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CONTENTS

National Officers

Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Conformal Mapping*

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Before we develop a definition of conformal mapping, let us first consider the general concept of mapping. A definite association between those points of the complex plane representing the values of z and those representing the values of w is established by the relation $w = f(z)$. For convenience it is customary to represent the z-points in one plane, called the Z-plane, and the w-points in another plane, called the W-plane. A relationship exists between these two planes which is somewhat similar to that possessed by the two co-ordinate axes in the consideration of functions of a real variable. As the point P traces any curve C in the Z-plane, the corresponding point Q will trace a curve S in the W-plane. The relation between the two curves C and S is expressed by saying that the curve C in the Z-plane is mapped upon the W-plane, thus providing us with the curve S in the W-plane. Usually it is convenient in discussing the general properties of mapping to speak of the mapping of the one plane upon the other rather than of the mapping of some particular configuration from the one plane upon the other.

Now that we have established the broad concept of a mapping let us narrow this concept to that of mapping which is conformal. Let C_1 and C_2 by any two smooth arcs through a point z_0 of the Z-plane having as images under the mapping $w = f(z)$ two smooth arcs S_1 and S_2 in the W-plane which pass through the point w_0, the image of z_0. If the angle γ at z_0 from C_1 to C_2 is the same, both in magnitude and sense, as the angle at w_0 from S_1 to S_2, the mapping $w = f(z)$ is said to be conformal at the point z_0.

Now that we have an understanding of what constitutes a conformal mapping I shall state the following useful theorem and indicate a proof.

THEOREM: The mapping of the Z-plane upon the W-plane by means of a function $w = f(z)$ is conformal at each point where $f(z)$ is analytic and the derivative $f'(z)$ is not zero.

Let $f(z)$ be a function which is analytic at a point z_0 of the Z-plane. Because f is analytic at z_0 the derivative of f at z_0 exists.

* A paper presented at a Regional Convention of KME at St. Mary's Lake, Michigan on April 28, 1962.

Moreover, if we assume that $f'(z_0) \neq 0$, the argument ϕ_0 $(0 \leqq \phi_0 < 2\pi)$ of $f'(z_0)$ in the polar representation

$$f'(z_0) = R_0 e^{i\phi_0}$$

has a unique value. If C is any curve through z_0, it can be shown that the directed tangent to C at z_0 is rotated through the angle ϕ_0 by the transformation $w = f(z)$. Because the angle ϕ_0 is determined only by the function $f(z)$ and the point z_0, it is the same for all curves through z_0; thus it follows that the mapping $w = f(z)$ is conformal at z_0.

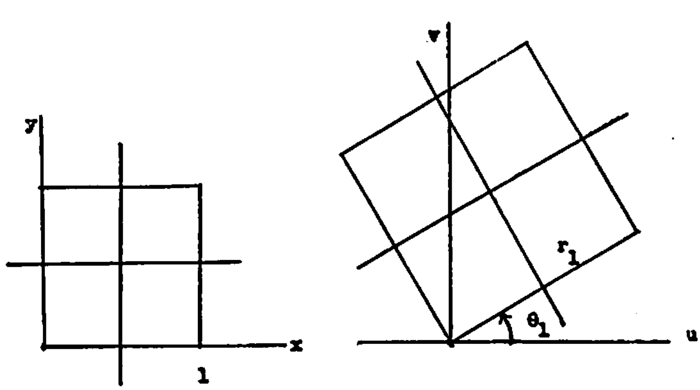
I shall now utilize the above theorem by choosing two analytic functions to serve as examples of conformal mappings. Let us first consider the transformation

$$w = cz,$$

where z and w are complex variables and c is a complex constant. Writing $w = \rho e^{i\phi}$, $z = re^{i\theta}$, and $c = r_1 e^{i\theta_1}$ the above equation becomes

$$\rho e^{i\phi} = r_1 e^{i\theta_1} r e^{i\theta} = r_1 r e^{i(\theta_1 + \theta)}.$$

This leads to $\rho = r_1 r$ and $\phi = \theta_1 + \theta$. As a result, the origin maps into the origin, all distances from the origin are multiplied by the constant r_1, and all straight lines which pass through the origin are turned through the constant angle θ_1. The following diagram serves to illustrate these points.



As a final example let us consider the mapping

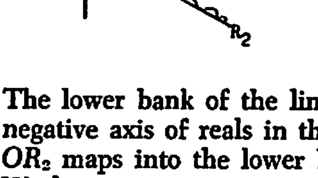
$$w = z^n.$$

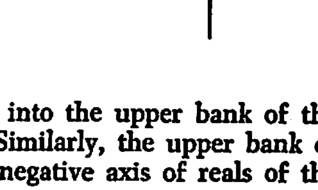
If we rewrite z and w so that $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\phi + i\sin\phi)$, the preceding equation becomes

$$\begin{aligned}\rho(\cos\phi + i\sin\phi) &= [r(\cos\theta + i\sin\theta)]^n \\ &= r^n(\cos n\theta + i\sin n\theta),\end{aligned}$$

from which we obtain the relationships $\rho = r^n$ and $\phi = n\theta$. Thus a circle about the origin in the Z-plane maps into a circle in the W-plane and a straight line through the origin in the Z-plane transforms into a straight line through the origin of the W-plane. The relation between θ and ϕ tells us that $(1/n)$th of the circle in the Z-plane maps into the whole of the circle in the W-plane; as a result, any sector bounded by two half-rays from the origin which make an angle of $2\pi/n$ radians with each other is transformed by the mapping $w = z^n$ into the entire W-plane. The values of ϕ corresponding to the chief amplitude of w lie in the interval $-\pi < \phi \leq \pi$.

As an illustration of this let us consider the sector bounded by OR_1 and OR_2, making the angles π/n and $-\pi/n$ respectively with the positive x-axis. This maps in a continuous, single-valued manner upon the entire W-plane.





The lower bank of the line OR_1 maps into the upper bank of the negative axis of reals in the W-plane. Similarly, the upper bank of OR_2 maps into the lower bank of the negative axis of reals of the W-plane.

As the above discussion and illustration point out, the mapping is not conformal at the origin of the Z-plane. But this does not contradict the theorem introduced earlier, since the derivative of the function $f(z) = z^n$ is zero at this point, provided that n is greater than 1. As is apparent, the mapping is conformal everywhere else.

This ends my discussion on conformal mapping. For further information as to the properties and applications of this interesting and useful concept I refer you to the references listed in the bibliography which follows. In concluding I wish to acknowledge the invaluable aid of Dr. Kaj L. Nielsen of the Mathematics Department of Butler University, who acted as my adviser on this project.

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Mathematical reasoning is deductive in the sense that it is based upon definitions which, as far as the validity of the reasoning is concerned (apart from any existential import), needs only the test of self-consistency. Thus no external verification of definitions is required in mathematics, as long as it is considered merely as mathematics.

—A. N. WHITEHEAD

The Rotation Group of a Tetrahedron

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"The rigid motions of a three-dimensional euclidean space constitute a group" [6, p. 41]*. Such a group is formed when a regular polyhedron is rotated so that symmetries result. "A symmetry of a geometrical object is any movement of that object which brings it into coincidence with itself" [3, p. 1]. These symmetries are the one-one transformations which preserve distances on the polyhedra, and "they are known as 'isometries' . . ." [2, p. 124]. In this paper I shall be particularly concerned with the isometries of a tetrahedron. It will be shown that rotations of this geometrical object constitute a group.

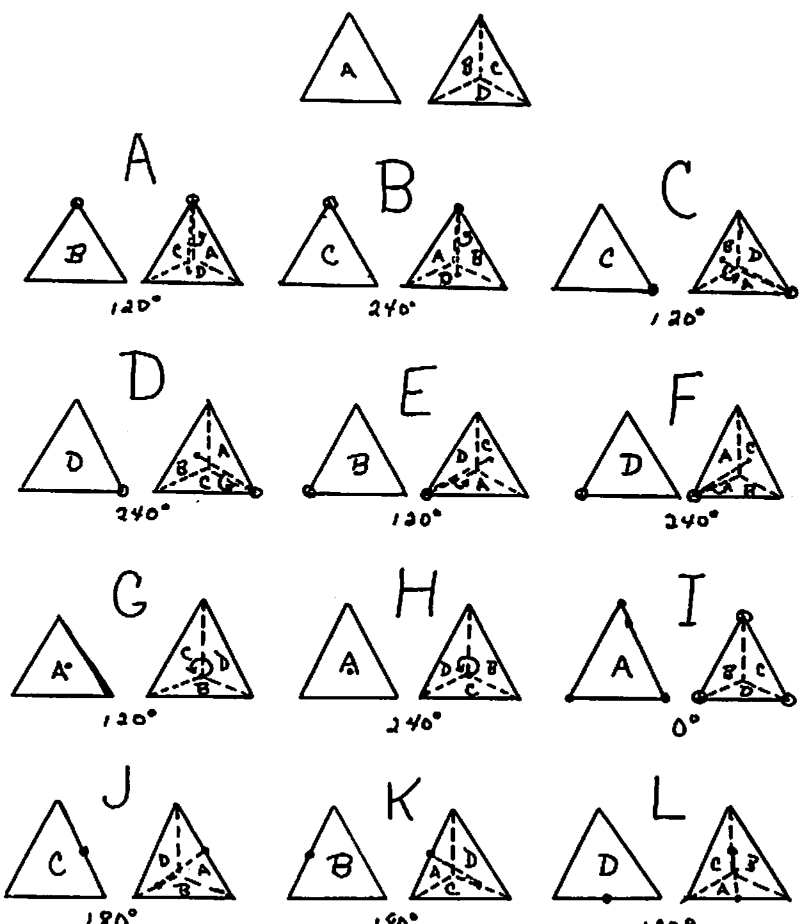
"A uniform triangular pyramid is a regular tetrahedron" [4, p. 102]. Having four faces, the figure has three sides per face with three faces meeting at each of the four vertices. There are six edges. By rotating the tetrahedron about each axis formed by connecting each vertex to the center of the opposite face, nine different symmetries may be formed with successive rotations of 120 degrees. Three more symmetries are obtained by rotating the tetrahedron 180 degrees on each of the three axes formed by joining the midpoints of the opposite edges, thus resulting in a total of twelve different symmetries. The resulting positions of the various faces of the tetrahedron under each of the rotations are shown in the figure on page 72. All rotations are positive, i.e., counterclockwise.

To show that the isometries of a tetrahedron constitute a group, we must show that the definition of a group is satisfied. A group is a non-empty set of elements, $G = \{g_1, g_2, g_3, \cdots, g_n\}$, with a single valued binary operation, $\cdot$, defined on it such the the following conditions are satisfied: 1) *Closure:* $(g_1 \cdot g_2)$ is contained in G for every g_1 and g_2 contained in G; 2) *Associative Law:* $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$; 3) *Existence of an Identity:* there is an element, I, contained in G such that $I \cdot g = g$, for all g contained in G; and 4) *Existence of an Inverse:* for every g contained in G there exists an element, g^{-1}, contained in G such that $g^{-1} \cdot g = g \cdot g^{-1} = I$ [5, p. 4]. If the group G satisfies the commutative law, i.e., $g_1 \cdot g_2 = g_2 \cdot g_1$ for every g_1 and g_2 contained in G, then the group is said to

* The first number in the brackets refers to the references at the end of this article; the second number denotes the page in that reference.

ROTATIONS

Initial Position



The circled vertices are the ones which do not change position during a given rotation.

be abelian. Not all groups are abelian, so it will be considered later in this article whether the rotation group of a tetrahedron is abelian or not.

We will now define the set T as the set of twelve isometries of a tetrahedron shown in the diagram, i.e., $T = \{A,B,C,D,E,F,G,I,H,J,K,L\}$. The single valued binary operation, $\cdot$, for the set T we will define to be a combination of successive rotations of the tetrahedron. For example, $A \cdot B$ is the single isometry that has the same effect on the tetrahedron as first doing A then following this with B. A table showing the results of the operation $\cdot$ on any two elements of the set of isometries of a tetrahedron follows:

$\cdot$	A	B	C	D	E	F	G	H	I	J	K	L
A	B	I	G	L	J	D	K	E	A	H	C	F
B	I	A	K	F	H	L	C	J	B	E	G	D
C	E	L	D	I	K	G	J	B	C	F	A	H
D	K	H	I	C	A	J	F	L	D	G	E	B
E	L	C	J	H	F	I	A	K	E	B	D	G
F	G	J	B	K	I	E	L	D	F	C	H	A
G	J	F	L	A	C	K	H	I	G	D	B	E
H	D	K	E	J	L	B	I	G	H	A	F	C
I	A	B	C	D	E	F	G	H	I	J	K	L
J	F	G	H	E	D	A	B	C	J	I	L	K
K	H	D	F	B	G	C	E	A	K	L	I	J
L	C	E	A	G	B	H	D	F	L	K	J	I

Now we must show that each of the conditions for a group is satisfied. First let us consider whether the operation $\cdot$ is closed for the set T. An examination of the table above quickly shows that this operation is closed for T, i.e., that (any element of T) $\cdot$ (any element of T) = (an element of T). This would be obvious without the table, as well, since the results of all possible rotations of a tetrahedron are elements of T.

Secondly we must show that the associative law holds for the operation $\cdot$ on set T. For example, $A \cdot (E \cdot K) = (A \cdot E) \cdot K$ if the associative law holds. $E \cdot K = D$ and $A \cdot E = J$, according to the table; therefore, $A \cdot D = J \cdot K$ if the associative law holds. A glance at the table shows that $A \cdot D = L$, and $J \cdot K = L$, so we have shown that the associative law does hold in this case. Similarly by examining all other cases in this manner, it can be shown that the associative law holds in every case. This, however, would be quite a task, since there are $12^3 = 1728$ possible cases. The number of cases to be examined could be reduced to $11^3 = 1331$ by noting that the associative law must be true for any one of them involving the identity element, but this still leaves a sizeable number of cases. Fortunately, the associative law may be proved to hold for any set of transformations, so we may apply this theorem without checking all the cases, and conclude that the associative law does hold for the operation $\cdot$ on T.

