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CONTENTS

Page

National Officers ---------------------------------- 2

Methods of Extracting Square Roots
By Judith Cretcher Enos ---------------------- 3

An Introduction to Cryptanalysis: Two Ciphers
By S. H. Sesskin ------------------------------ 16

Integral Right Triangles of Equal Area
By Harvey Fiola ------------------------------- 27

A Tangential Approach to the Area Under a Curve of Integration
By Charles R. Bonnell -------------------------- 30

Mathematical Notes, *By Charles H. Pearsall, Jr.* ---------- 34

The Area of Hyperbolic Sector, *By H. S. M. Coxeter* ------- 39

The Problem Corner --------------------------------- 40

The Mathematical Scrapbook ------------------------- 45

The Book Shelf ------------------------------------- 50

Installation of New Chapter ------------------------- 55

Brief History of Kappa Mu Epsilon -------------------- 56

Active Chapters of Kappa Mu Epsilon ------------------ 63

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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics in the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Methods of Extracting Square Roots

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1. Introduction. The average person knows but one or two ways of extracting square roots. The long division method as described by Short (24)[1] is commonly taught in the junior high school, and if a person is lucky he is introduced to the logarithmic method of extracting square roots in the high school. Actually there are many distinctly different methods for finding square roots, and for each different method there are variations and generalizations. My purpose in preparing this paper is to compile as many of these methods of computing arithmetic square roots as I have been able to find.

2. Short's Method. I will begin with the method most commonly taught in the public schools—the method described by Short (24). It will be noticed that this procedure can be used to find the *n*th root of arithmetical numbers. The extraction of square roots is only a specific application of a more general procedure. The steps, which I will illustrate in finding the square root of 2, are the following:

1. Beginning at the decimal point, separate the number into periods of n figures each.

$$\sqrt{2.00'00'00'}$$

2. Find the largest integer whose nth power is not greater than the leftmost period. This is the first figure of the root.

$$\begin{array}{l} \phantom{\sqrt{}}1 \\ \sqrt{2.00'00'00'} \end{array}$$

3. Raise the figure found in Step 2 to the nth power and subtract from the leftmost period.

$$\begin{array}{l} \phantom{\sqrt{}}1 \\ \sqrt{2.00'00'00'} \\ \phantom{\sqrt{}}\underline{1} \\ \phantom{\sqrt{}}1 \end{array}$$

4. Bring down the next period.

$$\begin{array}{l} \phantom{\sqrt{}}1 \\ \sqrt{2.00'00'00'} \\ \phantom{\sqrt{}}\underline{1} \\ \phantom{\sqrt{}}1\ 00 \end{array}$$

[1] The numbers in parentheses refer to the bibliography at the end of this paper.

5. Add a zero to the figure (or figures) representing the part of the root already found and raise to the $(n - 1)$th power.

$$10^1 = 10$$

6. Multiply this by n and use the product as a trial divisor.

$$\begin{array}{rl} & \phantom{\sqrt{}}1.4 \\ & \sqrt{2.00'00'00'} \\ & \phantom{\sqrt{}}1 \\ 10 \times 2 = 20) & \phantom{\sqrt{}}\overline{1\ 00} \end{array}$$

7. Divide the number found my means of Steps 3 and 4 by this trial divisor; place the quotient thus found as the next figure of the root; also use the quotient as directed in Steps 8 and 9.

8. In one of the $(n - 1)$ factors in Step 5, replace the zero by the new figure of the root and multiply all the factors.

$$14^1 = 14$$

9. Replacing one zero at a time, repeat Step 8 until all the zeros are used up.

10. Add the products in Steps 5, 8, and 9, and multiply their sum by the figure of the root last found.

$$(10 + 14) \cdot 4 = 96$$

11. If this product is smaller than the remainder, subtract and start anew with operation 5.

$$\begin{array}{rl} & \phantom{\sqrt{}}1.4 \\ & \sqrt{2.00'00'00'} \\ & \phantom{\sqrt{}}1 \\ 20) & \phantom{\sqrt{}}\overline{1\ 00} \\ & \phantom{\sqrt{}1\ }96 \\ & \phantom{\sqrt{}}\overline{4\ 00} \end{array}$$

12. If this product is larger than the remainder, the last figure of the root must be decreased and operations 7 through 11 repeated.

13. This procedure is continued until the desired degree of accuracy is obtained.

When a stated number of decimal places is desired, a short cut may be applied for finding the last few decimal places of a square root. The rule as given by Wentworth (27) may be stated, "One less than the number of figures already obtained may be found without error by division, the divisor to be employed being twice the part of the root already found." Since in the case above we have found two figures, we can find a third by dividing 2×14 into the number obtained by annexing a zero to the remainder. Dividing 40 by 28, we obtain 1 as the third significant figure.

3. Arleigh's Method. A somewhat simpler method has been devised by Arleigh (1). It is very adaptable for younger or less mathematically inclined students because of the ease with which it can be applied and remembered. However, the disadvantage in this method lies in the fact that approximations are obtained by guesswork which means that a great deal of trial and error may be necessary.

The theory behind Arleigh's method is easily explained and understood. It is based on the fact that

$$x^2 - y^2 = (x - y)(x + y),$$

or

$$x^2 = y^2 + (x-y)(x + y).$$

In using this identity, we let

y = an approximation to the required root

and x = a better approximation.

An example will make this method clear. In finding the square root of 2, we may take 1.5 as our first approximation. Then $y = 1.5$, and $1.5^2 = 2.25$, so 1.5 is evidently too large. Now letting $x = 1.45$ and $y = 1.5$, we proceed as follows:

$$\begin{array}{lrl}
 & 2.2500 & \\
-.05(1.5 + 1.45) = & \underline{-.1475} & y = 1.5,\ x = 1.45 \\
1.45^2 = & 2.1025 & \\
-.03(1.45 + 1.42) = & \underline{-.0861} & y = 1.45,\ x = 1.42 \\
1.42^2 = & 2.0164 & \\
-.01(1.42 + 1.41) = & \underline{-.0283} & y = 1.42,\ x = 1.41 \\
1.41^2 = & 1.9881 & \\
.004(1.41 + 1.414) = & \underline{.011296} & y = 1.41,\ x = 1.414 \\
1.414^2 = & 1.999396 \text{ etc.} &
\end{array}$$

4. Newton's Algorithm. Perhaps a better method than the preceding, requiring less trial and error and obtaining a greater degree of accuracy more quickly, is that commonly known as Newton's algorithm. It is based on the theory that, if a is a good approximation to the square root of x, then $\frac{1}{2}(a + x/a)$ is a better approximation. In words this says that the average of an approximation to the square root of a number and the quotient obtained by dividing that number by the first approximation gives a second approximation that is closer to the correct root. An example will make this clear. In finding the square root of 2, we have:

$x = 2 \qquad\qquad a = 1.5$
$a_1 = \frac{1}{2}(1.5 + 2/1.5) = \frac{1}{2}(1.5 + 1.33) = 1.41.$
$a_2 = \frac{1}{2}(1.41 + 2/1.41) = 1.41425.$
$a_3 = \frac{1}{2}(1.41425 + 2/1.41425) = 1.4142135.$

In general, if two decimal places are known, the next step will give an approximation correct to four decimal places. Similarly if three decimal places are known, the next step gives an approximation correct to six decimal places. This may be stated: "If n decimal places are known to be correct, an approximation correct to $2n$ places may be obtained by applying Newton's algorithm (12)."

It is interesting to note that Newton's algorithm, as stated, is only one of several formulas that are either extensions or variations of the same main formula. If we take a as an approximation to $\sqrt[n]{x}$ with e as the error in a, the binomial theorem will give the following expression for e:

$$x = a^n + na^{n-1}e + \tfrac{1}{2}\, n(n-1)a^{n-2}e^2 + \cdots + e^n.$$

Then, if we consider only the first two terms of the expansion, since we are dealing with the second degree, we obtain

$$\sqrt[n]{x} = a + (x - a^n)/na^{n-1}$$

(which is easily shown to be identical to Newton's algorithm when $n = 2$)

$$\text{or } \sqrt[n]{x} = a + a(x - a^n)/nx$$
$$\text{or } \sqrt[n]{x} = a + a(x - a^n)/[(n+1)a^n - x]$$
$$\text{or } \sqrt[n]{x} = a + a(x - a^n)/[(n-1)x + a^n]$$

etc.

James (16) showed that Newton's algorithm does not have to be restricted solely to finding square roots. The principle upon which it is based applies equally well to any root. In finding the nth integral root of a number, divide any approximation of the root into the number, then into the quotient, continuing until $n-1$ divisions have been made. Take for the new approximation the arithmetic mean of the $n-1$ divisors and the final quotient. For example, in finding the cube root of two we would have:

$a_1 = 1.2 \quad$ = first approximation
$2/1.2 = 1.66666 \cdots$
$1.66666/1.2 = 1.388888 \cdots$
$a_2 = (1/3)(1.2 + 1.2 + 1.38888) = 1.26$, etc.

Though Newton's algorithm obtains a reasonable degree of accuracy fairly rapidly, Dunkel (7) has derived a method by which he can obtain more correct decimal places faster. He shows that by using Newton's algorithm we get an approximation, the error e of which is thus limited:

$$(d - q)^2/8a < e < (d - q)^2/4(a + q),$$

where d = first approximation (the assumed divisor)

q = quotient obtained by dividing the number by d

a = the average of d and q.

In view of this, Dunkel contends that by subtracting the lower limit of the error from a we will obtain an approximation even closer to the actual root. In other words, $a - (d - q)^2/8a$ is a better approximation to the root than a.

Similarly the error e' in this approximation is bounded in this manner:

$$(d - q)^4/128a^3 < e' < (d - q)^4/32(a + q)^2.$$

Thus

$$[a - (d - q)^2/8a] - [(d - q)^4/128a^3]$$

is a better approximation than a. This series can be extended as far as desired, but for all practical purposes the first two terms are sufficient. An example will make the process clear.

$x = \sqrt{2},\ d = 1.4$
$q = 2/1.4 = 1.4285714$
$a = \frac{1}{2}(d + q) = 1.4142857$
$(d - q)^2 = (-.02857)^2 = .0008162$
$8a = 11.3142856$
$(d - q)^2/8a = .0000721$
$a - .000072 = 1.414214 = a_1$, which is correct to six decimal places.

5. Uspensky's Method. Another method closely connected to Newton's algorithm is that of Uspensky (26). He and R. F. Newton (20) have both developed the general formula, which applies to any degree root:

$$s_1 = s[(n + 1)a + (n - 1)s^n]/[(n - 1)a + (n + 1)s^n]$$

where

s_1 = new approximation
s = first approximation
n = degree of the root desired
a = number whose nth root is desired.

In the case of square roots, this formula resolves itself into the easier formula

$$s_1 = s(3a + s^2)/(a + 3s^2).$$

This could also be expressed as

$$s_1 - s = 2s(a - s^2)/(a + 3s^2).$$

James (16) shows this method and Newton's algorithm are related, both being special cases of a more general iterative formula; in fact, they merely use different methods of averaging.

As an example of Uspensky's method, we will find the square root of 2.

$$a = 2,\ s = 1.5$$
$$s_1 = 1.5(6 + 2.25)/(2 + 6.75) = 1.4143$$
$$s_2 = 1.4143(6 + 2.00024449)/(2 + 6.00073347) = 1.414213562.$$

This is correct to nine decimal places. In general each new approximation has about three times as many correct decimal places as the preceding approximation.

6. Continued Fractions. Second in importance to Newton's algorithm because of the number of other methods which can be derived from it is the method of continued fractions. The explanation of this method is given by Hall and Knight (12). I will illustrate the method by the example,

$x = \sqrt{2}$, first expressing $\sqrt{2}$ as a continued fraction.

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + 1/(\sqrt{2} + 1)$$
$$\sqrt{2} + 1 = 2 + (\sqrt{2} - 1) = 2 + 1/(\sqrt{2} + 1)$$
$$\sqrt{2} + 1 = 2 + (\sqrt{2} - 1) = 2 + 1/(\sqrt{2} + 1)$$

etc.

Thus,

$$\sqrt{2} = 1 + 1/[2 + 1/(2 + 1/\cdots)]$$

with successive convergents

$$1/1,\ 3/2,\ 7/5,\ 17/12,\ 41/29,\ 99/70 \cdots.$$

Each successive convergent is a better approximation to $\sqrt{2}$. Thus, $99/70 = 1.4143$; here the error is less than $1/70^2 = .0002$ so that the approximation is correct to three decimal places.

As can be seen, this is a rather slow method of obtaining any degree of accuracy. If, as shown by Garver (11) and Chapman (5), we use a combination of Newton's algorithm and continued fractions, we will approach the limit much more quickly. In fact, by applying Newton's algorithm to a convergent found by continued fractions, another convergent twice as far out in the series will be obtained. In the previous example, we found an approximation correct to three decimal places by using the sixth convergent. Now by applying Newton's algorithm, we get

$[(99/70) + (2 \times 70/99)]/2 =$

$99/140 + 70/99 = 19{,}601/13{,}860 = 1.4142135$

which is correct to six decimal places and is equal to the twelfth convergent.

7. Swift's Method. Another method, closely connected to continued fractions, is that of Swift (25). He shows that if a and c are positive integers existing under the conditions

$2a \geqq c(k - 1),\ a^2 < kc^2,$

where k is the number whose root is to be found, then

$m'/n' = (am + kcn)/(cm + an)$

where

$m'/n' =$ a new approximation.

$m/n =$ a first approximation to the root.

Any number of sets of values can be found for a and c. In the case of $k = 2$, we could have any one of the following:

$(a = 1, c = 1), (a = 1, c = 2), (a = 2, c = 3),$

$(a = 4, c = 3)$, etc.

It is interesting to note that by using the first set of values, a series is obtained identical to that resulting from continued fractions. However, the higher the values for a and c, the more quickly the series will converge. Therefore, in finding the square root of 2, we will use $a = 4$, $c = 3$, $m = 3$, and $n = 2$.

