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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Cryptography with Matrices*

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Cryptography is an art that dates back to the time when people first learned to read. Down through time people have developed ciphers they hoped were undecipherable; at least until the messages had gone through and the orders had been carried out. Cryptographers like to claim that there is no such thing as an insoluble cipher. But Roger Bacon, in the early Middle Ages, wrote a manuscript in a cipher that has yet to be broken. Fletcher Pratt in *Secret and Urgent,* says, "It is extremely probable that an insoluble cipher could be produced by mathematical means today. This is true, however, only if the production of an insoluble cipher and the recording of some relatively short message in it were the only end in view. All ciphers in actual use break down on repetitions."[1]

This paper is an attempt to show that matrices can be used to change the frequency that letters of the alphabet appear in a cryptogram. With a change in the frequency, the decipherer, or the enemy, will not be able to use the elaborate frequency tables to any great extent.

First a few definitions are necessary. Ciphering, or cryptography, is the process of expressing words, that convey an idea to everyone, in symbols that convey an idea only to the few persons who share the secret. The clear is the message before ciphering. A message written in cipher is called a cryptogram. In a substitution cipher, letters of the clear are replaced by letters, figures or symbols; in a transposition cipher, the letters remain the same as in the clear, but are shuffled according to a prearranged pattern. Breaking a cipher is discovering the system by which it was composed. Frequency tables are tables showing the relative frequencies of letters, pairs of letters, triplets, syllables, and words in normal text.

The average length of English words is 4.5 letters per word. When one message, or two or three messages written with the same key have two hundred or more letters, a reliable comparison can be made between the frequency of the letters or symbols in the crypto-

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* A paper presented at the 1961 National Convention of KME and first place by the Awards Committee.

1 Fletcher Pratt, Secret and Urgent (The Bobbs-Merrill Company, New York, 1939), p. 15.

TABLE I[2]

Frequency of Occurrence of Letters in English

Letter	Frequency of occurrence in 1000 letters
1. *E*	131.05
2. *T*	104.68
3. *A*	81.51
4. *O*	79.95
5. *N*	70.98
6. *R*	68.32
7. *I*	63.45
8. *S*	61.01
9. *H*	52.59
10. *D*	37.88
11. *L*	33.89
12. *F*	29.24
13. *C*	27.58
14. *M*	25.36
15. *U*	24.59
16. *G*	19.94
17. *Y*	19.82
18. *P*	19.82
19. *W*	15.39
20. *B*	14.40
21. *V*	9.19
22. *K*	4.20
23. *X*	1.66
24. *J*	1.32
25. *Q*	1.21
26. *Z*	.77

2 Pratt, op. cit., p. 252.

gram and the frequency of the letters in the alphabet. From this comparison, a decipherer can break the cipher, assuming the frequency has not been changed.

Matrix multiplication will be used in an attempt to change the frequency of the letters in a message. If the frequency is changed, frequency tables will be of little use and the decipherer will have to turn to a different method of breaking the cipher.

Before a message can be transformed by matrix multiplication, numbers will have to be substituted for the letters in the clear; numbers in the clear correspond to themselves in the substitution cipher. It is desirable to work in modular arithmetic and to choose a prime number for the modulus; a prime modulus implies that non-zero elements have multiplicative inverses. 23 has been chosen because only letters of the alphabet and numbers appear in the message to be ciphered; if number substitutes for symbols were necessary, 29 or some greater prime number could be used. *Q* and *U* will be represented by the same number because *Q* is always followed by *U*. *X*, *Y*, and *Z* will be represented by the same number because they are the last three letters of the alphabet. The substitution cipher is

A	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Q,U	*R*	*S*	*T*	*V*	*W*	*X,Y,Z*
17	18	19	20	21	22	0 .

A message ciphered by the above substitution cipher is

C	*R*	*Y*	*P*	*T*	*O*	*G*	*R*	*A*	*P*	*H*	*Y*	*W*	*I*	*T*	*H*
3	18	0	16	20	15	7	18	1	16	8	0	22	9	20	8

M	*A*	*T*	*R*	*I*	*C*	*E*	*S*
13	1	20	18	9	3	5	19.

The message will now be further ciphered by matrix multiplication. The number substitutes are broken up into groups of 3 and then transformed by multiplying each triple by a 3 x 3 matrix. Any integer greater than 23 can be reduced to its equivalent modulo 23 by taking the positive remainder after dividing by 23. Thus, modulo 23,

$$23 \equiv 0,\ 24 \equiv 1,\ 25 \equiv 2, \cdots, 45 \equiv 22,\ 46 \equiv 0, \cdots,$$
$$200 \equiv 16 \cdots.$$

A matrix is a rectangular array of elements chosen from some field. The field of integers modulo 23 will be used to form the ciphering matrices. The 3 x 3 matrix chosen is

$$A = \begin{pmatrix} 17 & 11 & 13 \\ 9 & 0 & 20 \\ 2 & 12 & 21 \end{pmatrix}.$$

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is the general ciphering matrix and

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is the general triple from the message to be transformed,

the new triple is formed by ordinary matrix multiplication.

$$AX = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix} =$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = Y.$$

Multiplying the first triple of the above example by the chosen 3x3 matrix, the triple is transformed into the first triple of the cryptogram to be sent.

$$\begin{pmatrix} 17 & 11 & 13 \\ 9 & 0 & 20 \\ 2 & 12 & 21 \end{pmatrix} \begin{pmatrix} 3 \\ 18 \\ 0 \end{pmatrix} = \begin{pmatrix} 51 + 198 + 0 \\ 27 + 0 + 0 \\ 6 + 216 + 0 \end{pmatrix} = \begin{pmatrix} 249 \\ 27 \\ 222 \end{pmatrix} \equiv \begin{pmatrix} 19 \\ 4 \\ 15 \end{pmatrix}.$$

Each triple is transformed in like manner. If the last triple of the message lacks one or two elements, zeros are added to complete the triple. Using the substitution cipher in reverse, letters are substituted for the numbers. The cryptogram is now ready to be sent and reads

SDO TGL HNL OFM TXT PZI KOH HPE.

The steps in the ciphering system can be put together for easy comparison. The first row is the clear; the second row is the substitution cipher; the third row is the message transformed by matrix multiplication; the fourth row is the cryptogram to be sent.

C	*R*	*Y*	*P*	*T*	*O*	*G*	*R*	*A*	*P*	*H*	*Y*	*W*	*I*	*T*	*H*
3	18	0	16	20	15	7	18	1	16	8	0	22	9	20	8
19	4	15	20	7	12	8	14	21	15	6	13	20	0	20	16
S	*D*	*O*	*T*	*G*	*L*	*H*	*N*	*L*	*O*	*F*	*M*	*T*	*X*	*T*	*P*

M	*A*	*T*	*R*	*I*	*C*	*E*	*S*
13	1	20	18	9	3	5	19
0	9	11	15	8	8	16	5
Z	*I*	*K*	*O*	*H*	*H*	*P*	*E*

The number 0 appears in the third row twice and is substituted in the fourth row by X the first time and Z the second time. The number 0 represents X, Y, and Z in the substitution cipher so X, Y, or Z can be used in the reverse process. The first *R* in the message was transformed into a *D*, the second into a *N*, and the third into an *O*; the first *T* was transformed into a *G*, the second into itself, and the third into a *K*. Comparisons for the rest of the letters can be made in like fashion. From this brief comparison, it is obvious that if *A* appears two times in the clear, there is no guarantee that it will appear two times in the cryptogram sent; it may appear more times as well as fewer times. It is apparent that frequency tables will be of little use in breaking this cipher.

The receiver of the message must be able to decipher the message. There is an identity matrix, I, for multiplication such that $IX = X$. This identity matrix is

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If the determinant of A is not zero, there is another matrix, A^{-1}, that can be found such that $A^{-1}A = I$. Then A^{-1} can be used to transform Y back to X since

$$A^{-1}Y = A^{-1}(AX) = (A^{-1}A)X = IX = X.$$

The calculated multiplicative inverse for the above 3x3 matrix is

$$A^{-1} = \begin{pmatrix} 4 & 7 & 4 \\ 9 & 1 & 11 \\ 12 & 13 & 12 \end{pmatrix}.$$

The cryptogram can now be deciphered back to the clear. First the substitution cipher is used to transform the letters into numbers. Next each triple is transformed by matrix multiplication. Transforming the first triple Y by A^{-1}

$$\begin{pmatrix} 4 & 7 & 4 \\ 9 & 1 & 11 \\ 12 & 13 & 12 \end{pmatrix}\begin{pmatrix} 19 \\ 4 \\ 15 \end{pmatrix} = \begin{pmatrix} 72 + 28 + 60 \\ 171 + 4 + 165 \\ 228 + 52 + 180 \end{pmatrix} = \begin{pmatrix} 164 \\ 340 \\ 460 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ 18 \\ 0 \end{pmatrix}.$$

Finally, using the substitution cipher to transform the numbers into letters, the decipher can now be read. It is now desirable to cipher a longer message for more accurate frequency changes.

The following paragraph from *Secret and Urgent* has been chosen as the message to cipher, using a 2x2, a 3x3, and a 4x4 matrix.

> The year 1880 is a key-date in the history of ciphers. The experience of the Franco-Prussian and the Russo-Turkish Wars had now confirmed what the American Civil War had foreshadowed—that an age of mass armies had come, in which it would no longer be possible for the general to keep his battle under observation and to control its course by aides carrying word-of-mouth orders. He must work from the map, and map strategy demands communications fast as lightning, fast as the electric telegraph, and secret as the grave. Ciphers had been raised from the status of something a soldier could have with advantage to something he must have.[3]

The process of ciphering the paragraph, using a 2x2, a 3x3, and a 4x4 matrix, is similar to the above described process using a 3x3 matrix. Any non-singular matrix may be used. The three chosen for the system and their multiplicative inverses are

3 Pratt, op. cit., p. 201.

TABLE II

Frequency of Occurrence of Letters
in the Clear
and in the Cryptograms

Letter or Number	Clear	2x2 Matrix	3x3 Matrix	4x4 Matrix	Expected in 516 Letters
E	60	27	19	25	68
A & 1	50	20	21	20	42
T	44	22	23	23	54
S	39	17	9	22	31
O & 0	39	19	34	30	41
R	37	33	21	23	35
H & 8	37	17	25	21	27
I	33	18	18	15	34
N	28	19	23	23	37
D	23	14	29	16	19
C	18	32	26	22	14
M	17	16	18	18	13
Q & *U*	14	18	23	25	13
F	13	12	26	23	15
G	12	12	27	21	10
L	12	22	21	29	18
W	10	20	19	20	8
P	9	17	19	20	10
X, Y, & Z	7	46	28	23	11
B	6	21	10	27	7
V	6	33	29	29	5
K	4	31	27	22	2
J	0	30	21	19	1

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 17 & 11 & 13 \\ 9 & 0 & 20 \\ 2 & 12 & 21 \end{pmatrix}, \quad \begin{pmatrix} 2 & 10 & 12 & 4 \\ 13 & 16 & 5 & 8 \\ 14 & 6 & 15 & 9 \\ 3 & 11 & 7 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 18 \\ 22 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & 7 & 4 \\ 9 & 1 & 11 \\ 12 & 13 & 12 \end{pmatrix} \text{ and } \begin{pmatrix} 21 & 12 & 9 & 15 \\ 6 & 8 & 1 & 18 \\ 10 & 10 & 21 & 13 \\ 8 & 13 & 22 & 11 \end{pmatrix}.$$

After the paragraph has been ciphered by each matrix, the frequency of the letters in the clear can be compared with the frequency of the letters in the three cryptograms transformed by each of the three matrices above.

A study of the frequencies (See Table II, page 9) shows that as the matrices increase in dimension, the range of the frequency has decreased. The ranges are:

2x2	3x3	4x4
12 - 46	9 - 34	15 - 30

It is a conjecture that as the dimension of a matrix increases, the letters in a cryptogram approach a uniform distribution for certain matrices. It is also possible to choose matrices for which the frequency will increase as the dimension increases.

A decipherer will quickly realize his elaborate frequency tables are of little use when confronted with one of the above cryptograms. But if he discovers that matrices have been used, it has been suggested that an electronic computing machine will quickly break the cipher. While the machine can calculate at very high speeds, this does not mean that the human work can be speeded up. Assuming the dimension of the transforming matrix to be 3x3, a relative large number of messages using this particular key have been intercepted, and it takes three seconds to feed material into the machine, have the machine type out the deciphered message and have a human read the message to see if it makes sense, it would take 84,000 years, on an average, to break the cipher. This does not take into consideration singular matrices for which the average time would be less since there would be no output and human work necessary.

Maximum possible matrices using modulo 23 1.8×10^{12}

Average matrices used before breaking the cipher	$9x10^{11}$
Seconds to decipher with each 3x3 matrix	$3x10^{0}$
Seconds in a year	$3.2x10^{7}$

$$\frac{3\ (9x10^{11})}{(3.2x10^{7})} = 8.4x10^{4} = 84{,}000 \text{ years}$$

Cryptography plays a vital part in our country; it is used in the Secret Service and by the Embassies as well as by business firms. Because of the secrecy of this work, the lay citizen has no knowledge of the specific ciphering systems used. But it is conceivable that matrices have been and will continue to be used. Two reasons for this are frequency of letters or symbols in a message can be changed and the cipher is not easily broken.

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Little can be understood of even the simplest phenomena of nature without some knowledge of mathematics, and the attempt to penetrate deeper into the mysteries of nature compels simultaneous development of the mathematical processes.

—J. W. A. YOUNG

Uncertain Inferences*

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In accordance with the title of this talk, I shall speak about inferences, or conclusions, which may be false. I have in mind situations in which—for better or for worse—some data have been obtained somehow, and the gatherer of the data wishes to draw important conclusions therefrom. This is neither an uncommon nor, necessarily, an unworthy activity. Warren Weaver, now with the Alfred P. Sloan Foundation in New York recently had an article in *Science* with the title: The *Disparagement of Statistical Evidence.* He comments on the widespread tendency to make slurring remarks about statistics and ends that article with the following statement: "Statistical evidence is, in essentially all nontrivial cases, the only sort of evidence we can possibly have." If he is right—and I tend to agree with him—it is apparent that uncertain inferences based on statistical evidence are both necessary and important.

Most all inferences based on observed data are necessarily uncertain, but the degree of confidence we can place in them is statistically measurable, if proper precautions have been taken during the planning of the study. Some inferences are unnecessarily uncertain, and could have been improved by a better statistical design. Still other classes of inferences are unnecessarily and completely, though honestly, uncertain; and are useless. A last sort of uncertain inference consists of deliberate attempts to be misleading; and this sort of inference is *worse* than useless. Unfortunately, this latter type of uncertain inference often is laid at the door of the statistician.

The general area of research and service denoted by the term "statistics" is something of a puzzle to many people. To some, statistics is just more mathematics; yet the field of statistics differs materially from mathematics in several respects. In some ways, statistical reasoning is a form of inductive reasoning; but all of the formal reasoning used in statistics is *de*ductive. From some points of view statistics is a new science; yet its roots go down to the beginning of recorded history. In the opinion of some folks, statistics is a respected and

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interesting science; but others regard statistics as a dubious tool of professional liars. As a matter of fact, a book has been written with the title, "How to lie with Statistics", and, of course, many of you have heard the quotation from the British statesman, Disraeli, which goes, "There are three kinds of lies; lies, damn lies, and statistics." Of course there also are the *good* and *wise* people who agree with H. G. Wells when he says: "Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write."

Why should the word, "statistics" have so many different connotations? The reasons for this, and for uncertain inferences, are better understood with the aid of a little history, because the name, "statistics", and the development of its companion science, mathematical probability, began many centuries ago.

