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Kappa Mu Epsilon, national honorary mathematics fraternity, was founded in 1931. The object of the fraternity is four-fold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievement in the study of mathematics in the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

HOW TO DRAW A STRAIGHT LINE: A LECTURE ON LINKAGES

A. B. KEMPE

ACKNOWLEDGMENT: The PENTAGON is grateful to the Macmillan Company for their gracious permission to reprint this classic lecture on linkages, first published in 1877. Unfortunately, our facilities did not permit the photographic reproduction of the original illustrations, and the charm and delicate detail of the drawings by H. R. Kempe, the author's brother, had to be sacrificed.

The great geometrician Euclid, before demonstrating to us the various propositions contained in his *Elements of Geometry,* requires that we should be able to effect certain processes. These *Postulates,* as the requirements are termed, may roughly be said to demand that we should be able to describe straight lines and circles. And so great is the veneration that is paid to this master-geometrician, that there are many who would refuse the designation of "geometrical" to a demonstration which requires any other construction than can be effected by straight lines and circles. Hence many problems—such as, for example, the trisection of an angle—which can readily be effected by employing other simple means, are said to have no geometrical solution, since they cannot be solved by straight lines and circles only.

It becomes then interesting to inquire how we can effect these preliminary requirements, how we can describe these circles and these straight lines, with as much accuracy as the physical circumstances of the problems will admit of.

As regards the circle we encounter no difficulty. Taking Euclid's definition, and assuming, as of course we must, that our surface on which we wish to describe the circle is a plane, (1)[1] we see that we have only to make our tracing-point preserve a distance from the given centre of the circle constant and equal to the required radius. This can readily be effected by taking a flat piece of any form, such as the piece of cardboard I have here, and pass-

[1] These figures refer to notes at the end of the lecture.

ing a pivot which is fixed to the given surface at the given centre through a hole in the piece, and a tracer or pencil through another hole in it whose distance from the first is equal to the given radius; we shall then, by moving the pencil, be able, even with this rude apparatus, to describe a circle with considerable accuracy and ease; and when we come to employ very small holes and pivots, or even large ones, turned with all that marvelous truth which the lathe affords, we shall get a result unequalled perhaps among mechanical apparatus for the smoothness and accuracy of its movement. The apparatus I have just described is of course nothing but a simple form of a pair of compasses, and it is usual to say that the third Postulate postulates the compasses.

But the straight line, how are we going to describe that? Euclid defines it as "lying evenly between its extreme points." This does not help us much. Our textbooks say that the first and second Postulates postulate a ruler (2). But surely that is begging the question. If we are to draw a straight line with a ruler, the ruler must itself have a straight edge; and how are we going to make the edge straight? We come back to our starting-point.

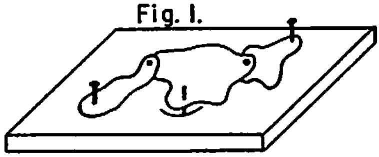
Now I wish you clearly to understand the difference between the method I just now employed for describing a circle, and the ruler method of describing a straight line. If I applied the ruler method to the description of a circle, I should take a circular lamina, such as a penny, and trace my circle by passing the pencil around the edge, and I should have the same difficulty that I had with the straight-edge, for I should first have to make the lamina itself circular. But the other method I employed involves no begging the question. I do not first assume that I have a circle and then use it to trace one, but simply require that the distance between two points shall be invariable. I am of course aware that we do employ circles in our simple compass, the pivot and the hole in the moving piece which it fits are such; but they are used not because they are the curves we want to describe (they are not so, but are of a different size), as is the case with the straight-edge, but because, through the impossibility of constructing pivots or holes of no finite dimensions, we are forced to adopt the best substitute we can for making one point in the moving piece remain at the same spot. If we employ a very small pivot and hole, though they be not truly circular, the error in the description of a circle of moderate dimensions will be practically infinitesimal, not perhaps varying beyond the width of the thinnest line which the tracer can be made to describe; and even when we employ large pivots and holes we shall get results as accurate, because those pivots and holes may be made by the employment of very small ones in the machine which makes them.

It appears then, that although we have an easy and accurate method of describing a circle, we have at first sight no corresponding means of describing a straight line and there would seem to be a substantial difficulty in producing what mathematicians call the simplest curve, so that the question how to get over that difficulty becomes one of a decided theoretical interest.

