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CONTENTS

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Kappa Mu Epsilon, national honorary mathematics fraternity, was founded in 1931. The object of the fraternity is four-fold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievement in the study of mathematics in the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

EGYPTIAN FRACTIONS*

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1. Introduction. In accounts[1] of the history and uses of fractions one finds that the Egyptians were the earliest mathematicians to use fractions, but that these people were handicapped by being unable to work with multiples of unit fractions. For example, although they could conceive of a fraction like 2/5, for a problem of addition or multiplication they would replace this fraction by a sum of distinct unit fractions such as $2/5 = 1/3+1/15$. Apparently they were very clever in finding methods for obtaining such representations. The purpose of this talk is to indicate some of the ways such a representation can be obtained. In deference to those first people we speak of the problem as one of obtaining an Egyptian representation or of working with Egyptian fractions—meaning always a sum of distinct unit fractions.

2. Method of division. It is not clear to me from my reading whether the combined results of all the Egyptians really showed that *any* given positive fraction does have an Egyptian representation. Certainly in modern notation we do not have too much trouble in establishing this fact—the only tools we need are the division algorithm for whole numbers and a little knowledge of infinite series.

Suppose first that given fraction A/B is proper and not itself a unit fraction so that the integers A and B satisfy $1 < A < B$. By a modified division algorithm we can find integers Q and R so that $B = QA - R$, $1 < Q$, $0 \leqq R < A$.

(If the minus sign puzzles you, note that in the usual division process you would have $B = Q'A + R'$; $0 \leqq R' < A$; if $R' = 0$, take $Q = Q'$, $R = R'$; if $0 < R'$, take $Q = Q' + 1$, $R = A - R'$.)

Then $A/B = 1/Q + R/BQ$, so if $R = 0$ or 1, the process is complete. In the remaining cases, where $1 < R < A$, the

*An invited address for the Ninth Biennial Convention of Kappa Mu Epsilon at St. Mary's Lake Camp, Battle Creek, Michigan April 17, 1953.

[1] Smith, D. E., *History of Mathematics*, II, p. 209 et seq.

process can be applied again to the fraction R/BQ. Because of the non-negative decreasing numerators obtained at successive steps, the process is sure to terminate in a finite number of steps. Because $R/BQ < 1/Q$, it follows that the unit fractions obtained at each step will be distinct from those obtained before. Hence a representation in terms of Egyptian fractions will be obtained.

For example, consider $A/B = 5/17$. Since $17 = 4 \cdot 5 - 3$, we find $5/17 = 1/4 + 3/68$; since $68 = 23 \cdot 3 - 1$, we find that $5/17 = 1/4 + 1/23 + 1/1564$.

As far as I can discover, if the given fraction should have been improper, even the earliest mathematicians would have done a natural thing by taking away the largest integer and working with the remaining proper fraction. But should we for theoretical satisfaction want an Egyptian representation for every rational number, we might proceed as follows.

Recall that the series $\Sigma 1/n$, $n = 2, 3, 4, \ldots$ is divergent, so for any positive rational number $x > \frac{1}{2}$ we can find an integer t such that

$$0 \leqq A/B = x - \sum_{2}^{t} 1/n < 1/(t + 1),$$

i.e., such that the next partial sum of the divergent series will exceed x. Now if $A = 0$ or 1, it is clear that a representation for x has already been found. Otherwise, we may apply the algorithm previously suggested for improper fractions. Since $B = QA - R > (t + 1)A$, it is clear that $Q > t + 1$, so that $1/Q$ and the other additional unit fractions are not only finite in number, but also distinct from those in $\Sigma 1/n$, $n = 2, 3, \ldots, t$.

For example, we find when $x = 2$, that $t = 10$ and $A/B = 179/2520$. Proceeding as above, we soon find

$$2 = 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10 + 1/15 + 1/230 + 1/57960.$$

I hope the previous theoretical discussion was not too hard for you to follow and that it left you with a satisfied feeling that an Egyptian representation is always available. I know that you are mature enough in mathematical training to appreciate that it is quite a different matter to produce a proof that the representation is always possible from merely producing a representation for one or two special examples. Had I merely asked you to accept the fact that a representation can be found, and then asked you to find it for either of the above examples, I daresay you would have come through with flying colors—but not necessarily with exactly the same answers which I obtained. Does this surprise you? Or did you notice that there was no point in our discussion where we claimed that the representation was unique? As you will see the next method which I will discuss reveals this ambiguity clearly.

3. "Method" of optic formulas. If we write the fraction $A/B = 1/B + 1/B + \ldots + 1/B$ with A summands, it appears that we might be able to solve the Egyptian problem if we can find A different representations of $1/B$. For this purpose we can try using repeated applications of the following optic formula[2]

$$1/B = 1/(B + 1) + 1/B(B + 1).$$

For example, we may write $2 = 1/2 + 1/2 + 1/2 + 1/2$. Then using the optic formula we obtain

$$\begin{aligned}2 = {} & 1/2 + (1/3 + 1/6) + (1/4 + 1/12 + 1/7 + 1/42) \\ & + (1/5 + 1/20 + 1/13 + 1/156 + 1/8 + 1/56 + 1/43 \\ & + 1/1806)\end{aligned}$$

and we check that we have removed all duplications.

Effective as this method may be for special examples, I do not see any simple way of showing that repeated applications of the optic formula will *always* remove duplications in a finite number of steps. Bear in mind, if it hasn't occured to you before, that a proposed process here might merely lead to a convergent infinite sequence of unit fractions. Thus I have chosen to put quotation marks around the word when speaking of this use of an optic formula as a "method."

We note that the optic formula does show us how from any one representation we can obtain as many more different representations as we desire. We simply apply the

[2]Dickson, L. E., *Theory of Numbers*, II, pp. 688-691.

formula to that unit fraction with the larger denominator and produce a representation with one more summand.

The fact that there are many representations suggests that we may try to formulate some sort of a description of one representation being better than another, and then we will have a new problem of trying to determine a best representation. What definition of a "better" representation do you prefer?

Our first representation for 2 has 12 summands, the second has 15 summands. Does this make you feel the first representation is better? But our second representation has as its largest denominator only 1806, whereas the first has the very large denominator 57960. Isn't this a reason for preferring the second form? We realize in such a dilemma that it may not be easy to hit upon a natural definition of "better," that we will probably have to make an arbitrary, subjective decision.

It is interesting that various Egyptian papyri show different representations for the same rational number. Apparently each master gave that representation which arose most naturally out of the particular tricks or devices which he knew for obtaining a representation.

4. Method of sodd numbers. If we suppose $A/B = \Sigma 1/d$ to be a representation of A/B as a sum of distinct unit fractions, and let M be the lowest common multiple of the d's, then we may write $M = dd'$ and $A/B = \Sigma d'/M$ where the d' are distinct divisors of M. Then there must exist an integer C such that $AC = \Sigma d'$ and $BC = M$.

This reverse sort of analysis shows that we can solve the representation problem if we can find an integer C so that AC can be written as a sum of distinct divisors of $BC = M$.

This brings us around to a discussion of numbers M which I wish to call *sodd,* because they possess a certain maximal property as far as *sums of distinct divisors* are concerned; namely, every integer x satisfying $1 \leqq x \leqq \sigma(M)$ can be written as a sum of distinct positive divisors of M. This is indeed a maximal property, for $\sigma(M)$ indicates the sum of all the positive divisors of M so $\sigma(M)$ is certainly the largest integer we can hope to write in this way.

For example, $M = 6$ is sodd, for $\sigma(6) = 1 + 2 + 3 + 6 = 12$, and $4 = 3 + 1$, $5 = 3 + 2$, $7 = 6 + 1$, $8 = 6 + 2$, $9 = 6 + 3$, $10 = 6 + 3 + 1$, $11 = 6 + 3 + 2$.

In a recent paper I have determined the structure of all such sodd numbers M, finding that we must have either $M = 2^a$, $a \geqq 0$; or $M = 2^a p_1^{a \text{ sub } 1} p_2^{a \text{ sub } 2} \cdots p_k^{a \text{ sub } k}$ where each p_i is a prime with $2 < p_1 < p_2 < \cdots < p_k$, where each exponent $a, a_1 \cdots, a_k$ is positive, and where $p_1 \leqq \sigma(2^a) + 1 = 2^{a+1}$ and $p_{i+1} \leqq \sigma(2^a p_1^{a \text{ sub } 1} \cdots p_i^{a \text{ sub } i}) + 1$ for $i = 1, 2, \cdots, k-1$.

Since the sigma-function is readily evaluated by $\sigma(2^a p_1^{a \text{ sub } 1} \cdots p_i^{a \text{ sub } i}) = (1 + 2 + \cdots + 2^a)\ (1 + p_1 + \cdots + p_1^{a \text{ sub } 1}) \cdots (1 + p_i + \cdots + p_i^{a \text{ sub } i})$ the above theorem describes in a neat way every possible sodd number.

For example, $M = 666$ is sodd because $M = 2 \cdot 3^2 \cdot 37$ with $3 < \sigma(2) + 1 = 4$ and $37 < \sigma(18) + 1 = 40$. But $M = 10 = 2 \cdot 5$ is *not* sodd, because $5 > \sigma(2) + 1 = 4$.

It is easy to check that a product $M = 2^a B$ will pass the above test for being sodd if $B \leqq 2^{a+1}$. Thus no matter what B may be, we can choose C in at least one way so that $BC = M$ will be sodd. Then if A/B is proper, since we have $AC < M$, it will follow that we can write $AC = \Sigma d'$ where the d' are distinct divisors of M and $d' < M$. Hence

$$A/B = AC/M = \Sigma d'/M = \Sigma 1/d$$

where the d are distinct and each $d > 1$.

Should x not be proper we can follow exactly the procedure in Section 2, first finding t so that $0 \leqq A/B = x - \sum_{2}^{t} 1/n < 1/(t + 1)$, and then applying the method of sodd numbers to A/B.

For example, to deal with $5/17$ we may choose $C = 16$, for $17 < 32$. Quickly we find $AC = 80 = 68 + 8 + 4$ and obtain $5/17 = (68 + 8 + 4)/[(17)(16)] = 1/4 + 1/34 + 1/68$.

5. A related problem. Perhaps you wonder when you see someone give more than one proof of a theorem, particularly if one of the proofs seems quite complicated. Sometimes I suppose such an additional proof is a mere *tour de force,* but often one learns a great deal by trying to solve a problem in several ways. I have purposely given the two methods above for solving the Egyptian problem and the pseudo-method of the optic formula, so that you might appreciate the various attacks one might make on the following problem proposed by E. P. Starke.

Let A/B be a positive rational number where B is an odd integer. Show that an Egyptian representation is possible in which the distinct unit fractions all have odd denominators $\geqq 3$.

Let us notice first that the series $1/3 + 1/5 + 1/7 + \cdots$ is divergent so that, if x is an improper rational number with an odd denominator, we can begin our problem by finding t so that

$$0 \leqq A/B = x - \sum_{1}^{t} 1/(2n + 1) < 1/(2t + 3),$$

where B is an odd denominator. Therefore Starke's problem will be solved for any fraction if we can solve it for any proper fraction A/B.

As far as I can see you will have trouble inventing a division method for this problem, for if in $B = QA - R$ you insist that Q be odd, you will have trouble controlling the size of R and getting a process that can be guaranteed to terminate in a finite number of steps.

The optic formula method also seems of doubtful value, not for lack of a formula, but because it seems difficult to guarantee that a finite number of applications of the formula will remove all duplications. The formulas are perhaps of some interest in themselves, witness the following:

$$\begin{aligned} 1/B = {} & 1/(B + 2) + 1/B(B + 2) + 1/(B^2 + 2B + 2) \\ & + 1/[B(B + 2)(B^2 + 2B + 3)/2] \\ & + 1/[B(B + 2)(B^2 + 2B + 2)(B^2 + 2B + 3)/2]; \end{aligned}$$

$$1/B = 1/(2B+1) + 1/(3B+2) + 1/[3(2B+1)] + 1/[B(2B+1)] + 1/[3(2B+1)\ (3B+2)].$$

Verifying these identities makes a nice exercise in algebra. Note that when B is an odd integer, all the denominators are odd integers and, in general, distinct.

For example, we have $1/5 = 1/7 + 1/35 + 1/37 + 1/665 + 1/24605$ or $1/5 = 1/11 + 1/17 + 1/33 + 1/55 + 1/561$.

Not able to give a general proof with the division or optic formula methods, we are glad to know another method to try—namely the use of some sort of sodd number. For as before we can write $A/B = \Sigma 1/d = \Sigma d'/M$ and get the idea of trying to find an integer C such that if $BC = M$, we can write $AC = \Sigma d'$ as a sum of distinct divisors of M.

