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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Biangular Coordinates*

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The two most commonly used coordinate systems for locating points in a plane form a progression which leads to a third coordinate system. In the Cartesian systems, two distances are measured to locate a point, while in the polar coordinate system, a distance and an angle are measured. The third system is one in which two angles determine a point.

In Fig. 1., the original line *AB* is placed horizontally with radius vectors from its endpoints to point P. The angles eta (η) and iota (ι) between the positive direction of *AB* and the radius vectors to P are the coordinates of the point P. It is evident that as P moves,

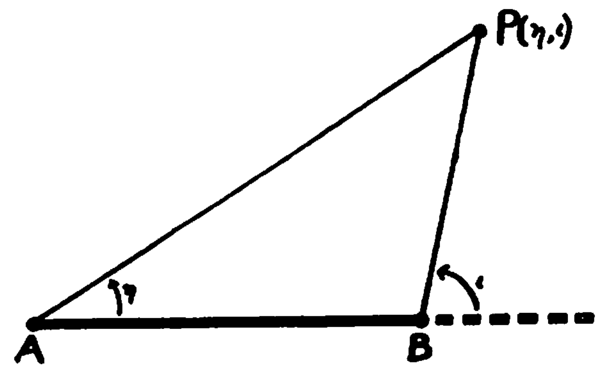


Fig. 1.

the locus of its motion can be represented by a fraction of η and ι, and conversely, any change in η or ι produces a change in the location of P. Several difficulties are apparent, however.

1. Different points on the original line can have the same coordinates.
2. The radius vectors do not necessarily intersect.
3. The coordinates of a point are multi-valued, since, for example, $(\eta + \pi, \iota)$ is the same point as (η, ι).

* A paper presented at the 1963 National Convention of KME and awarded first place by the Awards Committee.

Before proceeding to a study of loci in the new coordinates, four useful graphing tools will be developed: (A) equivalent coordinates of a point, (B) symmetry tests, (C) asymptote tests, and (D) coordinate transformation equations.

(A) *EQUIVALENT COORDINATES OF A POINT*

Two coordinate vectors are equivalent if they denote the same point. Thus

$$(a,b) \sim (c,d)$$

implies

$$c = a + n_1\pi$$
$$d = b + n_2\pi \ ,$$

where n_1 and n_2 are any integers.

(B) *SYMMETRY TESTS*

Since the image of (a,b) in the original line is equivalent to $(-a, -b)$, it follows that any $f(\eta,\iota) = 0$ is symmetrical about the original line if

$$f(\eta,\iota) \equiv f(-\eta, -\iota) \ . \qquad (1)$$

Example: $\eta^2 + \iota^2 = 1$ is symmetrical about the original line, since

$$\eta^2 + \iota^2 - 1 \equiv (-\eta)^2 + (-\iota)^2 - 1 \ .$$

Since the image of a pole in the perpendicular bisector of the line segment joining the poles is the other pole, $f(\eta,\iota) = 0$ is symmetrical about this bisector if

$$f(\eta,\iota) \equiv f(\iota,\eta) \ . \qquad (2)$$

Example: $\eta^2 + \iota^2 = 1$ is symmetrical about the bisector of the line joining the poles since

$$\eta^2 + \iota^2 - 1 \equiv \iota^2 + \eta^2 - 1 \ .$$

(C) *ASYMPTOTE TESTS*

In Fig. 2., let the circle represent an infinite circle, and let the curve $f(\eta,\iota) = 0$ be asymptotic to line PQ. The curve and the line thus intersect on the circle at Q. As point $P(\eta,\iota)$ goes out the asymptotic branch, it eventually reaches Q. Since the radius vectors now intersect the line PQ at infinity, these lines are parallel and

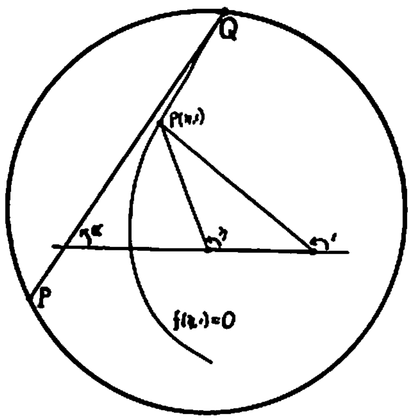


Fig. 2.

make equal angles at the transversal (the original line) and thus $\eta = \iota = \alpha$. This illustration indicates the following definition.

DEFINITION: Any locus has "asymptote angle α" if it satisfies the condition

$$(\eta,\iota) \sim (\alpha,\alpha) \tag{3}$$

for any α.

By the equivalent relation,

$$\begin{aligned} \eta &= \alpha + n_1\pi \\ \iota &= \alpha + n_2\pi \\ \eta - \iota &= (n_1 - n_2)\pi \\ \eta &= \iota + n\pi \,, \qquad \text{for } n \text{ any integer.} \end{aligned} \tag{4}$$

Any locus thus recedes to infinity in every direction for which a simultaneous solution of the equation of the locus and (4) exists. But the number of "asymptotes" is the same as the number of principal values of the asymptote angles represented.

Example: Let $\eta^2 + \iota^2 = 1$. Substituting (4),

$$(\iota + n\pi)^2 + \iota^2 = 1$$

Solving for ι,

$$\iota = [-n\pi \pm \sqrt{(2 - n^2\,\pi^2)}]/2$$

which is real for $n = 0$ only. Therefore

$$\alpha = \pm \sqrt{2}\ /2 \approx 40\tfrac{1}{2}^{\circ}$$

There are thus two asymptotes inclined at $\pm \sqrt{2}/2$ radians.

Of course, this method does not give the position of an asymptote if it exists, but only its orientation. It will yield asymptote angles whenever a curve goes to infinity, thus showing the direction of the curve at infinity. The definition of asymptote angle does not agree with the usual definition of asymptote in this respect.

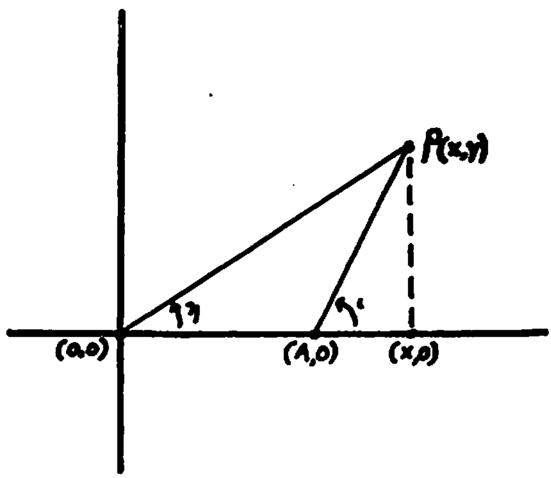


Fig. 3.

(D) *COORDINATE TRANSFORMATION EQUATIONS*

Let the η-pole coincide with the origin of a rectangular coordinate system, and let the ι-pole fall on the x-axis at the point $(x,\ y) = (A,\ 0)$ (see Fig. 3.). It is evident that

$$\tan \eta = y/x \qquad (5)$$

$$\tan \iota = y/(x - A) \qquad (6)$$

These relations, together with the rectangular-polar transformations can be interrelated to give Table I. Since the interpolar distance need not be specified in the definition of the biangular coordinates, the sole effect of the value of A is the determination of the scale size of the loci when transformed into other coordinates. Convenient values are $A = 1$ and $A = 10$.

TABLE I: Coordinate Transformations

From \ To	Rectangular	Polar	Biangular
Rectangular		$x = \rho \cos \theta$ $y = \rho \sin \theta$	$x = \dfrac{A \tan \iota}{\tan \iota - \tan \eta}$ $y = \dfrac{A}{\cot \eta - \cot \iota}$
Polar	$\rho = \sqrt{x^2 + y^2}$ $\theta = \arctan \dfrac{y}{x}$		$\rho = \dfrac{A}{\cos \eta - \cot \iota \sin \eta}$ $\theta = \eta$
Biangular	$\eta = \arctan \dfrac{y}{x}$ $\iota = \arctan \dfrac{y}{x - A}$	$\eta = \theta$ $\iota = \arctan \left(\dfrac{\rho \sin \theta}{\rho \cos \theta - A} \right)$	

Example:

$$\eta = 2\iota$$

$$\arctan [y/x] = 2 \arctan [y/(x - A)]$$

$$y/x = \frac{(2y)/(x - A)}{1 - (y^2)/(x - A)^2}$$

$$y(x^2 + y^2 - A^2) = 0$$

TYPICAL LOCI

(A) *THE N-SECTRIX*

Because of the biangular nature of these coordinates, the system adapts itself quite readily to a study of curves which can be used to fractionize angles, *i.e.*, curves such as the bisectrix, trisectrix, *etc.*

In fact, the first degree equation in these variables,

$$\eta = m\iota + \phi \text{ , } m \text{ and } \phi \text{ constants,} \tag{7}$$

represents the general solution to all such problems and thus is called the "N-sectrix". For example, both $\eta = 3\,\iota$ and $\eta = 1/3\,\iota$ can be used to trisect an angle, as can $\eta = 2/3\,\iota$.

While (7) is written in slope-intercept form, m and ϕ cannot be interpreted as the slope or intercept for these curves. For clarity, m and ϕ shall be referred to as the index and deferent respectively.

Specific curves of the family (7) will not be studied, but a general description of the curves will be given and the significance of the two constants will be investigated.

Case I: The N-sectrix with rational index.

Let $m = p/q$, an irreducible fraction with $q > 0$. (7) becomes

$$\eta = (p/q)\,\iota + \phi \text{ , } q > 0 \tag{8}$$

The Number of Asymptotes

In order to graph the N-sectrices, it is desirable to know the number of asymptotes as a function of the constants m and ϕ. A simultaneous solution of (8) with (4) yields the asymptote angle

$$\alpha_n = \frac{q\,(\phi - n\pi)}{q - p} \tag{9}$$

where α_n is the value of α associated with integer n. Let $k = q - p$ and (9) can be rewritten

$$\alpha_n = q\left(\frac{\phi}{k} - \frac{n\pi}{k}\right) \tag{10}$$

How many values of n are there such that the principal values of the asymptote angles will not repeat? The first value to repeat α_i is α_{i+k}, and there are thus $[(i + k) - i]$ or k asymptote angles. To avoid negative counts, the number of asymptote angles is therefore $|q - p|$.

Suppose, however, that $\phi = 0$, then

$$\alpha_n = nq\pi/k$$

so that when $n = 0, k, 2k, \cdots$, the original line is an asymptote. The line thus becomes part of the graph, but the number of asymptotes (in the usual sense of the word) is reduced to $[|q - p| - 1]$.

Another but perhaps more fruitful way of arriving at the same result is by means of a physical analogy. The derivative with respect to time of (8) gives

$$\frac{d\eta}{dt}\bigg/\frac{d\iota}{dt} = \frac{p}{q}.$$

In this form, it is apparent that the N-sectrices are the loci of intersection of diameters of two or more intermeshing gears; the ratio of their angular velocities is constant. But $\frac{d\eta}{dt}$ and $\frac{d\iota}{dt}$ have units of frequency (1/sec.) and thus the N-sectrices characterized by an index of p/q are traced by two gears, one of which has a rotational frequency of p, and the other, of q. These gears "reinforce" one another (have their directed diameters parallel) with a frequency which is the beat of their respective frequencies, hence

$$\text{beat frequency} = f_\eta - f_\iota = q - p.$$

Since negative frequencies are as meaningful as positive ones, the number of asymptotes is thus $|q - p|$ as before.

The Number of Branches

As with all curves, the total number of branches of each N-sectrix must be the same as the total number of principal values of the asymptote angle. Thus there are $|q - p|$ branches for the N-sectrix whose index is p/q. But the total number of rotations of

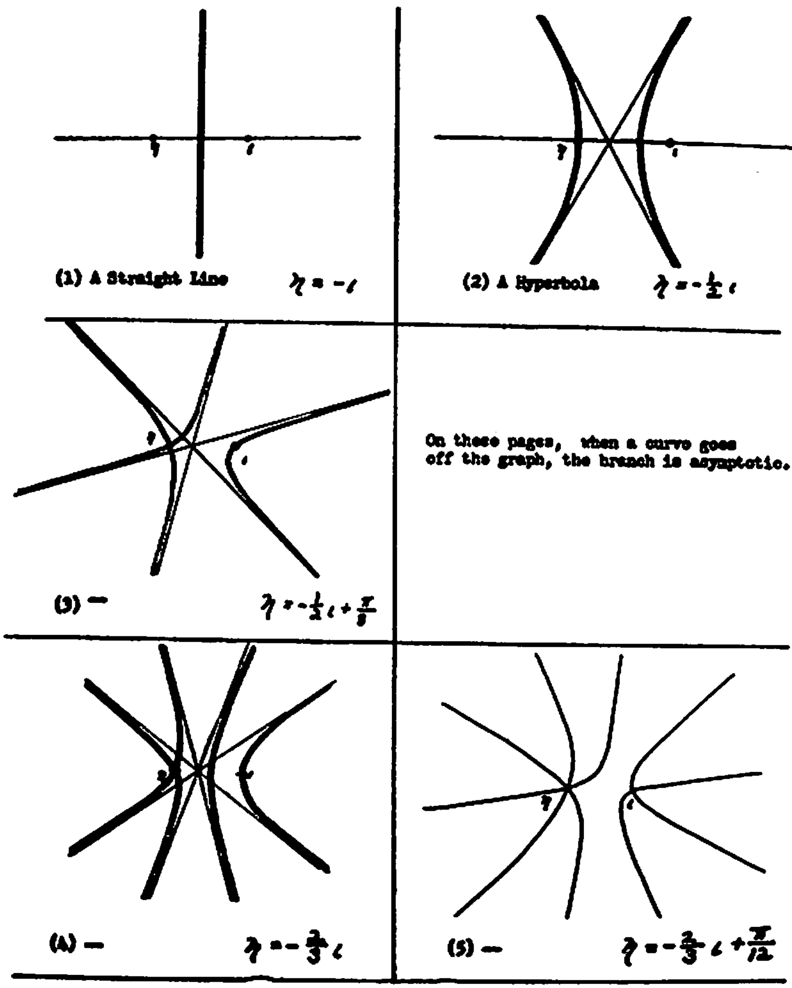


Fig. 4. Typical N-sectrices (Hyperoids)

both "gears" while producing the curve once is $q + |p|$ *. If p and q are large positive integers $q + |p|$ is a large number, while $|q - p|$ can be very small, *i.e.*, many rotations of the radius vectors may take place while the point describes only one branch of the curve. If $p > 0$, p or q loops will be formed, whichever is smaller. Thus,

* the sum of their frequencies

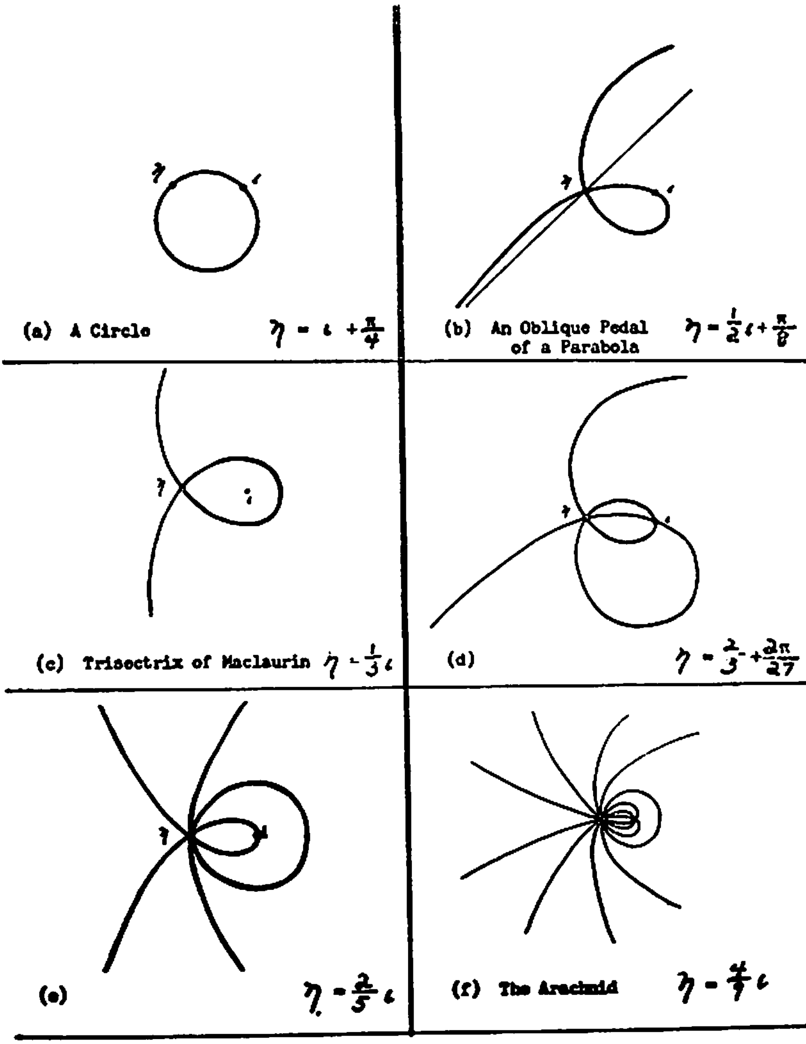


Fig. 5. Typical N-sectrices (Miscoids)

if the index of the N-sectrix is positive, it will contain loops—all such N-sectrices are referred to as "miscoids".

