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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Napierian Logarithms

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1. **Introduction.** In my mathematical experiences, I have perhaps been most amazed and awe-struck by the powerful calculating tool available in the tables of logarithms. From the time I first used these miraculous figures, I wondered time and again—"How were these computed?" and "Would I be able to understand their construction?" Consequently, this paper is not the mere completion of an assignment; it is, in part, the result of a personal challenge which I've looked forward to meeting since the time I learned that calculus would provide the answer to my queries.

Logarithms rank among the three most amazing discoveries in the field of mathematics, vying for honors with the Arabic number system and the introduction of zero as a number. Why? Stop and consider: By inspecting the series $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, etc., it does not seem too profound that a person of great ingenuity would reason that using the base 2, the exponent 2 with fractional increments would evolve the numbers between 4 and 8. But, when one considers the fact that exponential notation was not introduced into mathematics until some 33 years after the first publication of logarithm tables[1], and when one remembers that logarithms are exponents, how can he help wondering, as did Henry Briggs, "by what engine of wit or ingenuity you [John Napier] first came to think of this most excellent help in astronomy, viz., the logarithms."[2]

The motivation behind the discovery of logarithms came from what might seem to us today an unusual source—religion. To understand how religion gave impetus to Napier's delving mind, a brief description of the social organization of the time is necessary. The early seventeenth century was a period in which basic loyalties of the individual had not long been divorced from their century-old mater, the church, and aligned with the political entity, the state. No longer did religious loyalty ignore national boundaries. Religion still played a major role in the lives of men, but it tended to be subordinate to the patria. This new era witnessed the Protestant Reformation, and free-thinking men openly challenged the principles of scholasticism of the Roman Catholic Church whereby physical and natural laws were reconciled to religious dogma.

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[1] Introduced by Rene' Descartes in his **Geometrie**, 1637.
[2] Florian Cajori, **A History of Mathematics**, The Macmillan Co., London; 1938, p. 150.

Such was the case with Johann Kepler (1571-1630), official astronomer of the Holy Roman Empire, and with John Napier (1550-1617), Baron of Merchiston, in Scotland, a devout and ardent protestant and active church official in the Presbyterian faith of John Knox. Kepler, after studying a huge collection of data gathered from observations and applying his mathematical insight to it, was inclined to disbelieve the Ptolemaic geocentric theory (a basic theory of Roman scholasticism). For twenty-five years, Kepler labored with the mathematical proof of the Copernican heliocentric theory, a point of intense religious and metaphysical controversy, calculating endlessly the natural trigonometric functions involved. His prodigious task might not have been completed in his lifetime were it not for the interest of Napier in "the invention of methods for the diminution of the labour therein involved."[3]

How did Napier come about his idea of a logarithm? It has been conjectured that he first thought of substituting the easier processes of addition and subtraction for the more complicated operations of multiplication and division by examining the trigonometric identity

$$\cos A \cos B = (1/2)[\cos(A + B) + \cos(A - B)].$$

Others believe that because he had received much of his education on the continent, he was familiar with the work of Stifel, who wrote:

> *"Additio in Arithmeticis progressionibus respondet multiplicationi in Geometricis.*
>
> *"Subtractio in Arithmeticis progressionibus respondet in Geometricis Divisioni."*[4]

That is, addition in an arithmetic progression corresponds to multiplication in a geometric progression; while subtraction in an arithmetic progression is analogous to division in a geometric progression. This latter belief would seem to have more plausibility when one examines his method of establishing relationships between the natural sines and their logarithms. In fact, Napier reasoned intuitively that the logarithms of a series of numbers in geometric progression are themselves in arithmetic progression. This reasoning will be shown in a cursory development of his logarithms.

2. **Napier's construction of sines and their logarithms.** Apparently proceeding from the relationship between arithmetic and geometric series processes, Napier set about the task of constructing his sines in a decreasing geometric progression, and the adoption of this

[3] E. W. Hobson, **John Napier and the Invention of Logarithms**, 1614, Cambridge University Press; 1914, p. 13.

[4] J. L. Coolidge, **Mathematics of Great Amateurs**: Oxford, at the Claredon Press; 1949, p. 72.

premise led him to think of rates. He imagined two moving points describing straight lines, both commencing at the same velocity. One of the lines was of definite length and was to be the whole sine, i.e., sine of a 90° angle. The other line of indefinite length was to represent, i.e., segments of it, the logarithms of the sines. It should be clearly pointed out that his logarithms were for the natural sines only.

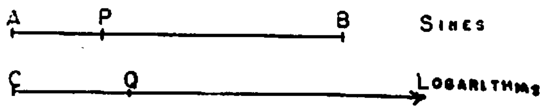

FIGURE 1

The point Q, representing the right terminal end of his logarithm, CQ, was assumed to move at a constant velocity of v linear units per unit of time. The point P, representing the left end of PB, a part of the whole sine (or sine of an angle less than a quadrant), was to begin its course at A with a velocity of v linear units per unit of time but was to continue at a decreasing velocity such that its rate during any time interval would be proportional to its distance from the ultimate goal at the beginning of that period of time. From these geometrical considerations he established his geometric and arithmetic series. He called CQ the logarithm of PB. If P, in the above figure, is a position of the moving point at the beginning of a time interval and P' its position at the end of that period of time, then the velocity of the point while traveling from P to P' was assumed to be $(PB/AB)v$.

Napier did not have the instantaneous velocity concept which was so baffling to earlier writers on the calculus; but, rather, imagined successive decreases in velocity—not continuous.

He reasoned that if the moving point P starting at distance r from its goal at rate v goes at a decreasing rate of speed such that its velocity at any instant during a time interval is proportional to its distance from the ultimate goal at the beginning of that time interval, then its distance from the ultimate goal at the end of that time interval would be equal to its distance at the beginning of the interval minus $1/r$ of that same distance. This may be shown as follows:

Let $r = AB$. Then during the first interval of time, the point (according to Napier) moved at a rate of $r/r \cdot v$ where v is the initial velocity and r/r is the ratio of the distance from the goal, B, at the beginning of the time interval to the whole distance. If the interval of time is chosen to be $1/v$ then, the point P travels a distance of $(r/r)\,(v)\,(1/v)$ or 1 linear unit during the first interval of time and consequently its distance from B at the end of that interval is $r - 1$ or $r(1 - 1/r)$. During the next interval of time the point would proceed at a rate proportional to its distance from B at the beginning of the interval (or the end of the preceding one); i.e., the velocity would be $[(r - 1)/r]v$. The distance traversed in the second interval of time is thus equal to $[(r - 1)/r](v)(1/v)$ or $1 - 1/r$ units. Now the point, at the second position considered, is $r - 1 - (1 - 1/r)$ or $r[1 - (1/r]^2$ units from B.

The distance of P from B at the end of the second interval of time is geometrically serial with its distance from B at the end of the first interval of time with a ratio $[1 - (1/r)]$. During the third interval, the velocity (being proportional to the remaining distance at the end of the second interval is $[r(1 - 1/r)^2/r] \cdot v$ and the point travels $[r(1 - 1/r)^2/r]v(1/v) = (1 - 1/r)^2$ units from the last position. Now the point is $r(1 - 1/r)^2 - (1 - 1/r)^2$ or $r(1 - 1/r)^3$ units from its ultimate goal—again in geometric series with the other positions. It can be established by mathematical induction that the point will be $r(1 - 1/r)^n$ units from B at the end of n intervals of time. The distance of the point P from B at the end of any interval of time will be its distance at the beginning of the interval less $1/r$ of the distance.[5] Since the intermediate points of any interval are not traversed by P at a rate actually proportional to their distances from the end of the line, Napier ignored them and chose only those at the ends of the intervals where the velocities decreased and did become proportional to the remaining distance. These points, were to be the sine values represented by the segment PB. For simplicity, if r is chosen as 10^n, each sine value can be determined by subtracting from the preceding sine value, $1/10^n$ of that same sine value. Napier made $r = 10^7$ and computed his sines as follows:

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[5] It is well to note here that Napier could actually reach the end of line AB and his method would make sine 0 finite, whereas we who now comprehend the instantaneous velocity concept know that if the velocity of P decreased proportionally to the distance from B at every instant, i.e., continuously, P would reach a condition such that its velocity would be infinitesimally small requiring an infinitely long time to reach B; and in that time, Q, the point describing his logarithm would have reached "infinity" so that as N approaches 0 the lim Nap. log N = ∞ as it should be. (Not $-\infty$, because his base is actually fractional.) This is of only academic significance, however, because he did not need to worry about sine 0, for multiplication by sine 0, (or plain zero if you please) results, of course, in a product of zero.

Whole sine, r	=	10,000,000.000 0000
$1/r$ of r	=	1.000 0000
$r(1 - 1/r)$	=	9,999,999.000 0000
$1/r$ of $r(1 - 1/r)$	=	.999 9999
$r(1 - 1/r)^2$	=	9,999,998.000 0001

etc.

As pointed out earlier, Napier had to reason that his logarithms increased arithmetically while the sines decreased geometrically.

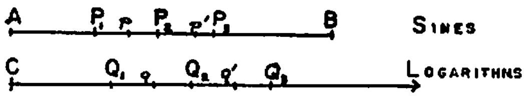

FIGURE 2

He said: let $P_1B/P_2B = P_2B/P_3B$, that is P_1, P_2, and P_3, were to be identifiable terminal points of sine segments constructed or determined in geometric series. Further, he said let p be any point within the interval P_1P_2 and p' be the corresponding point in P_2P_3. Since the velocity of the moving point at p bears a constant ratio to the velocity of the point p' and since Napier required that a constant velocity exist during each time interval, the time required for p to assume all positions between P_1 and P_2 would be equal to the time required for p' to assume all positions between P_2 and P_3. In the same interval of time, the points q and q' would have assumed all positions from Q_1 to Q_2 and Q_2 and Q_3 respectively so that $Q_1Q_2 = Q_2Q_3$, thus making his logarithms progress arithmetically. Although Napier assumed the converse to be true, i.e., that arithmetically progressing logarithms would result in geometrically progressing sines, he did not prove it.

3. **Instantaneous rates and Napier's notions.** Napier's logarithms were not what are known under the name of Napierian or natural logarithms today. In fact, Napier had no notion of bases or indices, and his logarithms are actually more closely related to the base 1/2 than to the natural base. This would seem apparent, i.e., the base being fractional, in order that the logarithm of a number should increase while the number itself decreased. The relationship using modern notation and the concept of instantaneous rates can

be shown as follows:

Let r = whole sine, $(r - x) = PB$, a partial sine where x is the difference between a whole sine and a partial sine. Let $y = CQ$, the Nap.log $(r - x)$; and let $ln\ N$ be the symbol for the natural logarithm of N.

$dx/dt = [(r - x)/r](v)$ or $d(r - x)/dt = [-(r - x)/r](v)$ where the minus sine indicates that $(r - x)$ is a decreasing function and $[(r - x)/r](v)$ is the rate, as Napier defined it, of P at the distance x from the beginning of the line. Hence,

$$d(r - x)/(r - x) = (-v/r)dt.$$

Integrating we obtain:

$$ln(r - x) = (-v/r)t + c.$$

To determine the constant of integration, we remember that when $t = 0, x = 0$.

Thus,

$$c = ln\ r$$

and

(1) $$ln(r - x) = (-v/r)t + ln\ r.$$

Now, consider the logarithm line CQ or Y.

$$dy/dt = v,$$

i.e., Q moves at the constant initial rate.

$$\therefore dy = v\ dt;$$

and integrating gives

$$y = vt + c_1 \cdot$$

But when $t = 0$, $y = 0$; so $c_1 = 0$ and $y = vt$. Thus

$$t = y/v.$$

Substituting this value for t above in (1) we get

$$ln(r - x) = (-v/r)(y/v) + ln\ r$$

or

$$y = r\ ln\ r - r\ ln(r - x).$$

But y also equals Nap. log $(r - x)$ as defined by Napier.

$\therefore$ Nap. log $(r - x) = r[ln\ r - ln(r - x)]$,

the relationship between Napier's logarithms and natural logarithms.

Using instantaneous rates and natural logarithms, it can be shown that if numbers increase geometrically, their logarithms increase arithmetically, and conversely. Let

$$dx/dt = (v)(AB - x)/AB \text{ or } dx/dt = k(AB - x)$$

where $k = v/AB$.

Therefore

$$-dx/(AB - x) = -k\ dt.$$

Integrating we obtain

$$+\ln(AB - x) = -kt + c, \text{ and } \ln(AB) = c$$

because when $t = 0$ then $x = 0$. Now, since $(AB - x) = PB$, $(AB - x_1) = P_1B$, $(AB - x_2) = P_2B \cdots$.

$$\ln(PB) - \ln(AB) = -kt = \ln(PB/AB)$$
$$\ln(P_1B) - \ln(AB) = -kt_1 = \ln(P_1B/AB)$$
$$\ln(P_2B) - \ln(AB) = -kt_2 = \ln(P_2B/AB)$$

By subtracting any two of the above equations of such a series, we get

$$\ln(P_{i+1}B/P_iB) = k(t_i - t_{i+1})$$

Thus, if $Q_1Q_2 = Q_2Q_3$ (*i.e.*, logarithms are arithmetically serial), $(t_i - t_{i+1})$ is constant; that is, equal time intervals transpire in the production of his logarithms of sines. Therefore $P_{i+1}B/P_iB$ equals a constant, and the sines will consequently be in geometric series. Conversely, if the sine values are to be in geometric progression, then $\ln(P_{i+1}B/P_iB)$ will be constant and hence $(t_i - t_{i+1})$ must be constant with the corresponding resultant of arithmetically progressing logarithms.

