1 & 0 \\ f & g & h \end{vmatrix} = 0,$$

$$hX - fZ = 0.$$

Solving $hX - fZ = 0$ and $Y = 0$ simultaneously, we get the coordinates of E which are $(f, 0, h)$. Similarly the coordinates of F and D will be found to be $(f, g, 0)$ and $(0, g, h)$ respectively.

The equation of EF is given by

$$\begin{vmatrix} X & Y & Z \\ f & 0 & h \\ f & g & 0 \end{vmatrix} = 0,$$

$$-\frac{X}{f} + \frac{Y}{g} + \frac{Z}{h} = 0.$$

Clearly the intersection of this line with $X = 0$ is the same as its intersection with the line

$$(6) \qquad \frac{X}{f} + \frac{Y}{g} + \frac{Z}{h} = 0.$$

Hence L lies on (6). The symmetry of this equation shows that the line passes through M and N also. Thus L, M, and N lie on the straight line whose equation is given by (6).

6. **The line at infinity.** If P is the centroid of the triangle ABC, then FE, FD, and ED become parallel to BC, AC, and AB respectively with the result that the points L, M, and N recede to infinity and we call the line LMN the line at infinity. Thus the polar of the centroid of the triangle of reference is called the line at infinity. Since the coordinates of the centroid are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, the equation of the line at infinity is given by

$$X + Y + Z = 0.$$

The above is a usual way of getting the equation of the line at infinity. We now give an alternative method for the same result using a simple property from elementary geometry.

Let ABC be the triangle of reference. $X + Y = 0$ is a line through C such that the X-coordinate of every point on it is equal in magnitude, but opposite in sign to the Y-coordinate of the point. Let CD represent $X + Y = 0$ (see Fig. 3). Since D is a point on this line we have

$$\frac{\triangle\, DBC}{\triangle\, ABC} + \frac{\triangle\, DCA}{\triangle\, ABC} = 0.$$

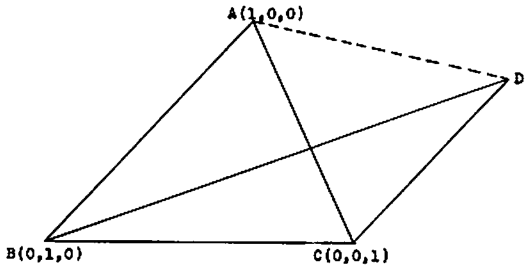

Fig. 3

Thus

$$\text{Area } (\triangle\, DBC) = -\text{ Area } (\triangle\, DCA)$$
$$= \text{Area } (\triangle\, DAC).$$

Since triangles DBC and DAC have equal areas, CD as a common base, and both of them lie on the same side of this base, CD must be parallel to BA. Evidently the point of intersection of CD and BA, which is a point at infinity, is $(1, -1, 0)$. Let us call it L. Similarly, the point of intersection of $X + Z = 0$ and $Y = 0$, which is another point at infinity, is given by $(1, 0, -1)$. Let us call it M. Then LM is the line at infinity and its equation will be given by

$$\begin{vmatrix} X & Y & Z \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 0$$

or

$$X + Y + Z = 0.$$

. . . .we cannot get more out of the mathematical mill than we put into it, though we may get it in a form infinitely more useful for our purpose.

—John Hopkinson

The Golden Section

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"Geometry has two great treasures: one is the Theorem of Pythagorus; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold, the second we may name a precious jewel."

—J. KEPLER (1571-1630)

As part of my teaching, I am always looking for interesting topics in mathematics that I can use in the classroom as a device to point out some of the truly interesting aspects of mathematics and how they affect our everyday lives. One such intriguing topic as this opens up under various names such as "The Golden Section", "The Divine Proportion", "The Section", "The Golden Ratio", and "The Golden Proportion", and branches off into areas such as "Phyllotaxis", "The Fibonacci Sequence", art, architecture, game theory, and some interesting puzzles.

I became interested in this topic during my first year of teaching, but until now I have not pursued this interest very far. I am sure that you will be as amazed as I was when I began to find out just how far this topic can lead one on.

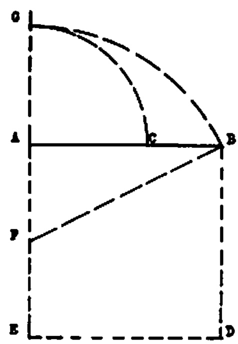
It is my intent in this paper to acquaint you with some of the far reaching applications of the golden section. I will not give any proofs of the materials unless the proof is reasonably short in length. If you wish to find out more about a particular proof than is given in this paper, the list of references in the bibliography will be helpful.

The golden section refers to a simple construction in Euclidean geometry. Given any line segment AB, there exists a unique point C such that the following proportion is correct:

$$\frac{AB}{AC} = \frac{AC}{BC}$$

In other words, the length of the original line segment AB is to the length of the segment AC as the length of the segment AC is to the length of the segment BC. When the point C has been located, the segment AB is said to be divided into a mean and extreme ratio.

The construction given below is the construction Euclid gives to locate the point C:



1. Construct square $ABDE$
2. Bisect side EA in F
3. Construct $FG = FB$
4. Construct $AC = AG$
5. Point C is the required point.

The proof of Euclid's construction can be shown by using the Pythagorean Theorem. This construction has been attributed to the Pythagoreans for two reasons: (1) Euclid has included this construction with other theorems and constructions that the Pythagoreans are responsible for, and (2) The fact that the Pythagoreans adopted the pentagram as a symbol of their organization. As you will see later, the pentagram contains a wealth of material related to the extreme and mean ratio.

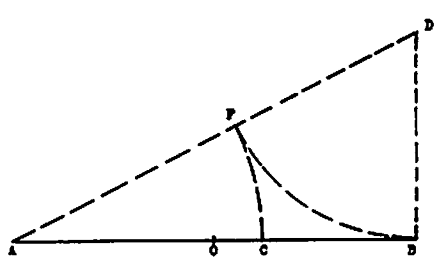
As with many other statements and theorems in mathematics that are attributed to the Pythagoreans, we really don't know for sure that Pythagorus was responsible for the discovery of the division of a line into extreme and mean ratio. Nevertheless, it was not until the time of Euclid that this problem gained in popularity. In Euclid's time, the division of a line in this manner was referred to simply as "$\tau o\mu'\eta$", meaning "the section."

After the fall of Greek mathematics, the section was left unnoticed for many years until it was rediscovered again during the Renaissance period. At this time, there was quite a to do about the value of this ratio as applied to nature, and the section was referred to as the divine proportion. This will be discussed later in the paper.

In 1509, Pacioda di Borgo wrote a treatise "De Divina Proportione" (Of the Divine Proportion) in which he describes, in thirteen chapters, the thirteen effects of the divine proportion. He refers to these effects in colorful language such as "The Seventh Most Excellent Effect" or "The Thirteenth Most Distinguished Effect." After his thirteenth chapter, Borgo states that the list most surely come to an end for the sake of salvation, since there were only thirteen persons present at the table of the Last Suppr. During this time, the divine proportion was used extensively in architecture and in great artist's works. Leonardo da Vinci referred to it as the golden section, and this seems to be the name used most often today. Kepler referred to it as the divine section.

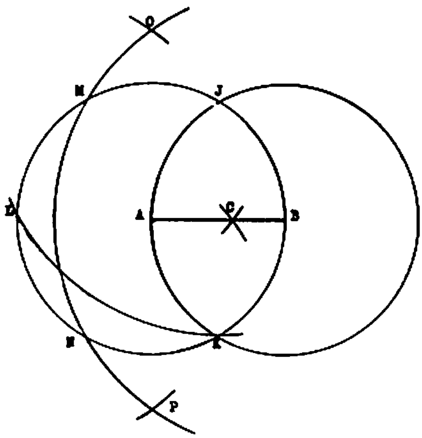
There are several possible constructions to obtain the point C. Two of them are included here. The first one is the one that I found most often in present day texts on geometry, and the second method locates the point C by using only a compass. The second construction is one of the so-called Mascheroni constructions.

CONSTRUCTION I.



1. Construct O so that $AO = OB$.
2. Construct BD perpendicular to AB so that $BD = BO$.
3. Construct $DF = DB$.
4. Construct $AC = AF$.
5. Point C is the required point.

CONSTRUCTION II.



The symbol, $A(AB)$, means a circle with center at A, and a radius $= AB$.

1. Construct $A(AB)$ and $B(AB)$ meeting in points J and K.
2. Construct $J(JK)$ and $A(AB)$ meeting in points K and L.
3. Construct $B(JK)$ and $A(AB)$ meeting in points M and N.
4. Construct $L(JK)$ and $B(JK)$ meeting in points O and P.
5. Construct $M(AO)$ and $N(AO)$ meeting in points C and some other point.
6. Point C is the required point.

Because this division of a line has the habit of appearing in some of the most unexpected places, the value of the ratio has been calculated and given a special name. Actually there is a little bit of confusion at this point, there are two names in use today and, depending on your way of thinking, there are two values associated with each name. You will find the golden section referred to as "τ" (tau) and "ρ" (phi). According to H.S.M. Coxeter, the name "τ" comes from the fact that τ is the first Greek letter of the word "τομ′η," meaning "the section." According to Martin Gardner, the name "ρ" was given to the golden section about fifty years ago by a man named Mark Barr because ρ is the first Greek letter in the name of the great Phildias who was believed to have used the golden section frequently in his art work. The two values associated with each name depend on how you evaluate the ratio. If τ, or ρ, is taken as the ratio of AB to AC, one value is obtained; if the ratio of AC to AB is evaluated, the other value is obtained. Throughout the rest of this paper we will refer to the value of the golden section as τ and will consider its value to be the ratio of AB to AC.

Now, the question is, what is the value of τ? If we let the distance AB be unity, then τ can be evaluated as follows:

i. $\tau = \dfrac{AB}{AC} = \dfrac{1}{AC}$

ii. $AC = 1/\tau \Longrightarrow BC = 1 - (1/\tau) = (\tau - 1)/\tau$

iii. $\dfrac{1}{(1/\tau)} = \dfrac{1/\tau}{(\tau - 1)/\tau}$

iv. $\tau = 1/(\tau - 1) \Longrightarrow \tau^2 - \tau - 1 = 0$

v. $\tau = \frac{1}{2}(\sqrt{5} + 1)$

By looking at equation (iv) one of the unique properties of τ becomes evident. Since,

$$\tau = \frac{1}{\tau - 1}$$

we can say that

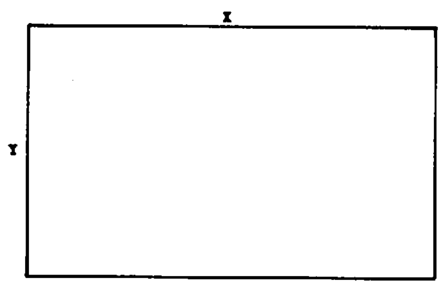
$$\frac{1}{\tau} = \tau - 1$$

In other words, to find the reciprocal of τ simply subtract one from τ. This fact can also be demonstrated as follows:

$$
\begin{aligned}
1/\tau = \frac{2}{\sqrt{5} + 1} = \frac{2(\sqrt{5} - 1)}{(\sqrt{5} + 1)(\sqrt{5} - 1)} &= \frac{2\sqrt{5} - 2}{4} \\
&= \frac{\sqrt{5} - 1}{2} \\
&= \frac{\sqrt{5} + 1 - 2}{2} \\
&= \frac{\sqrt{5} + 1}{2} - 1
\end{aligned}
$$

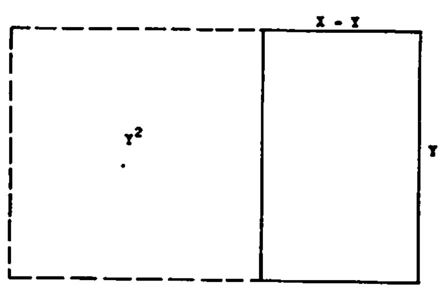
If you haven't already thought of this yourself, note that τ is the only positive real number that has this property.

I think that most people who are familiar with the golden section usually picture it in the form of the so-called golden rectangle. This is a rectangle in which the ratio of the length of the rectangle to the width is equal to τ. In the figure, $X/Y = \tau$.



A golden rectangle has many interesting properties. One of the most widely known properties is the fact that if a square (Y^2) is removed from the original rectangle, the newly formed rectangle with sides Y and $X - Y$ is still a golden rectangle. In other words,

$$\frac{Y}{X - Y} = \tau.$$



This fact can be proved as follows; (using basic rules for proportions)

$$\frac{X}{Y} = \tau \Rightarrow \frac{X - Y}{Y} = \tau - 1$$

$$\therefore \frac{X - Y}{Y} = \frac{1}{\tau}$$

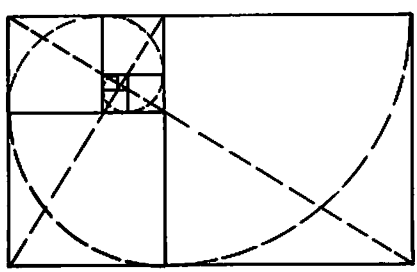
$$\therefore \frac{Y}{X - Y} = \tau$$

It is also true that if a square (X^2) is added to the original golden rectangle, the newly formed rectangle with length $X + Y$ and width X is also a golden rectangle. This can be shown as follows:

$$\frac{X}{Y} = \tau \Longrightarrow \frac{Y}{X} = \frac{1}{\tau} = \tau - 1$$

$$\therefore \frac{X + Y}{X} = \tau - 1 + 1 = \tau$$

By adding or subtracting squares, the golden rectangle regenerates itself over and over again. The "whirling squares" form a spiral arrangement, and if a quadrant of a circle is drawn in each square as shown below, the resulting curve is a very good approximation of a logarithmic spiral. This spiral, like the logarithmic spiral, has the property that its shape does not alter in any way as it spirals in or out. In other words, if you were to look at a very highly magnified picture of the spiral as it gets smaller and smaller, the picture would look the same as what you would see if you could stand somewhere out in space and look at a very large spiral.