The third condition which must be satisfied is that an identity element exists for the set and the operation $\cdot$. The identity element for the set T is I. I is the result of a rotation of zero degrees. Since it is clear that any element of T combined with a rotation of zero degrees is unchanged, we may conclude that $I \cdot$ (any element of T) = (any element of T) $\cdot I$ = (that element of T), i.e., I is the identity element for the operation $\cdot$ performed on the set T. Although this is evident without the use of the table by the argument presented above, an examinations of the tenth vertical and horizontal columns of the table verifies that I is truly the identity rotation.

Lastly, each element of T has an inverse, i.e., (any element of T) $\cdot$ (that element's inverse) = I. The truth of this statement may be established by remembering that each element of T may be obtained by starting with the tetrahedron in the initial position and performing some single rotation upon it (see the figure). We can also produce I with a rotation of 360°. If any element of T is used to transform the tetrahedron, its inverse will be the additional rotation about that axis needed to make a total rotation of 360°. Thus the combination of the two rotations has the same effect on the tetrahedron as I. An examination of the table shows that the following are the actual inverses for each element of T:

$A^{-1} = B$	$E^{-1} = F$	$I^{-1} = I$
$B^{-1} = A$	$F^{-1} = E$	$J^{-1} = J$
$C^{-1} = D$	$G^{-1} = H$	$K^{-1} = K$
$D^{-1} = C$	$H^{-1} = G$	$L^{-1} = L$

It has been verified from this discussion that the twelve symmetries of a tetrahedron constitute a group, since all the conditions of the definition of a group have been shown to be satisfied. Because $B \cdot C$ does not equal $C \cdot B$, the first equaling K and the second equaling L, the rotation group of a tetrahedron is not abelian. It may also be noted that this group is finite, consisting of only twelve elements.

For any group, a number of subgroups exist. "A subgroup of a group G is a set of elements of G which, by themselves, form a group with respect to the operation defined on G. Every group G contains the subgroup which contains the identity element alone, and also the subgroup which consists of the entire group G. The group G may or may not contain other subgroups called proper subgroups" [1., p. 91]. The rotation group of a tetrahedron does have several proper subgroups. Some of these are shown below. It may also be noted that this entire group is a subset of the Symmetric Group in the permutation of n objects.

•	*I*
I	*I*

•	*I*	*J*
I	*I*	*J*
J	*J*	*I*

•	*I*	*K*
I	*I*	*K*
K	*K*	*I*

•	*I*	*L*
I	*I*	*L*
L	*L*	*I*

•	*A*	*B*	*I*
A	*B*	*I*	*A*
B	*I*	*B*	*A*
I	*A*	*B*	*I*

•	*C*	*D*	*I*
C	*D*	*I*	*C*
D	*I*	*C*	*D*
I	*C*	*D*	*I*

•	*E*	*F*	*I*
E	*F*	*I*	*E*
F	*I*	*E*	*F*
I	*E*	*F*	*I*

•	*G*	*H*	*I*
G	*H*	*I*	*G*
H	*I*	*G*	*H*
I	*G*	*H*	*I*

•	*I*	*J*	*K*	*L*
I	*I*	*J*	*K*	*L*
J	*J*	*I*	*L*	*K*
K	*K*	*L*	*I*	*J*
L	*L*	*K*	*J*	*I*

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Mathematics is the gate and key of the sciences. . . . Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy.

—Roger Bacon

Solutions of Cubics and Quartics*

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Before ever attacking an equation to solve for its roots, it is good to know something about the nature of the roots and approximately where they lie. One means of doing this is to find the boundaries of the roots. The upper boundary or limit is the positive number M which gives all positive (+) signs when $x - M$ is divided synthetically into $f(x) = 0$. The equation $x^3-7x^2+49x-96$ has an upper limit of +8, because when the factor $(x-8)$ is removed from the equation by synthetic division, all the coefficients are positive.

$$\begin{array}{rrrr|l} 1 & -7 & 44 & -96 & \underline{8} \\ & 8 & 8 & 416 & \\ \hline 1 & 1 & 52 & 320 & \end{array}$$

The lower limit for positive roots would naturally be zero. For lower limits of negative roots, one can find the upper limit of the positive roots of $f(-x) = 0$ and then change the sign of the number found [5][1]. The lower limit is the number M_1 which results in alternating signs when $(x-M_1)$ is removed from $f(x)$.

After finding the limits of the roots, it is next advisable to narrow down the interval in which the roots lie. One method for accomplishing this is given by Descartes's rule of signs. The number of positive roots of $f(x) = 0$ can not exceed the number of variations in sign of $f(x)$ and differs from the number of variations by an even integer [5]. The number of negative roots equals the number of variations of sign of $f(-x)$ or differs from it by an even integer. Thus

$$3x^3-2x^2+x+1 = 0$$

may have either two positive roots or no positive roots, for there are two variations in sign in the equation. It has one negative root, for $f(-x)$ has one variation in sign.

By Rolle's Theorem, more about the roots can be discovered. If $f(x) = 0$ is an algebraic equation with real coefficients, between

* A paper presented at a Regional Convention of KME at St. Mary's Lake, Michigan on April 28, 1962.

1 The numbers in brackets refer to the Bibliography at the end of this article.

the consecutive real roots a and b of this equation, there is an odd number of roots to $f'(x) = 0$. In counting the roots here, a root of multiplicity m is counted as m roots [7]. Similarly, between any two consecutive real roots α and β of $f'(x) = 0$, there occurs at most one real root of $f(x) = 0$. If $f(\alpha)$ and $f(\beta)$ have opposite signs, then there is one real root between α and β, but there are no roots if $f(\alpha)$ and $f(\beta)$ have like signs. Another thing to remember is that at most, one real root of $f(x) = 0$ is greater than the greatest real root of $f'(x) = 0$, and at most one real root of $f(x) = 0$ is less than the least real roof of $f'(x) = 0$ [2]. By yet another theorem, if the coefficients of an algebraic equation $f(x) = 0$ are real, and if a and b are real numbers such that $f(a)$ and $f(b)$ have opposite signs, then the equation has at least one real root between a and b [7].

Thus an equation can be discussed and the nature of its roots and their approximate location can be found before the roots are ever actually solved for. For example

$$f(x) \equiv x^3 - 6x^2 + 3x + 7 = 0$$

has two changes of sign, so there are either zero or two positive real roots.

$$\begin{array}{lll} f(0) = + & f(1) = + & f(5) = - \\ f(-1) = - & f(2) = - & f(6) = + \end{array}$$

Therefore, there are roots between 0 and -1, 1 and 2, and 5 and 6.

An extension of the theorems concerning the intervals around roots is given by Sturm's Theorem. It enables one to locate exactly how many real roots are between two given numbers for equations without multiple roots. It states that the number of real roots between a and b of $f(x) = 0$, an algebraic equation with real coefficients and without multiple roots, and with a and b real numbers, $a < b$ and neither a root of the equation, equals $V_a - V_b$, where V_h denotes the number of variations in sign at $x = h$ of a certain set of functions. Sturm's functions are denoted by $f(x)$, $f_1(x)$, $f_2(x)$, $\cdots$, $f_n(x)$, where $f_1(x)$ is the first derivative of $f(x)$, $f_2(x)$ equals the negative remainder $(-R)$ of $f(x)/f_1(x)$, $f_3(x)$ equals $-R$ of $f_1(x)/f_2(x)$, and so on until a constant remainder is obtained or until the $-R$ equals $p(x-h)^2$, where h and p are real constants and h is not a multiple root of the equation. Then the next function can be designated by $+1$ if p is positive and -1 if p is negative. Since only the sign of the $f_i(x)$ is of importance, any one of the functions can be modified by removing a positive factor.

For the equation $f(x) = x^3 - 7x + 7$,

$$f_1(x) = 3x^2 - 7$$
$$f_2(x) = 2x - 3$$
$$f_3(x) = +1$$

Since f_3 will be a positive constant, it is merely designated by a 1. By setting up a table of values for x, the variations in sign may be counted.

x	f	f_1	f_2	f_3	V_x
2	+	+	+	+	0
1	+	−	−	+	2
0	+	−	−	+	2
−1	+	−	−	+	2
−2	+	+	−	+	2
−3	+	+	−	+	2
−4	−	+	−	+	3

From the chart, it is easy to see that there are 2 roots between 1 and 2 and one root between −3 and −4. Since the interval (1,2) has two roots, one might try to pick some numbers between 1 and 2 and see whether or not there was a change of sign.

x	f	f_1	f_2	f_3	V_x
2	+	+	+	+	0
1.5	−	−	0	+	1
1	+	−	−	+	2

This shows that there is one root in the interval (1,1.5) and one in the interval (1.5,2) [7].

Budan's theorem is easier to apply than Sturm's, but it also gives less information about the roots. Budan's functions are $f(x)$ plus all its derivatives, $f'(x)$ to $f^{(n)}(x)$. He denoted the variations in sign of the sequence $f(x)$, $f'(x)$, $f''(x)$, $\cdots$, $f^{(n)}(x)$ by V_c (where x has the value c and c is any real number). Then he said that the number of real roots of $f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ between a and b (a and b both real numbers, $a < b$, neither a root of the equation) is either equal to $V_a - V_b$ or is less than that quantity by a positive even integer [2]. By this theorem, an interval can be located in which there *may* be some roots. If there were three variations in sign for $f(0)$ and five variations for $f(1)$, then there would be either two real roots or no real roots in the interval. To determine the exact number, some other method must be used.

Other properties concerning roots of equations are also known. If a rational integral equation with rational coefficients has a root of $a + \sqrt{b}$, where a and b are rational numbers and b is not a perfect square, then the equation has $a - \sqrt{b}$ as a second root. Very similar to this is the theorem that if an algebraic equation with real coefficients has a root of $a + bi$ (a and b being real and $b \neq 0$), then $a - bi$ is also a root of the equation [1]. In other words, complex numbers always occur in pairs.

If a rational integral equation has all positive coefficients, then it can have no positive roots. Also, a rational integral equation with no missing powers and with its coefficients alternating in sign can have no negative roots. But is some power of x is missing, the situation is different. If an even number of terms, say $2m$, is lacking between two other terms, then the function has at least two imaginary roots. If the function has an odd number of terms missing, say $2m + 1$, between two other terms, then $f(x)$ has at least either $2m$ or $2m + 2$ imaginary roots, depending on the sign of the two terms. If they are alike, there are only $2m$ roots, but if they are opposite, there are $2m + 2$ roots [6].

By the factor theorem, if the value of $f(x)$, when x is replaced by r, is 0, then $x - r$ is a factor of $f(x)$ and r is a root of the function [7].

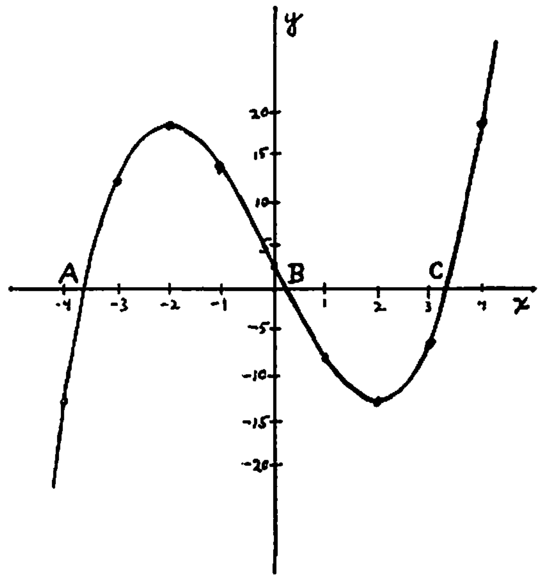
After finding out something about the roots of the equation, the next step is to solve for the roots themselves. One of the simplest ways of doing this is by graphing the equation. To graph

$$x^3 - 12x + 3 = 0$$

plot some points, find out about symmetry, intercepts, etc. The graph of $y = x^3 - 12x + 3$ is shown on page 81. Thus the roots are seen to be approximately -3.6, $.3$, 3.3 [5].

If $(x - r)^h$ is the highest power of $(x - r)$ and if r is real, then the graph of $f(x)$ crosses the axis at $x = r$ if h is odd. If h is an odd integer > 1, the graph on each side of $x = r$ is tangent to the X axis at $x = r$. But if h is an even integer, then the graph of $f(x)$ is entirely on one side of the X axis, near $x = r$, and tangent to the X axis at $x = r$.

Another method for finding roots of an equation involves the derivative of the function. If r is a root of $f(x) = 0$, then a neces-



sary and sufficient condition that r be a k-fold root of this equation is that it be a $(k-1)$ fold root of $f'(x) = 0$. One must find the highest common factor between $f(x)$ and $f'(x)$ and remove it from $f(x) = 0$ [4]. Thus the resulting equation is one of low enough degree that it can be handled without trouble. The equation

$$x^3 - 7x^2 + 15x - 9 = 0$$

has the roots 3, 3, 1. In order to find these roots, first take the first derivative of $f(x)$, which is

$$3x^2 - 14x + 15$$

and apply the Euclidean algorithm. If no common factor can be found between $f(x)$ and $f'(x)$, the equations are said to be relatively prime and $f(x)$ has no k-fold real root. The highest common factor obtained upon equating to zero the highest common factor of $f(x)$ and $f'(x)$ is $(x-3) = 0$. Thus, $x = 3$ is a double root of $f(x) = 0$. The equation remaining after removing the factor $(x - 3)^2$ is $x - 1 = 0$. Therefore, the third root is $+1$.

The above method works for finding roots of multiplicity but does not apply if there are n distinct roots. For this, other procedures must be followed. Anytime a rational root r of the equation is known, the factor $(x - r)$ should be removed by synthetic division. The resulting *depressed* equation contains the remaining roots.

If the coefficients in

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

are integral and real, then every rational root of the equation is of the form c/d. An extension of this is the following method. If a rational integral equation with integral coefficients has a rational root c/d, where c and d are relatively prime integers, then c is a divisor of a_n and d is a divisor of a_0 [6]. Another way to narrow down the candidates for roots is to find $f(1)$, for $c - d$ must divide $f(1)$.