Nor is the interest theoretical only, for the question is one of direct importance to the practical mechanician. In a large number of machines and scientific apparatus it is requisite that some point or points should move accurately in a straight line with as little friction as possible. If the ruler principle is adopted, and the point is kept in its path by guides, we have, besides the initial difficulty of making the guides truly straight, the wear and tear produced by the friction of the sliding surfaces, and the deformation produced by changes of temperature and varying strains. It becomes therefore of real consequence to obtain, if possible, some method which shall not involve these objectionable features, but possess the accuracy and ease of movement which characterize our circle-producing apparatus.

Turning to that apparatus, we notice that all that is requisite to draw with accuracy a circle of any given radius is to have the distance between the pivot and the tracer properly determined, and if I pivot a second "piece" to the fixed surface at a second point having a tracer as the first piece has, by properly determining the distance between the second tracer and pivot, I can describe a second circle whose radius bears any proportion I please to that of the first circle. Now, removing the tracers, let me pivot a third piece to these two *radial* pieces, as I may call them, at the points where the tracers were, and let me fix a tracer at any point on this third or *traversing* piece. You will at once see that if the radial pieces were big enough the tracer would describe circles or portions of circles on *them,* though they are in motion, with the same ease and accuracy as in the case of the simple circle-drawing apparatus; the tracer will not however describe a circle on the *fixed* surface, but a complicated curve.

Fig. 1.



This curve will, however, be described with all the ease and accuracy of movement with which the circles were described, and if I wish to reproduce in a second apparatus the curves which I produce with this, I have only to get the distances between the pivots and tracers accurately the same in both cases, and the curves will also be accurately the same. I could of course go on adding fresh pieces *ad libitum,* and I should get points on the structure produced, describing in general very complicated curves, but with the same results as to accuracy and smoothness, *the reproduction of any particular curve depending solely on the correct determination of a certain definite number of distances.*

These systems, built up of pieces pointed or pivoted together, and turning about pivots attached to a fixed base, so that the various points on the pieces all describe definite curves, I shall term "link-motions," the pieces being termed "links." As, however, it sometimes facilitates the consideration of the properties of these structures to regard them apart from the base to which they are pivoted, the word "linkage" is employed to denote any combination of pieces pivoted together. When such a combination is pivoted in any way to a fixed base, the motion of points on it not being necessarily confined to fixed paths, the link-structure is called a "linkwork:" a "linkwork" in which the motion of every point is in some definite path being, as before stated, termed a "link-motion." I shall only add to these expressions two more: the point of a link-motion which describes any curve is called a "graph," the curve being called a "gram" (3).

The consideration of the various properties of these "linkages" has occupied much attention of late years among mathematicians, and is a subject of much complexity and difficulty. With the purely mathematical side of the question I do not, however, propose to deal today, as we shall have quite enough to do if we confine our attention to the practical results which mathematicians have obtained, and which I believe only mathematicians could have obtained. That these results are valuable cannot I think be doubted, though it may well be that their great beauty has led some to attribute to them an importance which they do not really possess; and it may be that fifty years ago they would have had a value which, through the great improvements that modern mechanicians have effected in the production of true planes, rulers and other exact mechanical structures, cannot now be ascribed to them. But linkages have not at present, I think, been sufficiently put before the mechanician to enable us to say what value should really be set upon them.

The practical results obtained by the use of linkages are but few in number, and are closely connected with the problem of "straight-line motion," having in fact been discovered during the investigation of that problem, and I shall be naturally led to consider them if I make "straight-line motion" the backbone of my lecture. Before, however, plunging into the midst of these linkages it will be useful to know how we can practically construct such models as we require; and here is one of the great advantages of our subject—we can get our results visibly before us so very easily. Pins for fixed pivots, cards for links, string or cotton for the other pivots, and a dining-room table, or a drawing-board if the former be thought objectionable, for a fixed base, are all we require. If something more artistic be preferred, the plan adopted in the models exhibited by me in the Loan Collection can be employed. The models were constructed by my brother, Mr. H. R. Kempe, in the following way. The bases are thin deal boards painted black; the links are neatly shaped out of thick cardboard (it is hard work making them, you have to sharpen your knife about every ten minutes, as the cardboard turns the edge very rapidly); the pivots are little rivets made of catgut, the heads being formed by pressing the face of a heated steel chisel on the ends of the gut after it is passed through the holes in the links; this gives a very firm and smoothly-working joint. More durable links may be made of tinplate; the pivot-holes must in this case be punched, and the eyelets used by bootmakers for laced boots employed as pivots; you can get the proper tools at a trifling expense at any large tool shop.