$m'/n' = (4 \cdot 3 + 2 \cdot 3 \cdot 2)/(3 \cdot 3 + 4 \cdot 2) = 24/17$

$m''/n'' = (4 \cdot 24 + 2 \cdot 3 \cdot 17)/(3 \cdot 24 + 4 \cdot 17) = 198/140$

$= 1.4143$

The error in this case is less than .0001.

8. Camm's Method. Though the theorems already stated in connection with continued fractions are practical to a certain degree, Camm's method (3) is more practical in that it approaches its limit much more rapidly. However, without a computing machine the arithmetic would be rather arduous. This method is based on recurrence formulas and continued fractions. Camm shows that a surd of the form $\sqrt{m^2 - 1}$, where m is any number, can be expanded into the recurrence formula

$$\sqrt{m^2 - 1} = m - (1/q_0)[1 + (1/q_1)[1 + (1/q_2)[1 + \cdots]$$

where

$q_0 = 2m,\ q_r = q_{r-1}^2 - 2.$

Thus, the first four terms are

$$\sqrt{m^2 - 1} = m - (1/2m)[1 + (1/(4m^2 - 2))[1 + (1/(16m^4 - 16m^2 + 2))[1 + \cdots].$$

Similarly, surds of the form $\sqrt{m^2 + 1}$ can be expressed in the form

$$\sqrt{m^2 + 1} = m + (1/q_0)[1 - (1/q_1)[1 + (1/q_2)[1 + \cdots]$$

where

$q_0 = 2m,\ q_1 = q_0^2 + 2,\ q_{r+1} = q_r^2 - 2.$

In this case, the first four terms are

$$\sqrt{m^2 + 1} = m + (1/2m)[1 - (1/(4m^2 + 2))[1 + (1/(16m^4 + 16m^2 + 2))[1 - \cdots].$$

For $\sqrt{2}$, we apply this formula with $m = 1$. Thus,

$$\sqrt{2} = 1 + (1/2)[1 - (1/6)[1 + (1/34)[1 - \cdots]$$
$$= 1 + 1/2 - 1/12 - 1/408 = 1.4142157$$

which is correct to five decimal places.

By the application of continued fractions, all other roots of the form $\sqrt{m^2 + n}$ can be expressed in one of the above forms by using the formula

$$\sqrt{m^2 + n} = \sqrt{x_k^2 \pm 1}/y_k$$

where x_k/y_k is the kth convergent in the continued fraction expansion of $\sqrt{m^2 + n}$.

Either of the above forms converges very rapidly to the required square root, especially if m is large. In general, the sum of $r + 1$ terms will give the 2^rth convergent.

9. Escott's Method. Comparable to Camm's method for obtaining a high degree of accuracy with a few terms is Escott's method (10). It, too, is not practical without a calculating machine. Escott derives the formula

$$\sqrt{(x_1 + 2)/(x_1 - 2)} = [1 + 2/(x_1 - 1)] \cdot [1 + 2/(x_2 - 1)] \cdots$$

where

$x_{n+1} = x_n \cdot (x_n^2 - 3).$

If we are computing the square root of 2, we let

$(x_1 + 2)/(x_1 - 2) = 2$

whence $x_1 = 6.$

Then $x_2 = 6(33) = 198$

$x_3 = 198(39{,}201) = 7{,}761{,}798$

and $\sqrt{2} = (1 + 2/5)(1 + 2/197)(1 + 2/7{,}761{,}797)$

$= (1.4)(1.010152284)(1.0000002576722)$
$= 1.414213562373096$
which is correct to nine decimal places.

10. Method of Marchant Calculators, Inc. Marchant Calculators, Inc., has developed a method for computing square roots which is designed especially for use with a desk calculator. This method requires the use of a table of "square root divisors" (19). To illustrate, we again calculate $\sqrt{2.00}$.
The following is a brief extract from the Marchant table necessary for this problem:

A	Col. 1	Col. 2
196	2800000	8854377
199	2821347	8921883
202	2842534	8988882

To determine $\sqrt{N}$, we first select from Column A the number nearest to the left three significant figures of N. In this case we select 199. Adding 199 and 200, we obtain 399. This sum is divided by the entry in Column 1 or Column 2 opposite the number selected from Column A. Column 1 is used if there is an odd number of digits before the decimal point in N (as in our example) or an odd number of zeros immediately following the decimal point; if the number of digits before the decimal point is even, we use Column 2. To complete the proposed problem, 399 is divided by 2821347 giving the figures 141421. The decimal point is placed by the familiar rule: *Place the decimal point so there is one digit in the root each way from the decimal point for each corresponding pair of digits in* N. Thus, $\sqrt{2.00} = 1.41421$.

The intervals in the Marchant table are arranged so that the calculated square root differs from the true root by less than 5 in the sixth significant figure. Therefore, in general, the root obtained by this method is correct to five significant figures, though an error of 1 is possible in the fifth figure.

11. Crawford's Method. One of the most interesting methods is that by which square roots are obtained from a table of cosines. The formulas are easily proven and the work can be done mentally.

Crawford (6) shows that if

$k =$ the number whose square root is to be found
$x =$ that square root
$n =$ any number greater than zero, preferably a power of 10.
$A =$ an angle

and if k, n, A satisfy the formula

$\cos 2A = 2k/n^2 - 1$

then

$x = n \cos A.$

From these facts we can set up the following table for finding the square root of 2:

k	n	$2k/n^2 - 1$	$2A$	A	$\cos A$	$n \cos A$
2	10	$-.9600$	163°44.4′	81°52.2′	.141422	1.41422

As is evident, this gives an approximation correct to four decimal places.

12. Barrow's Method. Barrow's method (2) is based on the diophantine equation

(1) $y^2 - nx^2 = 1$

where n is any number not a perfect square. Now if two numbers x_0 and y_0 can be found which satisfy this equation, then

(2) $x_1 = 2x_0y_0,\ y_1 = 2y_0^2 - 1$

will also satisfy it. Furthermore, if x and y are sufficiently large numbers, the fraction y/x will be a good approximation to the square root of n. But we can make x and y sufficiently large by successive applications of (2) once we have found an initial solution to (1). Generally the initial solution can be found by inspection, though if that is not possible, Barrow has shown a rather complicated method by which it can be found. To illustrate, in finding the square root of 2 we have

$n = 2,$
$y^2 - 2x^2 = .1.$

For the initial solution we find

$x_0 = 2, y_0 = 3.$

Then by successive application of (2) we have

$x_1 = 2{\cdot}2{\cdot}3 = 12 \qquad y_1 = 2{\cdot}3^2 - 1 = 17$
$x_2 = 2{\cdot}12{\cdot}17 = 408 \qquad y_2 = 2{\cdot}17^2 - 1 = 577$
$x_3 = 2{\cdot}408{\cdot}577 = 470,832 \qquad y_3 = 2{\cdot}577^2 - 1 = 665,857$

whence

$$\sqrt{2} = 665{,}857/470{,}832 = 1.41421356.$$

This is correct to eight decimal places.

13. Table of Differences. The final method I will discuss is that most often used in setting up a table of square roots. It is the method, explained by Hutton (15), of using a table of differences. Any number whose square root is desired can be expanded thus

$$\sqrt{a^2 + n} = a + n/2a - n^2/8a^3 + n^3/16a^5 - \cdots$$

where a^2 is any number that is a perfect square. From this can be constructed the following table of values.

n	U_n
0	a
1	$a + 1/2a - 1/8a^3 + 1/16a^5$
2	$a + 2/2a - 4/8a^3 + 8/16a^5$
3	$a + 3/2a - 9/8a^3 + 27/16a^5$
4	$a + 4/2a - 16/8a^3 + 64/16a^5$

Successive differencing results in the following leading differences:

$$\Delta U_0 = 1/2a - 1/8a^3 + 1/16a^5$$
$$\Delta^2 U_0 = -1/4a^3 + 3/8a^5$$
$$\Delta^3 U_0 = 3/8a^5.$$

Given a^2, we can compute the values of the leading differences and then by simple addition obtain the square roots of

$$\sqrt{a^2 + n},\ n = 1, 2, 3, \cdots.$$

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An Introduction to Cryptanalysis: Two Ciphers

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The discussion in this paper will be pointed toward solution of the following cryptograms:

No. 1) IRHEL AGHOA VNGRT WSEEE SNRCL PHNGI PAASE NTEOO SXOAS TFMI.

No. 2) 64-39-75-47-54-68-76-60-76-59-75-49-94-39-47-59-66-39-67-57-58-56-76-59-54-57-47-58-54-67-44-47-77-57-78-47-58-57-77-59-77-56-44-59-66-39-65-26-87-48-46-26-74-66-48-58-58-35-75-28-66.

Cryptanalysis, truly, is a subject for mathematicians; for it requires logic, the patience, and the intuitive thinking that go into mathematics. But it is precisely because of this affinity for the subject that mathematicians being introduced to cryptography must be advised not to waste time inventing "unbreakable" ciphers, for these usually involve some more or less well-known systems or combinations of systems already treated in the serious literature of cryptography, or discarded in practice as too complicated.

To the cryptographer, such efforts by an amateur represent the same immaturity of viewpoint that mathematicians would see in efforts to find formulae for primes.

The material to be presented here is elementary from a cryptographic viewpoint but requires considerable background and some definition. These factors will be treated in cursory fashion because of space limitations so that the reader will have to supply extra thinking to make up the lack. However, those interested in going further are urged to get the heft of this fascinating subject through some fictional and factual reading.

A gem of cryptographic fiction is Poe's famous story *The Gold Bug,* also Conan Doyle's *Adventure of the Dancing Men.* An excellent fictional anthology, with a good quickie technical preface tossed in along with some interesting history, is *Famous Stories of Code and Cipher,* edited by Raymond T. Bond (Rinehart, 1947).

But for a real taste of the importance of the subject, one should read Herbert O. Yardley's *The American Black Chamber,* which deals with modern history and diplomacy, and Fletcher Pratt's *Secret and Urgent,* which selects important cryptographic incidents from the full range of history. *Gods, Graves and Scholars,* by G. W. Ceram, a more recent volume, has some excellent material on the archeological cryptographers and their work in deciphering, among other artifacts, the Babylonian tablets and the Rosetta stone.

Beyond a feeling for the subject—which seems to be as important in this field as it is in any mathematics—there is a need for specific knowledge dealing with the behavior of words and letters in language.

Cryptography is number theory in the world of letters, and just as the ABC of much of mathematics is 1, 2, 3, so the 1, 2, 3 of cryptography in English is E T A O N I R S H, the frequency table, a list almost as immutable as pi. As Parker Hitt in *Manual for the Solution of Military Ciphers* observes: "If ten thousand consecutive letters of a text be counted, and the frequency occurrence of each letter be noted, the number found will be practically identical with those obtained from any other text of ten thousand letters in the same language."

Of course, there may be greater deviations between specialized texts such as military text, or commercial text, and possibly telegraphic text.

We will not pursue this question of specific knowledge much further except to note that the annotated bibliography at the end will present sources of information for the reader who may be lured further along what the author has found to be a challenging trail for investigation.

The object of this paper will be to show the type of reasoning that goes into cryptanalysis. The demonstration will include one cryptogram from each of the two great classes of ciphers—*transposition,* apparently a Greek invention which uses the actual letters of the message but scrambles them according to some plan; and *substitution,* a Roman invention which substitutes other letters or symbols for actual letters of the message.

The first cipher to be treated is the Nihilist transpositions, technically a double transposition in which rows and columns of a matrix are scrambled according to the same plan.

The system requires a perfect square number of letters, with nulls such as X, J, or Z supplied to complete the square if necessary. Solution of this type depends on knowledge of a language invariant—uniform distribution of vowels throughout normal text. Vowels make up 40% of text, with or without y, with a 5% variation from the norm possible.

To illustrate the nihilist transposition plan and to show that uniform vowel distribution is general for the language and does not have to be cooked up, we will use part of the first sentence of the second paragraph of this article, "Cryptanalysis, truly, is a subject for mathematicians for it requires logic." The scrambling will be done according to alphabetic position of letters of *PENTAGON*.

The nihilist plan is illustrated in Figures 1, 2, and 3; although on a practical basis it can be done in one operation.

(1) message matrix:

	P	E	N	T	A	G	O	N
	7	2	4	8	1	3	6	5
7-	C	R	Y	P	T	A	N	A
2-	L	Y	S	I	S	T	R	U
4-	L	Y	I	S	A	S	U	B
8-	J	E	C	T	F	O	R	M
1-	A	T	H	E	M	A	T	I
3-	C	I	A	N	S	F	O	R
6-	I	T	R	E	Q	U	I	R
5-	E	S	L	O	G	I	C	X

Fig. 1

(2) columns transposed:

	A	E	G	N	N	O	P	T
	1	2	3	4	5	6	7	8
7-	T	R	A	Y	A	N	C	P
2-	S	Y	T	S	U	R	L	I
4-	A	Y	S	I	B	U	L	S
8-	F	E	O	C	M	R	J	T
1-	M	T	A	H	I	T	A	E
3-	S	I	F	A	R	O	C	N
6-	Q	T	U	R	R	I	I	E
5-	G	S	I	L	X	C	E	O

Fig. 2

(3) rows transposed:

	A	E	G	N	N	O	P	T
	1	2	3	4	5	6	7	8
1-	M	T	A	H	I	T	A	E
2-	S	Y	T	S	U	R	L	I
3-	S	I	F	A	R	O	C	N
4-	A	Y	S	I	B	U	L	S
5-	G	S	I	L	X	C	E	O
6-	Q	T	U	R	R	I	I	E
7-	T	R	A	Y	A	N	C	P
8-	F	E	O	C	M	R	J	T

Fig. 3

From this last block of letters, the cryptogram may be taken off in one of 641 ways, but practical considerations such as ease in remembering a pattern and flexibility restrict the taking-off to about 40 ways, including simple horizontal, alternating horizontal (forward on one line, backward on the next), simple or alternating vertical, simple or alternating diagonal, spiral clockwise or counterclockwise starting from the outside in or from the center out—all of these starting from any corner (except spiral from the center out).

