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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

The Four-Color Problem

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Many of you are familiar with the "handshake problem." The problem is to prove that the number of people who have shaken hands an odd number of times is even. The solution is derived from the fact that when two persons shake hands two handshakes are involved, and therefore the total number of handshakes must be even. Since the sum is an even number, it must contain an even number of odd addends.

Situations similar to this one were presented by the mathematics staff of the College of the University of Chicago in order to suggest methods of approaching a variety of problems. An example of a problem requiring a novel approach is the four-color map problem.[1]

This problem probably originated with British cartographers when it appeared that four colors were sufficient to color any map. The recognition of this as a mathematical problem is credited to Moebius in 1840. It came to the attention of Francis Guthrie who in turn brought it to the attention of the British logician and mathematician, De Morgan, in 1850. Cayley discussed the problem before the London Mathematical Society as early as 1878.

Just what is this problem? If we say, "How many colors are needed to color a map?" the answer might well be given, "No more than the number of regions in the map." So we need to state the problem in a more precise way, such as, "How many colors are both necessary and sufficient in order to color a map?" In addition to inserting the "necessary and sufficient" conditions, we need to define what we mean by coloring a map. We do this by saying no countries which have a common boundary—such as the United States and Canada—can be given the same color.[2]

However, if the boundaries of the two countries are common in

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[1] "Coloring Maps," The Mathematics Teacher, 50:546, (December, 1957).

[2] R. Courant and H. C. Robbins, What Is Mathematics? (New York, Oxford University Press, 1941), pp. 246-8, 264-7.

one or a finite number of points only, they may be given the same color. An example of two regions having a single common point is seen in the states of Colorado and Arizona. Also, although any area designated as an ocean must be considered as a separate region and given a color different from that of countries having a coastline, that part of a river or a lake lying within a region must receive the same color as the region in which it belongs. One further limitation is placed on the problem: the map must be on either a plane or spherical surface.[3]

The last limitation provides a very interesting feature of the problem. Although the problem remains to be solved for plane or spherical surfaces, it has been solved for some surfaces generally considered more complex. For the surface of a torus, for example, it has been proven that seven colors are both necessary and sufficient for the coloring of any map.[4]

While the problem has not yet been solved for the plane or sphere, efforts to solve it have met with some success. In 1879 a proof for the problem was given by Kempe, but in 1890 a flaw was found by Heawood in Kempe's proof. Heawood then proceeded to prove that five colors were sufficient to color any plane or spherical map. However, no one has been able to produce a plane or spherical map for which five colors are necessary.[5]

Heawood's solution was based on Euler's formula which states that for any simple polyhedron, regular or not, the number of vertices minus the number of edges plus the number of faces equals two. By a polyhedron is meant a solid figure whose surface consists of a number of polygonal faces. A simple polyhedron is one which contains no holes such as a doughnut does. This last limitation is necessary due to topological properties which Euler made use of in proving his formula. These properties are so important to the proof for the five-color theorem that a discussion of the theorem is usually found in a section on topology.[6]

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[3] The Mathematics Teacher, op. cit., p. 547.

[4] R. Courant and H. C. Robbins, op. cit., p. 248.

[5] Ibid., p. 247.

[6] Edward Kasner and James Newman, Mathematics and the Imagination, (New York, Simon and Schuster, 1956).

If we prove either the four-color or five-color theorem for the sphere, we have also proven it for the plane. To show this, imagine a spherical map on a beach ball. If we were to cut a small hole in one of the regions and pull the hole in all directions at once so as to enlarge it and the material did not tear, we would finally have a circular plane map which would be bordered by the region in which we cut the hole.[7]

The proof of the five-color theorem is dependent upon two more theorems. The first of these is that any spherical or plane map can be changed so that each vertex is the intersection of three edges or boundaries and in changing the map the number which is the sum of the number of vertices and the number of regions minus the number of edges ($V - E + F$) remains unchanged. To show this let us imagine the vertex as being a small circle which is part of one of the regions to which the vertex belongs. Now imagine this circle enlarged. We still have the same number of regions as before since the circular area belongs to one of the original areas. Also, we find the number of new vertices formed is one greater than the number of edges formed so our value of $V - E + F$ in Euler's formula remains unchanged since originally we had only one vertex and no edges.[8]

Having done this, it is possible to prove the second theorem which states that every spherical or plane map must contain at least one region of five sides or less. This proof is actually an algebraic rearrangement of Euler's formula resulting from the proof that each vertex is the intersection of three edges. If we let F equal the total number of regions, F_2 equal the number of regions with two edges, F_3 equal the number of regions with three edges, etc., then:

$$F = F_2 + F_3 + F_4 + \cdots.$$

Since each edge of a region has two ends and each vertex is the intersection of three edges, if we let E represent the total number of edges and V represent the total number of vertices, then

$$2E = 3V.$$

Also, since a region of n sides has n vertices and since each

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[7] R. Courant and H. C. Robbins, *op. cit.*, p. 247.

[8] Ibid., p. 264.

vertex belongs to three regions,

$$3V = 2F_2 + 3F_3 + 4F_4 + \cdots,$$

or

$$2E = 2F_2 + 3F_3 + 4F_4 + \cdots.$$

By Euler's formula we have $V - E + F = 2$ or $6V - 6E + 6F = 12$. If $2E = 3V$, then $4E = 6V$, and $4E - 6E + 6F = 12$ or $6F - 2E = 12$ or

$$6(F_2 + F_3 + F_4 + \cdots) - (2F_2 + 3F_3 + 4F_4 + \cdots) = 12,$$

or

$$(6-2)F_2 + (6-3)F_3 + (6-4)F_4 + (6-5)F_5 + (6-6)F_6 + (6-7)F_7 + \cdots = 12.$$

Since the right hand member of this equation is positive, at least one of the addends in the left member must be positive; and therefore, some region must have five edges or less.[9]

The remainder of the proof is an inductive process showing that if the number of regions in the map is reduced by one, and if this resulting map can be colored with five colors, then the original map can also be colored with five colors. This process of reducing the number of regions in the map is continued until five or fewer regions remain. Since any map containing not more than five regions can be colored with at most five colors, then the original map can also be colored with five colors. That the map can be reduced to not more than five regions is possible due to the proof that the map must contain one region of five or less edges. The proof not only shows five colors to be sufficient with which to color any map, but also provides a method of coloring the map. This is done by giving a color to each region in the map containing not more than five regions. Then as each boundary which was removed to reduce the number of regions in the map is replaced, a color can be found to use in the new area formed.[10]

Proofs of the four-color theorem for plane and spherical maps having up to thirty-eight regions have been found; so it is evident that a proof regarding a general number of regions would necessarily

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[9] Ibid., p. 265.

[10] Ibid., p. 265-6.

be quite complex. Considering the rate of progress being made today in various fields of mathematics, it is to be expected that entirely new areas are waiting to be discovered and explored. Quite possibly, the reasoning necessary for the understanding of these areas would seem as fantastic today as the idea of space satellites appeared to the average person a hundred years ago. If one simple new approach to mathematics is found by a study of the four-color problem, much has been gained; very possibly much more than can be appreciated at the present time. And perhaps someone will have his or her name placed among the greatest names in mathematics just by proving that four crayons are all that any schoolboy really needs in order to color his map![11]

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[11] Ibid., p. 247.

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"Coloring Maps," *The Mathematics Teacher*, 50:546 (December, 1957).

". . .A good notation has a subtlety and suggestiveness which at times makes it seem almost like a live teacher."

—Bertrand Russell

A Short, but Concentrated, Lesson on the Digital Computer

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Many teachers and students at our school have expressed curiosity about the operation of the digital computer. Working on the assumption that learning proceeds best by doing, I devised a little problem to give them a start. The reader is urged to give some time to it himself.

You will see that the problem starts with a brief description of the characteristics of the computer and then describes a mathematical problem broken down into steps that a computer is able to perform. You are then challenged to "play the part of computer" in solving the problem yourself. Instead of obtaining the entire answer, which is the type of lengthy routine job best suited to a computer, you are asked simply to describe the sort of answer that will be printed on cards.

Solving this problem will give more insight into the logical structure of this marvelous modern machine than almost any other approach. The answer is given for those who do not discover it for themselves.

A certain automatic electronic computer can:

a) store a number in any one of 1,000 numbered storage locations where it will remain until replaced by another. Transferring a number from an old to a new storage location leaves it in both locations and erases the previous contents of the new location.

b) perform arithmetical operations upon stored numbers and store the result or print it on a card.

c) follow a sequence of instructions, called a program, in which each instruction also directs the computer to the succeeding instruction.

d) be directed to proceed next to either of two alternate instructions, depending upon whether the contents of two storage locations are equal or unequal to each other.

The symbol, $C(x)$, will mean "contents of storage location x." For example, $C(104) = 1001$ means that the number 1001 is stored in location 104.

The problem:

The following set of numbers is stored in the computer:

C(100) = 37	C(102) = 2	C(104) = 0
C(101) = 3	C(103) = 31	

The following set of instructions is given:

No. Instructions

1. Store C(101) in location 150; proceed to (instruction) 2.
2. Divide C(100) by C(150); store remainder in location 151 and proceed to 3.
3. If C(151) = C(104), proceed to 7; otherwise proceed to 4.
4. If C(150) = C(103), proceed to 6; otherwise proceed to 5.
5. Add C(102) to C(150). Store sum in location 150 and proceed to 2.
6. Print C(100) on a card; proceed to 7.
7. Add C(102) to C(100); store result in C(100) and proceed to 1.

The human operator stops the machine as soon as it has printed 1369.

Requirement:

a) What information is printed on the cards?

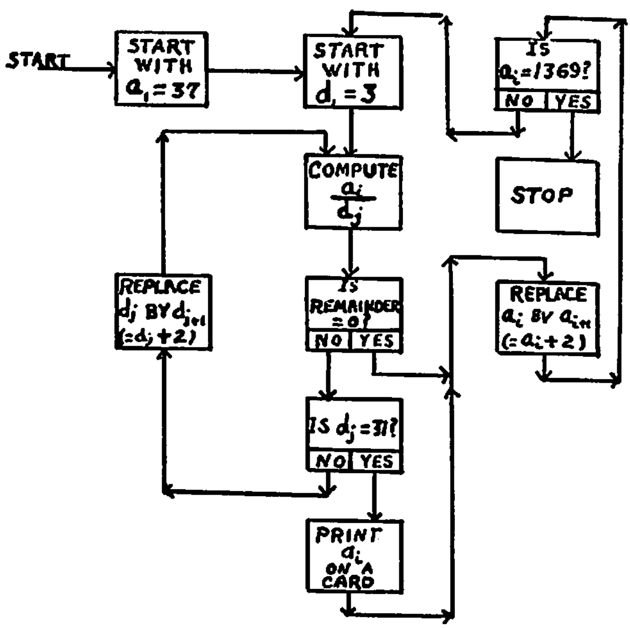
b) If comparable information beyond 1369 is desired, a change must be made in the stored information. What change must be made to obtain on cards comparable information up to 1,000,000?

Solution:

The computer will divide 37 by successive odd integers 3 through 31. If it encounters a zero remainder, the machine will replace the dividend by one that exceeds it by 2 and repeat the process. If no zero remainder occurs, the machine will print the dividend on a card, again add 2, and repeat the series of divisions. Note, for example, that the machine will print 37 but not 39 since division of 39 by 3 will give a zero remainder.

Therefore the numbers printed on cards will be primes so long as division by odd integers up to and including 31 will constitute a test for primes. It will fail for the first time at 37^2, or 1369. Thus the answer to requirement a) of the problem is that all the printed numbers are primes except for the last one, 1369.

To obtain primes up to 1,000,000, we must replace C(103) by 997, the largest prime less than 1,000.



FLOW CHART

Problems are generally programmed on "logical flow charts" before encoding for machine operation. The flow chart for this problem is reproduced here (see figure), except that the stopping of the machine by the human operator is replaced by an automatic "stop" order.

Electronic Analogue Computers

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An electronic analogue computer is a general-purpose problem-solving machine which is composed chiefly of electronic components but which may also include mechanical components. Variables of a problem are represented in the machine by voltages and mechanical displacements.

The heart of the computer is the operational amplifier. This is a simple type of electronic amplifier which multiplies the input voltage by $(-A)$; *i.e.*, for an input of e_{in} volts, the output voltage e_o is equal to $-Ae_{in}$ volts. A typical value for A may be 3×10^8. If an impedance (inductor, capacitor, resistor, or combination) is connected across the input and output, the output voltage will feed back through the impedance into the input. Since the output voltage has a sign opposite to that of the input, the output voltage will reduce the input voltage to an increment which will be almost zero. But the high-gain amplifier will multiply the increment to obtain a finite output voltage dependent on the input voltage.