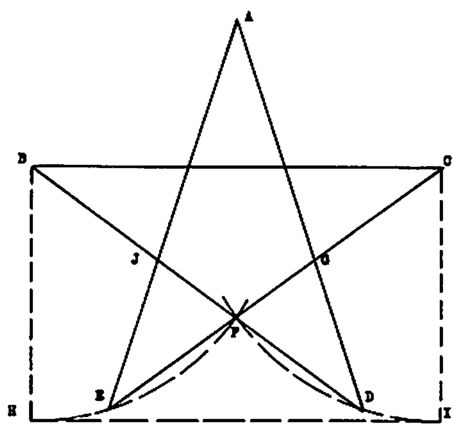


As this spiral winds inward, it keeps getting closer to the point formed by the intersection of the diagonals of two, shall we say, adjacent golden rectangles. Interestingly enough, these two diagonals are perpendicular to each other and divide each other into the golden section.

Since the golden rectangle has the property that adding or subtracting a square on one side yields another golden rectangle, a square is said to be a "gnomon" to the golden rectangle. A gnomon is that figure which added to or subtracted from a given figure makes another figure which is similar to the original figure.

The golden rectangle is also referred to as a "perfect squared rectangle of order infinity." A perfect squared rectangle of order n is one which can be divided up into n squares, no two of which have the same length side. As you can see, in the golden rectangle it is possible to keep on cutting off squares no matter how small the rectangle is. I think that the golden rectangle is the only perfect rectangle of order infinity, but I didn't find any material to back up my conjecture.

Now, the fun really starts. As mentioned earlier, the pentagram was the symbol of the Pythagorean society. The reason for this may be the fact that if you pick any segment on the pentagram and then take the next longer one or the next shorter one, these two segments will form a golden rectangle. In fact, triangle BCF (see the figure) is called the golden triangle since by moving sides BF to the position of BH and CF to CI such that BH and CI are perpendicular to BC and then connecting points I and H, you will always get a golden rectangle.

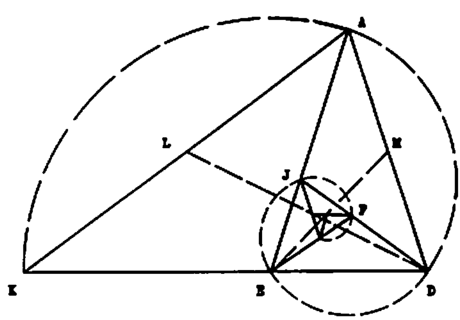


The proof of the statements concerning the above figure can be worked out by using the fact that $\tau = 2 \cos 36°$.

Going further, consider triangle ADE (next page). This is an isosceles triangle with base angles of 72°. The ratio of the equal sides to the base is equal to τ. The line segment DJ will bisect the angle EDA, and starting from this the following facts can be proved:

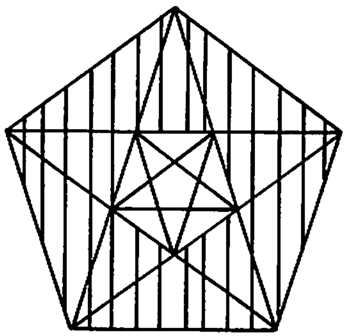
1. angle JED = angle DJE
2. angle JAD = angle JDA
3. In each triangle formed by cutting triangle ADE by DJ, the base and one of the equal sides will form a golden rectangle.

This triangle is called the "thrice-isosceles" triangle. You will also notice, that triangle ADE is similar to triangle JDE. Therefore, starting with triangle JDE, we can repeat the above procedure to get triangle JEF and so on. Also, we can construct triangle AEK such that the new triangle, triangle AKD is similar to triangle ADE. From this basis we can once again draw a spiral curve that closely approximates a logarithmic curve and has the same properties as the curve obtained from the whirling squares. The pole of the spiral formed

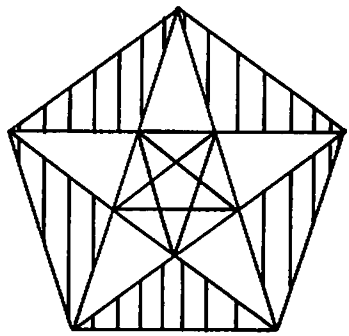


from the thrice-isosceles triangle is formed by the intersection of the two medians, DL and EM. Incidentally, these two medians cut each other into the golden section.

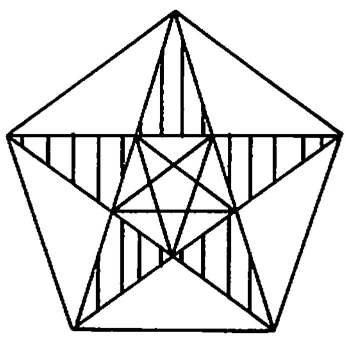
Starting with a pentagram inscribed in a pentagon, consider the shaded area in the figure on page 89. Let's call this area A. If you study the figure, you will see that this area is formed by five overlapping triangles. The shaded area in the second figure is formed by five more triangles, the next smaller size compared to the triangles in the first figure. This sequence is continued until you get to the fifth figure where you are back to a similar situation as in the first figure. The areas thus formed are related to each other by a constant multiplier, namely τ^{-1}. As you can see this process could be continued indefinitely in or out and the areas would form a geometric series with the ratio τ^{-1}.



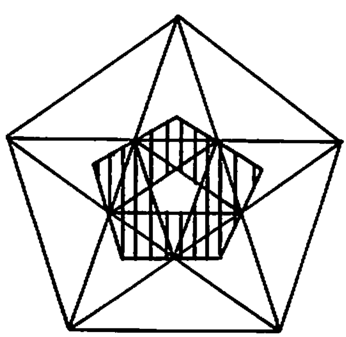
(1) A



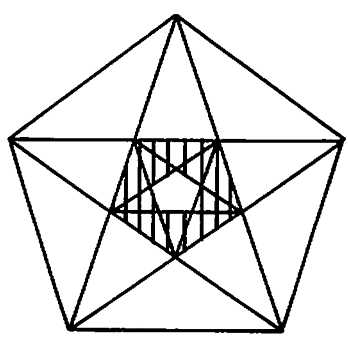
(2) $\frac{A}{\tau}$



(3) $\frac{A}{\tau^2}$



(4) $\frac{A}{\tau^3}$



(5) $\frac{A}{\tau^4}$

Let's consider more of the properties of τ as a real number. Since $\tau = \frac{1}{2}(\sqrt{5} + 1)$ and $\sqrt{5}$ is an irrational number, this means that τ is irrational, which means that τ can be expressed as some continued fraction. As I have shown before, $\frac{1}{\tau} = \tau - 1$. This can be changed around so that $\tau = 1 + \tau^{-1}$. Therefore,

$$\tau = 1 + \frac{1}{\tau} = 1 + \cfrac{1}{1 + \cfrac{1}{\tau}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\tau}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

This is the simplest continued fraction possible.

It is also possible to express τ very simply in another fashion. Since $\tau^2 - \tau - 1 = 0$, we can show that $\tau = \sqrt{1 + \tau}$. From this, the relation below follows quite readily.

$$= \sqrt{1 + \tau} = \sqrt{1 + \sqrt{1 + \tau}}$$

$$= \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}$$

Because of the fact that $\tau = 1 + \frac{1}{\tau}$, the powers of τ can be simplified very nicely, as follows:

$$\tau = 1 + \frac{1}{\tau} \Longrightarrow \tau^2 = \tau + 1$$

$$\begin{aligned} \tau^3 &= \tau^2 \cdot \tau = (\tau + 1)\,\tau = \tau^2 + \tau = \tau + 1 + \tau \\ &= 2\tau + 1 \\ \tau^4 &= \tau^3 \cdot \tau = (2\tau + 1)\,\tau = 2\tau^2 + \tau = 2\tau + 2 + \tau \\ &= 3\tau + 2 \end{aligned}$$

Since

$$\tau^{-1} = \tau - 1$$
$$\tau^{-2} = \tau^{-1}\tau^{-1} = (\tau - 1)^2 = \tau^2 - 2\tau + 1$$
$$= \tau + 1 - 2\tau + 1 = 2 - \tau$$
$$\tau^{-3} = \tau^{-2} \cdot \tau^{-1} = (2 - \tau)(\tau - 1)$$
$$= -2 - \tau^2 + 3\tau = 2\tau - 3$$

In the table below, I have given the values of τ^n for $n = \pm 10$. There is a definite pattern here that will allow you to find any power of τ that you wish. I will discuss this pattern a little later. Can you find it?

$\tau^0 = 1$	
$\tau^1 = \tau$	$\tau^{-1} = \tau - 1$
$\tau^2 = \tau + 1$	$\tau^{-2} = 2 - \tau$
$\tau^3 = 2\tau + 1$	$\tau^{-3} = 2\tau - 3$
$\tau^4 = 3\tau + 2$	$\tau^{-4} = 5 - 3\tau$
$\tau^5 = 5\tau + 3$	$\tau^{-5} = 5\tau - 8$
$\tau^6 = 8\tau + 5$	$\tau^{-6} = 13 - 8\tau$
$\tau^7 = 13\tau + 8$	$\tau^{-7} = 13\tau - 21$
$\tau^8 = 21\tau + 13$	$\tau^{-8} = 34 - 21\tau$
$\tau^9 = 34\tau + 21$	$\tau^{-9} = 34\tau - 55$
$\tau^{10} = 55\tau + 34$	$\tau^{-10} = 89 - 55\tau$

As we continue to study τ, we must mention its connection with another fascinating topic in mathematics — the Fibonacci Sequence. The Fibonacci Sequence is an additive series of numbers where each number in the sequence is found by taking the sum of the two previous numbers. This sequence is named after its founder, Leonardo of Pisa, who carried the nickname of Fibonacci. The Fibonacci Sequence, as given below, has a definite connection with τ.

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots, f_n, \ldots$$

where $f_n = f_{n-2} + f_{n-1}$. In the general term, n represents the number of the term and n takes on integer values greater than or equal to 0. For example, $f_0 = 0$, $f_3 = 1$, and $f_6 = 5$. If we consider the fraction $\frac{f_{n+1}}{f_n}$ for values of n greater than zero, as n goes to infinity, an interesting pattern developes.

$$\frac{1}{1} = 1$$

$$\frac{2}{1} = 1 + 1$$

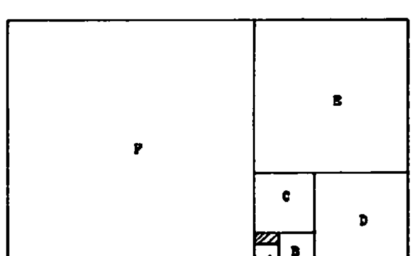
$$\frac{3}{2} = 1 + \frac{1}{2} = 1 + \frac{1}{1 + 1}$$

$$\frac{5}{3} = 1 + \frac{2}{3} = 1 + \frac{1}{\frac{3}{2}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + 1}}$$

$$\frac{8}{5} = 1 + \frac{3}{5} = 1 + \frac{1}{\frac{5}{3}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + 1}}}$$

As n gets larger and larger, you will notice that the fraction becomes closer and closer to the continued fraction of τ. Therefore, we can say that the value of f_{n+1}/f_n converges to τ.

Well, this is very nice, but we don't have to go to so much trouble to remember the Fibonacci Sequence. The fact is, if we build an additive sequence starting with any two real numbers, say a and b, this sequence will also produce fractions, g_{n+1}/g_n, like those of the Fibonacci Sequence in the sense that g_{n+1}/g_n will also converge to τ as n goes to infinity.



A neat illustration of the property discussed above can be worked out as follows. Let A and B (see figure on page 92) be any two squares you choose. Construct squares $C, D, E, F, \ldots$ as shown. As the size of the composite rectangle increases it will come closer to becoming a golden rectangle.

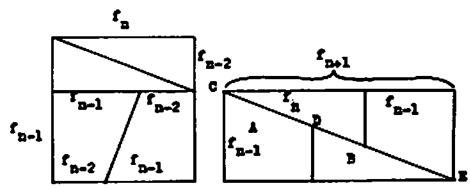
If we refer back to the table of powers of τ, we can see another pattern involving the Fibonacci Sequence. The table has been repeated below with some added guide lines to help you see the pattern.

$$
\begin{array}{llll}
\tau^0 = 1 & & & \\
\tau^1 = \tau & & \tau^{-1} = \tau - 1 & \\
\tau^2 = \tau + 1 & & \tau^{-2} = 2 - \tau & \\
\tau^3 = 2\tau + 1 & & \tau^{-3} = 2\tau - 3 & \\
\tau^4 = 3\tau + 2 & & \tau^{-4} = 5 - 3\tau & \\
\tau^5 = 5\tau + 3 & & \tau^{-5} = 5\tau - 8 & \\
\tau^6 = 8\tau + 5 & & \tau^{-6} = 13 - 8\tau & \\
\tau^7 = 13\tau + 8 & & \tau^{-7} = 13\tau - 21 & \\
\tau^8 = 21\tau + 13 & & \tau^{-8} = 34 - 21\tau & \\
\tau^9 = 34\tau + 21 & & \tau^{-9} = 34\tau - 55 & \\
\tau^{10} = 55\tau + 34 & & \tau^{-10} = 89 - 55\tau &
\end{array}
$$

Another property of the Fibonacci Sequence serves as a basis of an intriguing puzzle. This property is given in the equation:

$$(f_{n-1})(f_{n+1}) - f_n^2 = (-1)^n$$

In other words, a square of area f_n^2 is always one unit more or less in area than a rectangle with the dimensions f_{n-1} by f_{n+1}. If a square is cut up into four pieces as shown on the next page, there appears to be a change in area of one unit when the pieces are put together into a "rectangle" as shown, even though the same pieces are involved.



Some of you may have seen this same puzzle where the four pieces were put together to form a "triangle" instead of a rectangle. The answer to the problem of gaining or losing a unit of area, as you probably already know, is the fact that the pieces *A* and *B* do not fit together so that the line *CDE* is a straight line. Actually, there is one unit overlap along this line, or one unit is deleted along this line. As n becomes larger, it is very difficult to detect this difference along this line. If someone should comment about the fact that the pieces do not fit exactly, you can always toss that off by saying that it is due to crooked cutting.