Now as to the vowel invariant, note that half of the rows have the normal vowel frequency 3 (40% of 8), while the other half, although over or below the allowable 5% variation, have 2 or 4 vowels, the smallest variation possible for normality. Exceptions are extremely rare.

We will now attack Cipher No. 1, given two bits of information: (1) That the cipher is English. (2) That it is probably a nihilist transposition.

We will check the probability of transposition by applying certain basic subset tests for normalcy of text. In all of these subset tests, given at least 100 letters, there should be no more than a 5% variation from the normal. For short cryptograms such as the present one—49 letters—the variations might be greater.

Here are the tests:

1) The letters E T A O N I R S H, the nine most frequent in English, make up about 70% of normal text.
2) L N R S T show about 30%.
3) Vowels contribute about 40% of text.

A frequency count of cryptogram No. 1 shows:

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
5		1		6	1	3	3	3			2	1	4	4	2		3	5	3		1	1	1		

Applying test 1, we find the cryptogram has 36 E T A etc., which results in the fraction 36/49, a mite under 75%. Test 2 shows 17/49, about 35%. Vowels count up to 18, about 37%.

These statistics put all the subsets within normal range so that it is highly probable that our information about a transposition is correct. However, this is not certain. In cryptography, one must run every good probability to the ground; but good as any one seems, it must be ruthlessly discarded if it appears not to work. Solution is the only measure of certainty. In this connection, just to illustrate how devious and deceptive this subject is, Helen F. Gaines in her *Elementary Cryptanalysis* points out that "an apparent transposition with exactly 40% of vowels and 100% evenness of distribution is suspicious."

For a nihilist transposition, the primary problem is rebuilding the matrix from which the cryptogram was taken off. Here is where the vowel invariant shows its value. Let's write the cryptogram horizontally in a 7 x 7 matrix, and take vowel counts in each row and then again in each column (Fig. 4).

I	R	H	E	L	A	G	-3
H	O	A	V	N	G	R	-2
T	W	S	E	E	E	S	-3
N	R	C	L	P	H	N	-0
G	I	P	A	A	S	E	-4
N	T	E	O	O	S	X	-3
O	A	S	T	F	M	I	-3
2	3	2	4	3	2	2	

Fig. 4

Zero vowels in row 4 rules out the horizontal setup for it is extremely unusual to find more than five consonants in succession.

The vertical setup appears to be more likely. It definitely has uniform distribution; and in all cases but one (column 4), the frequency is normal. A uniform vowel distribution among the columns in this setup indicates a vertical take-off from the original matrix, so let's rewrite the cryptogram vertically (Fig. 5). Assume for the moment that it is the original matrix and go to work on it.

	1	2	3	4	5	6	7
1-	I	H	T	N	G	N	O
2-	R	O	W	R	I	T	A
3-	H	A	S	C	P	E	S
4-	E	V	E	L	A	O	T
5-	L	N	E	P	A	O	F
6-	A	G	E	H	S	S	M
7-	G	R	S	N	E	X	I

Fig. 5

From this point it is a question of anagramming the rows; and, thus revealing the key, to put the cryptogram in complete order. Anagramming is an almost instinctive procedure, for it is the recognition of the possibility that two or three letters are consecutive (the beginner may use bigram charts to check the most frequent bigrams).

The explanation of anagramming in Cryptogram No. 1 will be simplified in order to save space, but in practice logic and experience will invariably lead to the solution. Look at row 7 (Fig. 5). N G is a fairly frequent bigram. Placing their columns together (Fig. 6) reveals a column of extremely plausible bigrams.

In row 6 (Fig. 5) we have 2 S's, either of which might follow H A in Fig 6 to form "has." The S in column 6 would form N G X (might be a null) in row 7. More unlikely, it would form R R T in row 2. (In practice, this would have to be tested further for complete elimination.) Using the S in column 5 (Fig. 7) reveals good trigrams. C H P in row 3 might be the end of one word, beginning of another.

	4	1
1-	N	I
2-	R	R
3-	C	H
4-	L	E
5-	P	L
6-	H	A
7-	N	G

Fig. 6

	4	1	5
1-	N	I	G
2-	R	R	I
3-	C	H	P
4-	L	E	A
5-	P	L	A
6-	H	A	S
7-	N	G	E

Fig. 7

	4	1	5	2
1-	N	I	G	H
2-	R	R	I	O
3-	C	H	P	A
4-	L	E	A	V
5-	P	L	A	N
6-	H	A	S	G
7-	N	G	E	R

Fig. 8

Next, try the R in row 7 as a test for the very frequent bigram E R, and we get (Fig. 8). We are apparently on the right track for there are word suggestions already appearing. For example, in rows 1 (N I G H T ?); 4 (L E A V E ?); 5 (P L A N ?). Let's try "leave" in row 4 by adding column 3 (Fig. 9). Row 2 (R R I O W) looks somewhat implausible, but everything else looks very promising. Note in row 6, "has ge. . ."; "has *what*?" The only two letters left in that row are M and S. Probably "has gems." Let's see (Fig. 10). Everything looks good except row 2. However, if this setup is correct, the key is revealed in 4152376; all that is necessary to check it is to transpose the rows according to the same key (Fig. 11) and see if the message comes out. The message becomes "Leave tonight on plane for Rio. Watch passenger six. Has gems."

	4	1	5	2	3
1-	N	I	G	H	T
2-	R	R	I	O	W
3-	C	H	P	A	S
4-	L	E	A	V	E
5-	P	L	A	N	E
6-	H	A	S	G	E
7-	N	G	E	R	S

Fig. 9

	4	1	5	2	3	7	6
1-	N	I	G	H	T	O	N
2-	R	R	I	O	W	A	T
3-	C	H	P	A	S	S	E
4-	L	E	A	V	E	T	O
5-	P	L	A	N	E	F	O
6-	H	A	S	G	E	M	S
7-	N	G	E	R	S	I	X

Fig. 10

	4	1	5	2	3	7	6
4-	L	E	A	V	E	T	O
1-	N	I	G	H	T	O	N
5-	P	L	A	N	E	F	O
2-	R	R	I	O	W	A	T
3-	C	H	P	A	S	S	E
7-	N	G	E	R	S	I	X
6-	H	A	S	G	E	M	S

Fig. 11

The second cipher we will discuss is the nihilist number substitution using a "checkerboard" alphabet arranged normally in a 5 x 5 square with the I serving also for J. (Fig. 12). The order of reading coordinates is row-column. In our encipherment example, we will use the key word C O S I N E which has the numerical equivalents 13-34-43-24-33-15. For encipherment the key numbers are added to the text numbers thus:

	T	H	E	P	E	N	T	A	G	O	N	I	S	M	A	T	H
Text—	44	23	15	35	15	33	44	11	22	34	33	24	43	32	11	44	23
Key—	13	34	43	24	33	15	13	34	43	24	33	15	13	34	43	24	33
CIPHER—	57	57	58	59	48	48	57	45	65	58	66	39	56	66	54	68	56

The cryptogram: 57-57-58-59-48-48-57-45-65-etc.

	1	2	3	4	5
1-	A	B	C	D	E
2-	F	G	H	I	K
3-	L	M	N	O	P
4-	Q	R	S	T	U
5-	V	W	X	Y	Z

Fig. 12

This cipher is a periodic, which means that it is enciphered by a repeating key, each number of which gives rise to a different cipher alphabet. This type can be decrypted by general methods which apply to periodics. However, we will be concerned here only with a special method devised as a shortcut by M. E. Ohaver, a cipher expert, and published in *Elementary Cryptanalysis.*

The primary alphabet—obtained directly from the square—has only the digits 1-2-3-4-5. The key word numbers—also obtained directly from the square—use the same digits. If any same key word number were to be added to the primary alphabet numbers, the addition would not change the maximum difference of 4 which exists between any two primary alphabet numbers either in their units digits or their tens digits.

So, if a nihilist substitution contains two numbers 24 and 78, or 24 and 69,—with differences greater than 4 respectively in their tens digits or units digits—it is definitely established that those two numbers were not enciphered with the same key number.

However, if the two cryptogram numbers are 40 and 84, in which there appears to be a difference of only 4 in the tens digits, the zero must be counted as the addition of two 5's in the units column (since the key numbers, too, are restricted to the digits 1, 2, 3, 4, and 5). In this event, the 1 must be borrowed back from the tens digit, 4, making it 3, and increasing the difference in the tens digits to 5, making it impossible for the two numbers 40 and 84 to have been enciphered with the same key number.

There are some outstanding give-away numbers in this type of cipher. The digit 2 in a cipher indicates a 1 in the key, since 2 can have been formed only by 1 + 1. Zero is the result only of 5 + 5. Beyond this there are four other bingo numbers: 22 (indicates 11 in the key); 30 (indicates 15); 102 (51 in the key); and 110 (55 in the key).

In the light of the above discussion, we will now consider Cryptogram No. 2 at the beginning of this paper. It is immediately obvious that the cryptogram is not the result of a single addition throughout because the first two numbers 64-39 have a difference greater than 4 in the units digits.

A run-down on every alternate number reaches the 13th and 15th 94-47, which have a greater-than-4 difference in the tens digits. The cryptogram, therefore, is not the result of a repeating addition of two numbers. Period 2 is, therefore, eliminated.

Investigating period 3, we are brought up short by the 2nd and 5th numbers, 39-54, with the difference 5 in the units digits. Period 3 is out.

A check on period 4 reveals no difference greater than the maximum in any of the four alphabets. Figure 13 shows the setup for the supposed period 4.

```
64-39-75-47
54-68-76-60
76-59-75-49
94-39-47-59
66-39-67-57
58-56-76-59
54-57-47-58
54-67-44-47
77-57-78-47
58-57-77-59
77-56-44-59
66-39-65-26
87-48-46-26
74-66-48-58
58-35-75-28
66
```

Fig. 13

Now let's study each alphabet.

Alphabet 1: The tens half of the first cloumn has the maximum range 5-9. The smallest digit which can result in 9 is 4, and the largest which can result in 5 is 4. Therefore, the only digit which can produce all of the column's tens digits, 5-6-7-8-9, is 4. The units digits of the alphabet have the range 4-8. By similar reasoning, the key digit for units must be 3; and the entire key number which formed the first cipher alphabet must be 43 since it can have been no other.

Alphabet 2: Range of tens digits 3-6. Smallest digit for 6 is 1; largest for 3 is 2, so that the key letter must have either 1 or 2 as tens digit. Range within the units column, 5-9, with key digit 4. Alphabet 2 was keyed either by 14 or 24.

Alphabet 3: Range of tens digits, 4-7, indicates a key digit 2 or 3. Range of units digits, 4-8, indicates key digit 3. Key number for column is either 23 or 33.

Alphabet 4: Contains the give-away tens digit 2 and units digit 0. The key number is, therefore, 15.

The checkerboard alphabet may be written into the square in any order, but this will not affect the analysis. It simply means that another step, which will not be discussed here, has to be added to the solution.

However, in this case, granting that there was an orderly arrangement of the letters as in Figure 12, the replacement of letters for key numbers may suggest the key. Alphabet 1, key 43, is S; 2, key 14 or 24, is D or I; 3, key 23 or 33, is H or N; 4, key 15, is E. The only pronounceable combination from S (D or I) (H or N) E, is S I N E , which happens to be the key.

Another way of completing the analysis is to encipher the two alphabets for which you have definite keys and look for word suggestions, working backwards to find the key number that fits the column.

In any event, the rest is simple; merely subtract the key number from the cipher number, and use the resulting number to find the plain text message letters in the 5 x 5 square (Fig. 12).

The foregoing material is admittedly elementary; and the careful reader, no doubt, is busily thinking of ways to improve both these ciphers. It is really not worth the effort, for these two are just curios in a vast subject and were presented here to excite the interest of potential cryptographers and cryptanalysts.

In closing, the author again advising the beginner to avoid inventing "unsolvable" systems until he has studied cryptography and its techniques will quote the closing paragraph of "Secret and Urgent."

> "It is also worth noting that the Japanese ciphers for the occasion (deciphered by Maj. Yardley in 1922) were composed on a system of irrational bigram substitution, not violently different from the Great Cipher designed by Rossignol (a French mathematician) for Louis XIV in the seventeenth century. Jargon field codes and the Great Cipher again; not even in cryptography is there anything new."

Editor's note: Mr. Sesskin has offered to write another article on cryptanalysis for THE PENTAGON entitled "Modern Trends in Cryptanalysis" and dealing with so-called fractionated ciphers. Such an article will appear if there is enough reader interest. If you would like to see a discussion of this topic or of any other cryptographic topic, please inform the Editor.

ANNOTATED BIBLIOGRAPHY

The bibliography which follows is restricted to writings in English, mainly of an introductory nature. The opinions expressed in the annotations are those of the author.

Ball, W. W., *Mathematical Recreations.*

A very interesting chapter on ciphers includes material on substitution and transposition. Good for the beginner and quite handy since it is a standard volume usually included in mathematical libraries.

Friedman, W. F., "Cryptography" and "Codes and Ciphers," *Encyclopedia Brittanica.*

This provides excellent expository material. The classic Vigenere cipher and some of its offshoots are briefly explained. Also includes descriptions of some modern mechanical and transposition ciphers, as well as a description of code systems. The material may be somewhat advanced for the beginner, but the explanations are clear and a reading would be worthwhile.

Gaines, H. F., *Elementary Cryptanalysis,* (American Photographic Publishing Company).

Unfortunately, this is now out of print. An excellent treatise with complete explanations of theory and techniques for breaking most of the elementary types of ciphers. Some advanced stuff included. There are 167 problems of varying degrees of difficulty. This book is the official textbook of the American Crytogram Association. (Bibliography included).