When the index is negative, the curves are without loops and consist of branches which merely "go from infinity to infinity" along the asymptotes. For obvious reasons, the N-sectrices with negative indices are referred to as "hyperoids".

For both hyperoids and miscoids, if $\phi \neq 0$, all branches and loops cross the original line at the poles; there are q crossings through the η-pole and p crossings through the ι-pole.

If $\phi = 0$, the original line becomes a branch of the curve and thus passes through both poles. In addition, for hyperoids, one branch becomes detached from the other branches and from the pole for which the number of branches is smallest and instead crosses the original line such that it divides the interpolar distance in the ratio p/q, while remaining closest to its "home pole".

In the corresponding case of the miscoid, one loop detaches itself from its pole and places itself in a symmetrical position so as to surround the other loops and its pole. It crosses the original line so as to cut the interpolar distance externally in the ratio p/q or q/p, whichever is smaller.

Representative examples of miscoids and hyperoids appear in the illustrations.

TABLE II: Characteristics of N-sectrices

Number of branches	$\|q - p\|$
Number of asymptotes $(\phi \neq 0)$	$\|q - p\|$
$(\phi = 0)$	$\|q - p\| - 1$
Inclinations of asymptotes	$(q\phi - qn\pi)/(q - p)$
Number of loops (miscoids only)	p
Number of crossings of η-pole	q
Number of crossings of ι-pole	$\|p\|$
x intercept of independent branch if $\phi = 0$	$x = (qA)/(q - p)$

The Effect of the Deferent

To this point, concern has been shown for establishing the general shape of the curve when the index is known. But the deferent has an important influence in controlling the shape of the curves. These are summarized here.

Subcase I: $\phi = 0$

a. The symmetry test shows the locus is symmetrical about the original line.
b. The asymptote angle test gives the original line as an asymptote.
c. The original line is a branch of the curve.
d. A branch is detached from its pole.

Subcase II: $\phi \neq 0$

a. Since the asymptote angle is given by (9), the existence of ϕ rotates the asymptotes of the locus characterized by $m = p/q$ by $(q\phi)/(q - p)$ radians. The curve becomes unsymmetrical.
b. The locus is symmetrical about the original line if $\phi = mn\pi$, since

$$\eta = m(\iota + n\pi),\ n \text{ an integer,}$$

passes the symmetry test.

Case II: N-sectrix with Irrational Index

If m is irrational, the curve does not retrace itself, so that the locus becomes extremely complicated as $\eta \to \infty$. End of case.

A Special Miscoid—The Limacoid

The miscoids whose equations are

$$\eta = \frac{n}{n + 1}\,\iota \tag{11}$$

form a closely related group which are composed, except for the original line, entirely of loops. It is evident that the matrix of limacoids illustrated (See Fig. 6.) can be extended to the right and downward with preservation of the strophoidal relationship.

That many of the N-sectrices are interrelated as strophoids [3, p. 135] and inverses [1] and that they can be related to other curves [5, p. 163, "parabola" for example] is evident, but the author leaves this problem for the interested reader.

Since all the properties of any N-sectrix are explicable in terms of its index and deferent, its equation in biangular coordinates completely determines the curve. Positive identification of the N-sectrices may be made by means of a transformation to Cartesian or polar coordinates; none will be transformed here.

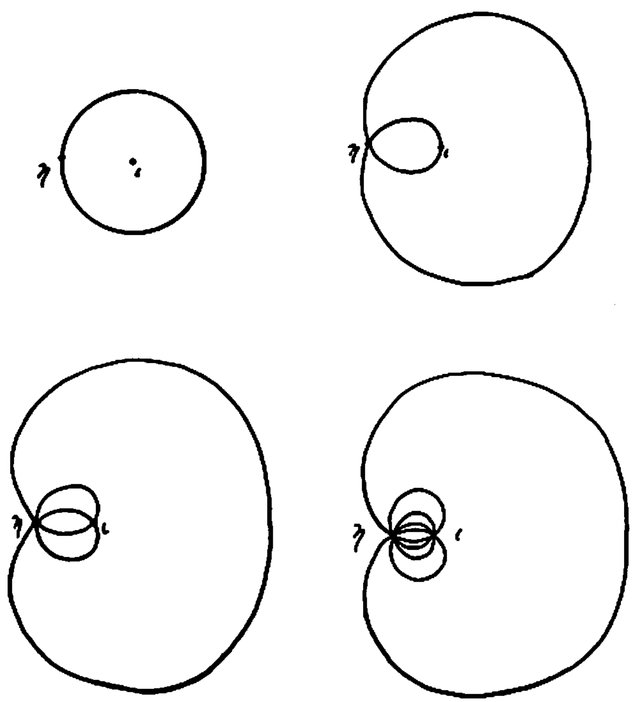
(B) *THE CENTRAL CONICS*

If any circle is placed so that the coordinate poles fall on the circumference, the circle is the locus of the vertex of a constant angle whose sides slide on the poles. The equation of the circle is

$$\eta = \iota - \beta \qquad \text{(an } N\text{-sectrix)}, \tag{12}$$

where β is the required angle. An interesting special case occurs when $\beta = 90°$; the locus is the circle for which the interpolar distance forms a diameter. The equation is

$$\eta = \iota - \pi/2 \tag{13}$$



(1) The Circle (Bisectrix)

(2) Trisectrix
a limacon

(3) Freeth's Nephroid
Strophoid of a circle

(4) Freeth's Supertrisectrix
Strophoid of Trisectrix

Fig. 6. Typical N-sectrices (Limacoids)

It may be rewritten thus

$$\tan \eta = \tan (\iota - \pi/2) = -\tan (\pi/2 - \iota)$$
$$= -\cot \iota$$

or

$$(\tan \eta)(\tan \iota) = -1 \tag{14}$$

What is the result if it is required that the product of the tangents of the angles be not -1 but any constant, *i.e.*, what is the locus of

$$(\tan \eta)(\tan \iota) = k\,? \tag{15}$$

If this equation is transformed to rectangular coordinates, the result is

$$\left(\frac{y}{x}\right)\left(\frac{y}{x - A}\right) = k.$$

After simplifying and completing the square, the result is

$$\left(x - \frac{A}{2}\right)^2 - \frac{y^2}{k} = \frac{A^2}{4} \tag{16}$$

Case I: $k > 0$

The locus is a hyperbola with the ends of the transverse axis at the poles. Using usual notation, the eccentricity is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{1 + k},$$

So $k = e^2 - 1$ and (15) may be rewritten

$$(\tan \eta)(\tan \iota) = e^2 - 1 \tag{17}$$

The asymptote angle relation gives

$$\alpha = \arctan (\pm \sqrt{e^2 - 1}) \tag{18}$$

Case II: $k = 0$

The locus is the original line—degenerate forms of a parabola.

Case III: $-1 < k < 0$

The locus is an ellipse with the ends of the major axis at the poles. As before, $k = e^2 - 1$ and (17) holds.

Case IV: $k = -1$

A circle as noted.

Case V: $k < -1$

The locus is an ellipse with the ends of the minor axis at the poles. In this case,

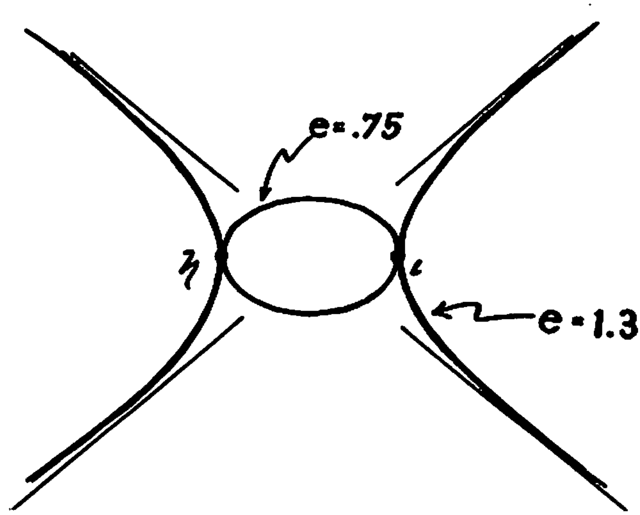
$$k = 1/(e^2 - 1) ,$$

so that the equation could be written

$$(\cot \eta)(\cot \iota) = e^2 - 1 \tag{19}$$

These cases produce what is perhaps an unusual definition of a conic—a conic is the locus of the vertex of a triangle such that the product of the tangents of the base angles is constant—and leads to speculation about the loci of equations involving products, ratios, sums, and differences of the other trigonometric functions.

The possibilities of biangular coordinates have not been ex-



$$(\tan \eta)(\tan \iota) = e^2 - 1$$

Fig. 7. The Central Conics

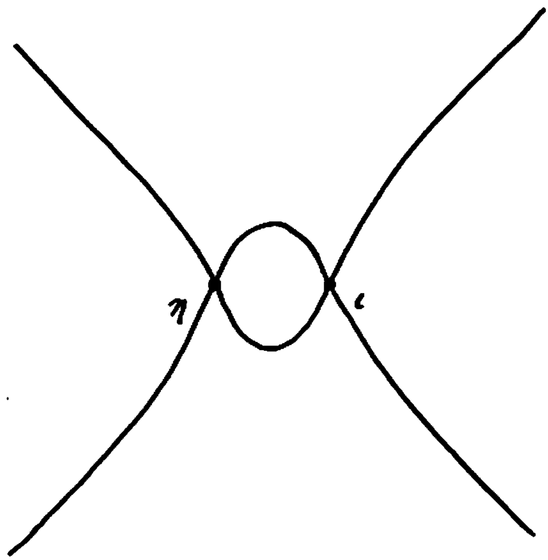


Fig. 8. An Equation of Higher Degree

hausted here, but this study of the N-sectrices and the central conics has indicated that further study can produce new curves with important properties and can explain more fully the properties of well-known curves. In addition, such a study gives added insight into the coordinate problem.

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A Study of Finite Geometry*

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I have come before you today to promote a cause which has long remained obscure under the false notion that Euclid is King. Withdrawing for the present from Euclid's "World of the Infinite", in which he produced his geometry, I shall enter into the realm of the finite to explore an intriguing geometry consisting of only 25 points. On your part, this adventure will demand close observation of a series of diagrams and geometric figures.

To unfold the "secrets" of this modern geometry, we will first characterize this mathematical system in terms of its undefined elements, unproved axioms, and its theorems. Thus, a finite geometry is a geometry which is based on a set of postulates, undefined terms, and undefined relations, and which limits the set of all points and lines to a finite number. In this system, you will notice that the names "point" and "line" are used somewhat differently from the Euclidean concept of point and line.

FIGURE # 1 Fundamental array of 25 "points"

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

In Fig. 1, we shall consider the array of 25 letters, *A* through *Y*. Each letter shall represent a point. A straight line shall mean any row or column of five distinct letters in the array, such as *AFKPU* or *ABCDE*. Two rows or two columns in the same array are *parallel* if they have no points in common. Therefore *ABCDE* is parallel to *KLMNO*. You will notice that there is no mention of the two lines meeting if sufficiently prolonged. No such extension of the lines is possible, since there exists no other points than those exhibited in the

* A paper presented at the 1963 National Convention of KME and awarded third place by the Awards Committee.

FIGURE # 2 Scheme for finding shortest distance between two "points".

A B C D E A B C D E

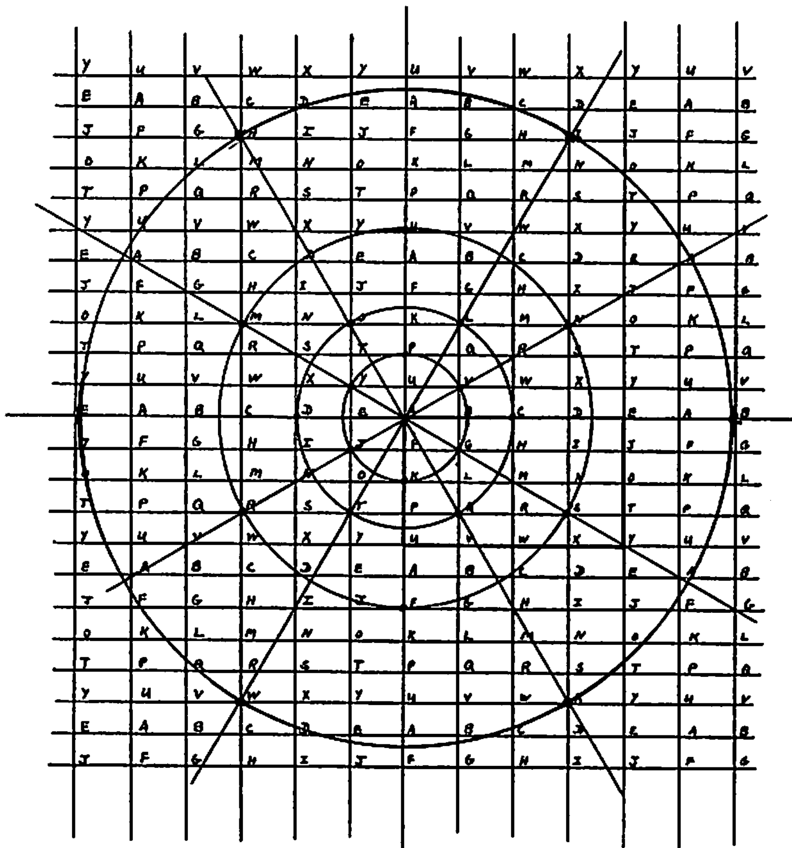
array. Any row and any column in the same array can be considered as *perpendicular*. For example, Row *ABCDE* is perpendicular to column *AFKPU* at the point *A*; and row *KLMNO* is perpendicular to column *AFKPU* at the point *K*. Distance between two points in the array is defined as the least number of steps separating the letters or points on the line which joins them. It is important to note that distance is measured only horizontally or vertically, not diagonally as from *A* to *G*. Therefore, the distance from *A* to *B* is one. Because the points on these lines are regarded as cyclically permutable, we can designate the distance from *E* to *A* also as one. It was mentioned in the definition that the distance was measured by the least number of steps. Look at the cycle in the second figure: *ABCDE ABCDE* . . . The shortest distance from *A* to *D* is two and is measured by the sequence of letters *D E A* rather than *A B C D*. Special emphasis is placed on the concept of distance because it will be used later to illustrate a geometric diagram of a miniature geometry.