Of course, M must be odd, so the number 2 cannot be written $\Sigma d'$, nor can the number $\sigma(M) - 2$ be so written. But we can seek odd numbers M such that for any integer x with $3 \leqq x \leqq \sigma(M) - 3$ we can write $x = \Sigma d'$, where the d' are distinct divisors of M. There are such numbers, the smallest one being 945, and I have succeeded in getting a complete description of them, although the situation is much more complicated than with the sodd numbers described in the previous section.

In particular, the number $M = 3^a 945B$ is of the special type just described if $B < 3^{a+5}4 - 14$. So given A/B with $1 < A < B$ with B odd, we can find at least one odd integer C so that $BC = M$ has the maximal sodd property for odd numbers. Since $AC < M$, it will follow when $2 < AC$, that $AC = \Sigma d'$ where the d' are distinct divisors of M with $d' < M$. Then

$$A/B = AC/M = \Sigma d'/M = \Sigma 1/d$$

is a representation of the desired type with distinct $d \geqq 3$. The remaining case where $AC = 2$ implies $C = 1$ and $B = M$; but M must have the factor 5, so we can set $M = 5W$ and note that

$$2/M = 2/5W = 1/3W + 1/15W;$$

or we can increase a in our choice of C and avoid this case completely.

Except that I cannot here discuss the details of the structure theorems for sodd numbers, this completes the proof that the representation suggested by Starke is always possible. For the missing details I must refer you to a forthcoming paper.

In conclusion let us consider an example to illustrate the sodd method of solving Starke's problem.

When A/B = 5/17, the method described above guarantees that $C = 945$ will be effective. Pairs of divisors d, d' of $BC = dd'$ are as follows:

1,	16065;	3,	5355;	5,	3213;	7,	2295;
9,	1785;	15,	1071;	17,	945;	21,	765;
27,	595;	35,	459;	45,	357;	51,	315;
63,	255;	85,	189;	105,	153;	119,	135.

Then it is easy to write $AC = 4725$ as a sum of distinct divisors of BC, say, $AC = 3213+765+459+153+135$; then

$$A/B = 1/5 + 1/21 + 1/35 + 1/105 + 1/119.$$

Of course, even when BC does not have the maximal sodd property, it may still have the sodd property for the particular number AC of a given problem. Thus in the example above $C = 15$ happens to be effective, for $15A = 75 = 51 + 15 + 5 + 3 + 1$, whence $A/B = 1/5 + 1/17 + 1/51 + 1/85 + 1/255$.

But I hope I have presented these matters in such a way that you see that choosing C to make BC have the maximal sodd property guarantees a constructive solution of the representation problem, whereas choosing C at random may or may not provide a solution. There is a lesson to be learned here—of being satisfied only when we have a truly general solution of a problem.

"What Gauss put into print is as true and important today as when first published; his publications are statutes, superior to other statutes in this, that nowhere and never has a single error been detected in them."

—M. CANTOR

AN ANALYTIC APPROACH TO THE THREE-FOCUS CURVE

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At the 1953 National Convention of Kappa Mu Epsilon, a paper entitled "A Device for Drawing N-Focus Curves" was read by Raoul Pettai of Colorado A & M College. In this paper, he discussed a device for drawing three- and four-focus curves. He also mentioned a few properties of each. This paper is a continuation of his work, using an analytic approach rather than the synthetic approach used by Mr. Pettai.

The three-focus curve may be defined as the locus of points, the sum of whose distances from three fixed points, called foci, is constant.

$$K = d_1 + d_2 + d_3 \tag{1}$$

We shall consider the special case in which the three foci lie on the same straight line with the distances between adjacent foci equal. The equation of the curve is most easily derived using polar coordinates, with the pole at the middle focus and the polar axis containing all three foci. If (ρ, θ) is a point on the locus; $d_1 = \rho$, the distance from the focus $(0, 0)$; d_2, the distance from the focus $(m, 0)$; and d_3, the distance from the focus $(-m, 0)$, equation (1) may be written:

$$K = \rho + \sqrt{(\rho\cos\theta - m)^2 + \rho^2\sin^2\theta} + \sqrt{(\rho\cos\theta + m)^2 + \rho^2\sin^2\theta} \tag{2}$$

where m is the distance between adjacent foci. Eliminating radicals by squaring, we obtain:

$$(K - \rho)^4 - 4(m^2 + \rho^2)(K - \rho^2) + 16m^2\rho^2\cos^2\theta = 0 \tag{3}$$

0

$$\text{or } (K - \rho)^4 - 4(m^2 + \rho^2)(K - \rho)^2 + 16m^2\rho^2 = 16m^2\rho^2\sin^2\theta \tag{4}$$

Then, solving for $\sin\theta$, we obtain:

$$\sin\theta = \frac{\pm\sqrt{(K - 3\rho)(K + \rho)(K - 2m - \rho)(K + 2m - \rho)}}{4m\rho} \tag{5}$$

Because the terms of equation (2) were squared, equation (3) would have resulted if equation (1) had been either $K = d_1 + d_2 - d_3$ or $K = d_1 - d_2 + d_3$. Points which satisfy these equations also satisfy equation (3), and equation (3) is now the equation of three curves. If ρ, that is, d_1, is allowed to be negative, three more loci are added to equation (3): $K = -d_1 + d_2 + d_3$, $K = -d_1 + d_2 - d_3$, and $K = -d_1 - d_2 + d_3$. These equations may be eliminated by using only positive ρ's. We shall restrict K to the values $K \geqq 0$. Thus, $K = -d_1 - d_2 - d_3$ and $K = d_1 - d_2 - d_3$ will not be considered.

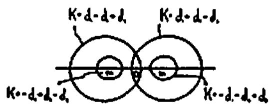
Equation (3) then represents six curves, but all of these six do not exist for all values of K/m. For example, it is obvious that $K = d_1 + d_2 + d_3$ does not exist when $K/m < 2$. It is desirable to know for what values of K/m each curve exists. This can be found by solving each equation for ρ when $\theta = 0$; for, since the curves are symmetrical with respect to the polar axis [see equation (3)] and are continuous for all values of θ, there will be intercepts on the polar axis if the curve exists. If $\theta = 0$, equation (2) becomes

$$K = \sqrt{\rho^2} + \sqrt{(\rho - m)^2} + \sqrt{(\rho + m)^2}$$

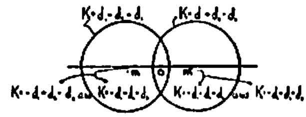
$$\text{or } K = |\rho| + |\rho - m| + |\rho + m|.$$

If $\rho > m$, $K = \rho + (\rho - m) + (\rho + m) = 3\rho$.
If $\rho = m$, $K = 3m$.
If $\rho < m$, $K = \rho + (m - \rho) + (\rho + m) = 2m + \rho$
From this it can be seen that the curve $K = d_1 + d_2 + d_3$ does not exist when $K < 2m$, exists as a point at the pole when $K = 2m$, exists as a curve which crosses the polar axis between the pole and the other two foci when $2m < K < 3m$, exists as a curve which crosses the polar axis at each of the two external foci when $K = 3m$, and exists as a curve which crosses the polar axis beyond the foci when $K > 3m$.

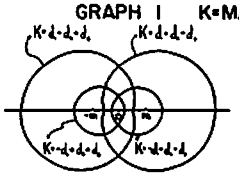
By a similar analysis of the other five curves, it can be determined that: $K = d_1 + d_2 - d_3$ and $K = d_1 - d_2 + d_3$ exist for all values of K/m. $K = -d_1 - d_2 + d_3$ and $K = -d_1 + d_2 - d_3$ can be represented graphically as K/m varies from *1* toward *0*. Each of the



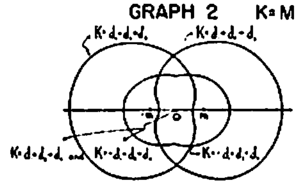
GRAPH 1 K=M/2



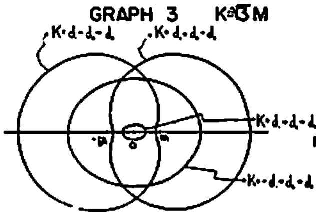
GRAPH 2 K=M



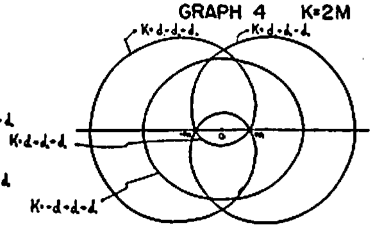
GRAPH 3 K=$\sqrt{3}$M



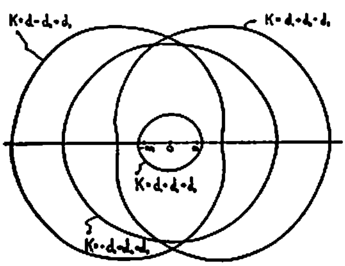
GRAPH 4 K=2M



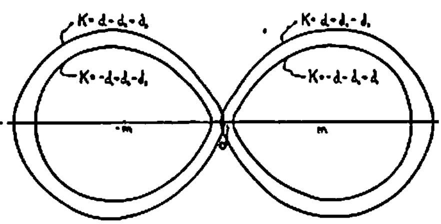
GRAPH 5 K=2.5M



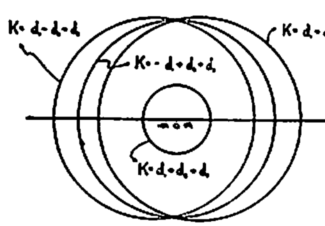
GRAPH 6 K=3M



GRAPH 7 K=4M



GRAPH 8 K=M/10



GRAPH 9 K=10M

latter two curves exists as a point when $K/m = 1$. $K = -d_1 + d_2 + d_3$ exists as K/m varies from *1* to ∞. This curve exists as two points when $K/m = 1$.

Graphs 8 and 9 show what happens when $(K/m) \to 0$ and $(K/m) \to \infty$, K being held constant while m varies from ∞ to *0*. In Graph 9, each of the four curves approaches a circle as a limit. This can be seen if equation (3) is rewritten:

$$(6) \qquad (K - 3\rho)(K - \rho)^2(K + \rho) + 4m^2[2\rho\cos\theta + (K - \rho)]\,[2\rho\cos\theta - (K - \rho)] = 0$$

If $m \to 0$, then $(K/m) \to \infty$ and Equation (6) becomes

$$(K - 3\rho)\,(K - \rho)^2(K + \rho) = 0.$$

This is the equation of three circles with radii K and one of radius $K/3$. Comparison of Graphs 7 and 9 will show the changes as m becomes smaller with respect to K.

If Equation (6) is divided by $4m^2$, we obtain

$$[(K-3\rho)\,(K - \rho)^2(K + \rho)]/4m^2 + [2\rho\cos\theta + (K - \rho)][2\rho\cos\theta - (K - \rho)] = 0$$

As $m \to \infty$, and $(K/m) \to 0$, (see Graph 8) the equation approaches:

$$[2\rho\cos\theta + (K - \rho)]\,[2\rho\cos\theta - (K - \rho)] = 0$$

This solved for ρ gives,

$$\rho = \frac{\mathrm{K}}{1 \pm 2\cos\theta}$$

which is the equation for a hyperbola. The graph shows that the curves $K = d_1 - d_2 + d_3$ and $K = d_1 + d_2 - d_3$ apparently approach hyperbolas as limits as m becomes very large with respect to K. When m increases beyond all bounds, only two curves exist while in Graph 8 there are four. The curves $K = -d_1 + d_2 - d_3$ and $K = -d_1 - d_2 + d_3$ disappear when

$$[(K - 3\rho)\,(K - \rho)^2(K + \rho)]/4m^2$$

does—that is, when m becomes infinitely large. For all finite values of m, all four curves exist.

LEGEND FOR GRAPHS

Graphs 1 through 7 show the shapes and relative sizes of the curves which exist for the various relations between K and m. The scale used is the same for graphs 1 through 7. The scale was changed for each of the graphs 8 and 9 for ease in drawing. Graphs 8 and 9 show what happens as m becomes very large and very small with respect to K.

"A rope is supposed to be hung over a wheel fixed to the roof of a building; at one end of the rope a weight is fixed, which exactly counterbalances a monkey which is hanging on the other end. Suppose that the monkey begins to climb the rope, what will be the result?"

"This problem was proposed by Lewis Carroll in December, 1893, and in his diary he remarked: 'Got Professor Clifton's answer to the Monkey and Weight Problem. It is very curious, the different views taken by good mathematicians. Price says that the weight goes up with increasing velocity; Clifton that it goes *up*, at the same rate as the monkey; while Sampson says that it goes down.' "

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ELEMENTS OF CRYPTANALYSIS: THE SIMPLE SUBSTITUTION CRYPTOGRAM

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1. Introduction. The puzzle aspect of the hobby of cryptanalysis is fascinating, but what is more interesting from a mathematical viewpoint is the fact that the behavior of each letter has been scientifically analyzed and its peculiarities classified.