Harris, F. A., *Solving Simple Substitution Ciphers,* (American Cryptogram Association, 603 Cleveland Avenue, NW, Canton, Ohio), 75 cents.

Very good for the beginner. The techniques involved in solving monoalphabetic ciphers are basic to the entire field.

The Cryptogram, bi-monthly journal of the American Cryptogram Association, ($3 a year includes membership).

Contains outstanding articles on elementary and advanced cryptographic topics. Special departments offer examples of all types of cryptograms for solution. The beginner would do well to join.

Wolfe, J. M., *A First Course in Cryptanalysis.* (Brooklyn College Press), $7.

An extensive work dealing with basic cryptanalytic techniques and methods. Professor Wolfe uses mathematical interpretations and terminology to present his material.

Other References

D'Agapeyeff, Alexander, *Codes and Ciphers.*
Lysing, Henry, *Secret Writing.*
Zim, Herbert S., *Codes and Secret Writing.*

"Some derive the same sort of stimulus from a technical monograph as others find in a detective novel—to which, in fact, it may bear some sort of esoteric resemblance. It is all a question of taste, and taste is a thing which no one person can decide for another."

—R. Curle

Integral Right Triangles of Equal Area*

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The Pythagorean relation is expressed algebraically by the equation

$$a^2 + b^2 = c^2$$

where a and b are legs of a right triangle and c the hypotenuse. Any set of integers satisfying this relation is called a *Pythagorean number triple*. It is well known that all *primitive Pythagorean number triples* can be obtained from the formulas

$$a = 2uv,$$
$$b = u^2 - v^2,$$
$$c = u^2 + v^2,$$

where $u > v$ and u and v are relatively prime integers of opposite parity, *i.e.*, one odd and the other even. If we do not wish to restrict ourselves to primitive triples for the sides a, b, and c of a right triangle, we can use any integers for u and v, as long as $u \neq v$, in the formulas above to obtain three sides of an integral right triangle. It is the purpose of this paper to discuss two methods of obtaining sets of integral right triangles of equal area.

Method 1. Suppose two right triangles T_1 and T_3 are not similar and have areas A_1 and A_3 respectively with $A_3 = k^2A_1$; the area A_1' of a third right triangle T_1', similar to T_1, but such that the ratio of the sides of T_1' to those of T_1 is k: 1, will also have the area $A_1' = k^2A_1$. T_1' and T_3 will then be two distinct right triangles with the same area. By the use of the formulas for determining sets of Pythagorean number triples it is possible to get the relation described above and thus obtain two non-congruent integral right triangles of equal area. The procedure to be used follows:

Designate by A_1 the area of an integral right triangle T_1 with sides $a_1 = 2uv$, $b_1 = u^2 - v^2$, and $c_1 = u^2 + v^2$ where u and v are any two unequal integers. Using the legs a_1 and b_1 of T_1 in the place of u and v, determine a new triangle T_2 with sides

$$a_2 = 2a_1b_1,\ b_2 = |a_1^2 - b_1^2|, \text{ and } c_2 = a_1^2 + b_1^2 = c_1^2.$$

Using the sides a_2 and c_2 of T_2 obtain a new triangle T_3 with sides

$$a_3 = 2a_2c_2,\ b_3 = c_2^2 - a_2^2 \text{ and } c_3 = a_2^2 + c_2^2.$$

If A_3 is used to represent the area of T_3, we have

* Adapted by the Editor from notes received from the author. The author is 19 years of age and works on his father's farm. He says, "I have computed so many right triangles that I see them in the heavens."

$$
\begin{aligned}
A_3 = \tfrac{1}{2}\,(a_3b_3) &= a_2c_2\,(c_2^2 - a_2^2) \\
&= a_2c_2b_2^2 = a_2c_1^2b_2^2 \\
&= 2\,a_1b_1c_1^2b_2^2 \\
&= 4\,A_1c_1^2b_2^2 \\
&= (2c_1b_2)^2\,A_1.
\end{aligned}
$$

Let T_1' reprseent a triangle similar to T_1 such that the ratio of the corresponding sides is $2c_1b_2{:}1$. If A_1' is the area of T_1' we have $A_1' : A_1 = (2c_1b_2)^2 : 1$ and thus $A_1' = (2c_1b_2)^2A_1 = A_3$.

It should be evident that continuation of this process, using T_3 as the initial triangle, will enable one to obtain a triangle T_5 such that $A_5 = k^2A_3$; and by multiplying each side of T_3 by k and T_1 by $k(2c_1b_2)$ we obtain new integral right triangles T_1'', T_3', T_5 such that

$$A_1'' = A_3' = A_5.$$

This process can be continued for any finite number of steps, and thus it is theoretically possible to obtain any finite number of integral right triangles with equal areas. The sides increase in size rather rapidly as we carry out this procedure.

For easy reference we list the formulas below.

T_1: $a_1 = 2uv$; $b_1 = u^2 - v^2$; $c_1 = u^2 + v^2$.

T_2: $a_2 = 2a_1b_1$; $b_2 = |a_1^2 - b_1^2|$; $c_2 = a_1^2 + b_1^2$.

T_3: $a_3 = 2a_2c_2$; $b_3 = c_2^2 - a_2^2$; $c_3 = a_2^2 + c_2^2$.

T_1': $a_1' = (2b_2c_1)a_1$; $b_1' = (2b_2c_1)b_1$; $c_1' = (2b_2c_1)c_1$.

We may go directly from T_1 to the two equal integral right triangles T_1' and T_3 by the transformations

T_3: $a_3 = 4a_1b_1(a_1^2 + b_1^2)$; $b_3 = (a_1^2 - b_1^2)^2$;
$c_3 = a_1^4 + 6a_1^2b_1^2 + b_1^4$

T'_1: $a'_1 = 2[(a_1^2 - b_1^2)c_1]a_1$;
$b_1' = 2[(a_1^2 - b_1^2)c_1]b_1$; $c'_1 = 2[(a_1^2 - b_1^2)c_1]c_1$

Example —

If $u = 2$ and $v = 1$,
T_1: $a_1 = 4$; $b_1 = 3$; $c_1 = 5$,
T_3: $a_3 = 1200$; $b_3 = 49$; $c_3 = 1201$,
T_1': $a_1' = 280$; $b_1' = 210$; $c_1' = 350$.

Repeating the process using a_3 and b_3 for u and v we obtain

T_5: $a_5 = 339{,}252{,}715{,}200$
$b_5 = 2{,}066{,}690{,}884{,}801$
$c_5 = 2{,}094{,}350{,}404{,}801$
T_3': $a_3' = 4{,}143{,}735{,}357{,}600$
$b_3' = 169{,}202{,}527{,}102$
$c_3' = 4{,}147{,}188{,}470{,}398$

Multiplying the sides of T_1' by the same factor used in obtaining T_3' from T_5 we get

$$\begin{aligned} T_1'': a_1'' &= 966{,}871{,}583{,}440 \\ b_1'' &= 725{,}153{,}687{,}580 \\ c_1'' &= 1{,}208{,}589{,}479{,}300 \end{aligned}$$

Triangles T_5, T_3', and T_1'' are three integral right triangles of equal area.

Method 2.[1] There are three right triangles of equal area, A, whose sides are $2xy$, $x^2 - y^2$, and $x^2 + y^2$ where

$$\begin{aligned} x &= 3r^2 + s^2 \\ y &= 4rs,\ (3r + s)(r - s), \text{ or } (3r - s)(r + s) \\ A &= xy(x^2 - y^2) = xy(x - y)(x + y) \end{aligned}$$

Let A_1 represent the area of the triangle when $y = 4rs$, A_2 represent the area of the triangle when $y = (3r + s)(r - s) = 3r^2 - 2rs - s^2$, and A_3 represent the area of the triangle when $y = (3r - s)(r + s) = 3r^2 + 2rs - s^2$. Then

$$A_1 = A_2 = (-A_3) = 4rs\,(r^2 - s^2)(3r^2 + s^2)(9r^2 - s^2)$$

Using T_i, $i = 1, 2, 3$, to designate the three right triangles and a_i, b_i, c_i, $i = 1, 2, 3$, to represent their sides we have

$$\begin{aligned} T_1:\ a_1 &= 8\,(rs)(3r^2 + s^2) \\ b_1 &= (r^2 - s^2)(9r^2 - s^2) \\ c_1 &= (3r^2 + s^2)^2 + (4rs)^2 \\ T_2:\ a_2 &= 2(3r^2 + s^2)(3r + s)(r - s) \\ b_2 &= 4(rs)(3r - s)(r + s) \\ c_2 &= (3r^2 + s^2)^2 + [(3r + s)(r - s)]^2 \\ T_3:\ a_3 &= 2(3r^2 + s^2)(3r - s)(r + s) \\ b_3 &= 4(rs)(3r + s)(r - s) \\ c_3 &= (3r^2 + s^2) + [(3r - s)(r + s)]^2 \end{aligned}$$

Example —

For $r = 2$ and $s = 1$ we have the three equal integral right triangles

$$\begin{aligned} T_1&: a_1 = 208,\ b_1 = 105,\ c_1 = 233, \\ T_2&: a_2 = 182,\ b_2 = 120,\ c_2 = 218, \\ T_3&: a_3 = 390,\ b_3 = 56,\ c_3 = 394. \end{aligned}$$

Concluding remarks. Using each of the triangles obtained in *Method* 2 as the T_1 of Method 1 and computing the sides of T_3 by *Method 1*, it is possible to obtain six integral right triangles of equal area. Continuing in this way the ascent can be made in sets of three to any desired multiple of three right triangles of equal area.

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[1] Editor's note—Euler gave this method in his *Opera Postuma* 1, 1862, pp. 250-2.

A Tangential Approach to the Area under a Curve by Integration *

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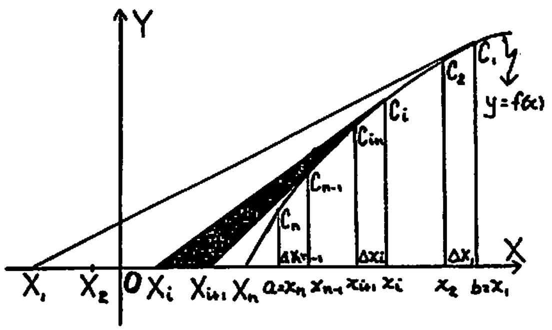
Modern text books make a good, sensible development of the area integral by the usual procedure of summing rectangular increments of area with the fundamental theorem of integration. The importance of the fundamental theorem could easily justify its use in another approach to the area under a curve. Although either development will lead to the same end point, the form of the result of the treatment in this paper can make basic principles more evident and permit the introduction of new concepts earlier.

Consider the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates at the points $x = a$ and $x = b$. The function $y = f(x)$ will be considered to be continuous throughout the interval $a \leqq x \leqq b$ and to have a derivative at every point of the interval.

An approximation for the area abC_1C_n can be obtained by starting with the area of triangle X_1bC_1 and subtracting from it the areas of the successive triangles beginning with $X_1X_2C_1$ and ending with X_naC_n. $X_1, X_2, \ldots, X_n$ are the points of intersections of the x-axis and tangents to $y = f(x)$ at points $C_1, C_2, \ldots, C_n$ respectively.



*Paper presented to the Mathematical Meeting May 8, 1954, at Hamline University.

The area of triangle X_1bC_1 is $\frac{1}{2}\ [b - X_1]f(b)$. The area of triangle X_naC_n is $\frac{1}{2}\ [a-X_n]f(a)$. One side of each of the triangles $X_iX_{i+1}C_i$, $i = 1, 2, \ldots, n$, is a tangent. It is necessary to have an expression for the x-intercepts of these n tangents. The equations of these tangents can be written as

1) $\quad y - f(x_i) = f'(x_i)(x - x_i), \qquad a \leqq x_i \leqq b.$

The x-intercepts of these tangents, designated by X_i, can be readily obtained in the form

2) $\quad X_i = x_i - f(x_i)/f'(x_i), \qquad f'(x_i) \neq 0$

The areas of triangles X_1bC_1 and X_naC_n can be written:

3) Area of triangle $X_1bC_1 = \frac{1}{2}\ (b - [b - f(b)/f'(b)]f(b)$
$= \frac{1}{2}[f^2(b)/f'(b)]$.

4) Area of triangle $X_naC_n = \frac{1}{2}\ (a - [a - f(a)/f'(a)])f(a)$
$= \frac{1}{2}[f^2(a)/f'(a)]$.

The areas of the triangles $X_iX_{i+1}C_i$, $i = 1, 2, \ldots, j, \ldots, n$, can be written as

5) Area of triangle $X_jX_{j+1}C_j = \frac{1}{2}(X_{j+1} - X_j)f(x_j)$
$= \frac{1}{2}\ [x_{j+1} - x_j + f(x_j)/f'(x_j) - f(x_{j+1})/f'(x_{j+1})]f(x_j)$.

Then twice the area of abC_1C_n, designated by $2A$, can be written as

6) $$2A = f^2(b)/f'(b) - f^2(a)/f'(a) - \lim_{n\to\infty} \sum_1^n [f(x_i)\Delta x_i - f(x_i)\Delta(f(x_i)/f'(x_i))].$$

By the fundamental theorem, 6) becomes

7) $$2A = f^2(b)/f'(b) - f^2(a)/f'(a) + \int_a^b f(x)dx - \int_a^b f(x)d[f(x)/f'(x)].$$

Integrating the last integral in 7) by parts produces

8) $$\int_a^b f(x)d[f(x)/f'(x)] = [f^2(x)/f'(x)]_a^b - \int_a^b f(x)dx.$$

Substituting 8) into 7), we obtain

9) $$A = \int_a^b f(x)dx \qquad \text{Q.E.D.}$$

An instructive form of 7) may be obtained by performing the differentiation $d[f(x)/f'(x)]$ in the integrand of the right member of 7) and substituting 9) in the left member of 7).

$$10)\ 2\int_a^b f(x)dx = [f^2(b)/f'(b)] - [f^2(a)/f'(a)] + \int_a^b [f^2(x)f''(x)dx]/[f'^2(x)]$$

It is clear that the definite integral of the right member of 10) can be expressed again by 10. Since this process may be repeated until the derivatives vanish or until an infinite series is formed, 10) states explictly that a definite integral is a function of its limits only. Furthermore, each term of the series will have the form of the product of an ordinate and a subtangent of the successive functions.