The 25-point geometry which will be developed is by no means the only miniature geometry. However, as compared, for example, to a 7-, 15-, or 31-point model, the 25-point geometry is the easiest to demonstrate and yet is complex enough to be non-trivial.

We will concentrate again on the 5 x 5 array of letters shown in Fig. 1. Since we will confine ourselves to these and only these letters, we can readily see that our geometry certainly has only 25 distinct points. However, the number of lines in our system is not so readily determined. By a transformation group of rotations, we can operate on the given array of letters and generate all possible lines in the 25-point geometry.

These transformations are most easily studied by showing them as rotations on a rectangular lattice as in Fig. 3. Close observation will reveal that this lattice is merely an extension of the original 5 x 5 array in which the cyclic order of the letters in both rows and columns is repeated. Appropriate circles have been drawn on the lattice so that we can follow the rotations of the points on these circles in a counter clockwise direction through an angle of 60 degrees.

FIGURE # 3



Device to show all possible transformations of 25 "points" by rotation through 60° in a counter-clockwise sense.

Now look ahead at Fig. 4. By using the lattice just described, we can translate the letters in array #1 into a new arrangement of letters, as shown in array #2. Let *A* remain as a fixed point throughout. Think of position notation, that is, first row, second column, and then notice, for example, that the point *B* in array #1 is replaced by *I*, as shown in array #2. Thus, in the second array, the letter *I* is written in the position previously occupied by *B*. Likewise, *C* is replaced by *L*, *D* by *T*, and *E* by *W*. The remaining points of array #2 are generated in the same manner. Using the lattice again, we operate on array #2 in the same way and produce a third distinct arrangement. Again, notice that *A* remains fixed, while *I* is replaced by *H*, *L* by *O*, *T* by *Q*, and *W* by X, and so on for the remaining 20 points, as well as the other three arrays shown in Fig. 4.

FIGURE # 4

1

A	B	C	D	E
F	G	H	I	J
K	L	$\boxed{M}$	N	O
P	Q	R	S	T
U	V	W	X	Y

2

A	I	L	T	W
S	V	E	H	K
G	O	R	U	D
Y	C	F	N	Q
$\boxed{M}$	P	X	B	J

3

A	H	O	Q	X
N	P	W	E	G
V	D	F	$\boxed{M}$	T
J	L	S	U	C
R	Y	B	I	K

4

A	E	D	C	B
U	Y	X	W	V
P	T	S	R	Q
K	O	N	M	L
F	J	I	H	G

5

A	W	T	L	I
M	J	B	X	P
Y	Q	N	F	C
G	D	U	R	O
S	K	H	E	V

6

A	X	Q	O	H
R	K	I	B	Y
J	C	U	S	L
V	T	M	F	D
N	G	E	W	P

Arrays produced by six distinct rotations through 60°.

To explain exactly how the rectangular lattice was used to determine these six arrays, we refer again to the lattice in Fig. 3, and consider the smallest inner circle. Find point P two units directly above the center point A. Then according to the transformation, P is rotated in a counter-clockwise direction through 60 degrees, and is replaced by Y; Y is replaced by J, J by K, K by G, G by V, and V by P. Now turn over to Fig. 4 again and follow the same translation of the letter P in the six arrays. Focus your attention on the circled letter in each array. P is replaced by Y; Y by J, J by K, K by G, G by V, and V by P. After six rotations, we return again to the original array.

You will recall that the purpose of these transformations was to determine the exact number of lines in the miniature geometry. Since two lines are regarded as identical only when both consist of the same five distinct letters, regardless of the order of these letters, it is evident that only the first three arrays in Fig. 4 represent lines distinct from those in any other array. The fourth array is the same as the first, except that the points in the rows and columns have been permuted. The same is true for the fifth and second arrays and the sixth and third arrays. Therefore, since we have 5 rows and 5 columns in each of the first three arrays, there is a total of 30 lines generated by the transformations and we have arrived at a 25-point, 30-line geometry.

The first three arrays of letters in Fig. 4 constitute a symbolic representation of this miniature geometry. Although these arrays have clearly shown the existance of the 25 points and 30 lines, it is possible to construct a geometric diagram which demonstrates other familiar ideas found in Euclidean geometry. This diagram is built on the concept of circle, tangency, and perpendicularity.

We shall define a *circle*, as usual, to be the locus of all points equi-distant from a fixed point. Let M be the fixed point or center. Then, from an examination of the first three arrays of letters in Fig. 4 and recalling that each line is cyclic, we see that the row points of distance one from M are pointed out to be: L and N in array #1; P and J in array #2; F and T in array #3. These, of course, are points equi-distant from M in the rows only. Later we will show that the points equi-distant from M in the columns yield a related geometric diagram.

In Fig. 5a, these six letters, which are circled for easy identification, are arranged on the circumference of a circle with center M, and in the clockwise order of L T P N F J. Please note that in

FIGURE # 5a Locus of row "points" - L, T, P, N, F, J - one unit distant from center M.

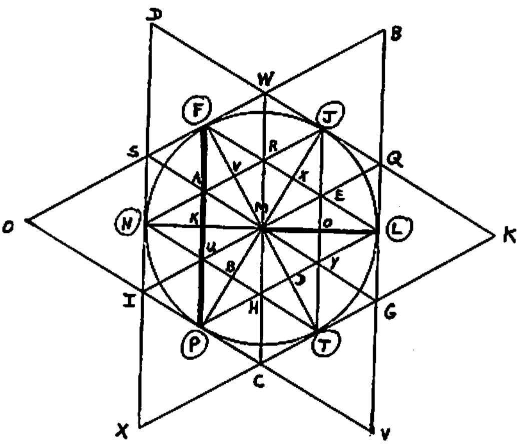


FIGURE # 5b
Table of Tangents

Tangent at L = BGLQV
" " T = XGTCK
" " P = IVOCP
" " N = DINSX
" " F = OWFSB
" " J = WKDQJ

FIGURE # 5c Fifteen column "lines" of basic arrays as found in the geometric diagram above.

F	B	W	X	J	G	V	L	T	D	N	L	O	I	K
A	Q	R	I	E	Y	C	E	H	W	A	Y	S	U	G
K	L	M	N	O	M	P	X	B	J	V	D	F	M	T
U	G	H	S	Y	A	I	R	U	Q	R	H	W	E	C
P	V	C	D	T	S	O	F	N	K	J	P	B	Q	X

FIGURE # 5d Three basic arrays.

1

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

2

A	I	L	T	W
S	V	E	H	K
G	O	R	U	D
Y	C	F	N	Q
M	P	X	B	J

3

A	H	O	Q	X
N	P	W	E	G
V	D	F	M	T
J	L	S	U	C
R	Y	B	I	K

referring to this figure, we will always name a sequence of letters in a clockwise sense.

Next, we want to find those lines which are tangent to the circle at each of the six points — *L T P N F J*. In Fig. 5a, the *tangent* at the point *L* is defined as that line which is perpendicular to the radius *LM*. In array #1 of Fig. 5d, at the bottom of the page, observe that the row *KLMNO* is perpendicular to the column *BGLQV* at the point *L*. Thus, the tangent line at *L* consists of the letters *BGLQV*, but arranged in a different order. Further, to determine the tangent at the point *T*, look at array #3 of Fig. 5d where there exists a row *VDFMT*. Then, the column *XGTCK* is perpendicular to the row *VDFMT* at the point *T*. We proceed in a similar fashion until we have determined all six tangents as described in the complete Table of Tangents of Fig. 5b. You will notice that we determine the tangents to the circle by referring to more than one array. Thus, although the geometric diagram does exhibit an ordinary circle, it is actually what you might call a *dismembered* figure, since the points from which it was constructed have been taken from three different sources. Strange as it may seem, however, such a procedure leads to a very logical structure.

Having determined the six tangent lines as given in Fig. 5b, we must decide the particular order in which to arrange these letters on the geometric figure. Consider the two successive tangency points, *L* and *T*. From the Table of Tangents, you will notice that the tangent lines through each of these points, *L* and *T*, have one letter, *G*, in common; that is, the lines intersect at *G*. Thus point *G* is determined on the figure. Considering the remaining pairs of tangent points, we can likewise specify, in Fig. 5a, the positions around the circle of *C*, *I*, *S*, *W*, and *Q*. The extreme points on the figure, *K*, *V*, *X*, *O*, *D*, and *B*, are determined by considering another relation between intersecting tangent lines. Thus, in the Table of Tangents, the lines at *L* and *P* meet at the point *V*.

The inner points of the circle are designated by again finding intersections between lines in the arrays. Your attention is drawn to the fact that the extreme points of the figure, *K V X O D B*, are actually projected to the inside of the circle where they are represented as members of other lines. This is necessary and logical since, in this geometry, every point lies on six distinct lines. For example, in Fig. 5d it is obvious that *A* lies on a row and a column in each of the three arrays; in other words, on six lines.

The figure just described shows 25 distinct points and 15 distinct lines. To show that the column lines of the arrays in Fig. 5d are actually represented in the geometric diagram of Fig. 5a, each

FIGURE # 6a Locus of column "points" - Y, H, U, A, R, E, - one unit distant from center M.

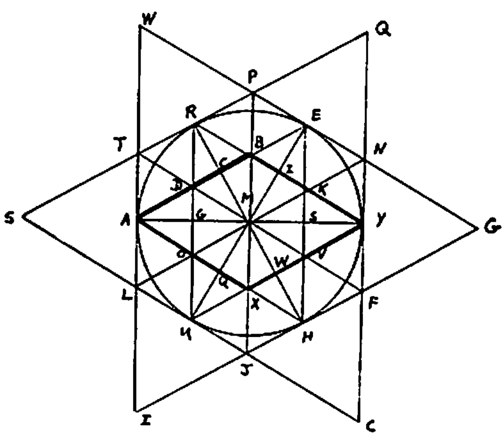


FIGURE # 6b
Table of Tangents

Tangent at Y = YCFNQ
" " H = FGHIJ
" " U = JLSUC
" " A = AILTW
" " R = PQRST
" " E = NPWEG

FIGURE # 6c Fifteen row "lines" of basic arrays as found in the geometric diagram above.

A	G	L	S	U	I	E	U	Q	P	H	W	F	C	R
D	F	O	T	X	L	K	O	N	B	X	P	V	J	B
C	H	M	R	W	A	S	G	Y	M	Q	E	M	U	I
B	J	K	P	V	T	V	D	F	X	O	N	D	L	K
E	I	N	Q	Y	W	H	R	C	J	A	G	T	S	Y

FIGURE # 6d Three basic arrays.

1

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

2

A	I	L	T	W
S	V	E	H	K
G	O	R	U	D
Y	C	F	N	Q
M	P	X	B	J

A	H	D	Q	X
N	P	W	E	G
V	D	F	M	T
J	L	S	U	C
R	Y	B	I	K

column of the arrays is taken and, in Fig. 5c, the points rearranged to correspond to the order of these points as drawn in the diagram. For example, in Fig. 5d, column one of array #1 — $A\ F\ K\ P\ U$ — is rewritten as $F\ A\ K\ U\ P$, which is the heavy vertical line in Fig. 5a. To represent the remaining 15 lines geometrically, we must again refer to the three basic arrays in Fig. 5d, and consider all those points in a column which are of distance one from the circled letter M. These are H and R in array #1, A and Y in array #2, and E and U in array #3. Following the same procedure as in the construction of the first geometric figure, we again determine the tangent lines, points of intersection and the positions of all letters so as to produce Fig. 6a. This new diagram shows all those lines which are actually rows of the three basic arrays, whereas the previous figure represented the columns of the arrays. In Fig. 6c, the order of the points of the arrays in Fig. 6d is again rearranged to correspond to the order of the same points as shown in the geometric diagram.

As mentioned earlier, these geometric diagrams of the 25-point miniature geometry are helpful because they employ such concepts as tangency, parallelism, perpendicularity, and intersection of lines in a familiar context, whereas these properties had to be particularly defined in the basic arrays. But one need not stop here. It is likewise possible to demonstrate equilateral and isosceles triangles, rectangles, and squares in this system by merely employing again the definitions of distance, perpendicular and parallel lines.

Having discussed the symbolic and geometric representations of this 25-point model, we will now list the basic axioms, which formally summarize what has already been demonstrated.

1. Every line contains 5 and only 5 points.
2. Not all points belong to the same line.
3. Every point lies on 6 and only 6 lines.
4. There are 30 and only 30 lines.
5. Two distinct lines meet in a unique point unless they are parallel.
6. There is one and only one line joining any two points.
7. Through any point, there is one and only one line parallel to a given line.
8. Through any point, there is one and only one line perpendicular to a given line.

Since no two statements of the system are contradictory, it is apparent that the axioms are at least consistent. However, we have not attempted to establish their completeness and independence.

Despite the simplicity of this miniature geometry, many theorems of Euclidean geometry hold here also. For example, we can demonstrate that the diagonals of a rhombus are perpendicular. In Fig. 6a, consider the heavily outlined rhombus, $A\ B\ Y\ X$. Recalling the definition of distance and referring to Fig. 6d, you will notice that $AB = XY = AX = YB = 1$. The line containing the points $A\ B$ in the first array is certainly parallel to the line containing the points X Y. In array #3, the line containing $A\ X$ is parallel to the line containing $B\ Y$. Thus, the opposite sides of the rhombus are parallel and equal, as one would expect. The diagonals of the rhombus consist of the points $B\ M\ X$ and $A\ G\ M\ S\ Y$. But in array #2 of Fig. 6d, notice that $A\ G\ M\ S\ Y$, written as column $A\ S\ G\ Y\ M$, is perpendicular at M to the row containing the points $B\ M\ X$. Certainly, then, the diagonals of the rhombus $A\ B\ Y\ X$ are perpendicular at the point M. This is but one of the many fundamental theorems which can be deduced from the basic assumptions.

In conclusion, then, we will all certainly agree that there is more to this finite geometry than first meets the eye.

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Monte Carlo Solutions to Waiting Line Problems*

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Lines! Lines! Lines! Everywhere man goes he is faced with waiting lines—be it at Disneyland, the corner bakery, or at a convention. To the average individual, the waiting line is a time wasting frustration to be avoided if possible. To the businessman, however, the line may be profitable, storing customers, so to speak, until they can be waited upon. On the other hand the line can be dangerous. For if it becomes too long, people will leave or may not patronize the man again, thus causing him to lose business he would otherwise have had. And if the line vanishes, though the customers may be happy, there is the needless expense of idle personnel and facilities. What then should be done to insure that the line will be such that profits will be maximized?

This is an important question and the investigations in this area have resulted in the field known as queuing theory. For the case of certain simple systems with a given number of servicing points and with specified distributions for arrival and service times, a direct mathematical analysis can be performed to obtain information about the behavior and probable states of the waiting lines. However, actual situations become extremely difficult to handle in this manner. The simple fact that a long line discourages some customers from stopping can complicate the analysis terribly. Hence, some technique short of experimenting with actual modifications of the system is necessary to determine how waiting lines in real-life situations can be made profitable.

The Monte Carlo method is one such technique. Through the use of Monte Carlo sampling procedures, the operation of the actual system can be simulated. Instead of observing arrivals and departures of actual units and watching the effect that various changes in the system have upon profits, the operation of a model which simulates the real system is observed to see what effects these changes have. The arrivals and servicings of units in the actual system can be studied and fitted to particular probability distributions. Then simulated populations may be set up that will also be described by these

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same probability distributions. A random sampling of the simulated populations will give arrivals and servicings like those of the real system. Through the use of these samples, the operation of the model will simulate the real system.