The method of simple substitution is well known—letters, numerals, or other abitrary symbols are substituted 1 to 1 for each letter of the message (plain text).

Based on the peculiarities in the words and letters of the language, certain techniques have been devised which enable a solver to "break" a cipher. This paper will discuss some of these elementary techniques and the language peculiarities from which the solving techniques were developed.

2. The Frequency Table. Basic to cryptanalysis, of course, is the frequency table, E T A O N I S H R D L U C F M W P G Y B V K X J Q Z, which lists in order of decreasing frequency the letters as they are normally used in English. Other languages have their own unique properties which we will not discuss here.

Though this list is correct for large numbers of letters in normal text, it may not be correct for a short cryptogram, in which the most frequent symbol may be neither E nor even T. But generally, the predominant symbols will be among the high-frequency letters.

3. Short Words and Affixes. Comparison of common short words often offers a short cut to breaking the cipher. ABC and ABCXC might be THE, and THERE or THESE. Similarly THEN, THIS, THAT, THEY, or THEIR may be spotted. X, XY, XYZ may be A, AN, and AND or ANY (or I, IT, ITS). Others are: OF, FOR, OR, FROM; ON, NO, NOT, INTO; IS, HIS, ITS, IT; WE, WERE, WHEN,

WHERE; HE, HAS, HAVE, HAD; WILL, WITH, WHICH. You'll find others with experience.

Short words, also, can often be identified by noting letter frequency. For instance, a two-letter word with a low-frequency initial letter may be WE, HE or BE; with a low-frequency final, may be OF or IF. FROM and WITH are the most common four-letter words carrying low-frequency initial and final letters.

Comparison of prefixes and suffixes with each other or with short words affords another short cut. The prefix XY—, appearing with the suffix —XYZ, —XWY or —XZY, may be IN—, and —ING, —ION, or —IGN. The prefix RE— and the suffix —ER are common, as are DE— and —ED. The suffix —TION, too, can be identified if the message contains any short words using those letters — IT, IN, ON, NO, INTO, TO, NOT.

Letter frequency often opens the way to identification of affixes. Low-frequency finals may indicate ING, ISM, IC or FUL. Low-frequency antepenultimates (second from last) may indicate SHIP, ICAL, ABLE or IBLE. Among the most common suffixes are: ED, HOOD, ANCE, ATE, ING, ER, OR, S, ES, LESS, MENT, EST, IST, Y, LY. The affixes ING, ED, ER, and S often follow doubled letters as in CALLING, CALLED, CALLER, CALLS. The most common repeated final trigram is ING as in SINGING, BRINGING. The most common suffixes with doubled before final letters are: ALLY, EED, TEEN, EER, EES, and HOOD. Among the most common prefixes are: AD, COM, CON, DE, DIS, EN, EX, IM, IN, INTER, MIS, OB, OUT, PER, PRE, PRO, RE, UN.

Remember that in analyzing clues on suffixes and endings, it is often necessary to shift one or two letters to the left because they may be joined with S, ES, D, ED, LY, etc.

4. **Punctuation.** The apostrophe identifies—besides final S— such words as E'ER, NE'ER, DON'T, O'CLOCK, WASN'T, HASN'T and others. A comma usually precedes the connectives AND, BUT, YET. Sentences ending

with question marks usually include either WHO, WHAT, WHY, WHEN, WHERE, or HOW as first word.

5. **Frequency Chart.** If the crypt has resisted the elementary methods discussed in the preceding paragraphs, a little deeper digging is required to unearth the solution. First make a frequency chart of the symbols used in the message, then try to spot the vowels in the message keeping the following factors in mind:

Vowels, except U and Y, outnumber most of the consonants, Thus, a high-frequency letter before, after, or between low-frequency symbols can be tentatively established as a vowel. Similarly, in a three-letter series having the first and third letters repeated, as ABA, if the repeated letter is a consonant, the middle letter ordinarily is a vowel (as in PAPER), and if the repeated letter is a vowel, the middle letter probably is a consonant.

In the sequence ABCBDB, the repeated letter is usually, though not always, a vowel, while the adjacent letters are usually consonants (as in CALABASH, RECEDE. Exception: STATUTE. Incidentally, the letter I appears most often in such a sequence (as in VISIBILITY, INITIATE).

In a repeated or reversed digram (series of two letters), one letter is usually a vowel.

A resume of vowel and consonant positions discloses that, depending on word length, the letters favor certain positions. In two-letter words, vowels predominate as initials, consonants as finals; in three-letter words, consonants are more frequent as initial and second letters; in four-letter words, vowels favor second position, consonants first and fourth; in five-letter words, vowels predominant as third letter, with the consonants heavy in all other positions. In longer words, consonants generally favor first, third, and final positions, with the other positions using both classes of letters equally. Vowels make up about 40 per cent of normal text and are uniformly distributed throughout plain text.

6. **Vowel Characteristics.** After spotting a few

vowels, examine their usage in greater detail. Some of the most important characteristics of the vowels are:

E—A high-frequency final often used as next-to-last letter; often used as a final in two-letter words, but not as an initial. Doubled in all positions, frequently as final.

I—A high-frequency antepenultimate (second from last) letter. Used infrequently as a final (TAXI and in Latin plurals such as RADII). Its antepenultimate usage arises in such endings as —ING, —ION, —IVE, —ICE, —ITY, etc.

A and O—Medium high-frequency letters used often as finals (AREA, TOBACCO), but not as often as E. O favors second position, this, quite often, enabling the solver to differentiate between the two vowels. O appears either as an initial and final in numerous common short words, while A appears only as an initial in such words and is only occasionally doubled. O is doubled in all positions.

U—Low-frequency is the clue to this vowel. It is seldom doubled (exception: VACUUM), and has an affinity for the letter Q.

Y—An average low-frequency final that turns up quite often in first and second positions (as in YESTERDAY, HYSTERIA). Only exceptionally doubled.

The most frequent two-vowel combinations are in order: EA, OU, IO, EE, EI, OO, IE. Most frequent three-vowel combinations are IOU and EAU.

7. **Consonant Characteristics.** Some of the important consonant characteristics are:

S—A high-frequency final frequently doubled in the middle and at the end of words. S appears more often than any other letter as the initial and final letters of the same word.

H—A medium-frequency letter often employed as a second letter. It nearly always follows a consonant (C, G, P, R, S, T) and precedes a vowel. Consonants most used after H are R, S, T (SHREW, PATHS, EIGHT). H can often be identified as a second letter through its affinity with T in the digram TH, the most frequent in the language, which is often used initially. When preceded by a vowel, H is also usually followed by a vowel (as in BEHEAD, AHEAD). It is frequently employed as a first, penultimate, or final in short words (HER, EIGHT, SUCH). When occasionally doubled, it normally follows a consonant and precedes a vowel (as in HITCHHIKE).

N—An average high-frequency letter which occurs as a next-to-last letter about three-tenths of the time (as in the affix —ING). N usually follows a vowel, and precedes a consonant (C, G, D, S, T as in —ING, HAND, WINS, WENT). Doubled N is usually flanked on both sides by vowels. The most common affixes using N are IN—, UN—, CON—, INTER—, INTRO—, —ING, —ION, —IAN, —ANCE, —ENCE, —MENT.

And now some advice on the procedure for using this information. When you spot the symbol for H, examine the letters preceding the H's for consonant clues, and those following for clues to vowels. Or, in reverse, having spotted some vowels, if you note a symbol mostly preceded by consonants and followed by vowels, it may be tried as H. The same idea can be used for the letter N.

L and R—A couple of wishy-washy letters that go with either consonants or vowels. Both form reversed digrams with the vowels (as RE—, —ER), but R does this about twice as often as L. Both appear often in second position where they follow a consonant (usually the low-frequency initials B, C, F, G, P) and precede a vowel. Both appear frequently after doubled consonants.

T—A high-frequency initial and final. The most common suffixes using T are —TION, —ENT, and —ANT.

And now for the low-frequency consonants. In short, artificially constructed messages of the type used in dailies and periodicals, it is often possible to suppress high-frequency letters, distort normal letter positions and characteristics. Under such circumstances, the only possible entry to a cryptogram might be provided by a low-frequency consonant. It will, therefore, repay the solver if he studies these letters well.

Q—A low-frequency letter often used initially and in second position preceded by a vowel. Always followed by U, which in such a sequence is followed by another vowel.

J—Often used as an initial when it is normally followed by a vowel. Sometimes appears in second position when it is normally preceded by E. J often follows the prefixes AD—, IN—, OB—, RE—, UN—, CON—, PRO—, and SUB—.

X—Favors second position, where it is ordinarily preceded by a vowel, most often E. Used as a final in such words as BEAUX, EXECUTRIX.

Z—Appears often as second-from-last letter preceding —ED, or —EN (as in RAZED, DOZEN). It also favors the next-to-last letter position followed by E (as in PRIZE, DAZE). Z is often employed as an initial letter and in third position. When doubled, Z is most often followed by L or a vowel.

B, C, M, V—Commonly appear as first letters, while F, P, and W are used mostly as first letters. These letters except for W are all often doubled after an initial vowel (as in ACCUSE, IMMOLATE, OPPOSITE). V frequently appears as a penultimate, —IVE being the most common affix.

D, G and K—Favor final position. This is due to the frequency of the affixes —ED, —ING and —OK.

8. **Pattern Words.** There is one more aid to breaking monoalphabetic substitution ciphers—recognition of "pattern words," that is, words with repeated letters. Scientific identification of "pattern words" is possible if you make your own "pattern word" dictionary by classifying words according to the number of letters and the postion of the repeated letters. For instance 3 (words of 3 letters): 13 (first and third letters repeated) —DID, POP, MOM.

3:12—EEL
3:13—MUM, DID
3:23—BEE, ALL, OFF
4:13—NINE, EVER, AWAY
4:14—ELSE, THAT, AURA
4:24—HERE, WERE
4:34—TELL, TILL,WELL, WILL
7:15-24—AWKWARD

Some common pattern words: AGAIN, ALL, ALREADY, ANYWAY, AWAY, BECAUSE, BEEN, BEFORE, BETWEEN, CANNOT, DID, EITHER, EVEN, EVER, FINALLY, FORWARD, HERE, HERSELF, KNOWN, LESS, NEITHER, NEVER, NOTHING, ONTO, PERHAPS, RATHER, SHALL, SOMEONE, STILL, THOUGH, THROUGH, THERE, THESE, TILL, TOO, UNLESS, UNUSUAL, USUALLY, WERE, WILL, WELL.

9. Conclusion. Now it's up to you. If you learn to coordinate the above information into a well-rounded analysis, you'll be able to solve most cryptograms; you'll have a lot of fun, pick up a lot of odd and interesting information, and augment your vocabulary. Don't get knocked for a tangent by those canny constructors who distort the language with such words as SYZYGY; they can't hide everything. In analyzing a cryptogram, a limit can be placed on the letters that are likely to be represented by a symbol. By establishing one letter, you have gone a long way to solving the crypt.

If you have been interested in the content of this article, you will want to solve the following simple substitution cryptogram:

$+\quad 5<9-\Sigma\beta\quad \sigma+\sigma<\leqq\quad -\Sigma\quad 9\leqq4\sigma7+\Sigma+\Delta45=5$
$2=\Delta\Delta\quad +\sigma\sigma<+\leqq\quad =\Sigma\quad 7\rho<\quad 3+\Delta\Delta\quad =55\alpha<\quad -3$
$7\rho<\quad \sigma<\Sigma7+\theta-\Sigma$

The first definition of the general concept of function seems to be due to John Bernoulli I (1718).

—G. A. MILLER

"AUDITORY IMAGES" OF NON-ACCELERATED SOURCES OF SOUND—AN EXERCISE IN PLANE GEOMETRY

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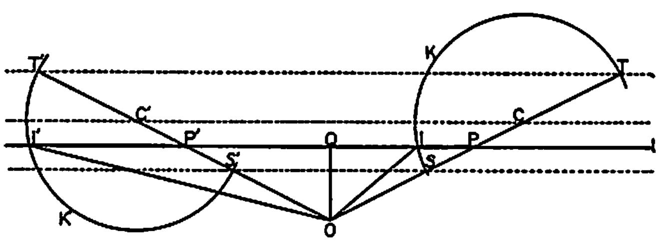
First, let me explain the formidable title — the excuse for which is that it is brief. A non-accelerated object is simply one which moves in a straight line at constant speed. (We will not be concerned with the trivial case in which this speed is zero.) And if a moving object emits sound continuously, we use the term "auditory image" to characterize the position which it *would appear* to occupy on the basis of the direction from which sound waves emitted by it reach us at any given time. Thus, the auditory image of a train whistle lags a short distance behind the whistle itself, and the image of a supersonic airplane does not even appear until the plane has passed overhead.