By rearranging 10) in the form

$$11)\ \int_a^b f(x)dx - \int_a^b [f^2(x)f''(x)dx/2f'^2(x)] = \tfrac{1}{2}[f^2(b)/f'(b) - f^2(a)/f'(a)]$$

it is interesting to note that the difference of the areas bounded by the two different functions, defined over the same interval, is always expressable as the difference in area of two right triangles with sides that are functions of the limits.

From 8) we obtain:

$$12)\ \int_a^b f(x)dx = f^2(b)/f'(b) - f^2(a)/f'(a) - \int_a^b f(x)d[f(x)/f'(x)]$$

and substituting $f(x)/f'(x) = Q(x)$ and $f(x)dx = f'(x)Q(x)dx$ we obtain the useful relation

$$13)\ \int_a^b f'(x)Q(x)dx = f(b)Q(b) - f(a)Q(a) - \int_a^b f(x)[dQ(x)].$$

Equation 13) reduces to two very fundamental relationships. If $Q(x) = x^n$

$$14) \int_a^b f'(x)x^n dx = f(b)b^n - f(a)a^n - n\int_a^b f(x)x^{n-1}dx.$$

and when $n = 0$

$$15) \int_a^b f'(x)dx = f(b) - f(a).$$

Conclusions—

If a function $f(x)$ is finite in the interval $a \leqq x \leqq b$ then equations 10) and 11) are proper if there are no maximum or minimum points in the interval or at the end points.

It is also shown that a triangulation rule can take its place along with Simpson's rule or the trapezoidal rule for approximating areas.

The special function $f(x) = cx$ has an exact solution in equation 11), but of more interest is the function of $f(x) = cx^2$. Equation 11) states that two right triangles each determined by a tangent and subtangent to the curve at the end points of the interval $a \leqq x \leqq b$ have a difference of ¾ of the area bounded by the curve $f(x) = cx^2$ and the x-axis in the interval. This simple fact has been applied to ballistic problems and has greatly reduced some tedious work.

Mathematical Notes

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Student, Hofstra College

A Note on Simple Areas and Volumes

If

$$(1)\ P(x) = a_0x^3 + a_1x^2 + a_2x + a_3$$

then from the calculus we have

$$\int_0^h P(x)dx = \Big[x(a_0x^3/4 + a_1x^2/3 + a_2x/2 + a_3)\Big]_0^h$$
$$= h\,(a_0h^3/4 + a_1h^2/3 + a_2h/2 + a_3).$$

By substitution, it may be verified that

$$(2)\ [P(O) + 4P(h/2) + P(h)]\,h/6 = \int_0^h P(x)dx.$$

Thus, if the condition $P(x)$ is satisfied by a function of x, then its integral over the interval from $x = 0$ to $x = h$ is given by (2).

If a solid has parallel cross sections that are cubic, quadratic, linear, or constant functions of the distance from the origin, the exact volume can be found by taking the sum of the areas of the upper and lower bases plus four times the area of the mid-section and multiplying the result by one sixth the height. This is the prismoidal formula from antiquity, which then was derived by non-calculus methods. It yields the exact volumes of ellipsoids, circular cylinders, wedges, prisms, cones, and pyramids or any solid for which (1) is satisfied by parallel cross sections.

It is remarkable that, in addition to being applicable to many common solids, the convenience of a mid-section is employed. There is indication that the extension of this method to quartic functions will not yield such convenience.

Equation (2) may be used to find the area under any curve $y = P(x)$. This is the basis of Simpson's rule.[1]

Some applications of (2) will now be given.

Example 1. Volume of ellipsoid of dimensions a, b, c. We assume knowledge of the area of an ellipse.

$$V = (2c/6)\,[0 + 4(\pi ab) + 0] = (4/3)\pi abc$$

Example 2. Volume of the intersection of two right elliptical cylinders with axes intersecting at right angles and with a common dimension a, which is normal to the axes.

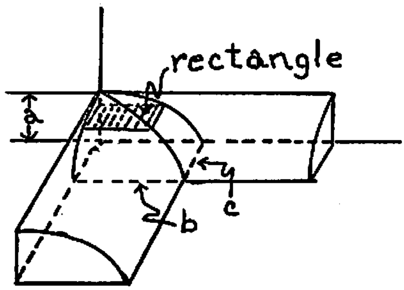
It can be established that the cross sections of the solid which

[1]See Granville, Smith, Longley, Elements of Differential and Integral Calculus, pp. 247-248.

are normal to the direction of the common dimension are rectangles and that condition (1) is satisfied.

Therefore,

$$V = (2a/6)[0 + 4\,(4bc) + 0]$$
$$= (16/3)\,abc.$$



By letting the dimensions a, b, and c in the above examples become a common dimension r, we have the volume of a sphere in Example 1 and the volume of the intersection of right circular cylinders in Example 2.

To find the area of the sphere, consider the volume divided into many small needlelike pyramids each containing a base on the sphere's surface and all having a common vertex at the center of the sphere. Applying the prismoidal formula to the needlelike pyramids, summing, and noting that in the limit the height of each pyramid becomes r, we have:

$$1/3\ r\,(\Sigma B_i) = 4/3\ \pi r^3$$
$$S = \Sigma B_i = 4\pi r^2$$

where $4\pi r^2$ is seen to be the desired area of the surface.

Similarly, since the center of the intersection of two right circular cylinders, exampe 2, is r distance from any plane tangent to the surface of the cylinders, we have the surface of the solid determined as

$$S_2 = \Sigma B_i = (16r^3/3)/[(1/3)r] = 16r^2$$

It is well known that the intersection of a plane and a circular cylinder is a curve called an ellipse. I leave it for the reader to prove that if this circular cylinder were cut open and laid flat, the ellipse would transform into a sine curve. This type of representation is used to visualize an interesting series of curves called Lissajous figures; the ellipse here corresponding to the case where the frequency ratio is 1:1.

If in the case of the two intersecting cylinders the radii are taken to be equal to one unit and if the common solid is viewed in the direction normal to the axes of the intersecting cylinders, it may be seen that there are eight 45° sections each of whose area is two units (see area determination above). These sections will transform into a plane figure bounded by the arc of a sine curve of unit amplitude and a straight line joining the end points of the arc. This result is consonant with the determination of the area under a sine curve by using the methods of the formal calculus.

Thus, we see that the prismoidal formula is not insignificant in its range of applicability.

A Note on Finding the Particular Solution to the Differential Equation $f(D)y = P(x)$ where $P(x)$ is a Polynomial

It is assumed here that the reader has only elementary knowledge of differential equations. The elementary method of finding the particular solution of the above equation (where it is understood that $f(D)$ is the notation used to indicate a function on the differential operator D where $D = d/(dx)$) is to employ the "method of undetermined coefficients". This method usually entails the formation of the correct undetermined coefficient equation which is then operated upon by $f(D)$ yielding results that must be grouped according to like variables; and then it is argued that, if the assumed particular solution is to be a solution, its coefficients are identical with those contained in the right hand side of the given differential equation; thus a set of simultaneous equations is obtained from which the coefficients are ultimately determined. Needless to say, the above method has many points where error can occur. The following method of attack on polynomial type will obviate the above procedure. In addition to minimizing the possibilities of error, it is superior with respect to time consumption. I call it the "Method of Successive Inspection" which as you shall see is an apt christening.

Consider first the case where $P(x)$ is a constant only. Then if $(a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = b_m$ is the given differential equation, its solution is seen to be by inspection $y_p = b_m/a_n$.

Next consider the case where $P(x)$ is the general polynomial represented by $P(x) = b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m$. To find the particular solution, we tentatively assume that it is ${}_1y_p = (b_0/a_n)X^m$ where in forming this function, we have considered only the coefficients of the highest power of $P(x)$ and the lowest differential power of $f(D)$. This is done simply because when this equation is substituted in the equation $f(D)y = P(x)$ and the indicated operations performed upon it the result has the highest power of x agreeing with $P(x)$. However, usually after this step has been performed, the lower powers of x will not agree with $P(x)$. We consider these lower powers of x as contributions toward our goal and assume another particular solution of lesser degree. In other words, by putting in the above tentative particular solution we have taken care of the highest power of $P(x)$ and have also obtained a total of m of the next highest power of x. But, if there were b_1 of this type in $P(x)$, then we must assume the next tentative particular solution as ${}_2y_p = [b_1 - m)/a_n]X^{m-1}$. By putting this in and performing the indicated operations upon it, the next highest power of x in $P(x)$will be satisfied. Furthermore, these portions are satisfied in descending powers of x, so that ultimately we have all that is required. The particular solution is then the sum of all the tentative particular solutions. This method is best illustrated by examples. Suppose that we are given the equation $(2D^2 + D - 2)y = -6x^2 + 10x + 8$, then our first tentative particular solution will be $y = 3x^2$. Note that coefficient is obtained by inspection, i.e., $-6/-2$ is 3. Applying $(2D^2 + D - 2)$ to $3x^2$, we have the result $-6x^2 + 6x + 12$. Note that the coefficient of x^2 agrees with that of $P(x) = -6x^2 + 10x + 8$ and hence no more need be said about it. But, when $3x^2$ is assumed, we obtain $6x$; and since we ultimately need $10x$ (from $P(x)$), this means we need an additional $4x$. To get this, we next assume a solution of $-2x$ (the coefficient being found from $4/-2 = -2$, by inspection). Operating on $-2x$ with the given operator, we have the result $4x - 2$. Thus the coefficient of x is satisfied. As a result of assuming these tentative particular solutions, we have so far a total constant term of $+12 - 2$ or 10. But we need only $+8$ as is seen by examining $P(x)$. Thus, our next assumption must be something that will yield -2 when operated upon. By inspection this is seen to be 1 for $-2/-2$ is 1. Therefore the total particular solution is the sum of the tentative particular solutions or $3x^2 - 2x + 1$.

The method described above is not very systematic. We now introduce certain methods for proper bookkeeping. For each tentative solution assumed, we have associated with it three columns. At the top of the leftmost column we place the tentative particular solution. We differentiate as many times as the highest differential degree of the given differential equation, listing these subsequent results in order in this column below the tentative solution. In the next column we place the coefficients of the $f(D)$ with the lowest exponent of D (usually zero) having its associated coefficient at the top of the column and the others following it in order. *Note: Be especially careful to include any zero coefficients as in the method of synthetic division.* Multiply corresponding quantities of the first two columns, listing their product in the third column. This column gives the result of applying the differential operator to our tentative particular solution. We have a check at this point. The results, if correct, will have the highest power of x agreeing with the highest power of x in $P(x)$. If this is not so, we made an error in our inspection. By considering the next highest power of x in the third column and comparing this with the needed amount in $P(x)$, we determine our next tentative solution. We draw another vertical line indicating a partition from our initial results and draw a set of three more columns from which we get additional results. Then, at each time by considering the amount we have in the third columns, we know how to assume next. Inspect the following example and then make up a problem and test the speed and accuracy with any method previously used.

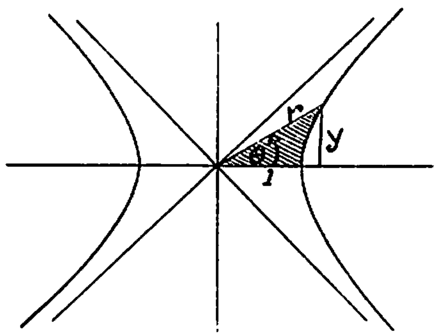
To find: y_p of $(D^2 - 2D + 4)y = 12x^3 - 2x^2 + 44$

$3x^3$	4	$12x^3$	$4x^2$	4	$16x^2$	9	4	36
$9x^2$	-2	$-18x^2$	$8x$	-2	$-16x$			
$18x$	1	$18x$	8	1	8			

$$y_p = 3x^2 + 4x^2 + 9$$

The Area of a Hyperbolic Sector

H. S. M. Coxeter
Professor, University of Toronto



The analogy between the circular and hyperbolic functions is most clearly seen in the fact that the parameter for the rectangular hyperbola

$$x = \cosh t, \qquad\qquad y = \sinh t$$

is equal to twice the area of the sector from (1, 0) to ($\cosh t$, $\sinh t$). It is therefore desirable to have a really simple proof for this well known fact.

Using Cartesian and polar coordinates simultaneously, we have

$$\begin{aligned}
&x = \cosh t, \qquad y = \sinh t,\\
&r = x \sec \theta = \cosh t \sec \theta,\\
&\tan \theta = y/x = \tanh t,\\
&\sec^2\theta \, d\theta = \operatorname{sech}^2 t \, dt,
\end{aligned}$$

so that twice the area of the sector is

$$\begin{aligned}
\int_0 r^2 \, d\theta &= \int_0 \cosh^2 t \sec^2\theta \, d\theta\\
&= \int_0 \cosh^2 t \operatorname{sech}^2 t \, dt\\
&= \int_0 dt\\
&= t.
\end{aligned}$$

The Problem Corner

EDITED BY FRANK C. GENTRY

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond the calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before March 1, 1955. The best solutions submitted by students will be published in the Spring, 1955, issue of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Frank C. Gentry, Department of Mathematics, University of New Mexico, Albuquerque, New Mexico.