4. **Napier's construction of logarithms of sines.** Before we finally examine his method of finding logarithmic values for particular sines, we will call your attention to a point of interest regarding the ultimate development of common logarithms:

As was shown earlier, Napier determined his sine values by successive subtractions of one ten-millionth of each sine from itself to arrive at the subsequent values. He continued this process up to $r(1 - 1/r)^{100}$. Having done so, he found that some of the values he was obtaining were worthless because they were not tabulated sine values. This, of course, is obvious, for the rate of change of the sine function (a decreasing function) increases as the angle approaches zero.[*]

Realizing he was making useless subtractions, his common ratio affecting only the seventh place, Napier increased his common ratio to $(1 - 1/10^5)$, which meant that instead of subtracting one ten-millionth part of the preceding sine to obtain the next sine fractional value, he subtracted one hundred-thousandth of it resulting in a change in the fifth place for sine values. Since $(1 - 1/10^5)^1$ was to be comparable to $(1 - 1/10^7)^{100}$, $(1 - 1/10^6)^2$ would be

[*] Consider functional values of $\cos x$, as x goes from $\pi/2$ to 0. These functional values are the ratio of change of $\sin x$ with respect to a change in x, since $[d(\sin x)]/dx = \cos x$.

comparable to $(1 - 1/10^7)^{200}$. Thus in one operation, he "lopped-off" ninety-nine of his previous subtractions which were not quickly enough producing tabulated values of the natural sine function. Note that I said $(1 - 1/10^5)^1$ was comparable to $(1 - 1/10^7)^{100}$, not equal to it.

$$\begin{aligned}(1 - 1/10^7)^{100} &= 1^{100} - (100)(1)^{99}(1/10^7)^1 \\ &\quad + [(99)(100)(1)^{98}(1/10^7)^2]/2 - \cdots \\ &= 1 - 100(1/10^7) + (50)(99)(1/10^{14}) - \cdots \\ &= 1 - (1/10^5) + (1/2)(1/10^{10}). \quad \text{(approximately)}\end{aligned}$$

This discrepancy of approximately $(1/2)(1/10^{10})$ between $1 - 1/10^7)^{100}$ and $(1 - 1/10^5)$ affected the eleventh place and was not serious enough to impair the utility of his method.

Again, after fifty of these operations, the process was not producing tabulated sine values rapidly enough. At this point he apparently became aware of the fact that he could produce sine values by any common ratio, and the only effect this would have on his logarithms would be a commensurate adjustment in the common addend. Hence, he began to "just make numbers", i.e., sets of numbers, in geometric progression with slightly varying common ratios and then select those which were virtual equals of tabulated natural sine values. It was this revelation that he passed on to his contemporary, Henry Briggs, who ultimately began computation of common logarithms of natural numbers.

Now, how did Napier compute his logarithms? Since no method was then available by which logarithms would be calculated to an arbitrary degree of accuracy, Napier obtained two limits between which his logarithms must lie and also established limits for the difference of the logarithms of two sines. These limits were the very basis on which his whole logarithm construction depended.[7]

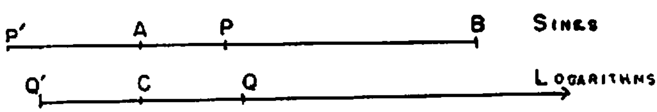

FIGURE 3

Since the velocities of P and Q at A and C are equal, that of P decreasing, it is clear that $CQ > AP$ when P lies beyond the first interval. Letting Q_1 move to the left of C for the same length of time that Q moves to the right of C, and letting P_1 be the point on AB

[7] The P' and Q' in Figure 3 should be P_1 and Q_1 respectively.

corresponding to Q_1 on the logarithm line, it is clear that since $Q_1C = CQ$, $P_1A > Q_1C$. Let $PB = x$. By definition Nap. log $x = CQ$. Therefore, recalling that Napier chose AB to be 10^7, Nap. log $x > AP$ or $(10^7 - x)$. Also, Nap. log $x = Q_1C < P_1A$. Consider now P_1B as the whole sine. Then $AB/P_1B = x/AB$ because we have already shown that P_1B, AB, and PB are in geometric progression. Consequently $10^7/P_1B = x/10^7$, AB being equal to 10^7. Therefore, $P_1B/10^7 = 10^7/x$ (by inversion) and $(P_1B - 10^7)/10^7 = (10^7 - x)/x$. $P_1B - 10^7 = P_1A$. So $P_1A/10^7 = (10^7 - x)/x$ or $P_1A = [10^7/x](10^7 - x)$. As a result Nap. log $x < [10^7/x](10^7 - x)$.

These are Napier's limits for the logarithm of a sine equal to x, i.e.,

$$(10^7 - x) < \text{Nap. log } x < (10^7/x)(10^7 - x).$$

Suppose the sine decreases from y to x. Napier's logarithm increases at a steady rate. The sine decreases at a decreasing rate. At y the sine decreases at a rate which is $y/10^7$ times the rate of increase of the logarithm, but at x the rate of decrease of the sine is $x/10^7$ times the rate of increase of the logarithm. Therefore

$$y - x < (y/10^7)(\text{Nap. log x} - \text{Nap. log y})$$
$$y - x > (x/10^7)(\text{Nap. log x} - \text{Nap. log y})$$

and thus

$$[10^7/y](y - x) < (\text{Nap. log } x - \text{Nap. log } y) < [10^7/x](y - x).$$

Accordingly, the logarithm of his first sine (9,999,999) is between 1.0000000 and 1.0000001. Napier chose the arithmetic mean, 1.0000005, as the logarithm.

5. **Conclusion.** We have thus seen how Napier developed his logarithms. We have barely touched on the development of his tables. However, enough has been done to show the clear thinking and analytical approach that Napier made to his problem. It is indeed a remarkable feat; I humbly bow to his genius.

"How is error possible in mathematics?"

—HENRI POINCARE

Modern Trends in Cryptography: The Fractionated Cipher

By S. H. Sesskin
Student, Hofstra College

The history of cryptography links the development of secret writing with the increase in literacy. Obviously an illiterate can read no message, hence any writing to him is secret writing and there is little need for ciphering. The need grows when knowledge spreads.

As the level of literacy rose, military, diplomatic—and criminal—activities dictated a need for more secure ciphers. Generally this led to more and more complicated ciphering procedures, most of which were outgrowths of the then current methods and were merely expansions of transposition or substitution ciphers.[1] Unfortunately for some of the so-called experts, complication often did not mean security, though many of their ciphers served well in earlier periods.

Today, with the high degree of literacy, the device which seems to be attracting most attention from cryptographers is a substitution-transposition technique.

Though these are the ciphers to be examined here, this discussion will be limited strictly to paper and pencil ciphers, and, of course, will not include ciphers coming from the newer electronic devices. (Despite these new devices paper and pencil ciphers will be studied as long as there are spies and criminals who cannot have access to such devices, and as long as wars are fought in the field where such devices not only would prove cumbersome, but would require the maximum protection from capture.)

The substitution-transposition cipher is not mere substitution and transposition piled on transposition upon retransposition. The rawest amateur can take any message, effect some kind of substitution, transpose it in some elementary fashion, and then continue to retranspose it until the message will, of course, be beyond solution; but it will also be beyond practicality and thus, useless, since an important consideration must be practicality.

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[1] See **Introduction to Cryptanalysis**, THE PENTAGON, Fall, 1954, for brief description of these two types of ciphers. Also bibliography.

Practicality presupposes a recipient, a human recipient with human failings; practicality requires that for safety's sake in the case of spies and criminals the key to the cipher be sure-fire and easily remembered by the recipient (no writing down of key); practicality requires that the method have only a few steps (the fewer the better) first because of the time factor, and secondly because of the error factor. A complicated method would result in compounding mistake upon mistake both in enciphering (putting the message into cipher) and in deciphering, so that the message could be lost in a maze of error. In this case the message would be secret to everyone—including the recipient.[2]

Practical ciphering has always been a compromise between space and time (the mathematician's fields, by the way) in an effort to obtain the maximum security in time at a minimum cost in words. And today it is more so than ever. In fact, today the balance is even more delicate, for the experts seek not so much an insoluble system, as one that will give security for a stated time.

The story is told that during a battle in North Africa in World War II when Rommel was winning, Nazi orders to tanks in action went out in clear language. Secrecy was not necessary, for the enemy was so disorganized they could do nothing even knowing the Nazi plans.

This is one extreme of ciphering; the amateur's cryptogram is the other.

The main point of this paper will be to show the mathematical type of thinking that goes into cipher analysis as exemplified in development of the cryptographic tools which led to the general solution of the classic fractionated cipher known as Delastelle's bifid.

Fractionation describes these ciphers accurately, as will be evident in an examination of two substitution-transposition types which will be made before studying Delastelle's bifid. There will be no attempt to analyze these; they are being presented simply as interesting illustrations of method.

The first example, suggested some years ago by M. E. Ohaver, a cipher expert, is a Morse Code fractionation which was explained in Helen F. Gaines' *Elementary Cryptanalysis*.[3] The second is the

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[2] Col. Parker Hitt in his **Manual for the Solution of Military Ciphers** devotes a chapter exclusively to errors and says: "In some cipher methods a mistake in enciphering one letter will so mix up the deciphering process that only one familiar with such errors can apply the necessary corrections."

[3] **Elementary Cryptanalysis** is scheduled for republication this year by Dover Press.

German Field Cipher of 1918, which was actually in use and which proved very effective for a time during World War I until solved by the French analytical genius, Georges Painvin.

The Continental Morse Alphabet Arranged by Group-Lengths

E .	S ...	H	B -...
T -	U ..-	V ...-	X -..-
	R .-.	F ..-.	C -.-.
I ..	W .--	ü ..--	Y -.--
A .-	D -..	L .-..	Z --..
N -.	K -.-	ä .-.-	Q --.-
M --	G --.	P .--.	ö ---.
	O ---	J .---	ch ----

Fig. 1

We will illustrate the Ohaver fractionation as applied to the message "The Cartesian system . . . ".

The message is first set up in groups of uniform length (seriated) and one-to-one substitutions are effected from the Morse Alphabet (Figure 1). A seriation of seven results in the following:

Plain:	T	H	E	C	A	R	T	E	S	I	A	N	S	Y	. . .
Morse:	-	....	.	-.-.	.-	.-.	-	.	...	..	.-	-.	...	-.--	
No. of elements:	1	4	1	4	2	3	1	1	3	2	2	2	3	4	

The transposition is effected by reversing the digits, retaining the sequence of Morse elements, but regrouping them according to the sequence of reversed digits. In effect, this simply changes the spacing:

Digits:	1	3	2	4	1	4	1	4	3	2	2	2	3	1
Morse:	-	...	..	-.-.	.	-.-.	-	....	...	--	..	..	-.-	-
Cipher:	T	S	I	C	E	C	T	H	S	M	I	I	K	T

This method received attention as recently as April, 1950, when it was discussed in *The Cryptogram* in connection with an improved system suggestion by Col. F. D. Lynch, USAF, ret.[4]

For the German Field Cipher the first step is to set up a checkerboard substitution square. In Figure 2 such a square has been set up on the key word PENTAGO(N), giving each letter two coordinates from among the substitution letters A D F G X, which were

[4] Indicating the high interest in these types is the fact that Col. Lynch's system is lately proving very popular with members of the American Cryptogram Association.

actually used by the Germans. For P we get the substitution AA (row first, column second); for B, DF.

	A	D	F	G	X
A	P	E	N	T	A
D	G	O	B	C	D
F	F	H	I	K	L
G	M	Q	R	S	U
X	V	W	X	Y	Z

Fig. 2

In Figure 3 the method has been applied to the message, "Attack hill ten at dawn." The substitutions are set up under the fractionation key TANGENT, each letter being numbered according to the following plan: 1 is assigned A; 2 to E (since BCD do not appear); 3 to G; 4 and 5, respectively, to the first and second N's from left to right, etc.

```
6143257
TANGENT
```

For clarity in the sample encipherment we have included the plaintext equivalents in lower case letters above and between their substitutes.

```
                6 1 4 3 2 5 7
                T A N G E N T
plaintext:       a   t   t   a
substitutes:    A X A G A G A
                   c   k   h
                X D G F G F D
                 i   l   l   t
                F F F X F X A
                   e   n   a
                G A D A F A X
                 t   d   a   w
                A G D X A X X
                   n
                D A F
```

Fig. 3

The cryptogram is now taken out by columns starting with column 1 (A), 2 (E) etc., and transmitted in five-letter groups thus:

XDFAG AAGFF AGFXA XAGFD DFGFX AXAXF GADAD AXX

In a preliminary analysis of this cipher in *The Cryptogram* of August, 1953, by Milton Harawitz, the following remarks were made, the terminology of which should be interesting to mathematicians:

"The development of the mathematical theory is not essential to the method of solution and will be omitted. The theory requires us to reconstruct the key. Since the order of the columns of both messages are mathematical functions of each other, we can form a cycle . . . and reconstruct the key."

As is now evident, fractionation requires a substitution alphabet with characters composed of two or more elements. Bifid would be two, trifid three elements, multifid many elements.[5]

We illustrate the classic bifid method on the message "Time, space, and motion . . .".

In Figure 4 we have the basic key square necessary to Delastelle's method, set up on the key word GEOM(E)TRY, with each letter in the key square identifiable by two number coordinates (row first, column second). Thus R is 21.

In Figure 5 we have used a seriation of seven to divide the message and have placed vertically beneath each letter its two coordinates from the key square.

In Figure 6, the transposition has been effected so that the coordinate numbers from Figure 5 have been read off horizontally, but replaced vertically in pairs. These vertical pairs give us the ciphertext, the top letters being read as the row coordinate of the cipher letter, the bottom letters as the column coordinate, and the cipher letter itself being taken from the appropriate intersection in the original key square. Decipherment is, of course, the inverse process.