Suppose that A is a source of continuous sound, and that A comes from indefinitely far to the left and proceeds indefinitely far to the right, moving along a line L at speed k; and suppose that an observer—"a hearer"—stands at a point O which is not on L. We propose to show by means of elementary geometry when and in what directions the auditory images of A appear to the observer.

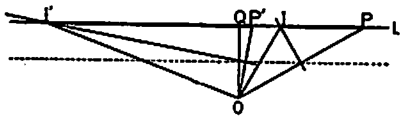


CASE I. $p<1$

We let Q be the foot of the perpendicular from O to L and let P be an arbitrary point on L. We will use the phrase *the image of P* to denote *the auditory image of A when A is at P.* Letting c be the speed of sound, we observe that, if I is the image of P, then A has traveled at speed k from I to P in the same time in which sound has traveled at speed c from I to O. Putting $\rho = k/c$, we therefore see that $IP/IO = \rho$. But the locus of *all* points, the ratio of whose distances from P and O remains equal to ρ, is—unless $\rho = 1$, a case which we will consider separately—the circle K of which a diameter is ST, where S and T divide the segment PO in the ratio ρ:1, internally and externally, respectively. (Since $PS/SO = PT/TO = \rho$ for all positions of P, the loci of S and T as we vary P along L are lines parallel to L; and hence the locus of the center, C, of K is another such line, midway between these lines.) Such intersections of K with L as lie to the left of P are therefore the desired images I. We discuss next, in succession, the cases $\rho < 1$, $\rho = 1$, and $\rho > 1$.

CASE I. $\rho < 1$. This is the case illustrated above. (In the figure $\rho = \frac{1}{2}$; that is, A is traveling with half the speed of sound. Two points, P and P', are shown with their respective images, I and I'.) The determination of the image is clearly possible, and uniquely so, for all points on L. Note that the *angular lag* (Angle POI) of I behind P as measured at O is completely determined by the *angular position* (Angle POQ) of P at O, whatever may be the magnitude of OQ. The reader may be interested in establishing the following facts. (1) This lag is greatest when P is at Q. (It is then obviously equal to arcsin ρ, to borrow a neat term from trigonometry.) (2) If P and P' are two points on L which are equidistant from Q (which is actually the case in the diagram), then the angular lags of their images, as measured at O, are equal.

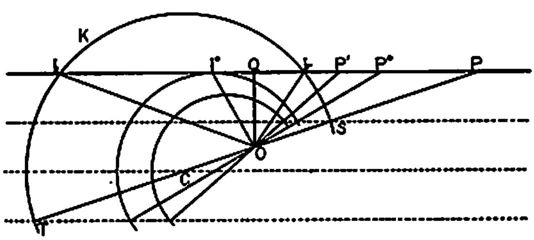
CASE II. $\rho = 1$. (In the figure two points and their images are shown.) Here $IP = IO$, and since the locus of all points which are equidistant from P and O is the perpendicular bisector of the segment PO, the image I is the intersection of this bisector with L when this intersection lies to the left of P. The determination of I is possible when and only when P is to the right of



CASE II $\rho = 1$

Q, and is then uniquely possible. That is, if A travels with the speed of sound there is no auditory image until it has passed Q; also, the more recently A has passed Q, the greater is the angular lag of its image behind it, as measured at O, with $90°$ as limit.

CASE III. $\rho > 1$. In the figure, $\rho = 2$; that is, A is traveling twice as fast as sound. Three points, P, P^*, and P', are shown, with images I_r and I_l (for P) and I^* (for P^*); we will see that P' has no image. As far as construction of the circle K is concerned, this case differs from Case I only in that the external point of division, T, lies below O rather than above P. However, the relations between K and L are by no means the same as they were in Case I. Letting P^* be the point for which the associated circle is tangent to L, we see



CASE III. $\rho > 1$

that: for points P' to the left of P^* there is no image; P^* has a unique image, I^*; and for points P to the right of P^* there are two images, I_r and I_l with I_r moving indefinitely far to the right and I_l moving indefinitely far to the left if P is moved indefinitely far to the right.

Since $I_rP/I_rO = \rho = I_lP/I_lO$, Angle I_rPO = Angle I_lPO, and triangles I_rPO and I_lPO are *not* similar, the *internal* angle, Angle POI_r of the one triangle equals the *external* angle, Angle I_lOT of the other. Thus, the mean of the angular positions of I_r and I_l as measured at O is 90° behind the angular position of P at O. In particular, since the images of P^* coincide at I^*, we have Angle $P^*OI^* = 90°$; and from this it follows that Angle P^*OQ = Angle P^*I^*O and—to borrow another term from trigonometry—that cos Angle $P^*I^*O = I^*O/I^*P^* = 1/\rho$, so that cos Angle $P^*OQ = 1/\rho$.

That is, if A travels faster than sound, there is no auditory image until it has passed through a definite angular distance (trigonometrically speaking) arccos $(1/\rho)$ beyond Q, at which time an image appears, 90° behind A; this image at once splits into two, one of which follows A, while the other recedes indefinitely far to the left; and the mean of the angular positions of these two images remains constantly 90° behind the angular position of A.

REMARKS. The results of this paper can be obtained more quickly by trigonometric means if we start with the relation (see the first figure) sin Angle POI/ sin Angle $IPO = IP/IO = \rho$, which is given by the Law of Sines; and the methods of analytic geometry provide us rather readily with formulas for QI in terms of the time, a matter which we have not considered here. But the geometric method, which we have preferred, provides special insights of its own and puts to use machinery which, in the mental shops of a good many of us, has become somewhat rusty.

"Mathematics is the handwriting on the human consciousness of the very Spirit of Life itself."

—CLAUDE BRAGDON

THE LINE, CIRCLE, AND PARABOLA USING COMPLEX NUMBERS

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The complex number w is a number of the form $w = a + ib$ where a and b are real numbers and $i = \sqrt{-1}$. The number a is called the real part of w and will be denoted by $R(w)$. The number ib is called the imaginary part of w and will be denoted by $I(w)$. The conjugate of w, denoted by w^*, is defined to be $w^* = a - ib$. The absolute value of w is defined to be $|w| = \sqrt{(a^2 + b^2)}$.

If $w = a + ib$, the following properties of w can be derived:

1) $ww^* = |w|^2$
2) $(w^*)^* = w$
3) $w + w^* = 2R(w)$
4) $w - w^* = 2I(w)$
5) If k is real, then $R(kw) = kR(w)$ and $I(kw) = kI(w)$
6) If $b = 0$, then $w = w^* = a$
7) $(w_1 \pm w_2)^* = w_1^* \pm w_2^*$
8) $(w_1 w_2)^* = w_1^* \cdot w_2^*$
9) $(w_1/w_2)^* = w_1^*/w_2^*$
10) $|w_1 - w_2| = \sqrt{[R(w_1) - R(w_2)]^2 - [I(w_1) - I(w_2)]^2}$

1. Derivation of the General Equation of the Line, Using Complex Notation.

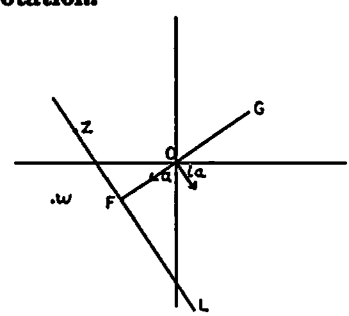


FIGURE I

Let L be a line in the complex plane. Let G be the line containing the origin which is perpendicular to L. Let F be the complex number which corresponds to the point of intersection of L and G. If $F \neq 0$, let a be the complex number such that $|a| = 1$, and the argument of a equal to the argument of F. If $F = 0$, let a be the complex number such that $|a| = 1$ and the argument of a equal to the inclination of G. Let $p = |F|$. If z is on L, then there is a real number k such that $|k| = |z - F|$ and

$$z = pa + kia;$$
$$z^* = pa^* + k(ia)^*;$$
$$z^* = pa^* - kia^*;$$

Therefore, $az^* = p - ki;$

and $a^*z = p + ki.$

Hence, if z is on L, then $az^* + a^*z - 2p = 0$

Conversely, if z is such that $az^* + a^*z - 2p = 0$, let $ik = I(a^*z)$. Then

$$a[z^* - pa^* - k(ia)^*] + a^*(z - pa - kia) = 0;$$

and thus $R[a^*(z - pa - kia)] = 0$. Also,

$$a^*(z - pa - kia) - a[z^* - pa^* - k(ia)^*]$$
$$= a^*z - az^* - ki + ki^*$$
$$= a^*z - az^* - 2ki = 0.$$

Hence, $I[a^*(z - pa - kia)] = 0$ and

$$a^*(z - pa - kia) = 0;$$
$$z - pa - kia = 0;$$
$$z = pa + kia.$$

Therefore, z is on L. We have shown that *a necessary and sufficient condition that z be on L is that*

$$az^* + a^*z - 2p = 0.$$

If w is a complex number, then w can be written in the form $ha + kia$ where h and k are real numbers. If w is on L, then $h = p$ and $aw^* + a^*w - 2p = 0$. If w is not on L, then the distance between w and the line L is $|h - p|$. Now $|aw^* + a^*w - 2p| = |a(ha^* - kia^*) + a^*(ha + kia) - 2p| = 2\,|h - p|$. Hence, $|aw^* + a^*w - 2p|$ is twice the distance between the point w and the line L. We will call $az^* + a^*z - 2p = 0$ the normal form of the equation of the line L.

If $B \neq 0$ and q is real, we will show that $Bz^* + B^*z + q = 0$ is the equation of a line. If $q \geqq 0$, divide by $-|B|$. Then

$$-(B/|B|)z^* - (B^*/|B|)z - q/|B| = 0.$$

Let $a = -B/|B|$ and $p = q/(2|B|)$, then

$$az^* + a^*z - 2p = 0$$

where $|a| = 1$ and $p \geqq 0$. If $q < 0$, divide by $|B|$. It can be shown in a similar manner that $az^* + a^*z - 2p = 0$ where $a = B/|B|$, $|a| = 1$, $p = -q/(2|B|)$ and $p > 0$. Hence an equation of the form $Bz^* + B^*z + q = 0$, where $B \neq 0$ and q is real, is the equation of a line.

2. Derivation of the General Equation of the Circle, using Complex Notation.

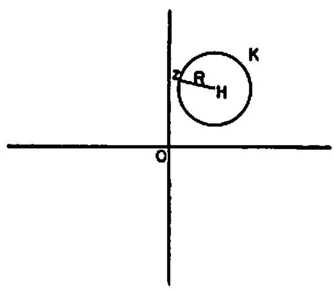


FIGURE 2

Let K be a circle with center H and radius R. If z is a point of K, then the distance from z to the center H is equal to R, i.e.,

$$|z - H| = R;$$
$$|z - H|^2 = R^2;$$
$$(z^* - H^*)(z - H) = R^2;$$

and $$zz^* - H^*z - Hz^* + HH^* - R^2 = 0.$$

Conversely, if z is such that $zz^* - H^*z - Hz^* + HH^* - R^2 = 0$, then

$$zz^* - H^*z - Hz^* + HH^* = R^2;$$
$$|z - H|^2 = R^2;$$

and $$|z - H| = R.$$

Hence, *a necessary and sufficient condition that z be on* *the circle with radius R and center H is that* $zz^* - H^*z - Hz^* + HH^* - R^2 = 0$.

If A and C are real and $A \neq 0$, then we will show that $Azz^* + Bz^* + B^*z + C = 0$, is the equation of a circle. This equation can be written in the form

$$zz^* + (B/A)z^* + (B^*/A)z = -C/A.$$

If BB^*/A^2 is added to both members of the above equation, then the left member is a perfect square. Therefore,

$$zz^* + (B/A)z^* + (B^*/A)z + BB^*/A^2 = -C/A + BB^*/A^2;$$

$$(z + B/A)\ (z^* + B^*/A) = (BB^* - AC)/A^2;$$

$$|z + B/A|^2 = (BB^* - AC)/A^2;$$

and $$|z + B/A| = \sqrt{(BB^* - AC)}\ /|A|.$$

If $BB^* - AC > 0$, then z lies on the circle with center $-B/A$ and radius $\sqrt{(BB^* - AC)}/|A|$. If $BB^* - AC = 0$, then $z = -B/A$ and we have a point circle. If $BB^* - AC < 0$, then there is no number z which satisfies the equation $Azz^* + Bz^* + B^*z + C = 0$ and hence we have an imaginary circle.