Problems Proposed

71. *Proposed by Charles Pearsall, Student, Hofstra College, Hempstead, New York.*

Given a triangle partitioned into n equal areas by lines parallel to the base. If the altitude is h show that: 1) the partitioning is independent of the length of the base; 2) the location of the lines of division as measured along the altitude from the vertex is given by the sequence: $h\sqrt{1/n}, h\sqrt{2/n}, h\sqrt{3/n}, \cdots, h\sqrt{j/n}, \cdots, h$. $(j = 1, 2, \cdots, n.)$

72. *Proposed by Frank Hawthorne, Hofstra College, Hempstead, New York.*

A rectangle piece of sheet metal with integral dimensions a and b has equal squares of side c cut from each corner. The sides are then bent up to form a rectangular box with no top. The value of c is chosen so as to make the volume of the box a maximum. If c is rational and if a triangle can be formed with sides a, b, c, and angle $C = 60°$, show that c is integral.

73. *Proposed by Victor Thebault, Tennie, Sarthe, France.*

On the sides BC, CA, AB of the triangle ABC, determine the points M, N, P such that $BM = MN = NP = PA$.

74. *Proposed by H. D. Larsen, Albion College, Albion, Michigan.*

From a five-place table of logarithmic sines, calculate the five-place logarithms of 2, 3, and 7.

75. *Proposed by Carl V. Fronabarger, Southwest Missouri State College, Springfield, Missouri.* (Source unknown)

If it is known that eleven of twelve objects have the same weight, show how it can be determined by three weighings on a pair of balances which object has a weight differing from that of the others and whether it is heavier or lighter.

76. *Proposed by S. H. Sesskin, Student, Hofstra College, Hempstead, New York.*

Show that the square of the sum $1 + 2 + 3 + \cdots + n$, for any n, diminished by the square of the same sum for $n - 1$ is equal to n^3.

Solutions

55. *Proposed by Norman Anning, University of Michigan, Ann Arbor, Michigan.*

Show that a triangle given at random is three times as likely to have an obtuse angle as not to have one.

Solution by Morris Rosen, Hofstra College, Hempstead, New, York.

Two angles will determine a triangle in species. For an angle to belong to a triangle it must have a measure between 0° and 180°. The probability that two such angles have a sum less than 180° is ½. The probability that a randomly chosen angle of a triangle be less than 90° is ½. The probability that a second angle also be less than 90° is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. The probability that the third angle is also acute, *i.e.*, the probability that the sum of the first two be between 90° and 180° is $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$. Hence the probability that one angle be obtuse is $\frac{1}{2} - \frac{1}{8} = \frac{3}{8}$. Thus the probability that one angle be obtuse is 3 times the probability that they are all three acute.

Also solved by Harvey Fiola, Forman, North Dakota.

65. *Proposed by C. W. Trigg, Los Angeles City College, Los Angeles, California.*

What is the probability that the ten's digit of the square of a randomly selected integer will be odd?

Solution by Charles Thornton, Capuchino High School, San Bruno, California.

A randomly selected integer N may be represented by $a10^n + \cdots + 100x + 10y + z$. Upon squaring this expression, the last two terms are found to be $20yz + z^2$. Thus the ten's digit of N^2 is equal to the sum of the unit's digit of $2yz$ and the ten's digit of z^2. Since $2yz$ is always even, the ten's digit of N^2 is odd if and only if the ten's digit of z^2 is odd. This is true in two of the ten possible digital values of z. Therefore, the probability that the ten's digit of N^2 is odd is $1/5$.

Also solved by Robert Swanstrom, Los Angeles City College; Richard T. J. Mahoney, Bowling Green State University; and by Charles Pearsall, Morris Rosen, and S. H. Sesskin, all of Hofstra College.

66. *Proposed by C. W. Trigg, Los Angeles City College, Los Angeles, California.*

If the elements of each of the columns (or rows) of a determinant of order greater than two when taken in order form an arithmetic progression, then the value of the determinant is zero.

Solution by Charles Pearsall, Hofstra College, Hempstead, New York.

Subtracting any row (or column) from any two other rows (or columns) yields a determinant with the elements of two rows (or columns) proportional. Thus the determinant has the value zero.

Also solved by Robert Swanstrom, Los Angeles City College; Richard T. J. Mahoney, Bowling Green State University; Morris Rosen and S. H. Sesskin, Hofstra College.

67. *Proposed by D. M. Morrison, St. Joseph, Missouri.* (From S. I. Jones, *Mathematical Clubs and Recreations*, p. 111).

The combined ages of Jane and Mary total 44. Now Jane is twice as old as Mary was when Jane was half as old as Mary will be when Mary is three times as old as Jane was when Jane was three times as old as Mary. How old are they?

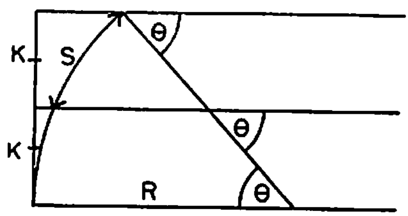
Solution by Larry Rubinoff, Wayne University, Detroit, Michigan.

Let x be Jane's present age and y Mary's present age. Then $x + y = 44$. Now Jane is twice as old as Mary was a years ago, *i.e.*, $x = 2(y - a)$, when Jane was half as old as Mary will be b years hence, *i.e.*, $(x - a) = (y + b)/2$, when Mary is three times as old as Jane was c years ago, *i.e.*, $(y + b) = 3(x - c)$, when Jane was three times as old as Mary, *i.e.*, $(x - c) = 3(y - c)$. Eliminating a, b, and c and solving for x and y leads to Jane's age as $27\frac{1}{2}$ and Mary's age as $16\frac{1}{2}$.

Also solved by Thomas W. Martin, Kansas State Teachers College, Pittsburg, Kansas; Hugh A. Mahoney, Rochester Institute of Technology, Rochester, New York; Charles Pearsall, Morris Rosen, and S. H. Sesskin, all of Hofstra College; and John M. Usry, University of New Mexico.

68. *Proposed by James Woods, Student, University of New Mexico, Albuquerque, New Mexico.*

How long is the arc S in the figure?



Solution by S. H. Sesskin, Hofstra College, Hempstead, New York.

Since the length of arc is equal to the product of the radius and the subtending angle, $S = R(\theta - x)$ where x is the angle subtending the remaining arc in the illustration. Now $\theta = Arc\ sin\ 2K/R$, $x = Arc\ sin\ K/R$. Therefore

$$S = R(Arc\ sin\ 2K/R - Arc\ sin\ K/R).$$

Also solved by Charles Thornton, Capachino High School, San Bruno, California; Larry Rubinoff, Wayne University; Charles Pearsall and Morris Rosen, Hofstra College.

69. *Proposed by the Editor.* (From Nicholson's, *Elements of Plane and Spherical Trigonometry,* The Macmillan Company, 1911).

On the bank of a river, there is a column 200 feet high supporting a statue 30 feet high; the statue to an observer on the opposite bank subtends an equal angle with a man 6 feet high standing at the base of the column. Required, the width of the river.

Solution by James N. Rogers, Butler University, Indianapolis, Indiana.

Let α be the angle of elevation of the top of the statue from a point on the opposite bank and let β be the angle subtended at that point by the statue and the man. Then

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{200}{x}$$

where x is the width of the river. Substituting for $\tan\alpha$ and $\tan\beta$ leads to

$$\frac{230/x - 6/x}{1 + (230/x)(6/x)} = \frac{200}{x}$$

Solving this equation for x leads to $x = 107$ feet approximately.

Also solved by Cameron Macdonald, Charles Pearsall, Morris Rosen, and S. H. Sesskin, all of Hofstra College.

70. *Proposed by the Editor.*

The point A is three feet from the center O of a circle of radius 5 feet. The point B is 4 feet from O and 5 feet from A. Locate the point P on the circle so that the distance $AP + PB$ may be as small as possible.

Solution by Charles Pearsall, Hofstra College, Hempstead, New York.

The minimum distance $AP + PB$ will occur when the lines AP and PB make equal angles with the tangent to the circle at P. Hence AP and BP make equal angles α with the radius OP. Let β be the angle BOP. Then in triangle AOP, $\cos\beta\,/\,AP = \sin\alpha/3$ and in triangle OBP, $BP\cos\alpha = 5 - 4\cos\beta$. Considering the area of the quadrilateral $OBPA$ as the sum of the areas of the two triangles in two ways we have $BP \cdot AP \sin 2\alpha + 12 = 15\cos\beta + 20\sin\beta$ from which, by substituting from the above expressions and simplifying, we obtain $12\cos 2\beta = 25\cos(\beta + \text{Arc}\tan 4/3)$. This equation may be solved for β approximately by Newton's Method. Its solution is $\beta = 10°\ 3.4'$. From triangle AOP, $\overline{AP}^2 = 34 - 30\sin\beta$ or $AP = 5.36$ feet. From triangle OBP, $\overline{BP}^2 = 41 - 40\cos\beta$, or $BP = 1.27$ feet. Hence the minimum distance is 6.63 feet.

Also solved by Morris Rosen of Hofstra College who set up the distance to be minimized as a function of the angle BOP, set its derivative equal to zero thus obtaining a sixth degree equation which he solved approximately by Newton's Method.

The Mathematical Scrapbook

EDITED BY H. D. LARSEN

Poesy has been called a creation, a making, a fiction; and the Mathematics have been called . . . the sublimest and the most stupendous of fictions.

—THOMAS HILL

=△=

In a number of years equal to the number of times a rhinoceros's mother is as old as the rhinoceros, the rhinoceros's father will be as many times as old as the rhinoceros is years old now. The rhinoceros's mother is twice as old as the rhinoceros will be when the rhinoceros's father is twice as old as the rhinoceros will be when the rhinoceros's mother is less by the difference in ages between the mother and father than three times as old as the rhinoceros will be when the rhinoceros's father is one year less than twelve times as old as the rhinoceros is when the rhinoceros's mother is eight times the age of the rhinoceros.

When the rhinoceros is as old as the rhinoceros's mother will be when the difference in ages between the rhinoceros's father and the rhinoceros is less than the age of the rhinoceros's mother by twice the difference in ages between the rhinoceros's father and the rhinoceros's mother, the rhinoceros's mother will be five times as old as the rhinoceros will be when the rhinoceros's father is one year more than ten times as old as the rhinoceros is when the rhinoceros is less by four years than one-seventh of the combined ages of his father and mother.

Find their respective ages. (For the purposes of this problem, the rhinoceros may be considered to be immortal.) —A.S.E. Ackermann in the *Mathematical Gazette.*

=△=

Why Teachers Get Gray

PROBLEM: $(x^6)(x^3) \div (x^3)^2 = ?$

"SOLUTION": $x^{18} \div x^6 = x^3$

"Happy the lot of the pure mathematician. He is judged solely by his peers and the standard is so high that no colleague can ever win a reputation he does not deserve. No cashier writes articles in the Sunday Times complaining about the incomprehensibility of Modern Mathematics and comparing it unfavourably with the good old days when mathematicians were content to paper irregularly-shaped rooms or fill bath-tubs with the wastepipe open. Better still, since engineers and physicists have occasionally been able to put his equations to destructive use, he is even given a chair in a State University."

—W. H. Auden

=△=

Cryptarithm

This "Three Threes" problem in multiplication has a unique solution.

$$\begin{array}{r} x\,x\,x \\ x\,x \\ \hline x\,x\,3\,x \\ x\,3\,x \\ \hline x\,x\,3\,x\,x \end{array}$$

=△=

Mantisa or *Mantissa,* according to Lewis and Short, is a Latin (Tuscan) word signifying a "worthless addition, a makeweight." The decimal part of the logarithm is evidently the excess or surplus. — *Mathematical Gazette*

=△=

In a well known university, the names of the president, a professor, an instructor, and a janitor are Mr. James, Mr. Jones, Mr. Haines, and Mr. Ross, but not necessarily in that order. In the same university there are four students, James, Jones, Haines, and Ross.

The student with the same name as the professor belongs to the fraternity of which Ross is a member.

The daughter-in-law of Mr. Jones lives in Philadelphia.

The oldest son of the president is seven years old.

The wife of the janitor has never seen Mr. Ross.

The father of one of the students always confuses Haines and

Jones in class but is not absent minded.

Mr. Haines is the father-in-law of the instructor and has no grandchildren.

Match the names.

$=\Delta=$

What is the angle between the faces of a tetrahedron?

$=\Delta=$

$$1 = 1^3$$
$$5 + 9 + 13 = 3^3$$
$$17 + 21 + 25 + 29 + 33 = 5^3$$
$$37 + 41 + 45 + 49 + 53 + 57 + 61 = 7^3$$

etc.

$=\Delta=$

How should we define x^n when n is a positive whole number? It is not correct, for example, to define x^5 as "x multiplied by itself 5 times." B. Evans suggests that x^5 be defined as 1 multiplied by x five times: $x^5 = 1 \cdot x \cdot x \cdot x \cdot x \cdot x$. There is some merit in this suggestion. The interpretation of x^0 as 1 follows rather naturally.

$=\Delta=$

Calculus

(To the tune of *Clementine*)

In a college, seeking knowledge,
Maids for mathematics pine.
'Tis no fable, Calculus's able
All grey matter to refine.

Chorus

Of my darling differentials,
Higher orders you combine.
Then you're lost and gone forever
Irrespective of your sign.

In a mathematics classroom
Differentiation's fine.
Why be floored? Why be bored?
Find the slope of every line.

(Chorus)

Maximum of f is called for.
Carefully derive f prime.
Be a hero. Equate to zero,
And you'll find it every time.
(Chorus)
Areas and also volumes
Are quite easy to divine.
Use some gumption; place the function
Underneath the integral sign.
(Chorus)
In a problem,there's no goblin.
Courage you must ne'er resign.
If convergent, not divergent,
Series may your f define.
(Chorus)

—CLARA E. SMITH

=Δ=

A Mathematical Game

What mathematical terms are suggested below?