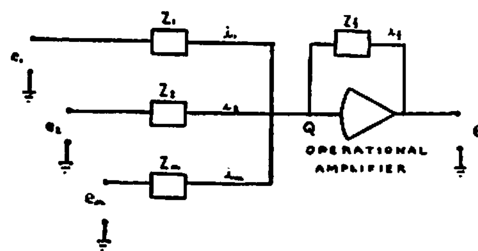


FIG. 1

If the voltage e_o in Figure 1 is to be no higher than 100 volts for the safety of the programmers, then the voltage at Q with respect to ground will be $(-1/A)e_o$, which is almost zero; and hence point Q may be considered to be at ground potential. The voltage across z_1 is then e_1, where e_1 is a function of time and a source of voltage. Thus

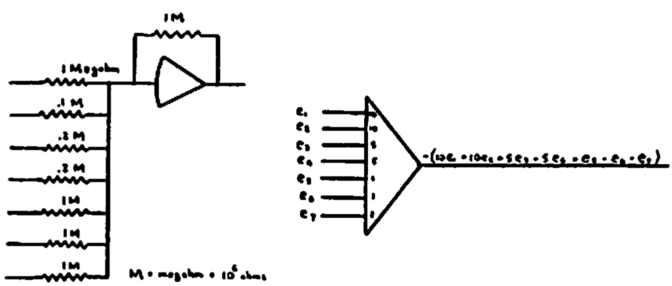
$$i_1(t) = [e_1(t)]/[z_1(t)].$$

The resistance of the operational amplifier may be considered infinite, hence negligible current flows through it. Therefore

$$(1)\quad i_f = \sum_{i=1}^{n} i_i \quad \text{or} \quad -e_o/z_f = \sum_{i=1}^{n} (e_i/z_i).$$

$$(2)\quad e_o = -z_f \sum_{i=1}^{n} (e_i/z_i).$$

If z_f is a resistor, one obtains an electronic summer (Figure 2) capable of multiplying by a constant.



CIRCUIT OF SUMMER SYMBOL FOR SUMMER

FIG 2

If z_f is a capacitor, one obtains an electronic integrator (Figure 3) capable of multiplying by a constant. Equation (1) becomes

$$i_f = dq/dt, \quad \text{or} \quad \sum_{i=1}^{n} (e_i/z_i) = -c(de_o/dt).$$

$$(3)\quad e_o = (-1/c)\int \sum_{i=1}^{n} (e_i/z_i)dt.$$

The feedback capacitor is usually kept fixed at $1\mu f$ in an integrator.

In the symbolic notation all grounds are deleted, but they are actually present in the machine. All voltages are with respect to ground.

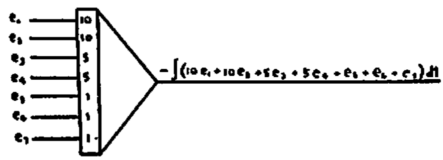
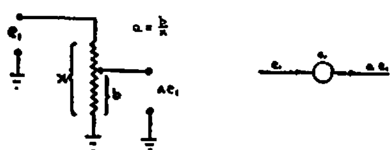


FIG. 3 SYMBOL FOR INTEGRATOR

Potentiometers (Figure 4), called *pots*, are generally used to multiply by constants less than one.



CIRCUIT OF POT POT SYMBOL

FIG. 4

There are several ways to observe the voltage variables. One method is to use an oscilloscope. For permanent recordings, pen recorders may be used. These are controlled by electromagnets. Positive voltages will displace the pen in one direction proportionally to the voltage applied, and negative voltages will displace the pen in the opposite direction. The pen inks or burns impressions on ruled paper rolling by it at a constant known velocity (*e.g.*, 1 mm./sec.). This supplies a calibration for the x or t-axis.

A description of the setting up of a simple problem may prove more interesting than the above discussion. Suppose a mass is suspended from a spring as shown in Figure 5. There will be a force on the mass which will be proportional to the displacement. When the mass is in motion, there will be a retarding force which is proportional to the velocity of the mass. This force is due to the fluid in the cylinder. Equating the forces on the mass, when it is displaced a distance $y = L$ and released, results in the differential equation,

$$d^2y/dt^2 = (-K/m)(dy/dt) - (c/m)y$$

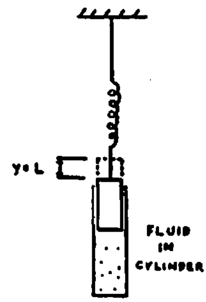


FIG. 5

Assume that d^2y/dt^2 is fed into integrator 1 in Figure 6. Then the voltage out of integrator 1 is $-\int(d^2y/dt^2)dt = -dy/dt$. Pot 1 is fed by (-100) volts and is set to feed integrator 2 the $(-L)$ volts which is the initial condition on y when time t is zero. Pot 3 multiplies y by c/m and amplifier 3 merely multiplies this by (-1).

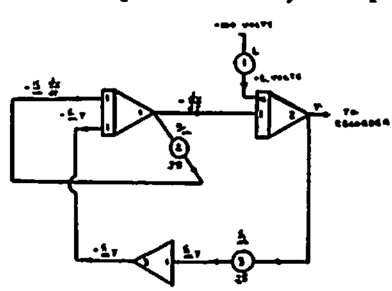


FIG. 6

Pot 2 multiplies $-dy/dt$ by K/m. The result is that $[(-K/m)(dy/dt) - (c/m)y]$ volts is fed into integrator 1, which is the voltage assumed to be fed into this integrator. Thus, when the patching together of the summer, integrators, and pots is finished and the switch on the computer is closed to operate, the voltages will begin to vary in time according to the differential equation. With $m = 2$ slugs, $C = \frac{1}{2}$ lb./ft., and $K = 1\frac{1}{2}$ lb./ft. sec., the displacement y represented by voltage will vary in sinusoidal manner with exponential decay.

Solving the above problem does not require an analogue computer. But the use of an analogue computer is particularly suitable for handling the more intricate problems involved in designing electronic brains for missiles and gun directors, where there may be ten parameters which will affect the speed of response to the target. Ten-digit accuracy is not always needed in this type of design work, and the lower cost of the analogue, as compared to the cost of the digital, warrants its use in this field.

Analogue computers are excellent tools which perform tedious calculations for man and leave him with extra time to do more creative work.

"Nature gets credit which should in truth be reserved for ourselves: the rose for its scent, the nightingale for its song, and the sun for its radiance. The poets are entirely mistaken. They should address their lyrics to themselves and should turn them into odes of self congratulation on the excellence of the human mind."

—Alfred North Whitehead

Mathematical Structures

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Historians trying to dramatize the vastness of the literature of mathematics have identified Poincaré as the last mathematician to view mathematics as a whole. Toward the close of the nineteenth century, Poincaré wrote philosophical treatises about mathematics and physics and published important research papers in many branches of mathematics. Most mathematicians since Poincaré have limited themselves to a few specialties within the broad field of mathematics.

What then of a student who sets for himself today the goal of understanding where the pieces of mathematics that he knows fit into the total picture of mathematics? Is it impossible to match the genius of Poincaré and thus to get up-to-date as of 1900? Even if the student duplicates Poincaré's performance, what about the mathematics of the twentieth century that has been said to exceed in bulk and significance all of the pre-twentieth-century mathematics?

This ambitious student should realize that modern mathematics has taken two main directions. It is true that twentieth-century mathematicians have created new branches of mathematics and extended older branches of mathematics far beyond the levels familiar to Poincaré. It is also true that twentieth-century mathematicians have learned much that will help the student of today to view mathematics as a whole. So, while they have multiplied the details of mathematics, twentieth-century mathematicians have simplified the total picture.

It was difficult, perhaps impossible, in 1900 to say just what mathematics was. Now, in 1958, many mathematicians would agree that *mathematics is a storehouse of structures.* Adopting this point of view, it may be possible to achieve a panoramic view of mathematics even in this short paper.

First, what is a mathematical structure? A structure involves a set of elements, some symbols, and some agreements about the manipulation of these symbols. In a mathematical structure the elements are abstract. Examples are numbers, points, and triangles. Mathematicians use symbols—a numeral like "2" for the number "two," a dot on the blackboard for a point, or a sketch of a triangle—to facilitate the discussion of the corresponding abstractions.

As an example of a structure, consider a set of elements symbolized by $S = \{a, b, c, \ldots\}$ and the agreements:

A 1) If x is an element of S then νx is an element of S (more briefly: $x \varepsilon S \rightarrow \nu x \varepsilon S$).

A2) $\nu(\nu x)$ is equivalent to x.

This means for each element x of S there is a corresponding element, νx, of S; moreover, the element, $\nu(\nu x)$, that corresponds to νx, is equivalent to x.

This may seem very remote from the world, that is, abstract. But many familiar-looking things are instances of this structure. Take, for example, a set of *statements*. A statement is a collection of words to which you can assign a truth value; that is, you can decide whether a statement is true or false. Now interpret the symbol, ν, as "It is not the case that." If the set S contains the negation of each of its elements, (A 1), then the set S is an instance of the structure; that is, ordinary English usage demands that

"It is not the case that (It is not the case that x)"

is true when x is true and false when x is false. Hence, it has the same truth value as x; that is, it is equivalent to x (A 2). Other examples are: a set of numbers that includes the negative of each number and the interpretation of ν as the ordinary negative sign; a set of triangles that includes the mirror image of each triangle and the interpretation of ν as "the mirror image of."

In some ways the structure which we described is typical of mathematical structures. It is abstract and has several interpretations that are, on the surface, quite different. But it is too broad to serve as a good example of mathematical structures. You would find it hard to prove an interesting and extensive list of theorems for this structure.

For the purposes of classification, the great structures of mathematics fall into three categories, as follows:

1) order structures,
2) algebraic structures,
3) topological structures.

The mathematical structure with which you have had most experience is a combination of an order structure and an algebraic structure. It is called an ordered field. The identities of algebra and skills like the solving of equations are really based upon the postulates of an ordered field. That is, the large number of seemingly isolated rules of algebra are properties of an ordered field. They can be proved as theorems from the postulates of an ordered field.

To begin with a simple order structure, consider a *linearly ordered set* defined as follows:

a set S: $\{x, y, z, \ldots\}$;
a connective phrase: $<$;
the postulates: 0 1. $(x \neq y) \rightarrow (x < y$, or $y < x)$,
0 2. $(x < y) \rightarrow (y \nless x)$,
0 3. $(x < y$ and $y < z) \rightarrow (x < z)$.

Examples of this structure are:

1) English words with "alphabetically preceeds,"
2) weights of packages with "is lighter than,"
3) real numbers with $x < y$ interpreted as "there exists a positive real number, p, such that $x = y + p$." (Notice that this is not the usual interpretation of "less than.")

Sample theorem:

To prove: $x \nless x$.
Proof: Suppose $x < x$.
Then $x \nless x$ (by 0 2).
Since the assumption that $x < x$ leads to a contradiction of the assumption, we must reject it and conclude that $x \nless x$.

Other theorems to prove:

1) $(x = y)$ or $(x < y$ or $y < x)$,
2) it is not the case that $(x = y$ and $x < y)$.

Notice that theorems proved for linearly-ordered sets apply to each instance of a linearly-ordered set. This suggests the power of an abstract approach. There is no need to develop a separate theory for each interpretation of the postulates for a linearly-ordered set.

To introduce algebraic structures, consider an *abelian group*:

a set G: $\{a, b, c, \ldots\}$;

a well-defined binary operation: $*$;

the postulates: G 1. $(x \varepsilon \text{G and } y \varepsilon \text{G}) \rightarrow$ (there is an element $z \varepsilon$ G such that $x * y = z$),

G 2. $(x \varepsilon \text{G and } y \varepsilon \text{G}) \rightarrow (x * y = y * x)$,

G 3. $(x \varepsilon \text{G}, y \varepsilon \text{G, and } z \varepsilon \text{G}) \rightarrow [x * (y * z) = (x * y) * z]$,

G 4. There is an element $e \varepsilon$ G, and $(x \varepsilon \text{G}) \rightarrow (x * e = x)$,

G 5. $(x \varepsilon \text{G}) \rightarrow$ (There is an element $x' \varepsilon$ G and $x * x' = e$).

Examples of abelian groups are:

1) the integers with $*$ as addition, $e = 0$, and $x' = -x$;
2) the nonzero remainders obtained in the division of natural numbers by 5, {1,2,3,4,}, with $*$ as multiplication, $e = 1$, $1' = 1$, $2' = 3$, $3' = 2$, $4' = 4$.
3) the rotations of the Euclidean plane, with $*$ as combining successive rotations, and x' as a rotation that undoes what the rotation x does.