Histories of statistics invariably start with a discussion of some form of gambling because both gambling and statistics depend heavily upon chance events, and hence upon the theory of probability. Games of chance were invented by man some time between 40,000 years ago and 3000 B.C. According to recent journal articles on the history of probability and statistics, the astrogalus or anklebone, of the sheep was an early basis for primitive games of chance. With a little help from man, the astragalus provided a four-sided solid, with two rounded ends, which could be thrown onto a flat surface to see which of the four flat sides would be on top when this object stopped rolling. *The interesting feature of this game was the element of chance.* There was roughly one chance in four that any particular flat side would be uppermost after a throw. One way or another, mankind seems always to have been intrigued by uncertain events, and usually has attached some form of gambling to them.

About 3000 B.C. some rather well-constructed dice of various sorts were being made from well-fired buff pottery; but the present-day six-sided die with the two-part partitions of the number 7—such as 2 and 5—on opposite sides appears not to have become common before 1400 B.C. It was nearly 2800 years later that playing cards also were used regularly in gambling.

In spite of the availability of gambling devices, an active interest in gambling, and the clear dependence of games of chance on the theory of probability, a calculus of probability was extremely slow to develop. Two reasons for this slow growth, as recently suggested by M. G. Kendall in *Biometrika,* were:

1) Gambling often was associated with vices, so a serious study of chance events lacked respectability.

2) The Church opposed scientific writings on probability as associated with chance events, apparently because the Church insisted that all events were pre-determined by God and hence were not really chance events at all.

Apparently, the first written treatise on probability theory was done by Gerolamo Cardano in the 16th Century A.D., long after gambling had become a popular sport. He was the illegitimate son of the geometer, Fazio Cardano. In a 1955 article in *Biometrika,* the British statistician F. N. David says of Gerolamo Cardano, "He was a physician, philosopher, engineer, pure and applied mathematician, astrologer, eccentric, liar, and gambler."; so you see that having a wide range of interests and being a liar appeared very early in the heritage of present-day statisticians.

The derivation of the word "statistics" apparently began at least as long ago as the 4th century B.C. Aristotle is reported by Westergaard to have written at least 158 descriptions of political states. These essays on what was called "matters of state" were almost entirely non-numerical, and certainly completely non-statistical in the modern sense of the word "statistical"; yet this pastime seems to have provided the origin of the word, "statistics through its connection with the word "state".

During the 17th and 18th centuries the Germans developed an intellectual society known as the Staatenkunde which specialized in philosophical discussions of matters of state. In other areas of Europe—notably Scandinavia and Britain— more highly mathematical studies of socio-political matters were being made. Such numerical records as numbers of births and deaths, incidence of disease, figures on agricultural production, volume of foreign trade, and censuses of human populations were being taken—all with a socio-political motivation. The British used the term "Political Arithmetic" for the study of these records, and this seems a more logical term; but for *some* reason the term "statistics" won out in general usage during the 19th century.

By the middle of the 19th century various forms of socio-political arithmetic were rather well-developed, and were generally called statistics. The theory of probability as it related to deductive reasoning had become highly developed with very little relation to what was called statistics. At about this same time, scientists were beginning to feel a need for a reliable procedure for making decisions upon the basis of observed data. For example, the biologist, Gregor Mendel was conducting his genetic studies of garden peas, was counting the offspring from certain crosses, and was trying to determine the mode of inheritance of traits which interested him. As a matter of fact, the idea of planned experimentation had been introduced by the middle of the 18th century, more than 100 years before Mendel was performing his experiments; but it was to be the 20th century before major progress in this area of inductive inference based on sample data was to become well-developed.

As is common in the history of important scientific discoveries, several people produced ideas which *could* have hastened the development of the field of statistics as we know it today; but something caused the spark only to smolder; or to go out. For example, during the early 19th century, the mathematician K. F. Gauss developed a method of least squares by a procedure now called the method of maximum likelihood. Unfortunately, he concentrated his attention on the problem of minimizing the harmful effects of errors of estimate, and thereby left it to R. A. Fisher to develop the Method of Maximum Likelihood a century later. When that method was developed, it not only opened a whole new field of statistical research, but this development included Gauss' work in least squares as a special case.

It is a puzzling fact that the mathematicians of the 18th, 19th and even some in the 20th, centuries have failed to realize the potentialities of their theory of probability in relation to inductive inferences. Uncertainty is an integral part of the concept of probability, but the nature and extent of that uncertainty can be specified rigorously.

In any event, as the 19th century closed, a rather complete theory of probability was available to scientists as a tool for *de*ductive reasoning; and a demand was developing for a process for making *in*ductive decisions on the basis of experimental data. De Moivre, Gauss, and many others had done extensive work on the normal distribution and the concept of errors of measurement. The mathematical expression for the normal curve was well-known, and the fact that it involved only two basic parameters was well-known; so the fundamental nature of what we now call statistical population of numerical measurements was also known. However, the following four important ideas were missing as yet from what we now call modern statistics, or were not sufficiently understood:

1) The manner in which one can, and should, identify a process of scientific experimentation with a process of

drawing randomized samples from statistical populations.

2) The manner in which one can estimate the parameters of a statistical population on the basis of information embodied in a relatively small sample; and can do so with mathematical rigor.
3) The manner in which one can test logical hypotheses about the parameters of a statistical population; and can do so with mathematical rigor.
4) The idea that the choice of a *best* method of estimating population parameters or of testing hypotheses about them requires new axioms; no one can *prove* a particular method is best.

The first half of the 20th century included the appearance of three particularly bright stars in the statistical sky. One of them, R. A. Fisher, is best known for a theory of estimation appropriate to the parameters of a statistical population. This theory essentially is based on the principle that the best way to estimate a population parameter is to so choose that estimate that it maximizes the likelihood of occurrence of that sample; hence the name, Method of Maximum Likelihood. Professor Fisher described such apparently good properties of estimators as consistency, efficiency, sufficiency, and invariance; and showed that maximum likelihood estimators have those properties, if they can be had in the situation being studied. Fisher's first major publication of the theory of estimation was entitled, "On the mathematical foundations of theoretical statistics", published in 1922. The analysis of variance, which sometimes is listed as his major contribution to statistics, seems to have been devised initially as a procedure for systematizing the algebra and arithmetic required when one wishes to compare one estimate of population variance with another. This interest in estimation apparently led Fisher into an incomplete and somewhat inadequate development of the theory of testing hypotheses about population parameters. However, in conjunction with his research into the theory of estimation, R. A. Fisher developed two basic principles which I would like to emphasize:

1) The samples to be used as the basis for statistical inference must be randomized samples, and
2) All the information that is in the sample should be used.

Adherence to those two principles avoids unknown biases and deliberate biases which distort the facts, makes the fullest possible use of the information in the sample, and permits the formulation of rigorous probability statements to accompany the uncertain inferences made from experimental data.

Chronologically, a second star in the statistical firmament of the 20th century is Jerzy Neyman who, with E. S. Pearson as junior author, published a sound mathematical basis for using sample data to test statistical hypotheses. With Neyman taking the lead, they introduced the idea of errors of the first and second kinds, and used the probabilities of those errors to determine best tests of statistical hypotheses, if such best tests are possible for a given situation. In 1928, Neyman and Pearson published their first paper on the general theory of two-valued decision functions known as the Neyman-Pearson theory of testing statistical hypotheses. Much of the statistical research since that time has been concerned with the application and generalization of that theory.

The third statistical star of this century, in chronological order of discovery, was Abraham Wald. Unfortunately, his tremendous contribution to several areas of statistics were cut short by an airplane crash in India in 1950. Probably his most outstanding contribution was a broad generalization of existing theory known as Decision Theory. Until 1939 when Wald began publishing on this topic, statistical theory and application were chiefly limited to one-stage experiments, to two-valued decisions regarding statistical hypotheses, and to point and interval estimates of one parameter at a time. In practice, this still is true to a considerable degree; but Wald's Decision Theory opened the way for multi-decision analyses which theoretically can come much closer to fitting actual situations than previous statistical theory had accomplished.

Another facet of modern statistics is embodied in the term "stochastic processes". A stochastic process is any process which includes a random element in its structure. The growth of a bacterial colony is a stochastic process; so is the production of fluctuating numbers of electrons and photons in a cosmic-ray shower. Time is quite commonly an added variable, and the population parameters are functions of time. Thus the science of statistics is becoming more complex, and also more flexible so that it can solve a wider range of problems. Nothing, however, can keep honest statistical inferences based on samples from being uncertain inferences.

I should like to devote the remainder of this talk to a brief mention of a few of the many fields of application of the theories of probability and statistics. Circumstances in this immediate area have determined that many of the applications of statistics have been associated with biological research. This may have caused some of you to believe that statistics is mostly a biometric science; but this is not true. Biological research merely was one of the earlier fields of application of statistical methods. Moreover, I believe biologists may more fully appreciate their need for statistics than physical scientists, who sometimes think they measure without error. To illustrate the generality of statistical theory and method, I would point out that Professor George W. Snedecor, a pioneer in biometry and former head of the Department of Statistics and Statistical Laboratory at Iowa State University is now, at 80 years of age, a statistical consultant at the Navy Electronic Laboratory at San Diego, California. As you see, old statisticians never die either; and they are mighty slow to fade away. It also can be pointed out that the five full-time staff members of the Department of Statistics and Statistical Laboratory at Kansas State University have, as a group, graduate credits in the following subjects in addition to probability and statistics: mathematics, genetics, physics, paleontology, economics, chemistry, animal nutrition, psychology, geology, and history.

I shall surprise a few of you by using an example from literature to illustrate how rigorous, though uncertain, inference can be made.

According to a 1959 publication in the Journal of the Royal Statistical Society of Britain, it is known that between the writing of *Republic* and the writing of *Laws*, Plato wrote *Critias, Philebus, Politicus, Sophist,* and *Timeaus;* but the order in which those five dialogues were written is not known. A Ph.D. candidate in the Department of Classical Languages, Birbeck College, University of London undertook a statistical study of this matter. The specific problem is to place those dialogues in chronological order, with due allowance for whatever errors are possible while making this decision. The decision already had been made to do a numerical, rather than a philosophical, study of this problem; so the first step was to choose a way to measure literary style numerically. Previous studies of literary style had used lengths of words in terms of letters, and lengths of sentences in terms of numbers of words; but this student, a Mr. Brandwood, chose to use the lengths of the last five syllables in each sentence as his measure of literary style. He further simplified the work by classifying the syllables merely as *short* or *long*. This meant that the last five syllables in any sentence always could be put into one of but 32 classes, ranging from all 5 syllables are "short" to all 5 syllables are "long".

The next step was to show that this method of measuring literary style actually will distinguish between different literary compositions. The purpose of Brandwood's study was to detect a difference in *time* of writing; so he must first show that his method will distinguish between *Republic* and *Laws*, which were known to have been written at different times, and in that order. It turns out that when random samples are taken from each dialogue, the stylistic score for *Republic* is a negative number which is so much smaller than the positive stylistic scores for *Laws*, that there is no reasonable doubt of the hypothesis that Plato's literary style—as measured this way—changed between the writing of those two dialogues.

Now that it is established that Brandwood's statistical procedure will detect a time spread in Plato's writing, it is in order to apply the same method to the other five dialogues under study. This was done. The end result was to give the following order to the seven dialogues which have been included in this study: *Republic, Timaeus,* then *Sophist* and *Critias* in either order, *Politicus,* and finally *Philebus* and *Laws* in either order. Of course it can not be proved that Brandwood's method is the best there is for detecting changes in styles of writing; nor can it be denied that someone else might devise a method for measuring literary style which would put those seven works of Plato in a different time ordering. Brandwood has used an interesting statistical method for studying Greek literature; and he has shown that it has some power to detect time spread among Plato's dialogues.

As noted earlier, another procedure for studying possible changes in literary style is to make a frequency distribution of the lengths of words, simply by counting the numbers of one-letter, two-letter, etc., words in a large random sample. When I applied that procedure to a 1936 publication by R. A. Fisher, and also to a 1956 publication by Fisher, I found that he did not change his writing style (as so measured) during that 20-year period; or else he changed back again. During approximately the same period of time, Jerzy Neyman's style of writing, as measured by word length, *did* change enough to be detected statistically. However, Neyman changed from a short-word style—as compared to Fisher—to a style which was essentially the same as Fisher's in 1956. Incidentally, when I applied these same methods to my own book published in 1954, I found that my literary style was almost identical to Neyman's 1939 style, with respect to word length.

If I wished to be somewhat journalistic and also to use the loose statistical thinking one often encounters, I should conclude the following from the above analyses:

1) As he grew older, Neyman became more eloquent, and began using longer words.
2) Fisher always was eloquent, and therefore did not change materially.
3) In 1954, Fryer was using an old 1939 style of writing, so it is no wonder his book was not a best-seller.

I would like to mention briefly one other application of probability and statistics to a current, important, and exceedingly complex problem. The development of cancer among individuals of the same species is such a variable process from individual to individual that it seems reasonable to assume that, *in general,* carcinogenesis involves a chance mechanism of some sort. Moreover, this chance mechanism surely changes with time, with type of environment, and with respect to several other variables. Statistically, this calls for the use of a complex stochastic model of carcinogenesis. A number of these stochastic models were presented and discussed at the Fourth Berkeley Symposium on Mathematical Statistics and Probability held last summer.

All such models must start with assumptions, and no perfect, or even nearly perfect, model yet has been devised; but some of those stochastic models have done remarkably well in describing carcinogenesis in future experiments. As Neyman notes in a recent article, the Arley-Iversen model predicted the distribution of cancer induction times very closely for a number of experiments in which different strains of animals were treated with different carcinogens. However, this model did quite badly for an experiment in which mice were irradiated with ultra-violet rays. Neyman has done extensive work on models for carcinogenesis, and has had considerable success in this work. However, he and others always encounter the same difficulty: To be able to handle the mathematics involved, they are forced to make partially unrealistic assumptions, with the result that their work lacks complete generality.

As I have tried to point out in this talk, statistical inferences

(Continued on page 70)

Graphical Integrations*

BRUCE BERNDT
Student, Albion College, Michigan

Occasions often arise in various areas of applied mathematics when the integration of a function is very tedious or perhaps impossible. Often too, the function is unknown, and only certain numerical values are given. Possibly we are faced with the problem of determining the area under a curve whose equation is unknown. To resolve these problems mathematicians have invented numerous methods of approximate integration. These methods may be grouped into three classifications: those using equidistant abscissas, those employing symmetrically spaced, non-equidistant abscissas, and finally certain methods of graphical integration. In this paper I will present three methods of graphical integration. The first method employs the familiar Cartesian coordinate system; the latter two the polar coordinate system. In each method I will attempt to approximate a curve by a series of connected straight lines. In Fig. 1, I have approximated a function $f(x)$ by straight lines alternately perpendicular and parallel to the x-axis to demonstrate how such an approximation can be done. In a similar manner we can approximate a function $\phi(\theta)$ in polar coordinates.

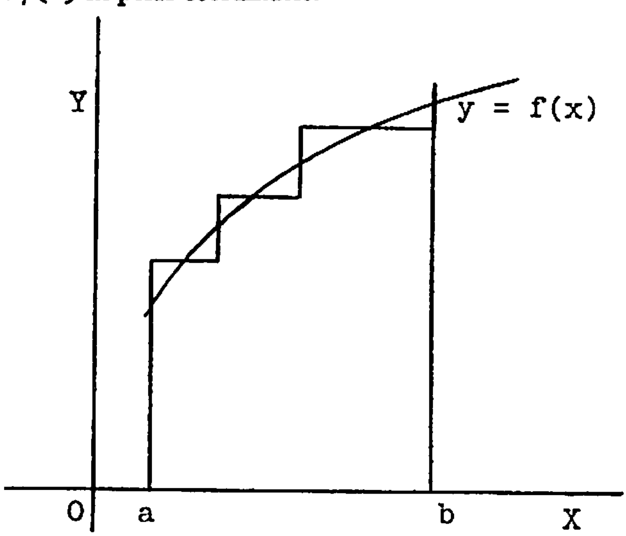

Figure 1.