1. A color + a slang word for gentleman.
2. Word used with either + noise + devoured.
3. Abbreviation for advertisement + a bird + a small coin.
4. Word used for many + name for a single term.
5. Abbreviation for absent + nickname for sister + *a*.
6. English word for Monsieur + a word meaning to choose.
7. Accident + a color + a sea bird.
8. Word for four + contraction for he would + word for hurry.
9. Second musical note in the musical scale + fourth letter of the alphabet + plural objective personal pronoun.
10. Word for four + rodent + noise of a clock.
11. Fifth letter of the alphabet + exterior of the mouth.
12. Word for two + girl's name + metric measure of weight.
13. Attempt + type of worm.
14. Slang word + *ome* + a part of a forest.
15. Boy's name + *ge* + mule talk.
16. Equal + short fooball pass.
17. Word for two + metric measure of length.

18. Prefix of former wife + great American poet + tool of a fisherman.
19. Boy's name + optical organ + flower.
20. Abbreviation for professional + last syllable of legislative body + avoid.

(Answers: tangent, ordinate, adjacent, polynomial, abscissa, circle, rectangle, tetrahedron, radius, quadratics, ellipse, parallelogram, triangle, geometry, algebra, equilateral, parameter, exponent, maximum, progression.)

=△=

If you do not expect the unexpected, you will not find it; for it is hard to be sought out, and difficult.

—HERACLITUS

=△=

"In mathematics, as in other fields, to find oneself lost in wonder at some manifestation is frequently the half of a new discovery."

—P. G. L. DIRICHLET

=△=

"But there is another reason for the high repute of mathematics; it is mathematics that offers the exact natural sciences a certain measure of security, which, without mathematics, they could not attain."

—ALBERT EINSTEIN

=△=

"Ah, but my computations, people say
Reduce the year to better reckoning—nay
'Tis only striking from the calendar
Unborn tomorrow and dead yesterday."

—OMAR KHAYYAM

=△=

"Mathematics is the class of all propositions of the form P implies Q."

—H. POINCARE

=△=

"Nature and Nature's Law lay hid in night
God said, 'Let Newton be', and all was light."

—POPE

=△=

"A mathematical science is any body of propositions arranged according to a sequence of logical deductions."

—J. W. YOUNG

The Book Shelf

EDITED BY FRANK HAWTHORNE

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Frank Hawthorne, Hofstra College, Hempstead, New York.

Squaring the Circle and Other Monographs by E. W. Hobson, Hilda P. Hudson, A. N. Singh, and A. B. Kempe, Chelsea Publishing Co., (Washington Bridge Station) New York 33, 360 pp. $3.25.

This book collects into a single volume four monographs which have already been published separately. Three of these are well within the range of the undergraduate mathematics major; the fourth, on non-differentiable functions, is on a higher level. Both teacher and student of undergraduate mathematics will find in these monographs a variety of interesting and instructive material.

The first of these monographs, "Squaring the Circle", by E. W. Hobson, traces the long history of man's effort to understand the nature of the number pi. The author divides this history into three periods. The first long period of empirical and geometrical attempts came to an end with the introduction of the calculus. The second period was characterized by the use of the calculus in its various applications to calculate a more accurate value of pi. It was not until the third period, when the mathematician began to suspect the existence of transcendental numbers and did something about it, that the true nature of pi emerged. This period, and the attempts of mathematicians to "square the circle", came to an end with the discovery of the transcendental nature of pi.

This monograph touches on many phases of elementary mathematics and is truly a rich source of interesting and inspirational material. Perhaps the reader as he contemplates the almost four thousand year siege of the problem will appreciate the author's observation that "the quality of the human mind, considered in its collective aspect, which most strikes us in surveying this record is its colossal patience."

The second monograph, "Ruler and Compasses", by Hilda P. Hudson, is a thorough treatment of the problem of construction possible by means of the Euclidean ruler and compass. The author develops early the analytical equivalents of ruler constructions and of ruler and compass constructions. As one might suspect, the impossibility of duplication of the cube and tri-section of the general angle are established. Both ruler constructions and ruler and compass constructions from given geometrical data are then considered in detail, the last chapters dealing with special conditions from which ruler and compass constructions may be done with just one of these instruments. This monograph could well form the content of a course in college geometry.

The third monograph, "The Theory and Construction of Non-differentiable Functions", by A. N. Singh, brings together the results in this field of investigation to a comparatively recent date (1935). The four chapter (lecture) headings, Functions Defined by Series, Functions Defined Geometrically, Functions Defined Arithmetically, Properties of Non-differentiable Functions, indicate the general content of this discussion. While the undergraduate will likely find this monograph above his level, he might be interested in scanning it to see the general trend. In doing so he will encounter many of the great names in mathematics in the last century.

Readers of THE PENTAGON will recall that the fourth monograph, "How to Draw a Straight Line", by A. B. Kempe, was published in its entirety in the Spring, 1952, issue. This monograph, originally a lecture to a group of English science teachers in 1876, was obviously inspired by the interest created by the introduction to English scientific circles by Sylvester in 1874 of Peucellier's linkage by means of which a straight line may be drawn. Besides this one, earlier linkages which approximate a straight line and others with interesting properties are described. Among these are the pantograph and the linkage by which an angle may be divided into n equal parts. Diagrams of the linkages are given and it is likely that a student with a mathematical and mechanical turn of mind would find construction of some of these models highly interesting and instructive.

—E. A. HEDBERG

Dictionary of Mathematical Sciences, Volume I (German - English), by Leo Herland, Frederick Unger Publishing Co. (105 E. 24th St.), New York 10, 235 pp. $3.25.

Note—Volume II (English-German) was received by The Book Shelf Editor too late to get it reviewed. It is available for $4.50.

In the preface it is claimed that this is the first modern bilingual dictionary centering about the major subjects of mathematics and geometry. It aims to cover a number of mathematical fields, some in considerable detail, such as mathematical logic, statistics, and commercial arithmetic. It does not claim to cover completely the highly specialized terms of the various branches of mathematics and geometry, although an attempt has been made to include all important terms. Suggestions for additions are solicited. The dictionary does not contain words in daily use in German, except in so far as they might have a technical mathematical meaning, i.e., *Kegel* cone is included but not *Kegel* pin.

When this reviewer, a teacher of German language and literature, conducted a class of undergraduate senior mathematics majors last year in reading *Mengenlehre* by H. Hausdorff (New York, Dover Publications), we used whatever dictionaries, literary and technical, were immediately available. As I now look over this book on the theory of sets with Herland's dictionary at hand, I realize how our slow and often painful work could have been greatly facilitated. While we did arrive at a satisfactory translation and understanding of the text in most places, we nevertheless often had to compromise and be satisfied with approximate terms and idioms instead of the preciseness of expression which mathematics requires. This dictionary seems to have the advantage over other technical dictionaries of giving exact English words and expressions within specific contexts.

I feel that this dictionary can be an individual aid not only to mathematics students preparing for a Ph.D. reading examination but, what is more important, for the student, teacher, or professional person who needs and wants to read some of the many articles and books which are unavailable in their original form in any other language but German.

—FREDERICK J. CHURCHILL

Math Can Be Fun by Louis G. Brandes, The author, Encinal H. S. (210 Central Ave.), Alameda, California. $1.50.

Here are collected a large number of tricks, puzzles, oddities, and other "off-the-beaten-track" items of mathematics by an experienced classroom teacher. These are presented for use as enrichment material at the upper grade and junior high school levels. Many readers of THE PENTAGON either are or soon will be teachers in schools of this type, and this book should be of considerable value to them.

The reviewer teaches in a junior high school where he made extensive use of this book during the past school year. His students understood and enjoyed this "different" mathematics. Some few were enthusiastic, most benefited, and surely no one suffered from its use.

Probably most good, experienced teachers gradually accumulate a store of such materials. The beginning teacher should find this convenient compilation especially helpful, and the "old hand" will find useful additions for his collection.

The style is somewhat unusual. Numerous rebus-like phrases occur (thus "for you to see" appears as "4 U 2 C"), and some students found this aspect challenging. The reproduction is by rotary duplicator and is not always as clear as it might be, but this does not detract seriously from the worth of this work.

—Robert Dever

Mathemagic by Royal Vale Heath, Dover Publications Inc. (1780 Broadway) New York 19, New York. Paper, 126 pp., $1.00.

In a sort of Nihil Obstat to this attractive reprint, we are told that it is "not detrimental to the interests of the Professional Magicians of America". The reviewer assumed this to mean that no revelations of procedures by means of which such persons make a living were disclosed. His first reaction was that a similar Imprimatur by the American Mathematical Society might well be in order.

However, after more sober thought, the very lack of proof noted throughout the book may well be the key to its value to the undergraduate. Thus, any college student who goes through this book and substantiates each of the arbitrary procedures contained in it will have had a valuable experience. For instance, the number 142857 is discussed with particular reference to the cyclic permutations of digits which result in its first six multiples, but no mention is made that this property is an "of course" result of the fact that these digits form the repeating portion of the decimal equivalent of 1/7.

Much of the material in this book is simple; some even trivial if looked at "the right way". It may be useful to the student who needs practice in "which way to look".

—F. HAWTHORNE

Books Received by the Bookshelf Editor

An Analytical Calculus, Vol. II and III, by E. A. Maxwell, Cambridge University Press, (32 E. 57th St.), New York 22, 270 pp. and 195 pp., $3.50 and $2.75.

Analytic Geometry by Edward S. Smith, Meyer Salkover, Howard K. Justice, John Wiley and Sons, (440 4th Ave.) New York 12, xiii + 306 pp., $4.00.

Exercises in Workshop Mathematics by Leslie Smith, Cambridge University Press, (35 E. 57th St.), New York 22, v + 89 pp., $.90.

Introduction to Mathematical Statistics by Paul G. Hall, John Wiley and Sons, (440 4th Ave.), New York 12, xi + 331 pp., $5.00.

Introduction to Mathematical Thinking by Frederick Waismann, Frederick Ungar Publishing Co., (105 E. 24th St.), New York 10, x + 260 pp., $4.50.

Relaxation Methods by D. N. deG. Allen, McGraw Hill, (330 W. 42nd St.), New York 10, ix + 251 pp., $7.50.

1955 BIENNIAL CONVENTION

It is hoped that every chapter is planning to send a delegation to the 1955 Biennial Convention. The convention will be held at Nebraska State Teachers College, Wayne, Nebraska, on May 6, 7, 1955.

Students are urged to submit papers for consideration for presentation at the convention. Complete directions for papers to be presented to the Selection Committee can be found in the Spring, 1954, issue of THE PENTAGON.

INSTALLATION OF NEW CHAPTER

EDITED BY J. M. SACHS

THE PENTAGON is pleased to report the installation of California Beta Chapter of Kappa Mu Epsilon.

CALIFORNIA BETA CHAPTER

Occidental College, Los Angeles

California Beta Chapter was installed and twenty-four charter members of that chapter were initiated in a ceremony at Occidental College on May 28, 1954. The ceremony was conducted by Professor Chester G. Jaeger of California Alpha. Professor Jaeger was assisted by Professor Mabel Barnes of Occidental, Professor Hugh Hamilton of Pomona and Professor Elmer Tolsted of Pomona. Professor Barnes, who was a charter member of Nebraska Alpha, will serve as corresponding secretary for California Beta.

The charter members of California Beta are: Jack R. Becker, James A. Berget, John F. Charnell, James E. Conel, Thomas L. Drouet, Claude D. Fiddler, David A. Ford, Paul M. Foster, Norman A. Harris, Kelly L. Hester, Kenneth M. Hoffman, Donald W. Johnson, Paul B. Johnson (faculty), Robert T. Koyamatsu, Raymond C. Libby, Barbara C. Matson, Kikuko Matsumoto, Donald S. Miller, Elmer E. Moots (faculty), Florence V. Reese, Richard E. Robertson, Christine A. West, Mary M. Wootan, Robert E. Learned.

A banquet at the Student Union followed the ceremony. Professor Chester G. Jaeger addressed the group on the topic of "Mathematical Characters".

We wish to extend a hearty welcome to California Beta. The staff of THE PENTAGON joins with the national officers in expressing our best wishes for success to our newest chapter.

" If a lunatic scribbles a jumble of mathematical symbols it does not follow that the writing means anything merely because to the inexpert eye it is indistinguishable from higher mathematics."

—E. T. BELL

Brief History of Kappa Mu Epsilon *

The rapid growth of universities and colleges in the United States in the latter part of the 19th Century led to the development of professional societies in every field. The fields of law, medicine, science, engineering, teaching, etc., developed societies with memberships numbering thousands. Local clubs were formed in larger educational institutions to promote interest in special departmental objectives. Desire for affiliation with other groups of similar ideals led to the organization of these local clubs into national and state societies. In mathematics, Pi Mu Epsilon became the national fraternity for instructors and advanced students who were in educational institutions offering graduate work in mathematics. The first fraternities open to mathematics students on the undergraduate level seem to have been primarily science fraternities. These organizations did not appeal very strongly to those whose interest was in symbolic thinking.

The need for a national mathematics fraternity which would appeal essentially to the undergraduate was recognized by both the instructors and students of mathematics. Dr. Emily Kathryn Wyant is considered the founder of Kappa Mu Epsilon, which was organized to fill this need. Dr. Wyant was a graduate of the University of Missouri and was a member of Pi Mu Epsilon. In the fall of 1930 she went to Northeastern Oklahoma State Teachers College as a professor of mathematics. She went to work with vigor and enthusiasm to transform the mathematics club there, which had been in existence since 1927, into the first chapter of a national fraternity. Professor L. P. Woods, who was head of the Department of Mathematics and Dean of Men, was a valuable co-worker in working out the many details pertaining to the project. He was largely responsible for the completed rituals used for the initiation of members and installation of officers.