Now, as I have said, the curves described by the various points on these link-motions are in general very complex. But they are not necessarily so. By properly choosing the distances at our disposal we can make them very simple. But can we go to the fullest extent of simplicity and get a point on one of them moving accurately in a straight line? That is what we are going to investigate.

To solve the problem with our single link is clearly impossible: all the points on it describe circles. We must therefore go to the next simple case—our three-link motion. In this case you will see that we have at our disposal the distance between the fixed pivots, the distances between the pivots on the radial links, the distance between the pivots on the traversing link, and the distances of the tracer from those pivots; in all six different distances. Can we choose those distances so that our tracing-point shall move in a straight line?

The first person who investigated this was the great man James Watt. "Watt's Parallel Motion" (4), invented in 1784, is well known to every engineer, and is employed in nearly every beam-engine. The apparatus, reduced to its simplest form, is shown in Fig. 2.

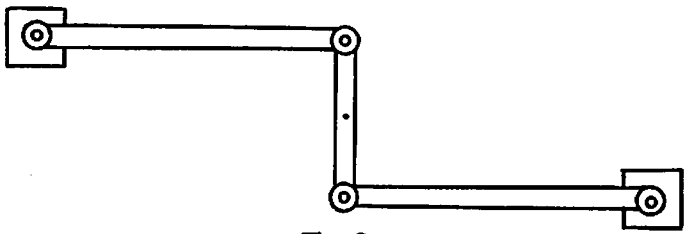


Fig. 2.

The radial bars are of equal length,—I employ the word "length" for brevity, to denote the distance between the pivots; the links of course may be of any length or shape,—and the distance between the pivots or the traversing link is such that when the radial bars are parallel the line joining those pivots is perpendicular to the radial bars. The tracing-point is situated half-way between the pivots on the traversing piece. The curve described by the tracer is, if the apparatus does not deviate much from its mean position, approximately a straight line. The reason of this is that the circles described by the extremities of the radial bars have their concavities turned in opposite directions, and the tracer being half-way between, describes a curve which is concave neither one way nor the other, and is therefore a straight line. The curve is not, however, accurately straight, for if I allow the tracer to describe the whole path it is capable of describing, it will, when it gets some distance from its mean position, deviate considerably from the straight line, and will be found to describe a figure 8, the portions at the crossing being nearly straight. We know that they are not quite straight, because it is impossible to have such a curve partly straight and partly curved.

For many purposes the straight line described by Watt's apparatus is sufficiently accurate, if we require an exact one it will, of course, not do, and we must try again. Now it is capable of proof that it is impossible to solve the problem with three moving links; closer approximations to the truth than that given by Watt can be obtained, but still not actual truth.

I have here some examples of these closer approximations. The first of these, shown in Fig. 3, is due to Richard Roberts of Manchester.

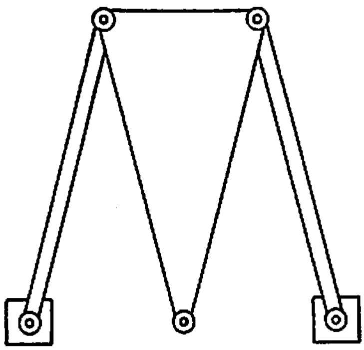


Fig. 3.

The radial bars are of equal length, the distance between the fixed pivots is twice that of the pivots on the traversing piece, and the tracer is situate on the traversing piece, at a distance from the pivots on it equal to the lengths of the radial bars. The tracer in consequence coincides with the straight line joining the fixed pivots at those pivots and half-way between them. It does not, however, coincide at any other points, but deviates very slightly between the fixed pivots. The path described by the tracer when it passes the pivots altogether deviates from the straight line.

The other apparatus was invented by Professor Tchebicheff of St. Petersburg. It is shown in Fig. 4. The radial bars are equal in length, being each in my little model five inches long. The distance between the fixed pivots must then be four inches, and the distance between the pivots of the traversing bar two inches. The tracer is taken half-way between these last. If now we draw a straight line—I had forgotten that we cannot do that yet, well, if we draw a straight line, popularly so called—through the tracer in its mean position, as shown

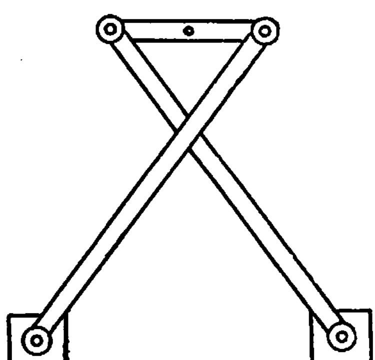


Fig. 4.