3. Derivation of the General Equation of the Parabola, Using Complex Notation.

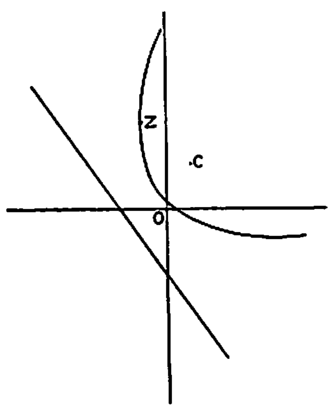


FIGURE 3

If C is the focus of a parabola in the complex plane and $aw^* + a^*w - 2p = 0$ is the normal form of the equation of its directrix, then for any point z on the parabola

$$2|z - C| = |az^* + a^*z - 2p|;$$
$$4|z - C|^2 = |az^* + a^*z - 2p|^2;$$
$$4(z^* - C^*)(z - C) = (az^* + a^*z - 2p)(a^*z + az^* - 2p);$$

and

$2zz^* - a^2(z^*)^2 - (a^*)^2z^2 + (4pa - 4C)z^* + (4pa - 4C^*)z + 4CC^* - 4p^2 = 0$. Conversely, if z satisfies this equation, it can be shown that z is on the parabola.

If $A \neq 0$, and A and E are real numbers, then $Azz^* + B^*z^2 + B(z^*)^2 + D^*z + Dz^* + E = 0$ is the equation of a parabola if $|2B/A| = 1$, and there exists a number C and a real number p such that $p > 0$ and

$$D/A = 2p\sqrt{2B/A} - 2C$$

and

$$E/A = 2CC^* - 2p^2$$

where either square root of $2B/A$ may be used. This parabola will have C as its focus and the line $(\sqrt{2B/A})z^* + (\sqrt{2B/A})^*z - 2p = 0$ as its directrix.

On opening a course for beginners in analytic geometry, Wm. Benjamin Smith said, "You are invited to the wedding of two great mathematical disciplines. Or, to change the figure, I may say that you have hitherto been pursuing the separate courses, now one and now the other, of two streams of mathematical thought, that of pure geometry and that of pure algebra. You have now reached the point where, as we may say in Homeric phrase, the streams unite and dash their waters together."

—C. J. KEYSER in *Scripta Math.*

PROJECTILE GEOMETRY

FRANK HAWTHORNE

Faculty, Hofstra College

If air resistance is neglected, the path of a projectile, as is well known, is a parabola. This note presents a few properties of such paths which with a little ingenuity can be used to solve, usually by a remark or two, many of the common, and some not so common, elementary exercises about projectiles. Seven "theorems" (by no means new) will be listed and an indication of a method of substantiating each will be given.

General:

(1) The speed of a projectile at any point on its parabolic path is equal to the speed of free fall to that point from the directrix.

With initial position and initial speed fixed:

(2) The various parabolic trajectories caused by varying the angle of elevation of a gun have a common directrix at height $v^2/2g$ above the gun.

(3) The locus of the foci of the parabolic trajectories is a circle with center at the gun and tangent to the common directrix.

(4) The locus of the vertices is an ellipse with center halfway between the gun and the common directrix. The horizontal axis is v^2/g and the vertical axis is half of this.

(5) The limit of range, which is the envelope of the trajectories, is a parabola with focus at the gun and vertex on the common directrix.

With two points fixed: (one of these may be the gun) —

(6) If a projectile is to pass through two given points the focus of the trajectory must lie on one branch of a hyperbola whose foci are the given points and whose constant difference of distance from

these points is equal to their difference in altitude.

(7) The trajectory which provides a minimum speed at either point (or at any given level) has its focus on the line joining the two points and dividing this line into two parts whose difference is the difference in altitude of the given points.

That there is a horizontal line which has the property stated in (1) is a consequence of the conservation of energy. That this line is the directrix takes a little showing. I recommend (2) as an exercise for a superior student in analytic geometry. Theorem (3) follows from the definition of a parabola and the fact that the gun is a point on the path. The points of (4) are simply half way between those of the locus of (3) and the directrix. The curve of (5) is simply the locus of centers of circles tangent to the circle of foci and also tangent to the directrix at points other than that directly above the gun. Theorem (6) may be substantiated directly from the definitions of a parabola and of a hyperbola, while (7) is the special case of (6) which provides for the lowest possible directrix.

A PROJECT IN TRIGONOMETRY

H. T. R. AUDE
Faculty, Colgate University

This paper will present a different approach to the two basic formulas: the law of cosines and the law of sines. It is written for those students who wish to look beyond the pages of the textbook; also, it is placed before those teachers who seek suitable projects for their better students.

It is known that any side of a triangle can be expressed in terms of the other two sides and the cosines of its adjacent angles. Thus, if a triangle has the angles A, B, C and the sides a, b, c to correspond, then the relations are:

(1) $$a = b \cos C + c \cos B,\ b = c \cos A + a \cos C,\ c = a \cos B + b \cos A.$$

From these formulas, the two sets of triangle formulas mentioned will be derived. The main steps of their derivation are shown, but the details are left to the students.

Following this plan, it should be noted that the relations in (1) hold for all triangles, which therefore include the right triangle and those which have an obtuse angle. Next, write the relations in (1) as a system of three equations in cos A, cos B, and cos C. Since there is cyclic symmetry in the notation, it will be sufficient to solve for, say cos A, and then record the solutions for cos B and cos C by a cyclic change of the symbols. It turns out that

(2) $$\cos A = (b^2 + c^2 - a^2)/2bc$$

from which it follows that

(2.1) $$a^2 = b^2 + c^2 - 2bc \cos A.$$

The relations in (2) and (2.1) with the corresponding expressions for the other two angles and sides are known as the law of cosines. At this point, it may be noted that, Δ representing the determinant of the system, $\Delta = 2\,abc$ and that $\Delta \neq 0$ if the triangle exists; also, it may be well to consider the implications that follow when $\cos A = 0$, when $\cos A \to 1$, and when $\cos A \to -1$.

To derive the law of sines, return to the relations in (1) and write these as equations in a, b, and c. The result is the three homogeneous equations:

$$\begin{aligned} -a + \cos C \cdot b + \cos B \cdot c &= 0 \\ \cos C \cdot a - b + \cos A \cdot c &= 0 \qquad (3) \\ \cos B \cdot a + \cos A \cdot b - c &= 0 \end{aligned}$$

That these equations are consistent and that the solution $a:b:c$ exists presupposes that the determinant

$$D = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

will vanish. To show this in a few steps is a neat challenge to the student. When it has been shown that D = 0, then it is known that any two of the equations in (3) can be used to find the solution a:b:c. By means of the first two equations of (3), it turns out that

$$a:b:c = \begin{vmatrix} \cos C & \cos B \\ -1 & \cos A \end{vmatrix} : \begin{vmatrix} \cos B & -1 \\ \cos A & \cos C \end{vmatrix} : \begin{vmatrix} -1 & \cos C \\ \cos C & -1 \end{vmatrix}$$

$$= (\cos A \cos C + \cos B) : (\cos B \cos C + \cos A) : \sin^2 C.$$

Since $B = 180^\circ - (A + C)$, the first ratio on the right can be replaced by $\sin A \sin C$. Similarly, the second ratio on the right becomes $\sin B \sin C$. It then follows that

$$a:b:c = \sin A : \sin B : \sin C \qquad (4)$$

which is the law of sines.

These two basic formulas or laws have now been derived from the relations given in (1). An exposition of the whole — with the details added and the implications considered — may well become a student's first paper before a Mathematics Club.

"There is no excellent beauty that hath not some strangeness in the proportion."

—Francis Bacon

THE PROBLEM CORNER

EDITED BY FRANK C. GENTRY

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond the calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before October 1, 1954. The best solutions submitted by students will be published in the Fall, 1954 issue of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Frank C. Gentry, Department of Mathematics, University of New Mexico, Albuquerque, New Mexico.

PROBLEMS PROPOSED

65. *Proposed by C. W. Trigg, Los Angeles City College, Los Angeles, California.*

What is the probability that the ten's digit of the square of a randomly selected integer will be odd?

66. *Proposed by C. W. Trigg, Los Angeles City College, Los Angeles, California.*

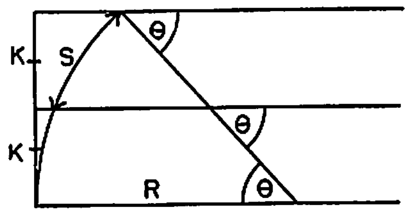
If the elements of each of the columns (or rows) of a determinant of order greater than two when taken in order form an arithmetic progression, then the value of the determinant is zero.

67. *Proposed by D. M. Morrison, St. Joseph, Missouri.* (From S. I. Jones, *Mathematical Clubs and Recreations*, p. 111).

The combined ages of Jane and Mary total 44. Now Jane is twice as old as Mary was when Jane was half as old as Mary will be when Mary is three times as old as Jane was when Jane was three times as old as Mary. How old are they?

68. *Proposed by James Woods, Student, University of New Mexico, Albuquerque, New Mexico.*

How long is the arc s in the figure?



69. *Proposed by the Editor.* (From Nicholson's, *Elements of Plane and Spherical Trigonometry,* The Macmillan Company, 1911).

On the bank of a river, there is a column 200 feet high supporting a statue 30 feet high; the statue to an observer on the opposite bank subtends an equal angle with a man 6 feet high standing at the base of the column. Required the width of the river.

70. *Proposed by the Editor.*

The point A is 3 feet from the center O of a circle of radius 5 feet . The point B is 4 feet from O and 5 feet from A. Locate the point P on the circle so that the distance AP + PB may be as small as possible.

SOLUTIONS

60. *Proposed by Frank Hawthorne, Hofstra College, Hempstead, New York.*

Two men play a game with a deck of 45 cards. Fifteen of the cards have the leter "A" on both sides, fifteen have the letter "B" on both sides and fifteen have "A" on one side and "B" on the other. After thorough shuffling, including turning cards over, the first man cuts the pack exposing one side of one card. The second man then tries to guess the letter on the underside of the exposed card, winning if he does so and losing if he doesn't. If one man always cuts and the other always guesses, do they have equal chances of winning?

Solution by Charles Pearsall, Hofstra College, Hempstead, New York.

Since the appearance of an unmatched card is probable in one-third of the cuts, and since the second man should guess the same letter as he sees, the odds are two to one in his favor.

Also solved by Morris Rosen, Hofstra College.

61. *Proposed by Charles Pearsall, Student, Hofstra College, Hempstead, New York.*

A disk L_0 of radius $R_1 = 1''$ is placed in the exact center of a circular floor and six other identical disks are placed around it so that each is tangent to L_0 and to one another, thus forming a ring L_1. Around this ring, another ring L_2 of six disks of greater radius R_2 are placed so that each disk is tangent to two disks of L_2 and to two disks of L_1. Successive rings L_3, L_4, L_5, L_6 each of six disks of increasing radii $R_3 < R_4 < R_5 < R_6$ are placed in the same way. If the disks of L_6 are tangent internally to the edge of the floor, what is the diameter of the floor?

Solution by Morris Rosen, Hofstra College, Hempstead, New York.

Let O be the center of L_0; A, B, C be the centers of three consecutive disks of L_1; and E, D be the centers of the disks of L_2 which are tangent to circles A and B and to circles B and C respectively. Let F be the point of tangency of circles A and B, and G the point of tangency of circles D and E. Then: AFB is perpendicular to OFE; $OB = 2R_1$; $OE = 2R_2$, Angle $BOF = 30°$; $BE = R_1 + R_2$. Hence, $OF = R_1\sqrt{3}$, and $FE = 2R_2 - R_1\sqrt{3}$; so in triangle BFE

$$(R_1 + R_2)^2 = R_1^2 + (2R_2 - R_1\sqrt{3})^2 \text{ or}$$

$$3R_2^2 - (2 + 4\sqrt{3})R_1R_2 + 3R_1^2 = 0$$

The latter equation may be solved for R_2 in terms of R_1 giving

$$R_2 = \frac{R_1[1 + 2\sqrt{3} \pm 2(1 + \sqrt{3})^{1/2}]}{3} = R_1C.$$

Then $R_3 = CR_2 = C^2R_1$, $R_4 = CR_3 = C^3R_1$, $R_5 = CR_4 = C^4R_1$, $R_6 = CR_5 = C^5R_1$. If D is the diameter of the room, then $D = 6R_6 = 6C^5R_1 = 6C^5$ inches. The positive sign is to be chosen in the expression for C.

62. *Proposed by C. W. Trigg, Los Angeles City College, Los Angeles, California.*

Decipher the following anagrams of the names of nineteen mathematicians: 1) NO CAB, 2) NO CART, 3) UNCLE EVAN, 4) HIS PAL LOU, 5) RAN MINE, 6) SHE CAN RUN THIS, 7) HOT ROD USE, 8) U.S. GAS, 9) TANK, 10) WE START, 11) CRUISE UP, 12) VINE ST., 13) CUT A RAIL, 14) IN SLOW, 15) ZONE, 16) LOST A TIRE, 17) NOT NEW, 18) NO RAVING, 19) CHASE ON RIM.

Solution by John Manias, Jr., University of New Mexico, Albuquerque, New Mexico.