Again, there is the interesting generality that *almost* any additive sequence starting with any real numbers a and b can be used as a basis for developing this puzzle in the same manner as the Fibonacci Sequence was used. In fact, every sequence that you can dream up except one will produce this puzzle. The question is, Which sequence will not yield this puzzle. In other words, under what circumstances will the square and the rectangle have exactly the same area? The answer can be seen by considering the following sequence in which the first number is 1 and the second number is τ.

$$1, \tau, 1 + \tau, 2\tau + 1, 3\tau + 2, 5\tau + 3, 8\tau + 5, \ldots t_n, \ldots$$

If you pick out any three consecutive numbers from this sequence, you will see that $(t_{n-1})(t_{n+1}) - t_n^2 = 0$. The reason for this depends on the fact that the sequence given above can be transformed into the sequence shown below by the use of the table giving the powers of τ.

$$1, \tau, \tau^2, \tau^3, \ldots \tau^n, \ldots$$

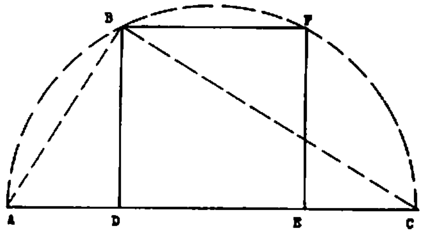
Now, when you pick any three consecutive numbers from this sequence, the numbers are represented by τ^{n-1}, τ^{n}, τ^{n+1}. From this, it is obvious that the following equation is true.

$$(\tau^{n-1})(\tau^{n+1}) = \tau^{2n}$$

Interestingly enough, the additive sequence described above is the only additive sequence where the ratio between any two consecutive terms is constant. Gardner refers to it as the "golden series" that all additive series strive to become.

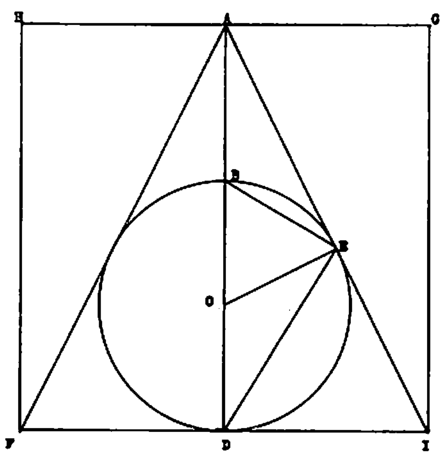
If a square is inscribed in a semicircle such that one side of the square is on the diameter, the similar right triangles formed will be such that the ratio of their legs will be τ. We have

$$\frac{CD}{BD} = \frac{BD}{AD} = \frac{CB}{BA} = \tau$$



The problem of the square in the semicircle could also be interpreted as an equilateral cylinder inscribed in a hemisphere, if you would like an application for the solid geometry class.

In the figure below, *HGIF*, is a square, *AI* = *AF*, and the

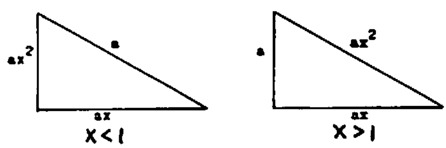


circle with center O is tangent to AF at E. From this the equations

$$\frac{AE}{AB} = \frac{AD}{AE} = \tau$$

can be shown to be correct. This problem could also be interpreted for the solid geometry class as a sphere inside a right circular cone inside an equilateral cylinder.

If you take any right triangle where the sides are terms of some geometric series, the following relations will hold. Let the sides be given by a, ax, and ax^2, where x is the common ratio.



If $x < 1$. Then $ax^2 < ax < a$.

$$a^2 = a^2x^2 + a^2x^4$$

$$1 = x^2 + x^4$$

$$x^2 = \tau^{-1}$$

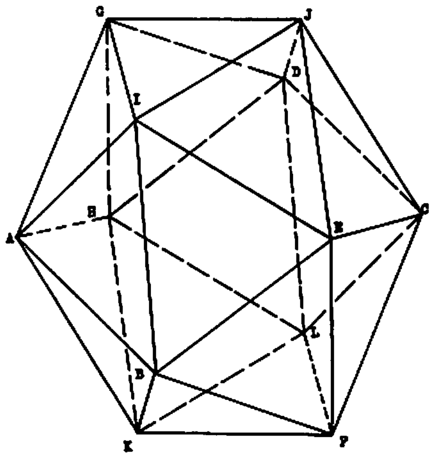
If $x > 1$. Then $a < ax < ax^2$.

$$a^2 + a^2x^2 = a^2x^4$$

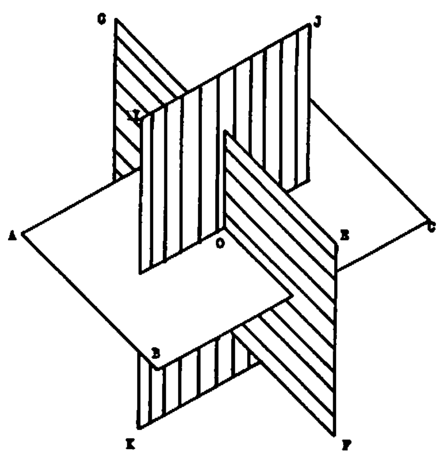
$$1 + x^2 = x^4$$

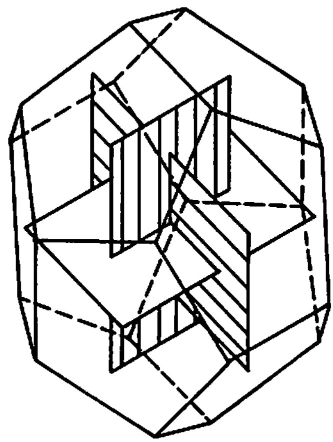
$$x^2 = \tau$$

In the area of solid geometry, the icosahedron and the dodecahedron are both involved with τ. In the icosahedron, there are 12 vertices. If the twelve vertices are connected as in the drawing on page 97, the result will be three golden rectangles, each of which is perpendicular to the other two, and all three intersect at their midpoints. The fact that the rectangles are actually golden rectangles can be shown by a careful study of the figure. Consider the pentagonal pyramid $GIECDJ$ formed by five of the faces of the icosahe-



dron meeting at one vertex. Since $GIECD$ is a pentagon, we know that the ratio $GE/GI = \tau$. But $GI = GH$. Therefore $GE/GH = \tau$ and $GEFH$ is a golden rectangle. Considering the point O to be the origin, and the planes to be the xy, yz and the xz planes, the coordinates of the vertices of an icosahedron of side length two turns out to be $(0, \pm 1, \pm\tau)$, $(\pm\tau, 0, \pm 1)$, and $(\pm 1, \pm\tau, 0)$.





The same type of situation will arise if you connect the midpoints of the faces of a dodecahedron as shown in the figure above. In this case, the coordinates given will locate these midpoints in three space. The golden rectangles of the icosahedron will fit here since an icosahedron can be inscribed in a dodecahedron. This can also be shown to be true since the rotational groups of the two solids are isomorphic to each other.

Referring back to the golden rectangle, we don't have to think of this as a single special kind of rectangle but as a special case of a set of rectangles with a common property. The common property involves the ratio y/x of the sides of a rectangle where y is the length and x is the width. We might ask what is the value of y/x such that if we remove n squares of dimension x by y from the original rectangle, the rectangle remaining will be similar to the original rectangle. In other words, the remaining rectangle with the dimensions $y - nx$ by x will be such that $y/x = x/(y - nx)$. If we let x be one unit, and solve for y, the result is that $y = \frac{1}{2}(n + \sqrt{n^2 + 4})$. Note, if $n = 1$, then $y = \tau$. We might also calculate the rceiprocal of y, which is:

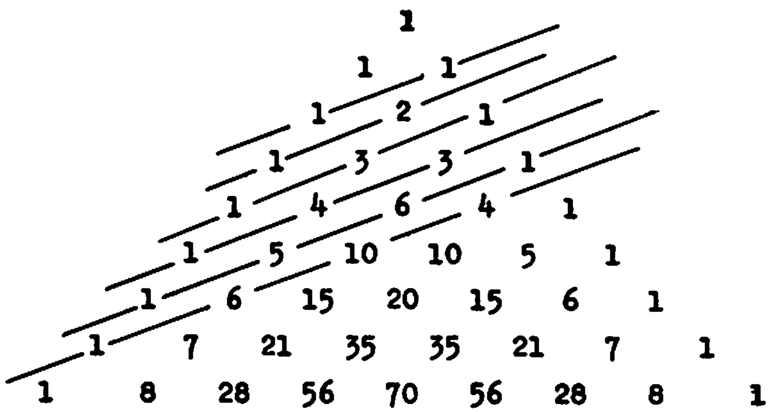
$$\frac{1}{y} = \frac{2}{n + \sqrt{n^2 + 4}} = \frac{1}{2}(n + \sqrt{n^2 + 4}) - n.$$

Again, if $n = 1$, $1/y = \tau - 1$.

If we consider the sequence given below, where the general term $x_j = x_{j-2} + nx_{j-1}$, this sequence will converge to y.

$$0,\ 1,\ n,\ n^2+1,\ n^3+2n,\ n^4+3n^2+1,\ n^5+4n^3+3n,\ \cdots$$

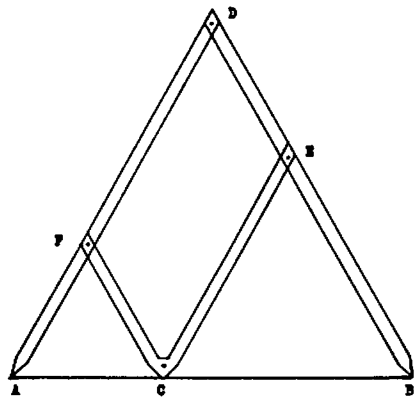
If $n = 1$, this sequence becomes the Fibonacci Sequence. The general sequence given above allows us to make an interesting observation concerning Pascal's triangle. If we take the numbers of Pascal's triangle by following the guide lines shown below, we will get the coefficients to the general sequence given above. If $n = 1$, you will notice that the sums of the numbers along the guide lines gives the Fibonacci Sequence.



For use in a demonstration, it would be nice to have a quick method of dividing a line into the golden section. The golden section sector compass can do the job very nicely and it is not very difficult to build. To use the compass, simply place the end points A and B on the ends of the line segment you wish to divide and the point C locates the golden section point. The illustration (next page) gives a picture of what the compass looks like and how they work along with a proof that the point C does actually divide the line into the golden section. The size of the model that you construct will limit its use in regards to the size of the line segments involved. The compass is constructed with

$$\frac{AD}{DF} = \frac{BD}{BE} = \tau$$

$$FC = AF \text{ and } EC = BE.$$



Now $AD/DF = \tau$ implies that $DF/AF = \tau$. Hence $DF/FC = \tau$. Since $DF = EC = EB$, then $EB/FC = \tau$ and $CB/AC = \tau$.

The golden section can also be the source of some interesting mathematical recreations. One such recreation is known as "Wythoffs game." This game is very similar to the game of Nim. To play this game, you need two sets of objects in two separate piles. The winner of the game is the player who can manuever in such a way so that he gets to pick up the last object or objects. Like Nim, there are so-called "safe" and "un-safe" conditions. A safe condition is one in which no matter what the other player does, you can win the game. For example (2,1) is a safe condition. No matter what your opponent does when he is faced with this situation, you can still win.

Obviously, if you can always fix it up so that your opponent is always faced with some safe condition, you will win the game. A series of safe conditions is given below.

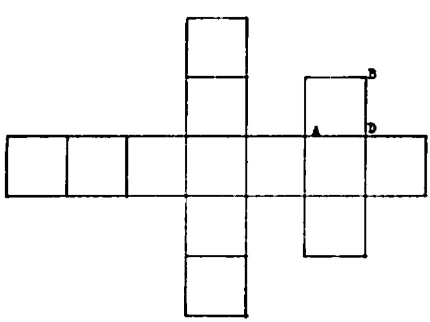
(1,2), (3,5), (4,7), 6,10), ...

Whatever your opponent does when he is faced with one of these safe conditions, it is always possible for you to either win the game on one move or get back to a safe condition on one move.

The question is, how do we know when we have a safe condition. The key is that the nth safe condition can be described by the number pair $([n\tau],[n\tau^n])$ where the brackets, $[x]$, mean the largest integer less than x. For example, $[3.7] = 3$.

Suppose, after your opponent has moved, there are (p,q) counters left, and suppose that $p < q$. If $p = q$ your move is to take all the counters and you win. If p is less than q, you must try to get back to a safe condition. A property of the sequence of number pairs described by $([n\tau],[n\tau''])$ is that every positive integer appears once in this series and the difference between the numbers of each pair is an integer that also appears only once in this series. Therefore, the number p will appear in some number pair representing a safe condition. Let this safe pair be (p,p'). If p' is less than q your next move is to remove x counters from q so that $q - x = p'$. If, however, p' is greater than q you must find the safe condition (a,b) such that $b - a = q - p$. When you find (a,b) then take away y counters from both p and q so that $p - y = a$ and $q - y = b$. Now, your opponent is faced with another safe condition and the game continues in your favor. I think you will agree, the easiest way to play this game is to memorize a few safe conditions and use them as much as possible. If the two beginning piles are in an unsafe condition and you are first, you can always win. If the beginning piles are in a safe condition and you are first, you will have to wait for your opponent to make a mistake so that you can get control of the game.

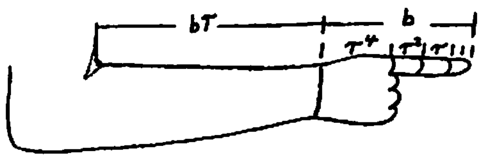
Before leaving the mathematical applications of the golden section, I would like to give you one last puzzle involving, naturally, the golden section. Consider the two beamed cross pictured below. The problem is to draw a straight line through A so that the area of the cross on one side of the line equals the area on the other side. (Hint: find a point C somewhere on the line segment BD and draw line CA. Solution on page 103.)



By this time you may be impressed with the wide range of applications of τ in mathematics. But, there is still more. In a non-mathematical sense, the golden rectangle has often been referred to as the most pleasing or the most beautiful rectangle. For this reason, many great artists have used this and other properties of the golden rectangle in their work. Such paintings as "Holy Family" by Michelangelo, "Magnificat" by Bottacellis, "Corpus Hipercubus" and "The Sacrament of the Last Supper" by Salvador Dali involve intricate use of the golden section and the properties of the golden rectangle.

The use of the golden rectangle in architecture goes back to the Parthenon in Rome. This building is a kind of monument to the golden rectangle. This fact is shown very nicely in Walt Disney's cartoon "Donald Duck in Mathemagicland." The golden section is also used extensively in the Cathedral Chartres and the Tower of Saint Jacques in Paris.