	1	2	3	4	5
1	G	E	O	M	T
2	R	Y	A	B	C
3	D	F	H	I	K
4	L	N	P	Q	S
5	U	V	W	X	Z

Fig. 4

Plaintext:	T	I	M	E	S	P	A	C	E	A	N	D	M	O	. . .
row coord:	1	3	1	1	4	4	2	2	1	2	4	3	1	1	
col. coord:	5	4	4	2	5	3	3	5	2	3	2	1	4	3	

Fig. 5

[5] Indicating the recent interest in fractionation ciphers is the fact that a trifid cipher was analyzed in **The Cryptogram** as recently as August, 1952. It is noteworthy that a trinumoral alphabet was invented and used by the Abbe Trithomo, 1499. Parker Hitt comments in the introduction to his **Manual** that "the ciphers of the Abbe Trithomo are the basis of most of the modern substitutions."

r. for cipher:	1	1	4	2	4	2	3		2	2	3	1	2	2	4
c. for cipher:	3	1	4	5	4	5	3		1	4	1	5	3	1	3
ciphertext:	O	G	Q	C	Q	C	H		R	B	D	T	A	R	P

Fig. 6

A cryptographic opinion in *The Cryptogram* doubted if this cipher's "fair degree of security is of enough value to compensate for the time required to encipher and decipher; a speed of about eight letters per minute."

This posed the cryptographic problem: A fair degree of security vs. lengthy encipherment and decipherment time. Could a speedier method be devised to make this cipher more worthwhile?

In this case the cryptographic and cryptanalytic problems were solved together. We will show the analysis of the latter problem, passing over for the moment the question of establishing seriation length, which is of primary importance in this cipher.

We'll start by establishing the precise relationship between the elements of cipher letters and those of plaintext. Since each letter has two elements—row and column—we will assign subscript 1 for row elements, 2 for column elements to each plaintext letter (Figure 7) and use the subscripted plaintext letters instead of number coordinates to perform the transposition (Figure 8), finally (Figure 9) placing the cipher letters from Figure 6 over their equivalents, and identifying each column of three letters by 1, 2, 3, etc.

T I M E S P A

$T_1I_1M_1E_1S_1P_1A_1$

$T_2I_2M_2E_2S_2P_2A_2$

Fig. 7

$T_1M_1S_1A_1\ I_2\ E_2P_2$

$I_1\ E_1P_1T_2M_2S_2A_2$

Fig. 8

1	2	3	4	5	6	7
O	G	Q	C	Q	C	H
T_1	M_1	S_1	A_1	I_2	E_2	P_2
I_1	E_1	P_1	T_2	M_2	S_2	A_2

Fig. 9

In Figure 9 we have, in effect, solved for each cipher letter in terms of plaintext, which is the enciphering method. Let us use Figure 9 to solve for each plaintext in terms of cipher letters, which is the deciphering method.

In Figure 9 we have from col. 1 that the row-element of cipher-letter O, that is O_1, is equivalent to T_1; and from col. 4 that the column-element of cipher C, that is C_2, is equivalent to T_2; so that (T_1T_2) which is plaintext T, is thus (O_1C_2); and (I_1I_2) from Figure 9 cols. 1 and 4 is (O_2Q_1). Solving similarly for all values, we get:

$$\begin{array}{ccccccc} T & I & M & E & S & P & A \\ O_1 & O_2 & G_1 & G_2 & Q_1 & Q_2 & C_1 \\ C_2 & Q_1 & Q_2 & C_1 & C_2 & H_1 & H_2 \end{array}$$

And noting that the horizontal sequence of row and column elements is the sequence of cipher letters from the cryptogram in Figure 6, (except that they are doubled), we set up the entire sample cryptogram similarly. (Figure 10)

group 1	group 2
T I M E S P A	C E A N D M O
$O_1O_2G_1G_2Q_1Q_2C_1$	$R_1R_2B_1B_2D_1D_2T_1$
$C_2Q_1Q_2C_1C_2H_1H_2$	$T_2A_1A_2R_1R_2P_1P_2$

Fig. 10

Now let us suppose that the cryptogram of Figure 10 is a fragment of a longer unknown cryptogram, and that the probable words, obtained through frequency examination and knowledge of the source of the cryptogram, are correct. We next attempt to build up the key square.

We have two types of cipher values, 1-2s and 2-1s. The 1-2 values give us direct relationships which can be manipulated algebraically to build up the key square thus:

In group 1, Figure 10, we have for example, that the row element of A, which is A_1, is equivalent to C_1, the subscript of which identifies it as the row element of C. Since AC have the same row element, they must be in the same row.

In group 2, A_1 is equivalent to B_1; therefore B is on the row with AC. In the same group, C_1 is equivalent to R_1; therefore R joins ABC.

The 2-1 values give indirect information which, nevertheless, can be manipulated.

In group 1, I_1 is equivalent to O_2, which means that I's row in the key square of Figure 4 equals O's column (3 in this case); and P_2 is equivalent to H_1.

In group 2 (a 1-2 value) O_2 is equivalent to P_2. Therefore, $P_2 = O_2 = I_1 = H_1$ which places H and I in the same row; and, of course, O and P in the same column of the key square.

These equations can be used to set up partial key squares from which other plaintext values may be obtained, the entire process building up to complete solution.

The dominant fact which emerged after years of research along this line is that the original number coordinates are unnecessary middlemen which can be eliminated in the ciphering process.

Figure 11 shows the ingenious device which resulted and which was fully explained in *The Cryptogram* of February, 1947, in an article by Herbert Raines.

					1-square				
					G	R	D	L	U
					E	Y	F	N	V
					O	A	H	P	W
					M	B	I	Q	X
2-square					T	C	K	S	Z
G	R	D	L	U	G	E	O	M	T
E	Y	F	N	V	R	Y	A	B	C
O	A	H	P	W	D	F	H	I	X
M	B	I	Q	X	L	N	P	Q	S
T	C	K	S	Z	U	V	W	X	Z
					basic square				

Fig. 11

The 1-square is composed of rows from the original basic square set up column-wise. The 2-square is composed of columns set row-wise. The basic square, of course, is the original square.

Now let's assume we are deciphering. Set up the cipher letters this way to save copying:

O G Q C   R B D T
C Q C H   T A R P

This, of course, is Figure 10 with the repetitions deleted. The 1-2 letters are the vertical OC, GQ, QC, CH, etc. The 2-1 letters are the diagonal OQ, GC, QH, etc.

We examine the 1-2s first in connection with Figure 11.

O_1C_2 is the letter at the intersection of the row containing O in the basic square and the column containing C in the basic square. It is T.

G_1Q_2 is the letter at the intersection of G-row and Q-column in the basic square—M. Q_1C_2 yields S; C_1H_2 yields A.

Now the 2-1s.

O_2Q_1 is that letter in the basic square which is at the intersection of the row governed by 2-square O, and the column governed by 1-square Q. It is I. For G_2C_1 find G in the 2-square, C in the 1-square. The required letter is at the intersection—E. Q_2H_1 yields P; etc.

For enciphering with this literal indices method, we can use the message directly. In Figure 9, for example, the first cipher letter is obtained from (T_1I_1); (M_1E_1) gives us the second cipher letter; the third is from (S_1P_1); (A_1T_2) gives us the fourth, with T starting a message repeat, and the fifth cipher coming from (I_2M_2); (E_2S_2), the sixth; and (P_2A_2) the seventh. Thus we have all the first group cipher letters from T_1 I_1 — M_1 E_1 — S_1 P_1 — A_1 T_2 — I_2 M_2 — E_2 S_2 — P_2 A_2. We now examine these in connection with Figure 11.

(T_1I_1) the intersection of the T-row in basic square, and the I-column in the 1-square—O.

(M_1E_1) find M in the basic square, E in the 1-square and at the intersection in the basic square is G. (S_1P_1) yields Q. (A_1T_2) a 1-2, yields C.

(I_2M_2) is that letter in the basic square which is at the intersection of the row governed by 2-square I and the column governed by basic-square M. It is Q.

(E_2S_2) find E in the 2-square, S in the basic square and at the intersection of their respective row and column in the basic square is C. (P_2A_2) yields H.

To recapitulate:

All "real" letters are obtained from the basic square. (By real letters we mean those which actually appear as plaintext when we are solving for plaintext—deciphering or decrypting; or those which appear as ciphertext when we are solving for ciphertext—enciphering. These are distinguished from the others which are being used as coordinates.)

In the notation (A B), the first value is always a row value, the second a column value.

For 1-2 values—All letters are in the basic square.

For 2-1 values—First letter in 2-square, second in 1-square.

For 1-1 values—First letter in basic square, second in 1-square.

For 2-2 values—First letter in 2-square, second in basic square.

An important point in recovery of the key square is the presence of the so-called "spine" letters, the five letters which appear in the same position in all three squares, along the diagonals from upper left to lower right.

If in building up the key square, we get the same letter in the 2-square and the basic square on the same row, then that letter is a diagonal letter. Similarly, if the same letter appears in the basic square and the 1-square in the same column, it is on the diagonal.

(An interesting exercise to pose at this point for those inclined to try would be to use the above technique to establish the method of enciphering and deciphering an evenly seriated message, as opposed to an oddly-seriated one which is being treated here.)

For the decrypting procedure (solving without knowledge of the key) we will take Figure 10 as our assumption, and seek to recover the key square. Set up a tentative square as in Figure 12.

			Q_1
	O_1	T	.
		C_2	.
O_2	.	.	I

Fig. 12

The square is built up in the following manner: 1-2 values and 2-1 values are transferred to the tentative squares in accordance with the properties outlined above. When letters are found on the same row in the basic square, they are added to the related columns in the 1-square. The inverse is also applied. When letters are found in the same column in the basic square, they are added to the related rows in the 2-square. Actually the three-square device automatically equates the values mentioned earlier.

From the two groups in Figure 10 all the relationships in Figure 13 can be recovered. The reader may try this as an exercise.

		B						
	M	R						
	E	A						Q
	G	C	P				H	S
R G D	G	E	M				.	.
	R	.	.	C	A	B	.	.
	D	.	.	.	.		.	.
M Q	.	.	.	.	.	.	P	.
		.	.	T	.	.	O	.
		.	.	.	H			.
P O	.	.	.	.	.	.	.	I
B	.	N	.					
			Q	S				

Fig. 13

One more item of great aid in decrypting is the appearance in the cipher of "naturals" and "half-naturals". These are cipher values in which actual message letters appear. For instance, if in Figure 10, group 2, we had established (B_2R_1) as N and had no other values, then the "half-naturals (B_1A_2) and D_1R_2), preceding and following N might suggest "and". A "natural" would be a doubled 1-2 letter such as (A_1A_2) equals A; but (A_2A_1) $\neq$ A unless A is a "spine" letter.

Seriation: We will illustrate without theory a purely statistical method of finding the period of a bifid which was discussed by Charles P. Windsor in *The Cryptogram* of June, 1946.

```
T I M E S P A   C E A N D M O
1 3 1 1 4 4 2   2 1 2 4 3 1 1
5 4 4 2 5 3 3   5 2 3 2 1 4 3
```

cipher:

```
13 11 44 25 44 25 33   21 24 31 15 23 21 43
O  G  Q  C  Q  C  H    R  B  D  T  A  R  P
```

Fig. 14

In Figure 14 the cipher letters O, G, and Q (group 1) and R, B, and D (group 2) are formed by combining exclusively the first (row) components of the plaintext; while Q, C, and H and A, R, and P result exclusively from the second (column) components. If we take separate frequency counts for the first component letters and second component letters for various possible periods, we shall find that that period which shows the greatest difference between these distributions generally is the correct period.

We will illustrate on the following sample bifid:

```
TTTYC  AIBXE  AMGBL  SMGML  NAIMD
SMGOE  OXNNV  LAEWI  TTLYA  SFSIF
TITYF  LIPWG  TOFTO  TSRPK  IICLN
RMLQW  IKDTR  MKBRK  ABHQR  NHLXM
MSBVF  MMTNL  SMGML  NNIKA  S.
```

For period 5, in group 2, A and I are first-component letters, B is mixed, and X and E are second-component letters. Following are the frequency counts for three different periods:

Period 5

	A	B	C	D	E	F	G	H	I	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1st:	5	1	–	–	–	1	–	1	6	2	2	9	4	2	–	–	1	6	7	–	–	–	1	–	–
2nd:	2	1	1	1	2	3	1	–	2	3	5	4	3	2	1	2	3	–	2	–	2	3	2	3	–

Period 7

	A	B	C	D	E	F	G	H	I	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1st:	3	2	1	–	3	1	1	1	5	1	5	8	–	3	1	2	1	1	7	–	–	2	2	–	–
2nd:	2	2	1	1	–	4	2	1	4	5	4	5	5	1	1	–	3	3	4	–	2	–	–	1	–

Period 9

	A	B	C	D	E	F	G	H	I	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1st:	3	2	1	1	3	2	3	1	2	1	4	5	2	3	1	1	3	5	6	–	1	1	–	1	–
2nd:	3	2	–	1	–	1	2	1	6	2	6	6	5	1	1	1	1	3	5	–	1	1	2	2	–

Karl Pearson's Chi-square test is used to establish the measure of divergency between the elements of these pairs. It is computed this way: Take the difference between the first and second component frequencies for each letter, square the result, then divide it by the sum of the two frequencies. For each period add the 26 quotients; the greatest sum indicates the most likely period.[6]

Results for the three examined periods are:

Period 5: 34.7; Period 7: 26.2; Period 9: 12.5. Period 5 happens to be correct. As a check, note in the sample cryptogram the characteristic formation of bifid repeats in groups 1 and 9: T T — Y; and in groups 3 and 4: M G — L. (An analysis of such repeats was used in earlier techniques of obtaining the period.)

In lieu of more material we give the following information about the sample cryptogram:

Starts "The beautiful . . . "; Probable words: Equations, theorems, mathematics.