Sample theorem:

To prove: $(x')' = x$.

Proof:	$x' * (x')' = e.$		(G 5)
	$x * [x' * (x')']$	$= x * e.$	($*$ is well-defined)
But	$x * [x' * (x')']$	$= (x * x') * (x')'$	(G 3)
		$= e * (x')'$	(G 5)
		$= (x')'.$	(G 2 and G 4)
And	$x * e$	$= x.$	(G 4)
Hence	$(x')'$	$= x.$	

Other theorems to prove:

1) $x' * x = x * x'$,
2) the inverse, x', of x is unique.

Notice that the postulates begin to look more familiar. You probably recognize the commutative law (G 2) and the associative law (G 3). There is a very extensive theory of abelian groups; that is, many theorems can be proved from these postulates. There is also an important theory of nonabelian groups in which the commutative law (G 2) is not assumed.

Typical of algebraic structures is the presence of binary operations, like the * of the group structure. You are most familiar with binary operations like addition and multiplication. But the binary operation of the group is abstract. Example 3) above suggests another kind of interpretation of the operation *.

As a second example of algebraic structures, consider a *commutative field*:

a set K: $\{a,b,c,\ldots\}$;
two well-defined binary operations: $+$ and $\cdot$;
the postulates:

for $+$

K is an abelian group under $+$. We designate the identity for $+$ by 0; the inverse of x with respect to $+$ by $-x$.

for $\cdot$

With the element 0 excluded, K is an abelian group under $\cdot$. We designate the identity for $\cdot$ by 1; the inverse of x with respect to $\cdot$ by $1/x$.

for $+$ and $\cdot$ (Distributive Law)

If x, y, and z are variables whose range is K, then $x\cdot(y+z) = x\cdot y + x\cdot z$.

Examples of commutative fields are:

1) the set of integers modulo 5 (that is, remainders obtained in the division of natural numbers by 5);
2) the set of rational numbers;
3) the set of numbers $a + b\sqrt{3}$ where a and b are rational numbers.

Sample theorem:

To prove: $x\cdot 0 = 0$.

Proof:	$a + 0 = a.$	(G 4 for $+$)
	$x\cdot(a+0) = x\cdot a,$	($\cdot$ is well-defined)
Hence	$x\cdot a + x\cdot 0 = x\cdot a.$	(Distributive Law)

Since $x \cdot a \in K$, there is an element $-(x \cdot a) \in K$, and $x \cdot a + [-(x \cdot a)] = 0$. (G 1 for $\cdot$ and G 5 for $+$)

Now

$$
\begin{aligned}
&-(x \cdot a) + (x \cdot a + x \cdot 0) \\
&\quad = \{x \cdot a + [-(x \cdot a)]\} + x \cdot 0 \quad &&(\text{G 3 for } +) \\
&\quad = x \cdot a + [-(x \cdot a)] + x \cdot 0 &&(\text{G 2 for } +) \\
&\quad = 0 + x \cdot 0 &&(\text{G 5 for } +) \\
&\quad = x \cdot 0. &&(\text{G 2 and G 4 for } +) \\
&-(x \cdot a) + x \cdot a = x \cdot a + [-(x \cdot a)] &&(\text{G 2 for } +) \\
&\quad = 0. &&(\text{G 5 for } +)
\end{aligned}
$$

Hence

$$x \cdot 0 = 0.$$

Other theorems to prove:

1) $(a \cdot b = 0) \longleftrightarrow (a = 0 \text{ or } b = 0)$.
2) $(-a) \cdot b = -(a \cdot b)$.
3) in a field an equation of the form $a \cdot x + b = c$ with $a \neq 0$ is equivalent to an equation of the form $x = (1/a)[c + (-b)]$.

Notice how the previous study of abelian groups makes it easier to describe the field structure. Notice that the identity elements, 0 and 1, suggest numbers but should be considered as abstract. Notice, also, that the operations $+$ and $\cdot$ suggest addition and multiplication but should be considered as abstract. Notice how the groups under $+$ and $\cdot$ (with the element 0 excluded) are tied together by the distributive law.

The set of rational numbers is the "smallest" field that contains the natural numbers as a subset. The set made up of positive numbers and zero is not a field; this set does not include the inverses for addition (such as -3, the inverse of 3). The set of integers is not a field; this set does not include the inverses for multiplication (such as $1/3$, the inverse of 3). This is, of course, the mathematical reason for inventing negative integers as additive inverses of positive integers and fractions as multiplicative inverses of nonzero integers.

As an example of combining order and algebraic structures, consider the postulates for *linear ordering of a commutative field*:

Postulates: K 0 1. $(x < y) \rightarrow (x + z < y + z)$.
K 0 2. $(x < y \text{ and } 0 < z) \rightarrow (x \cdot z < y \cdot z)$.
K 0 3. $(x < y \text{ and } z < 0) \rightarrow (y \cdot z < x \cdot z)$.

Taken together, the postulates for a linearly-ordered set, a commutative field, and linear ordering of a commutative field are a sufficient basis for much of the work in elementary algebra. These postulates do not completely characterize the set of real numbers. There is, besides, the order postulate related to *denseness*:

$$(x \in S,\ y \in S \text{ and } x < y) \rightarrow (\text{there is an element } b \in S \text{ such that } x < b < y);$$

that is, between each two distinct elements of a dense set, there is another element of the set. The set of rational numbers has this property.

The set of real numbers has, in addition, properties connected with continuity. The postulates that enable you to deal with "geometrically evident" ideas about continuity and limit are typical of topological structures. They introduce concepts like *upper bound* and *least upper bound*. You may be interested in studying the *axiom of continuity*:

$(S \in L$ and $S \neq \emptyset$ and S is bounded above$) \rightarrow ($there is an element $u \in L$ such that u is a least upper bound of $S)$

This means that you consider each subset, S, of a set L other than the empty set $\emptyset$. If for each such subset there is an element of L such that each element of S is less than or equal to this element of L (that is, there is an element $b \in L$ and for each x such that $x \in S$, $x \leqq b$), then the axiom of continuity states that there is, in L, a least upper bound of S (that is, among all the upper bounds there is a smallest one). An example will clarify the application of this postulate to numbers and illustrate the fact that it holds for real numbers but not for rational numbers.

Consider the set, S, of all numbers whose squares are less than 3. Then $1.7 \in S$, 1.8 is not an element of S, etc. An upper bound for S is 2; that is, each element of S is less than 2. Other upper bounds for S are 1.8, 1.74, 1.733, $\cdots$. In the set of rational numbers, the set S has no least upper bound. In the set of real numbers, the set S has the least upper bound $\sqrt{3}$.

These few examples of mathematical structures serve their purpose if they help you to understand the assertion that "mathematics is a storehouse of structures." Some of the advantages of thinking of mathematics in this way are immediately evident:

1) the student of mathematics who endeavors to fit isolated fragments of mathematics into the total picture has a framework in which to work;

2) the teacher who tries to decide which topics to include and which topics to exclude from the mathematics curriculum may use, as a guide, the question, "Will this topic help my students to gain insight into an important mathematical structure?"

3) the scientist who needs a mathematical model to apply to his observations can examine several mathematical structures. If he finds that the postulates for a given structure fit the situation with which he is working, he may proceed to use the theorems that have been proved for this structure with a high degree of confidence. If one or more of the postulates do not fit his problem, he can avoid the mistake of trying to apply the wrong mathematical model. If he is unsure as to whether or not a postulate holds in this situation, he has a direction for the further study of his problem.

4) the research worker in mathematics can organize his study of unsolved problems—either creating new structures or investigating further properties of old structures.

A word of caution may be appropriate in closing. There are a great many logically-possible mathematical structures. It is true that similarities appear in different structures and that the same structure may appear in many disguises. Hence, it seems fair to say that viewing mathematics as a storehouse of structures will be useful to students, teachers, and persons interested in applying known mathematics or discovering new mathematics. Whenever a frame of references is needed to relate the parts to the whole, this view of mathematics should be exploited.

A Note on Areas Under a Sine Curve

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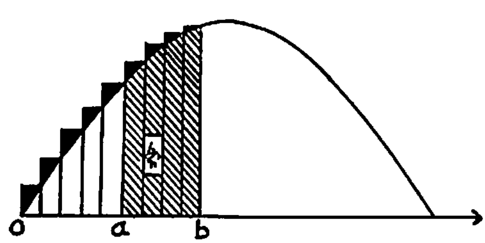
In this article I shall attempt to show how the area bounded by the x-axis, the lines $x = a$, and $x = b$, and the curve $y = \sin x$ may be found by a means other than the usual antidifferention of the sine function. I think that this method is more interesting from a mathematical standpoint although calculating the integral of $\sin x$ from a to b by antidifferentiation is quite easy.

Let us consider the curve $y = \sin x$ in the interval $(0,b)$. If this interval is divided into n segments (b/n), the area will be approximated by

$$A = \sum_{r=1}^{n} (b/n)\sin r(b/n);$$ r an integer. If we let $\theta = b/n$,

$$A = \sum_{r=1}^{n} (b/n)\sin r\theta = b \sum_{r=1}^{n} (\sin r\theta/n).$$



It follows, and is fairly simple to prove, that the area is given exactly by

$$A = \lim_{n \to \infty} b \cdot \sum_{r=1}^{n} (\sin r\theta/n).$$

Having stated this theorem, we may now derive an expression for the area. The method will be to find the required summation, take the limit as $n \to \infty$, and then multiply by b.

Let $S_n = \sum_{r=1}^{n} \sin r\theta$ and $S'_n = \sum_{r=1}^{n} \cos r\theta$.

If we add (iS_n) to S'_n we will get terms of the form $\cos r\theta + i\sin r\theta$, $(i^2 = -1)$; therefore, by De Moivre's theorem:

$$S'_n + iS_n = \sum_{r=1}^{n} (\cos r\theta + i\sin r\theta) = \sum_{r=1}^{n} (\cos \theta + i\sin \theta)^r.$$

This is, of course, a geometric series whose first term is $\cos \theta + i \sin \theta$, whose ratio is $\cos \theta + i\sin \theta$, and whose sum is given by

$$S'_n + iS_n = [\cos \theta + i \sin \theta - \cos (n + 1) \theta - i \sin (n + 1)\theta] /[1 - \cos \theta - i \sin \theta].$$

Since we have to find only S_n, it will only be necessary to take the imaginary part of the right-hand expression. Rationalizing the denominator,

$$S'_n + iS_n = [\cos \theta + i \sin \theta - \cos (n + 1) \theta - i\sin (n + 1)\theta] [1 - \cos \theta + i \sin \theta]/[(1 - \cos \theta)^2 - i^2\sin^2\theta]$$

The imaginary part of this expression $(= iS_n)$ is

$$[i\sin \theta - i\sin \theta \cos (n + 1)\theta - i \sin (n + 1)\theta + i\sin (n + 1)\theta \cos \theta]/[2 - 2\cos \theta].$$

Simplifying, since $\sin (n + 1)\theta \cos \theta - \cos (n + 1)\theta \sin \theta = \sin n\theta$,

$$\begin{aligned} S_n &= [\sin n\theta + \sin \theta - \sin (n + 1)\theta]/[2(1 - \cos \theta)] \\ &= [\sin n\theta + \sin \theta - \sin n\theta \cos \theta - \cos n\theta \sin \theta]/[2(1 - \cos \theta)] \\ &= [\sin \theta(1 - \cos n\theta) + \sin n\theta(1 - \cos \theta)]/[2(1 - \cos \theta)] \\ &= [\sin \theta(1 - \cos n\theta)]/[2(1 - \cos \theta)] + (\sin n\theta)/2. \end{aligned}$$

But $1 - \cos \theta = 2 \sin^2(\theta/2)$.

Remembering that $\theta = b/n$,

$$S_n = [(1 - \cos b) \sin (b/n)]/[4 \sin^2 (b/2n)] + (\sin b)/2.$$

which can be written in the equivalent form

$$S_n = [\sin (b/n)]/(b/n) \cdot [(1 - \cos b)(1/b)] \\ /[\sin^2 (b/2n)/(b^2/4n)] + (\sin b)/2.$$

We must now divide S_n by n since the summation is on $(\sin r\theta)/n$.

$$S_n/n = \frac{[\sin (b/n)/(b/n)][(1 - \cos b)/b]}{[\sin (b/2n)/(b/2n)][\sin (b/2n)/(b/2n)]} + \frac{(\sin b)}{2n}$$

The $\lim b/n = \lim b/2n = 0$, as $n \to \infty$.