An example of this in the equation

$$10x^4 - 13x^3 + 17x^2 - 26x - 6 = 0.$$

By the above theorem, c must divide -6, d must divide 10, and $c - d$ must divide -18. The candidates for x can be seen below.

$$\begin{aligned} c&: \quad 1, 2, 3, 6 \\ d&: \quad 1, 2, 5, 10 \\ c/d&: \quad -1, \pm 2, 3, 4, -5, \pm\tfrac{1}{2}, -\tfrac{1}{5}, \tfrac{1}{10}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{6}{5}, \tfrac{3}{2} \end{aligned}$$

At this point, it is a question of trial and error. The possible roots must be tried (the easiest way is by synthetic division) until a root is found. One root is $-1/5$, a second $3/2$. Upon removing the factors $(x + 1/5)$ and $(x - 3/2)$ from the original equation, $x^2 + 2 = 0$ is the remaining equation. Solving this, the last two roots are found to be $\pm\sqrt{2}i$.

Another method for solution involves the relationship between the coefficients of the equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$. The sum of the roots equals $-a_1/a_0$; the sum of the products of the roots taken two in a set, equals a_2/a_0; the sum of the products of the roots taken 3 in a set equals $-a_3/a_0$, and so on. The product of all the roots equals $-a_n/a_0$ when n is odd and a_n/a_0 when n is even. These relationship may be written

$$\begin{aligned} a_1/a_0 &= -(r_1 + r_2 + \cdots + r_n) \\ a_2/a_0 &= (r_1r_2 + r_1r_3 + \cdots + r_{n-1}r_n) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n/a_0 &= (-1)^n r_1r_2r_3 \cdots r_n \end{aligned}$$

The solution of a problem shows better how these relationships are used. Let $f(x) \equiv x^3 + 4x^2 - 9x + h = 0$ and $\alpha = -\beta$ where

α and β are two of the roots. Using γ for the third root, we have

$$\begin{aligned} a_1/a_0 &= -4 = \beta - \beta + \gamma \\ a_2/a_0 &= -9 = -\beta^2 - \beta\gamma + \beta\gamma \\ a_3/a_0 &= -h = -\beta^2\gamma \end{aligned}$$

With three equations and three unknowns, the roots may be solved for and h thereby determined [5].

Newton's method for integral roots is a method for testing roots of equations of any degree and would be appropriate for solving cubics and quartics. Let

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

be an equation with integral coefficients. In order for r to be a root of $f(x)$, $1/r$ must be a root of $x^nf(1/x)$. Let

$$f(x) \equiv x^3 - 7x^2 + 15x - 9 = 0.$$

Then

$$\begin{aligned} c&:\ 1,\ 3,\ 9 \\ d&:\ 1 \\ c/d&:\ \pm1,\ \pm3,\ \pm9. \end{aligned}$$

To test the roots by Newton's method, the coefficients of $f(x)$ are written in the order $a_0, a_1, a_2, \cdots, a_n$. In the synthetic division of $1/r$, start from the *right* and work *left*.

$$\begin{array}{rrrr|l} 1 & -7 & 15 & -9 & \underline{\tfrac{1}{3}} \\ -1 & 4 & -3 & & \\ \hline 0 & -3 & 12 & -9 & \end{array}$$

Thus the synthetic division shows that $1/r$ is a root of $x^nf(1/x)$, and, therefore, r must be a root of $f(x)$ [2].

Another method for integral roots is especially useful for equations whose constant term has a great many divisors. Let d be a divisor of the constant term and s be any integer whatsoever. If d is an integral root of $f(x) = 0$, $f(s)$ is divisible by $s - d$. Therefore, all candidates for roots must divide the constant and $s - d$ must divide $f(s)$. Let

$$f(x) \equiv x^3 - 35x^2 - x - 1260 = 0.$$

Also let $s = 2$, then

$$f(s) = -1394 = -2\cdot17\cdot41.$$

When a divisor of 1260 is found which when subtracted from s is also a divisor of -1394, then d may be tested by synthetic division to determine whether or not it is a root. If $s - d = \pm c$, where c is a positive divisor of $f(s)$, then $d = s \pm c$.

$$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \qquad s = 2 \qquad f(s) = -2 \cdot 17 \cdot 41$$

c	1	2	17	34	41	82	697	1394
$s+c$	3	4	*19*	36	*43*	84	*699*	*1396*
$s-c$	1	0	-15	-32	-39	-80	-695	-1392

By the above test, nine divisors are eliminated (those in italics) leaving only seven possible divisors. Upon testing those remaining, one finds that $x = 36$ is one root and that the other two are imaginary [2].

A special kind of equations, reciprocal equations, have an interesting method of solution. Although this method is used more for quintics and higher powered equations, it is still very useful for cubics and quartics. Let $f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, and let the coefficients be such that

$$a_0 = a_n, \quad a_1 = a_{n-1}, \quad a_2 = a_{n-2}, \quad \text{etc.}$$

An equation of this form is called a reciprocal equation of the first class. A reciprocal equation of the second class has its coefficients related thusly:

$$a_0 = -a_n, \quad a_1 = -a_{n-1}, \quad a_2 = -a_{n-2}, \cdots$$

In solving reciprocal equations, one should first reduce them to even degree and first class. To do this, one can use the fact that -1 is a root of a reciprocal equation of odd degree and first class, that $+1$ is a root of a reciprocal equation of odd degree and second class, and that 1 and -1 are both roots of a reciprocal equation of even degree and second class. To solve a reciprocal equation, use

$$a_0x^{2m} + a_1x^{2m-1} + \cdots + a_{2m} = 0$$

as the standard form. By dividing by x^m, grouping terms, and using

$$a_{2m} = a_0, \quad a_{2m-1} = a_1, \quad a_{2m-2} = a_2, \cdots$$

the equation can be written

$$a_0(x^m + 1/x^m) + a_1(x^{m-1} + 1/x^{m-1}) + \cdots + a_m = 0.$$

Letting $y = x + 1/x$, each of the parenthetical expressions can be written

$$x^2 + 1/x^2 = (x + 1/x)y - 2 = y^2 - 2$$
$$x^3 + 1/x^3 = (x^2 + 1/x^2)y - (x + 1/x) = y^3 - 3y$$

The general relation is seen to be

$$x^k + 1/x^k = (x^{k-1} + 1/x^{k-1})y - (x^{k-2} + 1/x^{k-2})$$

The original equation will be of degree m when written in terms of y. Thus, a reciprocal equation in standard form and of degree $2m$ can be reduced to an equation of degree m. By solving this equation and then substituting

$$y = x + 1/x$$

roots of the equation in y can be obtained and thus related back to the original equation by $x + 1/x = y$. Let

$$f(x) \equiv x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0.$$

Since it is of odd degree and second class, $+1$ is one root. Upon removing the factor $(x - 1)$ from $f(x)$, the resulting equation is

$$x^4 - 4x^3 + 5x^2 - 4x + 1 = 0.$$

Dividing through by x^2, one gets

$$x^2 - 4x + 5 - 4/x + 1/x^2 = 0$$

which, when regrouped, equals

$$(x^2 + 1/x^2) - 4(x + 1/x) + 5 = 0.$$

Substituting $y = x + 1/x$, the equation becomes

$$y^2 - 4y + 3 = 0.$$

Solving for y, one gets $y = 3$, $y = 1$. Putting them back in $y = x + 1/x$, x is found to equal [2]

$$1, \quad \frac{3 \pm \sqrt{5}}{2}, \quad \frac{1 \pm \sqrt{3}\,i}{2}$$

A method strictly for cubics is given by Cardan's formulas. In order to use them, the cubic

$$ax^3 - bx^2 + cx - d = 0$$

must be reduced to an equation without the x^2 term (a reduced cubic). To do this, use the relationship $x = y + b/3a$, thus getting the transformed equation $y^3 + py - q = 0$.
The roots of the reduced cubic are then

$$\begin{aligned} y_1 &= \sqrt[3]{A} + \sqrt[3]{B} \\ y_2 &= \omega \sqrt[3]{A} + \omega^2 \sqrt[3]{B} \\ y_3 &= \omega^2 \sqrt[3]{A} + \omega \sqrt[3]{B} \end{aligned}$$

where

$$\omega = \frac{-1 + \sqrt{3}\, i}{2}, \qquad \omega^2 = \frac{-1 - \sqrt{3}\, i}{2}$$

$$A = \frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \qquad B = \frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

If the discriminant, $4p^3 + 27q^2$, is positive, then there is one real root and there are two imaginaries. If it is negative, another method of solution should be used, for the roots would be imaginary. If $4p^3 + 27q^2$ is zero, all three roots are real [7].
Let

$$f(x) \equiv x^3 + 63x - 316 = 0.$$

$$y_1 = \sqrt[3]{\frac{316}{2} + \sqrt{\frac{(316)^2}{4} + \frac{(63)^3}{27}}} + \sqrt[3]{\frac{316}{2} - \sqrt{\frac{(316)^2}{4} + \frac{(63)^3}{27}}}$$

$$= 7 - 3 = 4$$

$$y_2 = \left(\frac{-1 + \sqrt{3}\, i}{2}\right)(-3) + \left(\frac{-1 - \sqrt{3}\, i}{2}\right) 7 = -2 - 5\sqrt{3}\, i$$

$$y_3 = \left(\frac{-1 + \sqrt{3}\, i}{2}\right) 7 + \left(\frac{-1 - \sqrt{3}\, i}{2}\right)(-3) = -2 + 5\sqrt{3}\, i$$

Cardan's solution may be used when the discriminant is negative, but when it is positive, it involves extracting cube roots of imaginary numbers. An easier method to use in this case is the trigonometric solution. The roots of an equation of the form

$$x^3 + 3Hx + G = 0$$

with a positive discriminant, equal

$$z = 2\sqrt{-H}\cos\frac{\theta + 2k\pi}{3}, \qquad k = 0, 1, 2$$

where

$$\cos\theta = -G/2\sqrt{-H^3}.$$

An example of a problem solved by the trigonometric method is [4]

$$x^3 - 36x - 72 = 0.$$

The discriminant $D = 4(3H)^3 + 27G^2 < 0$ since $H = -12$ and $G = -72$.

$$\begin{aligned} \cos\theta &= 72/2\sqrt{144 \cdot 12} = \sqrt{3}/2 \\ \theta &= 30^\circ \\ x_1 &= 4\sqrt{3}\cos(30^\circ + 0^\circ)/3 = 6.8227 \\ x_2 &= 4\sqrt{3}\cos(30^\circ + 360^\circ)/3 = -4.4533 \\ x_3 &= 4\sqrt{3}\cos(30^\circ + 720^\circ)/3 = -2.3695 \end{aligned}$$

Descartes's solution of the quartic

$$x^4 + bx^3 + cx^2 + dx + e = 0$$

is a way to find the roots when the x^3 term of a quartic is missing. To get this reduced quartic, use the relationship $x = z - p/4$. This leaves

$$z^4 + qz^2 + rz + s = 0. \qquad (1)$$

The left member of (1) can be written as the product of two quadratic factors. To determine the factors, equate the coefficients of like powers of z in (1) with the right side of the identity

$(z^2 + 2kz + j)(z^2 - 2kz + m) = z^4 + (j + m - 4k^2)z^2 + 2k(m - j)z + jm.$

Thus

$$j + m = 4k^2 + q, \qquad 2k(m - j) = r, \qquad jm = s.$$

If $k \neq 0$, the first two give

$$2j = q + 4k^2 - r/2k, \qquad 2m = q + 4k^2 + r/2k \qquad (2)$$

Inserting these values in $2j \cdot 2m = 4s$, one gets

$$64k^6 + 32qk^4 + 4(q^2 - 4s)k^2 - r^2 = 0. \qquad (3)$$

By solving this as a cubic in k^2, values can be found for j and m by substitution in (2). Then (1) becomes

$$z^2 \pm 2kz + \tfrac{1}{2}q + 2k^2 \mp r/4k = 0.$$

It is not necessary to find all the roots of (3), for any one of them will do. Let

$$f(z) \equiv z^4 - 2z^2 + 8z - 3 = 0.$$

Then by solving using the method suggested by Descaretes,

$$\begin{aligned} z^4 - 2z^2 + 8z - 3 &\equiv (z^2 + 2kz + j)(z^2 - 2kz + m) \\ &\equiv z^4 + (j + m - 4k^2)z^2 + 2k(m - j)z + jm \end{aligned}$$

$$-2 = j + m - 4k^2, \qquad 8 = 2k(m - j), \qquad -3 = jm$$

$$2j = -2 + 4k^2 - 8/2k \tag{4}$$

$$2m = -2 + 4k^2 + 8/2k \tag{5}$$

$$\begin{aligned} &(-2 + 4k^2)^2 - 64/4k^2 = -12 \\ &4 - 16k^2 + 16k^4 - 64/4k^2 = -12 \\ &64k^6 - 64k^4 + 64k^2 - 64 = 0 \qquad k = 1 \end{aligned}$$

By substituting $k = 1$ in (4) and (5), $m = 3$ and $j = -1$. The equation can then be written

$$(z^2 + 2z - 1)(z^2 - 2z + 3) = 0$$

and the roots can be found by [4]

$$\begin{aligned} &z^2 + 2z - 1 = 0 &\qquad &z^2 - 2z + 3 = 0 \\ &z = -1 \pm \sqrt{2} & &z = 1 \pm \sqrt{2}\,i \end{aligned}$$

A second method for solving quartics was given by Ferrari. The equation

$$x^4 + bx^3 + cx^2 + dx + e = 0 \tag{6}$$

must be written

$$x^4 + bx^3 = -cx^2 - dx - e.$$

The left member contains two terms of the square of $(x^2 + \tfrac{1}{2}bx)^2$. Completing the square, one gets

$$\begin{aligned} (x^2 + \tfrac{1}{2}bx)^2 &= \tfrac{1}{4}b^2x^2 - cx^2 - dx - e \\ &= (b^2/4 - c)x^2 - dx - e. \end{aligned}$$

Adding $(x^2 + \tfrac{1}{2}bx)y + \tfrac{1}{4}y^2$ to each member, one gets

$$(x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}y)^2 = (\tfrac{1}{4}b^2 - c + y)x^2 + (\tfrac{1}{2}by - d)x + \tfrac{1}{4}y^2 - e. \tag{7}$$

The second member is a perfect square of a linear function of x if and only if its discriminant equals zero.

$$(\tfrac{1}{2}by - d)^2 - 4(\tfrac{1}{4}b^2 - c + y)(\tfrac{1}{4}y^2 - e) = 0$$

may be written as

$$y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2 = 0. \tag{8}$$

Choose any root of this resolvent cubic equation. Then the right member of (7) is the perfect square of a linear function, say $mx + n$. Thus

$$(x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}y) = mx + n, \quad (x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}y) = -mx - n$$

Roots of these are the four roots of (7) and hence roots of (6). An example will be worked to demonstrate how the method unfolds under actual conditions. Let

$$f(x) \equiv x^4 + 2x^3 - 12x^2 - 10x + 3 = 0.$$
$$b = 2, \qquad c = -12, \qquad d = -10, \qquad e = 3$$

Hence, (8) becomes

$$y^3 + 12y^2 - 32y - 256 = 0.$$

The second, third, and fourth coefficients are divisible by 4, 4^2, and 4^3 respectively. Hence any integral root of y must be divisible by 4. Let $y = 4z$, where z is an integer, in the above equation. By removing 4^3 from the equation in z, one gets

$$z^3 + 3z^2 - 2z - 4 = 0.$$

The integral root must divide the constant 4.

$$z\colon \pm 1, \pm 2, \pm 4 \text{ — candidates for roots}$$
$$z = -1$$

Therefore, $z^2 + 2z - 4 = 0$

$$z = -1 \pm \sqrt{5}$$

Hence $y = 4z = -4$ is the only integral root.