---

[6] It is interesting to note from a mathematical viewpoint that more and more completely mathematical tools are being fashioned by members of the American Cryptogram Association for use in decryptment. Recent articles have dealt not only with the Chi-square test, but also with the Sigma test and Phi test. Furthermore, using the theory of coincidences, certain constants have been established for plaintext alphabets and random alphabets which have been found to be extremely useful.

See if you can solve it. If you cannot, the key square is based on L Q W H J U D O F (D O F) X (O X) V. This is Caesar's alphabet:

Plaintext:	A	B	C	D	etc.
Cipher	D	E	F	G	etc.

One final point to answer the obvious question: How would you start if you didn't know this was a bifid, or, generally, how would you tackle the unknown cipher?

There is no technique for this except judgment, experience, and a working knowledge of cryptography. The author has solved some few such unknown ciphers and can only say that starting such a problem is like being thrust into the blackness of a huge cavern with the knowledge that there is, somewhere, a tenuous thread that will lead the way out. One gropes and gropes and first finds all sorts of cobwebs that merely feel like threads. Perhaps one may not even find the thread, but there may be a wisp of a zephyr, a hunch, that might be followed to the light. Just keep groping till something turns up.

The advice given by the French cryptographer General Marcel Givierge in his *Cours de Cryptographie* is less impressionistic though more direct. General Givierge remarks that novices tend to recoil from the multiplicity of possibilities inherent in an unknown cipher and seem to be uncertain "which end to pick it up by". The best advice, the General adds, is to pick it up **somewhere** and do **something**, rather than be satisfied to sit all day long and admire the cryptogram!

"Yet we, who are borne on one dark grain of dust
Around one indistinguishable spark
Of star-dust, lost in one lost feather of light
Can by the strength of our own thought, ascend
Through universe after universe."

—Alfred Noyes

A Small Sample of Additive Number Theory

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1. **Introduction.** In number theory one of the types of problems which has excited a great deal of interest is the general question of writing each of a given set of integers as the sum of other integers where the summands are to be chosen from a particular class of integers. For instance one might ask the questions:

1. Which positive integers can be written as the sum of two or more consecutive positive integers?
2. Which positive even integers can be written as the sum of two prime integers?
3. Which positive integers can be written as a sum of distinct powers of 2 and in how many ways?
4. In how many ways can a given positive integer be written as a sum of positive integers, repetitions being allowed but order of summands irrelevant?

These are but a sample of many hundreds of questions which could be and are asked. These questions are not all easy to answer—despite the simplicity of their statements. In fact, of the four listed, the second has not yet been answered by the mathematical world. There is a conjecture—known as Goldbach's conjecture—to the effect that all even integers greater than 2 are indeed sums of two odd primes. The truth of the conjecture is unknown. On the other hand the answers to questions (1), (3), and (4) are known. The answers to (1) and (3) are quite easy to arrive at while that of (4) is a little more difficult.

2. **Positive integers as the sum of two or more consecutive positive integers.** The answer to question (1) is that all integers which have at least one odd factor greater than 1 and only those can be written as a sum of at least two consecutive positive integers.

To show this we proceed as follows. Let the number be n. Now $n = 2^t(2k + 1)$ for some $k \geqq 1$. Thus

$$\begin{aligned} n = \quad & k \quad + \quad (k+1) \\ & + (k-1) + (k+2) \\ & + (k-2) + (k+3) \\ & \quad \cdots\cdots\cdots \\ & + (k - 2^t + 1) + (k + 2^t). \end{aligned}$$

The numbers on the right are paired so that if the two numbers in a pair are added we arrive at the sum $2k + 1$. But there are 2^t pairs so the sum on the right is $2^t(2k + 1)$. Hence the equality is valid. Now starting at the bottom of the left column on the right of the equal sign and reading to the top and then reading down the right column we find

$$n = (k - 2^t + 1) + \cdots + (k - 2) + (k - 1) + k + (k + 1) + \cdots + (k + 2^t).$$

This gives n as a sum of consecutive integers. We leave it to the reader to finish the proof so as to have only positive integers in the sum.

As an example of the above we have

$$\begin{aligned} 12 &= 2^2 (2 \cdot 1 + 1) \\ &= 1 + 2 \\ &+ 0 + 3 \\ &- 1 + 4 \\ &- 2 + 5 \\ &= (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 = 3 + 4 + 5. \end{aligned}$$

We omit the proof that 2^n cannot be written as a sum of at least two consecutive integers.

3. **Positive integers as the sums of powers of 2.** The answer to question (3) is that every positive integer can be written as a sum of distinct powers of 2 (allowing the zero power to appear) and in only one way. A method of proof which has proved very valuable in answering questions of the kind expressed in (1) to (4) will be employed here. Let us consider the algebraic identities

$$(1 - x)(1 + x) = 1 - x^2,$$
$$(1 - x)(1 + x)(1 + x^2) = (1 - x^2)(1 + x^2) = 1 - x^4,$$
$$(1 - x)(1 + x)(1 + x^2)(1 + x^4) = (1 - x^4)(1 + x^4) = 1 - x^8,$$
$$(1 - x)(1 + x)(1 + x^2)(1 + x^4)(1 + x^8) = (1 - x^8)(1 + x^8) = 1 - x^{16},$$

etc.

We can rewrite these as follows:

$$1 + x = (1 - x^2)/(1 - x),$$
$$(1 + x)(1 + x^2) = (1 - x^4)/(1 - x),$$
$$(1 + x)(1 + x^2)(1 + x^4) = (1 - x^8)/(1 - x),$$
$$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8) = (1 - x^{16})/(1 - x),$$

etc.

By long division we find

$$(1 - x^4)/(1 - x) = 1 + x + x^2 + x^3,$$
$$(1 - x^8)/(1 - x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7,$$
$$(1 - x^{16})/(1 - x) = 1 + x + x^2 + x^3 + \cdots + x^{15},$$
etc.

Therefore we have

$$1 + x = 1 + x,$$
$$(1 + x)(1 + x^2) = 1 + x + x^2 + x^3,$$
$$(1 + x)(1 + x^2)(1 + x^4) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7,$$
$$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8) = 1 + x + x^2 + x^3 + \cdots + x^{15},$$
etc.

The $(n + 1)$th such expression is

$$(1 + x)(1 + x^2) \cdots (1 + x^{2^n}) = 1 + x + x^2 + x^3 + \cdots + x^{2^{n+1}-1}.$$

Thus in multiplying the factors $1 + x$, $1 + x^2, \cdots, 1 + x^{2^n}$ we get 2^{n+1} different powers of x, each only once. But in multiplying the left-hand side we get all p o s s i b l e summands of the form $x^a \cdot x^b \cdots x^l$ where a is 0 or 1, $b = 0$ or $2, \cdots, l = 0$ or 2^n. Now $x^a \cdot x^b \cdots x^l = x^{a+b+\cdots+l}$ and since x^m occurs once only on the right for each m from 1 to $2^{n+1} - 1$ we must have $m = a + b + \cdots + l$ for one and only one set of $a, b, \cdots, l$. These numbers $a, b, \cdots, l$, are powers of two or are 0, so we have proved that every positive integer from 1 to $2^{n+1} - 1$ can be written in one and only one way as a sum of distinct powers of 2. The reader should check and make sure that this really proves what we set out to prove.

In a similar vein using the identity

$$[(1 - x^3)/(1 - x)][(1 - x^9)/(1 - x^3)][(1 - x^{27})/(1 - x^9)][(1 - x^{81})/(1 - x^{27})] = (1 - x^{81})/(1 - x),$$

and observing that

$$(1 - x^3)/(1 - x) = 1 + x + x^2$$
$$(1 - x^9)/(1 - x^3) = 1 + x^3 + x^6$$
$$(1 - x^{27})/(1 - x^9) = 1 + x^9 + x^{18}$$
$$(1 - x^{81})/(1 - x^{27}) = 1 + x^{27} + x^{54}$$
$$(1 - x^{81})/(1 - x) = 1 + x + \cdots + x^{80}$$

we find that

$$(1 + x + x^2)(1 + x^3 + x^6)(1 + x^9 + x^{18})(1 + x^{27} + x^{54}) = 1 + x + \cdots + x^{80}.$$

Now dividing both sides by x^{40} we get

$$[(1 + x + x^2)/(x)][(1 + x^3 + x^6)/(x^3)]$$
$$[(1 + x^9 + x^{18})/(x^9)][(1 + x^{27} + x^{54})/(x^{27})]$$
$$= x^{-40} + x^{-39} + \cdots + x^{40}$$

or

$$(x^{-1} + 1 + x)(x^{-3} + 1 + x^3)(x^{-9} + 1 + x^9)(x^{-27} + 1 + x^{27})$$
$$= x^{-40} + \cdots + x^{40}.$$

This equation shows that all numbers from -40 to $+40$ can be written in one and only one way by using without repetition the numbers ± 1, ± 3, ± 9, ± 27. This result has application to the problem of finding the minimum number of weights needed to weigh all weights from $-n$ to $+n$ where the weights may be placed in either pan of a balance (Bachet's weight problem).

4. **Partitions of positive integers.** We now come to question (4). Again the answer is known. Here, however, we find that one can give two sorts of answers. The first is to give a formula by which one can by devoting enough time and patience calculate in a mechanical fashion the number in question. The second sort of answer is one that will give us a good approximation to the correct result with relatively little calculation. At present all methods of giving the second sort of answer are very difficult. In this paper we will evolve a formula which gives a satisfactory answer. That is, we will derive a formula which will give us a mechanical procedure for determining the result but which is rather useless for actual calculations. Our result will give a formula for the number of ways n can be written as a sum of positive integers in terms of the number of ways that $n - 1, n - 2, \cdots, 1$ can be so written. Such a formula is called a *recursion formula*. We will denote the number of ways by $p(n)$. For example $p(4) = 5$ since 4 can be written in the following five ways

$$1 + 1 + 1 + 1;\ 1 + 1 + 2;\ 1 + 3;\ 2 + 2;\ 4.$$

We will call each of these five ways of writing 4 a *partition* of 4. Then $p(4)$ is the number of partitions of 4 and in general $p(n)$ is the number of partitions of n. In the partition $1 + 3$ of 4 we call 1 and 3 parts. Now if we add up all parts of all partitions of 4 we will get

$$(1 + 1 + 1 + 1) + (1 + 1 + 2) + (1 + 3) + (2 + 2) + 4$$
$$= 4 \cdot p(4).$$

In general if we add up all parts of all the partitions of n we will get $n \cdot p(n)$ since the sum of the parts in a given partition is n and there are $p(n)$ partitions.

We shall now proceed to add up all parts in all partitions in a second way. We will do this by first adding together all the 1's which appear as parts in the partitions, then the 2's, etc. For instance in our example with 4 we have $4 \cdot p(4) = (1 + 1 + 1 + 1 + 1 + 1 + 1) + (2 + 2 + 2) + 3 + 4.$

Our first question is then—how many 1's are there altogether? Since $n = (n - 1) + 1$ we see that from any partition, say $a + \cdots + l$, of $n - 1$ we get a partition of n containing 1 by adding 1,

$$(a + \cdots + l) + 1 = (n - 1) + 1 = n.$$

Since every partition of n containing a 1 is of the form $1 + a + \cdots + l$ where $a + \cdots + l$ is a partition of $n - 1$ we see that the number of partitions of n which contain at least one 1 is equal to the number of partitions of $n - 1$, i.e., is equal to $p(n - 1)$. In a similar way we reason that the number of partitions of n which contain at least two 1's is equal to $p(n - 2)$. Continuing we find that the number of partitions of n having at least h 1's is equal to $p(n - h)$. Now the number having exactly one 1 is given by $p(n - 1) - p(n - 2)$. The number having exactly two 1's by $p(n - 2) - p(n - 3)$, etc. Thus the total number of 1's is given by

$$[p(n - 1) - p(n - 2)] + 2[p(n - 2) - p(n - 3)] + 3[p(n - 3) - p(n - 4)] + \cdots + (n - 1)[p(n - [n - 1]) - p(n - n)] + n\,p(0) = p(n - 1) + p(n - 2) + p(n - 3) + \cdots + p(0)$$

where $p(0) = p(n - n) = 1$ = number of partitions with n 1's. Our example with 4 gives $p(3) = 3$ partitions with at least one 1, $p(2) = 2$ partitions with at least two 1's, $p(1) = 1$ partition with at least three 1's, $p(0) = 1$ partition with at least four 1's. Thus the number with exactly one 1 is $p(3) - p(2)$. The number with exactly two 1's is $p(2) - p(1)$, with three 1's $p(1) - p(0)$ and with four 1's, $p(0)$. Hence the total number of 1's is

$$[p(3) - p(2)] + 2[p(2) - p(1)] + 3\,[p(1) - p(0)] + 4p(0) = p(3) + p(2) + p(1) + p(0) = 3 + 2 + 1 + 1 = 7.$$

In general the number of parts h occurring in all partitions is given by

$$p(n - h) + p(n - 2h) + \cdots + p(n - rh)$$

where r is the largest integer such that $rh \leqq n$. Thus $r \leqq n/h$ and $r + 1 > n/h$. One usually writes this $r = [n/h]$ = largest integer less than or equal to h. Hence the sum of *all parts h* is

$$h[p(n - h) + p(n - 2h) + \cdots + p(n - [n/h]h)] = h \sum_{i=1}^{[n/h]} p(n - ih).$$

Since there are parts 1, 2, $\cdots$, n we find that the sum of all parts of all sizes is then

$$n \cdot p(n) = \sum_{h=1}^{n} h \sum_{i=1}^{[n/h]} p(n - ih).$$

If we divide both sides of this equation by n we obtain

$$(1) \qquad p(n) = (1/n) \sum_{h=1}^{n} h \sum_{i=1}^{[n/h]} p(n - ih).$$

This formula (1) gives us a recursion formula for $p(n)$ and solves the problem as we phrased it. However for the purposes of greater symmetry we proceed to write (1) in an alternative way. To do this we ask for the coefficient of $p(0)$ on the right side of (1), then for the coefficient of $p(1)$, etc. Now $p(0)$ will occur with coefficient h whenever $n - ih = 0$, $1 \leqq i \leqq [n/h]$, i.e., when h is a divisor of n. Hence we will have on collecting all $p(0)$'s on the right of (1)

$$p(0)\Sigma h = p(0)\sigma(n) \quad , h \text{ a divisor of } n,$$

where we use the symbol $\sigma(n)$ to denote the sum of all divisors of n. E.g.,

$$\sigma(6) = 1 + 2 + 3 + 6 = 12.$$

Now $p(1)$ will occur with coefficient h whenever $n - ih = 1$, $1 \leqq i \leqq [n/h]$, i.e., when $n - 1 = ih$ or in other words when h is a divisor of $n - 1$. Hence we will have on collecting all $p(1)$'s together the coefficient $\sigma(n - 1)$. Similarly $p(k)$ occurs with coefficient h whenever $n - ih = k$ or when $ih = n - k$; hence when h divides $n - k$. Thus the coefficient of $p(k)$ is $\sigma(n - k)$. Now adding all up we get the final result

$$(2) \qquad p(n) = (1/n) \sum_{h=1}^{n} h \sum_{i=1}^{[n/h]} p(n - ih)$$

$$= (1/n) \sum_{i=0}^{n-1} p(i)\sigma(n - i).$$

As an example we compute $p(5)$ using this formula.

$$p(5) = (1/5) \sum_{i=0}^{4} p(i)\sigma(5 - i)$$

$$= (1/5)\,[p(0)\sigma(5) + p(1)\sigma(4) + p(2)\sigma(3) + p(3)\sigma(2) + p(4)\sigma(1)]$$

$$= (1/5)\,[1\cdot 6 + 1\cdot 7 + 2\cdot 4 + 3\cdot 3 + 5\cdot 1] = (1/5)(35) = 7.$$

These 7 partitions are

1+1+1+1+1, 1+1+1 +2, 1+1+3, 1+2+2, 1+4, 2+3, 5.