Since the first serious group of students of mathematics to be organized into a fraternity was the Society of Pythagoras, it was decided that the emblems of Kappa Mu Epsilon would be those of

*EDITOR'S NOTE: Material for this brief history of Kappa Mu Epsilon was taken from data furnished by Miss Laura Z. Greene, National Historian; national convention reports; the National Constitution; an article entitled "Early Years of Kappa Mu Epsilon" by J.A.G. Shirk which appeared in the Spring, 1942, issue of THE PENTAGON; and the obituaries of Dr. Wyant, Dr. Condit, and Dr. Shirk, which appeared respectively in the Fall, 1942; Spring, 1948; and Fall, 1951, issues of THE PENTAGON.

the Pythagoreans as nearly as possible. The emblems chosen for the new fraternity were the five-pointed star and the pentagon. Since $\rho = a \sin 5\theta$ (a five-leaved rose) fits into the pentagon, the wild rose which usually has five petals was chosen as the fraternity flower. The pink of the wild rose and the silver of the star were chosen for the colors. In making the crest it seemed advisable that the sciences using mathematics should be recognized, so five emblems were selected for these and placed around the star on the shield. The motto, translated into English, is "Unfold the glory of mathematics." The objective of the organization since its inception has been the fulfillment of this motto.

Dr. Wyant and Prof. L. P. Woods along with 22 other faculty members and students became charter members of Oklahoma Alpha, Northeastern Oklahoma State Teachers College, Tahlequah, April 18, 1931, thereby making the dream for the fraternity a reality. On the same day the national organization elected the following officers: President Pythagoras, Dr. Kathryn Wyant; Vice-president Euclid, Professor Ira S. Condit; Secretary Diophantus, Miss Lorene Davis; Treasurer Newton, Professor L. P. Woods; Historian Hypatia, Miss Bethel DeLay.

The following newspaper account was given of the transformation of the Mathematics Club at Tahleuqah into Kappa Mu Epsilon:

> "The king is dead, long live the king." This may be applied to the "Mathematics Club," of Northeastern. As "The Pentagon" the club is dead; as Kappa Mu Epsilon, it lives.
>
> Kappa Mu Epsilon had its Founders' Day banquet last Saturday evening, April 18, at the Hotel Thompson. At that time 20 people took the pledge and signed the constitution, thus becoming charter members of the Oklahoma Alpha chapter. The banquet room was decorated in pink and white. Wild flowers and red buds were in the corners of the room. Tall white candles and floor lamps gave a soft light. Rose nut cups and hard painted place cards added to the color of the room. The menu as written inside the place cards told of paraboloids, ruled surfaces, and even parallel lines that were to be eaten at the mathematical table.

Paul Lewis was the Radical Axis (toastmaster) of the evening. The program consisted of the fraternity song and the following talks: Parabolas (parables) by "Bus" Layton; Comic (conic) sections by Dr. Kathryn Wyant; Lipstick (elliptic) conditions by Clara Green and Transformations by Dean L. P. Woods.

The formal transformation from The Pentagon into Kappa Mu Epsilon was directed by Mr. Woods.

Dr. Wyant had carried on an extensive correspondence with faculty members at other colleges in regard to the founding of a national fraternity such as this. Among those with whom she corresponded were Dr. Ira S. Condit of Iowa State Teachers College, Cedar Falls, and Dr. J. A. G. Shirk of Kansas State Teachers College, Pittsburg. Dr. Condit participated in the preliminary negotiations for the founding of the fraternity and indicated such an interest that he was elected the first vice-president. The enthusiasm for this organization spread on his own campus with the result that the second chapter of Kappa Mu Epsilon, Iowa Alpha, was installed at Iowa State Teachers College, Cedar Falls, May 27, 1931. Kansas Alpha, the third chapter, was installed January 30, 1932, at Kansas State Teachers College, Pittsburg. Next was Missouri Alpha, Southwest Missouri State College, Springfield, May 20, 1932.

During the development of Kappa Mu Epsilon at Tahlequah, the Mississippi State College for Women and the Mississippi State College were progressing with their plans for inaugurating a mathematical fraternity. Officers of Kappa Mu Epsilon urged the groups to give up their contemplated organization and become affiliated with their society whose organization was entirely completed. They agreed to unite; Mississippi Alpha, Mississippi State College for Women, Columbus, was installed May 30, 1932, and the Mississippi Beta, Mississippi State College, State College, was installed December 14, 1932. This brought the membership to six chapters by the end of 1932.

Much of the early success of Kappa Mu Epsilon is attributed to the dynamic and inspiring leadership of Dr. J. A. G. Shirk. He succeeded Dr. Wyant as National President in 1935 and served in that capacity until 1939. "The Early Years of Kappa Mu Epsilon," an article which appeared in the Spring, 1942, issue of THE PENTAGON, was written by Dr. Shirk. Dr. Ira S. Condit helped formulate the policies of the organization and set up the first conventions. Dr. Wyant, Dr. Condit, and Dr. Shirk are now dead. However, they will always be remembered in Kappa Mu Epsilon for their many contributions in the development of the fraternity.

Another early leader in Kappa Mu Epsilon and one who has made great contributions to the organization over a long period of time has been Miss E. Marie Hove. She has served in an official capacity in the organization from 1933 to the present. From 1933 to 1937 she served as national historian and she has served as national secretary from 1937 to the present time.

Official business at the national convention is transacted by delegates elected by each chapter and members of the National Council—the National President, National Vice-president, National Secretary, National Treasurer, National Historian, and the immediate Past President. The convention is held every two years for the purpose of electing officers, voting on motions to amend the constitution or by-laws, voting on motions relating to establishment of new chapters, and deciding matters concerning the fraternity.

It is a noteworthy fact that there has been an ever-increasing number of student papers presented at the national conventions. This furnishes evidence that the fraternity is fulfilling its four-fold objective which is:

1. to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program;
2. to help the undergraduate realize the important role that mathematics has played in the development of the western civilization;
3. to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demand for logical and rigorous modes of thought;
4. to provide a society for the recognition of outstanding achievement in the study of mathematics in the undergraduate level.

The first convention was held at Northeastern State Teachers College, Tahlequah, Oklahoma, April 21 and 22, 1933. By this time there were eight chapters on the roll. In the interval from 1941 to 1947 no conventions were held due to conditions caused by World War II. However, during this time, the National Council met to handle the business of the fraternity. At a meeting in 1947 the Council voted to have the ritual revised, since the old ritual did not express the depth of purpose that many Kappa Mu Epsilon members felt it should. Dr. C. V. Newsom, who was then at Oberlin College, Oberlin, Ohio, was asked to prepare a draft of a revised ritual for the consideration of the Council. Using the draft prepared by Dr. Newsom, the Council revised the initiation ritual in 1948. This revision is the one in use at the present time.

The first issue of THE PENTAGON appeared in the fall of 1941; an official journal for Kappa Mu Epsilon had been authorized at the fifth biennial convention April 18 and 19, 1941. The task of planning the journal and formulating its editorial policy was entrusted to Dr. C. V. Newsom, who was at the University of New Mexico, Albuquerque. Dr. Newsom served as editor until his resignation in 1943. Dr. O. J. Peterson, who was then National President, said, "The publication of such a journal is probably the most significant project ever undertaken by the fraternity." This magazine was to cater to the needs of college students of mathematics and serve as a medium through which outstanding student papers could be published. Dr. Harold D. Larsen, University of New Mexico, followed Dr. Newsom as editor. The first issue under his editorship appeared in the fall of 1943. The Spring, 1953, issue of THE PENTAGON had as its editor Dr. Carl V. Fronabarger, Southwest Missouri State College, Springfield. Dr. Fronabarger is the present editor of the journal.

Since the first chapter was installed in 1931 with twenty-four members, the organization has grown to a membership of over 10,000—fifty-two chapters have been installed in twenty-three states. Dr. J. A. G. Shirk has aptly said, "History renders the ultimate verdict as to the value of any movement, and the growth and the influence of Kappa Mu Epsilon . . . give a portent of its greater contributions in the decades yet to come."

National Officers

President

Kathryn Wyant*	1931 - 1935
J. A. G. Shirk*	1935 - 1939
C. V. Newsom	1939 - 1941
O. J. Peterson	1941 - 1943
E. R. Sleight	1943 - 1947
Henry Van Engen	1947 - 1951
Charles B. Tucker	1951 -

Vice-president

Ira S. Condit*	1931 - 1935
O. J. Peterson	1935 - 1937
C. V. Newsom	1937 - 1939
E. H. Taylor	1939 - 1941
E. D. Mouzon	1941 - 1943
Fred W. Sparks	1943 - 1947
H. R. Mathias	1947 - 1949
Harold D. Larsen	1949 - 1951
Cleon C. Richtmeyer	1951 -

Secretary

Lorene Davis	1931 - 1933
J. A. G. Shirk	1933 - 1935
C. E. Smith	1935 - 1937
E. Marie Hove	1937 -

Treasurer

L. P. Woods	1931 - 1933
C. N. Mills	1933 - 1935
L. E. Pummill	1935 - 1939
Henry Van Engen	1939 - 1943
Loyal F. Ollmann	1943 - 1953
Leslie Madison	1953 -

Historian

Bethel De Lay	1931 - 1933
E. Marie Hove	1933 - 1937
Orpha Ann Culmar	1937 - 1943
Sister Helen Sullivan	1943 - 1947
Cleon C. Richtmeyer	1947 - 1951
Laura Z. Greene	1951 -

*Deceased

National Conventions

1. Northeastern State Teachers College Tahlequah, Oklahoma
April 21 and 22, 1933
2. Kansas State Teachers College Pittsburg, Kansas
April 26 and 27, 1935
3. Agricultural and Mechanical College State College, Mississippi
April 30 and May 1, 1937
4. Eastern Illinois State Teachers College Charleston, Illinois
April 28 and 29, 1939
5. Central Missouri State Teachers College Warrensburg, Missouri
April 18 and 19, 1941
6. Illinois State Normal University Normal, Illinois
April 11 and 12, 1947
7. Washburn University Topeka, Kansas
April 10, 11, and 12, 1949
8. Southwest Missouri State College Springfield, Missouri
April 26 and 27, 1951
9. St. Mary's Lake Camp Battle Creek, Michigan
Michigan Alpha, Michigan Beta, and Michigan Gamma were hosts.
April 17 and 18, 1953
10. Nebraska State Teachers College Wayne, Nebraska
May 6 and 7, 1955

ACTIVE CHAPTERS of KAPPA MU EPSILON*

Chapter	Location	Installation Date
Oklahoma Alpha	Northeastern State College, Tahlequah	April 18, 1931
Iowa Alpha	State Teachers College, Cedar Falls	May 27, 1931
Kansas Alpha	State Teachers College, Pittsburg	Jan. 30, 1932
Missouri Alpha	Southwest Missouri State College, Springfield	May 20, 1932
Mississippi Alpha	State College for Women, Columbus	May 30, 1932
Mississippi Beta	State College, State College	Dec. 14, 1932
Nebraska Alpha	State Teachers College, Wayne	Jan. 17, 1933
Illinois Alpha	Illinois State Normal University, Normal	Jan. 26, 1933
Kansas Beta	State Teachers College, Emporia	May 12, 1934
New Mexico Alpha	University of New Mexico, Albuquerque	March 28, 1935
Illinois Beta	Eastern Illinois State College, Charleston	April 11, 1935
Alabama Beta	State Teachers College, Florence	May 20, 1935
Alabama Gamma	Alabama College, Montevallo	April 24, 1937
Ohio Alpha	Bowling Green State University, Bowling Green	April 24, 1937
Michigan Alpha	Albion College, Albion	May 29, 1937
Missouri Beta	Central Missouri State College, Warrensburg	June 10, 1938
South Carolina Alpha	Coker College, Hartsville	April 5, 1940
Texas Alpha	Texas Technological College, Lubbock	May 10, 1940
Texas Beta	Southern Methodist University, Dallas	May 15, 1940
Kansas Gamma	Mount St. Scholastica College, Atchison	May 26, 1940
Iowa Beta	Drake University, Des Moines	May 27, 1940
New Jersey Alpha	Upsala College, East Orange	June 3, 1940
Tennessee Alpha	Tennessee Polytechnic Institute, Cookeville	June 5, 1941
New York Alpha	Hofstra College, Hempstead	April 4, 1942
Michigan Beta	Central Michigan College, Mount Pleasant	April 25, 1942
Illinois Gamma	Chicago Teachers College, Chicago	June 19, 1942
New Jersey Beta	State Teachers College, Montclair	April 21, 1944
Illinois Delta	College of St. Francis, Joliet	May 21, 1945
Michigan Gamma	Wayne University, Detroit	May 10, 1946
Kansas Delta	Washburn Municipal University, Topeka	March 29, 1947
Missouri Gamma	William Jewell College, Liberty	May 7, 1947
Texas Gamma	Texas State College for Women, Denton	May 7, 1947
Wisconsin Alpha	Mount Mary College, Milwaukee	May 11, 1947
Texas Delta	Texas Christian University, Fort Worth	May 13, 1947
Ohio Gamma	Baldwin-Wallace College, Berea	June 6, 1947
Colorado Alpha	Colorado A & M College, Fort Collins	May 16, 1948
California Alpha	Pomona College, Claremont	June 6, 1948
Missouri Epsilon	Central College, Fayette	May 18, 1949
Mississippi Gamma	Mississippi Southern College, Hattiesburg	May 21, 1949
Indiana Alpha	Manchester College, North Manchester	May 16, 1950
Pennsylvania Alpha	Westminster College, New Wilmington	May 17, 1950
North Carolina Alpha	Wake Forest College, Wake Forest	Jan. 12, 1951
Louisiana Beta	Southwest Louisiana Institute, Lafayette	May 22, 1951
Texas Epsilon	North Texas State College, Denton	May 31, 1951
Indiana Beta	Butler University, Indianapolis	May 15, 1952
Kansas Epsilon	Fort Hays Kansas State College, Hays	Dec. 6, 1952
Pennsylvania Beta	La Salle College, Philadelphia	May 19, 1953
California Beta	Occidental College, Los Angeles	May 28, 1954

* Listed in order of date of installation.