Also, $\lim\limits_{n \to \infty} (\sin b)/2n = 0$, and $\lim\limits_{x \to 0} (\sin x)/x = 1$.

Therefore as $n \to \infty$ the limit of the sum S_n/n is $(1 - \cos b)/b$.

$$A = \lim_{n \to \infty} b \cdot \sum_{r=1}^{n} S_n/n = b[(1 - \cos b)/b] = 1 - \cos b.$$

Therefore the area from 0 to b is equal to $1 - \cos b$. The area from 0 to a is similarly given by: $1 - \cos a$. Hence, the area under the curve from a to b is given by

$${}_aA_b = \cos a - \cos b.$$

The reader may generalize this procedure to curves of the type

$$y = r \sin mx \text{ and } y = r \cos mx.$$

"How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth."

—Sir Arthur Conan Doyle *(The Sign of Four)*

Newton's Method

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The method of approximating the real roots of equations, known as Newton's Method, is presented in almost every elementary calculus book. Although it is the method in common use, it is considerably different from the one actually devised by Newton.

The method is explained as follows:

Let $f(x)$ be a real-valued function of a real variable x. It is desired to find approximately a real root of the equation $f(x) = 0$. Consider the graph of the equation, $y = f(x)$, (Fig. 1). A first approximation of the root is selected. Denote this first approximation by a_1. At the point $[(a_1, f(a_1)]$ on the graph, draw the line tangent to the curve. Its equation is $y - f(a_1) = f'(a_1)(x - a_1)$. The x-intercept of this line is found to be $a_2 = a_1 - f(a_1)/f'(a_1)$. a_2 is then taken as a second approximation and the tangent line at the point $[(a_2, f(a_2)]$ is drawn and its x-intercept found. The new x-intercept, called a_3, is taken to be a third approximation to the actual root r. In order that the method work, it is assumed that the sequence of approximations thus obtained converge to r.

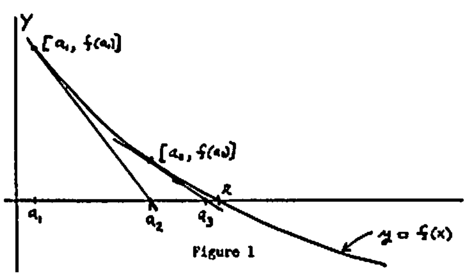

Figure 1

No calculus book that we have seen tells the student what assumptions are sufficient on the function $f(x)$, in order that the sequence, $a_1, a_2, a_3, \ldots$, converge to a root and no text presents anything that could be called a proof. Indeed, one text makes the statement that the approximation a_2 will "probably be better than the approximation a_1." It is quite easy to draw pictures which will illustrate situations in which the method will not work. It is also easy to see that at least we must not allow $f'(x)$ to be zero in too many places as we have $f'(a_1)$ in a denominator of our formula. We propose to state and prove rigorously the sufficient conditions for convergence of successive approximations obtained by Newton's Method to the real root of an equation.

We shall assume that $f(x)$ satisfies the condition that there exists an interval $a \leqq x \leqq b$ which contains one and only one root r of the equation $f(x) = 0$. We shall also assume that $f(x)$, $f'(x)$, and $f''(x)$ are continuous throughout the interval and that $f'(x)$ and $f''(x)$ are never zero and hence cannot change sign on this interval. Under these conditions there are four cases which may arise. These four cases are illustrated in Fig. 2.

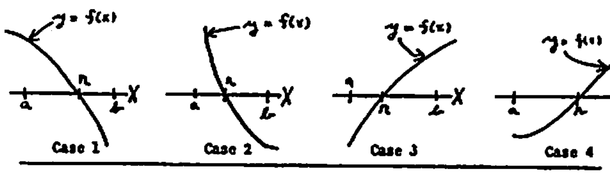

Figure 2

Case 1. If $f(a)$ is positive and $f(r)$ is zero, $f(x)$ is decreasing so that $f'(x)$ is negative and therefore negative throughout the interval. Here the curve is concave downward; that is, $f''(x)$ is negative.

Case 2. The same as Case 1, except that the curve is concave upward, so that $f''(x)$ is positive.

Case 3. If $f(a)$ is negative and $f(r)$ is zero, then $f(x)$ is increasing so that $f'(x)$ is positive and must remain positive throughout the interval. Here the curve is concave downward, so that $f''(x)$ is negative.

Case 4. The same as Case 3, except that the curve is concave upward, so that $f''(x)$ is positive throughout the interval.

The reader is reminded that the continuity of $f'(x)$ and $f''(x)$ together with the requirement that they be different from zero on the interval $a \leqq x \leqq b$ necessarily implies that they not change sign on the interval.

The conditions and case divisions which have been given were first obtained by G. T. Coate in "On the Convergence of Newton's Method of Approximations," *American Mathematical Monthly*, 44: 464, (1937). However, the proof which he presents is entirely geometrical and depends in a very real way on his figures. Also, his geometrical arguments obscure the real analysis which is behind the proof.

In each of the four cases mentioned, there is what we shall call a "right side" and a "wrong side" of the root. The right side is the side on which the function and the second derivative agree in sign and the wrong side is the one on which they disagree in sign. Since in each case the function changes sign and the second derivative does not, there is in each case a right side and a wrong side. In each case we select our first approximation from the right side, although it is easy to show that if the first approximation is selected on the wrong side, the second approximation will be on the right side. The "right side" is on the right for Cases 1 and 4 and on the left for the other cases. We shall present a proof only for Case 1, but the proof is entirely similar for each of the other cases.

Recall that the law of the mean says that if a function $f(x)$ is continuous on the interval $a \leqq x \leqq b$ and possesses a derivative on the interval $a < x < b$, then there exists a c, $a < c < b$, such that $f'(c) = [f(b) - f(a)/[b - a]$.

In Case 1 we have $a < r < b$, $f(a) > 0$, $f(r) = 0$, and $f(b) < 0$, $f'(x) < 0$, and $f''(x) < 0$. We shall take b as our first approximation. The equation of the line tangent to the curve at the point $[(b, f(b)]$ is $y(x) = f(b) + f'(b)(x - b)$. The x-intercept of this line is $b_2 = b - f(b)/f'(b)$ and b_2 is the second approximation. Since $f(b)$ is negative and $f'(b)$ is negative, $-f(b)/f'(b)$ is negative, hence $b_2 < b$. We may apply the law of the mean to the function $f(x)$ on the interval $r \leqq x \leqq b$ and find a c, $r < c < b$, such that $f'(c) = [f(r) - f(b)]/[r - b]$. Since $f''(x)$ is negative, $f'(x)$ is decreasing. Therefore, since $c < b, f'(c) = -f(b)/(r - b) > f'(b)$. That is, $-f(b) < f'(b)(r - b)$, since $(r - b) < 0$. Thus $0 < f(b) + f'(b)(r - b) = y(r)$. Now $y'(x) - f'(b) < 0$, so that $y(x)$ is a decreasing function. Since $y(b_2) = 0$ by definition and $y(r) > 0$, we have $r < b_2$. We now have $r < b_2 < b$, so that the second approximation is necessarily better than the first.

We now proceed by induction and suppose that this process has been repeated until we have an nth approximation b_n, satisfying the condition $r < b_n < b$. We then find b_{n+1} from b_n in the same manner that we found b_2 from b and the same proof will show that $r < b_{n+1} < b_n$. Thus we have $r < \cdots b_{n+1} < b_n < \cdots < b_2 < b_1$, and we see that each successive approximation is better than the preceding one.

The sequence $\{b_n\}$ is a strictly decreasing sequence bounded below by r. Such a sequence necessarily has a limit. Therefore, let

$\lim b_n = h$ as $n \to \infty$. Since h is the greatest lower bound of the sequence $h < b_n$ for every n. If $h < r$, we could take n so large that b_n would differ from h by less than $r - h$ and hence b_n would be smaller than r. Since this is impossible, $h \geqq r$.

$$b_{n+1} = b_n - f(b_n)/f'(b_n) \text{ or } f'(b_n)(b_{n+1} - b_n) = -f(b_n).$$

If we take the limit of both sides of the second equation as n approaches infinity and use the continuity of $f(x)$ and $f'(x)$, we have $f'(h)(h - h) = -f(h)$. Therefore $f(h) = 0$, and since r is the only root in this interval, $h = r$. Thus it is seen that the sequence of approximations converges to the root r.

It might be pointed out that the restrictions on the function $f(x)$ may be relaxed somewhat. All one really needs to assume is that $f(x)$, $f'(x)$, and $f''(x)$ are continuous and non-zero on the interval except perhaps at the root itself. These changes increase the applicability of the method.

Another method of producing the formula is to expand the function $f(x)$ into a Taylor Series about the first approximation, *viz.*,

$$f(x) = f(a_1) + f'(a_1)(x - a_1) + [f''(a_1)/2!][x - a_1]^2 + \cdots$$

It is then assumed that if one substitute $x = r$, that the terms involving $(r - a_1)^2$ and higher powers are so small that they may be neglected. Then the equation $0 = f(a_1) + f'(a_1)(r - a_1)$ is solved for r and the answer is called a_2, a second approximation.

While this method produces the usual formula, we have nowhere seen a proof of the convergence of the approximations using the Taylor Series. Certainly the Taylor Series requires much more hypothesis on $f(x)$ than does our proof.

It is our hope that the foregoing will prove helpful to the many calculus students who have wondered whether or not Newton's Method would always lead to correct answers.

Development of Orthogonal Functions

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1. The dot product of vectors as related to orthogonality.

The dot product or inner product of two vectors V_1 and V_2 is defined as $|V_1||V_2| \cos \theta$, where θ is the angle between the two vectors and $|V_1|$ is the absolute value of the length of the vector.

If θ is an angle of 90°, $\cos \theta = 0$; and the inner product is zero. Thus the two vectors are orthogonal (or perpendicular to each other) if and only if the inner product is zero.

2. Orthogonality in a space of three dimensions.

Let $g(r)$ or g denote a vector in three-dimensional space whose rectangular coordinates are $g(1)$, $g(2)$, $g(3)$. The square of the vector's length, called the vector's *norm*, will be written as $N(g)$. From the Pythagorean theorem we know that the norm is equal to the sum of the squares of the vector's components.

$$N(g) = g^2(1) + g^2(2) + g^2(3) \text{ or } \sum_{r=1}^{3} g^2(r).$$

If $N(g) = 1$, the vector $g(r)$ is a unit vector and is called a *normed* or *normalized* vector.

The inner product of two vectors, symbolized by (g_1, g_2), can also be expressed as the sum of the products of each vector's components on each axis.

$$(g_1, g_2) = g_1(1)g_2(1) + g_1(2)g_2(2) + g_1(3)g_2(3)$$
$$= \sum_{r=1}^{3} g_1(r)g_2(r).$$

(The subscripts differentiate between the vectors and the numbers in the parentheses indicate the components along the respective axes.)

If our vectors $g_1(r)$ and $g_2(r)$ are orthogonal,

$$(g_1, g_2) = 0,$$

or

$$\sum_{r=1}^{3} g_1(r)g_2(r) = 0.$$

3. Orthonormal sets of vectors.

If we have an orthogonal set of three vectors $g_n (n = 1,2,3)$, a set of unit vectors ϕ_n having the same directions are formed by dividing each component of g_n by the length of g_n. For example the components of ϕ_1 are $\phi_1(r) = g_1(r)/\sqrt{N(g_1)}$; $(r = 1, 2, 3)$.

This set of unit vectors is called an *orthonormal set.* Such a set can be described by means of inner products by writing

$$(\phi_m, \phi_n) = 0 \quad \text{if } m \neq n,$$
$$(\phi_m, \phi_n) = 1 \quad \text{if } m = n.$$

An example of an orthonormal set is the set of unit vectors along the three coordinate axes.

Every vector $f(r)$ in three-dimensional space can be represented as a linear combination of the three unit vectors of any orthonormal set.

Thus $f(r) = c_1\phi_1(r) + c_2\phi_2(r) + c_3\phi_3(r)$; $(r = 1,2,3)$.

To find the coefficients of the unit vectors $\phi(r)$, we can take the inner product of both members of the above equation by ϕ_1. For example to find c_1,

$$(f, \phi_1) = c_1(\phi_1, \phi_1) + c_2(\phi_2, \phi_1) + c_3(\phi_3, \phi_1).$$

But because the set of vectors $\phi_n(r)$ are orthonormal

$$(\phi_2, \phi_1) = 0,$$

and $$(\phi_3, \phi_1) = 0.$$

But $$(\phi_1, \phi_1) = 1,$$

thus $$(f, \phi_1) = c_1.$$

c_2 and c_3 are found by a similar process.