Unfortunately, this technique will not necessarily give the optimum solution. It will show only which of the tested variations is best. There is no way of determining whether the best tested form of the system is actually the optimum form or whether there exists some other as yet unthought of solution which would produce a greater profit. However, it is easy to try various modifications on the model. No capital expenditures are required, and there is no risk of losing customers, etc. By using an electronic digital computer to perform the sampling and to operate the model, answers can be obtained rapidly and easily. Thus, numerous possibilities may be tried.

One further word of caution is in order. The observation of the operation of a real business for one day gives no assurance that the next day's operations will be the same. There might be a 50% increase in the number of customers. The longer the period of observation though, the greater the assurance that the normal business pattern has been revealed. Likewise, a simulated day's operation of a model will not give "the" answer. Each day's operation yields a slightly different result, just as in the operation of a real system. However, the more simulated days that are run, the greater the probability that the results shown by the model will hold true for the real system. Still, no matter over how long a simulated time period the model is operated, the results are never 100.0% certain.

In order to illustrate the simplicity and flexibility of the Monte Carlo method on problems not suited to direct analysis, consider the following situation. A man owns 2 gasoline stations in the same town, located some 2½ to 3½ minutes apart by motor scooter. At both stations customer arrivals are on the average 3½ minutes apart. It takes an attendant an average of 5 minutes to service a car. It is estimated that only ½ the potential customers will stop if there are as many cars waiting as there are being serviced and that no one will stop if there are twice as many cars waiting as there are being serviced. On the average it takes the gross profit from 30 cars to cover an 8 hour day's depreciation and maintenance expenses for one station. It takes the gross profit from 40 cars to pay an attendant's wages. Given these conditions, how should the stations be staffed in order to maximize profits?

Although this problem falls short of being a real-life situation, it is simple enough to show clearly the use of the Monte Carlo technique. Once this method of solution is understood, further conditions, such as rush hours and part-time help, can easily be added to give the problem a greater resemblance to an actual situation.

Before the model can be constructed, a means must be devised for obtaining the simulated populations. To do this requires a slight digression to probability theory. We say that $f(t)$ is a probability density function if

a) $f(t) \geqq 0$ for all t,

b) $\int_{-\infty}^{\infty} f(t)dt = 1$,

c) Probability $[a \leqq t \leqq b] = \int_a^b f(t)dt$.

Next the distribution function, $F(x)$, is the probability of obtaining a sample value of t less than or equal to x. Therefore,

$$F(x) = \text{Prob}\,[t \leqq x] = \int_{-\infty}^{x} f(t)dt.$$

$F(x)$ is thus a monotone increasing function ranging from 0 to 1 as x ranges from $-\infty$ to $+\infty$.

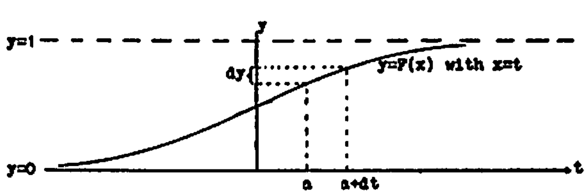
If $y = F(t)$, we see that, when dt is small,

$$\text{Prob}\,[a \leqq t \leqq a + dt] \sim dy.$$

But $dy = F'(t)dt = f(t)dt$, hence

$$\text{Prob}\,[a \leqq t \leqq a + dt] \sim f(t)dt.$$

Thus a random value of y between 0 and 1 will yield a value of t in accordance with the given distribution function since the probability



of obtaining any t is proportional to the density function at that point.

Now suppose a simulated population is required to fit a given density function, $f(t)$. We can use random numbers R, $0 < R < 1$, and the t values of the intersections of the line $y = R$ with the curve $y = F(t)$ will be the required simulated population of t's.

The length of time required to service a unit in a system often follows an exponential probability law that parameter μ, the mean service time. Assuming that this is the case for our gas station, the density function for service time is $\mu e^{-\mu t}$ for $t \geqq 0$ and is 0 for $t < 0$, since the service time can not be negative. The average service time at the station is 5 minutes, so the mean service rate is $1/5$ service per minute or $\mu = 1/5$. Since the graphical procedure for obtaining simulated populations is merely a means of solving for the variable in the distribution function when R is set equal to $F(t)$, these simulated populations may be obtained algebraically. Since

$$f(t) = \mu e^{-\mu t} \text{ for } t \geqq 0$$
$$f(t) = 0 \qquad \text{for } t < 0,$$

then

$$F(t) = \int_{-\infty}^{t} \mu e^{-\mu x} dx = 1 - e^{-\mu t} \text{ for } t > 0.$$

Letting $R = F(t) = 1 - e^{-\mu t}$, we solve for t in terms of R to obtain

$$t = \frac{1}{\mu} \ln \left(\frac{1}{1 - R}\right).$$

But if R is random on the interval $0 < R < 1$, $1 - R$ will also be random between 0 and 1. Hence, any number of simulated service times can be obtained from the expression $\frac{1}{\mu} \ln \left(\frac{1}{R}\right)$ or in our example, $5 \ln\left(\frac{1}{R}\right)$, where R is a random number, $0 < R < 1$.

Demands for service generally follow a Poisson probability law, so the number of cars arriving in a given time period, t, can be found by sampling according to the distribution of the Poisson density function which is

$$p(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \qquad \text{for } k = 0, 1, 2, \cdots$$

$$F(x) = \sum_{k=0}^{x} \frac{(\lambda t)^k e^{-\lambda t}}{k!}.$$

where λ is the mean arrival rate and k is the number of arrivals in time t. Assuming this is true for the gas station in our problem, with arrivals being an average of 3½ minutes apart, λ will equal 1/3.5. However, the time between each arrival is much more interesting than the number of arrivals in a given time period, for this permits each arrival time to be determined and thus the state of the waiting line to be known at all times. But, as shown in the next paragraph, the distribution function of the time between arrivals is merely the exponential distribution with parameter λ.

Let $F_A(t)$ be the distribution function of the time of the first arrival (or the next arrival if time is set back to 0 with each arrival). Then $F_A(t)$ = Prob [next arrival at time $\leqq t$] if $t \geqq 0$, and $F_A(t) = 0$ if $t < 0$ as the next arrival cannot come before the last arrival. Thus

$$\begin{aligned} 1 - F_A(t) &= \text{Prob [next arrival at time} > t] \\ &= \text{Prob [no arrivals in time } t] \\ &= p(0) \\ &= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} \\ &= e^{-\lambda t}. \end{aligned}$$

Hence

$$F_A(t) = 1 - e^{-\lambda t}$$

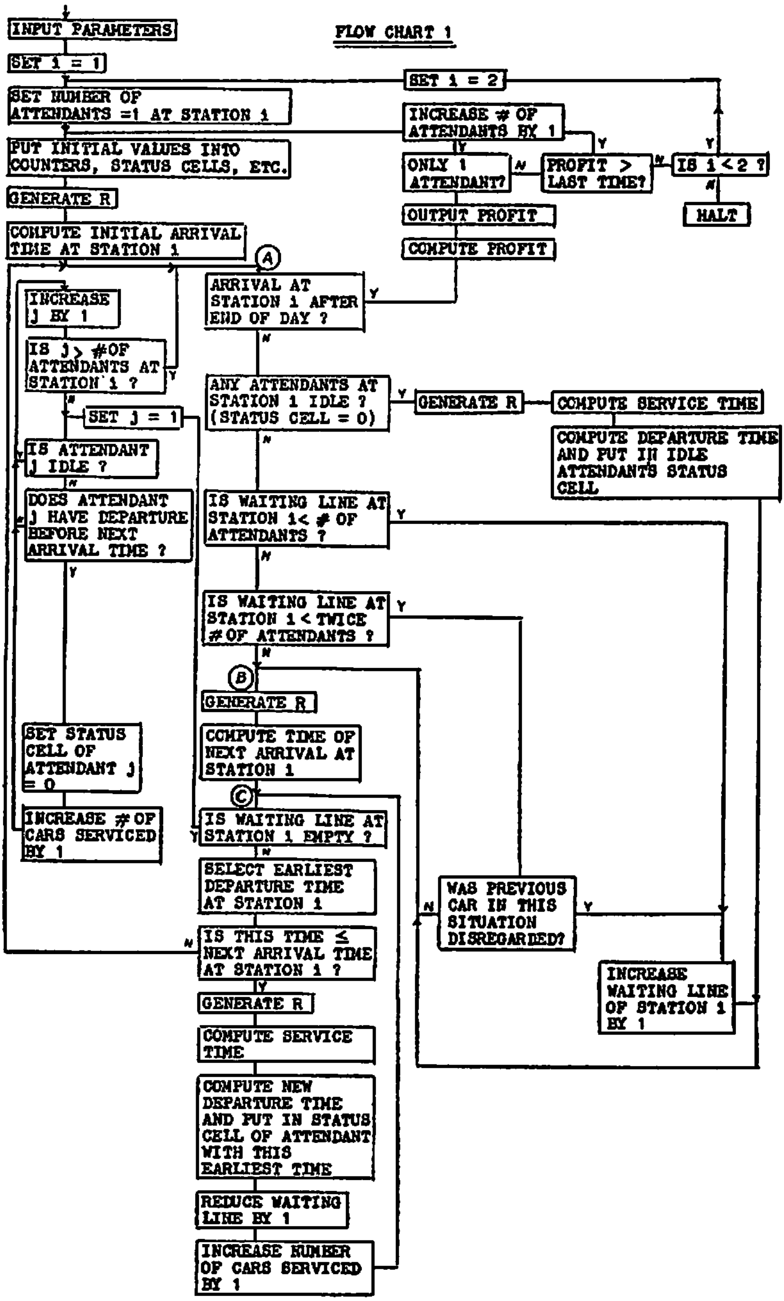
and $F_A(t)$ has an exponential distribution with parameter λ. Therefore, any number of simulated times between arrivals can be obtained from the expression $\frac{1}{\lambda} \ln\left(\frac{1}{R}\right)$, or in our case $3.5 \ln\left(\frac{1}{R}\right)$, where R is a random number as before. By using the first of these values as the time for the initial simulated arrival, each subsequent arrival time may be found by merely adding a time between arrivals to the previous arrival time. Since all these times are in minutes and decimal fractions thereof, the operation time of the station is best left in minutes, i.e., closing time is at 480.

The next step in the problem is to find a means of generating random numbers. Congruital methods, both multiplicative and additive are often used when very large computers are available. A method utilizing successive squaring and truncation of high and low order digits is also used. However, due to the basic capabilities of the computers to which I had access, I used a cubic formula developed by Wyant and Howell. An initial number between 0 and 1 is selected to be the source value. Each time a random number is to be generated, the previous source value is modified by one of two constants, one being positive, the other negative, so as to produce a new source value between 0 and 1. Using this value in the cubic expression yields a number between 0 and .38. Multiplying this by 1000 and truncating the 3 high order digits yields a random number R, $0 < R < 1$ as desired. The problem of generation of random numbers is a whole topic in itself. Hence, I will beg the question concerning how random my random numbers were and say merely that a sample of 4000 of these numbers was quite acceptable for the purposes of this problem.

The only remaining step now is to set up a model on the computer. The basic model will reflect the normal operation of the stations with a given number of employees. I designed my first model as a variation of the basic. It repeatedly modified itself so that it would simulate the system's operation with different numbers of attendants. This permitted the determination of the number of attendants it would be necessary to employ at each station in order to obtain the highest profit.

Memory locations in the computer were set aside to count the number of cars serviced, the profit made by each station, and the number of cars in each station's waiting line. It was assumed that the attendants wait on the cars in the order in which they arrive, regardless of which pumps they stop at, and thus a station may be considered to have only one waiting line. Additional memory locations were set aside to hold the time of the next arrival at each station and to show the status of each attendant—that he was idle by a 0 or that he was busy by the departure time of the car he was servicing. The first flowchart shows the program for this model.

The computer began operation in the upper left corner by accepting the input of λ, μ, and an initial source value for the random number subroutine. Station 1 with one attendant was considered


FLOW CHART 1
INPUT PARAMETERS
SET i = 1
SET i = 2
SET NUMBER OF ATTENDANTS =1 AT STATION i
INCREASE # OF ATTENDANTS BY 1
PUT INITIAL VALUES INTO COUNTERS, STATUS CELLS, ETC.
ONLY 1 ATTENDANT?
PROFIT > LAST TIME?
IS i < 2 ?
HALT
GENERATE R
OUTPUT PROFIT
COMPUTE INITIAL ARRIVAL TIME AT STATION i
COMPUTE PROFIT
A
INCREASE j BY 1
ARRIVAL AT STATION i AFTER END OF DAY ?
IS j > # OF ATTENDANTS AT STATION i ?
ANY ATTENDANTS AT STATION i IDLE ? (STATUS CELL = 0)
GENERATE R
COMPUTE SERVICE TIME
SET j = 1
COMPUTE DEPARTURE TIME AND PUT IN IDLE ATTENDANTS STATUS CELL
IS ATTENDANT j IDLE ?
DOES ATTENDANT j HAVE DEPARTURE BEFORE NEXT ARRIVAL TIME ?
IS WAITING LINE AT STATION i < # OF ATTENDANTS ?
IS WAITING LINE AT STATION i < TWICE # OF ATTENDANTS ?
B
GENERATE R
SET STATUS CELL OF ATTENDANT j = 0
COMPUTE TIME OF NEXT ARRIVAL AT STATION i
C
INCREASE # OF CARS SERVICED BY 1
IS WAITING LINE AT STATION i EMPTY ?
SELECT EARLIEST DEPARTURE TIME AT STATION i
WAS PREVIOUS CAR IN THIS SITUATION DISREGARDED?
IS THIS TIME ≤ NEXT ARRIVAL TIME AT STATION i ?
INCREASE WAITING LINE OF STATION i BY 1
GENERATE R
COMPUTE SERVICE TIME
COMPUTE NEW DEPARTURE TIME AND PUT IN STATUS CELL OF ATTENDANT WITH THIS EARLIEST TIME
REDUCE WAITING LINE BY 1
INCREASE NUMBER OF CARS SERVICED BY 1

first. All counters and status cells were set to 0, and the initial simulated arrival time was computed. The computer then entered the main loop of the program at *A*. If the time of the arrival was before the end of the day, the computer moved to point *B* via one of the 4 procedures for placing cars in the system. (1) If one of the attendants was idle, a departure time, found by adding a service time to the arrival time, was placed in that attendant's status cell. (2) If all the attendants were busy, but the number of cars waiting was less than the number being serviced, the number in the waiting line was increased by 1. (3) If the number waiting in line was greater than the number being serviced but less than twice that number, a flip-flop was multiplied by −1. If the value of this cell was then positive, the waiting line was increased by 1. If the value was negative, the computer went on to point *B*, just as if the car had been placed in the system. In other words, the car drove on. (4) If the number waiting in line was equal to twice the number being serviced, the arrival was disregarded. In this way the conditions for the arrival of cars were met.

Having placed the car in the system and arrived at *B*, the computer calculated the next simulated arrival time. Then, from point C, it proceeded to run the system up to this time. If there was a waiting line, the earliest departure time for a car being serviced was dedermined. If this departure time was later than the next arrival time, the computer went back to point *A* to check on the placement of the arrival in the system. Otherwise, the attendant involved began working on another car beginning at the time of the departure. Therefore, moving on down the column, a simulated service time was added to the existing departure time in the attendant's status cell. The waiting line was reduced by 1; the number of cars serviced was increased by 1. The computer then went back to point C to check if there was still a waiting line.