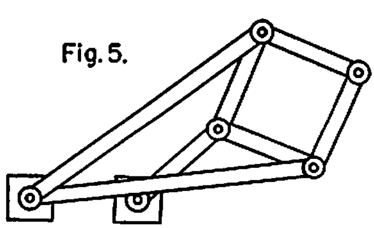
in the figure, parallel to that forming the fixed pivots, it will be found that the tracer will coincide with that line at the points where verticals through the fixed pivots cut it as well as at the mean position, but, as in the case of Robert's parallel motion, it coincides nowhere else, though its deviation is very small as long as it remains between the verticals.

We have failed then with three links, and we must go on to the next case, a five-link motion—for you will observe that we must have an odd number of links if we want an apparatus describing definite curves. Can we solve the problem with five? Well, we can; but this was not the first accurate parallel motion discovered, and we must give the first inventor his due (although he did not find the simplest way) and proceed in a strict chronological order.

In 1864, eighty years after Watt's discovery, the problem was first solved by M. Peaucellier, an officer of Engineers in the French army. His discovery was not at first estimated at its true value, fell almost into oblivion, and was rediscovered by a Russian student named Lipkin, who got a substantial reward from the Russian Government for his supposed originality. However, M. Peaucellier's merit has at last been recognized, and he has been awarded the great mechanical prize of the Institute of France, the "Prix Montyon."

M. Peaucellier's apparatus is shown in Fig. 5. It has, as you see, seven pieces or links. There are first of all two long links of equal length. These are both pivoted at the same fixed point; their other extremities are pivoted to opposite angles of a rhombus composed of four equal shorter links. The portion of the apparatus I have thus far described, considered apart from the fixed base, is a linkage termed a "Peaucellier cell." We then take an *extra* link, and pivot it to

Fig. 5.



a fixed point whose distance from the first fixed point, that to which the cell is pivoted, is the same as the length of the extra link; the other end of the extra link is then pivoted to one of the free angles of the rhombus; the other free angle of the rhombus has a pencil at its pivot. That pencil will accurately describe a straight line.

I must now indulge in a little simple geometry. It is absolutely necessary that I should do so in order that you may understand the principle of our apparatus.

In Fig. 6, QC is the extra link pivoted to the fixed point, Q, the other pivot on it C, describing the circle OCR. The straight lines PM and PM' are supposed to be perpendicular to $MRQOM'$.

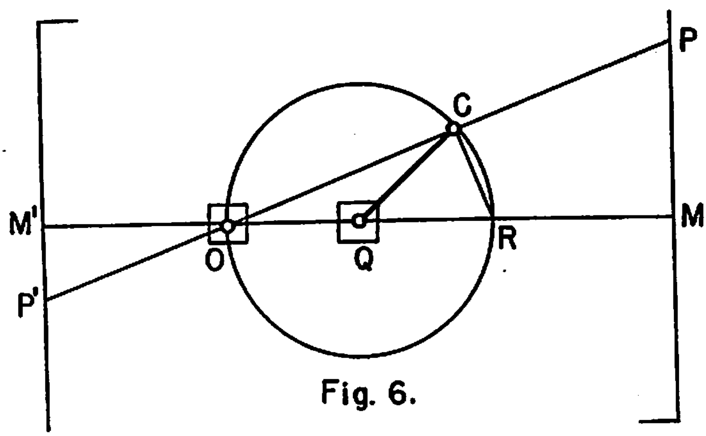


Fig. 6.

Now the angle OCR, being the angle in a semicircle, is a right angle. Therefore the triangles OCR, OMP are similar. Therefore,

$$OC : OR = OM : OP.$$

Therefore,

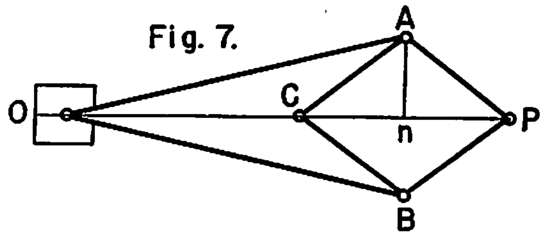
$$OC \cdot OP = OM \cdot OR,$$

wherever C may be on the circle. That is, since OM and OR are both constant, if while C moves in a circle P moves so that O, C, P are always in the same straight line, and so that $OC \cdot OP$ is always constant; then P will describe the straight line PM perpendicular to the line OQ.