The coefficients c_n are written as

$$c_n = (f, \phi_n) = \sum_{r=1}^{3} f(r)\phi_n(r); \; (n = 1,2,3).$$

Thus our vector $f(r)$ can be represented as

$$f(r) = (f, \phi_1)\phi_1(r) + (f, \phi_2)\phi_2(r) + (f, \phi_3)\phi_3(r)$$
$$= \sum_{r=1}^{3} (f, \phi_n)\phi_n(r); \; (n = 1,2,3).$$

The definitions and results just found can be extended to vectors in a space of k dimensions. Thus the index r would vary from 1 to k and the indices m, n which identify the different vectors of an orthonormal set, vary from 1 to k. Thus the definition of the inner product of the vectors g_1 and g_2 becomes

$$(g_1, g_2) = \sum_{r=1}^{k} g_1(r)g_2(r)$$

Likewise an extension to a space of countably infinite dimensions is possible. Indeed this extension is necessary in order that we might describe a relationship between orthogonal vectors and orthogonal functions.

4. Functions as vectors.

Any function $G(r)$ which has real values when $r = 1,2,3, \cdots, k$ will represent a vector in a space of k dimensions if it is agreed that these values are the components of the vector. This function may not be defined for any other values of r, in which case its graph would consist of k points.

If $G(r)$ is defined only at these points, it is determined by the vector. Graphically it is represened by k points whose abscissas are $r = 1,2, \cdots, k$ and whose ordinates are the corresponding components of the vector.

Now let $G(x)$ be a function defined for all values of x in an interval $a \leqq x \leqq b$. To consider this function as a vector, the components should consist of all the ordinates of the function's graph in the interval from a to b.

The argument x has as many values as there are points in the interval, so that the number of components is not only infinite but uncountable.

The norm of the function or vector $G(x)$, is defined as

$$N(G) = \int_a^b [G(x)]^2 dx.$$

The inner product of two functions $G_m(x)$ and $G_n(x)$ is defined as

$$(G_m, G_n) = \int_a^b G_m(x)G_n(x)dx.$$

Two functions are orthogonal if and only if $(G_m, G_n) = 0$ when $m \neq n$.

The norm may also be expressed as $N(G) = (G,G)$.

A set (or system) of functions $\{G_n(x)\}$, where $n = 1,2,3 \cdots$, is orthogonal in the interval a to b if $(G_m, G_n) = 0$ is true when $m \neq n$ for all functions of the set. The functions of the set are normed by dividing each function $G_n(x)$ by $\sqrt{N(G)}$, thus forming a set$\{\phi_n(x)\}$, $n = 1,2,3, \cdots$, which is orthonormal. An orthonormal set of functions $G_n(x)$ in the interval $a \leqq x \leqq b$ is then shown as follows:

$$(\phi_m, \phi_n) = 0 \quad \text{if } m \neq n; \qquad (m,n = 1,2,3, \cdots)$$
$$(\phi_m, \phi_n) = 1 \quad \text{if } m = n.$$

or $$\int_a^b \phi_m(x)\phi_n(x)dx = 0 \quad \text{if } m \neq n;$$
$$\int_a^b \phi_m(x)\phi_n(x)dx = 1 \quad \text{if } m = n.$$

5. The use of the orthogonal property of orthonormal functions.

Many times in physics, it is convenient to express the motion of some vibrating body with an infinite series of orthogonal functions as terms. One series commonly used is the Fourier sine series. For an example suppose we have found that, among other conditions, the body is vibrating in a single plane and is at a distance $y = f(x)$ above the x-axis of a coordinate system. Then it is possible in certain cases that $f(x) = A_1\sin x + A_2\sin 2x + A_3\sin 3x + \cdots + A_n\sin nx + \cdots$. In order to evaluate the coefficients, we can use the orthogonal properties of the sine functions whose fundamental interval is $-\pi \leqq x \leqq \pi$.

Our definition of an orthonormal set of functions is one where

$$\int_a^b \phi_m(x)\phi_n(x)dx = 0 \quad \text{if } m \neq n; \quad (m,n = 1,2,3, \cdots)$$
$$\int_a^b \phi_m(x)\phi_n(x)dx = 1 \quad \text{if } m = n.$$

It is necessary to recall that

$$\sin nx \sin mx = (1/2)[\cos(m - n)x - \cos(m + n)x]$$
$$\text{and} \qquad \sin^2 nx = (1/2)(1 - \cos 2nx).$$

$$\int_0^\pi \sin nx \sin mx \, dx = (1/2)\int_0^\pi [\cos(m - n)x - \cos(m + n)x]dx$$
$$= (1/2)[(\sin (m - n)x)/(m - n) - \sin (m + n)x)/(m + n)]_0^\pi$$

and because the sine of any multiple of π is zero the

$$\int_0^\pi \sin mx \sin nx \, dx = 0 \quad \text{if } m \neq n \quad (m,n = 1,2,3, \cdots)$$

If $m = n$ the above integral becomes

$$\int_0^\pi \sin^2 nx \, dx = (\tfrac{1}{2}) \int_0^\pi [1 - \cos 2nx] dx$$
$$= (\tfrac{1}{2})[x - \sin 2nx/2n]_0^\pi = \pi/2.$$

We can now use this property of the sine functions in the interval $0 \leqq x \leqq \pi$ to evaluate the coefficients of the Fourier sine series.

If we take the inner product of both sides of the equation

$$f(x) = \sum_{z=1}^{\infty} A_z \sin zx$$

by $\sin nx$ we get

$$\int_0^\pi f(x) \sin nx \, dx = \sum_{z=1}^{\infty} [A_z \int_0^\pi \sin zx \sin nx \, dx].$$

But we know that the integral on the right is nonzero only when $z = n$. When $z = n$ the integral is $\pi/2$ and so our equation now is

$$\int_0^\pi f(x) \sin nx \, dx = A_n \, \pi/2.$$

And thus we have found by the use of the inner product that our coefficient is

$$A_n = (2/\pi) \int_0^\pi f(x) \sin nx \, dx.$$

Here we see one application of orthogonal functions to determine the coefficients in a Fourier series.

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Sign of a Real-Valued Function of Real Variables

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1. Introduction.

In very few books written in English can one find a systematic treatment of sign of real functions of real variables. Aside from its being interesting, the indication of the sign of real functions of real variables is necessary for many problems of mathematics. For example, the sign of the first derivative is necessary to determine whether the function is increasing or decreasing. In this note we give some methods for obtaining the sign of a real-valued function of real variables for all possible values of the variables, and we give a few examples of the use of this sign. Intuitive methods are stressed. We hope that some of these ideas may be carried to the classroom for high-school and freshman college algebra.

Theorem 1: Let a and b, with $a \neq 0$, be real numbers, and let x be a real variable. Then $ax + b$ has the same sign as a for $x > -b/a$, and it has different sign from a for $x < -b/a$.

Proof: Suppose $f = ax + b$. Then $f/a = x + b/a$. Since $x = -b/a$ makes f/a equal to zero, $x > -b/a$ makes $f/a > 0$, *i.e.*, f and a have the same sign; and $x < -b/a$ makes $f/a < 0$, *i.e.*, f and a have opposite signs.[1]

Theorem 2: Let a, b, and c be real numbers with $a \neq 0$, and x a real variable.

1) If $b^2 - 4ac < 0$, then $y = ax^2 + bx + c$ has always the same sign as a.

2) If $b^2 - 4ac = 0$, then $y = ax^2 + bx + c$ has always the same sign as a, except for $x = -b/2a$ in which case $y = 0$.

3) If $b^2 - 4ac > 0$, let $r_1 < r_2$ be the two roots of $ax^2 + bx - c = 0$. Then $y = ax^2 + bx + c$ has the same sign as a when either $x < r_1$ or $x > r_2$ and y has opposite sign to a for $r_1 < x < r_2$.

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[1] Also, see The Pentagon, Spring, 1957, p. 88.

Proof: Clearly,
$y = ax^2 + bx + c = a[(x + b/2a)^2 - (b^2 - 4ac)/(4a^2)]$.
If $b^2 - 4ac < 0$, then $(x + b/2a)^2 - (b^2 - 4ac)/(4a^2) > 0$.
Therefore, y has always the same sign as a and 1) is proved.

If $b^2 - 4ac = 0$, then $y = a(x + b/2a)^2$. Here $(x + b/2a)^2$ is always positive except when $x = -b/2a$. Therefore y has always the same sign as a except for $x = -b/2a$ which establishes 2).

If $b^2 - 4ac > 0$, then $y = ax^2 + bx + c = a(x - r_1)(x - r_2)$. Let us use a table and carry the sign of each element on the table and keep in mind that the product of two numbers with the same sign is positive and the product of two numbers with opposite signs is negative.

By Theorem 1, $x - r_1$ is positive for $x > r_1$ and it is negative for $x < r_1$ and of course it is zero for $x = r_1$. A similar thing can be said about $x - r_2$. Therefore, we have Table 1.
This table shows that $(x - r_1)(x - r_2)$ is positive for $x < r_1$ and $x > r_2$ and negative for $r_1 < x < r_2$. If $a > 0$, then the signs of y and $(x - r_1)(x - r_2)$ are the same; if $a < 0$, then y and $(x - r_1)(x - r_2)$ have opposite signs; and the proof of 3) is complete.

2. Sign of a Rational Fraction.

Let $y = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$
$/(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0)$,

where a_i, b_j are real numbers for $i = 0, \cdots, n$ and $j = 0, \cdots, m$. To determine the sign of y we first obtain the sign of the numerator and the sign of the denominator; then observing that the sign of the ratio of two numbers of like sign is positive and the ratio of two numbers with different signs is negative, we obtain the sign of y for the values of x for which y is defined. So the problem reduces to one of finding the signs of two polynomials. Now to obtain the sign of a polynomial we factor it into linear factors and quadratic factors with real coefficients. Indicating the sign of each factor, we can get the sign of the polynomial. We shall discuss later the case for which there are not rational roots and factoring is not practical.

EXAMPLE: $y = (1 - x^3)/(x^4 + 2x^3 - 3x^2 - 8x - 4)$.

In order to determine the sign of y for different values of x, we note that $y = [(1 - x)(x^2 + x + 1)]/[x^2 - 4)(x^2 + 2x + 1)]$.

We display the signs of each factor in Table 2.

The result is:

$$y > 0 \quad \text{for } x < -2, \text{ and } 1 < x < 2.$$

$$y < 0 \quad \text{for } -2 < x < -1, \quad -1 < x < 1, \quad \text{and } x > 2.$$

3. An Application to Solving Inequalities.

EXAMPLE: Solve $9/(x - 3) > 16/(3x + 2)$.

Clearly this inequality can be replaced by $9/(x - 3) - 16/(3x + 2) > 0$, or $(11x + 66)/[(x - 3)(3x + 2)] > 0$. This step changes the problem to one of determining the sign of $y = (11x + 66)/[(x - 3)(3x + 2)]$ and choosing x where $y > 0$. By Theorems 1 and 2 we show the signs of $11x + 66$ and $(x - 3)(3x - 2)$ in Table 3.

Therefore the inequality is satisfied if $-6 < x < -2/3$ or $x > 3$.

4. Sign of a Nonalgebraic Real-Valued Function of a Real Variable.

In general, there may be some special devices that can be used in obtaining the sign of $y = f(x)$ in a certain domain; for example, the sign of $y = (2 - 3x) \tan x$, for $0 \leqq x \leqq 2\pi$. This problem is very easy and can be solved by methods similar to that used in the preceding examples. The actual solution is left to the reader. For some functions, if other devices fail, the graph of the function can be used to obtain the sign.

5. Sign of a Real-Valued Function of Two Real Variables.

Let $z = f(x, y)$ be a real function of two real variables x and y. It is intuitively clear that the surface $z = f(x, y)$ intersects the xy-plane in the locus $f(x, y) = 0$ and this locus may divide the plane $z = 0$ into different regions. For the regions where the surface is above the xy-plane clearly $z > 0$, and for the regions where the surface is below the xy-plane $z < 0$. So in this case the problem of discovering sign is one of determining the different regions in which the point (x, y) must be located in order to have $z > 0$ or $z < 0$. We will give an example and leave the formal work to the reader.