* A paper presented at the 1961 National Convention of KME and awarded second place by the Awards Committee.

In Fig. 2 below a rectangle of base b_1 and height h_1 is placed on the x-axis. Construct a line through point $(a,0)$ parallel to the line joining points $(0,h_1)$ and $(-1,0)$. The intersection of this line with h_1 will be designated by (x_1,l_1).

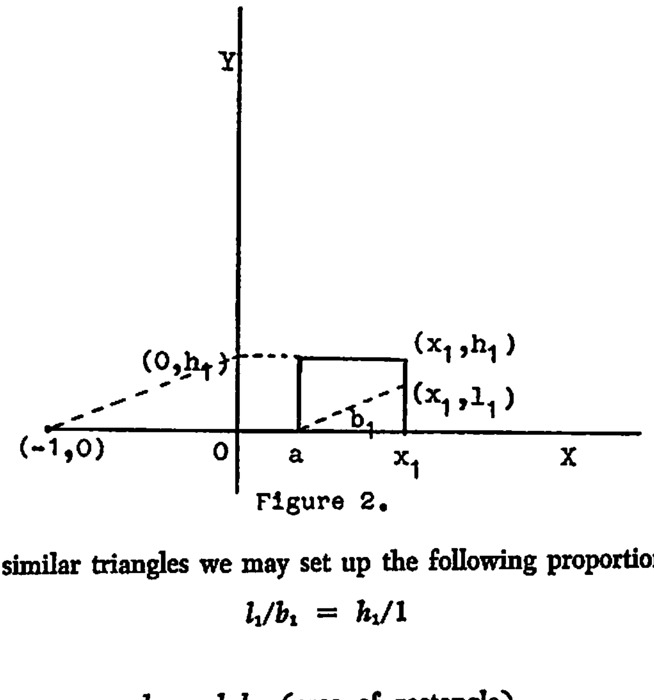

Figure 2.

From similar triangles we may set up the following proportion

$$l_1/b_1 = h_1/1$$

or

$$l_1 = b_1h_1 \text{ (area of rectangle)}$$

We see that there are the same number of units in l_1 as there are square units in the area of the rectangle.

Now construct another rectangle of base b_2 and height h_2 beside our first rectangle. In a similar manner construct a line through point (x_1,l_1) parallel to the line joining points $(-1,0)$ and $(0,h_2)$. The intersection of this line with h_2 is (x_2,l_2). Once again from similar triangles we may establish a proportion

$$(l_2-l_1)/b_2 = h_2/1$$

or

$$l_2-l_1 = b_2h_2 \text{ (area of 2nd. rectangle)}$$

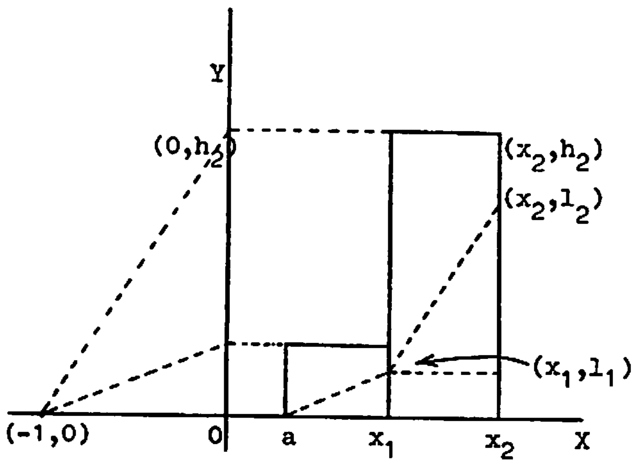

Figure 3.

Substituting in the value of l_1 we have

$$l_2 = b_1h_1 + b_2h_2 \text{ (sum of areas of both rectangles)}$$

Thus to integrate some function $f(x)$, we can approximate the area under its curve by the use of rectangles. These rectangles should be so constructed that the area of the rectangles above $f(x)$ is equal to the area not enclosed by the rectangles below $f(x)$. If we integrate $f(x)$ from $x = a$ to $x = b$, the sum total of the bases will equal $b-a$. The integral will then be

$$\int_a^b f(x)dx = l_n$$

where n designates the number of rectangles used. To obtain further accuracy we may approximate $f(x)$ by different systems of rectangles and then average the results.

Now let $\rho = \phi(\theta)$ be the equation of a curve AB in polar coordinates. We must find the area of the curve bounded by AB and two vectors from the pole O.

$$A = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} \phi^2(\theta)\, d\theta$$

We shall first find the area of OAB where AB is a straight line perpendicular to one of the vectors.

Construction (See Fig. 4):

Let OA be a constant a.
Draw OZ, an arbitrary axis.
On OZ lay off a unit length OK.
Also on OZ construct $OM = (1/2)AB$.
Draw KA.
Construct MN parallel to KA.
Let P be a point on OB such that $OP = ON$.

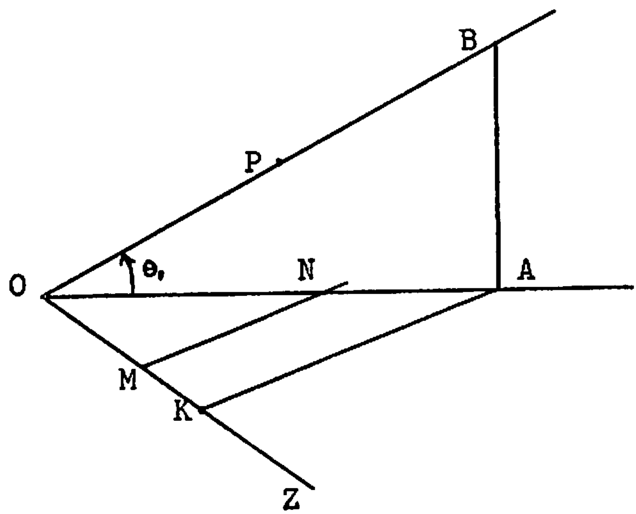

Figure 4.

OP gives a measure of the area of triangle OAB since

$$ON/OM = OA/OK.$$

or

$$ON = OA \cdot OM/OK = OA \cdot (\tfrac{1}{2})AB$$

The equation for AB is simply

$$\rho = a/\cos\theta$$

Thus for the area we have

$$A = \tfrac{1}{2}\int_0^{\theta_1} \frac{a^2 d\theta}{\cos\theta} = \tfrac{1}{2}a^2 \tan\theta_1$$

Let us alter the above method by letting AB be any straight line. The construction is similar to the previous example.

Construction (See Fig. 5):

Let OA be a constant a.
Draw OH perpendicular to AB.
Lay off an arbitrary axis OZ.
Construct a unit distance OK on OZ.
Construct $OM = (\tfrac{1}{2})AB$ on OZ.
Draw KH.
Construct a line MN parallel to line KH.
On OB again construct $OP = ON$.

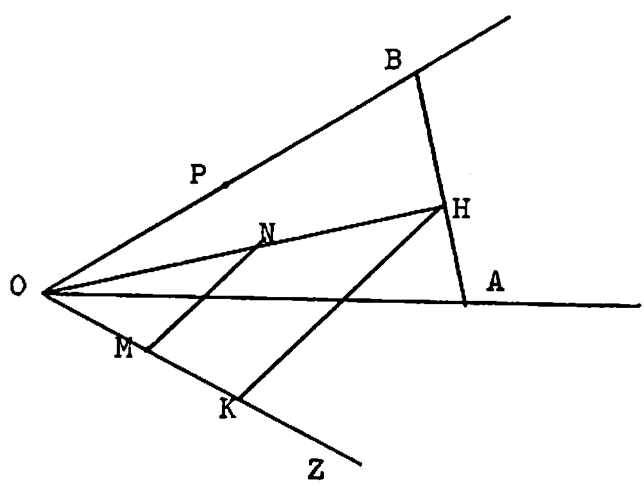

Figure 5.

OP again is a measure of the area OAB, for we have by similar triangles

$$ON/OM = OH/OK$$

or

$$ON = OH \cdot (\tfrac{1}{2})AB$$

To integrate some function $\phi(\theta)$ in polar coordinates, we can approximate its curve between A and B by either of the two methods outlined above or possibly a combination of the two. The area between two vectors and $\phi(\theta)$ will then be the sum of the areas of the triangles used, or the sum of the distances OP_1, $OP_2 \cdots\cdots OP_n$, where n designates the number of triangles used.

Suppose now we wish to find the area (Fig. 6) of an enclosed curve $ABCD$ where OA and OC are tangents to this curve. We can approximate the two portions of the curve, ABC and ADC, using the methods outlined above. The area of $ABCD$ will then be the difference in the sums of the OP's for the curves ADC and ABC.

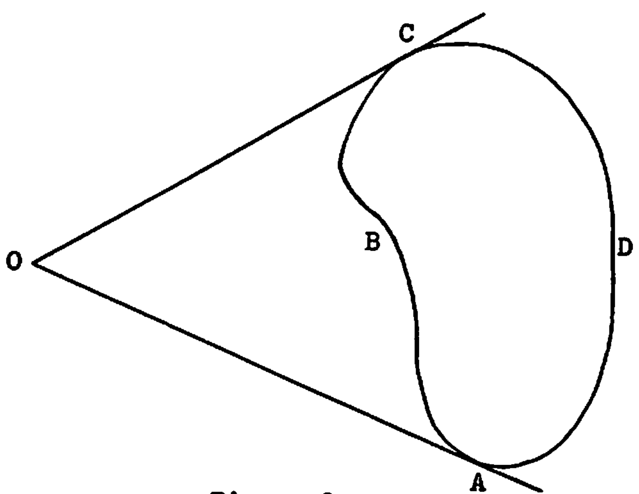

Figure 6.

The reason for the use of OP rather than ON will now be shown. Divide AB and OP each into n equidistant intervals. Construct lines OA_1, $\cdots$, OA_n where A_i is one of the constructed points on AB. Draw an arc of a circle with a radius of OB_1 where B_1 is the first of the constructed points on OB. The point of intersection of this arc and OA_1 will be a point on the integral curve of $\phi(\theta)$. An integral curve is simply the curve $F(x)$ represented by the integral

$$F(x) = \int f(x)dx$$

Similarly we can construct other points on the integral curve. If we use enough subdivisions on *AB* and *OB* we can establish a very accurate integral curve.

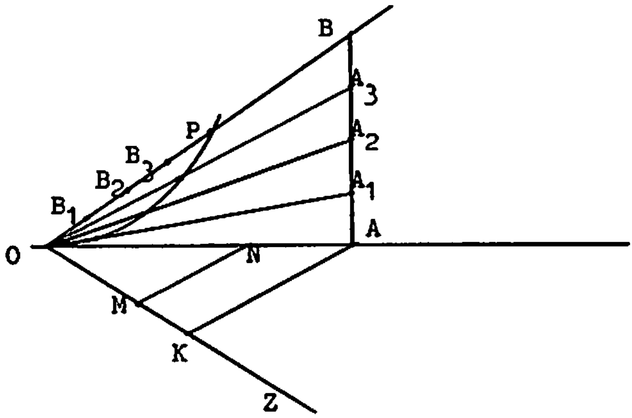

Figure 7.

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In Memoriam

Professor Harold D. Larsen of Albion College, Albion, Michigan died on August 9, 1961 at the age of 57. Dr. Larsen served as Editor of THE PENTAGON for nine years, 1943-1952, and as Business Manager from 1943 to 1953. After retiring as Editor, he continued as Associate Editor, writing the Mathematical Scrapbook until 1955.

Terminal Digits of $MN(M^2 - N^2)$ in the Scale of Five

CHARLES W. TRIGG
Faculty, Los Angeles City College

If M and N are integers, the unit's digit of

$$P = MN(M^2 - N^2) = MN(M + N)(M - N)$$

is dependent upon the unit's digits of its four factors. Let the unit's digits of M, N, and P be m, n, and p, respectively. These terminal digits, when treated as signless numbers, $m \neq 0 \neq n$, fall into two square arrays, one for $M \geqq N$, the other for $M \leqq N$.

In the scale of notation with base five*, p is zero if m, n, or $m - n$ is zero, or if m and n are complementary, as when

$$m + n = 2 + 3 = \text{five} = 10.$$

These zeroes fall along the diagonals of the square arrays.

$M \geqq N$

$n \backslash m$	1	2	3	4
1	0	1	4	0
2	4	0	0	1
3	1	0	0	4
4	0	4	1	0

$M \geqq N$

$n \backslash m$	1	2	3	4
1	0	4	1	0
2	1	0	0	4
3	4	0	0	1
4	0	1	4	0

These two arrays are closely related in that they are mirror images and one goes into the other by rotation about the cross-diagonal (lower left to upper right). Also, the corresponding elements in the two arrays are complementary.

Each array has the following properties:

1) The sum of the elements in each row and column is 10.

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*In the discussion of the arrays to base five, numbers in that scale are expressed as numerals. Those expressed by names are to the base ten.

2) Elements symmetrical to the main diagonals are complementary.
3) Non-zero elements symmetrical to the perpendicular bisectors of the sides are complementary.
4) The array goes into itself by rotation through 90°.
5) The non-zero elements occur at the vertices of a regular octagon. The sum of the elements on each side of the octagon is 10.
6) The non-zero elements consist of four 1's and four 4's. The 1's may be traversed by Knight's moves. These 1's lie at the vertices of a square. The same statements may be made about the 4's The 4's square occupies a position which is the reflection of the position of the 1's square.
7) The value of the array treated as a determinant is $-(30)^2$.
8) The array may be viewed as the composite of nine overlapping second order arrays. The second order arrays may be evaluated as determinants, e.g.,

$$\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} = -4, \quad \begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix} = 4.$$

The values of these second constitute a third order array, which vanishes when evaluated as a determinant. That is,

$$\begin{vmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{vmatrix} = 0.$$

9) The array may be viewed as the composite of four overlapping third order arrays. When evaluated as determinants, each of these third order arrays vanishes.

Mathematics is the science which draws necessary conclusions.
—Benjamin Peirce

An Analysis of Some of the Syllogisms Found in Alice in Wonderland*

RUTH GOODRICH
Student, Illinois State Normal University

Professor Smith begins his invitation to our April KME convention with

"The time has come," the Walrus said,
"To talk of many things;
Of sets—and subs—and syzygies
Of integrals—and rings—
And if the tangents to a curve—
Are secants' limitings."

Certainly the time has come to talk of many things in our world of complicated ideas and integrated subject fields. And it is becoming more apparent to the mathematician that his field is co-ordinated with many other basic and stimulating areas of learning.

As young people who are growing in mathematical knowledge, we of KME should always be alert for the appearance of and the co-ordination of mathematics in the physical sciences, the social sciences, literature, and certainly logic. We should be investigating the importance of mathematics in these fields and through our increased mathematical insight gain greater insights into these important areas. For example, our so-called "modern mathematics" with its sets and subsets, unions and intersections, universals and nulls should and does suggest a very basic tie with the age-old area of logic.

It is because of these apparent implications that I have written this simple exercise in logic. Even though simple, it is an exercise that could develop an attitude of exploration and maybe even a method for exploration into the more complicated similarities of mathematics and logic and, perhaps in the future, into other fields.

But in disagreement with our host, Mr. Smith, who says, "Oh, yes—my apologies to Lewis Carroll," I must say, "Thank-you," to Mr. Dodgson for the creative and stimulating tie he has emphasized between logic and mathematics.

* A paper presented at the 1961 National Convention of KME and awarded third place by the Awards Committee.