If there was no waiting line, the computer went to the lefthand column, checking each attendant for a departure that occurred before the next arrival time. For each such departure, the status cell of that attendant was set to 0 and the number of cars serviced was increased by 1. The computer then went back to *A* to check on the placement of the next arrival in the system.

After going through this procedure many times, a simulated time of next arrival finally exceeded 480. That last arrival was disregarded, and the day's profit was calculated for that station (cars serviced plus cars still in the system minus maintenance cost minus salary of all attendance). This profit, along with the station number and the number of attendants used, was then typed out by the computer. If the number of attendants was 1 or if the profit exceeded that made with 1 less employee, an additional attendant was added and another day's operation was begun. When the optimum staff size for the first station was determined, the cycle was repeated for the second station, beginning with 1 attendant.

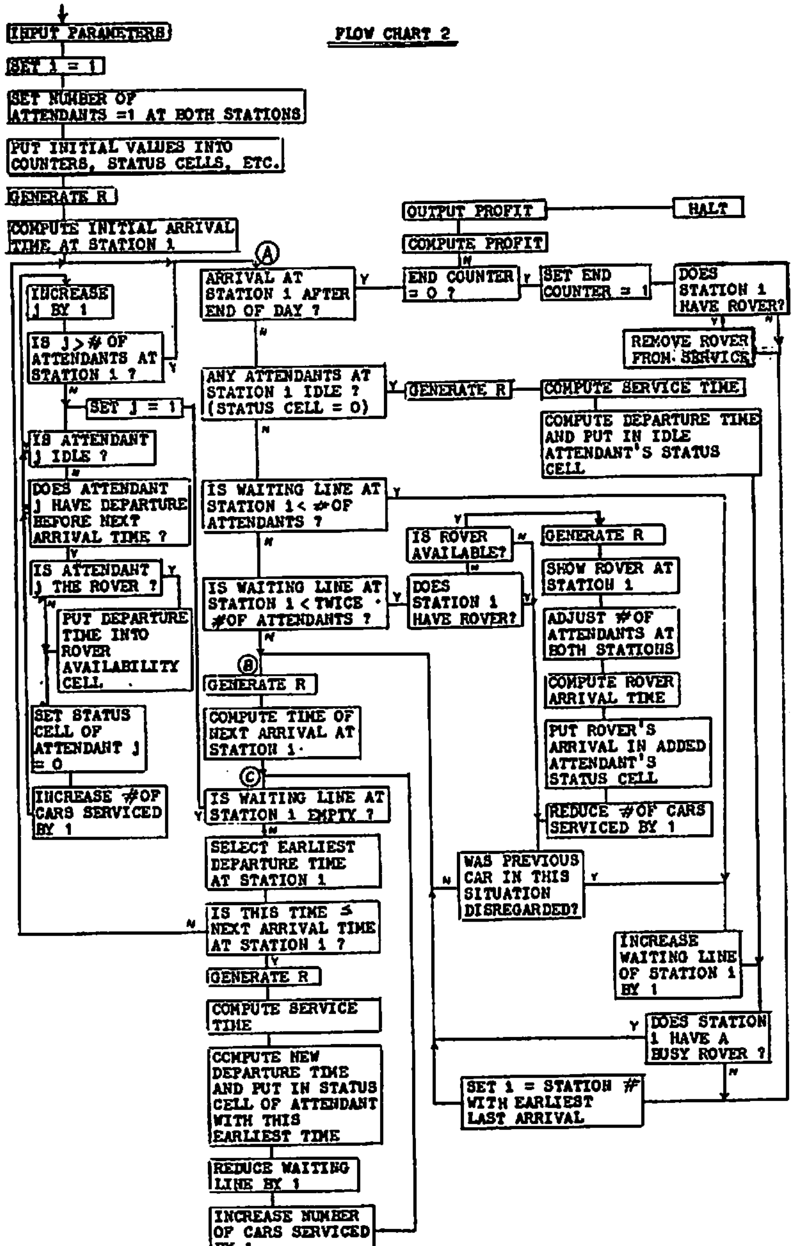
On an IBM 1620 computer it took approximately 4½ minutes to run one station for 8 simulated hours with a given number of attendants. After two weeks of simulated operation for each staff size, the greatest daily profit for the two station system, the gross from approximately 37 cars, was found to occur when 2 attendants were employed at each station. Although the two stations were set up identically, their highest average profit figures differed by about 1 car's gross per day. However, with a longer period of simulated operation, there is a greater probability that these figures would be closer to a common value.

But, to return to the question of maximizing profit, what would be the effect of having one employee shuttle from one station to the other as needed? This rover could be summoned by a buzzer arrangement but would not be called by a station unless there were more cars waiting than were being serviced. Otherwise, the conditions that gave rise to the call would more than likely be eliminated by the time the rover arrived. Also this rover would not be free to travel to the other station unless he was idle when summoned. Only a few changes need be made in the first model in order to reflect this modification. The second flowchart shows this, for it is merely the first flowchart with the necessary changes added on.

Additional memory locations had to be set aside to show at which station the rover was located and at what time he would next become idle. Instead of running one station for a day and then the other, the two had to be run semi-simultaneously so that the use of the rover could be coordinated. Each time a car was placed in the system, before the computer arrived at point *B*, operation was switched to the station whose previous arrival was the earliest. This station was then run up to its next arrival time, the arrival was put in the system, and the choice between stations was made again. Thus no station would inadvertantly use the rover because it didn't know that the other station would need the rover at that time.

However, if the rover was being used by a station, this choice was bypassed and the operation of that station was continued until

FLOW CHART 2


INPUT PARAMETERS
SET 1 = 1
SET NUMBER OF ATTENDANTS =1 AT BOTH STATIONS
PUT INITIAL VALUES INTO COUNTERS, STATUS CELLS, ETC.
GENERATE R
COMPUTE INITIAL ARRIVAL TIME AT STATION 1
A
ARRIVAL AT STATION 1 AFTER END OF DAY ?
END COUNTER = 0 ?
SET END COUNTER = 1
DOES STATION 1 HAVE ROVER?
REMOVE ROVER FROM SERVICE
COMPUTE PROFIT
OUTPUT PROFIT
HALT
INCREASE J BY 1
IS J > # OF ATTENDANTS AT STATION 1 ?
SET J = 1
IS ATTENDANT J IDLE ?
DOES ATTENDANT J HAVE DEPARTURE BEFORE NEXT ARRIVAL TIME ?
IS ATTENDANT J THE ROVER ?
PUT DEPARTURE TIME INTO ROVER AVAILABILITY CELL
SET STATUS CELL OF ATTENDANT J = 0
INCREASE #OF CARS SERVICED BY 1
ANY ATTENDANTS AT STATION 1 IDLE ? (STATUS CELL = 0)
GENERATE R
COMPUTE SERVICE TIME
COMPUTE DEPARTURE TIME AND PUT IN IDLE ATTENDANT'S STATUS CELL
IS WAITING LINE AT STATION 1 < #OF ATTENDANTS ?
IS ROVER AVAILABLE?
GENERATE R
SHOW ROVER AT STATION 1
IS WAITING LINE AT STATION 1 < TWICE #OF ATTENDANTS ?
DOES STATION 1 HAVE ROVER?
ADJUST #OF ATTENDANTS AT BOTH STATIONS
B
GENERATE R
COMPUTE ROVER ARRIVAL TIME
COMPUTE TIME OF NEXT ARRIVAL AT STATION 1
PUT ROVER'S ARRIVAL IN ADDED ATTENDANT'S STATUS CELL
C
IS WAITING LINE AT STATION 1 EMPTY ?
REDUCE #OF CARS SERVICED BY 1
SELECT EARLIEST DEPARTURE TIME AT STATION 1
WAS PREVIOUS CAR IN THIS SITUATION DISREGARDED?
IS THIS TIME ≤ NEXT ARRIVAL TIME AT STATION 1 ?
INCREASE WAITING LINE OF STATION 1 BY 1
GENERATE R
COMPUTE SERVICE TIME
DOES STATION 1 HAVE A BUSY ROVER ?
COMPUTE NEW DEPARTURE TIME AND PUT IN STATUS CELL OF ATTENDANT WITH THIS EARLIEST TIME
SET 1 = STATION # WITH EARLIEST LAST ARRIVAL
REDUCE WAITING LINE BY 1
INCREASE NUMBER OF CARS SERVICED BY 1

such time as the rover was idle. The departure time of the rover's last car was then put in the rover availability cell, and the choice between stations was again made by the computer. Thus, whenever the station without the rover was being operated, the availability cell would show correctly when the rover would be free. If the rover was needed and the availability time was earlier than the next arrival time, the rover was dispatched at once for the other station. If the rover was not free at that time, another check was made each time a car arrived, as long as conditions warranted the rover being called. Actually checks should have been made more often than this, but this procedure has the advantage of making the flowchart much simpler and thus easier to understand.

Since the rover's travel time may take any value between 2½ and 3½ minutes, it can be fit to a uniform probability law with density function equal to 1 for $2\frac{1}{2} \leqq t \leqq 3\frac{1}{2}$ and equal to 0 otherwise.

Thus

$$f(t) = \frac{1}{3.5 - 2.5} = 1 \qquad \text{for } 2.5 \leqq t \leqq 3.5$$

$$f(t) = 0 \qquad \text{otherwise}$$

and

$$R = F(t) = \int_{2.5}^{t} f(t)dt = t - 2.5$$

or

$$t = R + 2.5$$

This shows that a simulated travel time may be obtained from the expression $R + 2.5$ where R is a random number as before.

When the rover was available and was summoned, the memory cell showing the rover's location was changed, the number of attendants at one station was decreased by 1 while that number was increased by 1 at the summoning station. A computed simulated travel time was added to the arrival time of the car that gave rise to the summons, the sum being placed in the newly added attendant's status cell. The number of cars serviced was then decreased by 1, since the rover's arrival would be handled as the departure of a car from the extra attendant.

When the next arrival time at one station exceeded 480, a counter was set. If the rover was at that station, a time after closing was put in the availability cell, since the rover would not be available to the other station for the remainder of the day. The computer then went back into the loop to select the station with the earliest time of next arrival, etc. When a station's time of next arrival again exceeded 480, the computer, noting the condition of the counter, calculated the profit for the two station system and typed out this information.

The operation of this modified model for 3 simulated weeks, using a rover with 1 permanent attendant at each station, yielded an average daily profit of 40 cars's gross, 3 more than obtained with 2 attendants at each station and no rover. It is now the decision of the owner as to whether he wants to obtain the greater profit or to service 37 more cars per day for a slightly smaller profit in the hopes of building up trade. It is also the decision of the owner as to whether he wants to try to find another modification which would be still more profitable.

I hope that by the working of this one problem I have illustrated the great flexibility and versatility of the Monte Carlo technique as well as the speed and ease with which solutions may be tested. While this method will never give exact answers and while it is possible for the solutions to be incorrect, it does have a great usefulness and a definite applicability to real-life problems that do not lend themselves readily to direct mathematical analysis.

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[3] "Reference Manual IBM 1620 Fortran," San Jose, California: International Business Machines Product Publications, 1962

[4] Sasieni, Maurice, and others. *Operations Research — Methods and Problems*, New York: John Wiley and Sons, Inc., 1959

[5] Wyant, Linsley, and John M. Howell. *Intercom Programming for the Bendix G-15 Computer*, Dubuque, Iowa: William C. Brown Company, 1961

The Problem Corner

EDITED BY J. D. HAGGARD

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before March 1, 1964. The best solutions submitted by students will be published in the Spring 1964 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor F. Max Stein, Colorado State University, Ft. Collins, Colorado.

PROPOSED PROBLEMS

166. *Proposed by Phil Huneke, Pomona College, Claremont, California.*

Find all integers m and n which satisfy:

1. $m^n = n^m$
2. $n > m.$

167. *Proposed by Fred W. Lott, Jr., State College of Iowa, Cedar Falls.*

Identify the fallacy in the following "proof" that you are as old as Methuselah.

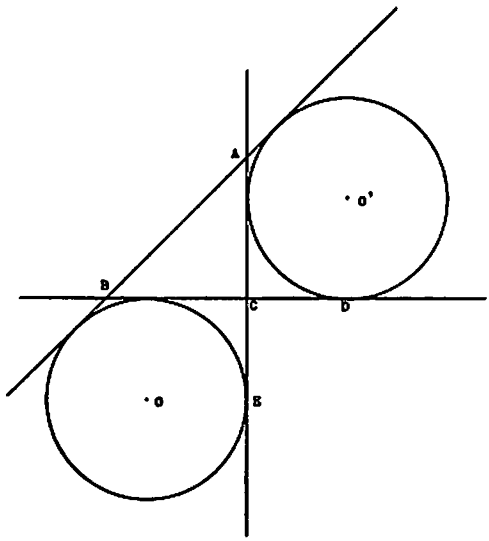
Let Y be your age, M be Methuselah's age, and $A = \frac{1}{2}(Y + M)$, be the average of the two ages. Then:

1. $2A = Y + M$
2. $2AY = Y^2 + MY$
3. $2AM = MY + M^2$
4. $Y^2 - 2AY = M^2 - 2AM$
5. $Y^2 - 2AY + A^2 = M^2 - 2AM + A^2$
6. $(Y - A)^2 = (M - A)^2$
7. $Y - A = M - A$
8. $Y = M$

168. *Proposed by Leigh R. James, State University of New York, Albany.*

Find all integers, greater than one, which are equal to the sum of the factorials of their digits.

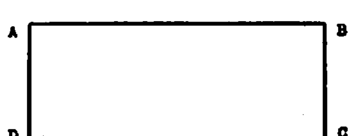
169. *Proposed by Joseph Dence, Bowling Green State University, Bowling Green, Ohio.*



In the above diagram the length of sides AC and BC of triangle ABC are both equal to 5 units, and $AB = 5\sqrt{2}$ units. Circles O and O' are escribed. Find the distance between the two points of tangency D and E.

170. *Proposed by the Editor.*

Square the rectangle $ABCD$. That is, construct a square with area equal that of the rectangle $ABCD$.



SOLUTIONS

161. *Proposed by Ann Penton, State University of New York, Oswego.*

Express the difference between the squares of two positive integers, x and y, as a sum of $|x - y|$ odd integers.

Solution by Phil Huneke, Pomona College; Claremont, California.

$$|x^2 - y^2| = (x + y)(x - y) = \sum_{i=1}^{|x-y|} (x + y + d_i)$$

where $d_i = 0$ if $(x + y)$ is odd, and $d_i = (-1)^i$ if $(x + y)$ is even.

Proof: If $x + y$ is odd,

$$\sum_{i=1}^{|x-y|} (x + y + d_i) = \sum_{i=1}^{|x-y|} (x + y)$$

= the odd number $(x + y)$ added $|x - y|$ times

$$= (x + y)(|x - y|)$$

$$= |x^2 - y^2|.$$

If $x + y$ is even,

$$\sum_{i=1}^{|x-y|} (x + y + d_i) = \sum_{i=1}^{|x-y|} (x + y + (-1)^i)$$

$$= \sum_{i=1}^{|x-y|} (x + y) + \sum_{i=1}^{|x-y|} (-1)^i.$$

But since $x + y$ is even, $x - y$ is even, thus

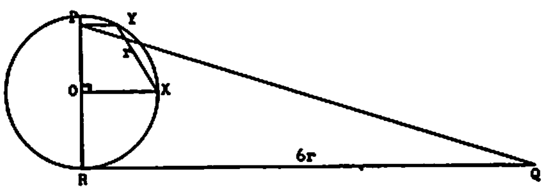
$$\sum_{i=1}^{|x-y|} (-1)^i = 0,$$

and by the first part of the proof

$$\sum_{i=1}^{|x-y|} (x + y) = |x^2 - y^2|.$$

162. *Proposed by J. F. Leetch, Bowling Green State University, Bowling Green, Ohio.*

In the June 1962 *Popular Science* appears the following construction for a segment approximating the length of the circumference of a given circle.