EXAMPLE: A projectile P is shot with the initial velocity v_0. Where in a vertical plane should a point (X, Y) be in order to be reachable by P? If we choose the coordinate system in the vertical plane with the x-axis horizontal, positive y-axis directed upward and

x	$-\infty$ $\qquad$ r_1 $\qquad$ r_2 $\qquad$ $+\infty$
$x - r_1$	− ··· − ··· − 0 + ··· + ··· + + + ··· + ··· +
$x - r_2$	− ··· − ··· − − − ··· − ··· − 0 + ··· + ··· +
$(x - r_1)(x - r_2)$	+ ··· + ··· + 0 − ··· − ··· − 0 + ··· + ··· +

Table 1

x	$-\infty$ $\qquad$ -2 $\qquad$ -1 $\qquad$ 1 $\qquad$ 2 $\qquad$ $+\infty$
$1 - x$	+ ··· + ··· + + + ··· + + + ··· + 0 − ··· − − − ··· −
$x^2 + x + 1$	+ ··· + ··· + + + ··· + + + ··· + + + ··· + + + ··· +
$x^2 - 4$	+ ··· + ··· + 0 − ··· − − − ··· − − − ··· − 0 + ··· +
$x^2 + 2x + 1$	+ ··· + ··· + + + ··· + 0 + ··· + + + ··· + + + ··· +
y	+ ··· + ··· + ∞ − ··· − * − ··· − 0 + ··· + ∞ − ··· −

* $(-\infty)$

Table 2

x	$-\infty$ $\qquad$ -6 $\qquad$ $-2/3$ $\qquad$ 3 $\qquad$ $+\infty$
$11x + 66$	− ··· − ··· − 0 + ··· + ··· + + + ··· + ··· + + + ··· +
$(x-3)(3x+2)$	+ ··· + ··· + + + ··· + ··· + 0 − ··· − ··· − 0 + ··· +
y	− ··· − ··· − 0 + ··· + ··· + ∞ − ··· − ··· − ∞ + ··· +

Table 3

the origin at the mouth of the gun, then the equation of the path of P is:

$$y = x \tan \alpha - (g/2v^2_0 \cos^2\alpha)x^2.$$

The point (X, Y) is on the path. Therefore,

$$Y = X \tan \alpha - (g/2v^2_0)(1 + \tan^2\alpha)X^2.$$

This is a second degree equation for $\tan \alpha$. In order to have any real root the discriminant must be non negative, *i.e.*,

$$X^2(v^4_0 - 2gv^2_0Y - g^2X^2) \geqq 0.$$

This implies that

$$v^4_0 - 2gv^2_0Y - g^2X^2 \geqq 0.$$

We easily see that $z = v^4_0 - 2gv^2_0Y - g^2X^2$ intersects $z = 0$ in the parabola

$$Y = v^2_0/2g - (g/2v^2_0)X^2,$$

and this parabola divides the xy-plane into two regions. One region makes $z > 0$ and contains $(0,0)$; the other makes $z < 0$. Therefore the answer to the problem is:

The point (X, Y) has to be either on the parabola

$$Y = v^2_0/2g - (g/2v^2_0)X^2$$

or in the region containing $(0,0)$, *i.e.*, below the parabola.

"The number is certainly the cause. The apparent disorder augments the grandeur."

—Edmund Burke (*On the Sublime and the Beautiful*)

The Problem Corner

EDITED BY J. D. HAGGARD

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond the calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before March 1, 1959. The best solutions submitted by students will be published in the Spring, 1959, issue of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to J. D. Haggard, Department of Mathematics, Kansas State Teachers College, Pittsburg, Kansas.

PROBLEMS PROPOSED

116. *Proposed by J. Max Stein, student, Colorado State University.*

While traveling in Iowa in the spring, I observed that there were several directions I could look across a field of check-planted corn with the hills of corn apparently lying along straight lines. Assuming that a hill of corn determines a point, that the distance between rows is the same as the distance between hills in a row, and that the field is infinite in extent, then by looking over any hill of corn will there be any direction I can look across the field such that my line of sight will contain no other hill of corn?

117. *Proposed by Frank Hawthorne, New York State Department of Education.*

(From a New York Regents examination in advanced high school algebra.)

A man traveled 60 miles by bus and 600 miles by plane, taking 6 hours for the trip. On the return trip the speed of the plane was reduced by 50 miles an hour, but the speed of the bus was increased by 10 miles per hour so that the return trip also took 6 hours. Find the average speed of the plane and the average speed of the bus.

118. *Proposed by Frank C. Gentry, University of New Mexico, Albuquerque.*

Find four integers a, b, c, d; $a < b < 10$, $c < 10$, $d < 10$, such that $(a/b)(10c + d) = 10d + c$.

119. *Proposed by Mark Bridger, High School of Science, Bronx, New York.*

(Taken from Hall and Knight's *Higher Algebra.*)

Find the sum of all numbers greater than 10,000 formed by using the digits 0, 2, 4, 6, 8, no digit being repeated in any number.

120. *Proposed by the Problem Corner Editor.* (Taken from the July, 1957, Preliminary Actuarial Examination.)

If $g'(x) = f(x)$ and $f(x)$ is continuous, evaluate $\int_a^b f(x)g(x)dx$.

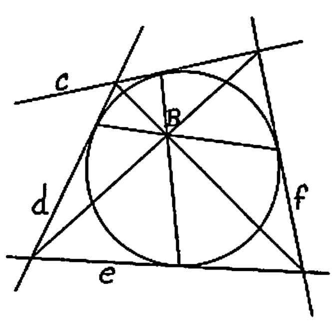
SOLUTIONS

85. *Proposed by Carl V. Fronabarger, Southwest Missouri State College, Springfield.*

In any quadrilateral circumscribed about a circle, the diagonals and the lines joining the points of tangency of opposite sides are concurrent.

Solution by Charles Pierson, University of New Mexico, Albuquerque.

Consider the following circle inscribed in the quadrilateral whose sides are c, d, e, f.



Specialize Brianchon's theorem and consider the 6 degenerate lines, c, c, d, e, e, f and also d, d, e, f, f, c.

cc, ee	
cd, ef	intersect in the Brianchon point B.
de, cf	

also,

dd, ff
de, cf intersect in the Brianchon point B'.
ef, cd

But two of the lines in each of these cases are the same two lines, thus B coincides with B' and all four lines go through the same point B.

107. *Proposed by the Problem Corner Editor.* (From a Russian secondary school examination.)

Prove that the g. c. d. of the sum of two numbers and their l. c. m. is equal to the g. c. d. of the numbers themselves.

Solution by Mark Bridger, High School of Science, Bronx, New York.

Let the two numbers be $x = ac$ and $y = bc$, with g. c. d. $= (x, y) = c$ and $(a, b) = 1$. Since the l. c. m. is the smallest number that is divisible by x and y, it is obvious that this number is abc. Since a and b are relatively prime, no factor of a is a factor of b nor is it a factor of $(a + b)$; thus $(ca + cb, abc) = c$. That is, the g. c. d. of the sum $(ca + cb)$ and their l. c. m., abc, is c; which is also the g. c. d. of the two numbers ac and bc.

111. *Proposed by C. W. Trigg, Los Angeles City College.*

The letters in (HI)(VE) = BBB represent distinct digits, some four of which are consecutive. Decode the equation.

Solution by Charles F. Waite, Pomona College, Claremont, California.

$(HI)(VE) = BBB = 111B$. Since the only factors of 111 of two digits or less are 3 and 37, $(HI)(VE) = (3)(37)(B)$, and since the right side of this equation is equal to the product of two-digit numbers, $B \geqq 4$. Let $B = (M)(N)$ where $M = (HI)/3$ and $N = (VE)/37$. Then since HI is two-digit, $M \geqq 4$, and similarly $N \leqq 2$. Since $4 \leqq B = MN \leqq 9$, then $1 \leqq N \leqq 2$ and $4 \leqq M \leqq 9$. This restricts E to be either 7 or 4. Now $E = 4$ implies $HI = 12$, $VE = 74$ and $B = 8$, which does not fulfill the consecutive requirement. $E = 7$ implies $V = 3$ and $H = 1$ or 2. Therefore E is not one of the consecutive digits and consequently $1 \leqq B \leqq 5$, but we have previously seen that $4 \leqq B \leqq 9$, thus B is either 4 or 5. For $B = 5$, $H = 1$ which does not fulfill the consecutive requirement. For $B = 4$, $HI = 12$, $VE = 37$, $BBB = 444$, which fulfills all the requirements of the problem.

Also solved by Mark Bridger, High School of Science, Bronx, New York; George Frank, Colorado State University, Fort Collins; Barry Campbell, William Jewell College, Liberty, Missouri; and Carol Ann Sexton, Southwest Missouri State College, Springfield.

112. *Proposed by the Problem Corner Editor. (From The American Mathematical Monthly.)*

For what positive values of a is $\log_a b < b$ for all positive b?

Solution by Bostwick F. Wyman, Massachusetts Institute of Technology, Cambridge.

In the inequality $\log_a b < b$, substitute in $b/\ln a$ for the left member and solve for the natural logarithm of a, resulting in $\ln a > (\ln b)/b$. Now the maximum value of $(\ln b)/b$ is determined by setting its derivative $(1 - \ln b)/b^2 = 0$. Thus $\ln b = 1$ and $b = e$. On substituting this value for b in the last inequality we get $\ln a > 1/e$ or $a > e^{1/e}$.

Also solved by Mark Bridger, High School of Science, Bronx, New York.

113. *Proposed by the Problem Corner Editor.* (From *The Mathematics Teacher.*)

Find two similar triangles which are noncongruent but have two sides of one equal to two sides of the other.

Solution by Mark Bridger, High School of Science, Bronx, New York.

Let the similar triangles have sides of a, b, c, and a', b', c', respectively, with $a/a' = b/b' = c/c'$. Since two sides of one triangle are equal to two sides of the other, take $a' = b$ and $b' = c$, thus by substitution we obtain $a/b = b/c = c/c'$ or $b = \sqrt{ac}$ and $c' = c^2/b$.

These two triangles will be similar since $a/\sqrt{ac} = \sqrt{ac}/c = c/(c^2/\sqrt{ac})$

In locating a numerical example we must remember that the sum of two sides must be greater than the third side. A set of two such triangles is:

$a = 4$	$a' = 2\sqrt{5}$
$b = 2\sqrt{5}$	$b' = 5$
$c = 5$	$c' = 5\sqrt{5}/2$

Also solved by Barry Campbell, William Jewell College, Liberty, Missouri; and George Frank, Colorado State University, Fort Collins.

114. *Proposed by the Problem Corner Editor.* (Taken from Robinson's *Mathematical Recreations,* 1851.)

Professor E. P. B. Umbugio has recently been strutting around because he hit upon the solution of the fourth degree equation which results when the radicals are eliminated from the equation:

$$x = (x - 1/x)^{1/2} + (1 - 1/x)^{1/2}$$

Deflate the professor by solving this equation using nothing higher than quadratic equations.

Solution by Calys Emanuel, Washburn University, Topeka, Kansas.

The given equation may be written

$$(x - 1/x)^{1/2} = x - (1 - 1/x)^{1/2}.$$

Squaring both sides we obtain

$$x - 1/x = x^2 - 2x(1 - 1/x)^{1/2} + 1 - 1/x$$

or

$$2(x^2 - x)^{1/2} = x^2 - x + 1.$$

Letting $x^2 - x = y$ this becomes $2y^{1/2} = y + 1$ which has $y = 1$ as a solution. Substituting $y = 1$ into $x^2 - x = y$ gives $x^2 - x - 1 = 0$, which has the two roots $1 \pm \sqrt{5}/2$. But only $1 + \sqrt{5}/2$ is a root of the given equation.

Also solved by Mark Bridger, High School of Science, Bronx, New York; Barry Campbell, William Jewell College, Liberty, Missouri; and George Frank, Colorado State University, Fort Collins.

The Mathematical Scrapbook

EDITED BY J. M. SACHS

Everything that the greatest minds of all times have accomplished towards the comprehension of forms by means of concepts is gathered into one great science, mathematics.

—J. F. HERBART

=Δ=

Mathematicians are like Frenchmen; whatever you say to them they translate into their own language and forthwith it is something entirely different.

—GOETHE

=Δ=

The calendar, in its past, present, and future, offers a fertile field for investigation and speculation. A number of items, historical and otherwise, connected with the calendar have crossed the desk of the Editor of the Scrapbook in the past few months. Some of them are old, some seem new.

=Δ=

Many of the earliest known calendars were based on a moon cycle of about 29½ days, with the year composed of 12 lunar months for a total of 354 days. This kind of rough approximation made necessary the insertion of an extra month from time to time to keep the same months in the same seasons.

The ancient Hebrews inserted the extra month seven times in a period of nineteen years. In nineteen of our years we have 6939¾ days. The nineteen Hebraic years plus the seven months would be a total of 6932½ days. Thus the Hebrew calendar would disagree with ours by 7¼ days in nineteen years.