There have also been many claims made about the golden section as the "number of our physical body." These claims involve such things as "Lonc's Relativity Constant." Lonc's constant is equal to τ and his claim is that the distance from the ground to your navel, multiplied by τ will give you your height. Of course, there is no way to prove this. Also, certain bones in our bodies are thought to be related by τ. The illustration given below gives this relationship as it is related to your hand and forearm.



Finally, τ indirectly serves as an organizer of nature. The Fibonacci numbers have an application in a branch of botany called Phyllotaxis. Phyllotaxis refers to the arrangement of leaves on a branch. Starting with some leaf, follow a spiral as you move from leaf to leaf until you come to another leaf directly above the leaf you started with. If you count the number of times the branch has been circled and divide this by the number of leaves needed to get from the first leaf to the one directly above it, this fraction seems to always work out to be a fraction obtained by taking two consecutive numbers from the Fibonacci Sequence. Coxeter points out that not all plants can be depicted as fitting into the Fibonacci sequence by giving two other sequences,

(1) 3, 1, 4, 5, 9, . . .
(2) 5, 2, 7, 9, 16, . . .

which are needed to describe the arrangement of the leaves of some plants. I would like to point out, that the sequence (1) could be written as:

$$3(1, 1, 2, 3, 5, 8, \ldots) - 2(0, 1, 1, 2, 3, 5, 8, \ldots).$$

Likewise, sequence (2) can be expressed as

$$5(1, 1, 2, 3, 5, 8, \ldots) - 3(0, 1, 1, 2, 3, 5, 8, \ldots).$$

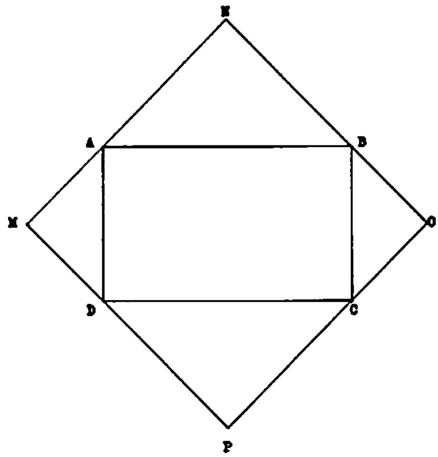
So, you could say that these plants can still be described by the use of the Fibonacci Sequence.

The Fibonacci Sequence can be used in a similar manner to code the spiral arrangement of plants such as the florets of a sunflower, the scales of a fir cone, the whorles of a pineapple and many more. If you look at a pineapple, you will notice that the hexagon shaped cells are arranged in rows in various directions: five parallel rows sloping gently up to the right, eight parallel rows sloping more steeply up to the left and thirteen rows sloping very steeply up to the right. Sometimes the slopes will be in the opposite direction.

Remember the spiral generated by the golden rectangle? This appears in nature also. The shell of the Chambered Nautilus is in the shape of a logaritmic spiral such as that of the whirling squares. One property of this type of spiral is that its shape is never altered, for this reason, as the animal grows and builds a new chamber in his shell, he always moves into an identical home.

We close this article with one more interesting fact about the golden rectangle. If the points A, B, C, and D (figure next page) are located so that they divide their respective sides of the square $MNOP$ into the golden section, then the rectangle formed by connecting these points is a golden rectangle.

(Solution to problem on page 101: Locate C so that $BD/BC = \tau$.)



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For a list of books, pamphlets, and articles, refer to The Mathematics Teacher. Vol. 43, page 426, 1950.

Linear Expression of the Greatest Common Divisor

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There is a theorem in most Modern Algebra texts fundamentally stating: If positive integer d is the greatest common divisor of two non-zero integers a and b, there exist integers x' and y' such that $d = x'\,a + y'\,b$, d being the smallest positive integer which is expressible in this form. This study is an elaboration on this concept of linear expression and a characterization of the integers x' and y'.

In order to discuss this, we must first understand the basic Euclidean Algorithm method of computing the greatest common divisor (g.c.d.). Given non-zero integers a and b (Let us consider a and b positive for ease of discussion since g.c.d is always positive regardless of signs, and the use of negative integers will change only signs not numerical values.) with $b > a$:

Divide b by a, yielding remainder r_0

$$b = q_0 a + r_0 \qquad a > r_0 > 0,$$

then divide a by r_0, yielding remainder r_1

$$a = q_1 r_0 + r_1 \qquad r_0 > r_1 > 0,$$

then divide r_0 by r_1 yielding remainder r_2

$$r_0 = q_2 r_1 + r_2 \qquad r_1 > r_2 > 0.$$

All r_i obtained by dividing the preceding divisor by remainder are non-negative integers, decreasing step by step, so that ultimately as we continue some $r_i = 0$;

$$r_l = q_n r_m + r_n \qquad r_m > r_n > 0,$$

$$r_m = q_p r_n + 0 \qquad r_n > r_p = 0.$$

The last positive remainder, r_n, can readily be proven to be the g.c.d of a and b and is written $r_n = (a, b)$.

The general process for finding an x' and y', and thus a linear expression of d in terms of a and b, is based on these divisions of the Euclidean Algorithm. Taking the equations for the Algorithm divisions, starting with the first, express each remainder in terms of a and b:

$$r_0 = b - aq_0 = (1)b + (-q_0)a$$

$$r_1 = a - r_0q_1 = a - (b - aq_0)q_1 = (-q_1)b + (1 + q_0q_1)a$$

$$r_2 = r_0 - q_2r_1 = [b + (-q_0)a] - q_2\,[(-q_1)\mathrm{b} + (1 + q_0q_1)a]$$
$$= (1 + q_1q_2)b + (-q_0 - q_2 - q_0q_1q_2)a.$$

(In actual practice the numbers which these letters represent can be combined into much simpler form.) Ultimately we get down to r_n which we find equal to $x'a + y'b$, x' and y' being sums and products of integers and thus integers themselves, as prescribed by the theorem.

An illustrative example of finding the g.c.d and a linear expression of it is shown here:

$$(a,b) = d$$

$$(24,110) = d$$

To find the g.c.d.:

$$110 = 4(24) + (14)$$

$$24 = 1(14) + (10)$$

$$14 = 1(10) + (\ 4)$$

$$10 = 2(\ 4) + (\ 2)$$

$$4 = 2(\ 2) + (\ 0)$$

Therefore $2 = (24,110)$.

$$d = ax' + by'$$

$$2 = 24x' + 110y'$$

To find the linear expression:

$$14 = 110 - 4(24) = b - 4a$$

$$10 = 24 - 14 = a - 14 = a - (b - 4a) = 5a - b$$

$$4 = 14 - 10 = (b - 4a) - (5a - b) = -9a + 2b$$

$$2 = 10 - 2(4) = (5a - b) - 2(-9a + 2b) = 23a - 5b$$

Therefore $2 = 23a - 5b$.

Now we should be able to compute the g.c.d of any two given nonzero integers and obtain a linear expression of it in terms of the given integers.

Looking at a simpler example, we know $(6,9) = 3$. The theorem says the g.c.d can be expressed in a certain linear form, which, by the method above, we find to be $3 = (-1)6 + (+1)9$. The theorem says there is always one such linear expression in a and b of their g.c.d. and we have written the one their method indicates. But it is also true that $3 = (8)6 + (-5)9$ and $3 = (17)6 + (-11)9$ and more. It would be reasonable to wonder, as I did when I noticed this, whether this were an unusual example or if g.c.d.'s could always be expressed in more than one way. I then sought to find out if all g.c.d.'s could be expressed in more than the one acknowledged way and to find a formula characterizing every possible way.

Looking at the problem algebraically, we have $d = (a,b)$ and $d = x'a + y'b$, and we wish to find more equations, if they exist, always in the $d = xa + yb$ form. Given integers a and b, integer d is determined, and x and y are left variable. Having one equation in two unknowns, we should realize any value for x determines a unique value for y, so there are many solutions to the equation. However, the difficulty here is that x and y must simultaneously be integers, and this makes the solution unusual.

Our equations must have the form of d on one side and only integral multiples of a and of b on the other. We take the equation $d = x'a + y'b$ to work from. In order to keep only d as the left member, we could only multiply the equation by one or add zero, and it would seem at first that neither of these processes would make any change in the equation. Multiplication by the number one seems unproductive any way that one looks at it. Addition of zero itself would not be beneficial; no addition of a term will be beneficial unless the term is like $(n)a$ or $(m)b$, so that the coefficient of a or of b is changed. We can add $(n)a$ though, only if we subtract $(n)a$ simultaneously, so the coefficient of a is still x', and we have made no progress. The quantity $(nab - nab)$ could be added also, but if we consider it $(nb)a - (nb)a$ or $(na)b - (na)b$, the coefficient of a or of b is still x' or y'. However, if we consider this quantity as $(nb)a - (na)b$ and add it, we get $d = (x' + nb)a + (y' - na)b$. In order that we keep the coefficients of a and b as integers, (na) and (nb) must always be integers. We can always be sure of this if n is one of the integers (Z), so for now we shall stipulate $n = z$. Now the equation is $d = (x' + zb)a + (y' - za)b$. When integer z is zero, we still have our original equation. For every other integer z, however, we get new coefficients of a and b and a completely new linear expression. Thus in every case of two integers, their g.c.d. can be expressed linearly, according to a particular form, in as many ways as there are integers. This is a far step from the one linear expression the Euclidean Algorithm procedure and the theorem guarantee.

Our next question could well be: Does this cover all possibilities? or Are there any coefficients in between those in the series we have already designated? If there are any coefficients of a and b between those in $d = (x' + zb)a + (y' - za)b$ and adjacent $d = [x' + (z + 1)b]a + [y' - (z + 1)a]b$, they must be a fraction of the difference between the present coefficients. In effect there we have added b to the coefficient of a and subtracted a from the coefficient of b. Fractions of these differences would appear as b/c and a/c if $c > 1$. Is it possible that we could add to a's coefficient such a quantity less than b and subtract from b's coefficient one less than a to get still more coefficients of correct form? We could if these two quantities, b/c and a/c are both integers, but they are only if division of c into both a and b yields integers. This would not be so if c were irrational, so we can restrict c to the rational numbers. A rational number c does not always have to be an integer to fulfill the condition that a/c and b/c be integers, but the largest c is integer d, the g.c.d. of a and b, with all other satisfactory c's being of the form d/z for every z. The larger the c (and d is the largest), the smaller the numbers a/c and b/c will be. Therefore a/d and b/d are the smallest amounts (integers) by which we can vary the coefficients. Consequently the formula $d = (x' + zb/d)a + (y' - za/d)b$, will yield every single possible linear combination of satisfactory form.

Looking at one example, we find the basic, or Euclidean Algorithm, equation for $6 = (12,18)$ to be $6 = (-1)12 + (+1)18$. Here a is 12, b is 18, d is 6, x' is -1, and y' is $+1$. Substituting in the formula, we get $6 = (-1 + 3z)12 + (+1 - 2z)18$. Any z then will give us an expression which holds, and $z + 1$ will give the adjacent expressions. Below we have $z = -1$ through $z = +3$ to illustrate five consecutive expressions in the series:

$$6 = (-4)12 + (+3)18$$
$$6 = (-1)12 + (+1)18$$

$$6 = (+2)12 + (-1)18$$
$$6 = (+5)12 + (-3)18$$
$$6 = (+8)12 + (-5)18.$$

Any random z (e.g. 100) will give an equally valid expression:

$$6 = (299)12 + (-199)18.$$

It could here be pointed out, parenthetically, that in practice if the g.c.d. is obvious without going through the divisions of the Algorithm and if one can see any linear combination, $d = x^*a + y^*b$, that satisfies all conditions prescribed, then $d = (x^* + zb/d)a + (y^* - za/d)b$ will give all other equations without the necessity for going through the work of finding x' and y'.

There certainly do "exist integers x' and y' such that $d = x'a + y'b$" as the original theorem stated. As a matter of fact there exists an infinite number of pairs of integers (x,y) in every case. The pairs, as characterized above as coefficients, are $(x' + zb/d, y' - za/d)$.

Graphic representations illustrate and add interest to these and other concepts. Taking an equation with which we have been working: $d = xa + yb$, and taking a specific a and b, and thus d, we are left with an equation in x and y. The equation is satisfied with the coordinates anywhere along the line $d = xa + yb$. Working with our theorem in mind, however, we are concerned with points where x and y are simultaneously integers. (See Figure 1). If we search

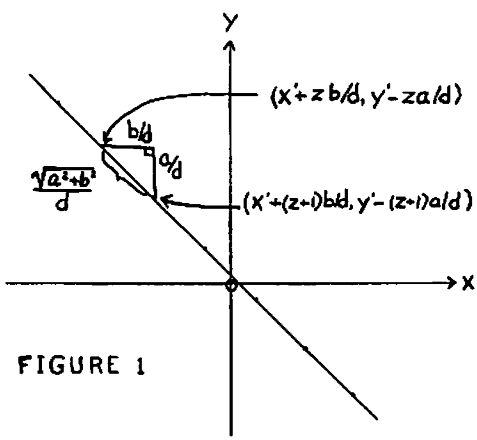

FIGURE 1

along the line we find them to be appearing at seemingly regular intervals and, checking, find them to be agreeing numerically with our characteristic coefficients $(x' + zb/d,\ y' - za/d)$, $[x' + (z+1)b/d,\ y' - (z+1)a/d]$, etc. It is because of the fact that we continuously change the coefficients by constant values ($\pm b/d$ and $\mp a/d$ respectively) that the points of interest are distributed evenly along the line. The distance between adjacent points is $(\sqrt{a^2 + b^2})/d$.

Looking at a typical example of this (see Figure 2.), we have $3 = 6x + 9y$ with the satisfactory values for x and y being specified as coordinates. By the formula for distance between two adjacent points, given above, we can see that the distance in this case is $(\sqrt{6^2 + 9^2})/3$, which is approximately 3.6.