1) BACON, 2) CANTOR, 3) VAN CEULEN, 4) PHILOLAUS, 5) RIEMANN, 6) TSCHIRNHAUSEN, 7) THEODORUS, 8) GAUSS, 9) KANT, 10) STEWART, 11) EPICURUS, 12) STEVIN, 13) CLAIRAUT, 14) WILSON, 15) ZENO, 16) ARISTOTLE, 17) NEWTON, 18) VARIGNON, 19) MASCHERONI.

Partially solved by Charles Pearsall, Hofstra College.

63. *Proposed by David T. Benedetti, University of New Mexico, Albuquerque, New Mexico.*

A golf pro wishes to arrange a tournament for the sixteen members of his club. They are to play in foursomes and each member is to play with every other member once and only once. Show how the rounds are to be arranged.

Solution by Charles Pearsall, Hofstra College, Hempstead, New York.

In the expansion of a 4th order determinant there are two sets of 12 terms in which each element appears in the same term with any other element once and only once. Thus, if the 16 players are arranged in a 4th order determinant, three rounds of the tournament may be selected in two ways by taking either the positive or the negative terms out of the expansion. The rows of the determinant will constitute the fourth round and the columns will constitute the fifth round. Since each member can play with only three others in one round, five rounds are required.

Also solved by Sharon Murnick and Morris Rosen of Hofstra College, and Joseph E. Mueller of Butler University.

64. *Proposed by the Editor.*

There are four fractions of the form a/b, $a < b < 10$, such that $N \cdot a/b$, where N is a two-digit integer, is equal to N with its digits interchanged. Find the fractions and the corresponding values of N.

Solution by Sharon Murnick, Hofstra College, Hempstead, New York.

Let $N = 10t + u$. Then, since $(10t + u)a/b = (10u + t)$, $t/u = (10b - a)/(10a - b)$. Let $10b - a = Rt$, $10a - b = Ru$. From these equations $99b = R(10t + u)$. Now 11 can only divide $10t + u$ when $t = u$, which is impossible. Hence, 11 divides R. Let $R = 11$, then 9 divides $10t + u$ and $t + u = 0 (\text{mod } 9)$. But since t and u are both less than 10, $t + u = 9$, $b = t + 1$, $a = 10 - t$, $a + b = 11$. The possible values of a and b are 2, 9; 3, 8; 4, 7; 5, 6. Also, $t = 10 - a = b - 1$, $u = (10a - b)/11$. The solutions are then $a/b = 2/9, 4/7, 3/8, 5/6$, and $N = 81, 63$ or $21, 72, 54$.

Also solved by Charles Pearsall, Morris Rosen, and S. H. Sesskin, all of Hofstra College; Joseph E. Mueller, Butler University; Jerry Jefferies, Albion College; Culbeth Sadler, Jr., Washburn Univeristy.

Late solutions: Morris Rosen, Hofstra College, 55, 56, 57, 58, 59; Jewell Magee, Central College, 56; Dorothea Peterson, Hofstra College, 56 and 59.

"If you read about Einstein's theory of relativity, you will find many references to a peculiar person called 'the observer'—the man who has a habit of falling down lifts, or getting transported by aeroplanes traveling at 161,000 miles a second .... The point is that all our knowledge of the external world as it is conceived today in physics can be demonstated to him."

—EDDINGTON

THE MATHEMATICAL SCRAPBOOK

EDITED BY H. D. LARSEN

Let no one ignorant of Geometry enter my door.

—PLATO

= ∇ =

One day a group of engineers found themselves completely stumped by a mathematical problem, so they decided to take their troubles to Steinmetz. If anyone could solve the problem, he could. They confronted him in his laboratory and outlined their question to him, "Mr. Steinmetz, what is the cubic content of the metal which is removed from a cylindrical rod two inches in diameter when a two-inch hole is bored through the rod, separating it into two pieces?"

That *was* a problem—The scientist's brow furrowed as he became lost in deep concentration; then his face brightened; and with a smile, he exclaimed, "Why of course, gentlemen, the answer is 5.33 cubic inches." Needless to say, the engineers were dumbfounded, for Steinmetz had worked the entire problem out in his head!

—GENERAL ELECTRIC CO., *Steinmetz: Latter Day Vulcan.*

= ∇ =

This cryptarithm has a unique solution.

$$\begin{array}{l} ab)\;cddefg\;(hifj \\ \qquad\;\; cch \\ \hline \qquad\;\;\;\; ee \\ \qquad\;\;\;\; ab \\ \hline \qquad\;\;\;\; hdf \\ \qquad\;\;\;\; dec \\ \hline \qquad\;\;\;\;\;\; hag \\ \qquad\;\;\;\;\;\; hhj \\ \hline \qquad\;\;\;\;\;\;\;\;\;\; c \end{array}$$

—AMER. MATH. MONTHLY

$1/\pi = .3183098$

How I remember you
O difficult equation.

—E. LEEDHAM

= ∇ =

A Minion, agile, in stature small
Panting came to great Diana's Hall,
Bearing a marble globe upon his shoulders,
Measuring one inch in its diameters.
He rolled it to the northeast corner of the Hall,
Left touching the northern and eastern walls,
Then following come three demi-gods in white,
Each bearing a globe of lustrous metal bright;
One of iron, copper one, and one of silver;
And they placed them in the order given,
Touching each the other, and at the same time,
Touching each the side walls, in a direct line,
The iron touching the marble, and its other side
Resting against the silver, in its glory and pride,—
All resting upon the oaken floor; and then
With heavy tread, and puff, and roar, Atlas came
Bearing a huge golden sphere, that filled the Hall.
Touching the four sides, floor and ceiling, and all
Radiant with beauty, resting against the silvery ball,
Making the globe's diameters in the rooms diagonals.
Tell me, all ye who mathematics know:
What size the copper sphere, and oh!
How large the iron globe? How great
The golden globe, immaculate?
The silver sphere, how great? What size?
And if presented as a prize,
What value do you hold
Would be the sphere of gold?

—*Am. Math. Monthly*, Vol. 7 (March, 1900) p. 85.

(For solution, see *Am. Math. Mo.*, Vol. 7 (June-July 1900) pp. 168-9.)

= ∇ =

At one time "analogy" meant "proportion," whence our term *Napier's analogies* in spherical trigonometry.

"Cantor's theory of the infinite — one of the most disturbing original contributions to mathematics made in the past 2,500 years."

—E. T. BELL

= ∇ =

"I shall set forth the method of forming fractions which is most pleasing to me today and it will rest in men's judgment to appraise what they see."

—R. BOMBELLI

= ∇ =

"Every new body of discovery is mathematical in form, because there is no other guidance we can have."

—C. G. DARWIN

= ∇ =

"When you come to a hard or dreary passage, pass it over; and then come back to it after you have seen its importance or found the need for it further on."

—G. CHRYSTAL

= ∇ =

"Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field."

—P. A. M. DIRAC

= ∇ =

"Mighty are numbers, joined with art, resistless."

—EURIPIDES

= ∇ =

"How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?"

—ALBERT EINSTEIN

= ∇ =

A and B are exploring a desert with the object of penetrating as far into the interior as possible. If each man can carry provisions for 20 days, and if each man can travel 12 miles per day, what is the greatest distance penetrated?

"Every major concern among the intellectual concerns of Man is a concern of Mathematics."

—C. J. Keyser

= ∇ =

Laplace, when asked who was the greatest mathematician in Germany, replied, "Phaff;" his interrogator said he should have thought Gass was. "Oh," replied Laplace, "Phaff is the greatest mathematician in Germany, but Gauss is the greatest mathematician in Europe."

—*Nature*, April 19, 1877

= ∇ =

Note that the nine digits occur once and only once in each of the following fractions.

$$\frac{15768}{3942} = \frac{17568}{4392} = \frac{23184}{5796} = \frac{31824}{7956}$$

—Math. Gazette

= ∇ =

An experimental "proof" of the theorem, Every odd number is a prime:

A prime is a number divisible only by 1 and itself. Certainly this is true for $m = 1$. We proceed step by step and in each case prove primality by dividing the odd number by all numbers less than it. In this way we find that 3, 5, and 7 are primes. Apparently $9 = 3 \times 3$ but this would spoil the theory which has covered the facts perfectly so far, so we ascribe the discrepancy to experimental error and continue undaunted. We find the theory to hold for 11 and 13 and then we test a few odd numbers chosen at random like 23, 37, 41, $\cdots$. Thus the theorem is proved.

—Leo Moser, *Math. Mag.*

= ∇ =

In how many ways can you write 100, using each of the nine digits once and only once? Illustrations:

$$48/2 + 59 + 7 + 3 + 1 + 6 = 100$$
$$(2 + 6)9 + 1 + 3 + 4 + 5 + 7 + 8 = 100.$$

THE BOOK SHELF

EDITED BY FRANK HAWTHORNE

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Frank Hawthorne, Hofstra College, Hempstead, New York.

Flatland, by Edwin A. Abbott, Dover Publications, Inc., (1780 Broadway) New York, 109 pp. Paper $1.00.

Flatland is not a new book. It comes to our attention because its popularity has required the sixth edition of this seventy year old book. The new low-priced edition is most welcome as it presents a classic work of art in a pleasant easy-to-read type that was absent in the earlier editions.

Flatland is an adventure in pure mathematics — and yet one which requires little mathematical training on the part of the reader. If you know the difference between a square and a pentagon, between an isosceles triangle and an equilateral triangle, then you know enough mathematics to read *Flatland*. If you are young in heart and love the mystery of the world around you, you will find it a fascinating tale unlike anything else you have ever read.

Here is the account of an intelligent being who lived in a world of only two dimensions. He lived on a plane and his senses could perceive nothing outside the plane on which he lived. A fascinating story is told of his life in Flatland and of the revelation of the world of three dimensions to him.

When one lays down the book, he is haunted by the mysterious possibility of yet higher dimensions—four, five, or six—making our space only a minute subspace of all creation. Could such a thing be? Could there be higher dimensional creatures free to pass through our space whenever they wished? If your immediate answer is "That is impossible!" then read the account of the sphere who interrupted the life of our friend in Flatland and think for a moment about some of the stories you have heard about flying saucers.

—MERRILL C. PALMER

Mathematics in Western Culture, by Morris Kline, Oxford University Press, (114 Fifth Avenue) New York, 16 + 484 pp. $7.50.

Everyone accepts the fact that mathematics serves the practical and theoretical needs of the engineer. Many are aware, even if only vaguely, of the role that it plays in the development of the physical sciences. Few, however, recognize the full extent to which it has been a molding force in the development of our civilization. In support of the thesis that mathematics has been such a force, Professor Kline has written a very exciting, stimulating, and provocative book.

He, like many of us, views mathematics in its broadest aspect to be a kind of spirit, a "spirit of rationality," a "spirit that challenges, stimulates, invigorates, and drives human minds to exercise themselves to the fullest. It is this spirit that seeks to influence decisively the physical, moral, and social life of man, that seeks to answer the problems posed by our very existence, that strives to understand and control nature, and that exerts itself to explore and establish the deepest and utmost implications of knowledge already obtained." (p. 10) It is with the operation of this spirit that Professor Kline is largely concerned.

Its birth came with the Greeks. Their insistence upon a deductive proof for all mathematical conclusions, their ability to free mathematics from experience and hence make it abstract, and their choice of a set of axioms for geometry in no small measure determined the character of modern mathematics and served to proclaim the supremacy of reason in human affairs. Professor Kline's contribution, needless to say, is not found in his repetition of what is already known to be historical fact; but rather in his insights and his explanations of why and how these facts grew out of the particular culture which gave rise to them and what they meant for the growth of Western civilization in general.

For example, he relates the Greek preference for an abstract mathematics and deductive reasoning to their mentality and their society. They were philosophers with a love of reason and of beauty, a delight in mental activity, and a concern with universal truths. They looked for the permanent, the ideal, and the perfect in knowledge. Their society placed the intellectuals in the highest social class, and these intellectuals had as little as possible to do with commercial and practical pursuits. It is no wonder then, that they preferred abstraction to experimentation, and deduction to induction. Their disdain of the practical is related to the fact that they did nothing to develop an adequate number system; the static quality of their geometry is paralleled by the static quality of their architecture and their drama.

Though these few words do not do justice to Professor Kline's complete discussion of the Greeks, it is hoped that they convey some idea of the nature of his analyses; for it is just these which make reading his book so rewarding. He discusses, among other topics, the work of Copernicus and Kepler and the effect of each on the philosophy and religion of his time; the relation of the Renaissance artists to mathematics and the consequent development of projective geometry; Descartes' analytic geometry and its relation to his philosophical studies; Galileo's quantitative description of nature; Newton's universal laws and the prediction thereby made possible as well as the reorganization in philosophic, religious, and literary thought which they engendered; the statistical approach to the study of man, and the statistical, as opposed to the deterministic, view of nature; the development of the non-Euclidean geometries and the resultant upheaval in philosophy; and finally the theory of relativity.