The ancient Greeks added three months in eight years. This would be a total of 2920½ days compared with 2922 days in eight of our years, a difference of 2½ days in eight years.

=Δ=

It seems clear that rather early in the development of civilization came the recognition of the fact that the year was approximately 365 days. The following ingenious method is probably responsible for the early use of the 365-day year. A prominent star, clearly identified, was watched for and the day was noted when this star appeared above the horizon just before sunrise, the star not having been visible in the sky on the previous day just before sunrise. A tally of days was made until this same star again appeared just before sunrise (heliacal rising). This tally led to the conclusion that the year consisted of 365 days.

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The Roman calendar also used the extra month but before the time of Julius Caesar the insertion of extra months seems not to have followed any regular pattern. With the aid of the Egyptian, Sosigenes, Caesar reformed the calendar. In the process the year 46 B.C. was lengthened to 445 days, with succeeding years to be 365 days except for leap years which were to be 366 days. This was the Julian Calendar.

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In 1582 Pope Gregory XIII adjusted the calendar again by ordering that the day following October 4 should be October 15, and that henceforth a century year would be a leap year only if the first two digits were divisible by 4. Most of the Catholic countries adopted the Gregorian Calendar before 1600. England and America did so in 1752 by having September 14 follow September 2. (Eleven days were lost instead of the 10 days in 1582 because 1700 was a leap year in the Julian Calendar and was not in the Gregorian.) Considerable distrust and suspicion followed this action in England with many people feeling that somehow their lives were being shortened and they were being made old before their times.

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In 1793 in France it was decreed that the calendar should commence with the formation of the First Republic on September 22, 1792. A year was to consist of twelve 30-day months, each month divided into three ten-day periods, with five festival days at the year's end dedicated to Virtue, Genius, Labor, Opinion, and Rewards, respectively. Every fourth year was to have an additional "Revolution Day". The century years were to be treated as in the Gregorian Calendar with a fine adjustment scheduled for the year 4000 which was not to be a leap year. Napoleon returned France to the Gregorian Calendar in 1806.

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Russia adopted the Gregorian Calendar in 1918.

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Some of the readers may wish to test the following conjectures

or answer the questions based on our present calendar.
In a non-leap year:

1) The year begins and ends on the same day of the week. (For a leap year this would mean that the ending day would be one day later in the week than the beginning day.)
2) There are 52 of every day of the week except the beginning and end day and 53 of this day. (For a leap year this would mean that there are 52 of each day except the beginning and end day. There are 53 of the beginning day and 53 of the end day.)
3) Put a pin through the 12 monthly pages of a monthly calendar so that a number is pierced on each page. If the January day of the month is N, the sum of the numbers pierced is $12N - 3$.
4) Every day in the week will be the first day of some month in the year.
5) What is the maximum sum possible under the conditions proposed in 3)?
6) What is the minimum sum possible under the conditions proposed in 3)?

=Δ=

Calendar reforms are still being discussed and many proposals have been made. One of these is the suggestion that the year be divided into equal quarters of 91 days or almost equal quarters. This proposal suggests further that the months be somehow equalized. At present a month may have as few as 24 or as many as 27 weekdays and either 4 or 5 Sundays.

=Δ=

The thirteen-calendar-month scheme suggests four-week months plus a Year Day in a non-leap year or two Year Days in leap years.

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The World Calendar Association proposal calls for four 91-day quarters—each quarter having, in order, months of 31, 30, and 30 days. An intercalary day, not part of either year, occurs between December 30 and January 1. This day would not have a day name such as Monday or Tuesday. There would be two such days in leap years. Holidays such as July 4 would always occur on the same weekday. For example, if January 1 were Sunday to begin with, then

January 1 would always be Sunday, July 4 would always be Wednesday, and Thanksgiving would always be on November 23.

The calendar square trick which follows is credited by Martin Gardner (*Scientific American*, January, 1957) to Mel Stover of Winnipeg. Suppose a square array of sixteen numbers is marked off on any calendar page. For example, suppose we have

3	4	5	6
10	11	12	13
17	18	19	20
24	25	26	27

Cross out any number in the array (say 25) and line out its row and column. Now cross out any number remaining (neither crossed out not lined out) such as 10 and line out its row and column. Do the same for any number remaining, say 5. The single number now left is 20. Cross it out. The sum of the cross-outs is twice the sum of the corner numbers on either diagonal. In our example, the sum of the cross-outs is $25 + 10 + 5 + 20 = 60$; and the sums of the corner numbers on the diagonals are $3 + 27 = 6 + 24 = 30$.

How could one prove this result for any 4×4 square from any calendar page? We could call the number in the upper-left corner n. Any element in the first column is thus $n + i \cdot 7$, $i = 0,1,2,3$; any element from the second column is $(n + 1) + j \cdot 7$, $j = 0,1,2,3$; any element from the third column can be written as $(n + 2) + k \cdot 7$, $k = 0,1,2,3$; any element from the fourth column can be written as $(n + 3) + m \cdot 7$, $m = 0,1,2,3$. If we choose elements according to the suggested scheme we will have exactly one element from each row and exactly one from each column. In this way the sum of the four elements can be expressed as,

$$n + i \cdot 7 + (n + 1) + j \cdot 7 + (n + 2) + k \cdot 7 + (n + 3) + m \cdot 7$$
$$= 4n + 6 + (i + j + k + m)7.$$

Since there is exactly one element from each row, $i + j + k + m = 0 + 1 + 2 + 3 = 6$. Thus the sum of the four cross-outs is $4n + 48$. The sum of the corner elements on either diagonal is $2n + 24$. Twice $2n + 24$ is $4n + 48$.

Gardner suggests that this trick has many ramifications as yet unexplored. Your Editor suggests the following for further investigation:

1) Any 2×2 calendar square has the sum of the cross-outs equal to the sum of the diagonal elements on either diagonal.
2) Any 3×3 calendar square has the sum of the cross-outs equal to $3/2$ times the sum of the corner elements on either diagonal.
3) Any $N \times N$ calendar square has the sum of the cross-outs equal to $N / 2$ times the sum of the corner elements on either diagonal.
4) What can you say about an $N \times N$ square of integers arranged so that the difference in consecutive elements in the same row is r and the difference between consecutive elements in the same column is c. (Let us agree that the elements increase as we move to the right and down.)
5) What is the largest possible value for N for a calendar square?
6) What are the possibilities for calendar cubes consisting of an $N \times N$ square marked on a page of the calendar and the same positions marked on the following $N - 1$ pages?

"Mathematics is a useful tool, but it is also something far greater, for it presents in unsullied outline that model after which all scientific thought must be cast."

G. St. L. Carson
Essays on Mathematical Education

The Book Shelf

EDITED BY R. H. MOORMAN

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor R. H. Moorman, Box 169A, Tennessee Polytechnic Institute, Cookeville, Tennessee.

Finite-Dimensional Vector Spaces, (2nd edition), by Paul R. Halmos, The University Series in Undergraduate Mathematics, D. Van Nostrand Company, Inc., (120 Alexander Street) Princeton, N.J., 1958, 193 pp., $5.00.

The appearance of this book is most welcome to those of us who were familiar with it in its earlier paperback form as Number 7 of the Annals of Mathematics Studies, published by the Princeton University Press in 1942. The new edition has been revised and extended, although the most noteworthy change is the addition of exercises.

The author's purpose is to make his work entirely self-contained. He includes fields, matrices, and determinants in this edition, material which was almost entirely absent from the earlier edition. A mathematics book at this level, however, cannot be entirely self-contained, and it may be assumed that most readers will have some background in algebra. Certainly, it seems strange to include a section on fields and to assume at least an elementary knowledge of groups.

The first chapter includes a discussion of fields and the definition and development of vector spaces. In this chapter the discussion of dual spaces seems particularly excellent. Although the discussion is confined to spaces of finite dimensions, the author's stated purpose of emphasizing theorems and proofs which have infinite-dimensional analogues is thoroughly realized.

The second chapter is concerned with transformations, thus supplying the author's motivation for the material in the first chaper. An excellent discussion of matrix theory is included in this chapter and much of the material on matrices would be accessible and valuable to a student even out of the context of the book. The author's lucid style is displayed to excellent advantage in this chapter.

The third chapter is concerned with inner products, normed spaces, and orthogonality. The first part of this chapter seems so much more accessible and easy than the second chapter that it might have been better pedagogy to present it first. The author's efforts to keep a simple geometric model before the eye of the reader are in evidence here. With this technique the reader is introduced to some difficult ideas almost without realizing it.

The fourth chapter connects the foregoing material with modern analysis. This chapter also illustrates the impossibility of a self-contained mathematics book. Surely no reader will appreciate this chapter who has not obtained a strong foundation in advanced calculus. The chapter contains a general discussion of convergence and concludes with an elegant proof of the ergodic theorem and a perhaps too-brief discussion of power series.

The fourth chapter is followed by an appendix which presents the Hilbert space to the reader as the most useful generalization of the previous material to infinite-dimensional spaces.

The addition of the exercises certainly makes the book more usable as a text. Although the exercises vary in difficulty from the trivial to the extremely difficult, this variation is not indicated in any way. Here, most of all, it is apparent that the book is not self-contained. The reviewer questions the advantage of presenting a problem to a student before the tools of solution have been developed. Possibly this technique would aid the development of a research capability in a graduate student, but more probably it would discourage a reader who might well be able to grasp most of the material. When a student masters a section of a mathematics book, he expects to be able to work the problems at the end of that section. If he is unable to work them, he becomes frustrated.

In general the publisher has done a good job in presenting the author's material. The expression "The University Series in Undergraduate Mathematics" which appears on the cover should not be an obstacle to its use as a text in a graduate course, for at many universities this material might be more suitable at that level.

Halmos' book is the best in its field and is one of the best-written mathematics books available. Mathematics students and teachers alike are indebted to Professor Halmos for making his work available in more useful form.

—W. M. Perel
Texas Technological College

Understanding Arithmetic, by Robert L. Swain, Rinehart and Company, Inc., (232 Madison Avenue) New York, 1957, xxi + 264 pp., $4.75.

For the student preparing to teach arithmetic in the elementary schools, an understanding of basic mathematical principles is essential. The title of this book is very appropriate, since it makes clear many of the basic principles of arithmetic. Many of the topics are introduced by giving a short historical background, stimulating both interest and understanding. There is other supplementary material, consisting of illustrations and comments, which the author includes as an integral part of the book but which is printed in smaller type.

The book has many excellent features, probably the most outstanding being its readability. The language is clear and concise, and the examples and illustrations are excellent. The illustrations are unusually effective because of the use of shaded areas and blocks. There is a large amount of enrichment material introduced with each topic. Some examples are: a paragraph on logic included in the chapter entitled "Sets and Numbers," and the paragraphs on duodecimal and binary number systems in Chapter VI entitled "Twelves and Twos."

The number system and operations are based on the mathematical concepts of sets and set operations. Subtraction and division are developed as inverse operations of addition and multiplication respectively. The traditional computational methods are also introduced, but the introduction appears to be with greater emphasis upon understanding rather than mere mechanical manipulation.

In the preface the author writes, "We propose with this book to help him (the adult who seeks to understand arithmetic) lay a new foundation and build more soundly." The reviewer believes that the author has accomplished his objective.

The chapters on fractions and decimals are well organized and interspersed with teaching suggestions.

The National Council of Teachers of Mathematics has published a pamphlet which expresses very well the present-day needs in arithmetic:

> The first half of the twentieth century has witnessed rapid developments and changes of point of view in the field of mathematics. The curricula of the schools and colleges have

> not been sufficiently influenced by these changes. Our mathematics program must be revised. Much of what we now have is good and must remain in the curriculum, but the time has come for careful selection of content and method related to present-day mathematical needs. The changes that must be made can be considered under three headings: content, presentation and requirements.[1]

The reviewer believes that the author has skillfully selected and interwoven the traditional mathematics usually taught in elementary schools with the most recent developments and applications of mathematics. The reviewer believes that this publication fulfills a long-felt need in arithmetic.

—IRENE NOLAN
Tennessee Polytechnic Institute

Arithmetic for Colleges, Revised Edition, by Harold D. Larsen, The Macmillan Company (60 Fifth Avenue) New York, 1958, xiii + 286 pp., $5.50.

This book is designed for a one-semester course in the principles and applications of elementary arithmetic. The exceedingly fine manner in which this book is written makes it invaluable to all students who seek proficiency in arithmetic as well as to those who are preparing to teach in the elementary schools.