One will have no difficulty in convincing himself that (2) is not very useful as a computational device.

One might feel that the recursion could be removed in (2) by replacing $p(n-1)$ by its value in terms of $p(n-2), \cdots, p(0)$ and then replacing $p(n-2)$ in the same way by its value in terms of $p(n-3), \cdots, p(0)$, etc. Carrying out this process gives the result

$$p(n) = (1/n)[\sigma(n) + A_1\sigma(n-1) + A_2\sigma(n-2) + \cdots + A_{n-1}\sigma(1)]$$

where the A's are given by

$$A_i = [1/(n-i)][\sigma(i) + A_1\sigma(i-1) + A_2\sigma(i-2 + \cdots + A_{i-2}\sigma(2) + A_{i-1}\sigma(1)].$$

Thus we see that such a process removes the recursion so far as the $p(i)$ on the right are concerned but gives us another recursion in terms of the A_i which is equally bad.

For further reading we recommend:

1. *Encyclopedia Brittanica*

2. *Notes on Summer Conference on Collegiate Mathematics, 1954*, by Hans Rademacher; obtainable from University of Oregon, Mathematics Department.

3. G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers*, Oxford University Press.

For other references consult Hardy and Wright.

"Arithmetical symbols are written diagrams and geometrical figures are graphic formulas."

—D. Hilbert

A Game Of Solitaire With Checkers

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Have you heard about the "drummer", who, about 35 years ago, traveled from town to town taking orders for the firm he represented? In addition to being a salesman, he served as an entertainer for the townspeople. He had a repertoire of tricks which he used to awe them and gain their respect. One game he introduced was solitaire with checkers.

The checkers were arranged on the board as shown in the diagram below. No moves were to be made, and the object was to clear the board of all except one checker by jumping the other checkers off the board. Jumps could be made along any diagonal as in a regular game of checkers and could be made either forward or backward. The individual won if after a sequence of jumps only one checker remained on the board.

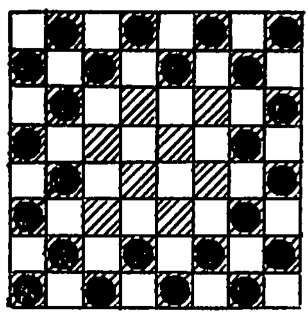

FIGURE I

Have you ever tried it? Why not get out your checkers and try your luck before going further?

After playing this game you are probably wondering if your lack of success was due to failure to make certain proper jumps. The number of sequences of jumps possible is so large that it would not be practical to determine empirically whether or not a solution is possible. It is the purpose of this paper to show a method by which we can determine whether or not it is possible to win.

Suppose the squares of the checker board are colored in the manner indicated in the following diagram where + is a symbol for red, − for green, and × for yellow.

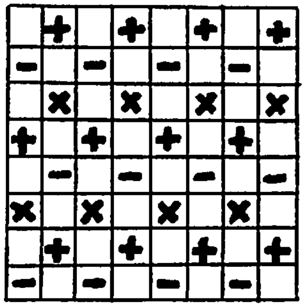

FIGURE 2

It may be observed that when a jump is made onto a square of any color, the number of checkers on squares of that color is increased by one, while the number of checkers on squares of each of the other two colors is decreased by one.

Let

R = the number of checkers on red squares at the beginning of the game.

G = the number of checkers on green squares at the beginning of the game.

Y = the number of checkers on yellow squares at the beginning of the game.

r = the number of jumps onto red squares.

g = the number of jumps onto green squares.

y = the number of jumps onto yellow squares.

R′ = the number of checkers on red squares after a sequence of r + g + y jumps.

G′ = the number of checkers on green squares after a sequence of r + g + y jumps.

Y′ = the number of checkers on yellow squares after a sequence of r + g + y jumps.

Since a jump onto a square of any color increases the number of checkers on that color by one and decreases the number of checkers on each of the other two colors by one, we can write the relations:

$$(1) \qquad \begin{aligned} R + r - g - y &= R' \\ G + g - r - y &= G' \\ Y + y - r - g &= Y'. \end{aligned}$$

By adding the above relations in pairs, we obtain

$$(2) \qquad \begin{aligned} (R + G) - 2y &= R' + G' \\ (G + Y) - 2r &= G' + Y' \\ (R + Y) - 2g &= R' + Y'. \end{aligned}$$

From the diagram it may be noted that at the start of the game $R = 10$, $G = 10$, and $Y = 4$, and hence, $R + G$, $G + Y$, $R + Y$, being the sum of even numbers are even. Also, $2r$, $2g$, $2y$, are even. Therefore $R' + G'$, $G' + Y'$, $R' + Y'$, being the difference of even numbers, must also be even.

The three ways of successfully completing this game of solitaire may be indicated as follows:

	G'	R'	Y'
Case 1	1	0	0
Case 2	0	0	1
Case 3	0	1	0

In Case 1, $R' + G'$ is not even; in Case 2, $G' + Y'$ is not even; in Case 3, $R' + Y'$ is not even. Therefore the impossibility of winning this game of solitaire has been demonstrated.

This paper has given an illustration of the fact that mathematics coupled with ingenuity can be used to solve rather easily problems which otherwise might prove more difficult.

"Mathematics is the science which draws necessary conclusions."

—B. Pierce

Length, Width, Height, and Then What?[1]

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Student, Central Missouri State College

What is the fourth dimension? Do we ask a physicist to define electricity? The mathematician cannot find a model of the fourth dimension which means anything to the physiological sense for we are strictly three-dimensional beings, at least in sense perception.[2]

It is impossible to form a mental picture of the fourth dimension. Nevertheless it is not an absurdity, but a useful mathematical concept with a well-developed geometry. In the realm of science the physicist says that it takes four or five dimensions, or perhaps even six to describe the atom. We have long heard of time as a fourth dimension in relativity.

The idea of geometries of n dimensions began to suggest itself to mathematicians about the middle of the nineteenth century. Gayley, Grassmann, Riemann, Clifford, and others introduced it into their mathematical investigations. Other mathematicians took it up in different ways as time passed. In the first volume of the *American Journal of Mathematics*[3] there is an article which shows that a sphere may be turned inside out in space of four dimensions without tearing, and in the third volume of the same journal a full description of the regular figures in space of four dimensions was given. Others have written on the theory of rotations and on the intersections and projections of different figures. Veronese has an extensive work on geometry of n dimensions with theorems and proofs similar to the proofs we use in school. In the last few years there have been many varied and interesting articles in popular magazines, and some books have been published to explain more particularly what the fourth dimension is.[4]

We do not speak of time as the fourth dimension for, mathematically speaking, time involves the unit of imaginaries, $\sqrt{-1}$, and is indeterminable.[5] We are concerned with the fourth dimension exactly like length, width, and height, standing at right angles to all

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[1] A paper presented at a joint meeting of Missouri and Kansas chapters of Kappa Mu Epsilon at Warrensburg, Missouri, Spring, 1954.
[2] Ingalls, A. G., "Hypergeometry and Hyperperplexity", **Scientific American**, 161:131, September, 1939.
[3] **American Journal of Mathematics**, (Johns Hopkins University; American Mathematical Society, 1878), Baltimore, Johns Hopkins Press.
[4] Manning, Henry P., **The Fourth Dimension Simply Explained**, Munn and Company, Inc., New York (1919), pp. 14-15.
[5] Farley, Ralph, "Visualizing Hyperspace", **Scientific American**, 160:148-149, March, 1939.

these. We must suppose a direction in which we can never point extending from every point of our space.

To gain a partial and symbolic idea of the meaning of a fourth dimension, we resort to analogy in four different processes. These processes are (1) comparison with lower dimensions, (2) bounding lines, (3) algebraic powers, and (4) symmetrical forms.

In comparison with lower dimensions we first consider a space of only one dimension. A person in such a space would be limited to a straight line, which he would conceive as extending infinitely in both directions. If he encountered another being, neither could pass the other because his only possible movement would be along this line. If he is really within a space like ours, although his perception is confined to one direction only, and if someone in our space should lift one of the two beings, and place him on the other side of the first, he would lose sight of the first as soon as the movement took place and it would be beyond his comprehension how the movement was effected.

Now think of a space of two dimensions, like a shadow. Beings in such a space could move around one another, but one of them completely surrounded by others would be imprisoned by them. If, as before, the two-dimensional space is within our space, and really depends on the limitation of the perceptive faculties of the beings in question, the imprisoned being could be lifted by a person in our space, and set down outside of the beings surrounding him. They would lose sight of him during this movement, and not understand how the change of position had been effected.[6]

From these suppositions of one-dimensional space and two-dimensional space, the inference is drawn that there may be a fourth dimension in our supposedly three-dimensional space, and that our lack of understanding of it arises only from the limitation of our perceptive faculties. We are told that there are light rays which are invisible to us, solely because our eyes are so constructed as to be unable to perceive them. Also we are told that there are tones so low or so high that we can never hear them because our ears are not attuned to them. Hence, it may be that a fourth dimension does exist but the limitations of some of our senses might operate to render us unable to perceive it.[7]

The straight line segment of one-dimensional magnitude, ends

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[6] Kaempffort, Waldemar, "What is the Fourth Dimension?" McClure, 42:222,225,226, November, 1913.
[7] Nikonow, J. P., "Is Space Curved?" Scientific American, 147:278-279, November, 1932.

in points; the square, a two-dimensional magnitude, is bounded by straight line segments which are one-dimensional; the cube, a three-dimensional object, is bounded by squares, which are two-dimensional. It is inferred by analogy that three-dimensional magnitudes bound four-dimensional magnitudes. The "four-dimensional cube" is named the "tessaract" and is said to be bounded by cubes.[8]

In the series of the successive powers of a number x, x^2, x^3, $x^4 \cdots x^n$, x may be represented graphically by a straight line, of which x is the length; x^2, by a square, with x as the length of a side; x^3, by a cube, with x as the length of an edge. It is inferred that if we keep on, there must be a configuration corresponding to x^4, and so on indefinitely up to x^n.

Students of geometry might picture this idea in a different way. An equation containing two "variables" may be considered as representing the locus of a series of points in a plane, so an equation of three variables is the locus of points in three-dimensional space. But since, as shown before, in explaining the word "dimensions" the coordinates fix definitely and exactly the position of a point, equations with more than three variables exceed the scope of our perceptual geometry, and require the use of analogies for their interpretation.[9]

In the use of symmetrical forms to explain the fourth dimension we connect the vertex of an isosceles triangle with the middle point of the base and divide the triangle into two equal triangles. If we were confined to two-dimensional space we could never prove them equal by superposition. Since we are in three-dimensional space we are not under this restriction and we turn one of the triangles a half revolution on one side, and then the two figures may be made to coincide. There are many symmetrical solids; for instance, the two hands, which can never be brought into identical shape. We cannot prove the left hand equal to the right by putting on the right-hand glove. If we turn the right-hand glove inside out it will fit the left hand. With two-dimensional figures we prove them equal by use of three-dimensional space. From this it is inferred that in four-dimensional space, not only the glove, but the hand within it, might be turned inside out and made to coincide with the other hand.[10] The right and left rotation is characteristically four-dimensional. Something similar to it occurs in nature. A beam of polarized light is ro-

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[8] Kaempffert, op. cit., p. 229.
[9] Hinton, C. Howard, **The Fourth Dimension**, Swan Sonnenschein and Company, London (1906), pp. 67-72.
[10] Kaempffert, op. cit., p. 226.

tated either to the right or the left on passing through certain substances. Dextrose and levulose (sugars) owe their names to the fact that one rotates a polarized beam to the right (dextra—"right hand"), the other to the left (laeva—"left hand"). In chemical constitution they are exactly the same. It is suggested that their contrasted properties are due to right and left reversal of their atoms, a four-dimensional movement in the minute particles of which they are built up.[11]

Certain snails, exactly alike in all other characteristics, have one difference; some are coiled to the right, others to the left. It is remarkable that their juices have a corresponding property of rotating a polarized beam to right or left. This suggests that their external form is an expression of an internal difference, a right or left twist of their atoms by a four dimensional force.[12]

There are other applications that would be possible in a four-dimensional space. For example, knots would only be loops or coils and could be untied by carrying one loop out of our space and bringing it back in a different place. Links of a chain would fall apart. The contents of a bottle could be taken out without removing a cork. A four-dimensional physician could see and touch the inner parts of the body without breaking the skin. A four-dimensional man could leave or enter a closed room without disturbing the walls. He could see any part of the interior of a solid no matter how dense or opaque to us. Our safes would lie open to him. He could take a diamond necklace from a locked box without opening it. This is four-dimensional robbery and would be easy to a four-dimensional bank robber.[13]

Geometry of four dimensions is of importance not only to the mathematician, but it is also of interest in certain other lines of study. It involves questions of space which concern the philosopher; efforts to understand it make use of space perceptions and so attract the attention of psychologists; and attempts to use the theories of higher space in the explanation of physical and other phenomena make the subject of interest to those working in other branches of science. However, no one can consider himself completely equipped as a mathematician without some knowledge of the geometries of higher dimensions.