The quartic can be written

$$(x^2 + x)^2 = 13x^2 + 10x - 3$$

Adding $(x^2 + x)(-4) + (4)$ to each member [4]

$$(x^2 + x - 2)^2 = 9x^2 + 6x + 1 = (3x + 1)^2$$
$$x^2 + x - 2 = \pm(3x + 1)$$
$$x = 3, -1, -2 \pm \sqrt{5}$$

A third way to solve a quartic has also been found. Let

$$x^4 + 4ax^3 + 6bx^2 + 4cx + d = 1$$

be the general equation of any quartic. Then let

$$g = a^2 - b, \quad h = b^2 + c^2 - 2abc + dg, \quad R = 4ac/3 - b^2 - d/3$$
$$j = \tfrac{1}{2}(h + \sqrt{h^2 + k^3})^{1/3} + \tfrac{1}{2}(h - \sqrt{h^2 + k^3})^{1/3}$$
$$u = g + j, \qquad v = 2g - j, \qquad w = 4u^2 + 3k - 12gj$$

and then the four roots of the quartic are [6]

$$x_1 = -a + \sqrt{u} + \sqrt{v + \sqrt{w}}, \quad x_2 = -a + \sqrt{u} - \sqrt{v + \sqrt{w}}$$

$$x_3 = -a - \sqrt{u} + \sqrt{v - \sqrt{w}}, \quad x_4 = -a - \sqrt{u} - \sqrt{v - \sqrt{w}}$$

The signs are said to be as written unless $2a^3 - 3ab + c$ is positive. If this is the case, then all radicals change in sign except $\sqrt{w}$.

Not all roots are real, so it is necessary to have ways of finding imaginary roots of the form $x + yi$ of $f(z) = 0$. To do this, first expand $f(x + yi)$ by Taylor's theorem

$$f(x) + f'(x)yi - f''(x)y^2/1\cdot 2 - f'''(x)y^3 i/1\cdot 2\cdot 3 + \cdots = 0.$$

Since x and y are to be real and $y \neq 0$

$$\begin{cases} f(x) - f''(x)y^2/2! + f^{(4)}(x)y^4/4! - \cdots = 0 \\ f'(x) - f^{(3)}(x)y^2/3! + f^{(5)}(x)y^4/5! - \cdots = 0 \end{cases} \qquad (9)$$

If $f(x) = 0$ is of degree four or less, the second equation of (9) is linear in y^2. By substituting the resulting value of y^2 in the first equation, $E(x) = 0$, real roots can be found by one of the preceding methods.

To illustrate this method, let

$$f(z) \equiv z^4 - z + 1 = 0.$$
$$x^4 - x + 1 - 6x^2y^2 + y^4 = 0, \qquad 4x^3 - 1 - 4xy^2 = 0$$
$$y^2 = x^2 - 1/4x, \qquad -4x^6 + x^2 + 1/16 = 0$$

A cubic in x^2 has a single real root [6]

$$x^2 = .528727$$
$$x = \pm .72714$$

Then $y^2 = .18492$ or $.87254$

$$z = x + yi = .72714 \pm .4301i$$
$$= -.72714 \pm .93409i$$

Newton's method of approximating irrational roots requires a good deal of figuring, but it is useful. Given an approximate value, a, or a real root, one can find a closer approximation of $(a + h)$ to the root by neglecting the h^2 and h^3 of a small number h in Taylor's formula.

$$f(a + h) = f(a) + f'(a)h + f''(a)h^2/2 + \cdots$$

and hence by taking

$$f(a) + f'(a)h = 0, \qquad h = -f(a)/f'(a).$$

Repeat this process with $a_1 = a + h$ in place of a. Thus, when

$$f(x) \equiv x^3 - 2x - 5 = 0 \tag{10}$$

the root is between 2 and 3 $[f(3) = +, f(2) = -]$. Replace x by $(2 + p)$ and

$$p^3 + 6p^2 + 10p - 1 = 0. \tag{11}$$

Since p is a decimal, p^2 and p^3 can be disregarded.
For $a = 2$

$$h = -f(2)/f'(2) = 1/10$$
$$a_1 = a + h = 2.1$$

Replace p by $(.1 + q)$ in (11) and get

$$q^3 + 6.3q^2 + 11.23q + .061 = 0.$$

Divide $-.061$ by 11.23. Get $-.0054$ as approximate value of q.

$$h_1 = -f(2.1)/f'(2.1) = -.061/11.23 = -.0054$$

Neglect the q^3 and replace q by $(-.0054 + r)$

$$6.3r^2 + 11.16196r + .000541708 = 0$$

Drop $6.3r^2$, and solve for r. Hence [6]

$$x = 2 + .1 - .0054 - .00004853 = 2.09455147.$$

The value of x in any algebraic equation may be expressed by an infinite series. Let the equation by of any degree. By dividing by the coefficient of the term containing the first power of x, let it be placed in the form

$$a = x + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 \cdots$$

Assume that x can be expanded in a positive series.

$$x = a + ma^2 + na^3 + pa^4 + \cdots$$

By inserting this value of x in the equation and by equating the coefficients of like powers of a, the values of m, n, etc., are found. Then

$$x = a - ba^2 + (2b^2 - c)a^3 - (5b^3 - 5bc + d)a^4 + (14b^4 - 21b^2c + 6bd + 3c^2 - e)a^5 + \cdots$$

is an expression of one of the roots of the equation. So that the series will converge, a must be a small fraction.

$$\text{Example: } x^3 - 3x + .6 = 0.$$

Reduced to the given form, this is

$$.2 = x - x^3/3.$$

$$a = .2, \qquad b = 0, \qquad c = -1/3$$

$$x = .2 + 1/3(.2)^3 + 1/3(.2)^5 + \cdots = .20277$$

which is its value correct to the fourth decimal place. This equation has three real roots, but the series gives only one. Others can be found if their approximate values are known. Thus, one root is about $+ 1.6$. By placing $x = y + 1.6$, there results an equation in y whose root by the series is found to be .0218, and hence $+ 1.6218$ is another root of $x^3 - 3x + .6 = 0$ [6].

A logarithmic method of solving equations involves the formation of an equation whose roots are higher powers of the roots of the given equation. To do this, an equation is first derived whose roots are the squares of the given equation, then one whose roots are the squares of the second equation or the fourth powers of those of the given equation, and so on. Logarithms help with a great part of the work. This method is especially useful when all the roots of a given equation are real and not equal to zero.

Let p, q, r, s denote the roots, each of which is to be a real negative number. Let $[p]$ denote $p + q + r + \cdots$, $[pq]$ denote $pq + qr + rs + \cdots$ Then the general algebraic equation is

$$x^n - [p]x^{n-1} + [pq]x^{n-2} - [pqr]x^{n-3} + [pqrs]x^{n-4} - \cdots \quad (12)$$

and the equation whose roots equal $p^2, q^2, r^2, \cdots$ is

$$y^n - [p^2]y^{n-1} + [p^2q^2]y^{n-2} + [p^2q^2r^2]y^{n-3} + [p^2q^2r^2s^2]y^{n-4} - \cdots$$

in which $[p^2]$ denotes $p^2 + q^2 + r^2 + s^2 + \cdots$, $[p^2q^2]$ denotes $p^2q^2 + q^2r^2 + \cdots$, etc. From this equation, another one may be derived having roots p^4, q^4, r^4, and then another $p^8, q^8, r^8, \cdots$ This process can be continued until an equation is derived whose roots are $p^m, q^m, r^m, \cdots$ where m is a power of two sufficiently high for the subsequent operations.

The equation is

$$z^n - [p^m]z^{n-1} + [p^mq^m]z^{n-2} - [p^mq^mr^m]z^{n-3} + \cdots$$

Now, let p be the root of (12) which is the largest in numerical value, q the next, r the next, etc. Then, as m increases, the value of $[p^m]$ approaches p^m, that of $[p^mq^m]$ approaches p^mq^m, that of $[p^mq^mr^m]$ approaches $p^mq^mr^m$, etc. Hence, when m is large, $[p^m]$ is an approximation to the value of p^m and $[p^mq^m/p^m]$ is an approximation to q^m. Accordingly, by making m sufficiently large, the values of $p^m, q^m, r^m, \cdots$, and hence those of $p, q, r, \cdots$, may be obtained to any required degree of accuracy. When two roots are nearly equal numerically, it will be necessary to make m very large; when equal roots exist, they should be removed by the usual method.

The example of this method is a quintic, but the principle would apply for cubics and quartics. Let

$$f(x) \equiv x^5 + 13x^4 - 81x^3 - 34x^2 + 464x - 181 = 0.$$

The general equation is

$$x^n - ax^{n-1} + bx^{n-2} - cx^{n-3} + dx^{n-4} - \cdots = 0$$

so $a = -13$, $b = -81$, $c = +34$, $d = 464$, $e = +181$.

The equation whose roots are squares of the given quintic is now found from

$$y^n - Ay^{n-1} + By^{n-2} - Cy^{n-3} + Dy^{n-4} - \cdots = 0$$

$$A = a^2 - 2b = 331, \qquad B = b^2 - 2ac + 2d = 8373,$$

$$C = c^2 - 2bd + 2ae = 71618, \qquad D = d^2 - 2ce = 202988,$$

$$E = e^2 = 32761$$

$$y^5 - 331y^4 + 8373y^3 - 71618y^2 + 202988y - 32761 = 0.$$

Take logs of the coefficients. The equation is then written

$$y^5 - (2.51983)y^4 + (3.92288)y^3 - (4.85502)y^2 + (5.30747)y - (4.51536) = 0$$

in which the coefficients are expressed by their logs in parentheses. The logs of the coefficients for the equation whose roots are fourth, eighth, and sixteenth powers of the roots of the given quintic are

$$z^5 - (4.96762)z^4 + (7.36364)z^3 - (9.24342)z^2 + (10.56243)z - (9.03072) = 0$$

$$w^5 - (9.93290)w^4 + (14.31934)w^3 - (18.14025)w^2 + (21.12363)w - (18.06144) = 0$$

$$v^5 - (19.86580)v^4 + (28.29778)v^3 - (36.13131)v^2 + (42.24726)v - (36.12288) = 0.$$

The coefficients of the second, fourth, and fifth terms in the equation for v are the squares of those of the similar terms in the equation for w. Hence, two roots are determined.

$$\log p^8 = 9.93290 \qquad \log p = 1.24161 \qquad p = 17.443$$

$$\log t^8 = 18.06144 - 21.12363 \qquad \log t = -.38277 \qquad t = .4142$$

These are the values of the largest and the smallest roots of the quintic, but the method doesn't show whether they are positive or negative. By Descartes and by trial and error, $p = -17.443$ and $t = +.4142$.

To obtain the other roots, the process continues until two successive equations are found for which the coefficients in the second are the squares of the coefficients of the first. Since in this case two roots lie near together, the process does not end, with five place logs, until the 512th powers are reached. The three remaining roots are [6]

$$q = 3.230, \qquad r = 3.213, \qquad s = -1.4142.$$

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Letter to the Editor

Dear Editor:

I noticed the article by Andrew Robert Gale on "A Relationship Between Determinants and Progressions" on p. 25 of the Fall 1962 PENTAGON.

It seems to me that the more general results are: (1) If the first three elements of each row of a determinant of order n ($n \geqq 3$) form an arithmetic progression the determinant is zero, and (2) If two rows of a determinant of order n ($n \geqq 2$) form geometric progressions with a common ratio the determinant is zero.

A more general statement than (2) is that if each row of a determinant is a geometric progression the determinant is the product of the elements in the first column and the Vandermonde determinant

$$\begin{vmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^{n-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1} \\ 1 & r_3 & r_3^2 & \cdots & r_3^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1} \end{vmatrix}$$

$$= (r_n - r_1)(r_n - r_2) \cdots (r_n - r_{n-1})(r_{n-1} - r_1) \cdots (r_2 - r_1)$$

Marion T. Bird
San Jose State College
San Jose, California

A Note on Modular Systems in the Plane

HAROLD C. TRIMBLE
Faculty, State College of Iowa

In recent years students of the curriculum in mathematics have made frequent use of modular number systems. From the "clock arithmetic" of the elementary school to the "finite fields" of modern algebra, sets of remainders from divisions of the integers by a fixed positive integer m have provided neat examples to illustrate several important mathematical structures.

In the light of this successful experience with modular systems in one dimension, it is surprising that nothing seems to have been said about the possibilities of modular systems in two dimensions. This note is written in the hope of stimulating interest in the extension of modular systems to two or more dimensions.

Think of a set G of ordered number pairs (a,b), $0 \leqq a < 6$, $0 \leqq b < 5$. These are the lattice points on or inside the rectangle of Figure 1.