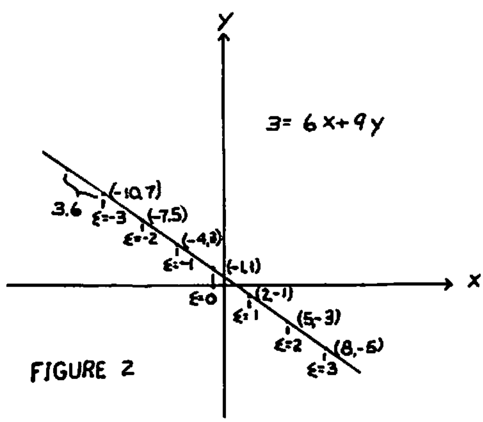

FIGURE 2

The graph in Figure 2 also calls attention to something else interesting. If we add the abscissa to the ordinate at every point and make a note of these values, we see that these sums, designated Σ (sigma), form an arithmetic progression as we move along the line. Two adjacent points have coordinates as shown in Figure 1. Taking the sum of the coordinates at each of the two adjacent points, we get $[x' + y' + (zb - za)/d]$ and $[x' + y' + (zb + b - za - a)/d]$. The d i f f e r e n c e between these sums, $\Delta\Sigma$ (delta sigma), is $(b - a)/d$, the common difference in the arithmetic progression. The existence of such a progression could supply a method of checking against omission or incorrect identification of coordinates, as an omitted or extraneous term in the progression would indicate error.

The graphs we have just considered were graphs in variables x and y. Let us turn to graphs in a and b. This means that instead of putting into $d = xa + yb$ specific values for a and b and d, we shall put in d and the sets of x's and y's we found in a particular case. Looking at the equation $d = x_j a + y_j b$, one of the many we found for some two numbers, a and b, we see that the b-intercept will be d/y_j. If we take another equation, $d = x_k a + y_k b$, this will have b-intercept d/y_k. The two lines graphed of these equations will intersect at a point, (a,b), the particular a and b from which d and the x's and y's were derived. This intersection shows that this a and b are the only two integers with g.c.d., d, with these coefficients in their linear combinations! All the other lines graphed in a and b for the chosen series will pass through this same point.

We might expect, with the infinite number of lines we have in any series, that all the lines would radiate all around this point, but it turns out that all the lines lie in a very restricted area. (See Figure 3.). If y_l is the smallest positive coefficient of b, all other

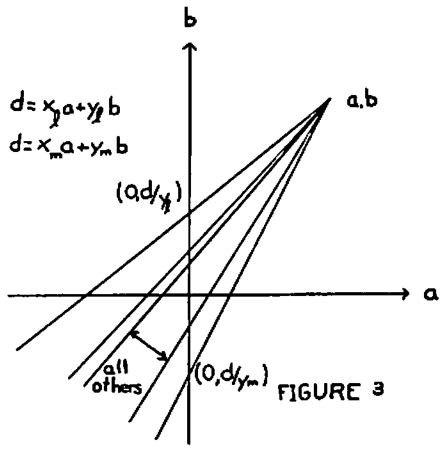

FIGURE 3

positive coefficients of b (y_p) will be larger, d/y_p will always be less than d/y_l, and the lines with these coefficients will intersect nearer the origin. Similarly if y_m is the largest negative coefficient (smallest in absolute value) of b, all the other lines with smaller negative coefficients (y_n) will intersect nearer the origin. Summarizing, y_l and y_m are the two coefficients of b nearest zero, one on each side of it. The b-intercepts of the lines with these coefficients, d/y_l and

d/y_m, form the outer limits of all b-intercepts, as every line in the series intercepts the b-axis in the small segment they determine.

Let's clarify this with an example (See Figure 4.). The g.c.d. of 6 and 9 is 3, and the two equations with b-coefficients nearest

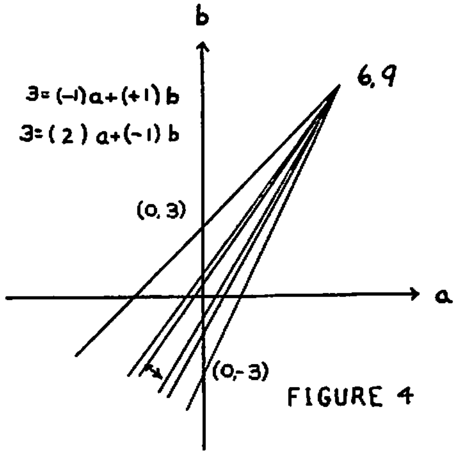

FIGURE 4

zero are $3 = -1a + 1b$ and $3 = 2a - 1b$. Thus d/y_l is $3/+1$ or 3, and d/y_m is $3/-1$ or -3. These two lines and all others from this set will intersect at (6,9), and all b-intercepts lie in the interval from $+3$ to -3. It should be noted that one of the "boundary" lines is the graph of the basic equation, $3 = -1a + 1b$. This will happen every time that $0 < |y'| < |a/d|$. We must be cautious of one case when we consider both the basic equation and the outer limiting b-intercepts; when $a = (a,b)$, the long process for finding the basic equation is inapplicable, but the logical basic equation is $d = 1a + 0b$. Thus, with $y' = 0$, there is no finite b-intercept, and the graph is a line parallel to the b-axis. The boundaries for b-intercepts, however, are still d/y_l and d/y_m.

Through graphic and algebraic presentations, we have seen that the greatest common divisor of two non-zero integers can always be expressed appropriately in infinitely more ways than just one and that, with the orderly distribution of these expressions, this subject proves a fascinating one to explore.

Differences Between Certain Properties of Sets and Sequences[1]

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In the modern classroom set theory has become a platform upon which many of the concepts of mathematics may be introduced. This paper will explore the differences between the concepts of dense point and cluster point of sets and the concepts of limit and limit point of sequences, and will show how sequences can be introduced in terms of sets with the use of a particular class of mappings. We will assume throughout the paper that sets are infinite unless otherwise indicated.

In the general framework of sets we have the following definitions:

DEFINITION 1. Let X be a set. L is a dense point of $X \Longleftrightarrow X$ is infinite and every neighborhood of the point L contains all but a finite number of points of X.

DEFINITION 2. Let X be a set. P is a *cluster* point of $X \Longleftrightarrow X$ is infinite and every neighborhood of the point P contains an infinite number of points of X [2, p. 179].

With these definitions the following theorems can be proved:

THEOREM 1. In a Hausdorff space a dense point of a set is unique.

Proof: Assume we have a Hausdorff space X with two dense points L and L'. Since L is a dense point, every neighborhood of it contains all but a finite number of points of X. And, similarly, every neighborhood of L' contains all but a finite number of points of X. Since X is Hausdorff, there exist neighborhoods M of L and N of L' such that $M \cap N = \emptyset$. But this is a contradiction and $L = L'$.

THEOREM 2. In a Hausdorff space, if a set has more than one cluster point, then it has no dense point.

Proof: Assume we have a Hausdorff space X with two cluster points P and P', $P \neq P'$. Since both P and P' are cluster points, every neighborhood M of P and every neighborhood N of P' contains an infinite number of points of X. And since X is Hausdorff there

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exist neighborhoods M' of P and N' of P' such that $M' \cap N' = \emptyset$. Then neither P nor P' can be dense points of X since there exists an infinite number of points outside of M' and an infinite number of points outside of N'. Similarly, no other point of X can be a dense point.

Observe that the proofs of these theorems rest quite strongly upon the assumption of a Hausdorff space. The following example and next two theorems show that this assumption is not only important but also essential in the above proofs.

EXAMPLE 1. Let $X = \{(x,y) \mid x \in R \text{ and } y \in R \ (R = \text{reals})\}$. Define the neighborhoods of X in the following way [3, p. 30]. A neighborhood of a point (a_0,b_0) is any set of points (x,y) which satisfies the following conditions:

1. for any $e > 0$, $|x - a_0| < e$, and
2. $y \in R$.

With neighborhoods defined in this way, all the points (x_0,y) with x coordinate the same lie in the same system of neighborhoods and for any neighborhood M of (x_0,y) and any neighborhood N of (x_0,y'), $M \cap N \neq \emptyset$.

THEOREM 3. In a non-Hausdorff space a dense point of a set is not necessarily unique.

Proof: Let $X = \{(x,y) \mid x = 1/n, \text{ for } n \text{ an integer} > 1, \text{ and } y = b \ (b \text{ fixed})\}$ and let neighborhoods be defined as in Example 1. The point $(0,b)$ is a dense point of X since every neighborhood of it contains all but a finite number of points of X. But the point $(0, b + 1)$ is also a dense point of X for the same reason.

THEOREM 4. In a non-Hausdorff space, if a set has more than one cluster point, then it may have a dense point.

Proof: Let X and neighborhoods be defined as in Theorem 3. Then X has both $(0,b)$ and $(0, b + 1)$ as cluster points. But these points are also dense points since every neighborhood of each point contains all but a finite number of points of X.

Now the concepts of limit and limit point of a sequence can be introduced as analogous to the concepts of dense point and cluster point of a set. We do this by introducing a particular class of mappings of the type $f\colon I \to A$ (we shall refer to this class as type S) where I is the set of all positive integers and A is any set. [1,p.16] If for each element n of I we associate the unique point $f(n) \in A$ and agree to call the associated point u_n, then we say that the mapping has determined an indexing operation between the points of A and the set of all positive integers. The set of all points $\{u_n\}$ of A which have been indexed by elements of I we will call the sequence associated with the set A by the particular mapping $f\colon I \to A$. Since we do not require the mappings of type S to be one-to-one (although we do require them to be onto), each point in A may be associated with more than one positive integer. This allows us to define a sequence on any finite as well as an infinite set. And since we may make use of an infinite number of mappings of type S to define a sequence on a set A, there are an infinite number of sequences that can be associated with any given set. Consider the following finite set.

EXAMPLE 2. Let $A = \{a,b\}$. Let $f(n) = a$ for n odd and $f(n) = b$ for n even. This particular mapping of type S gives rise to the following sequence:

$$a,b,a,b,a,b,\ \cdots$$

On the same set A we might define a different mapping of type S. Let $f(n) = a$ for $1 \leq n \leq 4$ and $f(n) = b$ for $n > 4$. This mapping of type S gives rise to the following sequence:

$$a,a,a,a,b,b,\ \cdots$$

Thus by using the infinite possibilities of mappings of type S, an infinite number of sequences can be associated with any set.

By using a mapping of type S we can associate a sequence with a set and define the limit and limit point of the sequence as follows:

DEFINITION 3. Let X be a Hausdorff space, $A \subset X$, and let $f\colon I \to A$ be a mapping of type S giving rise to a sequence $\{u_n\}_1^\infty$. The sequence $\{u_n\}_1^\infty$ has a limit $L \Longleftrightarrow$ every neighborhood of the point L contains all the u_n except for a finite number of values of n. (Note: this definition becomes the usual one for the limit of a sequence if the Hausdorff space is the real line.)

DEFINITION 4. Let X be a Hausdorff space, $A \subset X$, and let $f\colon I \to A$ be a mapping of type S giving rise to a sequence $\{u_n\}_1^\infty$. The sequence $\{u_n\}_1^\infty$ has a limit point $P \Longleftrightarrow$ every neighborhood of the point P contains u_n for an infinite number of values of n. (Note: this definition becomes the usual one for the limit point of a sequence if the Hausdorff space is the real line.)

With these definitions, theorems analogous to Theorems 1 through 4 can be proved:

THEOREM 5. In a Hausdorff space, the limit of a sequence of points is unique.

Proof: Let X be a Hausdorff space, $A \subset X$, and let a sequence $\{u_n\}_1^\infty$ be defined on A by a mapping $f: I \to A$ such that both L and L' are limits of $\{u_n\}_1^\infty$. Since L is a limit of $\{u_n\}_1^\infty$, every neighborhood M of L contains all the u_n except for a finite number of values of n. And since L' is a limit of $\{u_n\}_1^\infty$, every neighborhood N of L' contains all the u_n except for a finite number of values of n. But since X is Hausdorff, there exist neighborhoods M' of L and N' of L' such that $M' \cap N' = \emptyset$. But then it is impossible for both L and L' to be limits of $\{u_n\}_1^\infty$ for there would have to be a finite number of the u_n outside both M' and N'. This is a contradiction and $L = L'$.

THEOREM 6. In a Hausdorff space, if a sequence of points has more than one limit point, then it has no limit.

Proof: Let X be a Hausdorff space, $A \subset X$, and let a sequence $\{u_n\}_1^\infty$ be defined on A by a mapping $f: I \to A$ such that P and P' are both limit points, $P \neq P'$. Since P is a limit point, every neighborhood M of P contains u_n for an infinite number of values of n. And, similarly, for P'. But since X is Hausdorff, there exist neighborhoods M' of P and N' of P' such that $M' \cap N' = \emptyset$. Then, P is not a limit of $\{u_n\}_1^\infty$, since there are an infinite number of points outside M'. Similarly, P' is not a limit. And no other point can be a limit for the same reason.

THEOREM 7. In a non-Hausdorff space, the limit of a sequence of points is not necessarily unique.

Proof: Let neighborhoods be defined as in Example 1. Let A be the set $\{1 - 1/n \mid n \text{ is a positive integer}\}$, and let a sequence $\{u_n\}_1^\infty$ be defined on A by a mapping of type S, $f(n) = 1 - 1/n$ for n a positive integer. Then both $(1,0)$ and $(1,1)$ are limits of $\{u_n\}_1^\infty$ since every neighborhood of each point contains all the u_n except for a finite number of values of n.

THEOREM 8. In a non-Hausdorff space, if a sequence of points has more than one limit point, then it may have a limit.

Proof: Let neighborhoods, A, and $\{u_n\}_1^\infty$ be defined as in Theorem 7. Then both $(1,0)$ and $(1,1)$ are limit points since every neighborhood of each point contains u_n for an infinite number of values of n. But by Theorem 7, $(1,0)$ and $(1,1)$ are both limits.

Due to the obvious analogy of Theorems 1 through 4 with Theorems 5 through 8 and the obvious analogy of the underlying definitions, one might believe that the distinctions between the concepts of dense point and cluster point of a set and the concepts of limit and limit point of a sequence associated with a set is not an important one. However, a set can have a certain property without every sequence defined on it having the same property. Consider the following examples.

EXAMPLE 3. Let $X = \{x \mid x = n \text{ or } x = 1/n, \text{ for } n \text{ a positive integer}\}$. X is Hausdorff and has 0 as a unique cluster point (note that 0 is not a dense point). But define a mapping of type S in the following way:

$$f(n) = 1 \qquad \text{if } n = 3k - 2$$

$$f(n) = \frac{n+1}{3} \qquad \text{if } n = 3k - 1$$

$$f(n) = \frac{3}{n} \qquad \text{if } n = 3k$$

for $k = 1,2, \cdots$, and the limit points of this sequence are 1 and 0. Thus, if a set has a unique cluster point, not all sequences defined on it have a unique limit point.