It is also clear that if we take the point P' on the other side of O, and if $OC \cdot OP'$ is constant P' will describe the straight line $P'M'$. This will be seen presently to be important.

Now turning to Fig. 7, which is a skeleton drawing of the Peaucellier cell, we see that from the symmetry of

Fig. 7.



the construction of the cell, O, C, P, all lie in the same straight line, and if the straight line An be drawn perpendicular to CP—it must still be an imaginary one, as we have not proved yet that our apparatus does draw a straight line—Cn is equal to nP. Now

$$OA^2 = On^2+An^2$$
$$AP^2 = Pn^2+An^2$$

therefore,

$$\begin{aligned} OA^2 - AP^2 &= On^2-Pn^2 \\ &= (On-Pn)\,(On+Pn) \\ &= OC \cdot OP. \end{aligned}$$

Thus since OA and AP are both constant $OC \cdot OP$ is always constant, however far or near C and P may be to O. If then the pivot O be fixed to the point O in Fig. 6, and the pivot C be made to describe the circle in the figure by being pivoted to the end of the extra link, the pivot P will satisfy all the conditions necessary to make it move in a straight line, and if a pencil be fixed at P it will draw a straight line. The distance of the line from the fixed pivots will of course depend on the magnitude of the quantity OA^2-OP^2 which may be varied at pleasure.

I hope you clearly understand the two elements composing the apparatus, the extra link and the cell, and the part each plays, as I now wish to describe to you some modifications of the cell. The extra link will remain the same as before, and it is only the cell which will undergo alteration.

If I take the two linkages in Fig. 8, which are known as the "kite" and the "spear-head," and place one on the

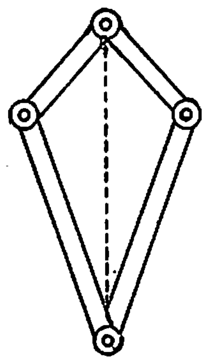
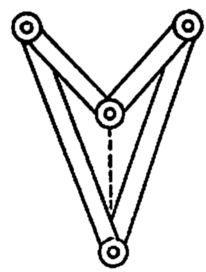


Fig. 8.



other so that the long links of the one coincide with those of the other, and then amalgamate the coincident long links together, we shall get the original cell of Figs. 5 and 7. If then we keep the angles between the long links, or that between the short links, the same in the "kite" and "spear-head," we see that the height of the "kite" multiplied by that of the "spear-head" is constant.

Let us now, instead of amalgamating the long links of the two linkages, amalgamate the short ones. We then get the linkage of Fig. 9; and if the pivot where the short

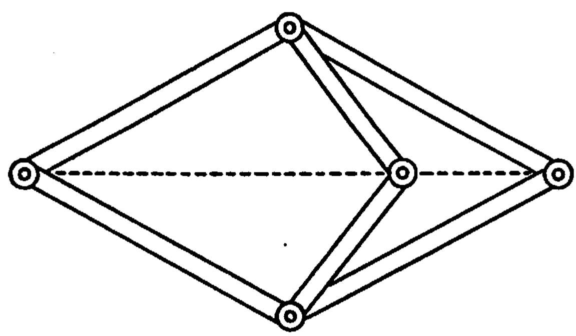


Fig. 9.

links meet is fixed, and one of the other free pivots be made to move in the circle of Fig. 6 by the extra link, the other will describe, not the straight line *PM*, but the straight line *P'M'*. In this form, which is a very compact one, the motion has been applied in a beautiful manner to the air engines which are employed to ventilate the Houses of Parliament. The ease of working and absence of friction and noise is remarkable. The engines were constructed and the Peaucellier apparatus adapted to them by Mr. Prim, the engineer to the Houses, by whose courtesy I have been enabled to see them, and I can assure you that they are well worth a visit.