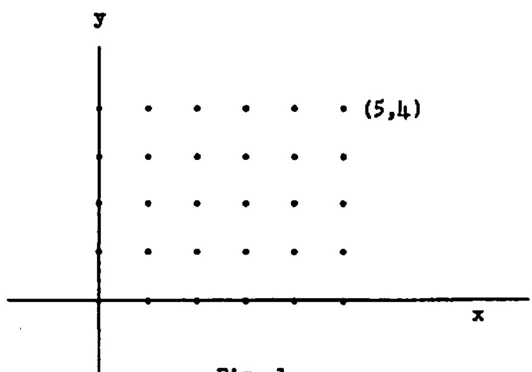


Fig. 1

Now apply the operation of vector addition to the elements of this set. Geometrically this amounts to using the parallelogram law to add two vectors. Figure 2 provides a reminder of the way to add $(2,3)$ and $(5,1)$. Physically the vector $(7,4)$ is the "resultant" of the vectors $(2,3)$ and $(5,1)$. Algebraically the equation

$$(2,3) + (5,1) = (7,4)$$

is an application of the definition

$$(a,b) + (c,d) = (a + c,d + d)$$

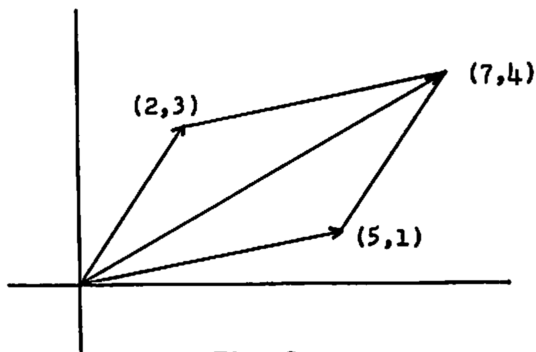


Fig. 2

of addition of vectors. The symbol **+**, a bold face plus sign, was chosen to avoid confusion with ordinary addition in a number system, yet preserve the idea that this is a generalization of ordinary addition.

To achieve closure under addition for the set G we now reduce each number pair by reducing the first component mod 6 and the second component mod 5. Thus (7,4) becomes (1,4) since $1 \equiv 7 \bmod 6$ and $4 \equiv 4 \bmod 5$. Geometrically this amounts to counting horizontally in multiples of 6 and vertically in multiples of 5 until a point on or inside the rectangle of Figure 1 is reached. Algebraically it amounts to defining equality of pairs as follows:

$$(a,b) = (c,d) \text{ if and only if } c = a + 6m \text{ and } d = b + 5n$$

where m and n are integers.

With these agreements it is easy to show that the system $(G,\mathbf{+})$ is an Abelian group. In fact, there is no difficulty in generalizing to moduli r and s to avoid the restriction to the moduli 6 and 5.

The reader should first check to be sure that vector addition is well defined relative to the given definition of equality of pairs. The proof is outlined below for those who are not well acquainted with this question of the uniqueness of an operation:

Suppose

$$(a,b) = (c,d) \text{ and } (e,f) = (g,h)$$

This means, by definition,

$$c = a + rm_1,\ d = b + sn_1,\ g = e + rm_2,\ h = f + sn_2.$$

Hence

$$\begin{aligned}(c,d) + (g,h) &= (c + g, d + h)\\ &= [a + e + r(m_1 + m_2), b + f + s(n_1 + n_2]\\ &= (a + e, b + f)\\ &= (a,b) + (e,f)\end{aligned}$$

In the previous steps liberal use was made of the familiar properties of addition and multiplication in the modular systems with the moduli r and s. The definitions of equality of pairs and of addition of pairs were also used. The conclusion is that "equals added to equals yields equal results," that is, that "addition of these pairs is well defined."

The reader should now verify the statements that follows:

(1) Closure is a consequence of the definition of equality of pairs, and the division algorithm.

(2) The associative property is a consequence of the associativity of systems of integers modulo r and s under addition.

(3) The identity element is $(0,0)$.

(4) The inverse of (a,b) is $(r - a, s - b)$, since $(a,b) + (r - a, s - b) = (r,s) = (0,0)$.

(5) The commutative property is a consequence of commutativity of systems of integers modulo r and s under addition.

The fact that $(G,+)$ is an abelian group is interesting but rather trivial. What is more exciting is to introduce a form of scalar multiplication of these vectors and to investigate the resemblance of the resulting system to a vector space.

First consider the set of I of integers, positive negative and zero, and, for $u \varepsilon I$, the product $u \bullet (a,b)$ defined as follows:

$$u \bullet (a,b) = (ua, ub)$$

This is the operation of scalar multiplication, denoted by the bold face symbol • to maintain the connection with the ordinary product, often symbolized by a raised dot. When the first and second components of the "product" vector are reduced modulo r and s, closure under scalar multiplication is obvious. That is,

$$\begin{aligned}ua &= q_1r + v_1 \qquad & 0 \leqq v_1 < r\\ ub &= q_2s + v_2 \qquad & 0 \leqq v_2 < s\end{aligned}$$

yield

$$(ua,ub) = (v_1,v_2)$$

where (v_1,v_2) belongs to G.

The reader is now invited to check the following properties of a vector space to see whether or not they hold for the system $\{G, +; I, +, \cdot; \bullet\}$:

Scalar multiplication is well defined.

(6) The set G is closed under scalar multiplication by elements of I.

(7) Two distributive laws hold, namely,

$$u \bullet [(a,b) + (c,d)] = u \bullet (a,b) + u \bullet (c,d)$$
$$(u_1 + u_2) \bullet (a,b) = u_1 \bullet (a,b) + u_2 \bullet (a,b)$$

(8) There is an associative law for scalar multiplication, namely,

$$(u_1 \cdot u_2) \bullet (a,b) = u_1 \bullet [u_2 \bullet (a,b)]$$

(9) There is an identity for scalar multiplication, namely, the integer 1, that is,

$$1 \bullet (a,b) = (a,b)$$

Once he assures himself that all of the nine properties hold in the system, he may conclude that it is a close relative of a vector space. It fails only in the fact that the set I is not a field. I does not contain the inverse for multiplication of each of its elements. For example, given the integer 3, there is no integer x such that $3x = 1$.

The maneuver of constructing a finite field from the set I by replacing equality of integers by congruence of integers modulo a prime integer p is well known. That is, the system

$F = \{0, 1, 2, \cdots, p - 1\}$ with addition and multiplication modulo p is a finite field. This raises the question as to whether the system $\{G, +; F, +, \cdot; \bullet\}$ is a vector space.

An investigation of Property (8), above, yields an intriguing result. This associative law does not hold in general for the system under study. For example, if $r = 6$, $s = 5$, and $p = 7$:

$$
\begin{array}{lll}
(3 \cdot 6) \bullet (2,3) & = 4 \bullet (2,3) & \text{(since } 18 \equiv \text{ mod } 7) \\
& = (2,2) & \text{(since } 8 \equiv 2 \text{ mod } 6, \text{ and} \\
& & \quad 12 \equiv 2 \text{ mod } 5) \\
3 \bullet [6 \bullet (2,3)] & = 3 \bullet (0,3) & \text{(since } 12 \equiv 0 \text{ mod } 6, \text{ and} \\
& & \quad 18 \equiv 3 \text{ mod } 5) \\
& = (0,4) & \text{(since } 0 \equiv 0 \text{ mod } 6, \text{ and} \\
& & \quad 9 \equiv 4 \text{ mod } 5)
\end{array}
$$

The fact that $(2,2) \neq (0,4)$ contradicts Property (8).

Now what if $r = s = p$? For example, what if r, s, and p are all replaced by the prime 5? A careful check will reveal that the system $\{G, +; F, +, \cdot; \bullet\}$ is a vector space when both r and s are equal to the prime p.

The systems in which the restriction $r = s = p$ is not made lack some of the other characteristics of the more familiar vector spaces. One might anticipate that there would be at least two linearly independent vectors in such a system. That is, that there would exist two elements (a,b) and (c,d) of the system such that

$$u_1 \bullet (a,b) + u_2 \bullet (c,d) = (0,0)$$

would require $u_1 = u_2 = 0$. Not so! Consider, for example, $r = s = 5$ and $p = 7$. Then

$$5 \bullet (a,b) + 5 \bullet (c,d) = (0,0)$$

for every choice of a, b, c, and d. If $r = s = 7$ and $p = 5$,

$$\begin{aligned} 2 \bullet (a,b) + 2 \bullet (c,d) &= 7 \bullet (a,b) + 7 \bullet (c,d) \\ &= (0,0) \end{aligned}$$

for every choice of a, b, c, and d.

If, however, $r = s = p$, then the vectors $(1,0)$ and $(0,1)$ are linearly independent. For example, if $r = s = p = 5$,

$$u_1 \bullet (1,0) + u_2 \bullet (0,1) = (u_1,u_2) \neq (0,0)$$

unless $u_1 \equiv u_2 \equiv 0 \bmod 5$.

The extension of these ideas to spaces of 3 or more dimensions, and the search for further modular systems in the plane is left to the reader. It remains only to repeat the purpose of this note, "to stimulate interest in the extension of modular systems to two or more dimensions."

There has not been any science so much esteemed and honored as this of mathematics, nor with so much industry and vigilance become the care of great men, and labored in by the potentates of the world, viz. emperors, kings, princes, etc.

—Benjamin Franklin

The Problem Corner

EDITED BY J. D. HAGGARD

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before October 1, 1963. The best solutions submitted by students will be published in the Fall, 1963, issue of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to J. D. Haggard, Department of Mathematics, Kansas State College, Pittsburg, Kansas.

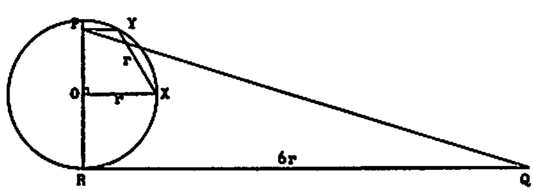
PROPOSED PROBLEMS

161. *Proposed by Ann Penton, State University of New York, Oswego.*

Express the difference between the squares of two positive integers, x and y, as a sum of $|x - y|$ odd integers.

162. *Proposed by J. F. Leetch, Bowling Green State University, Bowling Green, Ohio.*

In the June 1962 *Popular Science* appears the following construction for a segment approximating the length of the circumference of a given circle:



Construct XY of length r, PY parallel to OX, and RQ parallel to OX and of length $6r$. PQ is then "within a hair" of the circumference.

If a "hair" is assumed to be .001 in. wide, find the circles for which this approximation is correct.

163. *Proposed by V. E. Hoggatt, San Jose College, San Jose, California.*

If $x < y < z$ solve

$$\sin x + \sin y + \sin z = 0$$
$$\cos x + \cos y + \cos z = 0$$

164. *Proposed by Phil Huneke, Pomona College, Claremont, California.*

Find the positive integers, greater than one, for which the integer is equal to the sum of the cubes of its digits.

165. *Proposed by Fred W. Lott, Jr., State College of Iowa, Cedar Falls.*

Without using tables, determine which is larger, e^π or π^e.

SOLUTIONS

156. *Proposed by V. E. Hoggatt, San Jose State College, San Jose, California.*

Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $Q^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then show that for all natural numbers n, that $a + b + c + d$ is one of the Fibonacci numbers $1, 1, 2, 3, 5, \cdots$ where $F_1 = 1$, $F_2 = 1$, and $F_m = F_{m-1} + F_{m-2}$ for $m \geqq 3$.

Also show that $a^2 + b^2 + c^2 + d^2$ is one of the Lucas numbers $1, 3, 4, 7, \cdots$ where $L_1 = 1$, $L_2 = 3$, $L_m = L_{m-1} + L_{m-2}$ for $m \geqq 3$.

Solution by Norman Nielsen, Pomona College, Claremont, California.

Let $J_n = a + b + c + d$ where $Q^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then since $Q^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $J_1 = 3 = F_4$.

Similarly since $Q^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $J_2 = 5 = F_5$.

Now $Q^{n+1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a & b \end{pmatrix}$,

$J_{n+1} = (a+b+c+d) + (a+b)$, and $Q^{n+2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a+c & b+d \\ a & b \end{pmatrix}$

$$= \begin{pmatrix} (a+c)+a & (b+d)+b \\ a+c & b+d \end{pmatrix},$$

$J_{n+2} = (a+c)+a + (b+d)+b + (a+c) + (b+d) = J_{n+1} + J_n.$

Then if $J_m = F_x$ and $J_{m+1} = F_{x+1}$, then $J_{m+2} = F_{x+2}$. But for Q^1, $J_1 = F_4$ and for Q^2, $J_2 = F_5$; therefore by finite induction $J_n = F_{n+3}$ for all natural numbers n, and thus J_n is a Fibonacci number.

Let $K_n = a^2 + b^2 + c^2 + d^2$, then $K_1 = 1^2 + 1^2 + 1^2 + 0^2 = 3 = L_2$ and $K_2 = 2^2 + 1^2 + 1^2 + 1^2 = 7 = L_4$.

Now $K_{n+1} = (a+c)^2 + (b+d)^2 + a^2 + b^2$ and $K_{n+2} = (a+c)^2 + 2a(a+c) + a^2 + (b+d)^2 + 2b(b+d) + b^2 + (a+c)^2 + (b+d)^2 = 3K_{n+1} - K_n$.

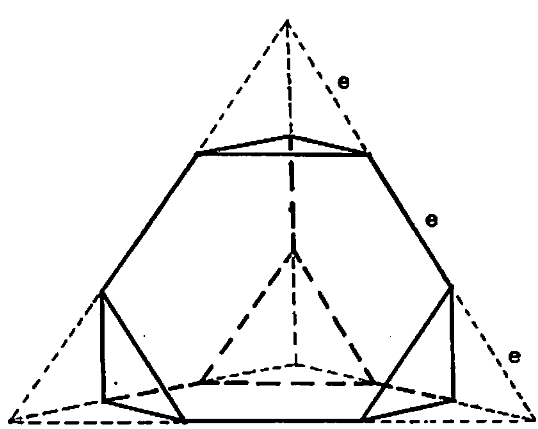
From the definition of Lucas numbers, $L_m = L_{m-1} + L_{m-2}$ or $L_{m-1} = L_m - L_{m-2}$ and $L_{m+1} = L_m + L_{m-1}$, $L_{m+2} = L_{m+1} + L_m$. Thus $L_{m+2} = 3L_m - L_{m-2}$. Then if $K_n = L_{m-2}$ and $K_{n+1} = L_m$, then $K_{n+2} = L_{m+2}$. But for Q^1, $K_1 = L_2$ and for Q^2, $K_2 = L_4$, therefore by induction $K_n = L_{2n}$ for any natural number n, thus K_n is a Lucas number.