EXAMPLE 4. Let $X = \{(x,0) \mid x = 1/n \text{ for } n \text{ a positive integer}\} \cup \{(0,1)\}$. For $x \in [0,1]$ define neighborhoods as $N_x = (x_1,1]$ for $0 \leq x_1 < x \leq 1$, $N_0 = [0,1]$ and $N_{(0,1)} = \{(x,y) \mid 0 \leq x < e < 1 \text{ and } y \in R\}$. Then X has 0 and $(0,1)$ as dense points. Define the following sequence on the set X:

$$(0,1), 1, \tfrac{1}{2}, 1, \tfrac{1}{2}, \tfrac{1}{3}, 1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots$$

Then this sequence has a unique limit 0 since $N_0 = [0,1]$ contains every point except $(0,1)$. The point $(0,1)$ is not a limit of the sequence since there are an infinite number of points of the sequence outside every neighborhood of it. Thus, a set can have two dense points, and a sequence can be defined on it having a unique limit.

EXAMPLE 5. Let $X = \{a,b\}$. X has neither dense points nor cluster points since these concepts are defined only on infinite sets. But define $f(n) = a$ if n is odd and $f(n) = b$ if n is even and this sequence has both a and b as limit points. And define $f(n) = a$ for $1 \leq n \leq 2$ and $f(n) = b$ for $n > 2$ and this sequence has point b as a limit. Thus, a set may have neither dense points nor cluster points and a sequence can be defined on it with either a pair of limit points or a unique limit.

These three examples show the differences between the properties of sets and sequences discussed in this paper. A sequence is more than just a set of points, it is a set of points indexed in a certain way by a mapping of type S. As a further difference, note that in the proof of Theorem 7 we were able to consider the set A as a sequence. The following theorem presents conditions under which a set may be considered as a sequence.

THEOREM 9. If a set X is T_1, and satisfies the first axiom of countability, then every infinite subset that has a dense point is a sequence.

Proof: Let A be an infinite subset of X with a dense point L. A is a sequence if there can be assigned a point in A to every positive integral value of n. Since X satisfies the first axiom of countability, there is a countable basis for the complete system of neighborhoods at L. Let this system be designated by $U_1 \supset U_2 \supset U_3 \supset \cdots$. Since L is a dense point, U_1 contains all but a finite number of points of A. These points can be assigned to the positive integers $1, 2, \cdots, n_1$. Similarly, U_2 contains all but a finite number of points of A. The points not already assigned to the positive integers $1, 2, \cdots, n_1$ can be assigned to the positive integers $n_1 + 1, n_1 + 2, \cdots, n_2$. Continuing this process, all the points of A will be assigned to a positive integer. For assume there exists an $x \varepsilon A$ which was not assigned to a positive integer. Then $x \varepsilon U_i$ for $i = 1,2, \cdots$. Given a neighborhood N of L, there exists a neighborhood U_k of L such that $U_k \subset N$. Hence $x \varepsilon N$, but then no neighborhood of L exists which does not contain x, and this contradicts the fact that X is T_1. Therefore, A is a sequence.

To sum up, this paper has discussed the differences between the concepts of dense point and cluster point of sets and limit and limit point of sequences. And while there are obvious analogies there are some basic differences.

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What Is Mathematics?

DANIEL B. LLOYD
Faculty, District of Columbia Teachers College

That field of study which is commonly spoken of as mathematics has been variously defined. Mathematicians themselves are far from being agreed as to its nature, its definition, or its specific purpose. Some of them like to define it vaguely as that subject matter which is of interest to mathematicians.

Numerous interesting and fanciful definitions have proceeded from the pen of laymen, poets and essayists. For instance, we find the following: "Mathematics is the grammar of size and order" (L. Hogben); "Mathematics is the science that uses easy words for hard ideas" (E. Kasner and J. Newman); "Mathematics is the language of definiteness, the necessary vocabulary of those who know." (White); "Mathematics is the science of how not to compute" (H. Maschke); "Mathematics is the science of drawing necessary conclusions" (B. Pierce); "Mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true" (Bertrand Russell); "Mathematics is the handmaiden of the sciences" (Eric T. Bell); "Mathematics is the queen of the sciences" (Karl Friedrich Gauss). It appears from the last two definitions that her sex is agreed upon, although her social position is still in doubt.

One of the most delightful elementary modern books concerning the nature of mathematics is that of Courant and Robbins entitled, *What is Mathematics?* Nowhere therein do the authors attempt to tell us what mathematics is; they are content to let us *see* what it is. Their only remark on this is the following succinct comment appearing in the preface: For scholars and laymen alike, it is not philosophy, but active experience in mathematics itself, that alone can answer the question: "What is mathematics?"

Throughout its history mathematics has defied any established definition. By its very nature it has been subject to diverse interpretations by investigators. A particular definition reflects the predilections of its proponent. Perhaps no final definition of mathematics will ever be given, as long as it continues to expand in scope, direction and emphasis.

Historically and classically mathematics was conceived as the basic integrating factor of all learning. "Mathesis" as the Greeks used the word, meant "all learning." The same word, "Mathesis" with the same spelling, was used by the Romans. Other words used were "litterae" (letters) and "scientia" (meaning all knowledge),—our word "science."

The modern inflated curriculum separates mathematics from the natural sciences, and all exact sciences from an enlarged list of the humanities and social studies.

Just where on the spectrum of knowledge mathematics really belongs is still an open question. All along this wide spectral band from physical, sensory, or vicarious experience, through the semi-abstract, or abstract symbol, and on into the ultra-abstract notion, or idea, we find it allocated by various thinkers. But its placement at any definite point encounters difficulty. Having now tracked mathematics to its nesting place from different directions, it might be of ultimate interest to ascertain whether it really exists at all. Maybe the nest is empty. However, we shall leave this to other inquisitors!

Installation of New Chapters

EDITED BY SISTER HELEN SULLIVAN

The Illinois Epsilon Chapter of Kappa Mu Epsilon was installed on May 22, 1963, at North Park College, Chicago, Illinois. Professor J. M. Sachs, Dean of Chicago Teachers College, North, was the installing officer. An initiation banquet was held at which Dr. Sachs gave a lecture on "Analytic Projective Geometry".

The charter members include the following students: Dan Akerlund, Richard Becker, Thomas Formeller, Peter Frisk, James Martins, Albert Morris, Peter Olson, David Schlichting, Virginia Sundberg, David Swanson, James Swanson, Jean Zobus. Three faculty members, John Bramsen, C. A. Jacokes, and Alice Iverson were also initiated.

New officers of Illinois Epsilon are:

President	Peter Olson
Vice President	Albert Morris
Secretary	Jean Zobus
Treasurer	Tom Formeller
Corr. Secretary	Alice Iverson

The Problem Corner

EDITED BY F. MAX STEIN

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on seperate sheets before October 1, 1964. The best solutions submitted by students will be published in the Fall 1964 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor F. Max Stein, Colorado State University, Fort Collins, Colorado.

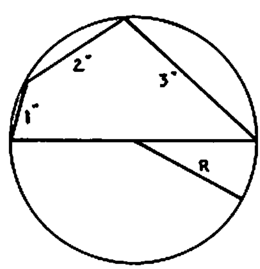
PROPOSED PROBLEMS

171. *Proposed by Robert A. Bruce, Colorado State University, Fort Collins, Colorado.*

Find x such that $x^{x^{\cdot^{\cdot^{\cdot}}}} = 2$. Similarly solve the equation $x^{x^{\cdot^{\cdot^{\cdot}}}} = 4$ for x compare the results. (A problem of this nature is found in Knopp, K., *Theory and Application of Infinite Series*, 2nd Eng. ed., Hafner Publishing Co., New York, 1947.)

172. *Proposed by James F. Rasmussen, Wayne State College, Wayne, Nebraska.*

Determine the radius of the circle shown below. (From *Analytic Geometry* by John W. Cell, 1950, Third Edition, John Wiley and Sons, Inc., New York.)



173. *Proposed by J. Frederick Leetch, Asst. Professor of Mathematics, Bowling Green State University, Bowling Green, Ohio.*

If G is a finite cyclic group of order n, generated by a, then the product of the n distinct elements is either a^n (if n is odd), or $a^{n/2}$ (if n is even).

174. *Proposed by George Tzelepis, Ulster County Community College, Kingston, New York.*

Consider the equations

$$(A) \qquad x^3 + 3mx + 2k = 0$$

$$(B) \qquad x^2 + 2mx + k = 0$$

Suppose that $k \neq 0$.

a. Find a relation between m and k, such that equations (A) and (B) have a common root.

b. Express k as a function of m.

c. Find the least positive, even, integer m, such that k is rational.

d. Solve equation (A) for x and solve equation (B) for x completely, using the values that you found for m and k.

175. *Proposed by the Editor.*

It is known that integers 0 through 112 can be expressed by using exactly four 4's and the opperation of addition, subtraction, multiplication, division, extracting the square root, factorial, decimal and powers. Show how to write integers 0 through 20 in this manner.

SOLUTIONS

166. *Proposed by Phil Huneke, Pomona College, Claremont, California.*

Find all integers m and n which satisfy:

1. $m^n = n^m$
2. $n > m$.

Solution by James F. Rasmussen, Wayne State College, Wayne, Nebraska.

Solution 1: $2^4 = 4^2$

Solution 2: $(-4)^{-2} = (-2)^{-4}$

Proof:

Assume for the present that $n > m > 0$; then $m \ln n = n \ln m$. Hence $n = m \ln n/\ln m$. Now there exists no integer such that $m/\ln m$ is an integer, hence $\ln n/\ln m$ is an integer and therefore n is some integral power of m. If we let $x = m$, then $n = x^p$, where p is some integer and $p > 0$. Therefore, $x \ln x^p = x^p \ln x$ or $p = x^{p-1}$. The following table gives x and p for the first few p's.

p	x
1	$-\infty < x < \infty$
2	2
3	$\sqrt{3}$
4	$\sqrt[3]{4}$

We note that for $p = 1$, we may assign any value to x; but then $m = n$, contrary to the original proposition. Hence $m = x = 2$ is the only permissible value—this corresponding to $n = 4$, and $2^4 = 4^2$.

Now we note that only if m and n are both even or both odd will $(-m)^{-n} = (-n)^{-m}$. In this case, both are even and therefore $m = -4$ and $n = -2$ represents second and final solution.

Also solved by Delia Hope Zelenko, Hofstra University, Hempstead, New York; Yeuk-Laan Chui, Anderson College, Anderson, Indiana; and the proposer.

167. *Proposed by Fred W. Lott, Jr., State College of Iowa, Cedar Falls.*

Identify the fallacy in the following "proof" that you are as old as Methuselah.

Let Y be your age, M be Methuselah's age, and $A = \frac{1}{2}(Y + M)$, be the average of the two ages. Then:

1. $2A = Y + M$
2. $2AY = Y^2 + MY$
3. $2AM = MY + M^2$

4. $Y^2 - 2AY = M^2 - 2AM$
5. $Y^2 - 2AY + A^2 = M^2 - 2AM + A^2$
6. $(Y - A)^2 = (M - A)^2$
7. $Y - A = M - A$
8. $Y = M$

Solution by James French, Anderson College, Anderson, Indiana.

From 6, $(Y - A) = \pm (M - A)$.

For the $(+)$ case, $Y = M = A$, the fallacy. To arrive at the second case, start with

$$A = \tfrac{1}{2}(M + Y)$$

then

$$2A = Y + M, \text{ or}$$

$$Y - A = -(M - A),$$

which is the $(-)$ case.

Also solved by Phil Huneke, Pomona College, Claremont, California and Yuek-Laan Chui, Anderson College, Anderson, Indiana.

168. *Proposed by Leigh R. Janes, State University of New York, Albany.*

Find all integers, greater than one, which are equal to the sum of the factorials of their digits.

Solution by Phil Hueneke, Pomona College, Claremont, California.

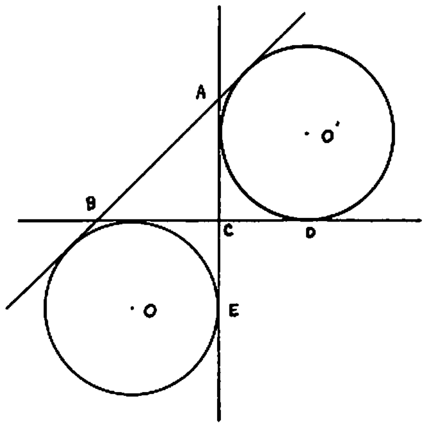
Solution: The integer 145.

Proof: $145 = 1! + 4! + 5! = 1 + 24 + 120 = 145$.

Editor's Note: By considering various properties which limit the number of possible solutions and checking the remaining numbers, Mr. Hueneke thought the above solution was unique. However, the proposer has given the additional solutions of $2! = 2$ and $40585 = 4! + 0! + 5! + 8! + 5!$, the latter discovered by the use of a computer. Can any reader either find more solutions or prove these three are the only ones possible?

169. *Proposed by Joseph Dence, Bowling Green State University, Bowling Green, Ohio.*

In the diagram below the length of sides AC and BC of triangle ABC are both equal to 5 units, and $AB = 5\sqrt{2}$ units. Circles O and



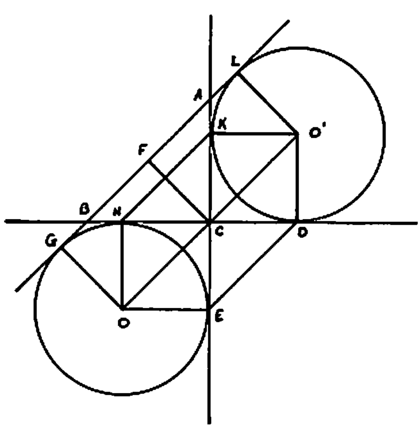
O' are escribed. Find the distance between the two points of tangency D and E.

Solution by Thomas L. Smith, Florence College, Florence, Alabama.

Solution: $DE = 5$ units.

Proof: Construct lines and letter intersections as in the figure below.

Since $AC = BC = 5$ units and $AB = 5\sqrt{2}$ units, then $\triangle ABC$ is a right isosceles triangle.