Charles Lutwidge Dodgson, the logician and mathematician, is only known to many as Lewis Carroll, the nonsense-novelist and poet. However, much of his nonsense writing is based on logic. Even

non-Euclidean space appeared possible to Dodgson. He often mused on what would happen if our fundamental assumptions about the universe did not hold good. But it can be seen that his *Symbolic Logic* and *Euclid and His Modern Rivals* have much in common with his well-known *Alice in Wonderland.*

As a child, I read through the dream-like pages of Lewis Carroll's *Alice in Wonderland,* first imagining myself in Alice's place, living all those curious moments with her, and even talking with all those strange animals. Then I tried to understand what those animals said and what she said to them. It seemed so strange that they could reason the way they did and come up with their convincing conclusions. In the words of Alice, I grew "curiouser and curiouser." Now that I am old enough to understand what this ingenious author created with his ever-experimenting mind, I want to analyze some of the propositions set before Alice by her strange friends to see if their conclusions are sound and do make sense. I feel, that in constructing this analysis, I will get a broader background for the logic of mathematics and maybe get an insight into the mind of a mathematician whose thoughts were way ahead of his day.

There are so many things that can be analyzed about Lewis Carroll's subtle and not too subtle logic. Analysis of the various propositions alone would appear very interesting. Even assigning examples from the story to the various forms and terms of logic would be very rewarding and interesting. But I have chosen to analyze a few of the more obvious syllogisms the characters in Alice's dream use as they appear before her and the syllogisms she uses as she wanders among them.

The basis for my analysis will be the very enjoyable and instructive introduction to logic, *Logic: An Introduction.*[1] The symbols, definitions, and rules followed will be taken from this work. I will also include very brief and limited definitions and descriptions of mathematical terms that apply. The symbols, rules, and accepted or well-defined steps, that are so important to a mathematician, become very important tools in this analysis.

A syllogism may be termed an argument of two premises and a conclusion. The categorical type, with which I am concerned, is an

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1 Ruby, L., Logic: An Introduction (J. B. Lippincott Co., 1950) pp. 123-218.

argument made of three categorical propositions that contain among them three and only three terms. The *middle term* is found in both premises as either the subject or the predicate. The *major term* is the predicate of the conclusion, as well as being found in one premise; and the *minor term* is the subject of the conclusion, as well as appearing in a premise.

The *subject* of the proposition is the thing or entity of which something is asserted and the *predicate* is that which is asserted of the subject. The two terms or the subject and the predicate are joined by a *copula,* which is a form of the verb "to be." The last part of the proposition is the quantifier or words such as all, some, no, or none, which indicate the extent to which we refer to the members of the subject term.

Using the symbols A, B, C for these three terms, the syllogism form may then be written as follows:

All A are B.
All C are A.
Therefore, All C are B.

Using the familiar

All men are mortals.
Socrates is a man.
Therefore, Socrates is a mortal.

it is seen that the *middle term* is man; the *major term* is mortal, and the *minor term* is Socrates. The *subjects* are men, Socrates, and Socrates; the *predicates* are mortals, man, and a mortal; the *copulas* are is and are; and the only *quantifier* is all.

Each subject and predicate is categorized as to the class relation shown between them. A class of things is a group of things having some characteristic in common, either by a natural grouping or by an arbitrary act of selection. The subject is included in the class named by the predicate.[2] A negative belongs to the copula so that the presence of the words no or not excludes a subject from a certain class entirely.

The *universal propositions* use the quantifier all when referring to the members of the subject class or they have a single thing or person as subject. The *particular propositions* use the word some.

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2 Looking at the term set that the mathematician uses, a comparable definition is found. A set is a well-defined collection of objects. And if the subject is included in the class of the predicate, that would be the same as saying if X is a member of set Y, then X is a proper subset of Y.

Thus the difference here is one of quantity. To make this more clear, consider the following statements:

1. All the citizens of Kansas are citizens of the US—Universal-affirmative.
2. All the citizens of Canada are not citizens of the US—Universal-negative.
3. Some smart people are math majors—Particular-affirmative.
4. Some smart people are not math majors—Particular-negative.

These propositions may be placed categorically in these arbitrary forms:

Universal-affirmative	*A* form
Universal-negative	*E* form
Particular-affirmative	*I* form
Particular-negative	*O* form

The following table will give a complete picture of the four types of propositions, with little explanation needed:

Types of Propositions	Traditional Form	Class Terminology
A Universal-affirmative	All S is P.	All $S < P$
E Universal-negative	No S is P.	All $S \nless P$
I Particular-affirmative	Some S is P.	Some $S < P$
O Particular-negative	Some S is not P.	Some $S \nless P$

The class terminology provides a symbolic shorthand to indicate relation between the class determined by the subject and that determined by the predicate. The symbol "$<$" indicates class inclusion, i.e., members in the subject are included in the predicate class, while "$\nless$" denotes class exclusion, i.e., members of the subject are excluded from the predicate.

The accompanying Venn diagrams or circles will give the four in a picture form.

A E I O

Of importance too, are the terms *distributed* and *undistributed*. The precise meaning of distributed is to say that all of the members of the class are designated by the term. When referring to only part of the class, then the term is undistributed. The following table summarizes these relationships with respect to the four forms.

Form	Subject	Predicate	Traditional Form	Class Terminology
A	*d*	*u*	All *Sd* is *Pu*.	$Sd < Pu$
E	*d*	*d*	No *Sd* is *Pd*.	$Sd \nless Pd$
I	*u*	*u*	Some *Su* is *Pu*.	$Su < Pu$
O	*u*	*d*	Some *Su* is not *Pd*.	$Su \nless Pd$

Now these four steps can be taken in the analysis:

1. Identify the three propositions in terms of the four classifications.
2. Symbolize each term with a letter.
3. Gather the symbols, stating them in the class terminology.
4. Put in the signs of distribution.

Evidence or reasons should be sufficient to prove the conclusions. Thus validity is conclusive proof; and that is what I will try to determine in the analysis. However, it must be remembered that "If it is impossible, granted the truth of the premises, that the conclusion should be false, then the argument is valid. . . .an invalid argument may be composed of true statements, and a valid argument may be composed of false statements."[3]

To test for validity, these five rules[4] can be used:

1. The middle term must be distributed at least once.
2. A term which is distributed in neither premise must not be distributed in the conclusion.
3. No conclusion is necessitated by two negative premises.
4. If either premise is negative, then the conclusion must be negative.
5. A negative conclusion cannot be drawn from two affirmative premises.

If one or more are not true, then the conclusion is invalid.

3 Ruby, op. cit., p. 151, 153.
4 Ibid., p. 175.

One of the big problems in the analysis of *Alice in Wonderland* is seeing the syllogism in the standard form. Since people do not talk or write in this form, the syllogisms must be reworded so that they will be in this form. The following simple steps are used as guides:

1. Make necessary grammatical revisions so that subject and predicate are clearly indicated.
2. Supply missing quantifiers if they can be detected from the meaning of the sentence.
3. Add missing complements to adjectives and other phrases so that the class is clearly understood and named.
4. Add the missing copulas, form of the verb "to be," usually through sentence revision.

Finally, two other forms likely to appear are the *enthymeme,* the incompletely stated syllogism through the omission of one or two of the propositions, and the *sorities*, the series of syllogisms forming one argument by leaving out unstated conclusions.

Now by following these well-defined steps the analysis becomes very easy, that of putting the syllogism in the standard form through grammatical revision, of following the four steps of identification and classification using symbols and signs of distribution, and finally of testing for validity.

THE ANALYSIS

Chapter 2—*"The Pool of Tears"*

"I'm sure I'm not Ada," she said, "for her hair goes in such long ringlets, and mine doesn't go in ringlets at all; . . ."

Ada (A) is a girl who has curly hair (H).	A	$Ad < Hu$
I (I) am not a girl who has curly hair (H).	E	$Id \nless Hd$
I (I) am not Ada (A).	E	$Id \nless Ad$

The argument is valid, no violation of the five rules.

Chapter 12—*"Alice's Evidence"*

"If you didn't sign it," said the King, "that only makes the matter worse. You must have meant some mischief, or else you'd have signed your name like an honest man."

An honest man (H) is a man who would have signed (M).	A	$Hd < Mu$
You (Y) are not a man who would sign (M).	E	$Yd \nless Md$

You (*Y*) are not an honest man (*H*). $E \quad Yd \nless Hd$

The argument is valid, no violation of the five rules.

Chapter 6—"*Pig and Pepper*"

"But I don't want to go among mad people," Alice remarked.

"Oh, you can't help that," said the Cat; "we're all mad here. I'm mad. You're mad."

"How do you know I'm mad?" said Alice.

"You must be," said the Cat, "or you wouldn't have come here."

A person who would come here (*C*)
is a mad person(*M*). $A \quad Cd < Mu$

(This is an enthymeme, assumed but not spoken.)

You (*Y*) are a person who came here (*C*). $A \quad Yd < Cu$

You (*Y*) are a mad person (*M*). $A \quad Yd < Mu$

The argument is valid, no violation of the five rules.

Chapter 6—"*Pig and Pepper*"

". . .And how do you know that you're mad?"

"To begin with," said the Cat, "a dog's not mad. You grant that?"

"I suppose so," said Alice.

"Well then," the Cat went on, "you see a dog growls when it's angry, and wags its tail when it's pleased. Now I growl when I'm pleased, and wage my tail when I'm angry. Therefore, I'm mad."

Here is a soritie. The first conclusion is left out and used as the proposition in the second syllogism. These two might be as follows:

A dog (*D*) is a creature that growls when it's
angry and wags its tail when it's pleased (*C*). $A \quad Dd < Cu$

I (*I*) am not a creature that growls when it's
angry and wags its tail when it's pleased (*C*). $E \quad Id \nless Cd$

I (*I*) am not a dog (*D*). $E \quad Id \nless Dd$

The argument is valid, no violation of the five rules.

A dog (*D*) is not mad (*M*). $E \quad Dd \nless Md$

I (*I*) am not a dog (*D*). $E \quad Id \nless Dd$

I (*I*) am mad (*M*). $A \quad Id < Mu$

The argument is invalid, a) no conclusion is necessitated by two negative premises and b) if either premise is negative, then the conclusion must be negative.

Chapter 5—*"Advice From a Caterpillar"*

"Well! What are you?" said the Pigeon. "I can see you're trying to invent something!"

"I—I'm a little girl," said Alice, rather doubtfully, as she remembered the number of changes she had gone through that day.

"A likely story indeed!" said the Pigeon, in a tone of deepest contempt. "I've seen a good many little girls in my time, but never one with such a neck as that! No, No! You're a serpent; and there's no use denying it. I suppose you'll be telling me next that you never tasted an egg!"

"I have tasted eggs, certainly," said Alice, who was a very truthful child; "but little girls eat eggs quite as much as serpents do, you know."

"I don't believe it," said the Pigeon; "but if they do, why, then they're a kind of serpent; that's all I can say."

Serpents (*S*) are creatures who eat eggs (*E*).	*A*	$Sd < Eu$
Little girls (*G*) are creatures who eat eggs (*E*).	*A*	$Gd < Eu$
Little girls (*G*) are serpents (*S*).	*A*	$Gd < Su$

The argument is invalid, undistributed middle term.

Chapter 6—*"Pig and Pepper"*

"I didn't know that Cheshire cats always grinned; in fact, I didn't know that cats could grin."

"They all can," said the Duchess; "and most of 'em do."

"I don't know of any that do," Alice said, very politely, feeling quite pleased to have got into a conversation.

"You don't know much," said the Duchess; "and that's a fact."

People who don't know of any cats that grin (*P*) are people who don't know much (*D*).	*A*	$Pd < Du$
You (*Y*) are a person who doesn't know of any cats that grin (*P*).	*A*	$Yd < Pu$
You (*Y*) are a person who doesn't know much (*D*).	*A*	$Yd < Du$

The argument is valid, no violation of the five rules.

Chapter 9—*"The Mock Turtle's Story"*

"Very true," said the Duchess; "flamingoes and mustard both bite. And the moral of that is—'Birds of a feather flock together.'"

Overlooking the truth or falsity of the propositions, it might appear as if the Duchess reasoned within her warped mind as follows:

A flamingo (F) is a bird (B).	A	$Fd < Bu$
A flamingo (F) is a creature that bites (b).	A	$Fd < bu$
A bird (B) is a creature that bites (b).	A	$Bd < bu$

This argument is invalid, the term "bird" is not distributed in the premise but is distributed in the conclusion.

A bird (B) is a creature that bites (b).	A	$Bd < bu$
Mustard (M) is a creature that bites (b).	A	$Md < bu$
Mustard (M) is a bird (B).	A	$Md < Bu$

The argument is invalid, undistributed middle term.

Mustard and flamingoes (M and F) are birds that bite (b).	A	M and $Fd < bu$
Birds that bite (b) are birds of a feather (f).	A	$bd < fu$
Mustard and flamingoes (M and F) are birds of a feather (f).	A	M and $Fd < fu$

This argument is valid, no violation of the five rules.

Mustard and flamingoes (M and F) are birds of a feather (f).	A	M and $Fd < fu$
Mustard and flamingoes (M and F) are a group of birds (G).	A	M and $Fd < Gu$
Birds of a feather (f) are a group of birds (G).	A	$fd < Gu$

The argument is invalid, "birds of a feather" is not distributed in the premise but is distributed in the conclusion.

Birds of a feather (f) are a group of birds (G).	A	$fd < Gu$
Groups of birds (G) flock together (T).	A	$Gd < Tu$
Birds of a feather (f) flock together (T).	A	$fd < Tu$

The argument is valid, no violation of the five rules.

It is strange that the Duchess could begin with a true statement and end with such an incongruous valid conclusion.

Of course, this is a simple exercise or analysis, but it is an example of what we as young people in mathematics should be doing. We should be exploring the fields related to mathematics, applying our analytical powers to other problems, and most important of all, gaining insight and knowledge into these other fields that will help us mature in our chosen field of mathematics.

The Problem Corner

EDITED BY J. D. HAGGARD

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before March 1, 1962. The best solutions submitted by students will be published in the Spring, 1962, issue of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to J. D. Haggard, Department of Mathematics, Kansas State College, Pittsburg, Kansas.