Also solved by Phil Huneke, Pomona College, Claremont, California; Roger Richards, Westminster College, New Wilmington, Pennsylvania; John W. Torbett, Southern Methodist University, Dallas, Texas.

157. *Proposed by C. W. Trigg, Los Angeles City College.*

Four regular hexagons and four equilateral triangles constitute the faces of an octahedron. Find its volume in terms of an edge e.

Solution by Norman Nielsen, Pomona College, Claremont, California.



The figure is the large equilateral tetrahedron of edge $3e$, minus the four smaller equilateral tetrahedrons of edge e. Therefore the volume is

$$(3e)^3 \frac{\sqrt{2}}{12} - 4e^3 \frac{\sqrt{2}}{12} = \frac{23}{12} e^3 \sqrt{2}.$$

Also solved by Phil Huneke, Pomona College, Claremont, California; Ann Penton, State University of New York, Oswego; Roger Richards, Westminster College, New Wilmington, Pennsylvania.

158. *Proposed by Clinton L. Wood, Colorado State University, Ft. Collins.*

Solve the following cryptogram by finding the appropriate replacements for the x's.

```
                x x x x x
x x x  | x x x x x x x x
           x x x
           -----
             x x x x
               x x x
             -------
                 x x x x
                 x x x x
                 -------
```

Solution by Karen L. Smith, State University of New York, Albany.

We will make reference to the following diagram.

```
                _ 0 _ 0 _
          ____________________
_ _ _ _|  1 0 0
          _ 9 9 _ _ _ _ _
          -------
              1 0
              _ 9 9 _
              -------
                  _ _ _ _
                  _ _ _ _
                  -------
```

There are 0's in the second and fourth place of the quotient, since there are no multiplications shown for those places. The first digit in the dividend and in the remainder of the first subtraction is 1, because there is no remainder in the hundred's place after the subtraction of the 3-digit numbers from the 4-digit numbers. In the first subtraction only 100 - 99 would give 1. In the second subtraction only 10 - 9 gives 0 provided 100 is borrowed in the minuend. Then the second digit in the second subtraction must be 9 as in the first subtraction.

The divisor must be greater than 111, because 111 will not multiply by any digit and produce a four digit product as appears in the third multiplication.

Let d be the last digit of the first product, making it $99d$, where d may be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Then we get the following ten possibilities for $99d$: 990, 991, 992, 993, 994, 995, 996, 997, 998, 999. These 10 possibilities factor as indicated:

1 ×	(990,	991,	992,	993,	994,	995,	996,	997,	998,	999)
2 ×	(495,		496,		497,		498,		499	)
3 ×	(330,			331,			332,			333)
4 ×	(		248,				249			)
5 ×	(198,					199				)
6 ×	(165,						166			)
7 ×	(				142					)
8 ×	(		124							)
9 ×	(									111)

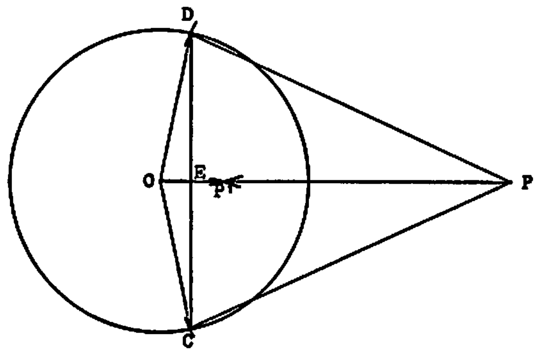
The 3 digit numbers shown are possible divisors, except 111, and the single digit is the first and third digit in the quotient. The last digit in the quotient can be any digit which when multiplied by the divisor yields a 4-digit number. There are 166 possible answers.

Also solved by S. F. Cooney, St. Meinrod College, St. Meinrod, Indiana; Phil Huneke, Pomona College, Claremont, California; Norman Nielsen, Pomona College, Claremont, California; Dale Oldham, Washburn University, Topeka, Kansas; Ann Penton, State University of New York, Oswego; Roger Richards, Westminster College, New Wilmington, Pennsylvania.

159. *Proposed by the Editor.*

Given a circle O of radius r and a point P outside the circle. With compasses construct the inverse of P. That is, find P' so that $OP \cdot OP' = r^2$.

Solution by Phil Huneke, Pomona College, Claremont, California.



Construction: With P as center and OP as radius draw an arc cutting the given circle in points D and C. Using D as center draw a circle with radius r. Using C as center draw a circle with radius r. These arcs intersect in O and in a second point P', which is the desired point.

Proof:

$$\overline{OP} = \overline{DP},\ \overline{OD} = \overline{OC} = r$$

$$\overline{OD}^2 = \overline{OP}^2 + \overline{DP}^2 - 2\overline{OP}\cdot\overline{DP}\cos(\angle DPO), \text{ by the cosine law.}$$

$$r^2 = 2\overline{OP}^2 - 2\overline{OP}^2\cos(\angle DPO)$$

or

$$\cos(\angle DPO) = \frac{2\overline{OP}^2 - r^2}{2\overline{OP}^2}$$

Segment DC will be perpendicular to OP at a point E. Then

$$\cos(\angle DPO) = \frac{EP}{DP} = \frac{OP - OE}{OP}.$$

Therefore

$$\frac{2\overline{OP}^2 - r^2}{2\overline{OP}^2} = \frac{\overline{OP} - \dfrac{r^2}{2\overline{OP}}}{\overline{OP}} = \frac{\overline{OP} - \overline{OE}}{\overline{OP}}.$$

Thus

$$\frac{r^2}{2\overline{OP}} = \overline{OE} = \tfrac{1}{2}\,\overline{OP'} \text{ and } \overline{OP'}\cdot\overline{OP} = r^2.$$

Also solved by Joseph B. Dence, Bowling Green State University, Bowling Green, Ohio; Robert Eldi, Hofstra College, Hempstead, New York; Roger Richards, Westminster College, New Wilmington, Pennsylvania.

Editors note: No one of the four solutions proved that points O, P′ and P are on a line.

160. *Proposed by Perry Smith, Albion College, Albion, Michigan.*

State a rule by means of which any repeating decimal can be written as the sum of two rational numbers.

Solution by Phil Huneke, Pomona College, Claremont, California.

Let x be a repeating decimal, k be the number of digits in the series which repeats, and p be the number of digits in x to the right of the decimal point and left of the first digit of the repeating series.

Then

$a = 10^p(10^k x - x)$ in an integer.

$$x = \frac{a}{10^p(10^k - 1)} = \frac{a}{2 \cdot 10^p(10^k - 1)} + \frac{a}{2 \cdot 10^p(10^k - 1)}$$

Since p is an integer and $p \geqq 0$; also k is an integer and $k \geqq 1$, thus $b = 2 \cdot 10^p(10^k - 1)$ is an integer. Thus $x = \frac{a}{b} + \frac{a}{b}$.

Also solved jointly by Wellington Engel and Robert Lohman, Kansas State College of Pittsburg; Norman Nielsen, Pomona College, Claremont, California; Andrea West, James Lick High School, San Jose, California.

Do not imagine that mathematics is hard and crabbed, and repulsive to common sense. It is merely the etherealization of common sense.

—W. THOMSON

The Mathematical Scrapbook

EDITED BY J. M. SACHS

I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery. (This is the concluding sentence in La Geometrie.)

—RENE DESCARTES

=△=

If $p = 1 + 2 + 2^2 + \cdots + 2^n$ is a prime then $2^n p$ is perfect, that is $2^n p$ is equal to the sum of all of its proper divisors. Let us examine a few cases.

$p = 1 + 2 = 3$; 3 is a prime; $2 \cdot 3 = 6$ is perfect.

$p = 1 + 2 + 2^2 = 7$; 7 is a prime; $2^2 \cdot 7 = 28$ is perfect.

$p = 1 + 2 + 2^2 + 2^3 = 15$; 15 is not a prime.

$p = 1 + 2 + 2^2 + 2^3 + 2^4 = 31$; 31 is a prime; $2^4 \cdot 31 = 496$ is perfect.

Can you make a general proof? Begin by asking for all proper divisors of $2^n \cdot p$ if p is a prime. Apply knowledge of geometrical progressions to sum of all divisors.

Cardan (1501-1576) suggested that all perfect numbers constructed as above must end in 6 or 8 and that there must be one such between any two successive powers of 10. What do you think about the truth of these suggestions? Can you prove them? (This method of constructing perfect numbers is called Euclid's Rule since it was included in his elements.)

=△=

A mathematician like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

—G. H. HARDY

=△=

Form any six digit integer by a repetition of a three digit pattern. This integer will always be divisible by 7, 11, and 13.

Examples:

531,531 = 7·75933, 531,531 = 11·48321, 531,531 = 13·40887.
174,174 = 7·24882, 174,174 = 11·15834, 174,174 = 13·13398.

Can you make a proof that any such six digit integer must be divisible by 7, 11, and 13?

=△=

It should be noted that a mathematical system as just described becomes merely an abstract form, and, as already implied, the same system may occur frequently as the underlying pattern in many diverse real and ideal situations. Thus there is validity in the assertion that the mathematician is less concerned with the solution of specific problems than he is with the development of general patterns that have widespread applicability in the study of particular situations.

—C. V. NEWSOM

=△=

Consider two digit integers with three digit squares. How many can you find such that the square when the digits are interchanged consists of the digits in the original square in reverse order? How could we attack such a problem? One clue lies in the fact that both squares are three digit integers. What limitations does this place on the two original digits? Is it also true the sum of the digits in either square is equal to the square of the sum of the digits in the original two digit number? If so, can you prove this must be so under our conditions?

=△=

The great notion of Group, . . . though it had barely merged into consciousness a hundred years ago, has meanwhile become a concept of fundamental importance and prodigous fertility, not only forming the basis of an imposing doctrine—the Theory of Groups—but therewith serving also as a bond of union, a kind of cerebrospinal system, uniting together a large number of widely dissimilar doctrines as organs of a single body.*

—C. J. KEYSER

* In case the reader doubts the arithmetic, let me hasten to add the date of this quotation is 1908.

=Δ=

A chain consists of heavy links. You wish to separate the chain into component links by cutting links, bending them at the cut, and thus sliding an uncut link free from a cut one. What is the minimum number of cuts needed for a chain of five links, the ends not fastened? How about six links? Does it change the problem any if the links form a loop with the formerly free ends fastened together? Suppose instead of single cuts we have a metal cutter which cuts both links where two are linked together. How does this change the picture?

=Δ=

It is clearly and unmistakably understood that in official interrogations, a spy always lies and a non-spy always tells the truth. You overhear a very deaf judge asking *A*,*B*,*C*, and *D* questions as follows:

J: (to *A*) Are you a spy?

A: (*softly*) No.

J: (to *B*) What did *A* say?

B: *A* said, "No."

J: (to C) What did *B* say?

C: (*softly*) *B* said that *A* said "Yes."

J: (to *D*) What did C say?

D: C said that *B* said that *A* said, "Yes."

On the basis of this much information can you identify each man as a spy or non-spy? Can you tell how many can be spies?

=Δ=

In some way or other, openly or hidden, even under the most uncompromising formalistic, logical, or postulational aspect, constructive intuition always remains the vital element in mathematics.

—R. Courant

=Δ=

In reading about Diophantus, your editor's attention was drawn to problems of the following type:

To find an integer which when added to 23 and 31 makes both sums squares:

$$x + 23 = n^2$$
$$x + 31 = m^2$$

Subtracting we get

$$8 = m^2 - n^2 \quad \text{or} \quad (m+n)(m-n) = 8.$$

We can find a solution by picking factors for 8. For instance we might choose $(m+n) = 8$ and $m-n = 1$ which would yield $m = 4.5$ and $n = 3.5$ and give us $x = -10.75$. On the other hand, we might choose $m+n = 4$, $m-n = 2$ since the only condition is that the product be 8. In this case we will get $m = 3$ and $n = 1$ which leads to the solution $x = -22$.

Under what conditions on the original integers will there be an integral solution for x? The reader might consider the possibility that m^2-n^2 is a prime or that it is a composite odd or a composite even. Under what conditions will x be positive? What would $x = 0$ mean?

=Δ=

To seek for proof of geometrical propositions by an appeal to observation proves nothing in reality except that the person who has recourse to such grounds has no due apprehension of the nature of geometrical demonstration. We have heard of persons who convince themselves by measurement that the geometrical rule respecting the squares on the sides of right-angle triangle was true; but these were persons whose minds have been engrossed by practical habits, and in whom speculative development of the idea of space had been stifled by other employments.

—W. Whewell

=Δ=

The advancement and perfection of mathematics are intimately connected with the prosperity of the State.

—Napoleon I.

=Δ=

Many arts there are which beautify the mind of man; of all other none do more garnish and beautify it than those arts which are called mathematical.

—H. Billingsley

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Harold E. Tinnappel, Bowling Green State University, Bowling Green, Ohio.

Elementary Technical Mathematics, F. L. Juszli and C. A. Rodgers, Prentice-Hall, New Jersey, 1962, 522 pp., $10.60.

Elementary Technical Mathematics is designed as a "careful, nonrigorous, practical, and graphic approach" to the mathematical needs of a first-year, post-secondary technical program. The teacher looking for a rigorous approach will be displeased with this text. However, from the viewpoint of a user of mathematics, at the level for which this book is intended, the authors have admirably succeeded.

As intended, this text is for technical programs. The authors introduce the idea of standard notation, i.e., scientific notation, for numerals in the first chapter and make use of the notation throughout. The ideas of significant digits and the use of the slide rule are well presented.

The numerous illustrations and solved examples will be quite effective in teaching the operations with "signed numbers". Applications are freely drawn from the various technical fields and will make the text more appealing to the student desiring a technical training. He will be able to see the answer to his inevitable questions, "What can this be used for?"