The book is not merely a view of arithmetic. The topics are presented from an advanced point of view, giving the reader greater knowledge of the fundamentals of arithmetic.

The thirteen chapters of the books are very logically arranged. The first chapter starts with the systems of notation and takes the reader through the history of numerology. The chapters which follow deal with addition, subtraction, multiplication, division, common fractions, decimal fractions, and percentage. The author has written these chapters in such a manner as to create great interest in short methods of calculation. The chapters on approximate numbers and the slide rule are clearly and carefully presented in considerable detail.

Throughout the book historical and recreational items are included which can be valuable aids to the teacher of elementary arithmetic. Methods of checking which emphasize the casting out of nines and elevens are used.

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[1] The National Council of Teachers of Mathematics, "As We See It," 1201 Sixteenth Street NW, Washington 6, D.C., 1958, pp. 3-4.

There are three main additions to the revised edition which were not included in the first edition: (1) two simple tests for casting out sevens, (2) a "bridging" rule is described which simplifies the casting out of elevens, (3) a simple, but little-known, theorem concerning the greatest common divisor of two numbers is described and then applied to the reduction of fractions to lowest terms.

A fine feature of this book is the supplementary exercise at the end of each chapter. These exercises and an extensive bibliography make it possible to extend the material covered in the book.

—F. W. Block
Tennessee Polytechnic Institute

"Certain characteristics of the subject are clear. To begin with, we do not, in this subject, deal with particular things or particular properties: we deal formally with what can be said about "any" thing or "any" property. We are prepared to say that one and one are two, but not that Socrates and Plato are two, because, in our capacity of logicians or pure mathematicians, we have never heard of Socrates or Plato. A world in which there were no such individuals would still be a world in which one and one are two. It is not open to us, as pure mathematicians or logicians, to mention anything at all, because, if we do so we introduce something irrclevant and not formal."

—Bertrand Russell

Installation of New Chapter

EDITED BY MABEL S. BARNES

THE PENTAGON is pleased to report the installation of

CALIFORNIA GAMMA CHAPTER

California State Polytechnic College
San Luis Obispo, California

On May 23, 1958, California Gamma Chapter was installed at California State Polytechnic College. Mr. Dana R. Sudborough of San Jose State College, San Jose, California, and formerly business manager of THE PENTAGON, conducted the installation ceremony. A banquet was held on the campus, and, following it, Mr. Sudborough addressed the new chapter on "Some Aspects of Kappa Mu Epsilon."

Student charter members are Paul W. LeVier, Henry C. Miller, Joe K. Bryant, Albert C. Dandurand, Gilbert C. Myers, Jr., Ray Kitaguchi, George Wells, Ted W. Miller, Jr., Robert Y. Minami, Donald L. Snider, Jerre J. Zimmerman, Thomas H. Schultz, Ross C. Higbee, Richard M. Bird, William C. Ring, William L. Lockwood, Richard Eckerman, and J. Byron Culbertson. Faculty charter members are Robert D. Gordon, Milo E. Whitson, Chester H. Scott, Oswald J. Falkenstern, George R. Mach, John Manning, Vol A. Folsom, Michael L. Hall, Olive M. Anderson, and Charles A. Elston.

The following officers were installed: Joe K. Bryant, president; Robert Y. Minami, vice-president; Jerre J. Zimmerman, secretary; Richard Eckerman, treasurer; George R. Mach, corresponding secretary.

We extend a warm welcome and best wishes to our new western chapter.

Kappa Mu Epsilon News

EDITED BY FRANK HAWTHORNE, HISTORIAN

Alabama Beta sponsored a coffee hour at the annual Homecoming on October 25. Representatives from ten different initiation years attended. Their regular programs featured outstanding speakers from Redstone Arsenal and local industries.

The spring activities of **California Beta** included a field trip to Electrodata Corporation in Pasadena and a lecture on "The Shoemaker's Knife" by Dr. Leon Bankoff, a Los Angeles dentist, who has mathematics as a hobby. Dr. Wayne E. Smith, a charter member of **California Alpha,** has joined the Mathematics Department as an assistant professor.

The following members of **California Beta** have entered graduate schools: Hugh Lawrence, Columbia; Charles James Pearson, Yale; and Fred Weiler, Purdue. Robert Emmerling, George Engelke, and Arthur Stacy have completed the 3-2 program in engineering and have received degrees from both Occidental College and the California Institute of Technology. Steve Ahrens, Edward Gehle, Anthony Grande, and John Stene are at the California Institute of Technology and Dorothy Dirks and James Manson are at Columbia on the 3-2 program. Jack Prestwich received the Kappa Mu Epsilon Award for achievement in freshman mathematics.

Merl Kardatzke, last year's president of **Indiana Gamma,** has been awarded a Woodrow Wilson Fellowship for graduate work in mathematics at the University of Chicago this year. The president and vice-president of this chapter are automatically the senior and the junior with the highest averages in mathematics courses. It is of interest that this year's officers are brother and sister, Myron and Myrna Williams.

"Mathematics in Three Dimensions" is the theme which **Kansas Gamma** has selected for the year's program. It was voted to have each meeting consist of three sections: a) mathematical theory underlying games and tricks; b) biographical sketch of a famous mathematician; c) a magazine report on current mathematical news. Both members and pledges will cooperate in the presentation of the semi-monthly programs. The first meeting of the year, October 6, was an orientation meeting in which the "actives" explained various aspects of chapter activities to the twenty new pledges. Formal pledge induction is scheduled for October 20.

A chili supper honoring the newcomers to **Kansas Gamma** will be held October 23, 1958. Plans are being made to feature two guest speakers during the coming academic year. Professor William R. Scott of Kansas University will speak on "Game Theory" in mid-November. Professor Robert Gaskill, a visiting lecturer sponsored by the *Mathematical Association of America*, will be on the Mount campus February 27, 1959.

Highlighting the spring activities of **Missouri Alpha** was the conferring of two annual awards. Charles Atwater received the Freshman Award which is bestowed annually upon the first-year student whose achievement in mathematics is most outstanding. Chapter President Howard Hufford was the recipient of the Merit Award which is given to the chapter member making the greatest contribution to the fraterniy during the school year. Programs presented by students included "Theory of Games" by Robert Pearce and "Kaleidoscopic Geometry" by Howard Hufford. Alumnus Harold Steenbergen presented an interesting program on the qualification test for registered engineers in Missouri.

Wedding bells will ring for Barbara Rentchler, **Missouri Beta** Chapter President, '57-'58, on Thanksgiving Day at Clinton, Missouri. The chapter sponsored tours through the I.B.M. Midwest Research Institute and Linda Hall Library as well as a trip to the regional convention at Emporia, Kansas.

New Jersey Beta won an originality award for its booth, a maze, in Montclair State's annual carnival.

Ohio Alpha sponsored a series of four lectures last year which were well attended by university students and high school students and teachers from the area of northwest Ohio. The lectures were:

"Some Basic Ideas in the Theory of Transonic Flow" by Dr. Karl Gottfried Guderley, Chief of Applied Mathematics Research Branch at Wright-Patterson Air Force Base.

"Principles and Applications of Analog Computers" by Roger A. Gaskill, Ford Engineering and Research Center.

"Your Telephone, 1965" by Ralph T. Riefenstahl, Public Activities Supervisor of the Ohio Bell Telephone Company.

"Mathematical Recreations" by Dr. Harry Langman, Chairman of the Mathematics Department, Ohio Northern University.

All of last year's officers of this chapter are now either in school or teaching.

An unusual feature of the October program at **Ohio Gamma** was the reports by students of their summer work. Dave Kaiser spoke of his work as a statistician at the National Carbon Laboratory; and Marlene Brown, of her experience as computer for the NACA (now NASA) Laboratory. Mr. Gino Coviello spoke in November on analog computers, and in January a demonstration of the differential analyzer was given by Dr. Dean L. Robb and Dave Kaiser. At subsequent programs, Mr. Robert Schlea spoke on Laplace transforms; and Dr. Gordon Grant, on "Radio Satellite Tracking."

Wisconsin Alpha again sponsored a mathematics contest for high school seniors. One hundred twenty contestants from twenty-six schools participated. Madison West High School placed first and was awarded a plaque. David Peterson from Madison West and Stephen Andrews from Riverside High School, Milwaukee, tied for first place. This year's pledges were required to work the problems that had been posed for the high school contest as a part of their initiation.

KANSAS-MISSOURI-NEBRASKA REGIONAL CONVENTION

On Saturday, May 10, 1958, the Kappa Mu Epsilon chapters from Kansas, Missouri, and Nebraska met at Kansas State Teachers College, Emporia, for a regional convention. This was the third regional convention covering this area. Kansas Beta was the host.

Eleven chapters were represented, and a group was present from a campus having no chapter, with the following registrations:

Chapter	Faculty	Students	Total
Kansas Alpha—Pittsburg	4	15	19
Kansas Beta—Emporia	6	20	26
Kansas Gamma—Mt. St. Scholastica	2	10	12
Kansas Delta—Washburn	2	8	10
Kansas Epsilon—Fort Hays	1	4	5
Missouri Alpha—Springfield	3	9	12
Missouri Beta—Warrensburg	3	3	6
Missouri Gamma—William Jewell	2	3	5
Missouri Epsilon—Central College	1	10	11
Nebraska Alpha—Wayne	3	11	14
Kearney, Nebraska		6	6
	27	99	126

PROGRAM

8:00-9:00 a.m.	Registration - - - Music Hall
9:30 a.m.	First General Session - Music Hall Auditorium Charles Trauth, Kansas Beta, Presiding Address of Welcome, President John E. King, Kansas State Teachers College, Emporia Business Meeting Roll Call
10:00 a.m.	Student Papers - Music Hall Auditorium 1. "Development of Orthogonal Functions," Ernest Milton, Jr., Kansas Epsilon, Fort Hays. 2. "Continued Fractions," Marilyn T. Houston, Kansas Epsilon, Fort Hays. 3. "Solving the Cubic Equation," David Pool, Kansas Beta, Emporia. 4. "Relations Between Hyperbolic and Circular Functions," Richard Franke, Kansas Epsilon, Fort Hays. 5. "Economics and Linear Programming," Marion Rowin, Missouri Gamma, William Jewell.
12:00 noon	Banquet - Student Union Ballroom Address, "What is New in Mathematics?" Professor O. J. Peterson, Kansas State Teachers College, Emporia.
1:30 p.m.	Let's Exchange Ideas Faculty Section - Music Hall Room 102 Professor Charles B. Tucker, Chairman Professor Helen Kriegsman, Secretary Student Section Charles Trauth, Chairman
2:30 p.m.	Student Papers - Music Hall Auditorium 6. "Kaleidoscopic Geometry," Howard Hufford, Missouri Alpha, Southwest Missouri State College.

7. "The Seven Bridges of Konigsberg," Francis Botts, Missouri Gamma, William Jewell.
8. "A Discussion of the Four-Color Problem," Vernon W. Powers, Kansas Alpha, Pittsburg.

3:30 Business Session - Music Hall Auditorium

Professor Helen Kriegsman, Kansas Alpha, reported for the faculty group. The problems discussed were:

1. The number of activities and other pressures that interfere with attendance at meetings. Two chapters require attendance with excused absences. The quality of the program is the largest factor.
2. Changes: Membership requirements with anticipated higher-level students entering college. Ten chapters felt a year of college work should be required. Three chapters felt students should be admitted on reaching a defined level of maturity in mathematics even if freshmen.
3. Need for material and problem solutions for THE PENTAGON: Use of THE PENTAGON for programs, and even solution of problems might well be meeting activities. Many solutions currently come from other than KME members.

Miss Nancy Bowman, Missouri Beta, reported for the student group.

1. Social activities at meetings: picnics, banquets, carnivals.
2. Program: Speakers from the outside.
3. Fund-raising activities: Carnival, selling tables and slide rules, shows, and dues.
4. General activities: Term papers, student speeches, tours.
5. Time of meetings: Scheduled, on call, with mathematics or science club.

6. Expenses: Scholarships, banquets, delegates to convention, newsletters.
7. Attendance at meetings: Required by some chapters, inactive status for those who do not attend.

At the final business meeting several chapters, Kansas Epsilon, Missouri Epsilon, and Missouri Beta extended invitations for the next regional convention two years from date. It was moved and approved that the National Secretary, Miss Laura Green send out a ballot for the selection of the next regional convention site.

"Through and through the world is infested with quantity: To talk sense is to talk quantities. It is no use saying the nation is large—How large? It is no use saying that radium is scarce—How scarce? You cannot evade quantity. You may fly to poetry and music, and quantity and number will face you in your rhythms and your octaves."

—Alfred North Whitehead