PROPOSED PROBLEMS

146. *Proposed by Fredric Gey, Harvey Mudd College, Claremont, California.*

Sum the following double series:

$$\sum_{k=n}^{\infty} \sum_{l=n}^{\infty} \frac{a^{l-n} x^{l}}{2^{k-n}(l-n)!}$$

147. *Proposed by Rickey M. Turkel, Hofstra College, Hempstead, New York.*

How many different paths can be traced through the following diagram, spelling the word "TRIGONOMETRY"?

```
                      T
                    T R T
                  T R I R T
                T R I G I R T
              T R I G O G I R T
            T R I G O N O G I R T
          T R I G O N O N O G I R T
        T R I G O N O M O N O G I R T
      T R I G O N O M E M O N O G I R T
    T R I G O N O M E T E M O N O G I R T
  T R I G O N O M E T R T E M O N O G I R T
T R I G O N O M E T R Y R T E M O N O G I R T
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148. *Proposed by C. W. Trigg, Los Angeles City College.*

Two numbers whose three digits are consecutive integers have the property that the number and a permutation of its digits can each be represented as the sum of two cubes. Identify the permutations.

149. *Proposed by Jim Brooking, State University of New York, Albany.*

Find any solution to the equation

$$\frac{\arcsin x}{\arccos x} = \arctan x$$

150. *Proposed by the Editor.*

The last proposition of the ninth book of Euclid's Elements states that "If $2^n - 1$ is a prime number, then $2^{n-1}(2^n - 1)$ is a perfect number." Show that a necessary condition for $2^n - 1$ to be prime is that n be prime.

SOLUTIONS

141. *Proposed by Tom Wood, University of Missouri, Columbia.*

Find the natural numbers $a, b, c, d, e, g, h, i, j$.

Given:

$$\begin{array}{r} a\ b\ c \\ -\ c\ b\ a \\ \hline d\ e\ f \\ +\ f\ e\ d \\ \hline g\ h\ i\ j \end{array} \quad \text{and} \quad \begin{array}{l} b = e - i \\ j = a + c \\ f = 7 \\ a > c \end{array}$$

Solution by Fredrick Carty, Hofstra College, Hempstead, New York.

Rewrite $\begin{array}{r} a\ b\ c \\ -\ c\ b\ a \\ \hline d\ e\ f \end{array}$ as $100a + 10b + c - (100c + 10b + a) = def$

giving $99(a - c) = def$. Since $a > c$, $c - a < 0$, thus $10 + (c - a) = f = 7$, giving $a - c = 3$ and $def = 297$.

$$\begin{array}{r} d\ e\ f \\ +\ f\ e\ d \\ \hline g\ h\ i\ j \end{array} \quad \text{becomes} \quad \begin{array}{r} 297 \\ 792 \\ \hline 1089 \end{array}$$

Therefore $d = 2, e = 9, f = 7, g = 1, h = 0, i = 8, j = 9$.
Since $a + c = j = 9$ and $a - c = 3$, $a = b$, $c = 3$.
$b = e - i = 9 - 8 = 1$.

Thus the complete solution is

$$\begin{array}{r} 613 \\ -316 \\ \hline 297 \\ +792 \\ \hline 1089 \end{array}$$

Also solved by Joseph Dence, Bowling Green State University, Bowling Green, Ohio; Dee Fuller, Davidson College, Davidson, North Carolina; Fredric Gey, Harvey Mudd College, Claremont, California; Robert Kurosaka, State University of New York, Albany; Elizabeth Ann O'Connell, San Jose State College, San Jose, California; Perry Smith, Albion College, Albion, Michigan; Dean Strenger, Nebraska State Teachers College, Wayne, Nebraska; Rickey Turkel, State University of New York, Albany; Patricia Woinoski, State University of New York, Albany.

142. *Proposed by R. C. Weger, William Jewell College, Liberty, Missouri.*

m and n are digits in a number system whose base is b, $m < b$, $n < b$. Find the largest $k < b$ such that

$$mn \equiv nm \pmod{k}$$

Solution by Fredric C. Gey, Harvey Mudd College, Claremont, California.

Since the base of the number system is b, $m \cdot b + n \equiv n \cdot b + m \pmod{k}$

or $$b(m - n) + (n - m) \equiv 0 \pmod{k}.$$

Factoring we obtain $(m - n)(b - 1) \equiv 0 \pmod{k}$.

Now since the left hand side must be divisible by k, and m and n are each less than b, then $k = b - 1$ is the greatest possible $k < b$.

Also solved by Fredrick Carty, Hofstra College, Hempstead, New York; Dee Fuller, Davidson College, Davidson, North Carolina; Perry Smith, Albion College, Albion, Michigan.

143. *Proposed by Ronald L. Hammett, San Jose State College, San Jose, California.*

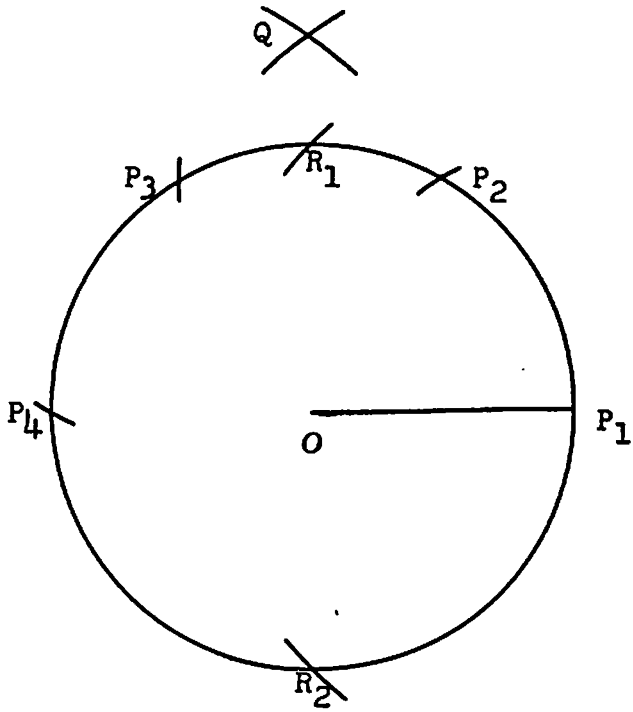
With compasses only, find the vertices of a square inscribed in a given circle.

Solution by Fredrick Carty, Hofstra College, Hempstead, New York.

Assume the center O of the given circle is known. Take any point P_1 on the circumference as one of the four vertices of the desired square. Now locate P_2, P_3 and P_4 consecutively on the circumference such that

$$P_1P_2 = P_2P_3 = P_3P_4 = OP_1$$

Each of these points P_i, $i = 2, 3, 4$ can be located on the given circle with a compass by using the radius of the circle and the point P_{i-1} as center, and continuing around the given circle.



Now locate the point Q which is P_2P_4 distance from both P_4 and P_1. Next draw the circle with radius OQ and center P_1. The points of intersection R_1 and R_2 of this circle with the given circle are two of the desired vertices, with P_1 and P_4 being the other two.

Proof of the above construction: Since P_1 is one vertex, another is the intersection of OP_1 with the given circle. Since P_1, P_2, P_3, P_4 are vertices of the regular inscribed hexagon, P_4 is a second vertex of the desired square. From this hexagon we also observe:

$$P_4Q = P_1P_3 = P_4P_2 = P_1Q = OP_1\sqrt{3}$$

Since $P_4Q = P_1Q$, triangle P_4QP_1 is isosceles and OQ is perpendicular to P_4P_1. From right triangle OP_1Q

$$(P_1Q)^2 = (OQ)^2 + (OP_1)^2$$

but $P_1Q = OP_1\sqrt{3}$, therefore $OQ = OP_1\sqrt{2}$, and since the side of an inscribed square is $\sqrt{2}$ times the radius this makes OQ equal to a side of the desired square. But $P_1R_1 = P_1R_2 = OQ$, therefore R_1 and R_2 are the other two vertices.

Also solved by Perry Smith, Albion College, Albion, Michigan.

144. *Proposed by Paul Chernoff, Harvard University, Cambridge, Massachusetts.*

Let $a_1(x) = x$, $(x \geqq 0)$ and $a_{n+1}(x) = \sqrt{a_n(x) + 1}$ for all $n \geqq 1$. Show that $\lim_{n \to \infty} a_n(x)$ exists for all $x \geqq 0$ and compute its value.

Solution by Perry Smith, Albion College, Albion, Michigan.

Since $a_n(x) = \sqrt{a_{n-1}(x) + 1}$ and $a_1(x) = x \geqq 0$, it is evident that $a_n(x) \geqq 0$. If we set $u = \sqrt{u + 1}$, we obtain the positive root $\dfrac{1 + \sqrt{5}}{2}$. We shall show that $r = \dfrac{1 + \sqrt{5}}{2}$ is the desired limit.

We examine three cases; either $x = r$, $x < r$ or $x > r$.

Case I. $x = r$

Since $r = \sqrt{r + 1}$ and $a_1(x) = r$, we have $a_2(x) = \cdots = a_n(x) = \cdots$ and the limit is obviously r.

Case II. $x < r$. We shall need two lemmas.

Lemma 1. If $a_n(x) < r$, then $a_{n+1}(x) < r$.

Proof: Let $a_n(x) = u$, $u < r$. Then $u + 1 < r + 1$ so that $a_{n+1}(x) = \sqrt{u + 1} < \sqrt{r + 1} = r$. Therefore r is an upper bound to the sequence.

Lemma 2. If $a_n(x) < r$, then $a_{n+1}(x) > a_n(x)$.

Proof: $x - \sqrt{x + 1}$ is easily shown algebraically or graphically to be negative for $0 \leqq x < r$. Thus $\sqrt{u + 1} > u$ or $a_{n+1}(x) > a_n(x)$.

Therefore the sequence is monotone increasing, and every such sequence which is bounded above has a limit; thus the given sequence has a limit.

We now show that this limit is r. We now know that

$$0 < \lim_{n \to \infty} a_n(x) \leqq r.$$

Assume the limit is L, where $0 < L < r$. Thus we could find terms in the sequence arbitrarily near (but never greater than) L. But we will show that there is a $\delta > 0$, such that $\sqrt{u + 1} > L$, whenever $u > L - \delta$, thus the limit cannot be L and must be r. Take $\delta = (1 + L - L^2)/2$. By lemma 2, $L < \sqrt{L+1}$, or $L^2 < L + 1$, thus $\delta > 0$. Also $\delta < 1 + L - L^2$ or $L - \delta + 1 > L^2$ and $\sqrt{L - \delta + 1} > L$. But from u $> L - \delta$ we have $\sqrt{u+1} > \sqrt{L - \delta + 1}$ so that $\sqrt{u+1} > L$. Thus the limit exists and is equal to r.

The proof for Case III is practically identical to Case II.

Also solved by Fredric Gey, Harvey Mudd College, Claremont, California, and Dee Fuller, Davidson College, Davidson, North Carolina.

145. *Proposed by C. W. Trigg, Los Angeles City College.*

If three alternate primes are in arithmetic progression, show that the difference between any two of them is greater than 5.

Solution by Dee Fuller, Davidson College, Davidson, North Carolina.

We divide the problem into three cases:

Case I. Suppose the common difference between the alternate primes is 1. Then they become x, $x + 1$, $x + 2$, but x is odd and thus $x + 1$ is even and not a prime. The same argument applies in the event the common difference is 3 or 5.

Case II. Suppose the common difference is 2. Then the alternate primes are x, $x + 2$, $x + 4$. Thus $x + 1$ and $x + 3$ must also be primes, but this is impossible by Case I.

Case III. Suppose the common difference is 4. Then the alternate primes are x, $x + 4$, $x + 8$. Since these are alternate primes, $x + 2$ and $x + 6$ are likewise primes.

Now any prime number gives a remainder of 1 or 2 when divided by 3. Now suppose that when the prime number x is divided by 3 the remainder is 1, then $x + 2$ would be equally divisible by 3, contradicting that it is prime. Next we suppose the remainder, when x is divided by 3, is 2. Then $x + 4$ would be equally divisible by 3, contrary to its being prime. Thus the common difference between three alternate primes cannot be 4.

Since in the various cases we have shown that the common difference between three alternate primes cannot be 1, 2, 3, 4 or 5, it is therefore greater than 5.

Also solved by Fredrick Carty, Hofstra College, Hempstead, New York; Fredric Gey, Harvey Mudd College, Claremont, California; Arie Paldervaart, University of New Mexico, Albuquerque; Perry Smith, Albion College, Albion, Michigan.

The Mathematical Scrapbook

EDITED BY J. M. SACHS

Mathematics as a science, commenced when first someone, probably a Greek, proved propositions about *any* things or about *some* things, without specifications of definite particular things. . .

—A. N. WHITEHEAD

=Δ=

As a variation of a popular problem in probability consider the following. How many people, minimum number, must be in a room in order that the probability that at least two of them were born on the same day of the week is greater than one-half? What is the minimum number needed so that the probability that at least three of them were born on the same day of the week is greater than one-half? The first question can be easily answered by looking at the probability that each new person admitted to the room was born on a day of the week different from the birthdays of those already in the room. The probability for the second entrant is thus 6/7, for the third entrant 5/7, for the fourth entrant 4/7, etc. The product (6/7)(5/7) is greater than one-half and the product (6/7)(5/7)(4/7) is less than one-half. Thus when there are four people in the room, the probability that at least two of them were born on the same day of the week is greater than one-half. How about the second question? If you find that too easy ask the same questions about sharing birth dates during the year.

=Δ=

The scientist is immediately struck by the way Einstein has utilized the various achievements in physics and mathematics to build up a coordinated system showing connecting links where heretofore none was perceived. The philosopher is equally fascinated by a theory which, in detail extremely complex, shows a singular beauty of unity in design when viewed as a whole.

—B. HARROW

Try this one on your non-mathematical friends and see if they think there is enough given information: Mr. *A* hired a crew of college boys to work for him for a week. The boys all earned the same amount. When Mr. *A* went to the bank he said to the teller, "I will pay the boys a total of $462 and you better give it to me entirely in singles because the boys are flat broke and won't be able to make change." Some of the boys pooled their entire earnings and bought a used car for $252. What is the smallest possible number of boys in the work crew and the smallest possible number sharing in the purchase of the car?

=Δ=

Triangular numbers are those capable of being represented by dots in a triangular array with one dot in the first row, two in the second, etc. and as many rows as there are dots in the last row.

1 3 6 10

In about 100 *A.D.* it was proved that if any triangular number was multiplied by 8 and then 1 added to the product, the result was a square. Can you prove this algebraically? Can you make a geometric argument based on the assembling of eight triangular arrays of dots and one extra dot?

=Δ=

The most important questions of life are, for the most part, really only problems of probability. Strictly speaking, one may even say that nearly all of our knowledge is problematical; and in the small number of things which we are able to know with certainty, even in the mathematical sciences themselves, induction and analogy, the principle means for discovering truth, are based on probabilities, so the entire system of human knowledge is connected with this theory.

—LAPLACE

=Δ=

The distinction between a mathematician and a statistician, according to students of statistics, can be made at the undergraduate level by the following test. The student is brought into a room in which is seated a very beautiful girl. He is told that he must stand on a chalk line and that he can cross the room in stages, each stage being half the remaining distance between himself and the girl. If he is a mathematician he will say, "Alas, I can never reach her!" If he is a statistician he will call back over his shoulder as he leaps forward, "I can get close enough for all practical purposes!"

=Δ=

Mathematics is an art because it creates forms and patterns of pure thought that exhibit the noblest achievement of the human mind. It has become one of the great humanities because it is a method of expressing, explaining and communicating man's total behavior. It still reigns as queen of the sciences in its clear, rigorous, logical structure, and in doing so serves as an ideal and goal for the perfection of the other sciences, as researchers in those fields seek to discover the laws of the physical, biological and social phenomena of our universe.

—H. F. Fehr

=△=

Can you prove that the square of any integer is equal to twice the sum of all the integers less than the original integer plus the original integer? $2[1+2+3+\cdots+(n-1)] + n = n^2$

=△=

Little Susy told her teacher that $4 \div 2 = 2$ but could not let well enough alone and went on to explain that she got the answer by subtraction. Poor Susy had chosen a special case in which $\frac{a}{b} = a-b$. Are there any other integral solutions besides $a=4$, $b=2$? What about rational solutions? What about real solutions? What about complex solutions?

=△=

In view of the current stress on fundamental research, the following quotation, more than fifty years old, is quite pertinent: "When the time comes that knowledge will not be sought for its own sake, and men will not press forward simply in a desire of achievement, without hope of gain, to extend the limits of human knowledge and information, then, indeed, will the race enter upon its decadence."

—C. E. Hughes

=△=

The *regula duorum falsorum* or The Double False Position Rule of Diophantus was applied as follows. Given an equation $f(x) = C$. Choose two values of x at random, say $x = a$ and $x = b$. Let $f(a) = A$ and $f(b) = B$. Then

$$x = \frac{a(B-C) - b(A-C)}{B-A}.$$

If $f(x)$ is a linear function of x this can be shown to provide a solution for $f(x) = C$ as follows. Let $y = f(x) - C$. Then $(a, A - C)$ and $(b, B - C)$ are coordinates of two points on the line whose equation is $y = f(x) - C$. If we write the equation of the line determined by these two points and ask for the value of x when y is zero we get the expression above.

What does this expression for x yield if $f(x)$ is taken as a quadratic function of x, e.g., $f(x) = Dx^2 + Ex$? If this is regarded as an approximation for x, how will you get a closer approximation? Can you show that the approximations you obtain converge to a value which satisfies the equation $f(x) = C$?

=△=

What's wrong here?

$$\frac{n}{1-n} = n + n^2 + n^3 + \cdots \text{ and } \frac{n}{n-1} = 1 + \frac{1}{n} + \frac{1}{n^2} + \cdots,$$

so

$$\cdots + \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \cdots = \frac{n}{n-1} + \frac{n}{1-n} = 0.$$

Does this puzzle you? If so you can feel comforted by the knowledge that the illustrious Euler fell into this trap, paradoxically in an essay in which he was urging great caution in use of divergent series.