If a line CF bisects $\angle ACB$, then $AF = BF = 2.5\sqrt{2}$ units and a right isosceles triangle, $\triangle BFC$, is formed.

$$BF = CF = 2.5\sqrt{2} \text{ units.}$$

A line drawn between O and O' passes through C. Since perpendicular lines between two parallel lines are equal,

$$O'L = CF = OG = 2.5\sqrt{2} \text{ units.}$$

Also

$$OG = OE = 2.5\sqrt{2} \text{ units.}$$

By the rule previously stated, $HC = OE = 2.5\sqrt{2}$ units.

HK is parallel to AB ($OO'KH$ is an isosceles trapezoid). Therefore $\triangle HKC$ is a right isosceles triangle with $CH = CK$. Hence $HK = 5$ units. But $DE = HK$. Thus $DE = 5$ units.

Also solved by Yeuk-Laan Chui, Anderson College, Anderson, Indiana; Thomas Dence, Bowling Green State University, Bowling Green, Ohio; Joe Dreisbach, North Texas State University, Denton, Texas; James French, Anderson College, Anderson, Indiana; Phil Hueneke, Pomona College, Claremont, California; Ann M. Penton, State University of New York, Oswego, New York; James Rasmussen, Wayne State College, Wayne, Nebraska; W. Carroll Reed, East Tennessee State University, Johnson City, Tennessee; Thomas A. Selby, Central Michigan University, Mt. Pleasant, Michigan and Michael Symons, Bowling Green State University, Bowling Green, Ohio.

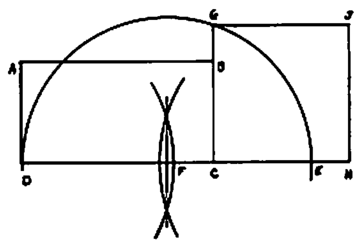
170. *Proposed by the Editor.*

Square the rectangle $ABCD$. That is, construct a square with area equal that of the rectangle $ABCD$.

Solution by Yeuk-Laan Chui, Anderson College, Anderson, Indiana.

To construct the figure (on the next page):

(1) Extend DC to E and $CE = CB$.



(2) Draw a half circle with DE the diameter.

(3) Extend CB to G where CB intercepts the half circle.

(4) Use CG as one side to form a square, $GCHJ$.

(5) Then $\overline{CG}^2 = \overline{DC} \cdot \overline{CE}$ or $\overline{CG}^2 = \overline{DC} \cdot \overline{BC}$.

(6) Thus Area $ABCD$ = Area $GCHJ$.

Also solved by James French, Anderson College, Anderson, Indiana; Phil Huneke, Pomona College, Claremont, California; James F. Rasmussen, Wayne State College, Wayne Nebraska; and W. Carroll Reed, East Tennessee State University, Johnson City, Tennessee.

Mathematics Teachers Needed Overseas

The Peace Corps estimates that during 1964 more than 5,000 teachers will be required to meet the requests coming to it from 48 countries throughout Latin America, Africa and Asia. These teachers will instruct on the elementary, secondary and college levels. More than 1,000 of these teachers have been requested to teach on the secondary and college levels in the fields of science and mathematics—650 in general science, physics, biology, chemistry, botany and zoology, and 350 in mathematics. The major requests have come from Bolivia, Ethiopia, Ghana, India, Liberia, Malaysia, Nigeria, Philippines, Sierra Leone and Turkey.

Teachers who can qualify and desire to secure one of these interesting overseas posts at the end of the current school year should file an application at an early date. Full details and an application form may be secured by writing the Division of Recruiting, Peace Corps, Washington, D.C., 20525.

The Mathematical Scrapbook

EDITED BY J. M. SACHS

Your editor has been rereading some of the Herbert Ellsworth Slaught Memorial Papers of the American Mathematical Monthly, particularly those devoted to geometry and topology. He recommends these publications not only as the source of more detailed discussions of some of the following items but as excellent orientation reading for undergraduate students.

If we examine an axiomatic approach to a Riemannian Type Geometry we can accept the following axioms due to David Gans (No. 4 of the H. E. Slaught Memorial Papers, Vol. 62, Number 7, American Mathematical Monthly):

1. A straight line is a closed line (of finite length) not intersecting itself.
2. Each pair of straight lines meet in exactly two points.
3. Among the lines joining two points there is one (or more) whose length is least. (This least length line is called a segment and its length is called the distance between the two points.)
4. A straight line is divided into two lines by each two of its points, and at least one of these lines is a segment.
5. Any given segment joining two points is contained in some straight line through the points.

What is the simplest model you can visualize for these axioms? Can you visualize other models as well? In what essentials do they differ? Can you prove the following theorems from these axioms alone?

I. If the two lines into which a straight line is divided by two of its points are unequal, the lesser is a segment joining the points; if equal, both are segments.

II. There is a straight line through each two points.

III. A unique straight line goes through two points if there is a unique segment joining them.

If we add an additional axiom we can examine a few more theorems.

6. A straight line through two points is the only straight line through them if, and only if, the points are non-antipodal

on that line. (Two points that divide a straight line into equal segments are called *antipodal* points of the line.)

IV. There is a unique segment joining two points if there is a unique straight line through them.

V. Two points cannot be antipodal on one straight line and non-antipodal on another.

VI. All straight lines bisect each other and have the same length.

How do these theorems fit your models? What other theorems can you develop?

=△=

The tremendous outpourings of geometrical knowledge in the 19th century . . . has not affected our basic approach to classical geometry. The new subjects form a sort of historical addition to the edifice—it has not been rebuilt in light of them. Higher geometry is not an outgrowth of basic geometry, but a more or less related subject studied primarily by analytic methods . . . Surely by the middle of the 20th century, a serious attempt should be made to rethink basic geometry in the light of the great 19th century advances, in order to achieve at least a minimum of conceptual attractiveness. If we default or fail in this attempt, geometry may disappear as an autonomous branch of mathematics and be reduced to a graphical way of describing certain results in the algebra of n variables.

—W. PRENOWITZ

=△=

If the preceding quotation is upsetting to those who love geometry for its own sake, consider the following fragment from the writing of G. Birkhoff. In this he admits to, ". . . the disturbing secret fear that geometry may ultimately turn out to be no more than the glittering intuitional trappings of analysis." What are your feelings? How does geometry appear to you?

=△=

In this same geometrical vein we can examine Number 8 in the H. E. Slaught Papers. This geometrical approach to topology by R. H. Bing has a great deal of stimulating material in a brief presentation. In reading the following definition, "A neighborhood in the

Euclidean plane E^2 is the interior of a circle. If a point lies on the interior of a circle, this interior is called the neighborhood of the point," a number of questions occurred to me.

1. Under what conditions is the intersection of two neighborhoods of a point again a neighborhood of this point?
2. What would be the consequences of using interiors of triangles or squares as neighborhoods?
3. If you used triangles, what would the intersection of two neighborhoods be?
4. If you used squares, what would the intersection of two neighborhoods be?
5. Would the intersection of two neighborhoods, in any one of the three cases, ever be a finite collection of non-intersecting neighborhoods? Would the original point always have to be interior to one of them?

=Δ=

The anxious precision of modern mathematics is necessary for accuracy, . . . it is necessary for research. It makes for clearness of thought and for fertility in trying new combinations of ideas. When the initial statements are vague and slipshod, at every subsequent stage of thought, common sense has to step in to limit applications and to explain meanings. Now in creative thought common sense is a bad master. Its sole criterion for judgment is that the new ideas shall look like the old ones, in other words it can act only by suppressing originality.

—A. N. Whitehead

=Δ=

Any triangle can be inscribed in a circle. Suppose a given set of three points, non collinear, are used to determine a triangle and then this triangle is inscribed in a circle. If we choose a fourth point to make the configuration a quadrilateral, under what conditions will the quadrilateral be inscribed in the circumcircle of the triangle? There are a number of ways of expressing this condition. Can you express it in terms of the angle formed at the fourth point and the relationship of this angle to the angles of the triangle? How many other ways can you express this condition? Does a coordinate system help in any of these ways?

=Δ=

. . . when such a history (of mathematics in the United States) is attempted, . . . the historian will doubtless be impressed by the tremendous influence of one man, E. H. Moore. In the late 1890's and early 1900's, the history of mathematics in this country is largely an echo of Moore's successive enthusiasms at the University of Chicago. Directly through his own work, and indirectly through that of the men he trained, Moore put new life into the theory of groups, the foundations of geometry and of mathematics in general, finite algebra, and certain branches of analysis as they were cultivated in America. Moore's interests frequently changed, and with each change, mathematics in this country advanced. His policy (as he related it shortly before his death) in those early years of his great career, was to start some thoroughly competent man well off in a particular field, and then, himself, get out of it. All his work, however, had one constant direction: he strove unceasingly toward the utmost abstractness and generality obtainable.

—E. T. BELL

=Δ=

Consider the problem of finding the dimensions of right triangles whose sides have lengths measured by consecutive integral numbers of units. The obvious first example is (3, 4, 5). Another solution is (20, 21, 29). How many solutions can you find, i.e., how many Pythagorean triples have the form (a, a + 1, c)? Fermat approached this problem as a special case of Pythagorean triples of the form $(a, a + d, c)$. What can you discover if $d = 2$? Is it true that if $(a, a + 1, c)$ is a triple of integers in a Pythagorean relation that $(2a, 2a + 2, 2c)$ is also a Pythagorean triple? What about the converse?

=Δ=

This must conclude my survey of the splendid accomplishments of American mathematics . . . I have felt as a traveler in a beautiful and unexplored country might feel who had taken his companions to some vantage points familiar to him so that they might enjoy the prospects which he happened to know, all the while realizing that on the morrow they would journey together towards more grandiose mountain peaks glittering along the horizon.

—G. D. BIRKHOFF

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of *The Pentagon*. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Harold E. Tinnappel, Bowling Green State University, Bowling Green, Ohio.

Mathematical Discourses: The Heart of Mathematical Science, Carroll V. Newsom, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964, 121 pages, $5.00.

This is a treatise written for the non-mathematician. Its five parts contain material which the author believes is essential to literacy regarding modern mathematics.

Part One discusses the earliest mathematical ideas. Included here are such topics as counting, bases for numeration systems, measurement, fractions and roots. The role of the Babylonian, Egyptian, and Chinese cultures in these developments is presented.

Part Two describes the contributions of the ancient Greeks (800 B.C. - 600 A.D.). A major portion is devoted to Euclid, including some details of his writings and comments on his imperfections. It is in Euclid's work that we see the first illustration of a mathematical discourse (system) i.e., the statement of axioms and the proving of theorems.

Mathematical discourse in the development of mathematics from the fall of Alexandria to the present is the subject of Part Three. First the Arabs, then the Italians and other Europeans work on the ideas initiated by the Greeks. The development of non-Euclidean geometries is presented in some detail. The evolution of discourses in the area we know as "modern algebra" is considered briefly.

Brief samples of discourses which should be familiar to college mathematics students comprise Part Four. The subtitles are Simple Order, The Group, Plane Projective Geometry, Boolean Algebra, and A Complete Ordered Field. In each of these a set of primary propositions is stated and a number of secondary propositions are stated and proved.

The final part includes three examples of how mathematical discourse may be applied to a specific situation. These are followed by a discussion of applying a mathematical theory (discourse plus interpretation) to Nature. The author suggests that man's attempt to explain his own mental processes is possibly the most important effort in this area.

The reviewer is pleased to see a work of this size and content available. KME members of all categories should find enjoyment and profit in reading it, despite the author's intended audience.

—J. Frederick Leetch
Bowling Green State University

The Gentle Art of Mathematics, by Dan Pedoe, The MacMillian Co., New York, 1959, 143 pages, $3.95.

"This book is intended for the many people who would like to know what mathematics is about, especially modern mathematics". This is an ambitious project for any book, especially one of this size. However, within the limitations of a reasonable well prepared reader who is not a "Niddy-Noodle", the author does touch on many amusing and instructive examples.

Chapter one discusses number representations, including the game of Nim and a familiar coin weighing problem. Chapter two is concerned with elementary probability, and has an historically interesting exchange of letters between Isaac Newton and Samuel Pepys on a dice problem. Chapter three introduces the concept of finite sets and their seeming paradoxes with the rationals and the real numbers as examples. Elementary set theory and logic of propositions are introduced in chapter four, and chapter five has a good discussion of popular combinatorial topology with some simple models to construct.

Rules of play (chapter six) is concerned with algebraic systems, and has a good discussion of symmetry to illustrate the group concept. Infinite series and some classical problems and paradoxes related to them are presented with amusing anecdotes in chapter seven. Chapter eight devotes only eight pages to logical paradoxes and the foundations of mathematics, but does touch on the leading questions. The book concludes with a brief discussion of "What is Mathematics" and what is mathematical discovery.

The author does a fine job of presenting many of the facets of modern mathematics in an informal, often whimsical way, and perhaps his only fault is that he makes it seem too much a game.

—R. N. Townsend
Bowling Green State University

New Directions in Mathematics, edited by Robert D. Ritchie, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963, 124 pp., $4.95.

A conference having the same title as this volume was held at Dartmouth College in November, 1961. The proceedings were recorded and transcribed with very little editing to form the contents of the book.

The principal speakers were Messrs. R. C. Buck, S. Eilenberg, L. Henkin, M. Kac, I. Kaplansky, P. Lax, E. E. Moise, H. O. Pollak, W. E. Slesnick, J. L. Snell, and A. W. Tucker. Each was encouraged to be visionary and incautious in his predictions. The program consisted of four panels which dealt with new directions in secondary school, college, applied, and pure mathematics.

Some of the topics presented and discussed were the completion of present second year graduate work by eighteen years of age, general college mathematics requirements if such predictions become a reality, a computer appreciation course, the desirability of breadth in mathematical interest, the role of physics courses in mathematics education, and the use of abstract and concrete approaches in teaching.

The informal nature of the proceedings, the lack of editing, and the candid photographs of the speakers give the reader an opportunity to become better acquainted with some of the leaders of today's mathematics.

—J. Frederick Leetch
Bowling Green State University

BOOKS RECEIVED

A Survey of Basic Mathematics, Fred W. Sparks, McGraw-Hill Book Company, Inc., New York, 1960, 257 pp., $3.95.

Introduction to Differentiable Manifolds, Serge Lang, John Wiley & Sons, Inc., New York 16, 1962, 126 pp., $7.00.

Boundary and Eigenvalue Problems in Mathematical Physics, Hans Sagan, John Wiley & Sons, Inc., New York 16, 1961, 381 pp., $9.50.

Perturbation Theory and the Nuclear Many Body Problem, Kailash Kumar, John Wiley & Sons, Inc., New York 16, 1962, 235 pp., $9.75.