Construct XY of length r, PY parallel to OX, and RQ parallel to OX and of length $6r$. PQ is then "within a hair" of the circumference.

If a "hair" is assumed to be .001 in. wide, find the circle for which this approximation is correct.

Solution by Joseph Dence, Bowling Green State University, Bowling Green, Ohio.

Segment OP is equal in length to the altitude of the equilateral triangle OXY, and hence has length $r\sqrt{3/2}$. Thus $PR = r + r\sqrt{3/2}$, and since triangle PRQ is a right triangle. $PQ =$

$r\sqrt{37.75 + \sqrt{3}}$, which to four decimal places is $6.2835\ r$.

The circumference of circle O to four decimal places is $6.2832\ r$. Thus for the required approximation to hold, we must have $(6.2835 - 6.2832)\ r \leqq 0.001$ inch, or $r \leqq 3.33$ inch.

Also solved by Phil Huneke, Pomona College; Claremont, California.

163. *Proposed by V. E. Hoggatt, San Jose College; San Jose California.*

If $x < y < z$ solve:

$$\sin x + \sin y + \sin z = 0$$
$$\cos x + \cos y + \cos z = 0$$

Solution by Phil Huneke, Pomona College; Claremont, California.

Rewrite the equations as follows:

$$\sin x + \sin y = -\sin z$$
$$\cos x + \cos y = -\cos z$$

Squaring each member we obtain

$$\sin^2 x + 2 \sin x \sin y + \sin^2 y = \sin^2 z$$
$$\cos^2 x + 2 \cos x \cos y + \cos^2 y = \cos^2 z$$

Adding the two equations we obtain

$$1 + 2 [\sin x \sin y + \cos x \cos y] + 1 = 1$$

or

$$2 \cos (y - x) = -1$$

Therefore, $y - x = \pm 120° = 120°$ or $240°$.

From the symmetrical role of x, y and z

$$z - y = 120° \text{ or } 240°$$
$$z - x = 120° \text{ or } 240°$$

and since $0 < x < y < z$, then, for $x = \alpha° > 0$, either

$$y = 360°n + 120° + \alpha°$$
$$z = 360°m + 240° + \alpha°$$

or

$$y = 360°n + 240° + \alpha°$$
$$z = 360°m + 480° + \alpha°$$

where m and n are any integers such that $0 \leqq n \leqq m$.

164. *Proposed by Phil Huneke, Pomona College; Claremont, California.*

Find the positive integers, greater than one, for which the integer is equal to the sum of the cubes of its digits.

Solution by the proposer.

Case I. The integer has one digit, a; then $a^3 = a > 1$, which is impossible.

Case II. The integer has two digits a and b. Denote the number by $a + 10b$. Thus $a^3 + b^3 = a + 10b$ or $a(a^2 - 1) = b(10 - b^2)$. Since $a(a^2 - 1) = a(a + 1)(a - 1)$, $a(a^2 - 1)$ is even and positive; therefore $b(10 - b^2)$ is even and positive, thus b is even and $b \leqq 3$. Therefore $b = 2$ and $b(10 - b) = 12 = 2 \cdot 2 \cdot 3 = (a - 1)(a)(a + 1) = a(a^2 - 1)$ which is impossible for any digit a.

Case III. The integer has three digits a, b, c. Denote the number as $100c + 10b + a$. Thus $a^3 + b^3 + c^3 = 100c + 10b + a$ or $a(a^2 - 1) + b(b^2 - 10) + c(c^2 - 100) = 0$. We construct a table for exploring the last sum with various digits.

x	$x(x^2 - 1)$	$x(x^2 - 10)$	$x(x^2 - 100)$
0	0	0	0
1	0	-9	-99
2	6	-12	-192
3	24	-3	-273
4	60	24	-336
5	120	75	-375
6	210	156	-384
7	336	273	-357
8	504	432	-288
9	720	639	-171

From the above table we want to add one element from each column to have a total of zero. We discover the four desired integers are:

$$
\begin{aligned}
100c + 10b + a &= 153 = 1^3 + 5^3 + 3^3 \\
&= 370 = 3^3 + 7^3 + 0^3 \\
&= 371 = 3^3 + 7^3 + 1^3 \\
&= 407 = 4^3 + 0^3 + 7^3
\end{aligned}
$$

Case IV. The integer has four digits a, b, c, d. Denote the number by $1000d + 100c + 10b + a$. Thus $a^3 + b^3 + c^3 + d^3 = 1000d + 100c + 10b + c$ or $a(a^2 - 1) + b(b^2 - 10) + c(c^2 - 100) = d(1000 - d^2)$. We know the integer must be less than or equal to $9^3 + 9^3 + 9^3 + 9^3 = 4 \cdot 9^3 = 2916$. Therefore the integer must be less than or equal to $2^3 + 9^3 + 9^3 + 9^3 = 2195$. If the integer is over 2000 then it is less than $2^3 + 1^3 + 9^3 + 9^3 = 1467$. Therefore it must be less than 2000. That is $d = 1$ or $a(a^2 - 1) + b(b^2 - 10) + c(c^2 - 100) = 999$. From the table above we can see that no a, b, and c satisfy this condition. Thus there are no desired integers of 4 digits.

If the integer has five digits it must be less than $9^3 + 9^3 + 9^3 + 9^3 + 9^3 = 5 \cdot 9^3 = 3645$, which has only 4 digits.

Therefore the only desired integers are: 153, 370, 371, and 407.

165. *Proposed by Fred W. Lott, Jr., State College of Iowa; Cedar Fall, Iowa.*

Without using tables, determine which is larger, e^π or π^e.

Solution by Phil Huneke, Pomona College; Claremont California.

Let **R** be a relationship, either "<", ">", or "=". We want to find **R** where e^π **R** π^e. This is equivalent to

$$\log_e e^\pi \ \mathbf{R} \ \log_e \pi^e$$

$$\pi \log_e e \ \mathbf{R} \ e \log_e \pi$$

$$\frac{\log_e e}{e} \ \mathbf{R} \ \frac{\log_e \pi}{\pi}$$

Consider the equation $y = \dfrac{\log_e x}{x}$. Then:

1) $\dfrac{dy}{dx} = \dfrac{1 - \log_e x}{x^2}$

2) $\dfrac{dy}{dx}$ is continuous for $x > 0$

3) $y = 0$ when $x = 1$

4) $y > 0$ when $x > 1$

5) $\displaystyle\lim_{x \to \infty} \frac{\log_e x}{x} = 0$

We can find a relative maximum of $y = \dfrac{\log_e x}{x}$ by setting

$\frac{dy}{dx} = 0$ and obtaining $x = e$. Therefore $\frac{\log_e x}{x}$ is maximum for $x = e$.

Thus: $\frac{\log_e e}{e} > \frac{\log_e \pi}{\pi}$ and "R" is ">" and $e^\pi > \pi^e$.

Also solved by Joseph Dence, Bowling Green State University; Bowling Green, Ohio.

Installation of New Chapters

EDITED BY SISTER HELEN SULLIVAN

The Maryland Alpha Chapter of Kappa Mu Epsilon was installed on May 22, 1963, at the College of Notre Dame of Maryland in Baltimore. Professor Loyal F. Ollman of Hofstra College, National President of Kappa Mu Epsilon was the installing officer. The charter members include the following fourteen students. Leslie Blyth, Dorothy Ann Carlin, Patricia Chaney, Carolyn Crosby, Sue Goodman, Mary Elaine Hershfeld, Barbara Hollin, Ottilie Hurd, Lynn Keene, Sue Leisher, Anita MacMullin, Beth Reese, Elizabeth MacMullin, Joyce Roesh. Four faculty members, Sister Marie Augustine SSND, Sister Mary Cordia SSND, Sister Mary Paula SSND, and Mr. John W. Marvin were also initiated.

New officers for Maryland Alpha are:

President	Joyce Roesh
Vice President-Treasurer	Mary Elaine Hershfeld
Secretary	Terry Bracken
Corr. Secretary	Sister Marie Augustine
Faculty Sponsor	Sister Mary Cordia

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of *The Pentagon*. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Harold E. Tinnappel, Bowling Green State University, Bowling Green, Ohio.

Linear Algebra and Geometry, Nicolaas H. Kuiper, North-Holland Publishing Company, Amsterdam. Translated by A. van der Sluis. John Wiley & Sons, Inc., Interscience Division (440 Park Avenue South), New York 16, 1962, 284 pp., $8.25.

This book is far from ordinary both in style and content. The approach to vector spaces, and their use in applications, is quite different from that found in most "Linear Algebra" texts. The author makes heavy use of modern concepts from abstract algebra, topology, and geometry, many of which are rarely found in such texts. It is the opinion of the reviewer that the introduction of new terms and concepts has been carried to excess and that insufficient explanation is sometimes given to bring out the significance of new ideas. The student must be very able and mature, or he must have expert guidance.

The selection, organization, and development of material is excellent and the writing is clear and concise. After a brief introduction to the terminology of sets, mappings, and functions, the author introduces "vector space" as an algebraic system with properties defined by a set of axioms. The theorems developed are those one normally sees, but the methods of proof and development are unusual. In addition, application of tools developed in vector spaces is made to the proofs of some well-known theorems from geometry by treating affine spaces as mappings of vector spaces.

The function concept is emphasized throughout. Homomorphisms are introduced early and matrices are introduced as homomorphisms on a vector space. Quadratic and hermitian forms on a vector space are considered, and corresponding quadratic curves and surfaces are illustrated as level sets of these functions. Euclidean spaces are then introduced as vector spaces with a positive definite quadratic function. Square matrices, as endomorphisms, are classified and the Jordan canonical form is developed.

In addition to these topics usually found in a text, chapters are inserted giving applications to statistics, projective geometry, and "motion" considered as a one-to-one mapping leaving distance invariant.

For use as a text on "Linear Algebra", this book is probably too ambitious for the average first course in abstract algebra. New concepts are very well defined but, while beautiful figures are given of geometric illustrations in some places, few examples are given in some places to illustrate use and meaning. The book will make a fine reference book for a bright student who wants introduction to a multitude of modern concepts and methods and who has the imagination and originality to develop examples from axioms and definitions.

—C. J. PIPES
Southern Methodist University

Via Vector to Tensor, W. G. Bickley and R. E. Gibson, John Wiley & Sons, Inc., (440 Park Avenue South) New York 16, 1962, 152 pp., $4.50.

This little book is an introduction to the methods of vector and tensor analysis. The text is 133 pages long, and it is divided into nine chapters. The first four chapters constitute Part I, an introduction to vectors, while the next five chapters constitute Part II, an introduction to tensors. The book is one of a series of textbooks in applied mathematics intended for the undergraduate and graduate level. This book will provide excellent background material for the reading of more advanced treatises in applied mathematics; for example, the recent book *Matrix-Tensor Methods In Continuum Mechanics* by S. F. Borg.

The book was written primarily for non-mathematical specialists who are interested in the applications of mathematics. Moreover, as the authors emphasize, their primary aim is understanding on the part of the reader. In view of these facts, the authors have chosen to present their material in a heuristic fashion. Thus, the reader does not find axioms, definitions, theorems and proofs as in books on pure mathematics. In the opinion of the reviewer, the authors have succeeded in producing a book which may be useful to the aforementioned class of students. On the other hand, he does not believe that the book will prove useful for the student of pure mathematics, as the authors had hoped. Indeed, a pure mathematician never really "understands" a subject until he has seen it developed beginning with a suitable set of axioms and undefined concepts. Moveover, it is the opinion of the reviewer that vector analysis should be approached via differential forms. Then, and only then, does one realize that the general Stoke's formula is an extension of the fundamental theorem of calculus to functions of several variables! It may be argued, however, that these criticisms are not justified in view of the audience for whom the book is intended.

—Richard E. Dowds
Butler University

Finite-Difference Methods for Partial Differential Equations, George E. Forsythe and Wolfgang R. Wasow, John Wiley and Sons, (440 Park Avenue South), New York 16, N. Y., 1960, 444 pp., $11.50.

This book, as the authors indicate, is being directed to (i) pure and applied mathematical analysts, (ii) those interested in using machines to solve partial differential equations, (iii) programers, and (iv) interested graduate students. A basic knowledge of advanced calculus and matrix theory is assumed.

The authors present a rather connected account of many of the important ideas, results, and methods now available in the solving of partial differential equations using finite difference approximations.

Various special methods for solving some non-linear, as well as most linear, hyperbolic, parabolic, and elliptic equations, and systems thereof are presented. More than half of the book is devoted to elliptic equations. Some of the methods presented are illustrated with certain physical problems. Special care has been given in the formulation of the difference equations problems, especially, in regard to the elliptic difference equations where much of the theory is developed using irregular nets.

In several places only methods, ideas and names are "dropped"; however, for those who wish to pursue the subject further the authors have included a seventeen page bibliography.

—S. Elwood Bohn
Bowling Green State University

Mathematical Methods for Digital Computers, Edited by Anthony Ralston and Herbert S. Wolf, John Wiley and Sons, Inc. (601 West 26th Street), New York 1, 1960, 293 pp., $9.00.

This book consists of twenty-six chapters, each of which is an independent unit. Each of the editors has contributed two chapters and there are twenty-two other contributors. "Each chapter has been written by someone in close contact with the latest development in his respective area." (preface).

The patch-work effect that so often results from multiple authorship has been greatly lessened by the adoption of a standard format for the chapters. With one or two exceptions, the chapters have the following organizations:

1. Function. A formulation of the problem.
2. Mathematical Discussion.
3. Summary of the Calculation Procedure.
4. Flow Chart.
5. Description of the Flow Chart.
6. Subtractions.
7. Sample Problem.
8. Memory Requirements.
9. Estimation of the Running Time.
10. List of References.

The only marked deviation from this format occurs in the first chapter, which is devoted to the generation of elementary functions. The coding of the problem is very properly not included, since the coding varies with the specific machine to be used. This uniform format greatly increases the usefulness of the book as a reference book. The presentations have achieved a commendable compromise between the requirements of brevity and clarity and the book is, in general, quite readable for those who have a moderate acquaintance with the field.

The Chapters are as follows:

1. Generation of Elementary Functions
2. Matrix Inversion and Related Topics by Direct Methods
3. The Solution of Linear Equations by the Gauss-Seidel Method
4. The Solution of Linear Equations by the Conjugate Gradient Method
5. Matrix Inversion by the Method of Rank Annihilation
6. Matrix Inversion by Monte Carlo Methods
7. The Determination of the Characteristic Roots of a Matrix by the Jacobi Method
8. Numerical Integration Methods for the Solution of Ordinary Differential Equations
9. Runge-Kutta Methods for the Solution of Ordinary Differential Equations
10. The Numerical Solution of Boundary Value Problems

11. The Solution of Ordinary Differential Equations with Large Time Constants
12. The Numerical Solution of Parabolic Partial Differential Equations
13. Iterative Methods for the Solution of Elliptic Partial Differential Equations
14. A Monte Carlo Method for the Solution of Elliptic Partial Differential Equations
15. The Numerical Solution of Hyperbolic Partial Differential Equations by the Method of Characteristics
16. The Solution of Hyperbolic Partial Differential Equations by Difference Methods
17. Multiple Regression Analysis
18. Factor Analysis
19. Autocorrelation and Spectral Analysis
20. Analysis of Variance
21. The Numerical Solution of Polynomial Equations
22. Methods for Numerical Quadrature
23. Multiple Quadrature by Monte Carlo Methods
24. Fourier Analysis
25. The Solution of Linear Programming Problems
26. Network Analysis

The editors of this book state that it is designed as a reference text for those workers in the field of numerical analysis who have acquired an understanding of the mathematics involved in the solution of problems on a digital computer.