This is an explanation of the term "fourth dimension". The

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[11] Ibid., p. 230.
[12] Ibid., p. 230.
[13] Manning, op. cit., pp. 188-189.

persistent reader will perhaps repeat the question, "Is there a fourth dimension?" If by this question he means, does a four-dimensional world exist physically, all we can say is that it is highly improbable. As a mental concept, the fourth dimension exists, but the world of our physical experience includes only the three dimensions.

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"I have found a great number of exceedingly beautiful theorems."

—FERMAT

The Problem Corner

EDITED BY FRANK C. GENTRY

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tool beyond calculus. Although new problems are preferred, old problems of particular interest are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets of paper before October 1, 1955. The best solutions submitted by students will be published in the Fall, 1955, number of THE PENTAGON with credit being given for other solutions received. To obtain credit a solver should affirm that he is a student and give the name of his school. Address all communications to Frank C. Gentry, Department of Mathematics, University of New Mexico, Albuquerque, New Mexico.

PROBLEMS PROPOSED

72. *Proposed by Frank Hawthorne, Hofstra College, Hempstead, New York.*

(Note: An error occurred in the wording of this proposal in the Fall, 1954, number. It should have read as follows:)

A rectangular piece of sheet metal with integral dimensions a and b has equal squares of side x cut from each corner. The sides are then bent up to form a rectangular box with no top. The value of x is chosen so as to make the volume of the box a maximum. If x is rational and if a triangle is formed with sides a and b and the angle C between them is 60°, show that the side c of this triangle is integral.

77. *Proposed by Paul W. Healy, University of New Mexico, Albuquerque, New Mexico.*

Find all numbers less than 10,000 such that each may be divided by 2 by prefixing the 2 and erasing the last digit. What is the form of all such numbers?

78. *Proposed by Victor L. Osgood, Oceanport, New Jersey.*

From the Pythagorean Theorem $2^2 + 3^2 = 13$. Find 3 more pairs of rational numbers, the sum of whose squares is 13.

79. *Proposed by Frank Hawthorne, Hofstra College, Hempstead, New York.*

(Adapted from a problem in the *American Mathematical Monthly.*)

An Englishman had walked one-third of the distance across a railroad bridge 7920 feet long when he heard a train coming behind him at 45 miles per hour. He could just escape by running at uniform speed to either end of the bridge. What was his name?

80. *Proposed by the Editor.* (From Rietz and Crathorne, *College Algebra*, Third Edition)

An aeroplane flying at 75 miles per hour and following a long straight road, passed an automobile going in the opposite direction. One hour later it overtook a second automobile. The automobiles passed each other when the aeroplane was 100 miles away. If both automobiles travel at the same speed, how far apart were they when the aeroplane passed the second one and what was their speed?

81. *Proposed by the Editor.* (From the First William Lowell Putnam Mathematical Competition, April, 1938)

A can buoy is to be made of three pieces, namely, a cylinder and two equal cones, the altitude of each cone being equal to the altitude of the cylinder. For a given area of surface, what shape will have the greatest volume?

SOLUTIONS

71. *Proposed by Charles Pearsall, Student, Hofstra College, Hempstead, New York.*

Given a triangle partitioned into n equal areas by lines parallel to the base. If the altitude is h, show that: (1) the partitioning is independent of the length a of the base; (2) the location of the lines of division as measured along the altitude from the vertex is given by the sequence:

$$h\sqrt{1/n},\ h\sqrt{2/n},\ h\sqrt{3/n},\ \cdots,\ h\sqrt{j/n},\ \cdots,\ h,$$
$$(j = 1, 2, \cdots, n).$$

Solution by the Proposer.

Take the vertex of the triangle as the origin of a rectangular system of axes, the altitude along the positive x-axis so that the equations of the sides will be $y = m_1x$ and $y = m_2x$ and the equation of the base $x = h$. Number the partitioned areas from the vertex from 1 to n. Then the area of the j^{th} area

$$A_j = \int_{x_{j-1}}^{x_j} (m_1 - m_2)x\,dx = [(m_1 - m_2)/2][x_j^2 - (x_{j-1})^2]$$
$$(j = 1, 2, 3, \cdots, n);\ x_0 = 0.$$

Equating these areas leads to $x_1^2 = x_2^2 - x_1^2 = x_3^2 - x_2^2 = \cdots = x_j^2 - (x_{j-1})^2 = (x_{j+1})^2 - x_j^2 = \cdots = h^2 - (x_{n-1})^2$. From this follows the recursion formula $(x_{j+1})^2 = 2x_j^2 - (x_{j-1})^2$. Hence $x_j^2 = jx_1^2$ and $h^2 = n\ x_1^2$. Consequently $x_1 = h/\sqrt{n}$ and

$x_j = h\sqrt{j/n}$. Since the base a was not used it is evident that the partitioning is independent of it.

Also solved by S. H. Sesskin, Hofstra College; Robert Anderson, Wayne University.

73. No general solution received. S. H. Sesskin of Hofstra College discussed a number of special cases.

74. No solution received.

75. *Proposed by Carl V. Fronabarger, Southwest Missouri State College, Springfield, Missouri.* (Source unknown).

It is known that eleven of twelve objects have the same weight; show how it can be determined by three weighings on a pair of balances which object has a weight differing from that of the others and whether it is heavier or lighter.

Solution by S. H. Sesskin, Hofstra College, Hempstead, New York.

Divide the 12 objects into 4 groups of 4, 4, 3, and 1 objects respectively and let these numbers designate the particular groups. Let x be the object sought and let R be any object established to not be x. Weigh the two 4's. (A) They balance, hence they are R's and x is in 3 or 1. (a) Weigh 3 against 3 R's. (1) They balance and x is 1 and a third weighing of x against an R will determine whether it is an L (lighter) or an H (heavier). (2) They do not balance, x is in 3 and is an L or an H according as 3 is lighter or heavier than the 3 R's. Weigh any two objects of 3 against each other. If they balance, the unweighed object is x, if they do not balance then x is the L or H one as established above. (B) The two 4's do not balance. Let objects in heavy group be H and others L. (a) Weigh 2 H and 1 L against 2 H and 1 R (from the 3 or 1). (1) If they balance, x is one of the remaining L's and can be determined as above. (2) If 2 H and 1 L is heavier, then x is one of the two H's and can be determined in the third weighing. (3) If 2 H and 1 L is lighter, then either x is the L or x is one of the two H's in 2 H and 1 R. Weigh these two H's. If they balance x is the L and is lighter, if they do not then x is the heavier.

Also solved by Harvey Fiala, State School of Science, Wahpeton, North Dakota; Robert Anderson, Wayne University. Sesskin pointed out that this problem, with another solution, appears in Kraitchik's *Mathematical Recreations.*

76. *Proposed by S. H. Sesskin, Student, Hofstra College, Hempstead, New York.*

Show that the square of the sum $1 + 2 + 3 + \cdots + n$, for any n, diminished by the square of the same sum for $n - 1$ is equal to n^3.

Solution by Harvey Fiala, North Dakota State School of Sciences and Robert Anderson, Wayne University. (Same solution)

Since the series are arithmetic, we have

$$[(n[n+1])/2]^2 - [([n-1]n)/2]^2 = n^2/4\,[(n+1)^2 - (n-1)^2] = (n^2/4)(4n) = n^3.$$

Also solved by Glen E. Swain, Central Missouri State College, Warrensburg, Missouri.

Editorial Note: Corrections listed below should be made to the Fall 1954 issue of THE PENTAGON.

Page 38, line 20, the equation should be

$$(D^2 - 2D + 4)y = 12x^3 - 2x^2 + 2x + 44$$

Page 38, line 24, should be

$$y_p = 3x^3 + 4x^2 + 9$$

The Mathematical Scrapbook

EDITED BY H. D. LARSEN

Mathematics is queen of the sciences and arithmetic the queen of mathematics.

—GAUSS

=△=

Carl Friedrich Gauss

The year 1955 marks the centenary of the death of Carl Friedrich Gauss, one of the greatest mathematicians the world has known. In commemoration of this event, Germany has issued a large-sized postage stamp bearing a portrait of Gauss.

"Wonderful was his richness of ideas; one thought followed another so quickly that he had hardly time to write down even the most meagre outline. At the age of twenty Gauss had overturned old theories and old methods in all branches of mathematics; but little pains did he take to publish his results, and thereby to establish his priority. He was the first to observe rigour in the treatment of infinite series, the first to fully recognize and emphasize the importance, and to make systematic use of determinants and of imaginaries, the first to arrive at the method of least squares, the first to observe the double periodicity of elliptic functions."

—F. CAJORI

"When scarcely three years old Gauss, according to an anecdote told by himself, followed mentally a calculation of his father's relative in regard to the wages of some workmen, who were to be paid for overtime in proportion to their regular wages, and, detecting a mistake in the amount, he called out, 'Father, the reckoning is wrong, it makes so much,' naming the exact amount. The calculations were repeated and it turned out that the child was correct, while all who witnessed the performance were greatly surprised. He retained an extraordinary ability for mental calculations throughout life and remembered the first few decimals of the logarithms of all numbers, so that he was able to use the data of a logarithmic table in his mental calculations, and hence he possessed a mental slide rule—a unique possession.

"Gauss was not only one of the greatest mental calculators on record, but he excelled equally in all branches of pure and applied mathematics. At the age of twenty he discovered the first rigorous proof of the fundamental theorem of algebra, which affirms that every algebraic equation has as many roots as its degree, and at the age of twenty-four he published his great work on the theory of numbers under the title *Disquisitiones Arithmeticae.*"

— G. A. MILLER

=△=

Peter the Great, of Russia (went so far as to prohibit any nobleman from marrying until he had passed an examination in geometry, arithmetic and navigation.

—TOYNBEE

=△=

Epitaph

Interred here are:

2 Grandmothers with their 2 granddaughters,
2 Husbands with their 2 wives,
2 Fathers with their 2 daughters,
2 Mothers with their 2 sons,
2 Maids with their 2 mothers,
2 Sisters with their 2 brothers.

How many are buried here?

=△=

Are you familiar with the lives of the great mathematicians? Then answer the following.

1. Who was the greatest Irish mathematician?
2. Who created projective geometry in a Russian prison?
3. Who advocated lying in bed until a late hour every morning?
4. Who created non-Euclidean geometry?
5. Who contributed much to mathematics, yet considered it merely as recreational?
6. Who was the Prince of Mathematicians?
7. Who did Napoleon designate as "the lofty pyramid of mathematical sciences?"
8. Who, though blind, continued his mathematical work?
9. Who was the most prolific mathematical writer in history?
10. Who insisted on the choice of "10" as the base for the metric system?

ANSWERS: Hamilton, Poncelet, Descartes, Lobachevski, Newton, Gauss, Lagrange, Euler, Euler, Lagrange.

=△=

"Infinite series were used by many mathematicians before they had any clear idea of what we call convergence and divergence."

—E. C. Titchmarsh

=△=

"It was Sir William Thomson (before he became Lord Kelvin) who first introduced the practice of writing a large or small number in the form (a number lying between 1 and 10) x (a power of 10)."

—R. F. Muirhead

=△=

Show that the curve $y = mx^2 - (1 + 8m)x + 4(4m + 1)$ passes through a fixed point A and determine its coordinates.

=△=

Circles A, B, and C of radii a, b, c are mutually tangent. Let D be a larger circumscribed circle tangent to each of the three smaller circles. Show that its radius is

$$\frac{abc}{2\sqrt{abc(a + b + c)} - (ab + bc + ac)}$$

=△=

A merchant marked his merchandise in code, replacing each digit by a preassigned letter. His new bookkeeper made out the following bill using the code. What were the correct figures in the bill?

e gimcracks at ib¢ ------------	$h.if
c baubles at ha¢ -------------	.dh
g GeeGaus at j¢ -------------	.bg
Total ------------	$i.ed

—Am. Math. Month.

=△=

There are five men in a poker game—Brown, Jones, Perkins, Turner, and Riley. Each smokes a different brand of cigarettes—Chesterfields, Old Golds, Camels, Luckies, and Raleighs—not necessarily respectively. They had at the start of the game, 3, 6, 8, 15, and 30 cigarettes apiece, again not necessarily respectively. At a certain moment during the game, when each man had just finished a cigarette—

a) Riley, who has smoked one-half of his cigarettes, has smoked one less than Turner.

b) The man who smokes Chesterfields had at the start twice as many cigarettes as he now has, plus one-half as many as he now has, plus 2½ cigarettes.

c) The Camel smoker asks Brown to pass him a match.

d) Each man has at least 2 cigarettes left.

e) One man absent-mindedly lights the tipped end of his fifth cigarette.

f) The Lucky smoker has smoked more than Perkins. In fact, the Lucky Smoker has smoked two more than the next heavier smoker.

g) To the number of cigarettes Turner had originally, add the number Riley smoked and the number of Chesterfields left, then subtract one more than the number of Luckies smoked. The result is the number of cigarettes Brown had originally.