=△=

Algebra is generous, she often gives more than is asked of her.

—D'Alembert

=△=

Before the introduction of the Arabic notation, multiplication was difficult, and the division even of integers called into play the highest mathematical faculties. Probably nothing in the modern world could have more astonished a Greek mathematician than to learn that, under the influence of compulsory education, the whole population of Western Europe, from the highest to the lowest, could perform the operation of division for the largest numbers. This fact would have seemed to him a sheer impossibility. . . .Our modern power of easy reckoning with decimal fractions is the most miraculous result of a perfect notation.

—A. N. Whitehead

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Harold E. Tinnappel, Bowling Green State University, Bowling Green, Ohio.

Electronic Computers, Principles and Applications, Second Edition, T. E. Ivall, Philosophical Library Inc., (15 East 40th Street) 1960, 263 pp., $15.00

This is the second edition of a book published in 1956. This edition is brought up to date on some recent developments on computer circuits and programming. Diagrams and pictures of some of the modern electronic computer equipment are introduced in this edition.

The book is intended as a non-mathematical and non-technical introduction to the computer field. The book is not intended to be a textbook for introducing the subject of electronic computers; and since it gives a fairly general and elementary description of the circuit techniques and construction of digital and analogue computers, it is not a book written for the computer experts.

A layman with a beginning interest in computers will find Chapters 1. Evolution of the Computer, 2. General Principles of Computing, 6. Applications of Analogue Computers, 12. Applications of Digital Computers, and 14. Computers of the Future readable and informative. However, since Chapters 3. Analogue Computing Circuits-1, 4. Analogue Computing Circuits-2, 5. Equipment of Analogue Computers, 7. Digital Computer Circuits-1, 8. Digital Computer Circuits-2, 9. Storage Systems, and 13. Recent Developments are devoted to describing the circuit techniques and construction of analogue and digital computers which employ various electronic devices, the student will find a knowledge of electronic and circuit techniques a necessity.

Some of the many applications of computers in science, research, commerce and industry are outlined, and the uniformity in the design of general-purpose computers and the basic units of such computers are explained in fairly non-technical terms.

The author has done a good job in carrying out the purpose of this book, and the students interested in electronic computers will find it worth reading. However, the same students may find the price of the book beyond their means.

—J. A. JENSEN
State College of Iowa

Modern Computing Methods, Second Edition, Staff of Mathematics Division, National Physical Laboratory, Teddington, Middlesex, Philosophical Library, Inc. (15 East 40th Street), New York, 170 pp., 1961, $6.00.

This book is regrettably anonymous but was actually written by C. W. Clenshaw, E. T. Goodwin, D. W. Martin, G. F. Miller, F. W. J. Olver, and J. H. Wilkinson, members of the staff of the Mathematics Division of the British National Physics Laboratory. It is intended as a working manual for all scientists and engineers who engage in computations which are not of a trivial nature. It would also provide an excellent basis for courses in numerical analysis at universities and colleges of technology.

The well-merited success of the first edition (1958) and the rapid advances in the field of automatic computation were the reasons for this complete revision. The book has been largely rewritten and expanded and is now in hard covers. The object of the revision has been to bring the material up to date, particularly with regard to methods suitable for automatic computation. The chapter on Relaxation Methods has been replaced by one on Linear Equations and Matrices: Iterative Methods, while the chapter on Computation of Mathematical Functions has been expanded into two chapters entitled Evaluation of Limits: Use of Recurrence Relations, and Evaluation of Integrals. Most of the other chapters have been rewritten with new material added. There are also entirely new chapters on Linear Equations and Matrices: Error Analysis, and Chebyshev Series. Other chapters treat zeros of polynomials, finite-difference methods, ordinary differential equations, partial differential equations, and the tabulation of mathematical functions. The first five chapters are concerned with linear equations and matrices. A comprehensive and fully annotated bibliography of 217 items is appended.

There is as yet no textbook for a modern integrated version of a pure mathematical field and its numerical application. However, of all the numerical analysis texts published in recent years, this is the only one which seems really concerned with the problems of automatic digital computation.

—David M. Krabill
Bowling Green State University

Electronic Business Machines, J. H. Leveson (Editor), Philosophical Library, Inc. (15 East 40th Street), New York 16, 1960, 272 pp., $15.00.

This book is based on a series of lectures delivered at Dundee Technical College during June and July, 1958. These lectures have been rewritten and arranged in a manner which provides a study of the scope, characteristics, operation, and application of computers.

The nineteen chapters have been grouped into three main divisions. The first group is entitled "Programming for Business Purposes," and is designed to explain what computers are and how they work. This group contains seven chapters and begins with a discussion of the instruction codes used with computers and how these may be used to construct programs. A good basic foundation for computer programming is laid here, with several examples of business problems which might be solved by the computer and an explanation of the actual step-by-step programming of each problem. The whole process of programming, including coding, flow-charts and check-out procedures, is covered adequately in these first seven chapters.

The second division of chapters is entitled "Business Management and Electronic Data Processing," and includes seven more chapters. Here are treated the considerations which a businessman must bear in mind when considering the use of a computer in his business. First we find an examination of business procedures and problems as they might appear from the computer's standpoint. Other topics, such as the relationship between the size of the organization and the size of the computer which it might require, are discussed. Also two chapters pertain to the selection and training of personnel for computer teams. In the final two chapters the economic aspect of computers is examined along with some of the problems of auditing and control.

The last division in the book is "Computer Equipment and Applications." In this division the main topics of discussion deal with actual physical aspects of the computer, such as input and output equipment, the relationship between computers and magnetic tape.

All of the usual input and output devices such as punched paper tape, punched cards, magnetic tape, and on-line printers are described along with some of the advantages and disadvantages of each.

For the reader who is interested in reading more about some particular phase of computers, perhaps programming or computer design, there are lists of references appearing at the ends of several of the chapters. The example programming problems which appear at the ends of Chapters 2 and 3, should provide the interested reader with an opportunity to test his understanding of program writing.

The list of contributors is quite impressive and consists of men from many different areas using computers. As was mentioned in the preface, since there are a number of different contributors, it is inevitable that a certain amount of repetition occur, but this has been kept at a minimum.

—Jimmy M. Rice
Fort Hays Kansas State College

Statistical Theory and Methodology in Science and Engineering, K. A. Brownlee, John Wiley & Sons, Inc., (440 Fourth Avenue) New York 16, 1960, 570 pp., $16.75.

This book gives a theoretical background for statistical methods considered to be of importance to scientists and engineers. With only concise reference to probability, it progresses from a brief but careful development of basic mathematical theory of statistics, including discrete and continuous distribution to ideas of statistical inference and point estimation. Nonparametric tests, analysis of variance, linear regression, analysis of covariance, simple experimental design, and multiple regression are among the topics covered with an emphasis on mathematical development.

Models for standard statistical methods are included in the various topics, and considerable attention has been given to assumptions involved in their use. Reasons for using a particular method are stated and interpretation of the results of its use are given. Pitfalls of incorrect application of the method are also discussed.

A feature of the book, especially in its use as a reference for theory behind the application of a statistical method, is the careful documentation of sources. One gains the impression that reference to the citations would give very adequate development of the theory involved, should one wish to pursue a more comprehensive theoretical discussion.

The author characterizes the book as an elementary work on statistical methods. The mathematical techniques required for using the book are presumably not much beyond the college algebra level. There would seem to be a distinct possibility that such a minimal background would be less than adequate to make for efficient use of the book by a student. Considerable facility with the notation seems necessary on the part of a reader, and as the author implies elementary calculus can be helpful in certain sections. None of this reflects on the excellent quality of the presentation however.

—EARLE L. CANFIELD
Drake University

An Introduction to Mathematical Statistics, H. D. Brunk, Ginn and Company, Boston, Massachusetts 17, 1960, xii + 403 pp., $6.50.

This book is prepared for a full year's work in probability and statistics. It gives the student the essentials of the first year's course in statistics and prepares him for more advanced work in this field. In its approach to both fields, probability and statistics, the text is quite clear and fundamental. The whole text is unique and comprehensive.

The book begins with 92 pages given to the subject of probability. These chapters give the elementary notions of probability necessary for the understanding of statistics. The fundamental concepts are clearly stressed and reinforced with appropriate illustrations. The idea of the mathematical model is used throughout these chapters. It seems that these 92 pages with some supplemental work would make a good three hour semester course in probability alone. Elementary Probability Spaces, General Probability Spaces, Random Variables, Combined Random Variables, The Algebra of Expectation are the five chapter headings, all of which are adequately discussed.

This section of the book is followed by 264 pages, treating the regular topics of elementary statistics. These chapters are very carefully developed by mathematical methods. While this is supposed to be an elementary treatise on statistics, it does lead the student into advanced thinking on this subject. And it gives a very broad viewpoint of elementary statistics.

The organization of the material is such that the instructor can adapt it to the special needs of the students. The chapter on regression is very well covered and treated in a more general manner than is usual. Random sampling as well as the other usual topics of statistics are treated at length. A knowledge of mathematics through integral calculus is necessary on the part of the student before attacking this text.

Another feature which makes the text more flexible is the introduction of additional material. This is done by means of starred chapters and sections. These sections may be used or omitted according to the wish of the instructor or to the length of time devoted to the course.

The book is well edited. The pages are attractive. The symbols and mathematical formulas are printed in a pleasing manner. The illustrations are clear. There are in each chapter lists of well chosen exercises.

—J. E. Dotterer
Manchester College

Time Series Analysis, E. J. Hannan, (Methuen's Monographs on Applied Probability and Statistics), John Wiley & Sons, Inc., (440 Fourth Avenue) New York 16, 1960, 152 pp., $3.50.

This book is one in a new Methuen series of monographs; the existing Methuen series in physics and biology has had as its object to provide short, inexpensive treatments to serve as "an introduction to, or revision of, specialized topics."

The new series deals with specialized topics in the theory and application of probability and statistics. The monograph under review here serves as an admirable introduction to the probabilistic approach to statistical analysis of time series. What might be termed the "traditional" approach, at least as practiced in economic time series analysis, involves such procedures as trend removal, seasonal analysis, moving averages, etc., but no underlying probability model from which statistical inferences can be drawn. Thus, the probability approach to the problems involved in time series analysis provides a more suitable framework than previously available.

This approach requires a background knowledge of probability and statistics of about the level embodied in Cramer's *Mathematical Methods of Statistics* plus some knowledge of infinite dimensional vector spaces, which Hannan eases by the manner in which he introduces the subject.

In order to restrict the monograph to small size and introductory nature, the discussion proceeds only through the case of a univariate time series consisting of a time-dependent mean plus a stationary component.

Estimation of the correlogram and of the spectral density, hypothesis testing and confidence intervals are considered. In addition, in the chapter on processes containing a deterministic component, the estimation of the regression coefficients, testing for departure from independence and the effect of trend removal on the analysis of the residuals are considered. An appendix gives proofs of the main theorems.

Of course, in a book of this nature, the emphasis is on an explanation of the main body of theory. However, the examples which are used illustrate very well the theory and the wide variety of possible applications.

In summary, *Time Series Analysis* is a concise but well-written introduction to modern theory in this field. The necessary mathematical background will be difficult for average college students of today, but not for such students in the future, as the curriculum moves to meet our needs and interests.

—PAUL D. MINTON
Southern Methodist University

Stochastic Processes, Problems and Solutions, Lajos Takacs, Methuen Monograph of Applied Probability and Statistics, John Wiley and Sons, Inc., (440 Fourth Avenue) New York 16, 1960, 137 pp., $2.75.

This is the third in the very welcome Methuen series of monographs on Probability and Statistics, a series designed as short introductions to specialized topics in these subjects.

This volume is a 62-page introduction to the theory of stochastic processes and applications of this theory. The applications are presented in the form of problems with solutions, which take up the remainder of the book.

Three chapters cover Markov chains (transition probabilities and limits, classification, limiting distributions, continuous state space and stationary stochastic sequences), Markov processes (Poisson process, Markov processes with finite or denumerably infinite states, with continuous transition, mixed Markov processes) and Non-Markovian processes (recurrent processes, stationary stochastic processes, secondary stochastic processes generated by a stochastic process). Definitions are given and theorems stated for these topics.