In a field as large and expanding as rapidly as the computer field, no one book can hope to cover the entire field and any book is at least somewhat out-dated before it comes off the press. Within these inherent limitations, this book does a competent job. It should be useful to anyone working with digital computers.

—Dean L. Robb
Baldwin-Wallace College

A Programming Language, by Kenneth E. Iverson, John Wiley & Sons, Inc., (440 Park Avenue South), New York 16, 1962, 286 pp., $8.95.

According to the preface, it is the central thesis of this book that the descriptive and analytic power of an adequate programming language amply repays the considerable effort required for its mastery. This thesis is developed by first presenting the entire language and then applying it in later chapters to several areas of application. The chapters which deal with applications are independent of one another and are designed to illustrate the universality of the language.

The first chapter is devoted to the development of prerequisites for the remaining six chapters which deal with applications. The chapter headings are as follows: 1. The Language; 2. Microprogramming; 3. Representation of Variables; 4. Search Techniques; 5. Metaprograms; 6. Sorting; 7. The Logical Calculus.

The author states that the material was developed in a graduate course given for several years at Harvard and in a later course presented at the IBM Systems Research Institute in New York. The book should be suitable for a two-semester course at the senior or graduate level.

A valuable summary of notation is given at the end of the book. At the end of each chapter is a list of references which supply ample encouragement for the reader to delve further into the subject or the background material. The text contains many exercises of varying difficulty which provide practice in handling the methods and techniques treated in the text. Some examples and many illustrative programs are given in the text. The style, printing, and general layout of the book are excellent.

—Jens A. Jensen
State College of Iowa

An Introduction to Linear Programming and the Theory of Games, by Abraham M. Glicksman, John Wiley and Sons, Inc. (440 Park Avenue South) New York 16, 1963, 131 pp., Paper - $2.25, Cloth - $4.95.

This volume is intended to be suitable for "high school 'honors' programs, for college students, for teachers, and for interested laymen." The reviewer feels that both the content and the exposition lend themselves to the achievement of this aim.

The topics presented include the Fundamental Extreme Point Theorem for convex sets in the plane, the Simplex Method for solving the linear programming problem, 2 x 2 matrix games, and the Duality Theorem. The explanations are frequently made with plausible reasoning and examples quite suitable for the intended readers. KME chapters might find some good program material in this book.

—J. Frederick Leetch
Bowling Green State University

Essays on Probability and Statistics, M. S. Bartlett, John Wiley & Sons, Inc., (440 Park Avenue South), New York 16, 1962, 124 pp., $4.50.

The eight papers constituting this small book were delivered by the distinguished British professor of statistics over the period 1949 to 1956. They represent collectively a consistent and comprehensive view of the unity and coherence of statistical theory wherever applied.

Bartlett is a mathematical statistician and he submits that the theory of statistics is of complementary importance to the statistical data since our arithmetic is useless unless we are counting the right things.

Five of the papers have been previously published but they had not been reviewed or abstracted and the author collected these eight lectures into one volume in order to make available his general philosophy. He considers probability theory the essential mathematical basis of statistical theory and would not distinguish between them except for the tendency to regard the theory of probability as a branch of pure mathematics and the theory of statistics as the application of this mathematical theory of statistical phenomena.

—David M. Krabill
Bowling Green State University

Graphics with an Introduction to Conceptual Design, ed. by A. S. Levens, John Wiley & Sons, Inc., (440 Fourth Ave.), New York 16, 1962, 743 pp., $9.50.

The principal objective of this book is to provide for the student a modern treatment of graphics and an introduction to conceptual design that will help him become "graphically literate", so that with confidence he can employ graphics—a powerful mode of expression—to the synthesis, analysis, and solution of problems that arise in the fields of design, development, and research.

It is assumed that the student has already developed a reasonable degree of proficiency with respect to lettering, use of drawing instruments, geometric construction, etc., in high school or college non-credit prerequisite courses. Appendix material included might provide ample subject matter for this elementary work.

The chapter content and sequence has been designed to provide the student with a clear and understandable development of the subject matter and also enable him to appreciate the significant role of graphics in engineering, research, development, and design. It emphasizes the importance of free-hand sketching as a powerful means for expressing new ideas and design concepts, for recording analyses of space problems, and for effective communication among engineers, scientists, and technicians. Stress is placed on the fundamental principles of orthogonal projection and their application to the analyses and solutions of space problems that arise in both engineering and science.

The power of graphical analysis and graphic methods of computation is set forth in the material on applications of the fundamental principles of orthogonal projection, vector quantities, and graphical mathematics. While the importance of algebraic methods in solving engineering problems is recognized, it is shown that in numerous instances these problems are best solved by use of graphical methods.

The last portion of the book deals with Conceptual Design which is the creative art of conceiving a physical means of achieving an objective. (This is the first and most crucial step in an engineering project.) This part affords the student an opportunity "to be on his own" and background based on past course work in mathematics, chemistry, physics, and actual job experience will have a direct bearing on progress with design problems and problems with many solutions. Many opportunities are provided to help develop creative thinking and "imagineering". The chapters on pictorial drawing, sections and conventional practice, fasteners, dimension specifications, and dimensioning for precision and reliability constitute additional background for the important final chapters Conceptual Design and Developing Creativity.

The book is well written, concise, and the examples are well chosen from the areas of application. The material is an important part of the education of engineers and scientists and is not intended for training of draftsmen. It should be readily available to all engineering students and is adequate for consideration as a text on the college level.

—Robert W. Innis
Bowling Green State University

The Mathematical Scrapbook

EDITED BY J. M. SACHS

For twenty-five centuries mathematicians have been in the habit of correcting their errors—and seeing their science enriched rather than impoverished thereby. This gives them the right to contemplate the future with serenity.

—N. BOURBAKI

$=\Delta=$

$$
\begin{aligned}
1 &= 1\cdot 1^2 + 1\cdot 0^2 + 2\cdot 0^2\\
3 &= 1\cdot 1^2 + 1\cdot 0^2 + 2\cdot 1^2\\
5 &= 1\cdot 2^2 + 1\cdot 1^2 + 2\cdot 0^2\\
7 &= 1\cdot 2^2 + 1\cdot 1^2 + 2\cdot 1^2\\
&\vdots\\
61 &= 1\cdot 7^2 + 1\cdot 2^2 + 2\cdot 2^2
\end{aligned}
$$

Is every odd positive integer expressible as $a^2 + b^2 + 2c^2$ where a, b, and c are integers? In 1748 Leonard Euler made a conjecture that the double of any odd integer is a sum of three squares and then proved that the relationship above folows. Can you arrive at his conclusion with the conjecture as an assumption? Do you see any patterns for finding the integers to square for given odd positive integers?

$=\Delta=$

A regular hexagon is constructed by using segments of the three sides of an equilaterial triangle and the three lines joining the nearest endpoints of such segments. How does the length of the side of the hexagon compare with the length of the side of the triangle? Can you make a similar investigation for a square and an octagon? What about a pentagon and a decagon? Is there any pattern in this type of relationship? Are these investigations aided in any way by the use of a coordinate system?

$=\Delta=$

The preceding problem was suggested to a student who promptly came up with what looks like a harder one. (This is an interesting device—I am not sure the student was able to solve my problem but he was extremely successful in sidetracking me.) Remove the conditions of regularity. Can anything be said about the maximum perimeter hexagon for a given triangle? What about minimum perimeter? Is a hexagon with a side of length zero still a hexagon? Is the problem more complicated for the square and octagon?

=Δ=

The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given; even the most rigorous expositions of the differential calculus do not base their proofs upon continuity but, with more or less consciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner • • •. It only then remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity.

—R. Dedekind

=Δ=

Consider the sequence 1, 1, 1, 2, 3, 5, 11, 26, • • •. Each term, beginning with the fourth, is the sum of the preceding term and the product of the two terms just preceding the preceding term, that is

$$a_n = a_{n-1} + a_{n-2}a_{n-3}.$$

Can you find the sum of the first n terms? Consider $n \geqq 3$.

Can you do the same for the sequence

$$a, a, a, a^2 + a, 2a^2 + a, a^3 + 3a^2 + a, \cdots$$

=Δ=

Solving problems is a practical art, like swimming or skiing, or playing the piano; you learn it only by imitation and practice.

—G. Polya

=Δ=

The great masters of modern analysis are Lagrange, Laplace, and Gauss who were contemporaries. It is interesting to note the marked contrast in their styles. Lagrange is perfect both in form and matter, he is careful to explain his procedure, and though his arguments are general they are easy to follow. Laplace on the other hand explains nothing, is indifferent to style, and, if satisfied that his results are correct, is content to leave them either with no proof or with a faulty one. Gauss is as exact and elegant as Lagrange but even more difficult to follow than Laplace, for he removes every trace of the analysis by which he reached his results, and studies to give a proof which while rigorous shall be as concise and synthetical as possible.

—W. Ball (written in 1901.)

=Δ=

To find integral solutions for the equation $ax + c = by$, form the two sequences

$$c + a, c + 2a, c + 3a, \cdots \quad \text{and} \quad b, 2b, 3b, 4b, \cdots$$

If a common term is found in these two sequences and it is the nth term of the first and the mth term of the second, then $x = n$ and $y = m$ is a solution of $ax + c = by$.

As an illustration, consider $13x + 8 = 19y$. In this example $a = 13, b = 19, c = 8$. The two sequences are

$$21, 34, 47, 60, 73, 86, 99, 112, 125, 138, 151, 164, 177, 190, \cdots$$
$$19, 38, 57, 76, 95, 114, 133, 152, 171, 190, \cdots$$

The common term is the 14th in the first sequence and the 10th in the second sequence. Thus $x = 14$, $y = 10$ is a solution.

This method of construction of a solution is due to John Kersey and was published in his Elements of Algebra in 1673. Can you justify it? Does it work for all such equations? Will it work with negative coefficients?

=Δ=

Any theory of mathematics must account both for the power of mathematics, its numerous applications to natural science, and the beauty of mathematics, the fascination it has for the mind.

—W. W. Sawyer

Kappa Mu Epsilon News

The Fourteenth Biennial Convention of Kappa Mu Epsilon was held April 8, 9, 1963 with Illinois Alpha at Illinois State Normal University, Normal, Illinois, as host chapter. Forty-five chapters were represented. The total registration was 307.

MONDAY, APRIL 8, 1963

The meetings were held in the University Union. Professor Carl V. Fronabarger, Missouri Alpha, National President of Kappa Mu Epsilon, presided. President Robert G. Bone of Illinois State Normal University welcomed the delegates to the campus. Professor Harold E. Tinnappel, National Vice-President responded for the Society. The following chapters, installed since the last national convention were welcomed:

Missouri Zeta, Missouri School of Mines and Metallurgy, Rolla
New York Epsilon, Ladycliffe College, Highland Falls
Nebraska Gamma, Nebraska State Teachers College, Chadron

A petition for a chapter of Kappa Mu Epsilon at North Park College was presented and approved.

Professor Harold E. Tinnappel presided during the presentation of the following papers:

1. *Pythagorean Arithmetic,* Jacquelyn Chamberlain, Michigan Beta, Central Michigan University.
2. *What Emphasis is New in Elementary Collegiate Mathematics,* Mary Martha Lutte, Missouri Beta, Central Missouri State College.
3. *Biangular Coordinates,* Myron Effing, Indiana Delta, Evansville College.
4. *Projective Geometry in Relation to Art,* Dorothy Gilman, New York Epsilon, Ladycliffe College.
5. *The Use of Probability in the Disintegration of Radioactive Elements,* Mary Ann E. Lapinske, New York Epsilon, Ladycliffe College.

After lunch in Hamilton-Whitten Hall the faculty members and students met separately in two sections, "Let's Exchange Ideas". The entire group reconvened at 2:30 p.m. and after reports from the two sections the following papers were read:

6. *Non-Euclidean Geometry, its Background and Development by Lobachevsky,* Joan M. Richardson, New York Epsilon, Ladycliffe College.
7. *A Study of Finite Geometries,* Martha F. Heidlage, Kansas Gamma, Mount St. Scholastica College.
8. *Rocketry, Single State, Solid Fuel,* Frederick J. Blume, Nebraska Beta, Nebraska State Teachers College at Kearney.
9. *Group Modulon,* Joe Wayne Fisher, Nebraska Beta, Nebraska State Teachers College at Kearney.

The convention banquet was served in the Ballroom of the University Union with Mr. Patrick J. Bibby, Illinois Alpha, as Master of Ceremonies. Mr. James R. Downing gave the invocation. Dr. Franz Hohn, University of Illinois was the guest speaker His subject was "Infinity."

TUESDAY, APRIL 9, 1963

The program started at 8:30 a.m. with the following student papers:

10. *A Determinant Expression for the Volume of a Tetrahedron,* Karen Smith, New York Beta, State University of New York at Albany.
11. *Linear Expression of Greatest Common Divisor,* Ann M. Penton, New York Beta, State University of New York at Albany.
12. *Monte Carlo Solutions to Waiting Line Problems,* Norman Nielsen, California Alpha, Pomona College.
13. *The Theorem of Helly,* Ronald Donnell, Missouri Alpha, Southwest Missouri State College.

The following papers were listed by title:

1. *Selected Proofs of the Pythagorean Theorem,* Thomas P. Kromer, Michigan Beta, Central Michigan University.
2. *Cantor's Contributions to Irrationals,* Lura Cusick, Missouri Beta, Central Missouri State College.
3. *Applications of Group Theory,* Laura Connolly, New York Epsilon, Ladycliffe College.
4. *The Lore of Numbers,* Jeanne O'Rourke, New York Epsilon, Ladycliffe College.

5. *Gentleman, Soldier, Mathematician, Rene Descartes,* Frances Vassalo, New York Epsilon, Ladycliffe College.
6. *Appplications of Set Theory to Trigonometry,* Agnes Walsh, New York Epsilon, Ladycliffe College.
7. *The Role of Ring Structure in Mathematics,* Jeanne Beyer, Kansas Gamma, Mount St. Scholastica College.
8. *Linear Programming,* Dolores Stieferman, Kansas Gamma, Mount St. Scholastica College.
9. *Mathematical Magic,* Charles Sayre, Kansas Gamma, Mount St. Scholastica College.

At the second general business session, reports of the national officers were read as well as the report of the auditing committee.

Colorado Alpha, Indiana Alpha, Indiana Delta, Kanssa Beta, Kansas Gamma, Kansas Delta, Kansas Epsilon, Louisiana Beta, Missouri Beta, Missouri Zeta, New York Gamma, New York Epsilon, and Pennsylvania Beta extended invitations for the 1965 Convention. The site will be selected by the National Council.

Dr. C. C. Richtmeyer reported for the nominating committee. There were no nominations from the floor and the following list of national officers were elected for 1963-1965.

President	Dr. Loyal F. Ollmann Hofstra College
Vice President	Dr. Harold Tinnappel Bowling Green State University
Secretary	Professor Laura Greene Washburn University of Topeka
Treasurer	Professor Walter C. Butler Colorado State University
Historian	Dr. J. D. Haggard Kansas State College of Pittsburg

Dr. R. G. Smith, Kansas Alpha, chairman of the awards committee made the following awards to the students named for papers presented during the convention.

First Place	Myron Effing, Indiana Delta
Second Place	Norman Nielsen, California Alpha
Third Place	Martha F. Heidlage, Kansas Gamma

Sister Helen Sullivan, Kansas Gamma, reported for the resolutions committee. The following resolutions were adopted.