What brand did each man smoke? How many cigarettes did each man have at the start? How many does each man have left?

—SCRIPTA MATHEMATICA

=△=

"Walking the other day to take the air,
(Bright shone the sun, the weather very fair.)
At distance I a dismal cloud did spy,
Which (as I thought) against the wind did fly.
While I upon my watch did look to see,
How time did pass away; Lo! instantly,
A dreadful flash of lightning pierc'd the cloud;
Just fourteen seconds after which aloud
The thunder roar'd: now I inform'd would be
How many feet the cloud did burst from me?"

=△=

"The majority of ideas we deal with were conceived by others, often centuries ago. In a great measure, it is really the intelligence of other people that confronts us in science."

—D. MACH

The Book Shelf

EDITED BY FRANK HAWTHORNE

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Frank Hawthorne, Hofstra College, Hempstead, New York.

Mathematics and Plausible Reasoning, G. Polya; Volume I: *Induction and Analogy in Mathematics;* Volume II: *Patterns of Plausible Inference,* Princeton University Press (Princeton, New Jersey), 1954, Volume I, 16 + 280 pp., $5.50; Volume II, 10 + 190 pp., $4.50; the set, $9.00.

In these two volumes, Professor Polya continues the development of a theme with which he has been concerned for many years and to which he has devoted a number of articles and two other books. Readers of THE PENTAGON will undoubtedly be familiar with Polya's *How to Solve It* which appeared several years ago. The *Aufgaben and Lehrsatze ans der Analysis* written jointly with G. Szego and one of the reviewer's favorite mathematical books, was an earlier excursion, albeit in a more advanced and more restricted field, into the territory with which the volumes at present under review are concerned; namely, the art of mathematical discovery and proof.

To understand Polya's purpose fully, let us recall the distinction between the science of mathematics and the art of practicing it. Great discoveries of our age are that mathematics is the collection of mathematical sciences and that a mathematical science is an abstract postulational system in which one begins with a system of postulates and, by the use of a logic, deduces the consequences of these postulates. Euclid's *Elements* is the earliest, though a somewhat imperfect, example of such a postulational system. However, it took the discovery two millenia later by Bolyai and Lobachevsky of the possibility of non-Euclidean geometries to shock mathematicians into the beginning of an understanding of the true nature of their science. The next century saw the understanding develop more fully and was highlighted by the publication of Hilbert's *Foundations of Geometry.* It is perhaps difficult for us today to appreciate fully the impact of Hilbert's book and the novelty to mathematicians at that time of what is today a commonplace of mathematical thinking.

The witty wife of a college dean once described to the reviewer her recollection of a college course in mathematics as being one in which the instructor said: "So and so and so and so and so. See?" This is indeed the structure of a mathematical demonstration. To prove that the postulates, which in conjunction form a sentence which we may represent by P, imply a consequence Q, one forms a sequence of propositions $P_1, P_2, \cdots, P_n$ such that P implies P_1, P_1 implies $P_2, \cdots, P_n$ implies Q. The art of constructing proofs in this manner is one of the two elements of the art of mathematics to which Polya gives enthusiastic and inspired attention. The second element is mathematical discovery: the art of conjecturing propositions Q which, as one hopes to demonsrate, are implied by P.

The art of mathematical discovery employs a type of reasoning different from that used in formal deductive reasoning. Polya calls this plausible reasoning. In Volume I, inductive and analogical reasoning are examined and profusely illustrated. The roles of generalization, specialization, and analogy in plausible reasoning are considered and illustrated by a wealth of examples chosen from geometry, the theory of numbers, and what Polya calls "physical mathematics". The reader who follows the discussion attentively and sharpens his wits on the examples at the conclusion of each chapter (solutions are given in an appendix) will have added considerably to his capacity for practicing the art of mathematics.

Volume II is concerned with the explicit formulation of various patterns of plausible reasoning suggested by the examples of the first volume. An analysis of the relation between the calculus of probability and the logic of plausible reasoning leads Polya to the conclusion that these patterns "are general points that, according to the usage of good scientists, are admissible in a scientific discussion, with a view to reasonably influencing the credibility of the conjecture discussed." The book ends with this exhortation: "The result of the mathematician's creative work is demonstrative reasoning, a proof, but the proof is discovered by plausible reasoning, by guessing • • •. I address myself to teachers of mathematics of all grades and say: Let us teach guessing!"

In the opinion of this reviewer, neither teacher nor student of mathematics can do better in perfecting his grasp of the art of guessing than to read carefully Polya's work on *Mathematics and Plausible Reasoning.*

—MAX CORAL

Relaxation Methods, D. N. de G. Allen, McGraw-Hill Book Company, Inc. (330 West 42nd St.) New York, 1954, 246 pp., $7.50.

This book is a lucid exposition of numerical techniques, recently developed and in common use with practicing engineers and mathematicians both here and in Great Britain. While no proofs are given for the convergence or even the derivation of the useful iterative methods contained, the book fills a definite and long felt need of engineers and technical students by compiling and organizing the basic techniques of the "relaxation method", the means of preparation of mathematical problems for the execution of the methods, and finally the application of the procedure to an entire series of old and new problems.

The method of relaxation derived its unique name from a literal description of the physical process involved. In its original form, as applied to the deformation of redundant structures under load, the method consisted of rigidly fixing each unknown in an arbitrarily constrained fashion, while momentarily "relaxing" one variable and adjusting this value to satisfy the local conditions imposed by the equation under study. In this manner, by successively permitting each variable, or unknown, to undergo slight adjustments, keeping all others fixed, an overall solution was eventually arrived at. This crude method, first proposed by R. V. Southwell and his associates proved to be so successful that it became universally adopted in the field of numerical analysis. This trend of widespread acceptance has continued in spite of the fact that little or no work has been done to place the theory on a firm and rigorous foundation.

Starting with an elementary application to the solution of a pair of simultaneous algebraic equations in two unknowns, the author proceeds to develop the full usefulness of the method by considering larger systems of equations, ordinary and partial differential equations, characteristic vector problems, etc. Several of the common pitfalls are explored and illustrated. Methods are presented to increase the rate of convergence of a solution, how to overcome a "divergent" process resulting from the iteration technique, and finally methods for increasing accuracy. Since the method can only be applied to a system of simultaneous linear equations, every problem must first be translated into this form before a solution is obtainable. Consequently a good portion of the book deals with setting up problems so that the method can be applied. For example, a differential equation is first replaced by an equivalent difference equation (equivalent in the sense that the solution obtained at a finite number of points approaches the true solution in the limit as the number of points is increased without limit). The difference equation is now a system of algebraic equations, with the boundary conditions acting as the nonhomogeneous vector and the unknowns being the values of the solution at the given finite number of points. While no mention is made of the application to the field of integral equations, the extension is readily made.

The book is well recommended for correlative reading in courses in engineering, applied mathematics, and numerical procedures. The book reads easily and should prove interesting and even stimulating to students in linear algebra, differential equations, and physics.

—SAMUEL PINES

Introduction to Mathematical Thinking, Friedrich Waismann, Fredrick Unger Publishing Company (105 East 24th Street) New York 10, 245 pp. $4.50.

The majority of text books used today by the students of undergraduate mathematics are designed primarliy to give them techniques of operations and efficiency in these techniques. The uninterested and undiscerning student is satisfied with a mere proficiency in manipulations and rarely does he question the basis or validity of the operations he executes. To the discerning student many questions arise regarding the foundations on which the science rests. Unfortunately, often he too will "proceed with faith", hoping "understanding will follow". The purpose of this book is to answer questions which arise such as, the principle of complete induction, the necessity of the Bolzano-Weierstrass theorem, and many others. As Karl Menger remarks in the preface, the reader "will gain a fundamental insight into the methods of dealing with some very basic questions, above all such that are of interest to the philosopher."

The method used by the author in treating material is descriptive. There are very few proofs contained in the work and these are given descriptively and not formally as in Wilder's *Foundations of Mathematics* or Weyl's *Philosophy of Mathematics and Natural Science*. Hence the book will not discourage those who are dealing with these "foundations" for the first time. In general, the order of contents follows that of most classical works on the calculus, approaching the calculus from the concept of number.

The contents are well integrated. To the reviewer this is remarkably evident in the chapter on "remarkable curves". This chapter would seem to be of special interest to the undergraduate. Unfortunately the appendix to this chapter, "What is Geometry?", is far too short. The chapter on the various schools of mathematical philosophy, although excellently organized, is also too short. At first reading some of the ideas of the first chapters seem to be out of their logical place, but considering that the author is writing primarily for a "philosophically minded observer" and not a trained mathematician, they fit in well with the overall purpose of the book.

According to the experience of the reviewer this book would be interesting and stimulating to an undergraduate, especially one who has completed a basic course in the calculus. It could also very well serve as a basic text book for a mathematics course in the liberal arts program. There is much material in this relatively short book and further development of some of the ideas mentioned therein could be instructive topics for seminars.

—DAMIAN CONNELLY

BOOKS RECEIVED BY THE BOOK SHELF EDITOR

The Elements of Probability Theory. Harald Cramer, John Wiley and Sons, Inc., (440 Fourth Avenue) New York, 281 pp., $7.00.

An Introduction to Deductive Logic, Hugues Leblanc, John Wiley and Sons, Inc., (440 Fourth Avenue) New York, 9 + 574 pp., $4.75.

Mathematics for Technical and Vocational Schools, Samuel Slade and Louis Margolis, John Wiley and Sons, Inc., (440 Fourth Avenue) New York, 9 + 574 pp., $4.48.

"When I use a word, it means just what I choose it to mean— neither more, nor less."

—L. CARROLL (C. DODGSON)

Installation of New Chapter

EDITED BY J. M. SACHS

THE PENTAGON is pleased to report the installation of Virginia Alpha Chapter of Kappa Mu Epsilon.

VIRGINIA ALPHA CHAPTER

Virginia State College, Petersburg, Virginia

Virginia Alpha Chapter of Kappa Mu Epsilon was installed and nineteen charter members were initiated at Virginia State College on January 29, 1955. Professor E. Marie Hove of Hofstra College, National Secretary of Kappa Mu Epsilon, served as the installing officer assisted by Professor Louise S. Hunter and Professor R. D. McDaniel of Virginia State College.

A banquet followed the installation with guests representing the faculties of the schools and divisions of the college, the Walter Johnson Mathematics Club, and the other honor societies of the college. Mr. Benjamin Williams, Chapter President, acted as toastmaster. Speeches of welcome were made by representatives of the other honor societies and the responses for Kappa Mu Epsilon were made by Miss Lottie J. Griffin, Chapter Vice-President. Professor E. Marie Hove addressed the group on "The History of Kappa Mu Epsilon."

The charter members of Virginia Alpha are Dorothy Batts, Emma D. Breedlove, Gladys M. Brown, College President R. P. Daniel (honorary member), Rheba Galloway, Lottie Griffin, Dorothy J. Harris, Dr. J. M. Hunter, Dr. Louise S. Hunter, Serelda I. James, James H. Johnson, Mary Hill Johnson, H. M. Linnette, B. S. Lowe, Dr. R. R. McDaniel, Dr. A. M. Myster, C. A. Taylor, Marcelle M. Walker, and Benjamin Williams.

The chapter officers are Benjamin Williams, President; Lottie J. Griffin, Vice-President; Emma D. Breedlove, Secretary; Gladys M. Brown, Treasurer; Dr. Louise S. Hunter, Corresponding Secretary, and Dr. R. R. McDaniel, Faculty Sponsor.

We wish to extend a most hearty welcome to Virginia Alpha and to wish them great success in our fellowship.

Kappa Mu Epsilon News

EDITED BY LAURA Z. GREENE, HISTORIAN

Alabama Beta held its annual Homecoming for members in October. The meeting was well attended and they feel it is a most worthwhile and enjoyable undertaking.

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Colorado Alpha holds an annual picnic in the spring. Members of the freshman class who have shown a special interest in mathematics during the year are special guests.

Dr. George Polya, Professor Emeritus of Stanford University was a guest of **Colorado Alpha** recently. He gave a series of lectures to members of the mathematics classes.

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Dr. C. N. Mills retired last August from teaching at Illinois State Normal University after twenty-nine years of service. It was under his direction that the **Illinois Alpha** Chapter was installed in 1933. Dr. Mills served as National Treasurer of Kappa Mu Epsilon, 1933-35. At the present time he is teaching mathematics at Augustana College.

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Illinois Delta presented "Apologia Mathematica" written by Bernadine Arseneau on a radio broadcast from W.J.O.L., Joliet, February 22.

Sister M. Ursuline is now advisor of the **Illinois Delta** Chapter.

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Twelve members were initiated at the annual Kappa Mu Epsilon banquet for the **Indiana Alpha** Chapter. Mr. Lester M. Rouch spoke on "The Use of Mathematics in the Problems of Everyday Life."

Indiana Alpha holds five regular meetings a semester.

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Indiana Beta will sponsor a public lecture by Dr. Frank Edmondson, head of the Astronomy Department of Indiana University, to be held in the new Holcomb Observatory and Planetarium, April 14. This will follow the annual dinner of the chapter.

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Kansas Beta is planning a newsletter to be sent to all alumni. This practice was followed before the war, and is now being revived.

Carol Law, a freshman, received a book, *Standard Mathematical*

Tables, as the Underclassman Award for having the highest scholastic standing.

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Kansas Gamma published a special edition of *The Exponent* which gave detailed information of Kappa Mu Epsilon, its purposes, requirements, and advantages.

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George Jones, a member of **Kansas Delta**, received the 1954 award for the highest scholastic standing in the freshman mathematics classes.