Problems with solutions illustrating applications of this theory are selected from a wide variety of fields. Some examples may indicate the scope: random walks with absorbing and reflecting barriers, waiting-time problems, cascade process in an electron multiplier, chain reaction in a nuclear reactor, nuclear decay, emission from electron tubes, spatial distribution of stars, distribution of telephone exchange calls, machine interference problem, birth and death process, diffusion process, drifting of stones on river beds, lifetime of machine components, random scaling, shot-noise and many others.

—PAUL D. MINTON
Southern Methodist University

Commutative Algebra, Vol. II, Oscar Zariski and Pierre Samuel, D. Van Nostrand Company, Inc., Princeton, New Jersey, 1960, 414 pp., $7.75.

This second volume gives a detailed and sophisticated analysis of valuation theory, polynomial and power series rings, and local algebra, and completes the first systematic treatment of commutative algebra since the 1935 Monograph *Idealtheorie* by W. Krull. It uses the more classical material of Volume I to present those topics of commutative algebra which are of a more advanced nature and a more recent vintage.

Some interesting features of this volume are "Instructions to the Reader", in which the authors indicate which sections could be skipped during a first reading; a detailed "Index of Definitions; and seven appendices which treat special topics of current interest (e.g. Relations Between Prime Ideals, Valuations in Noetherian Domains, Valuation Ideals, Complete Modules and Ideals, Complete Ideals in Regular Local Rings of Dimension 2, Macaulay Rings, and Unique Factorization in Regular Local Rings).

Since much of the material appears here for the first time in book form, and since a good deal of the material is new and represents current or unpublished research, this book meets the need for a text in an advanced graduate course in commutative algebra, as well as providing a current standard reference work in this area. Since the algebro-geometric connections and applications of the purely algebraic material are constantly stressed, this volume can also be used as an introduction to the arithmetic foundations of algebraic geometry.

This book should probably not be attempted by anyone who has not completely mastered Volume I or its equivalent.

—G. OLIVE
Anderson College

An Introduction to the Theory of Numbers, Ivan Niven and Herbert S. Zuckerman, John Wiley and Sons, Inc., (440 Fourth Avenue) New York 16, 1960, 250 pp., $6.25.

According to the preface, this book is intended for seniors and beginning graduate students and can be used for either a one semester or a full year course in the Theory of Numbers. The author has succeeded in writing a book which is well suited for students at this level and which certainly contains enough material for a full year course.

Assuming a knowledge of the real number system the author develops the basic theory in the first four chapters. The titles of these chapters; Divisibility, Congruences, Quadratic Reciprocity, and Some Functions of Number Theory, indicate their content. Later chapters deal with more specialized topics such as Diophantine Equations, Farey Fractions, Continued Fractions, Distribution of Primes and the Partition Function.

The development of the theory is clear and concise. All basic theorems are stated precisely and proofs are given for those theorems which present any real difficulty. This reviewer was especially impressed by the well-graded problem lists. These lists contain problems of all degrees of difficulty, from those requiring numerical substitution into a formula, to those requiring real mathematical insight. The student will be challenged by these exercises and will derive considerable satisfaction from their solution. Answers are given where appropriate.

This book represents a real addition to the literature in the field of number theory. Those colleges that plan to offer such a course for prospective secondary teachers will do well to consider it.

—W. Toalson
Fort Hays Kansas State College

In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure.

—Hermann Hankel

Installation of New Chapters

Edited by Sister Helen Sullivan

MISSOURI ZETA CHAPTER

Missouri School of Mines and Metallurgy, Rolla, Missouri

Missouri Zeta Chapter was installed on Friday, May 19, 1961 at Missouri School of Mines and Metallurgy, Rolla, Missouri. Dr. Carl V. Fronabarger, National President of Kappa Mu Epsilon, was the installing officer.

The installation took place at 7:00 p.m. Charter members are as follows: Leroy H. Alt, Andrew T. Aylward, Thomas B. Baird, Roger A. Barney, Richard W. Bolander, Albert E. Bolon, John S. Bosnak, John R. Burrows, Donald Eugene Burton, Carl S. Cave, David Dautenhahn, Curtis W. Dodd, Arthur Addison Duke, Dickran H. Erkiletian, Larry E. Farmer, Gerald O. Finne, August J. Garver, Robert Alfred Harris, Jr., Gary W. Havener, Joseph H. Hemmann, Lowell Lee Henson, Lawrence Linden Hoberock, Gerald Huck, John David Jarrard, Charles A. Johnson, James Walter Joiner, William Roy Jones, Jr., Walter Fred Kern, Jr., Richard Harry Kerr, John B. Kincaid, Jesse Wayne Knaust, Karl R. Kniele, John Virgil Knop, Frederic L. Kurz, J. Richard Leach, Ralph E. Lee, William Lawrence May, Charles C. McPheeters, Virgil E. Meredith, James R. Miller, Terry Lee Mills, Jimmie C. Morgen, Robert H. Nau, John A. Nelson, Richard E. Oeffner, S. J. Pagano, Daniel N. Payton, III, Fred Plassman, Alfred R. Powell, Howard D. Pyron, Rolfe M. Rankin, John A. Reagan, Dennis B. Redington, Ronald J. Rozell, Robert Olto Schwenker, Jack Mason Scrivner, Allan Noel Sheppard, W. Wayne Siesennop, Bert L. Smith, Lyman T. Smith, Raymond Francis Smith, Ellis Speicher, III, Glenn Earl Stoner, F. David Utterback, Ronald Wayne Walters, Frank Garnett Walters, Daniel Ralph White, William John Wolf, Jr., John J. Zenor, Robert E. Thurman.

Officers installed at the Ceremonies were: Arthur Duke, President; James Miller, Vice-President; Virgil Meredith, Recording Secretary; John Bosnak, Treasurer; Clellen McPheeters, Historian; and Professor D. H. Erkiletian, Corresponding Secretary. Dr. Charles Johnson was elected as Faculty Sponsor.

Following the installation Dr. Fronabarger gave an address to the group on the subject "Game Theory".

Kappa Mu Epsilon News

EDITED BY FRANK C. GENTRY, HISTORIAN

The Thirteenth Biennial Convention of Kappa Mu Epsilon was held April 21, 22, 1961 with Kansas Beta Chapter at Kansas State Teachers College, Emporia as host chapter. Thirty-four chapters were represented by forty-eight faculty members and guests and two hundred-eight students. Two institutions without chapters were represented by two faculty members and five students. This gave the Convention a total attendance of 263.

FRIDAY, APRIL 21, 1961

The meetings were held in the Terrace Room of the Student Union. Professor Carl V. Fronabarger, Missouri Alpha, National President of Kappa Mu Epsilon, presided. President John E. King of Kansas State Teachers College welcomed the delegates to the campus. Professor R. G. Smith, National Vice-President responded for the Society. The following chapters, installed since the last national convention were welcomed:

Virginia Beta, Radford College
Nebraska Beta, Nebraska State Teachers College at Kearney
Ohio Delta, Wittenberg University
Ohio Epsilon, Marietta College
Florida Alpha, Stetson University
Alabama Delta, Howard College
New York Delta, Utica College

A petition for a chapter of Kappa Mu Epsilon at the Missouri School of Mines and Metallurgy at Rolla was presented. It was approved unanimously.

Professor R. G. Smith presided during the presentation of the following papers:

1. *How to Code with Matrices,* Patricia R. Swope, Kansas Gamma, Mount St. Scholastica College.
2. *Cryptography with Matrices,* Barbara Mann, Iowa Alpha, State College of Iowa.
3. *An Analysis of Some of the Syllogisms Found in Alice in Wonderland,* Ruth Goodrich, Illinois Alpha, Illniois State Normal University.
4. *The Symmetries of Geometric Figures,* Ronald Weger, Missouri Gamma, William Jewell College.

After lunch in the student cafeteria the faculty members and students separated into two sections to discuss the proposed "Handbook for K.M.E.". The entire group reconvened at 2:00 p.m. and, after reports from the two sections, the following papers were read:

5. *Prime Numbers,* Robert Kennedy, Missouri Beta, Central Missouri State College.
6. *Graphical Integrations,* Bruce Berndt, Michigan Alpha, Albion College.
7. *Graphical Solution of Quartic Equations,* S. McDowell Steele, Jr., Kansas Alpha, Kansas State College of Pittsburg.
8. *Desirable Properties of Axiomatic Systems,* Leroy Adams and William Ted Stout, Missouri Alpha, Southwest Missouri State College.

The convention banquet was served in the Colonial Ballroom of the Student Union with Dr. O. J. Peterson as Master of Ceremonies. Dr. C. C. Richtmeyer, Past President of K.M.E. gave the invocation. Dr. H. C. Fryer, Head, Department of Statistics and Director of the Statistical Laboratory at Kansas State University was the guest speaker. His subject was "Uncertain Inferences".

SATURDAY, APRIL 22, 1961

The program started at 9:00 a.m. with the following student papers:

9. *Elliptic Integrals,* Arnold Hammel, Michigan Beta, Central Michigan University.
10. *An N Focus Locus,* John O. Kork, Colorado Alpha, Colorado State University.
11. *Mathematics and Literature,* Douglas A. Swan, Kansas Beta, Kansas State Teachers College, Emporia.
12. *Comments on Pascal's Triangle,* Robert M. Myers, Illinois Gamma, Chicago Teachers College.
13. *The Special Role of Cauchy Sequences in the Construction of the Number Systems,* Patricia Leary, Kansas Gamma, Mount St. Scholastica College.

At the second general business session, the reports of the national officers were read as well as the report of the auditing committee. President Fronabarger reported that he and Miss Greene, the National Secretary, had filed a Certificate of Amendment of the Certificate of Incorporation stating that Kappa Mu Epsilon is a non-profit organization and that any assets of the corporation at such time as it might be dissolved would be turned over to the Mathematical Association of America.

Illinois Alpha, Iowa Beta, Kansas Epsilon, Nebraska Beta, Ohio Delta, and Pennsylvania Gamma extended invitations for the 1963 Convention. The site will be selected by the National Council.

Richard Lewis, Kansas Alpha, reported that the student committe appointed to study the minimum national standards would investigate the matter more carefully, study the chapter requirements and bring their report to the regional meetings for discussion.

Dr. C. C. Richtmeyer reported for the nominating committee. There were no nominations from the floor and the following list of national officers were elected for 1961-63.

President	Dr. Carl Fronabarger Southwest Missouri State College
Vice President	Dr. Harold Tinnappel Bowling Green State University
Secretary	Miss Laura Greene Washburn University
Treasurer	Prof. Walter C. Butler Colorado State University
Historian	Dr. Frank C. Gentry University of New Mexico

Douglas R. Bey, Illinois Alpha, chairman of the awards committee made the following awards to the students named for papers presented during the convention.

First Place	Barbara Mann, Iowa Alpha
Second Place	Bruce Berndt, Michigan Alpha
Third Place	Ruth Goodrich, Illinois Alpha

Sister Helen Sullivan, Kansas Gamma, reported for the resolutions committee. The following resolutions were adopted.

Whereas this Thirteenth Biennial Convention of Kappa Mu Epsilon has been a most successful and profitable conference, be it resolved that we express our appreciation:

1. The the host chapter, Kansas Beta, and to Kansas State Teachers College for their fine hospitality, the use of their comfortable facilities, the excellent food and for all the other intangibles that contribute to the success of such a meeting as this.

2. To each of the national officers whose untiring efforts and continual inspiration and direction are responsible for the growth in numbers and prestige of our society. To Professor Carl Fronabarger, our national president, for arranging this very fine convention. To Vice President R. G. Smith for his special work in connection with student papers. To Professor Charles Tucker and his mathematics staff for the very splendid arrangements that have made our stay here so very pleasant. To all those both on the spot and behind the scenes who have in any way contributed to the smooth functioning of this conference.

3. To the editor and staff of The PENTAGON for the continued publication of the very fine magazine which reflects the best efforts of the society members.

4. To the thirteen students who prepared and presented excellent papers at these sessions as well as to all the other students who contributed by their presence and their scholarly attitude to the convention.

5. To all here present for the warm spirit of fellowship and courtesy that makes these meetings so memorable.

REPORT OF THE NATIONAL PRESIDENT

We are living today in a period of great ferment and change in the field of mathematics. Increased interest in this area, together with skyrocketing college and university enrollment, has caused an upsurge in the number of college and university mathematics societies or clubs which have petitioned for affiliation with Kappa Mu Epsilon. Since the time of the last Biennial Convention, held two years ago at Bowling Green University, Bowling Green, Ohio, six new chapters of Kappa Mu Epsilon have been established: Ohio Delta, Wittenburg University; Florida Alpha, Stetson University; Indiana Delta, Evansville College; Ohio Epsilon, Marietta College; Alabama Delta, Howard College; New York Delta, Utica College. We now have a total of more than sixty active chapters. A chapter has been approved for Pan American College, Edinburg, Texas, and it is expected that the installation of this chapter will be held early in the fall. By your action at this convention, the petition for a chapter at Missouri School of Mines and Metallurgy, Rolla, Missouri, has been approved, and installation of this chapter will take place as soon as arrangements can be made.

In 1958, one of the chapters made inquiry as to whether

Kappa Mu Epsilon has been declared exempt from Federal Income Tax by the Internal Revenue Service. If the society was tax-exempt, a gift was going to be made to help pay the expenses of a student to the 1959 Biennial Convention. Examination of the records revealed that the Articles of Incorporation were not written in such a manner as to give the organization tax-exempt status. As a result of much correspondence between the National Officers and the Internal Revenue Service we have now to come to the position of being able to achieve tax-exempt status. As a result of the passage, by the delegates to the convention, of the resolutions pertaining to this matter, the National Officers will be able to take the final steps necessary to make Kappa Mu Epsilon exempt from Federal Income Tax. It should be noted that this tax-exempt status applies to the national organization and not to the local chapters. Subordinate units should file separate applications for exemption.

Development of regional conventions for the even numbered years is making progress. In the spring of 1960, regional conventions were held at Washburn University of Topeka and Illinois Normal University. Eleven chapters for Kansas, Missouri, and Nebraska participated in the Regional Convention at Washburn University, seven chapters form Illinois, Indiana, Michigan, and Wisconsin participated in the Regional Meeting at Illinois Normal University. It is anticipated that a regional meeting will be held in 1962 in the New York, New Jersey, Pennsylvania area. The National Council is anxious to see the development of more of these regional meetings. To encourage this development, the National Organization, in the event that at least three chapters participate, will allow the host institution $100 to help defray the expenses of a regional convention.

As will be noted from the financial report of the treasurer, the organization is in a sound financial condition. We are not a profit-making body and to maintain our tax-exempt status we should not increase our assets to an unreasonable amount. Underwriting numerous regional conventions in the even numbered years would probably use up most of our gain during a biennium. The National Officers should also give consideration to the esablishment of scholarships or other awards.