Whereas this Fourteenth Biennial Convention of Kappa Mu Epsilon has been a very successful and profitable conference, be it resolved that we express our appreciation:

1. To the host chapter, Illinois Alpha, and to Illinois State Normal University for their fine hospitality, the use of their comfortable facilities, the efficient arrangement of all registration details, and for all the other factors (too numerous to mention) that contribute so definitely to the success of meetings such as this.
2. To each of the national officers whose unceasing efforts and continual inspiration and encouragement are responsible for the advancement of our society both in prestige and in membership. To Professor Douglas Bey and his mathematics staff, to Mr. Bruce Kaiser of the Student Union, and to Miss Elizabeth Terrill in charge of housing arrangements for the very splendid planning that has been responsible for our pleasant stay. To all those both on the spot and behind the scenes who have in any way contributed to the smooth functioning of this conference. To Professor Harold E. Tinnappel, vice-president, for his work in organizing the program of student papers.
3. To the editor and staff of *The Pentagon* for the continued publication of the very fine magazine which reflects the best efforts of the society members.
4. To the thirteen students who prepared and presented excellent papers at these sessions, to the nine students whose papers were listed by title, as well as to all the other students who contributed by their presence and their scholarly attitude to the convention.
5. To all here present for the warm spirit of fellowship and courtesy that makes these meetings so memorable, and again to Illinois Alpha for making this return visit to their expanded campus so very pleasurable.

REPORT OF THE NATIONAL PRESIDENT

The explosion in college enrollments and the tremendous increase in interest in the fields of science and mathematics have been reflected in the interest in and growth of Kappa Mu Epsilon.

During the last biennium, new chapters have been installed at: Missouri School of Mines and Metallurgy, Rolla; Ladycliffe College, Highland Falls, N.Y.; and at the Nebraska State Teachers College, Chadron, Nebraska. A chapter has been approved for Notre Dame College of Maryland, Baltimore; arrangements are now being made for the installation of this chapter. By your action at this convention, a chapter has been approved for North Park College in Chicago. It is my hope that arrangements for the installation of this chapter can soon be finalized.

Approximately 40 colleges and universities, during the past biennium, have made inquiries with respect to the establishment of chapters of Kappa Mu Epsilon. In response to each inquiry, the following items were sent: a copy of the National Constitution; a recent issue of *The Pentagon*; a letter indicating that the National Council's interest in the possibility of their becoming affiliated with Kappa Mu Epsilon, pointing out the fact that the National Council will be interested in evidence that the petitioning club is well-established as functioning organization with student participation in the activities of the club by presenting papers at regularly scheduled meetings; and a copy of the petition form which must be completed by each petitioning chapter. Some of the chapters have responded by saying that they do not feel that they are yet in a position to file the petition for a chapter but that knowledge of the expectations with respect to a petitioning chapter gives them a goal to work toward. I anticipate that during the next biennium some of these clubs will petition to become active chapters.

Evidence that such care in the selection of chapters of Kappa Mu Epsilon has paid off is indicated by the fact that very few chapters of Kappa Mu Epsilon have ever become inactive and by the fact that such a large proportion of the chapters are represented at this Biennial Convention; 45 out of 64 active chapters are represented.

As the number of active chapters increases, we should give increased attention to regional meetings in the spring of even-numbered years. In the spring of 1962 two such regional meetings were held; The Kansas-Missouri-Nebraska-Oklahoma Regional Conference was held at Mount St. Scholastica College, Atchison, Kansas with 10 chapters represented and 7 student papers presented; The Illinois-Indiana-Michigan-Ohio-Wisconsin Regional Conference was held at St. Mary's Lake, Michigan with 15 chapters represented and 8 student papers presented. I recommend that other regional conferences be set up in the spring of 1964. The National Council has approved the allotment of $100 to the director of each such regional conference to help defray the expenses.

I wish to express my appreciation to all of those persons who have contributed time and energy to Kappa Mu Epsilon during the past biennium; to Past President Richtmeyer for his advice and assistance with respect to the duties of the office of President; to the Vice President Tinnappel, who has had the primary responsibility with respect to the solicitation and selection of papers to be presented at this Fourteenth Biennial Convention; to the National Secretary Greene, for the long hours and tedious work that has been her lot as a result of the office she holds; to the National Treasurer Butler for the efficient and capable manner in which he has carried out the duties of his office; to the National Historian Gentry, for ably performing his duties; to all those sponsors, corresponding secretaries, chapter officers, and all others who have been responsible for the effective functioning of the local chapters; to those who have organized and conducted regional meetings; to those students who have prepared papers for regional and/or this Biennial Convention; to Mr. Bey and his associates who have made the local arrangements for this Convention; and to those who have served on the Convention committees.

To Mr. Lott and Mr. Waggoner, Editor and Business Manager respectively of *The Pentagon,* I wish to express sincere appreciation both personally and as President of Kappa Mu Epsilon. *The Pentagon* is a potent factor in welding our organization together and in promoting the objectives of Kappa Mu Epsilon.

It is my hope that expanding enrollments and changing curricula, with their demands upon the time and energy of the faculties and of the students will not be allowed to interfere with the objectives of Kappa Mu Epsilon. I hope that the organization will continue to be devoted to the development of an interest in and an appreciation of mathematics in addition to its role as an honor society. Encouragement of students to engage in undergraduate research and opportunities to present well-written papers has been part of our tradition and must be maintained. As we look forward to the next biennium, I trust that your enthusiasm, your efforts, and your loyalty to Kappa Mu Epsilon will enable the organization to continue to achieve its purposes and ideals.

—CARL V. FRONABARGER

REPORT OF THE NATIONAL VICE-PRESIDENT

The primary function of the vice-president is to make arrangements for the program of student papers. A preliminary announcement describing the procedure for submitting papers for the Fourteenth Biennial Convention appearing in the Spring, 1962 issue of *The Pentagon* was followed by a second invitation appearing in the Fall, 1962 issue.

The response from the chapters was quite good. Twenty two papers were submitted by students from ten different chapters. From these twenty two papers the Selection Committee chose thirteen to be presented at this convention. The Committee regrets that it could not schedule a greater number of the fine papers submitted. The additional nine papers are listed by title on your program.

It is felt that the benefits to the student who submits a paper extends beyond the valuable experience he gets in searching for a topic, collecting notes and preparing a paper. In addition to his presentation before the audience attending the convention, he probably has previously presented his paper before the local chapter, and thereby incidentally enriched the local program.

—HAROLD TINNAPPEL

REPORT OF THE NATIONAL SECRETARY

Since April of 1961, three chapters of Kappa Mu Epsilon have been installed, making a total of sixty-four active chapters. Representatives from forty-five chapters are attending this convention.

Kappa Mu Epsilon is thirty-two years old this month. The first chapter was formed at Northeastern State College at Tahlequah, Oklahoma. Five of the sixty-nine chapters that have been installed are now inactive. The total membership in the sixty-nine chapters is more than 18,000. Approximately 2,500 new members have been added during the last biennium.

—LAURA Z. GREENE

REPORT OF THE NATIONAL HISTORIAN

In the past biennium the files have been brought up to date by preparing a new folder for each chapter of the Society and acquiring a new transfer file box to hold them. Some of the folders contain a great deal of material and some contain little but a copy of the original petition of the chapter. If corresponding secretaries have installation programs or other materials relative to the history of their chapters, they would be welcomed. I have reinstated the custom of mailing a form asking for specific information to each corresponding secretary once a year. I believe this custom was inaugurated by Sister Helen Sullivan and continued by Miss Greene but had not been used for several years. As a result we received responses from nearly 50 of the chapters this spring. It is possible that more information should be requested on these forms, such as number and names of initiates, number of meetings and list of topics presented. On the other hand if such information is called for, the number of responses will be reduced.

There is one job which I undertook but did not complete. I leave it to my successor. There is little material in the file with regard to some of the National Conventions. Even the minutes for the Twelfth are missing. If there are programs or other materials concerning these meetings to be found, they should be in the files. Also, I believe that a corrected and dated list of the corresponding secretaries of the chapters should be prepared each year and a copy placed in the files. If copies of such lists from by-gone years are available, they should also be filed.

I am sorry that I have not had more time to devote to this office. I have great admiration for those National Historians who, in the early years of the Society, so meticulously prepared scrapbooks of each chapter's activities. It has been a pleasure to work with Professor Lott on *The Pentagon*. He has been most considerate of my delays.

—Frank C. Gentry

REPORT OF THE EDITOR OF THE PENTAGON

The publication of our fraternity's journal *The Pentagon*, is made possible by the voluntary contributions of many people. At this time I would like to express my appreciation and that of the entire organization to all those whose time-consuming and effective work have contributed to producing the magazine.

For the past biennium, our National Historian, Frank C. Gentry of the University of New Mexico, has edited the KME News section. Jerome Sachs, Chicago Teachers College, is the editor of the Mathematical Scrapbook. The Book Shelf is edited by Harold Tinnappel, Bowling Green State University, our National Vice-President. The

Problem Corner editor is J. D. Haggard, Kansas State College of Pittsburg, and Sister Halen Sullivan, Mount St. Scholastica College, edits the reports of the Installation of Chapters. Substantial contributions to producing *The Pentagon* are made by the Business Manager, Wilbur Waggoner of Central Michigan University, and the National Secretary, Laura Greene of Washburn University. I would like to commend Irwin M. Campbell, manager of the Central Michigan University Press where *The Pentagon* is printed, and his staff for their excellent work. Finally, we are indebted to all those persons who have written the articles published, contributed to the Problem Corner, or written reviews for The Book Shelf.

Three issues of *The Pentagon* have been published for this biennium. The manuscripts for the fourth issue have been sent to the printer and you should receive it sometime in May. In addition to chapter news, book reviews, the Problem Corner, and the Mathematical Scrapbook, there have been twenty articles published in these four issues. Sixteen of these papers were written by student authors and four by faculty members.

The Pentagon is a mathematics magazine for students and it is appropriate and in keeping with the ideals of Kappa Mu Epsilon to encourage the publication of student papers. While it takes a great amount of effort and attention to minute detail to prepare an article for publication, the rewards of seeing your work in print are great. I hope that faculty members at our various chapters will encourage and help students in the preparation of such manuscripts. In addition, articles by faculty members and others are welcome, particularly in those areas of mathematical interest where students, due to lack of background and experience, are not likely to make contributions. Any article that is of interest to undergraduate students in mathematics is solicited.

The journal of our fraternity should serve as a means of communication between the chapters, a magazine where interesting articles and thought-provoking problems may be found, and a place where young men and women entering the field of mathematics may express their mathematical ideas in print. I welcome your comments, suggestions, and criticisms to carry out these objectives. Above all, I appeal to you to contribute to *The Pentagon* either with articles for publication or with problems and solution to the Problem Corner.

—Fred W. Lott, Jr.

REPORT OF THE BUSINESS MANAGER OF THE PENTAGON

This report concerning the duties and activities of the Business Manager for *The Pentagon* to the fourteenth Biennial Convention of Kappa Mu Epsilon is the third such report I have given. During these past six years, I have seen the circulation of the official journal of Kappa Mu Epsilon increase greatly. Two thousand nine hundred fifty copies of the Fall 1962 issue of *The Pentagon* were printed. This is an increase of nearly fifty per cent over the number of journals printed for Spring, 1957. All of the Fall 1962 *Pentagons* have been mailed except for a small reserve which is used to fill requests for back issues. The office of the Business Manager of *The Pentagon* is able to fill orders for past issues except for Spring 1942, Fall 1943, Fall 1946, Spring 1947, and Fall 1947.

The distribution of the Fall 1962 printing was of interest to me. Copies of this *Pentagon* were mailed to every state in the union except Vermont and Alaska. The Fall 1962 *Pentagon* went to eleven foreign countries in Asia, Europe, Africa, North America, South America, and Australia. The only continent which did not receive our national journal was Antarctica.

For the first time in making this Business Manager's Report, I am not able to announce that more *Pentagons* were mailed to Kansas addresses than to any other state. The most frequent address for Fall 1962 *Pentagons* was New York. New York was followed by Missouri, Illinois, Kansas, California, Texas, and Indiana in frequency of journals mailed to addresses in that state. Each of these states received over one hundred twenty copies of our national journal.

A few remarks about my duties as Business Manager of *The Pentagon*. My first responsibility is to see that each person who is a subscriber receives each issue. To meet this obligation I keep two card files of subscribers. In one file, the cards are filed alphabetically, and in the other by subscription expiration date. When each subscriber receives the last issue for which he has paid, he will find inserted at the table of contents a subscription expiration notice with information on how to renew the subscription. During the 1961 and 1962 calendar years about 400 subscription renewals were received. During the same two years nearly forty dollars was spent paying postage on *Pentagons* which were returned for insufficient or wrong addresses. This represents approximately five hundred magazines which were returned because subscribers did not report their change of address or reported wrong addresses on their original subscription card. Printed on the inside of the front cover of each *Pentagon* is a statement that copies lost because of failure to notify the Business Manager of change of address cannot be replaced. I would appeal to each member of this convention to carry the message back to your home chapter that the Business Manager would appreciate a change of address when you move.

Each speaker presenting a paper at this convention will automatically have his subscription extended for two years. This is with the compliments of the National Council. Each author who has an article published in *The Pentagon* receives five complimentary copies of the issue in which the article appears.

Dr. Lott and the associate editors provide us with an excellent journal. I consider it a privilege to aid in the distribution of *The Pentagon* to our many readers.

—WILBUR J. WAGGONER

One of the most fascinating characteristics of mathematics is the surprising way in which the ideas and results of the different parts of the subject dovetail into each other. During the discussions . . . we have been guided merely by the most abstract of pure mathematical considerations; and yet at the end of them we have been led back to the most fundamental of all the laws of nature, laws which have to be in the mind of every engineer as he designs an engine, and of every naval architect as he calculates the stability of a ship. It is no paradox to say that in our most theoretical moods we may be nearest to our most practical applications.

—A. N. WHITEHEAD

FINANCIAL REPORT OF THE NATIONAL TREASURER

April 14, 1961 to April 1, 1963

Cash on hand April 14, 1961			$ 6,273.47
Receipts			
Initiates (2534 at $5.00)	$12,670.00		
Miscellaneous (Supplies, Installation, etc.)	1,142.00		
Total Receipts from Chapters		$13,812.00	
Miscellaneous Receipts			
Interest on Bonds	167.26		
Balfour Company (Commissions)	273.00		
The Pentagon (Surplus)	120.72		
Total Miscellaneous Receipts		560.98	
Total Receipts			14,372.98
Total Receipts Plus Cash on Hand			$20,646.45
Expenditures			
National Convention, 1961			
Paid to Chapter Delegates	$ 2,069.68		
Officers Expenses	525.42		
Miscellaneous (Prizes, Host Chapter, Expenses, Programs, Speaker, etc.)	276.75		
Total National Convention		2,871.85	
Balfour Company (Memberships, Certificates, Stationery, etc.)		2,106.80	
Pentagon (Printing and mailing four issues		5,737.68	
Installation Expense		115.36	
National Office Expense		634.75	
Legal Expense		191.00	
2 Regional Conventions		200.00	
Total Expense			11,857.44
Cash Balance on Hand April 1, 1963			8,789.01
Total Expenditures Plus Cash on Hand			$20,646.45
Bonds on Hand April 1, 1963	3,000.00		
Savings Account + 177.97 int.	2,966.40		
		$ 5,966.40	
Total Assets as of April 1, 1963			14,755.41
Total Assets 1961			12,061.90
Net Gain For Period			2,693.51

—WALTER C. BUTLER

14th Biennial KME Convention
ILLINOIS STATE NORMAL UNIVERSITY
Normal, Illinois
April 8 & 9, 1963