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An Introduction to the Theory of Stationary Random Functions, A. M. Yaglom, Prentice-Hall, Englewood Cliffs, New Jersey, 1962, 229 pp., $7.95.

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Dynamic Programming and Markov Processes, Ronald A. Howard, The Massachusetts Institute of Technology Press and John Wiley & Sons, Inc., New York 16, 1960, 136 pp., $5.75.

Readings in Mathematical Psychology, Vol. I, edited by R. Duncan Luce, Robert R. Bush, Eugene Galanter, John Wiley & Sons, Inc., New York 16, 1963, 535 pp., $8.95.

Handbook of Mathematical Psychology, Vol I, edited by R. Duncan Luce, Robert R. Bush, Eugene Galanter, John Wiley & Sons, Inc., New York 16, 1963, 491 pp., $10.50.

Handbook of Mathematical Psychology, Vol. II, edited by R. Duncan Luce, Robert R. Bush, Eugene Galanter, John Wiley & Sons, Inc., New York 16, 1963, 606 pp., $11.95.

Fifteenth Biennial Convention

April 25-27, 1965

According to an announcement from National President Loyal F. Ollmann, the fifteenth biennial convention of Kappa Mu Epsilon will be held on the campus of Colorado State University, Ft. Collins, Colorado on April 25-27, 1965. Students are urged to prepare papers to be considered for presentation at the convention. Papers should be submitted to Professor Harold E. Tinnappel, National Vice-President, Bowling Green State University, Bowling Green, Ohio, before February 1, 1965. See pages 114-115 of the Spring 1962 issue of **The Pentagon** for directions with respect to the preparation of such papers.

Kappa Mu Epsilon News

EDITED BY J. D. HAGGARD, HISTORIAN

Alabama Beta, Florence State College, Florence.

Sixty members and guests from eighteen different years, including five charter members, attended a coffee hour sponsored by the chapter during homecoming.

Five members attended the fourteenth national biennial convention.

Programs this year have featured such student papers as: "Our Research Program is Korea", and "Job opportunities in Mathematics", and a film "Modern Mathematics." Social activities included a Christmas party and an annual picnic.

James Weatherbee, a 1963 graduate, is a graduate assistant in mathematics at the University of Kentucky.

Alabama Delta, Howard College, Birmingham.

Dr. R. D. Anderson of Louisiana State University gave the principal address at an initiation held on February 25.

The chapter has conducted a number of programs for the Mathematics Club this year.

California Gamma, California State Polytechnic College, San Louis Obispo.

We are conducting monthly meetings and have student, faculty, and guest speakers. Together with the Mathematics Club, we are sponsoring a series of programs on "New Math" for the parents of school children in the county. We will again be helping our mathematics faculty to conduct an annual mathematics contest for high school seniors in the state of California.

Illinois Alpha, Illinois State University, Normal.

The activities of the chapter have proven quite successful so far this year. In September we initiated ten pledges into active membership and accepted twenty-five new students as pledges. In October we were very fortunate to have Dr. Paul Weichcel from the University of Illinois speak to us on "Comma Free Codes: Nature's Dictionary". Also, in October, KME participated in homecoming activities by building a float in conjunction with the mathematics club, which received a sixth-place prize, and by having a homecoming breakfast. Over sixty persons attended the breakfast which featured Dr. Drew as the main speaker. In November two pledge papers were presented, and found to be stimulating. Throughout the semester there was considerable discussion concerning the academic standards of the Illinois Alpha Chapter of KME but we decided not to change them. In December the annual Christmas party was held and in January a book sale was conducted. We are looking forward to continued success throughout the year.

Illinois Delta, College of St. Francis, Joliet.

We have devoted our meetings this year to discussions concerning the teaching of mathematics, organizing mathematics clubs, and the "New Modern Mathematics Program." Our chapter consists of all future mathematics teachers.

Illinois Epsilon, North Park College, Chicago.

Our programs this year have included the following: "The Mathematics of Fretted Instruments" by Carl Geis, student; "Strict Implication and Multivalued Logic" by Peter Frisk, student; "Fibonacci Numbers" by C. A. Jacokes, faculty; "On the Geneology of Families of Lines" by a guest speaker, Dr. A. Seybold, Chairman of the Department of Mathematics, North Central College, Naperville, Illinois; "An Introduction to Linear Programming" by Peter Olson, student.

We are anticipating the visit to our campus on April 27, 28, 1964, of Dr. Arno Jaeger, lecturer for the Mathematical Association of America. Dr. Jaeger will give three lectures and be honored at two coffee hours during his visit to our campus. The initiation banquet for new members will be held during this time.

Several members are planning to attend the regional conference at Bowling Green, Ohio.

Indiana Gamma, Anderson College, Anderson.

Two of last years graduates have received graduate appointments. John Howland is an Assistant Instructor of Mathematics at Ohio State University and Owen Kardatzke has an Atomic Energy Commission Fellowship at the University of Maryland.

Kansas Alpha, Kansas State College of Pittsburg, Pittsburg.

Joe Jenkins and William Livingston were the recipients of the Robert Miller Mendenhall award as the oustanding senior mathematics major for 1962-63 and were presented a KME pin each. Both of these students are continuing their studies at the University of Illinois where they have graduate assistantships.

Robert Lohman and Clark Engel, graduate assistants at KSC, have received graduate assistantships for 1964-65 at the University of Iowa and the University of Kansas, respectively.

Representatives from Phillip Petroleum Co., Bartlesville, Oklahoma, presented a panel discussion on their work in linear programming at the December meeting. Three of the participants were KSC alumni, including KME member Eddie Grigsby.

Kansas Delta, Washburn University, Topeka.

Ceremonies were conducted on December 11, at which time twenty-nine new members were initiated, including the following faculty: Lewis Huff, Leroy Moffitt, and Dr. Harold Sponberg. Dr. Harold Sponberg, Washburn president, was the principal speaker.

Kansas Gamma, Mount St. Scholastica College, Atchison.

Major papers presented this semester have been: "An Application of Groups to the Theory of Equations" by Miss Martha Heidlage (a product of her Summer Undergraduate Research at the University of Oklahoma); "Linear Programming" by Misses Jeanne Beyer and Virginia Voigt (a product of summer work experience at A. T. & T.); "Infinite Sets" by Miss Ann Daly; "Factorial Analysis" by Miss Sheila Catrambone.

Dr. George Springer was guest lecturer in December, when he presented three major lectures.

Kansas Gamma Chapter also took a trip early in November to the United States Government Storage Caves located on the outskirts of Atchison, Kansas.

Maryland Alpha, College of Notre Dame of Maryland, Baltimore.

During the year 1963-64, the members of Maryland Alpha visited the Computer Center of Bendix Radio Corporation in Baltimore. At the monthly meetings the following topics were discussed by the members: "Mathematics in Art," "Mathematics in Music," "Quaternions as Matrices," "Set Theory in Plane Euclidean Geometry," and "The History of Early Chinese and Japanese Mathematics." Dr. John P. LaSalle, co-director of the Mathematics Division of RIAS spoke in February. In April, Mr. William Gerardi, Mathematics Supervisor for Baltimore City schools will speak on the "Human Values of Mathematics."

Michigan Alpha, Albion College, Albion.

Some of the programs presented during this year include: "Spherical Trigonometry and Applications," by Miss Aurora Reyes, visiting teacher from the Philippines; "Analog Computers," a film and talk by Robert Parritt, a senior student, doing independent study on our analog computer; "Measure Theory and Lesbegue Integration" by Miss Lynn Marshall; "Some Fundamental Theorems of the Theory of Games," by Miss Dorothy Cooper.

Initiation was held in February at which time a number of pledge papers were presented in partial fulfillment of the requirements for membership.

Michigan Beta, Central Michigan University, Mt. Pleasant.

During this school year the chapter is sending out teams of members to meet with interested high school students in their local schools.

Mississippi Gamma, Mississippi Southern College, Hattiesburg.

New members initiated during the Fall Quarter of 1963 include: Katie L. Duncan, Andrea F. Ford, Mary F. Hicks, John R. Hicks, Bill Owen, Linda C. Smith, Ronald C. White, Gert Winter.

Missouri Alpha, Southwest Missouri State College, Springfield.

We initiated the largest group this fall in the history of the chapter.

Our eligibility requirements have been raised; new members must now have a "B" in the first calculus course, or must have completed 8 hours beyond this course. In either case an overall "B" average must be maintained in all mathematics courses.

The featured speaker this semester was Dr. Carl V. Fronabarger, past national president of Kappa Mu Epsilon, whose topic was "Isoparametric Equations."

Missouri Beta, Central Missouri State College, Warrensburg.

Dr. Claude H. Brown gave an address at the October meeting on "The Nature of Proof." Dr. Hemphill discussed graduate programs in several adjoining states at the November meeting. Initiation ceremonies were held January 21. A program will be presented later in the semester by the Computer Department.

Nebraska Beta, Kearney State College, Kearney.

Miss Ann Christensen was awarded the annual KME scholarship. Sixteen new members were initiated into the Nebraska Beta Chapter on November 21, 1963. In April, Dr. Drury W. Wall of Iowa University will give several lectures on some new topics in modern algebra. The annual spring banquet will also be held in April.

New Mexico Alpha, University of New Mexico, Albuquerque.

Beginning this semester all new initiates will be required to present a solution to a problem from a list compiled by KME members prior to initiation.

We are sponsoring a contest for the best paper in mathematics. The only requirement to enter is to be enrolled at the University of New Mexico. Only previously unpublished work (of the quality published in the Pentagon) may be entered, and on any subject pertaining to mathematics. There is a $25 first prize, and a $15 prize for second.

New York Alpha, Hofstra University, Hempstead.

The New York Alpha Chapter is honored by the election of Prof. Loyal F. Ollmann to National President of Kappa Mu Epsilon. Prof. Ollmann is Chairman of the Mathematics Department at Hofstra University and has been very active in the Alpha Chapter since its inception at Hofstra.

New York Epsilon, Ladycliff College, Highland Falls.

The first important KME event of the year at Ladycliff College was a mathematics symposium on the numeration systems. The speakers, Edith O'Connor and Patricia Maher, discussed the structure of the base ten system as well as the structure of other systems of numeration, and then demonstrated the fundamental operations in these other systems. Kathleen Fabish, Joanne Ranft, and Maryanne Pascale will conduct the symposium in Advanced Euclidean Geometry. They will discuss the theory and applications of Ceva's Theorem. The final symposium of the year will be concerned with Computer Systems. In addition to furthering the growth of mathematics, the symposiums are designed to arouse interest and enthusiasm in the annual MATHEMATICS FAIR held here on the Ladycliff College Campus. Invitations to participate are extended to secondary and elementary schools in the surrounding area.

The New York Epsilon Chapter of KME will formally initiate ten new members on April 13, 1964. The present members of the Chapter are planning to invite the graduate members of the Chapter back for the Ceremony of Initiation as well as for the banquet which follows.

New York Gamma, State University College, Oswego.

The New York Gamma Chapter of Kappa Mu Epsilon has revised the initiation procedure. Formerly, the pledges on the night of their initiation presented a solution to a problem they had been given. This year the society voted that a paper concerning some phase of mathematics be handed in prior to the initiation. It was also decided that these papers be at least three pages long and such that they could later be presented in a meeting of the society. Some of the topics that were developed into papers are as follows:

Mathematics in Chemical Bonding
Theory of GOPS (Games of Personal Strategy)
Fibonacci Numbers
Intelligent Machines

All who are interested, from the student body and the faculty, have been invited to attend these meetings at which the papers are being presented. The programs have brought about a decided increase in interest among the members.

Ohio Gamma, Baldwin-Wallace College, Berea.

Ohio Gamma accepted thirteen new candidates into membership in October, bringing its total of collegiate actives to 31. (Grand total: 240 since induction in 1947).

At each of our monthly meetings, student papers are presented. Recent speakers and their topics include: Susan Mueller, "History of Notation," Kenneth Planisek, "Bolzano's Theorem," Richard Early, "Mathematics of Sonic Booms," Thomas Burnett, "Digital Computers," John Skurek, "Probability and Gambling," and William Sigworth, "Modern Math."

Oklahoma Alpha, Northeastern State College, Tahlequah.

Mr. Mike Reagan joined the mathematics faculty at Northeastern State College in September, 1963, and immediately assumed the duties of sponsor of Kappa Mu Epsilon. Mr. Reagan became a member of Kappa Mu Epsilon in 1950 and served as president during the school year 1951-52. He attended the Eighth Biennial Convention in Springfield, Missouri, in 1951 as a representative of the Oklahoma Alpha Chapter. He obtained a M.Ed. degree from Oklahoma University in 1953, and has since done graduate work in mathematics and physics at the University of Vermont.

Pennsylvania Beta, La Salle College, Philadelphia.

Some of the papers presented at meetings this year include: "Eigenvalue and Eigenvectors" by John Palitowski, "Ideal Theory" by Richard Glascow, "Tensor Products" by Donald Savokinas, "Affine Geometry" by Nicholas Tavani, "Determinant Theory" by John Brophy.

Frank Testa and William Mayer, 1963 graduates of the chapter, have assistantships at Purdue University where they are continuing their study of mathematics. Peter Lang, of the same class, has a Danforth Scholarship to study at the University of Chicago.

Tennessee Beta, East Tennessee State University, Johnson City.

On January 8, 1964, the following six people were initiated into Kappa Mu Epsilon: Bill Buckles, Fredrick Denney, Mary R. Hurst, Barbara J. Leonard, Schery Lodter, and Ronald C. Marcum. This brings the total membership (active and alumni) in the Tennessee Beta Chapter to 145.

At the regular meeting, January 16, Prof. Stanford Johnson, Director of E.T.S.U. Computing Center, gave an interesting talk. Two KME members, Clarence Green and Tommy Martin, gave a demonstration of prablem solving on the computer.

Virginia Beta, Radford College, Radford.

Election of officers to serve during the spring quarter and to continue next year has been held. Initiation of new members has been conducted. It was conducted in March.

Wisconsin Alpha, Mount Mary College, Milwaukee.

Some of the programs this year include: October 30, Initiation of 12 new members into Kappa Mu Epsilon; November 13, PSSC Physics by Mr. Peacock from Wauwatosa East; December 11, "The Story of John Glenn," a movie loaned to the chapter by NASA. A mathematics contest was sponsored on March 21.