Chapter three on Dimensional Analysis should make life easier and more meaningful to the student who is asked to express his results in some recognizable unit. This chapter seems to be exceptionally well prepared and many beginning students of physics would do well to study it.

Since the equations of a line are presented before the work on trigonometry, the slope of a line is defined in terms of change in ordinate divided by change in abscissa, thus as rise/run; rather than as the tangent of the angle of inclination.

This reviewer was pleasantly surprised to find the answers to many of the problems printed with the problems, rather than in an answer section in the back. It is realized that many teachers may object to this feature, as it is rarely done in a beginning text.

The section on approximation of roots of an equation is well handled and is supported by a number of problems taken from engineering. Logarithms are introduced as an outgrowth of the work with exponents, thoroughly illustrated by diagrams and examples. Interpolation is illustrated by graphic means to show the linear interpolation as used and its relation to the true values of the function.

The authors have selected, from the many forms which might be used for the solution of oblique triangles, the sine law and the cosine law. However, these are introduced only after a careful presentation of many solutions using the right triangle. The right triangle has been used in resolution and composition to obtain results which might otherwise have been left to special formulas.

Graphs are used as needed. They are especially well presented in the section on periodic functions, so a student should understand the distinction between the amplitude and the period.

The authors show the engineering influence in the use of the letter j for $\sqrt{-1}$. The section concerning this operator, j, and its usefulness in vector applications, is well handled.

There are adequate tables, and the appendix contains a number of convenient illustrations for areas and volumes.

Any review of this book would be incomplete without reference to the extensive and well selected examples and exercises.

—John M. Burger
Kansas State Teachers College

Foundations of Analysis, Edward J. Cogan, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1962, 221 pp., $7.95.

As described in the preface, this book serves as an introduction to modern mathematics. It is designed for study by the teachers of secondary schools and others who wish to learn about the new approach to advanced mathematics.

The book contains three chapters which divide its subject matter into three parts. The first chapter discusses the use of logic and applies the notions here developed to sets and functions. The second chapter develops the real number system by the use of logic from Peano's axioms. The third chapter is concerned with real functions of one argument and introduces the reader to the ideas of continuity, derivative, and integral.

The second chapter is more formally organized than the others. The first chapter depends upon intuition for the introduction of new and perhaps unfamiliar concepts. By the time the reader has studied to the third chapter, he should be able, if desired, to supply the formalism which is omitted.

Numerous and extensive exercises are scattered throughout the text. These exercises serve to fix the concepts in the mind of the reader and to prepare the way for advanced material. In this way, the subject matter is well unified. The teachability of the book could be judged only after actual classroom experience, perhaps as the publishers suggest, in a National Science Foundation summer institute.

The reader of this text, however, should be aware of some seeming deficiencies. First, there are no answers at present to the shorter problems so that the reader who is attempting self-study is handicapped in finding how well he understands the material covered. Second, the notation, especially in the third chapter, is probably unfamiliar; hence the book would need to be studied in proper sequence; it would not serve as a ready reference book.

Third, the text, being a brief 200-odd pages, plunges into fundamental concepts very rapidly. This would require close attention and a certain mathematical maturity on the part of the student. It might even require supplemental study in more elementary texts.

Fourth, there are some minor errors such as wrong reference numbers. A more careful proofreading would have prevented the omission of letters from the middle of words. There is a wrong formula for the number of the n-ary connectives appearing on page 22 and in problem 12 on page 25.

The fifth point is most important and hence requires some extended discussion. The author follows the present trend in presenting his discussion of addition and multiplication of the natural numbers. This, the reviewer believes, is more fashionable than logical as will now be explained.

Every deductive system is composed of four discrete sets of elements. The first set is composed of the undefined terms; the second set is composed of those terms whose meaning is given by definitions involving the undefined terms and terms previously defined. The third set is composed of the assumptions, also labeled as axioms or postulates; these assumptions express relations between the defined and undefined terms and are taken without proof. The fourth set is composed of the theorems which express additional relations between these terms derived by the rules of deductive logic. The author is also following the modern trend when he labels all elements of the last two sets as theorems (p. 36), or true statements, and then distinguishes between the two types by whether they are accepted without proof or after proof.

Of the four sets above, only the set of definitions is not logically necessary. This set could be disposed with completely, yet all theorems stated in terms of the undefined terms alone would still hold true. This is the acid test for a definition. However, definitions are a great convenience in expression and an almost indispensible aid to thinking. Thus the need for definitions is human, not logical.

Now in common with most other writers, the author purports to define "addition" (p. 93) by two equations. This use of the word "define" is not consistent with its use as when we say that a triangle is defined as a polygon with three sides. Furthermore, if these equations are suppressed, then none of the theorems on addition can be derived. These facts should demonstrate beyond any doubt that these two equations are actually two assumptions about the properties of addition on the basis of which the other properties can be derived.

Pairs of statements similar to these are generally called inductive definitions. An accurate logical designation would label these statements as assumptions to which the deductive axiom could be applied. Similar so-called definitions appear throughout the modern treatment of the number system. Since modern mathematics places so much emphasis upon the logical development of a topic, is it asking too much that the names attached to all statements also be logically correct?

—CECIL G. PHIPPS
Tennessee Polytechnic Institute

A Primer of Real Functions, ed. by Ralph P. Boas Jr., The Mathematical Association of America, Quinn and Boden Company Inc., Rahway, New Jersey, 1960, 189 pp., $4.00.

This book is number 13 in a series of Carus Mathematical Monographs published by The Mathematical Association of America. The text is divided into two chapters. The first chapter covers sets and discusses such topics as countable and uncountable sets, metric spaces, compactness, convergence, and completeness to mention only a few. The exposition in the first chapter exceeds that of the second chapter, but in general it is good throughout the book.

The second chapter deals with functions. Partial contents include: continuous functions, sequences of functions, uniform convergence, derivatives, monotonic functions, and convex functions. The author takes time to construct examples, such as the Cantor Set or a continuous curve that passes through every point in a plane area, which are often mentioned by other sources but seldom discussed.

Throughout the book a different size type is used to indicate a more difficult proof or construction, or a feature not directly related to discussion. Definitions, theorems, and new terms are presented in italics when they are first stated in the text.

The list of exercises is a very good feature of this book. The exercises appear in the text as the occasion arises and vary from statements of theorems to requests for examples to illustrate the material just discussed. The answer to each exercise is provided in a list appearing in the back of the book for whatever use it may be needed.

The author states in the preface that the reader should have had at least a course in calculus. While this would seem adequate those with a much stronger background would still find it interesting. This book would be a valuable addition to the library of anyone interested in mathematics.

—JAMES S. BIDDLE
Bowling Green State University

Lie Algebras, Nathan Jacobson, Interscience Publishers, a division of John Wiley and Sons, New York, 1962, ix + 331 pp., $10.50.

In this extremely well-written book, Professor Jacobson has presented an interesting and significant body of material on Lie algebras, much of which was either unavailable before or was obtainable only to those with access to the many scholarly mathematics journals in which results have been published.

To the undergraduate reader of *The Pentagon*, a little background information on the subject of this book may be worth while. The term "Lie algebra" (pronounced like 'lee') was first introduced about thirty years ago by Herman Weyl at the Institute for Advanced Study, Princeton. The name honors the Norwegian mathematician, Marius Sophus Lie (1842-1899), who contributed much to the development of group theory. Although a handful of mathematicians began the study of Lie algebras before the turn of the century, much of the research in this area has been done within the last ten to fifteen years. Of the over 160 research papers listed in the excellent bibliography, at least three-fourths of them have been published since 1950.

What is a Lie algebra? The student will recall that to be a linear algebra or a non-associative algebra, the following rather extensive list of conditions must hold: There exists a set A of elements $(x, y, z, \cdots)$, called vectors, on which operations of addition and multiplication are defined such that A is an abelian group under addition, A is closed under multiplication, and such that the distributive laws hold. Further, there exists a field F whose elements $(a, b, c, \cdots)$ are called scalars. The scalars and vectors are brought together by means of an operation, called scalar multiplication, which is such that if $a \in F$ and $x \in A$, then $ax \in A$. Finally, for all $a,b \in F$ and $x, y \in A$, the following properties must be satisfied: $1x = x$ (where 1 is the unity of F), $(ab)x = a(bx)$, $a(xy) = (ax)y = x(ay)$, $a(x + y) = ax + ay$, and $(a + b)x = ax + bx$. If all the preceding conditions hold, then A is said to be a non-associative algebra over F. A Lie algebra is a non-associative algebra A for which the two additional properties hold: $x^2 = 0$, and $(xy)z + (yz)x + (zx)y = 0$, for all $x,y,z \in A$. From the fact that $(x + y)^2 = 0$, it immediately follows that $xy = -yx$, and hence that a Lie algebra is necessarily non-commutative.

A graduate student who uses *Lie Algebras* as a textbook should have previously had at least two or three semester courses in abstract algebra, including a course in linear algebra, all of the "unwatered-down" variety. Most undergraduates lack the mathematical preparation and maturity necessary to cope with a book of this caliber, but the exceptional senior could gain a general understanding of the subject and increase his vocabulary considerably by studying the first chapter, pp. 1-30, on basic concepts.

The concepts introduced in Chapter I are necessary for the structure theory presented in Chapters II-IV, these chapters being concerned with solvable and nilpotent Lie algebras, Cartan's criterion and its consequences, and split semi-simple Lie algebras. Chapter V, on universal enveloping algebras, introduces the concepts necessary for an understanding of the representation theory developed in Chapters VI-VIII. These latter chapters present the theorem of Ado-Iwasawa, classification of irreducible modules, and characters of the irreducible modules. The material of Chapter IX, on automorphisms of semi-simple Lie algebras over an algebraically closed field of characteristic zero, is applied in Chapter X to the problem of classifying the simple Lie algebras over an arbitrary field of characteristic zero. Each of the ten chapters concludes with a set of challenging exercises.

The richness of results and the clarity of presentation found in *Lie Algebras* will, no doubt, encourage many mathematics departments to introduce a graduate course in Lie algebras. The book is certainly a welcome addition to the library of any advanced mathematics student who is seriously interested in abstract algebra.

—VIOLET HACHMEISTER LARNEY
State University of New York
at Albany

Transmission of Information, Robert M. Fano, The Massachusetts Institute of Technology Press and John Wiley and Sons, Inc. (440 Park Avenue South) New York 16, 1961, 389 pp., \$7.50.

This book is written for graduate students and engineers who are interested in electrical communications and who are well versed on probability theory. There is an emphasis upon the points of view and methods of analysis which are likely to prove useful to these people in their future work.

The book is concerned with the branch of communication theory that is based upon the work of Claude Shannon and is sometimes called "Information Theory". The scope of the book is indicated by its chapter headings: the transmission of information, a measure of information, simple message ensembles (an ensemble is a space to which a probability distribution is assigned), discrete stochastic sources (ergodic, Markov), transmission channels (discrete, constant, time-continuous, Gaussian), channel encoding and decoding, encoding for binary symmetric channels, multinomial distributions, and encoding for discrete, constant channels.

The dependence upon a previous knowledge of probability theory is extensive and provides the basis for the development of special mathematical techniques in Chapter 8. The dependence upon the reader's knowledge of physics is minimal and should not deter anyone who has mastered the extensive mathematical prerequisites. Basically the book is concerned with a statistical theory of communications and is so subtitled. The primary emphasis is upon the use of statistical theories. Many theorems are developed relative to the theory of communications. At the end of the book there are a total of eighty suggested problems for the various chapters. This is a scholarly book developing a very interesting application of statistics and intelligible only to those who have previously done extensive work in statistics.

—BRUCE E. MESERVE
Montclair State College

Mathematical Discovery, Vol. I, George Polya, John Wiley & Sons, Inc. (440 Park Avenue South), New York 16, 1962, 216 pp., $4.75.

In the preface to this text the author states: "The preparation of high school mathematics teachers is insufficient." He goes on to assert that the "foremost duty of the high school in teaching mathematics is to emphasize methodical work in problem solving." The teacher must impart "know-how" to his students; he must show them how to solve problems.

This text, the first of two volumes, is an attempt to develop problem solvers and to improve the preparation of high school mathematics teachers. To this end, part of the book deals with a variety of patterns which may be effectively used as models that can be imitated in solving similar problems.

The book is divided into two parts. In Part One the reader is exposed to a variety of these patterns. Thus we find patterns dealing with loci, equations, the concept of recursion, etc. This first part also contains an excellent set of interesting problems which are difficult but which require very little knowledge beyond that of high school mathematics. For many of these the solution is given in detail, with emphasis on patterns useful for other situations. For others, the reader is given direction or hints and allowed to proceed to discover a solution on his own.

Part Two contains two chapters and is to be completed in Volume II of this series. The two chapters included are part of a section entitled "Toward a General Method" and represent the beginning of a search for a general method of solution applicable to all sorts of problems, an aim which the author recognizes as ambitious but "quite natural: although the variety of problems we may face is infinite, each of us has just one head to solve them, and so we naturally desire just one method to solve them." The reader will have to await the publication of Volume II to see this phase of problem solving completed.

Those who are familiar with Polya's earlier works, *How to Solve It* and *Mathematics and Plausible Reasoning,* will welcome this latest contribution. These various texts complement one another and provide rich insights into the whole area of mathematical discovery.

Although the book is designed primarily for current and prospective high school mathematics teachers, it is worth while reading for all serious college students of mathematics as an excellent aid to the study of the solution of problems.

—MAX A. SOBEL
Montclair State College

So You're Going to be a Teacher, Robert L. Filbin and Stefan Vogel, Barron's Educational Series, Inc. (343 Great Neck Road), Great Neck, New York, 1962, 138 pp., $1.25.

Mr. Filbin is a principal in a non-graded elementary school and Mr. Vogel, at the time of writing, had just finished his first year of teaching in the same school. As a team, they are able to appreciate the problems of the beginning teacher from the view-points of the new teacher and the administrator.

This book gives practical suggestions to the student in teacher-training in regard to checking on certification requirements for the state in which he wishes to teach. General principles for smoothly conducting sessions the first day, for attaining good home-school relations, and for working with supervisors and colleagues should prove worth-while for the beginning teacher. Wisely enough, the book does not pretend to give all the answers. Young teachers are assured that supervisors are happily available for consultation and assistance when problems arise.