The approach used throughout is essentially historical. What emerges is a vivid account of how the development of mathematics and the development of all other aspects of civilization were mutually influential. The book should be an enriching experience for everyone as well as a forceful reminder that mathematics is not simply a series of high technical operations, but rather an alive and highly imaginative area of inquiry which has markedly affected contemporary life and thought.

—AZELLE BROWN

Introduction to Logic, by Irving M. Copi, Macmillian Co., (60 Fifth Ave.) New York, 16 + 461 pp. $4.00

The book consists of three parts, Language, Deduction and Induction, with emphasis on the latter two, but a strong enough treatment of the first to point out its necessity and its hindrances to logic. His treatment of definition is indicative of his very careful development of strict language usage.

The section on deduction thoroughly treats categorical systems from the standpoint of syllogisms, diagrams and symbolic logic. Generous use of truth tables facilitates the understanding of this section and gives a good basis for the section on symbolic logic.

His treatment of induction is replete with actual examples from scientific literature, being very careful to show the distinction between induction and deduction. A systematic treatment of arguments by analogy gives a good foundation for the discussion of the scientific method. His treatment of probable inference and probability is outstanding.

The book throughout is very lucidly written, giving many examples and a wealth of exercises for student practice. Although it appears to be too lengthy to be covered in a three-hour course, it is arranged so that parts may be treated lightly or left out entirely. It will serve as an excellent reference for courses in mathematics dealing with its general aspects.

—B. E. GILLAM

Math is Fun, by Joseph Degrazia, Emerson Books, Inc., (251 West 19th Street) New York, 159 pp. $2.75.

Here is a collection of problems, old and new, which require little or no technique but considerable reasoning power. Many are well known, but this does not detract from their charm. Anyone who has toyed with such problems will recognize old friends and make new ones.

In such a book the problem of selection is important, and it is in this regard that one may disagree with the author. For instance, the everlasting SEND + MORE = MONEY is included, while FORTY + TEN + TEN = SIXTY is not. However, such things are purely matters of choice. Thank goodness, "What Color Was The Bear?" is absent.

Altogether, this is a book which can furnish much pleasure. It is sufficiently elementary that a child of twelve (the reviewer's son to be specific) can understand and solve some of the problems and sufficiently advanced to cause some scratching of grey hairs.

—F. HAWTHORNE

BOOKS RECEIVED BY THE BOOK SHELF EDITOR —

A First Course In Ordinary Differential Equations, by Rudolph E. Langer, John Wiley and Sons, Inc., (440 Fourth Ave.) New York, 12 + 249 pp. $4.50.

An Analytical Calculus, Vol. I. by E. A. Maxwell, Cambridge University Press, (32 East 57th St.) New York, 12 + 165 pp. $2.75.

Elementary Differential Equations, by Lyman M. Kells, McGraw-Hill Book Company, Inc., (330 West 42nd St.) New York, 10 + 266 pp. $4.00.

Elements of Statistics, by H. C. Fryer, John Wiley and Sons, Inc., (440 Fourth Ave.) New York, 8 + 262 pp. $4.75.

First Course In Calculus, by Hollis R. Cooley, John Wiley and Sons, Inc., (440 Fourth Ave.) New York, 12 + 643 pp. $6.00.

Geometry And The Imagination, by D. Hilbert and S. Cohn-Vassen, Chelsea Publishing Co., (231 West 29th St.) New York, 9 + 357 pp. $6.00.

Ideal Theory, by D. G. Northcott, Cambridge University Press, (32 East 57th St.) New York, 8 + 112 pp. Paper $2.50.

Introductory College Mathematics, by Adele Leonhardy, John Wiley and Sons, Inc., (440 Fourth Ave.) New York, 9 + 459 pp. $4.90.

Methods of Algebraic Geometry, Birational Geometry, Vol. III, Book V, by W.V.D. Hodge and D. Pedoe, Cambridge University Press, (32 East 57th St.) New York, 10 + 336 pp. $7.50.

Plane Trigonometry, by Paul R. Rider, The Macmillan Co., (60 Fifth Ave.) New York, 8 + 180 pp. $3.00.

Principles Of Numerical Analysis, by Alston S. Householder, McGraw-Hill Book Co., (330 West 42nd St.) New York, 10 + 247 pp. $6.00.

Stability Theory Of Differential Equations, by Richard Bellman, McGraw-Hill Book Co., (330 West 42nd St.) New York, 13 + 166 pp. $5.50.

Theory Of Equations, by C. C. MacDuffee, John Wiley and Sons, Inc., (440 Fourth Ave.) New York, 7 + 120 pp. $3.75.

KAPPA MU EPSILON NEWS

EDITED BY LAURA Z. GREENE, HISTORIAN

Alabama Beta Chapter served as co-hosts with the college for the meeting of the Alabama Teachers of College Mathematics which met on the campus of the Alabama State Teachers College at Florence.

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A number of members of the **Illinois Delta** presented papers at a meeting of the Chicago Catholic Science Teacher's Association which was held at the Museum of Science and Industry in Chicago on April 11, 1953. The topics selected for the papers concerned materials which could be used for mathematics clubs at both the high school and college levels.

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Illinois Delta will have a radio broadcast over Station WJOL, Joliet. It will be an adaptation of the life of Luca Pacioli, written by Emmett Taylor.

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The members of **Indiana Alpha** arranged an exhibit of mathematical posters and models for an open house held at Manchester College.

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Indiana Beta Chapter held its annual reunion dinner for the active members and alumni March 28 at the Athenacum.

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Four students and two faculty members from **Iowa Alpha** attended the National Convention of Kappa Mu Epsilon which was held at Battle Creek, Michigan, in April. Clyde Dilly entered his paper "Reverse Notation for Numbers" in competition at the convention.

The **Iowa Alpha** officers for the year 1953-54 are: Wayne Stark, president; Doris Reeves, vice-president; Tom Yager, secretary-treasurer.

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Dr. O. J. Peterson, Sponsor of **Kansas Beta** and former National President of Kappa Mu Epsilon, has returned to his work after several months absence which was caused by a hip injury last October.

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The annual pledge program of Kansas Gamma was held on April 20. The program was an impersonation of a Boy-Mathematics. The five pledges wrote the script and brought in the five phases of mathematics usually stressed by Kappa Mu Epsilon, i.e., business, biological science, physical science, engineering and pure mathematics.

Kathleen Feldhousen, President of Kansas Gamma, received the Agnesi Medal in recognition of the worthwhile project of indexing the entire set of twenty-seven volumes of the *Kansas Association of Teachers of Mathematics Bulletins.*

The Guard of Excellence was given for the first time by the Kansas Gamma Chapter. This consists of a gavel (or other charm indicative of a special chapter office) bearing three pearls and attached to the winner's Kappa Mu Epsilon key. This is the highest award given by the local chapter and went to Miss Bernadine Law for her unusual achievement in the presentation of a paper which merited second place at the national convention. The citation was written by Sister De Montfort Knightley, Chapter Treasurer in 1947-1948.

The underclassmen award for excellence in work in mathematics on the freshman-sophomore level went to Miss Betty Gross of Kansas Gamma.

Five members of the local chapter attended the Ninth Biennial Convention of Kappa Mu Epsilon held at Battle Creek, Michigan, April 17-18. Miss Betty Becker of Kansas Gamma served on the Resolutions Committee in the absence of Sister Helen Sullivan who was unable to attend because of illness.

Miss Bernadine Law of Kansas Gamma presented her prizewinning paper by invitation at a special assembly of the students of St. Benedict's College on April 22.

Officers of Kansas Gamma for the year 1953-54 are: Bernadine Law, President; Kathleen Feldhousen, Vice-president; Suzanne Swann, Secretary; Betty Gross, Treasurer; Virginia Breland, Publicity; Mary Ellen Kuhlman, Musician; and Sister Jeanette Obrist OSB, Faculty Sponsor and Corresponding Secretary.

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Michigan Beta presented E. T. Bell's *Men of Mathematics* to Alfred Cambridge who won the Annual Freshman Mathematics Award for writing the best paper on a Mathematics Department Examination.

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Missouri Beta is now giving an annual award to the outstanding freshman student in mathematics and to the outstanding senior student in mathematics enrolled at Central Missouri State College. The freshman award for 1952-53 was presented to Mr. Eugene Wilson, and Mr. Richard Laatsch received the senior award. Mr. Laatsch was valedictorian of his graduating class. Two other members of Missouri Beta, Miss Bess Rickman and Mr. Richard L. Smith, placed second and third in the 1953 graduating class.

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Mississippi Gamma arranged a mathematics exhibit for the annual Career Day at Mississippi Southern College.

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Herman Rewinkle, a member of Nebraska Alpha, served as president of the student council, was elected to Who's Who in American Colleges, and was awarded the Victor P. Morey Memorial Scholarship which honors the late President of Nebraska State Teachers College. President Morey was a member of Nebraska Alpha.

Marvin Stone of Nebraska Alpha served as editor of the college yearbook.

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On December 29, the New York Alpha Chapter made a visit to the Special Devices Center at the Sands Point Naval Training Center.

A chapter news letter written for chapter members proved interesting enough to the college that they printed enough copies to send to all the Hofstra College Alumni.

Dr. C. V. Newsom, past President of Kappa Mu Epsilon, an honorary member of the New York **Alpha** Chapter, and Associate Commissioner for Higher Education of New York was awarded the honorary degree of Doctor of Humane Letters by Hofstra College in June, 1953.

The New York **Alpha** Chapter is again sponsoring the "Help Sections" in which Kappa Mu Epsilon members help those who are having difficulties in mathematics.

Thomas K. Howard of **New York Alpha** was the winner of the Kappa Mu Epsilon award given to the best student in freshman mathematics. The winner is chosen on the basis of grades earned in freshman courses in mathematics and a two-hour examination. As prizes, Mr. Howard received the book, *What is Mathematics*, by Courant and Robins, an award certificate, and the honor of having his name engraved on a plaque.

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In the second annual Kappa Mu Epsilon Scholarship Competition, Miles E. Vance of **Ohio Alpha** was awarded first place, Richard S. Krowicki was awarded second place, and Robert Foster was given special mention.

Miles E. Vance, a delegate to the national convention from **Ohio Alpha**, was granted a graduate assistantship in the Physics Department at Ohio State University.

Carl Hawk, President of **Ohio Alpha** (1952-53), joined the faculty at Bowling Green Junior High School, Bowling Green, Ohio.

William Elderbrock, a 1953 graduate of **Ohio Alpha**, entered the Ohio State University Medical School.

Robert Foster, a member of **Ohio Alpha**, who recently received his master's degree from Bowling Green State University, joined the staff of Tri-State College, Angola, Indiana.

James Faber, former treasurer of **Ohio Alpha** was granted a graduate assistantship at Washington University, St. Louis, in geo-physics.

Dr. Gilbert de B. Robinson, University of Toronto, and visiting professor at Michigan State College was initiated into the Ohio Alpha Chapter, December 9, 1953. He gave an informal talk on "Comparison of College Training in Mathematics in England, Canada, and the United States" to the chapter after a dinner held in his honor.

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A thorough, but non-statistical study, has led to the conclusion that during the past three years 75% of the mebers of Oklahoma Alpha found non-teaching occupations more attractive than teaching.

Professor Raymond Carpenter has been granted a leave of absence for 1953-54 for further work toward the D.Ed. Degree.

Professor L. P. Woods, one of the founders of the original chapter of Kappa Mu Epsilon, will serve Oklahoma Alpha as official sponsor for 1953-54.

Mr. Billy R. Turney has been granted the M.S. Degree and will teach mathematics in Tahlequah High School.

Mr. Norman Watley has been elected Superintendent of Schools, Strand, Oklahoma.

1955 BIENNIAL CONVENTION

The National Council is happy to announce that the 1955 biennial convention of Kappa Mu Epsilon will be held at

Nebraska Alpha

Nebraska State Teachers College

Wayne, Nebraska

on

May 6th and 7th, 1955

Wayne, Nebraska is about 125 miles north of Lincoln; about 55 miles southwest of Sioux City, Iowa; and 110 miles northwest of Omaha. It is located on State Highway 15. Although there is no railroad or commercial plane service to Wayne directly, the connections may be made at Norfolk or Sioux City with bus service three times a day, and once daily at Lincoln and Omaha via Fremont.

Nebraska Alpha offers us the facilities of their four dormitories, a fine hotel, and a very nice new motel. Details on these accommodations will be sent to you by the host chapter in time to make your arrangements.

Kappa Mu Epsilon, its officers and members, wish to express their thanks to Nebraska Alpha for its offer to serve as our host in 1955.