I wish to express my appreciation to all those who have contributed time and energy to the organization during the past biennium: to Past President C. C. Richtmeyer for his advice and assistance with respect to the duties of the office of President; to Vice-President R. G. Smith who has accepted the primary responsibilities in connection with solicitation and selection of student papers to be presented at this convention; to Secretary Laura Z. Greene for the long hours and tedious work that has been her lot as National Secretary; to Treasurer Walter C. Butler who has, in addition to his regular duties, made many extra reports in connection with obtaining tax-exempt status for Kappa Mu Epsilon; to Historian Frank C. Gentry for ably performing his duties; to all those sponsor, corresponding secretaries, chapter officers, and others who have been responsible for the effective functioning of local chapters; to those who have organized and conducted regional meetings, to those students who have prepared papers for regional conventions or this National Convention; to Professor Charles B. Tucker and his associates who have made the local arrangements for this Thirteenth Biennial Convention; and to those who have served on other committees in connection with this convention.

Special thanks are due to Fred W. Lott, Jr., Editor, and Wilbur J. Waggoner, Business Manager of THE PENTAGON, for their work in producing and distributing the official magazine of the organization. This magazine is a potent force in establishing and maintaining fraternal ties among the chapters and in promoting the objectives of the society.

To all of you I wish to express my sincere appreciation for your efforts in making this a successful biennium for Kappa Mu Epsilon. It is my hope that expanding enrollment and changing curriculum, with their demands upon the time and energies of faculty and students alike, will not be allowed to turn our organization into strictly an honor society. Encouragement of students to engage in undergraduate research and opportunities to present well-written papers has been a part of our tradition that must be maintained. As we look forward to the next biennium, I trust that your enthusiasm, your efforts, and your loyalty to Kappa Mu Epsilon will enable the organization to continue to achieve its purposes and ideals.

—Carl V. Fronabarger

REPORT OF THE NATIONAL VICE-PRESIDENT

The principal responsibility of the vice-president is to plan and arrange for the program of student papers. Thanks to the fine response from students, as well as the support and cooperation of chapter sponsors, this assignment was reduced to a minimum.

An invitation to present papers printed in the Pentagon was followed on December 6, 1960, by a letter to chapter sponsors reminding them of this invitation and directions and rules for papers to be presented. As a result a sufficient number of papers of high quality were received on or about February 1, 1961.

It seems to me, and with no credit to myself, that the organization and excellent presentation of these papers has set a new high record. Although the thirteen papers on the program were prepared on twelve campuses, therefore representing independent selection of topics, there was almost no duplication of material. Where two papers were closely related by topic selection, one seemed to support or complement the other.

Particularly do I wish to mention the fine degree of timing adhered to by each student, showing hours of preparation and rehearsal. The least needed part of our planning and arrangements was the time-keeper, Dr. J. D. Haggard.

The toughest assignment undoubtedly is to serve on the Awards Committee.

—RONALD G. SMITH

REPORT OF THE NATIONAL SECRETARY

Thirty years ago, April 18, 1931, the first chapter of Kappa Mu Epsilon, Oklahoma Alpha, was established at Northeastern State College at Tahlequah. A month later, May 27, 1931, the second chapter, Iowa Alpha was established at Iowa State Teachers College, at Cedar Falls. New chapters were installed so that ten years later there were twenty-five chapters and in 1951 there were forty-eight chapters. Today, we have sixty-seven chapters with sixty-three of them active. During the past biennium eleven new chapters were installed making a total membership of more than fifteen thousand.

The permanent record card of each member is filed in the office of the secretary. Orders for membership certificates, jewelry orders and orders for charters for chapters are all approved and copies filed in the secretary's office. The initiation report that the corresponding secretary of the local chapter makes is the basis on which we order your membership certificates. Each correct report saves time for us and for you. I want to thank all of you for the care given these and other reports. You people make the work of the secretary a pleasure.

—LAURA Z. GREENE

FINANCIAL REPORT OF THE NATIONAL TREASURER

April 4, 1959 to April 14, 1961

Cash on hand April 4, 1959			$5797.94
Receipts			
Initiates (2153 at $5.00)	$10765.00		
Miscellaneous (Supplies, installations, etc.)	674.49		
Total receipts from chapters		$11439.49	
Miscellaneous receipts			
Interest on bonds	$160.40		
Balfour Company (Commissions)	273.35		
The Pentagon (Surplus)	247.32		
Total Miscellaneous Receipts		681.07	
Total Receipts			12120.56
Total receipts plus cash on hand			$17918.50
Expenditures			
National Convention, 1959			
Paid to Chapter Delegates	$2560.64		
Officers Expenses	612.72		
Miscellaneous (Prizes, host chapter, expenses, programs, speaker, etc.)	190.39		
Total National Convention		$3363.75	
Balfour Company (Memberships, certificates, stationery, etc.)		1916.35	
Pentagon (Printing and mailing four issues)		5180.91	
Installation Expense		264.65	
National Office Expense		919.37	
Total Expense			$11645.03
Cash Balance on Hand April 14, 1961			6273.47
Total Expenditures Plus Cash On Hand			$17918.50
Bonds On Hand April 14, 1961	$3000.00		
Savings Account + $161.81 int.	2788.43		
Total		$5788.43	
Total Assets as of April 14, 1961			$12061.90
Total Assets 1959			11425.19
Net Gain for Period			$636.71

—WALTER C. BUTLER

REPORT OF THE NATIONAL HISTORIAN

Since there are now so many chapters of Kappa Mu Epsilon, approximately 60, it has become necessary to edit the news items sent in for the Kappa Mu Epsilon News section of THE PENTAGON. This may cause some disappointment on the part of members who fail to see their names listed. I have tried to select those items which seemed to me to carry the most interest for the most people. The original reports sent me by the corresponding secretaries of the various chapters are filed in the historical file.

An examination of the file reveals a wide variation in content. There is a complete file of THE PENTAGON. The files of some of the early chapters—Alabama Alpha, Alabama Beta, Kansas Alpha, Kansas Beta, Kansas Gamma, and Nebraska Alpha, to mention a few—are bulging with newspaper clippings and other materials all neatly pasted on sheets of colored paper. The early historians—Miss Lay, Miss Hove, and Miss Culmar—must have worked many hours at this task. On the other hand, many of the folders of more recently installed chapters are completely empty, not even containing a copy of the original petition. It is my intention to try to see that every folder, representing an active chapter, contain a copy of the petition, a copy of the installation program, and periodic reports from the corresponding secretary. In order to implement this program, I propose to reinstate a custom started by Sister Helen Sullivan when she was National Historian. I would furnish each corresponding secretary a blank each fall and each spring on which he is asked to list the officers of his chapter and the programs for that half-year together with any news of special events concerning his chapter or its members. These forms will be used in compiling the materials for the Kappa Mu Epsilon News section of THE PENTAGON and then filed in the chapter's folder. Thus a continuous history of each chapter would accumulate in the national historian's files. I have already asked the National Secretary to help in this matter and she has kindly consented to do so. It may be necessary to ask assistance from other of the national officers in the next few months.

I am sorry that I am unable to attend the National Convention. I have asked Miss Mitchell to represent me at the Council Meeting, to present this report, and to bring me copies of all materials that should be in my files.

—Frank C. Gentry

REPORT OF THE EDITOR OF THE PENTAGON

A publication such as THE PENTAGON cannot exist without the voluntary contribution of many people. I would like to express my appreciation and that of Kappa Mu Epsilon to all who have so ably assisted in making our journal possible. Jerome Sachs of Chicago

Teachers College has edited the Mathematical Scrapbook. The Book Shelf Editor for the first issue of this biennium was R. H. Moorman, Tennessee Polytechnic Institute, while Harold Tinnappel of Bowling Green State University has continued in this capacity for the major portion of the biennium. Our National Historian, Frank C. Gentry, University of New Mexico, edits the KME News section. The Problem Corner Editor is J. D. Haggard, Kansas State Teachers College, Pittsburg. The reports of the Installation of Chapters have been edited by Mable Barnes of Occidental College. Considerable assistance has been received from the Business Manager of THE PENTAGON, Wilbur Waggoner at Central Michigan University, and from the National Secretary, Laura Greene, Washburn Municipal University. We are indebted to all of these persons for their time-consuming and effective work.

There have been three issues of THE PENTAGON published since I became the editor in 1959. The fourth, the 1961 Spring issue, is now in the hands of the printer and you should receive it early in May. In addition to chapter news, book reviews, the Problem Corner, and the Mathematical Scrapbook, there have been nineteen articles published in these four issues. Twelve of these papers were written by student authors and seven were written by faculty members and others.

THE PENTAGON is a mathematics magazine for students and it is appropriate and in keeping with the ideals of Kappa Mu Epsilon to encourage the publication of student papers. While it takes a great amount of effort and attention to minute detail to prepare an article for publication, the rewards of seeing your work in print are great. I hope that faculty members at our various chapters will encourage and help students in the preparation of such manuscripts. In addition, articles by faculty members and others are welcome, particularly in those areas of mathematical interest where students, due to lack of background and experience, are not likely to make contributions. Any article that is of interest to undergraduate students in mathematics is solicited.

The journal of our fraternity should serve as a means of communication between the chapters, a magazine where interesting articles and thought-provoking problems may be found, and a place where young men and women entering the field of mathematics may express their mathematical ideas in print. I welcome your comments, suggestions, and criticisms to carry out these objectives. Above all, I appeal to you to contribute to THE PENTAGON either with articles for publication or with problems and solutions to the Problem Corner.

—FRED W. LOTT, JR.

REPORT OF THE BUSINESS MANAGER OF THE PENTAGON

It is a pleasure for me on the occasion of this the thirteenth Biennial Convention of Kappa Mu Epsilon to be able to report to you as Business Manager of THE PENTAGON. I would like to take this opportunity to give you some data of interest concerning your national magazine. It is truly a national magazine. The Fall, 1960, issue of the PENTAGON was mailed to every state in the Union except to the following five states: North Dakota, Alaska, Montana, Vermont, and Idaho. THE PENTAGON is also an international magazine. The last issue was mailed to fifteen foreign countries, all over the world. Our national journal goes to Europe, Africa, Asia, South America, and to such well known islands as New Zealand, Japan, and Formosa. We also have subscribers residing in the District of Columbia and the Canal Zone.

Perhaps a few facts concerning the number of PENTAGONS mailed to the various states would be of interest. I reported at our last Biennial Convention that more PENTAGONS were mailed to Kansas than any other state. This statement is also true at this convention. One hundred eight-five Fall, 1960, Pentagons were mailed to Kansas addresses. In descending order of frequency, the next five states which received more than one hundred magazines were Illinois, California, New York, Missouri, and Michigan.

To give you some idea of the growth of THE PENTAGON in a period of less than six years, there were eighteen hundred copies of the Spring, 1955, PENTAGON printed, while last Fall's issue had a run of two thousand six hundred fifty copies or almost a fifty per cent increase during this time. All of these two thousand six hundred fifty copies have been mailed except a reserve of one hundred copies which will be used to fill requests for back issues during the next several years. I still receive requests for copies of the first PENTAGON, Fall, 1941 printed over twenty years ago.

During this past biennium the office of Business Manager has processed two thousand eighty-one new subscribers received from the national secretary as initiates in Kappa Mu Epsilon. Many new subscribers were also gained from publication agencies, high-school and college libraries, and from individuals.

The office of the Business Manager of THE PENTAGON is not one in which large sums of money are involved. Total receipts for the calendar years of 1959 and 1960 were \$802.00. This revenue is primarily received for new and renewal subscriptions, sale of reprints, and sale of back issues. Expenditures during this two year period for clerical help, postage, mailing and office supplies, and bank charges were \$504.68. The balance of the money not used for expenses is sent to the National Treasurer. A word of explanation I think is in order concerning this financial report. The National treasurer receives all two dollar subscriptions for the PENTAGON from new initiates. This officer in turn pays all the expenses of printing THE PENTAGON, and pays the postage on our bulk mailing at the time of printing an issue of our national journal.

I would like to close with a plea that has been made by every Business Manager in reporting to this convention. Shortly after you return home you will be receiving a copy of the Spring, 1961 PENTAGON. That is you will, if the address on file with the Business Manager, is your correct address. Every Fall and Spring, many copies of our journal are returned to me marked seven and one-half cents postage due because of incorrect addresses. This is expensive for the fraternity, but more important someone who should be reading a fine magazine will not be doing so. Please, if you change your address let me know so I can bring your subscription card up to date.

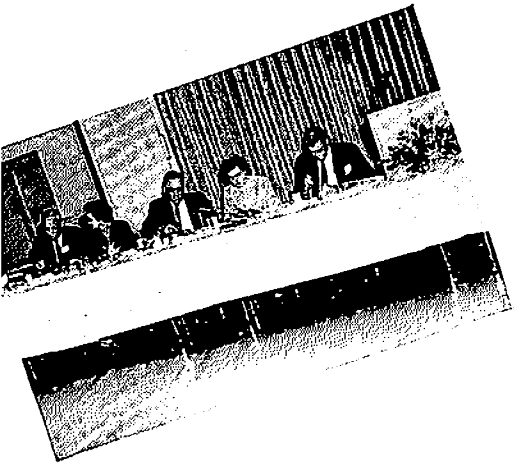
It has been, for me, a pleasure and a rewarding experience to serve you as Business Manager of our national journal over these past four years.

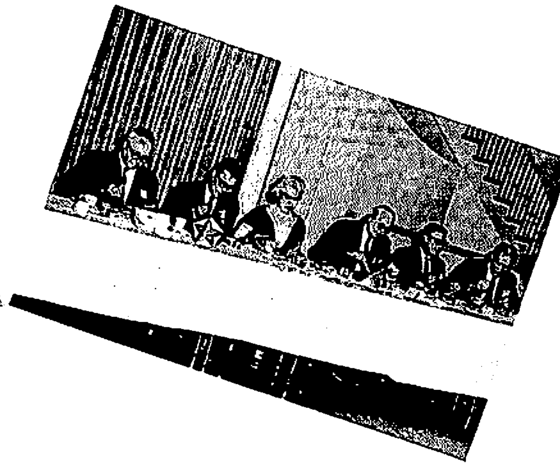
—Wilbur J. Waggoner

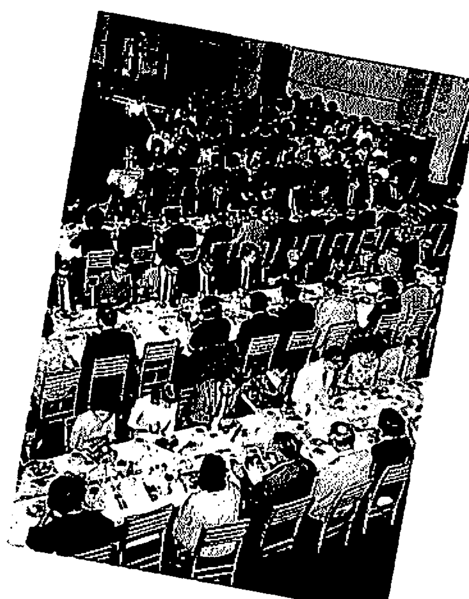
(Continued from page 20)

are necessarily uncertain inferences because they deal with chance events; but we can control some of the uncertainty by good experimental design and we can describe the degree of uncertainty rigorously if we use the best available statistical theory. I hope I also have convinced you that statisticians work on important, complex and challenging problems which come from virtually all areas of human interest and research activity.

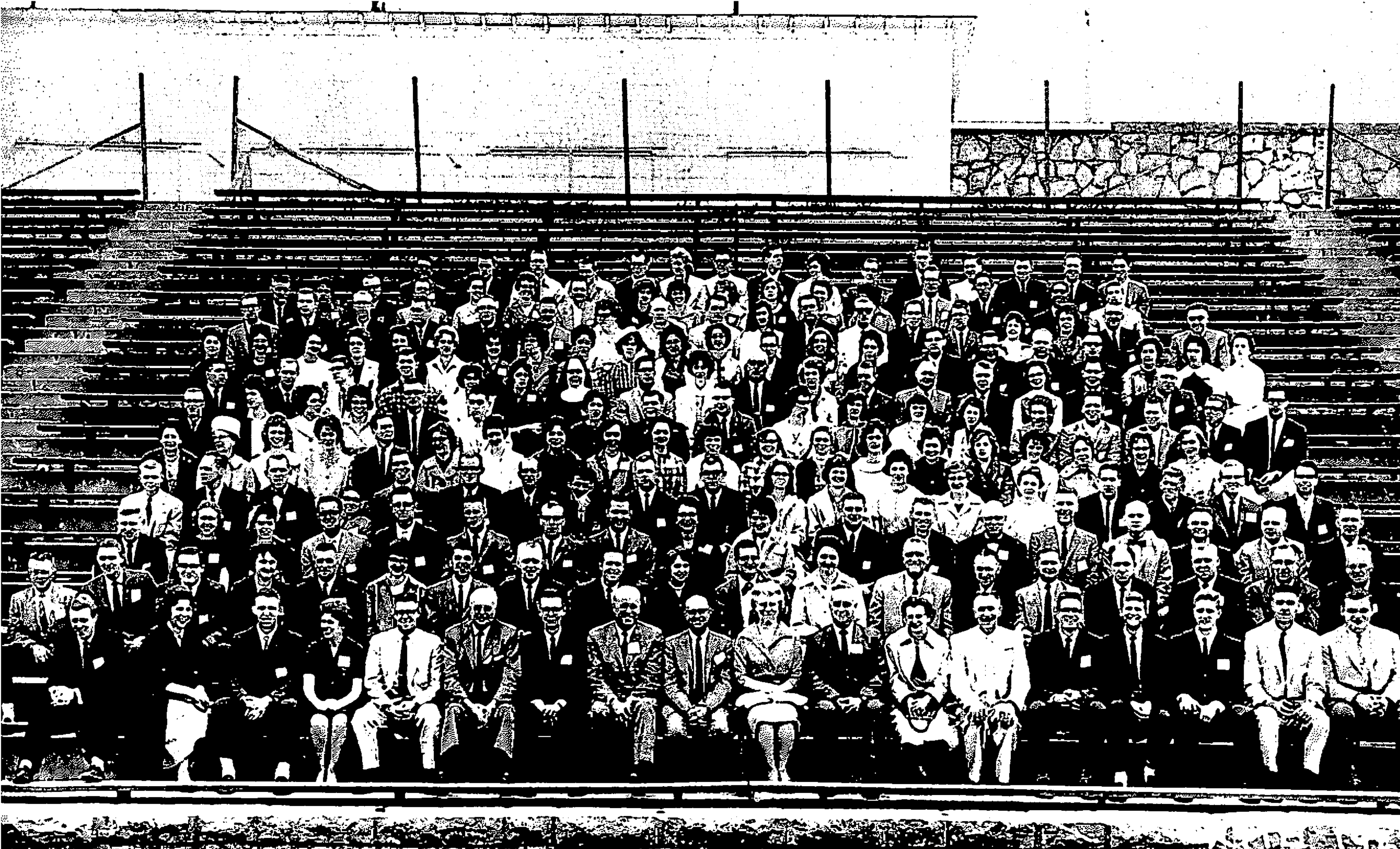
There is, however, no reason to believe that *non*-statisticians will cease to use so called statistics to mislead people. Hence, there probably is no good reason for statisticians to believe that people will cease to quote a *personalized* form of Disraeli's statement, and imply that there are three kinds of liars: plain liars, damn liars, and statisticians.







K.M.E. Banquet
National Convention, Emporia, Kansas
April 21, 1961



Kappa Mu Epsilon Convention, April 21-22, 1961.