—ESTHER D. KRABILL
Bowling Green State University

Classics in Logic, ed. by Dagobert D. Runes, Philosophical Library, Inc., New York, 1962, 818 pp., $10.00.

This is a big book, exceeding eight hundred pages and there is something for everybody. As in any anthology, the careful reader will miss some of his favorites, either favorite authors or favorite passages from authors who are represented. This, however, cannot be accounted a fault, since any editor has the responsibility of exercising judgement. Runes' choices indicate both breadth and depth in his knowledge of the literature of logic and of epistemology.

In the opinion of this reviewer, a chronological sequence rather than an alphabetical one would have enhanced the usefulness of the book. Such an organization would make it easier to trace the influence of each philosopher on his successors. It would also help us to understand the relations to philosophy of the external events contemporary with the writers studied and to consider mutual interaction. An expansion of the short notes prefacing each selection also would be helpful if more information as to the relations among the writers were supplied, especially where they take antogonistic positions. A treatment of the role played by Aristotle's Law of the Excluded Third would be of value, particularly with regard to sorting out modern writers as to their Aristotelean or non-Aristotelean posture. Mathematicians might like to see something from the neo-intuitionist school of Brouwer. The mutual impact of the computer and computer-related devices with symbolic logic seems to be neglected.

On the whole, this book will be of value to the student of logic and epistemology, furnishing a lead to more concentrated study of those authors who are found to be of interest to a particular individual.

—F. C. Ogg
Bowling Green State University

Treasury of World Science, ed. by Dagobert D. Runes, Philosophical Library, Inc., (15 East 40th Street) New York 16, New York, 1962, xxii + 978 pp., $15.00.

Emphasis in this volume, a philosophically oriented anthology of classic papers, is on the scientists and pioneers who forged new paths in their particular fields. The importance of the classics—even in science—is stressed by Dr. Wernher von Braun in his all too brief three page introduction when he says that classics "provide light in the study of science."

The book consists of excerpts from papers of 99 renowned scientists, covering such areas as physics, chemistry, mathematics, geology, medicine, and others. Inspection shows that the largest number of excerpts comes from the area of physics; papers by Ampere, Archimedes, Einstein, Maxwell, Newton, and Schrodinger are included. The next largest number of papers comes from the field of chemistry; excerpts from the works of Mendeleev, Ostwald, Lavoisier, Priestley, and von Baeyer are included. Biology includes such pioneers as Beaumont and Vesalius; medicine such pioneers as Lamarck and Lister.

My greatest criticism of the book is the fact that too few papers by pioneers in mathematics are included. The only excerpts included in this book are a discussion of pure mathematics by Bertrand Russell and portions from Euclid's *The Elements,* including his proof of the Pythagorean theorem. Gauss is relegated to a discussion of the theory of the motion of heavenly bodies, and such notables as Euler, Cauchy and the Bernoullis are conspicuously missing.

In view of the fact that the editor's purpose is to acquaint the reader with the philosophical implications of the selections which are included, rather than to introduce the reader to a maze of scientific knowledge, I would recommend the book as enjoyable "armchair reading."

—Joseph B. Dence
Bowling Green State University

Forces and Fields, Mary B. Hesse, Philosophical Library, Inc., 15 East 40th Street., New York 16, N.Y., 1961, 318 pp., $10.00.

One of the attributes of an educated person is a knowledge of the history of ideas. To be a true scholar and student of science and mathematics one must certainly know, at least in broad outline, the history of scientific thought which has preceded him. Even the scientific layman of today has become so science-oriented that he tends to accept a principle or law of science without knowing its origin. He doesn't question the thought processes which covered hundreds of years and eventually led some genius to the insight necessary to state an adequate description of a physical phenomenon which we now call a law or principle of nature.

The author has sub-titled this book *The Concept of Action at a Distance in the History of Physics.* This sub-title is indeed appropriate, for the author has traced for us, through the history of scientific thought, one of the most mystifying and least understood phenomenon in science. Action at a distance formed a part of the over-all attempt of the ancient Greek natural philosophers to describe the world as they observed it. While these philosophers were not "scientists" in the 20th century meaning of the word, their thinking was not abandoned by later scientists. Instead, it was modified, tested, and elaborated. It led 17th century genius into the beginning of our modern scientific thinking.

The first two chapters of the book discuss the logical status of physical theories and primitive analogies. The reader is thus given a background into the approaches which physical theories might take and an idea of how models have been used to portray these theories. This is the attempt which one makes to describe natural phenomena in terms of analogies drawn from other processes which are more familiar and felt to be better understood.

Chapters III and IV treat at some length what, in hindsight, could be considered the beginning of scientific thought processes. Pre-seventeenth century natural philosophers concerned themselves with attempts to describe how things which they observed fitted into a system. While their arguments could be considered metaphysical or even mythological, nevertheless, they were the beginning.

Chapters V, VI, VII, and VIII are devoted to a careful historical account of what might truly be called the scientific method of investigation which began during the 17th century. These chapters specifically treat the development of the concept of action at a distance and field theory as they were proposed by many of the famous scientists and mathematicians whose names we recognize, and whose works we study, today.

Chapter IX is concerned with the theory of relativity as it was developed to describe the results of the Michaelson-Morley experiment. Chapter X is entitled "Modern Physics" and leads the reader to an appreciation of the vast amount of thinking which has preceded the apparent revelations of modern science.

Chapter XI is devoted to the metaphysical framework of physics and the book ends with two mathematical appendices and a bibliography.

Among the books which are concerned with the history of science *Forces and Fields* seems to fill an extremely important position. An undergraduate student in science and mathematics usually cannot hope to be at the same time a thorough student in the history (and philosophy) of science and mathematics. *Forces and Fields* is not an exhaustive history of the subject, but an intermediate step which could be the goal of any serious student. This book is not to be considered 'light' reading. A background of some study in physics would be desirable if one is to appreciate the significance of the work.

—Edgar B. Singleton
Bowling Green State University

Kappa Mu Epsilon News

EDITED BY FRANK C. GENTRY, HISTORIAN

Alabama Beta, Florence State College, Florence.

At the Coffee sponsored by KME during Homecoming we had representatives from fifteen years of chapter history. We initiated 19 new members in March bringing our total membership to 370. Mr. James Hooper, a former member of our faculty, spoke on "The Role that Calculators Can Play in Industry." Dr. W. C. Royster, of the University of Kentucky, representing the Mathematical Association of America spoke to us on "The Modern Aspects of Mathematics".

Alabama Delta, Howard College, Birmingham.

Dr. Wimberly Royster, visiting lecturer for the Mathematical Association of America, was guest speaker at our initiation banquet. His subject was "Opportunities in Mathematics."

California Alpha, Pomona College, Claremont.

We used guest lecturers for our meetings this year with the lectures open to the public.

California Gamma, California State Polytechnic College, San Louis Obispo.

Together with our Mathematics Club we are sponsoring a series of lectures on the Trachtenberg Speed System of Arithmetic. The lectures are attended by many teachers and parents. We will also help sponsor a statewide high school mathematics contest as a part of the spring open house on the campus.

Colorado Alpha, Colorado State University, Fort Collins.

Our chapter is raising its standards for admission to membership. New members will be required to have completed the calculus, to have had one mathematics course beyond the calculus, to have a grade-point average of 3.0 out of a possible 4 in all mathematics courses and an overall grade-point average of 3.0. We are helping to develop a mathematics club for students not eligible for membership in KME. We plan to be represented at the Convention.

Florida Alpha, Stetson University, De Land.

We expect to initiate twenty or more new members this spring. We are planning a field trip to Cape Canaveral.

Illinois Alpha, Illinois State Normal University, Normal.

Our eighty members are looking forward to being hosts to the Fourteenth Biennial Convention, April 8-9, 1963.

Illinois Beta, Eastern Illinois University, Charleston.

Our chapter plans to give a prize in honor of the late Professor Lester VanDeventer to be given annually to an outstanding calculus student.

Illinois Delta, College of St. Francis, Joliet.

Our meetings this year have been coordinated with a seminar sponsored by the Argonne National Laboratory of Lemont, Illinois.

Indiana Alpha, Manchester College, North Manchester.

For one of our programs this year we had a representative of the Lincoln Life Insurance Company of Fort Wayne speak on the mathematical opportunities in the insurance field.

Iowa Beta, Drake University, Des Moines.

Professor Lawrence O'Toole, of the College of Business Administration spoke to us on "A Comparison of Calculus of Finite Differences with Calculus of Limits". We visited the Computing Center of Pioneer Hybrid Company of Des Moines.

Kansas Alpha, Kansas State College of Pittsburg, Pittsburg.

The Robert Miller Mendenhall memorial award for outstanding seniors in the field of mathematics was received by Joe Jenkins and William Livingston. Each was presented a KME pin. We initiated 36 new members this year.

Kansas Beta, Kansas State Teachers College, Emporia.

We initiated 29 new members this year. Members are learning to use the IBM 1620. About twenty-two students and faculty members plan to use the college bus to attend the Convention at Normal.

Three of our members, Margaret Leary, Patricia Swope, and Collette Chang are engaged in undergraduate research sponsored by the National Science Foundation. Margaret Leary will read a paper on "Topological Spaces" at the Kansas Sectional Meeting of the Mathematical Association of America. Dr. Kenneth May, of Carleton College, will be our guest in April. He will speak on "Hamilton—His Life and Work."

Michigan Beta, Central Michigan University, Mt. Pleasant.

We hope to have three or four cars going to the National Convention this year.

Mississippi Gamma, University of Southern Mississippi, Hattiesburg.

Mr. Larry Parks, Base Mathematics Services Laboratory, Eglin Air Force Base, was our guest this year. He spoke on "Real Time Data Reduction."

Missouri Alpha, Southwest Missouri State College, Springfield.

Dr. Lawrence E. Pummill, beloved Professor Emeritus of Mathematics and former Head of the Department, died January 26, 1963. Dr. Pummill was a charter member of our chapter and had served as National Treasurer of Kappa Mu Epsilon.

Missouri Beta, Central Missouri State College, Warrensburg.

We had as a guest speaker this year, Miss Blanche Longshore, Helping Mathematics Teacher, Kansas City School System, who spoke on "Teaching in Kansas City." We are planning to be represented at the Convention.

Missouri Zeta, Missouri School of Mines and Metallurgy, Rolla.

Dr. Henry C. Thacher, Jr., of Argonne National Laboratory, was guest speaker at our fall initiation banquet. He spoke on the subject, "Are Numerical Analysts Necessary?" At our spring initiation we expect to have Dr. John Olmsted, of Southern Illinois University. His subject is to be "Space Filling Curves." We will be represented at the Convention.

Nebraska Beta, Nebraska State Teachers College, Kearney.

We are sending five members to the National Convention.

New Jersey Beta, Montclair State College, Montclair.

Our chapter is sending all of its student officers and two other members to the National Convention. Professor Paul C. Clifford spoke to us on his experiences teaching the course, "Probability and Statistics" on "Continental Classroom". His subject was "Be Prepared".

New York Alpha, Hofstra University, Hempstead.

The president of our chapter last year has a graduate fellowship in chemistry at M. I. T. this year.

New York Beta, State University of New York at Albany.

We have initiated 45 new members this year. One member of our delegation to the National Convention is submitting a paper.

New York Gamma, State University College, Oswego.

Our members are planning on traveling to the National Convention by car so as to take as many delegates as possible.

New York Epsilon, Ladycliff College, Highland Falls.

Our members participate in the publication of an annual, Tau Mu Gamma, and in a newsletter, The Delta Function, which is published three times each year. We sponsor symposia for the benefit of teachers and students of the surrounding area. Our Mathematics Fair will be extended to include junior high school students this year. A large group of members and non-members is planning on attending the National Convention.

Ohio Alpha, Bowling Green State University, Bowling Green.

We have initiated 44 new members in the past year. Our chapter holds help sessions, with 2 members furnishing the help, one night each week. These sessions have an average attendance of 15 students and seem to be quite successful.

Ohio Delta, Wittenburg University, Springfield.

Mr. E. L. Godfrey, of Wright-Patterson Air Force Base, spoke to us on "Matrices in Statistics". We initiated 13 new members this spring.

Ohio Gamma, Baldwin-Wallace College, Berea.

All of our programs are given by student members this year.

Oklahoma Alpha, Northeastern State College, Tahlequah.

In recognition of the founding of the society on April 18, 1931, our chapter holds a Founders Day banquet each year. Many former members return each year for this event.

Pennsylvania Gamma, Waynesburg College, Waynesburg.

Our fall meetings were given over to student papers this year. We initiated fourteen new members in November.

Tennessee Beta, East Tennessee State University, Johnson City.

Mr. Robert Brown, Statistician at Tennessee Eastman Company, spoke to us on "Applications of Statistics to Quality Control."

Papers by outstanding mathematics students will be presented at the time of the initiation banquet in the spring.

Texas Epsilon, North Texas State University, Denton.

Our chapter initiated sixteen new members in November. We had several joint meetings with the Student Section of the American Institute of Physics and the Astronomy Club. Professor Paul D. Minton, Director of the Department of Experimental Statistics at Southern Methodist University described SMU's graduate program in statistics.

Virginia Beta, Radford College, Radford.

Each of our members spends one hour one afternoon each week giving special assistance to mathematics students. We feel that we greatly benefit from this service.

Wisconsin Alpha, Mount Mary College, Milwaukee.

Mr. John Bruce, Manager of Industrial Engineering and Quality Control at Kearney and Trecker, Milwaukee, spoke to us on "Mathematics in Industry." Mr. LeRoy Dalton, Past President of Wisconsin Mathematics Council, spoke on Mu Alpha Theta, national high school and junior college mathematics club. Ten of our members are attending a series of lectures on modern mathematics given by professors from Marquette University and the University of Wisconsin. We expect about 150 contestants for our annual mathematics contest for high school students in April.

Mathematics, once fairly established on the foundation of a few axioms and definitions, as upon a rock, has grown from age to age, so as to become the most solid fabric that human reason can boast.

—Thomas Reid