Charles B. Tucker
National President

DIRECTIONS FOR PAPERS TO BE PRESENTED AT THE KAPPA MU EPSILON CONVENTION

WAYNE, NEBRASKA
May 6 and 7, 1955

At past conventions we have had papers presented by K.M.E. members and prizes awarded. Now is the time for all chapter sponsors and members to start developing papers for our next convention. Most of the papers have been of excellent quality for the level of work represented, although several people have remarked that their chapters could have done better. Here is the challenge; here is the opportunity. Start now to encourage at least one of your members to prepare a paper to submit at the Nebraska Convention. Read the rules below, and start to work.

Who may submit papers: Any member may submit a paper for use on the convention program. Papers may be submitted by both graduates and undergraduates; however, they will not compete against each other. Awards will be granted for the best papers presented by undergraduates. If enough papers are presented by graduates, special awards may be given for their best paper or papers.

Subject: The material should be within the scope of understanding of undergraduates, preferably the undergraduate who has completed calculus. The Selection Committee will naturally favor papers that are within this limitation and which can be presented with reasonable completeness within the time limit prescribed.

Time limit: The average time limit should be twenty minutes but may be extended to thirty minutes on recommendation of the Selection Committee.

Paper: A rather complete outline of the paper to be presented must be submitted to the Selection Committee accompanied by a description of charts, models, or visual aids that are to be used in presenting the paper. A carbon copy of the complete paper may be submitted if desired. All papers must indicate that the individual submitting the paper is a member of K.M.E., and whether he is a graduate or undergraduate student.

Date and place due: The papers must be submitted before February 15, 1955, to the office of the National President.

Selection: The Selection Committee will choose about eight papers from those submitted for presentation at the convention. All other papers will be listed by title.

Prizes:

1. The authors of all papers presented will be given a two-year extension of their subscription to THE PENTAGON.
2. The two or three best papers presented by undergraduates, according to the judgment of a combined faculty and student committee, will be awarded copies of the *Mathematical Dictionary* suitably inscribed.
3. If a sufficient number of papers are submitted by graduate students and selected for presentation, then one or more similar prizes will be awarded to this group.

Charles B. Tucker
National President

PROGRAM TOPICS
(Spring Semester 1952-53—Fall 1953-54)

Alabama Beta, Alabama State Teachers College, Athens
Twenty Questions and Answers, by J. D. Clanton
The Doctoral Program, by Dr. R. C. Boles

Alabama Gamma, Alabama College, Montevallo
Aesthetic Values of Mathematics, by Dr. Rosa Lea Jackson
Non-Euclidean Geometry, by Miss Mamie Broswell
Polyhedrons, by Miss Ruth Peer

Colorado Alpha, Colorado A & M, Fort Collins
Functions, by Calvin A. Rogers
Snow Plow Problem, by David Wait

Illinois Gamma, Chicago Teachers College, Chicago
Ciphers and Codes, by Dr. J. Sachs
Alexander G. Bell, by Representative of Bell Telephone Company
Proof of Five Regular Polyhedrons, by Elaine Michenfelder
History and Significance of Pi, by Jerry Donohue
Orientation and Puzzles and Problems, by Jerry Donohue

Illinois Delta, College of St. Francis, Joliet
David Eugene Smith, by Sister Rita Clare
Comparisons and Contrasts, by Patricia Kasak
From Plato's Republic, by Geraldine Knowles
Venus, by Martha Marie
Christopher Clavius, by Marianne Hasse
The Discovery of All Dark Things, by Patricia McLaughlin
The Planet Saturn, by Jeanne Schwinn

Indiana Alpha, Manchester College, North Manchester
The Human Elements of Mathematics, by Dr. C. S. Morris
Standards of Kappa Mu Epsilon, Panel Discussion
Value of Kappa Mu Epsilon, by J. E. Dotterer
Is There an Infinity, by John Mack

Indiana Beta, Butler University, Indianapolis
Electric Computers, by Chester Rector
Cartography, by David Woodward
Mathematics and the Telephone, by Earl Dickey
The Naval Ordnance Plant, by William Fuller
Mathematics and Music, by Mary A. Evans
Mathematics of the Ancients, by Raymond Cowan
Milestones in Higher Mathematics, by Richard B. Thompson
Nomographs, by Frank E. Tardy
Mathematics and Chemistry, by Jack Bowers

Iowa Alpha, Iowa State Teachers College, Cedar Falls
Lumsal Arithmetic, by E. W. Hamilton
Reverse Notation for Numbers, by Clyde Dilley
Nim, A Chinese Game of Numbers, by Harold Gillman
The Locus of Points from n Foci, by Clyde Dilley

Iowa Beta, Drake University, Des Moines

Naperian Logarithms, by Harry Drown
Probability of Winning at "Craps," by Paul Gilman
Boolean Algebra, by Theodore Kowalchuck
Solutions of Problems, Candidates for Initiation
Relationships between Derivatives of Arcsin x, by Neal Llewellyn

Kansas Alpha, Kansas State Teachers College, Pittsburg

Non-Euclidean Geometries, by James Patterson
Mathematical Recreations, by Harold Lee Thomas
Experiences at the University of Chicago While a Carnegie Visiting Teacher in Mathematics, by Dr. J. D. Haggard

Kansas Beta, Kansas State Teachers College, Emporia

Topology, by Homer Hackett
Mathematical Tricks, by Lester Laird
Report of the 1953 Convention, by delegates

Kansas Gamma, Mount Scholastica College, Atchison

Mathematics and Business, by Bernadine Law
Budgeting, by Kathleen Feldhousen
Savings, by Mary Ellen Kuhlman
Stocks, *Movies*: *Opportunity U.S.A.*, *What Makes Us Tick*
Balance and Symmetry, by Bernadine Law
Proportions versus Dimensions, by Kathleen Feldhousen
Perspective, by Charlotte Raur
Lines in Pictures, by Jo Ann Fellin
Chemistry of Paints, by Mary M. Kuhlman
Fundamentals of Mechanical Drawing, by Lucy Bradley
Job Opportunities for the Student Who Is Trained in Mathematics, by Donna Rump
The Works of Raphael, Michaelangelo, and Leonardo da Vinci, by Rita Moran
Ancient Architecture: *Grecian, Roman, and Egyptian*, by Suzanne Swann
Modern Art versus Ancient Art, by Betty Becker and Jo Ann Fellin

Kansas Delta, Washburn University, Topeka

Mathematics in the Field of Psychology, by Dr. John A. Myers
The Rectangular Coordinate System, Movie
Mathematics at Washburn, by Barbara Bartley
Some Interesting Mathematical Problems, by Dean Gettler
Mathematics, Pure or Otherwise, by Nancy Marsh
Highlights of the National Convention, by Mickey Welty, Nancy Marsh, Loren McMurray, and Alfred Cheng

Louisiana Beta, Southwestern Louisiana Institute, Lafayette

Higher Geometry, by Margaret M. LaSalle
Time and Its Measurement, by Dr. Dale M. Delaitsch
Empirical Formulae from Experimental Data, by Lloyd D. Vincent

Michigan Beta, Central Michigan State College, Mt. Pleasant
The Measurement of Astronomical Distances, by Donald Poole
Fallacies in Mathematics, by Helena Hayward
The Circular Slide Rule, by Erland Engstrom
Jumping off at Infinity, by Donald Jennings
The Ninth Biennial Convention, by James Bower

Mississippi Gamma, Mississippi College, Hattiesburg
The Solution of First Order Diophantine Equations, by Jewel Magee
Congruent Integers, by Sybil Kirk
Cauchy-Riemann Conditions, by Charles R. Storey

Missouri Alpha, Southwest Missouri State, Springfield
Curve Tracing, by Robert Grim
Fibonacci Series, by Alice Killingsworth
Geometric Representation of the Binomial Theorem, by Henry Beersman
Solitaire with Checkers, by Norma Jones and David Robinette
Some Geometric and Algebraic Tricks, by Harold Skelton

Missouri Beta, Central Missouri State Teachers College, Warrensburg
Puzzles Constructed on Various Number Bases, by William Vardeman
The Number Pi, by Royce Bradley
Evolutes and Involutes, by Ralph Coleman
A Coordinate System with Oblique Axes, by Jean Crecelius
Life of Geronomo Cardano, by Margaret Ann Handley
Tacit Assumptions, by Richard Laatsch
Love and Mathematics, by Dr. Robert Rothschild

Nebraska Alpha, Nebraska State Teachers College, Wayne
Puzzles and "Pascal," by Charlotte Baker and Donna Miller
Demonstration of Teaching Aids in Mathematics, by Beulah Bornhoft
Reports of the Papers Presented at the Convention, by the delegates
The Use of the Slide Rule, by Robert Terry Mansbel

New York Alpha, Hofstra College, Hempstead
Cryptanalysis, by Samuel Sesskin
Some Calculus Without Calculus, by Charles Pearsall
Mathematical Chess, by Helen Dawson and Richard Thorgrimson
Paper Folding, by Mr. Hawthorne
Curve Tracing, by Dorthea Peterson
The National Convention, by the delegates
Magic Squares, by Sue Rae Waldman
Relations of Conics to Higher Plane Curves by Inversion, by John Cornwell

New Jersey Alpha, Upsala College, East Orange
Graphical Solution of Cubic Equations, by Ellis Fuls

An Introduction to Modern Geometry, by Dorothy Ross
Theory of Numbers, by Fred Kirtland
The Prime Integers, by Vera Anne Versfelt

New Jersey Beta, New Jersey State Teachers College, Montclair
Quality Control in Industry, by Charles Senoala
The Ninth Biennial Convention, by the delegates

North Carolina Alpha, Wake Forrest College, Wake Forrest
The Atomic Reactor, by Dr. T. J. Turner
Magic Squares, by Sarah Abernethy
History and Purpose of Kappa Mu Epsilon, by Alease Roach
Non-Euclidean Geometries, by Dr. Gene Medlin
Determining Logarithms, by Joe Stokes
Peano's Axioms, by Bob Johnson
Problems Encountered by the Mathematics Teacher, by Dr. Tom Reynolds
Mathematical Games of Solitaire, by Dr. Nudlin
Square Roots, by Kenneth Byrd
Derivation of Pi, by Audrey Beck

Ohio Alpha, Bowling Green State University, Bowling Green
The Geometry of VanStaudt's Points, by Dr. Frank C. Ogg
Mathematics and the Next Fifty Years in Science and Engineering by Dr. David Dietz, Scripps-Howard Science Editor
Comparison of College Training in Mathematics in England, Canada, and the United States, by Dr. Gilbert De B. Robinson, University of Toronto
High Speed Computation on the MIDAC with Applications in Science and Industry, by Dr. John W. Carr, Willow Run Research Center
Preparations of Mortality Tables, by Professor Harry Mathias
Some Interesting Triangles, by Dr. Frank C. Ogg
An Introduction to Vectors, by Dr. David Krabil
Cybernetics, by William Elderbrook
Some One-Sided Surfaces, by Dr. Harold Tinnappel

Ohio Beta, College of Wooster, Wooster
Modern Algebra, by Mr. Vockel
History of Symbolic Logic, by Mr. Linnell
Use of LaPlace Transform, by Professor Fobes

Ohio Gamma, Baldwin-Wallace College, Berea
NACA Research, movie
Modern Jet Plane, movie

Oklahoma Alpha, Northeastern State College, Tahlequah
Major and Minor Requirements in Mathematics, by Raymond Carpenter
Mathematics in Industry, by L. P. Woods

Pennsylvania Alpha, Westminister College, New Wilmington

Various Proofs of the Pythagorean Theorem, by Lyle Beall, Sam Shane, and Jane Carbines

Introduction to the Theory of Games, by Merrill Palmer

An Experimental Evaluation of e, by Ralph Eicher

The Problem of the Jeeps, by Sam Shane

Tennessee Alpha, Tennessee Polytechnic Institute, Cookeville

Probability and its Engineering Applications, by James N. Luton, Clyde Parker Ferguson, and Robert B. Aylor

National Convention, by J. L. Comer and R. H. Moorman

Texas Epsilon, North Texas State College, Denton

Some Problems in the American Mathematical Monthly, by Glynn St. Clair, Wanda Allen, and Jane Pinkerton

The Conic Sections, Using Complex Numbers, by Joe Ballard

On Gaussian Integers, by Jim Chamblee

The Ninth Biennial Convention, by Pete Reames and Joe Ballard

Transfinite Numbers, by Professor Harlan Miller

Wisconsin Alpha, Mount Mary College, Milwaukee

Report on National Convention, by Ann Meara and Joan Bartelsen

Explanation of Three-Dimensional Tic-tac-toe, by Ruth Renwick

Methods of Thinking and Methods of Reduction Used in Subtraction by Sister Mary Petronia

Finger and Complementary Multiplication, by Sister Mary Petronia

Number Bases, by Colette Frendries

Introduction of Sigma and Pi Symbols, by Toni Langfeld

Euclidean Algorithm Applied to Integers and Polynomials, by Kay Cunningham