Terry McAdam, corresponding secretary of **Kansas Delta** Chapter, is the author of *Very Much Alive*, a recent publication of Houghton-Mifflin. It is the story of the reactions of several paraplegics in a veteran's hospital where Mr. McAdam spent several months following an accident which paralyzed him permanently.

Margaret E. Martinson, instructor of mathematics at Washburn University, is on a leave of absence for the year 1954-55 and is studying at the University of Wisconsin.

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Michigan Beta again sponsored the Freshman Mathematics Award. The prize, *Mathematics and the Imagination*, was given to Tom Lee.

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Mississippi Gamma reports the following officers for the year 1954-55: President, Charles Young; Vice-president, Dorothy Proctor; Secretary-Treasurer, Shannon Clark; Reporter, Spurgeon Bradley; Faculty Adviser, Professor Harold L. Leone; and Corresponding Secretary, Professor Jack D. Munn.

A custom which helped to maintain interest in the chapter is the requirement that all members give a paper during their senior year. **Mississippi Gamma** Chapter feels that this provides very worthwhile programs.

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Missouri Beta sponsored a Kansas-Missouri meeting of Kappa Mu Epsilon May 1, 1954. The chapters participating were: **Kansas Alpha, Kansas Beta, Kansas Gamma, Kansas Delta, Missouri Alpha, Missouri Beta, Missouri Gamma, Missouri Epsilon.** Ninety members attended. Eight student papers were presented. Professor Tucker gave the address, "Whither Mathematics?", at the luncheon meeting. Both student and faculty members met in discussion groups.

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New Mexico Alpha initiated 30 members at the last meeting in the spring of 1954.

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Professor Morris Kline of New York University will be the speaker at the annual banquet on March 25 of **New York Alpha.**

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The **Wisconsin Alpha** Chapter of Kappa Mu Epsilon held a mathematics contest on April 10, 1954, under the auspices of Mount Mary College. Senior students from high schools in Milwaukee and vicinity were invited to participate and fourteen schools responded, ten of which were represented by a team of three.

The examination consisted of twenty problems involving arithmetic, algebra, and geometry, arranged in five groups of four problems each. Each set was timed according to relative difficulty and each set was checked for correct solutions while the next set was being solved. In this way it was possible to determine the winners very shortly after the last set had been finished.

Riverside and Messmer High School teams, both from Milwaukee, tied in the contest, and after three more bouts Riverside won by one point. The winning team was awarded a plaque with the year and the name of its school engraved on it. It is to be kept by that school until the next contest, which we hope to hold next year.

The six highest individual contestants each received a medal. These were Ronald Shaefer of Messmer with highest score; Ann Scherr of Messmer, second; and Thomas Howell and Charles Franchino, both of Riverside, and John Martin and Karen Gustafson of Immaculate Conception High School, Elmhurst, Illinois, tied for the next four prizes.

All the participants seemed to enjoy the challenge given them in this contest and felt that they had gotten a real inspiration. **Wisconsin Alpha** hopes to make this an annual affair to encourage and stimulate high-school students in their study of mathematics.

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Program Topics

(Spring Semester 1953-54 — Fall 1954-55)

Colorado Alpha, Colorado A and M, Fort Collins
Snow Plow Problem, by Keith Gardels
Maximum and Minimum, by David Wait
Three Problems in Analysis, by Peter T. Work
Transcendental Numbers, by Ted Speiser
Special Lectures, by Dr. George Polya, Leland Stanford University

Illinois Beta, Eastern Illinois State College, Charleston
The Binary System, by Miss Hendrix
The Use of Binary Numbers in Electric Computers, by Mr. Carl Willam
The Placement of Graduates with Training in Mathematics, by Dr. Zeigel
Rectangles, Continued Fractions, and Fibonacci Numbers, by Dr. Ringenberg and Mr. David Brown
Problems were proposed and solutions presented by students, Miss Joan Wyack, Jess Orvedahl, Bob Thrash

Illinois Gamma, Chicago Teachers College, Chicago
Discussion of the problem, *"The Case of the Playful Children"*
Screwballs and Inertia in Mathematics, by Dr. Goldsmith
Codes and Ciphers, by Dr. Sachs

Illinois Delta, College of St. Francis, Joliet
Christopher Clavius, by Mary Ann Hasse
The Discovery of All Dark Things, by Patricia McLaughlin
Magic Squares, by Anne Rademacker
Point Set Theory, by Sister M. Claudia, O.S.A.
History of the Club, by Sister M. Ursuline, O.S.A.
Pentagon Review, by Betty Anselmino
Book Review "The Diary and Sundry Observations of Thomas Edison" by Runes, Sister M. Rita Clare, O.S.A.
Periodical Review from Mathematics Teacher, by Joan Nahas

Indiana Alpha, Manchester College, North Manchester
Recent Solar Eclipse, showing slides which he took in the path of totality at Minneapolis, by George Arnold
Some Mathematical Fallacies, by David Neuhouser
The Teaching of Arithmetic During the Past Century, by Professor Merritt
On the Mathematics Books in our Library—A challenge to our KME members to use these books, by Professor Dotterer

Indiana Beta, Butler University, Indianapolis
Tour of New Holcomb Observatory and Planetarium Astronomy, by Dr. Harry Crull
Mathematics in Medicine, by Robert Crawford
Some Topological Properties of Continuous Functions, by Dr. R. H. Oehmke
Boolean Algebra, by Richard Thompson
Statistical Astronomy, by Dr. Frank Edmondson
Curve Fitting and Empirical Equations, by James Rogers

Iowa Alpha, Iowa State Teachers College, Cedar Falls
The Seven Bridges of Konigsberg, by Diane Sorenson

A Condensation of the Cartesian Coordinate System, by John De Jong
La Courbe du Diable, by Jame Lockwood
Mnemonics, Magic, and Mathematics, by Wayne Stark
The Geometry of Rene Descartes, by Lyman C. Peck
The Game of Nim, by Deloy Benson
Pivotal Condensation Methods of Evaluating Determinants and Solving Linear Systems of Equations, by Fred W. Lott

Iowa Beta, Drake University, Des Moines
Probability of Winning Craps, by Paul Gilman
Boolean Algebra, by Ted Kowalchuk
How Our Numerals Developed, by Walt Whitman
Magic and Numbers, by Richard Haun
Mathematical Relations in Music, Bill Manning
Number Systems, by Dr. Basil Gillam
Mechanical Solution of a Cubic Equation, Dean Dunsworth

Kansas Alpha, Kansas State Teachers College, Pittsburg
Probability, by Don Arnold
Logic, by Edgar Henry
Leonardo Da Vinci as a Mathematician, by Miss Helen Kriegsman
Quality Control, by William Goodwin
Introduction to Elementary Logic, by Tome Martin
Uses of Mathematics in a Ballistics Laboratory, by J. D. Haggard

Kansas Beta, Kansas State Teachers College, Emporia
Topology, by Homer Hackett
Mathematics of Primitive Americans, by Vernie Witten
Mathematical Oddities, by Professor Lester Laird
Grinding Telescope Lenses, by Dr. O. J. Peterson

Kansas Gamma, Mount St. Scholastica College, Atchison
Social Security, by Bernadine Law
Insurance, by Jo Ann Fellin
Taxation, by Donna Rump
Movie on Home Ownership, *"Every Seventh Family"*
Credit Buying and Small Loans, by Virginia Breland and Betty Gross
Mathematical Recreations, by Bernadine Law
Job Possibilities, by Virginia Breland
Teaching of Mathematics, by Jo Ann Fellin
Pi and Probabilities, by Carol Law
System of Winning at Games of Chance, by Joan Carvalho

Kansas Delta, Washburn University, Topeka
Mathematics in the Field of Engineering, Mr. William Mains
Counting Infinities, by Kirk Romary
History of the Theory of Numbers, by Barbara Bartley
Some Mathematical Fallacies, by Dick Admussen
From Oog to Googool, by Darrell Parnell
How to Get a Bright Idea, by Dr. W. C. Foreman, Baker University
The IBM Computers, by Mr. Beverly Brown
Some Topics in Topology, by Dr. R. H. Bing, University of Wisconsin

Louisiana Beta, Southwestern Louisiana Institute, Lafayette
Formation of Empirical Formulae and Equations, by Mr. Lloyd Vincent
Fluid Dynamics Developments, by Dr. P. A. Chieri
Plane Collineations, by Dr. Merlin M. Ohmer
Roots of an Algebraic Equation, by Mr. Gene P. LeBlanc

The Electronic Analogue Computer, by Dr. Z. L. Loflin
Affine Trigonometry, by Dr. W. L. Duren, Jr.
The Special Theory of Relativity, by Dr. Paul A. Delaup

Michigan Beta, Central Michigan College, Mt. Pleasant
The Circular Slide Rule, by Erland Engstrom
Jumping Off at Infinity, by Donald Jennings
Mathematics and Religion, by James Bower
Computing Machines, by Jerre Moore
The Fourth Dimension, by Helena Hayward
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"In my opinion a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy—an opinion, moreover, which has been expressed by many philosophers."

—Henri Lebesque

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Manuscripts prepared for use in THE PENTAGON should be prepared in a manner to lessen the editorial burden. The author should accept the responsibility of making many of the decisions on punctuation, wording, and organization which too frequently fall upon the editor.

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4. **References** and **footnotes** should be numbered consecutively and placed at the bottom of the page. Prepare footnotes according to the following style:

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[1]Florian Cajori, *A History of Mathematics,* (London: The McMillan Co., 1938), p. 150.

To a periodical article:

[2]J. P. Nikonow, "Is Space Curved?" *Scientific American,* 147:278-279, November, 1952.

[3]Ralph Farley, "Visualizing Hyperspace", *Scientific American:* 160:148-149, March, 1939.

To a technical bulletin, pamphlet, or similar publication:

[4]*Guidance Pamphlet in Mathematics,* The National Council of Teachers of Mathematics, Washington, D.C., 1953, p. 16.

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Sesskin, S. H., "An Introduction to Cryptanalysis", *The Pentagon,* 14:16-26, Fall, 1954.

ACTIVE CHAPTERS of KAPPA MU EPSILON*

Chapter	Location	Installation Date
Oklahoma Alpha	Northeastern State College, Tahlequah	April 18, 1931
Iowa Alpha	State Teachers College, Cedar Falls	May 27, 1931
Kansas Alpha	State Teachers College, Pittsburg	Jan. 30, 1932
Missouri Alpha	Southwest Missouri State College, Springfield	May 20, 1932
Mississippi Alpha	State College for Women, Columbus	May 30, 1932
Mississippi Beta	State College, State College	Dec. 14, 1932
Nebraska Alpha	State Teachers College, Wayne	Jan. 17, 1933
Illinois Alpha	Illinois State Normal University, Normal	Jan. 26, 1933
Kansas Beta	State Teachers College, Emporia	May 12, 1934
New Mexico Alpha	University of New Mexico, Albuquerque	March 28, 1935
Illinois Beta	Eastern Illinois State College, Charleston	April 11, 1935
Alabama Beta	State Teachers College, Florence	May 20, 1935
Alabama Gamma	Alabama College, Montevallo	April 24, 1937
Ohio Alpha	Bowling Green State University, Bowling Green	April 24, 1937
Michigan Alpha	Albion College, Albion	May 29, 1937
Missouri Beta	Central Missouri State College, Warrensburg	June 10, 1938
South Carolina Alpha	Coker College, Hartsville	April 5, 1940
Texas Alpha	Texas Technological College, Lubbock	May 10, 1940
Texas Beta	Southern Methodist University, Dallas	May 15, 1940
Kansas Gamma	Mount St. Scholastica College, Atchison	May 26, 1940
Iowa Beta	Drake University, Des Moines	May 27, 1940
New Jersey Alpha	Upsala College, East Orange	June 3, 1940
Tennessee Alpha	Tennessee Polytechnic Institute, Cookeville	June 5, 1941
New York Alpha	Hofstra College, Hempstead	April 4, 1942
Michigan Beta	Central Michigan College, Mount Pleasant	April 25, 1942
Illinois Gamma	Chicago Teachers College, Chicago	June 19, 1942
New Jersey Beta	State Teachers College, Montclair	April 21, 1944
Illinois Delta	College of St. Francis, Joliet	May 21, 1945
Michigan Gamma	Wayne University, Detroit	May 10, 1946
Kansas Delta	Washburn Municipal University, Topeka	March 29, 1947
Missouri Gamma	William Jewell College, Liberty	May 7, 1947
Texas Gamma	Texas State College for Women, Denton	May 7, 1947
Wisconsin Alpha	Mount Mary College, Milwaukee	May 11, 1947
Texas Delta	Texas Christian University, Fort Worth	May 13, 1947
Ohio Gamma	Baldwin-Wallace College, Berea	June 6, 1947
Colorado Alpha	Colorado A & M College, Fort Collins	May 16, 1948
California Alpha	Pomona College, Claremont	June 6, 1948
Missouri Epsilon	Central College, Fayette	May 18, 1949
Mississippi Gamma	Mississippi Southern College, Hattiesburg	May 21, 1949
Indiana Alpha	Manchester College, North Manchester	May 16, 1950
Pennsylvania Alpha	Westminster College, New Wilmington	May 17, 1950
North Carolina Alpha	Wake Forest College, Wake Forest	Jan. 12, 1951
Louisiana Beta	Southwest Louisiana Institute, Lafayette	May 22, 1951
Texas Epsilon	North Texas State College, Denton	May 31, 1951
Indiana Beta	Butler University, Indianapolis	May 15, 1952
Kansas Epsilon	Fort Hays Kansas State College, Hays	Dec. 6, 1952
Pennsylvania Beta	La Salle College, Philadelphia	May 19, 1953
California Beta	Occidental College, Los Angeles	May 28, 1954
Virginia Alpha	Virginia State College, Petersburg	Jan. 29, 1955

* Listed in order of date of installation.