Another modification of the cell is shown in Fig. 10. If instead of employing a "kite" and "spear-head" of the same dimensions, I take the same "kite" as before, but use a "spear-head" of half the size of the former one, the angles being however kept the same, the product of the heights of the two figures will be half what it was before, but still constant. Now instead of superimposing the links of one figure on the other, it will be seen that in Fig. 10 I fasten the shorter links of each figure together, end to end. Then, as in the former cases, if I fix the pivot at the point where the links are fixed together, I get a cell which may be used, by the employment of an extra link, to describe a straight line. A model employing this form of cell is exhibited in the Loan Collection by the Conservatoire des Arts et Métiers of Paris, and is of exquisite workmanship; the pencil seems to swim along the straight line.

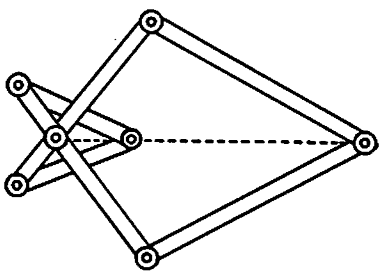


Fig. 10.

Mr. Peaucellier's discovery was introduced into England by Professor Sylvester in a lecture he delivered at the Royal Institution in January, 1874 (5), which excited very great interest and was the commencement of the consideration of the subject of linkages in this country.

In August of the same year Mr. Hart of Woolwich Academy read a paper at the British Asoociation meeting (6), in which he showed that M. Peaucellier's cell could be replaced by an apparatus containing only four links instead of six. The new linkage is arrived at thus.

If to the ordinary Peaucellier cell I add two fresh links of the same length as the long ones I get the double, or rather quadruple cell, for it may be used in four different ways, shown in Fig. 11. Now Mr. Hart found that

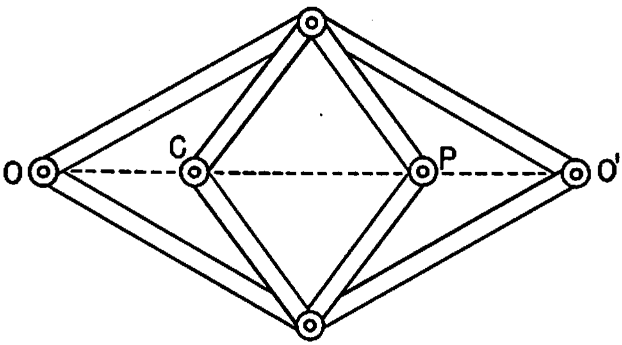


Fig. 11.

if he took an ordinary parallelogrammatic linkwork, in which the adjacent sides are unequal, and crossed the links so as to form what is called a contra-parallelogram, Fig.

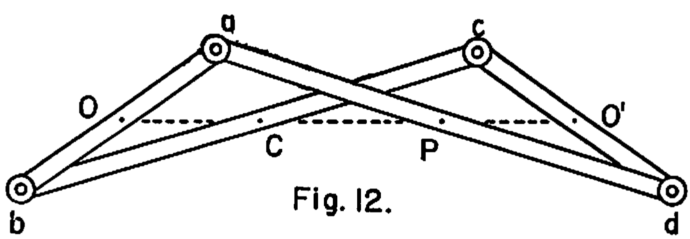
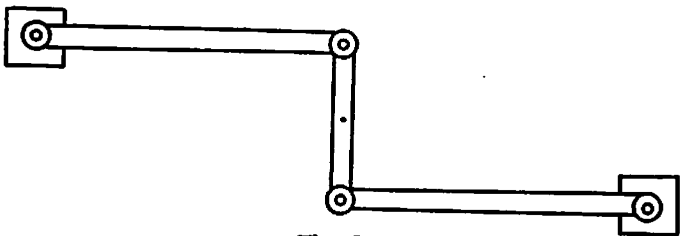


Fig. 12.

12, and then took four points on the four links dividing the distances between the pivots in the same proportion, those four points had exactly the same properties as the four points of the double cell. That the four points always lie in nearly every beam-engine. The apparatus, reduced to its simplest form, is shown in Fig. 2.



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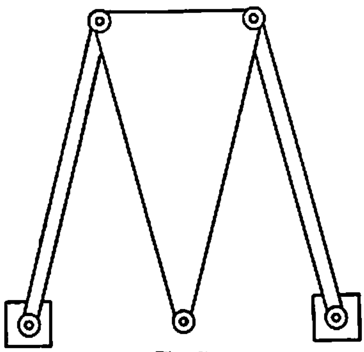

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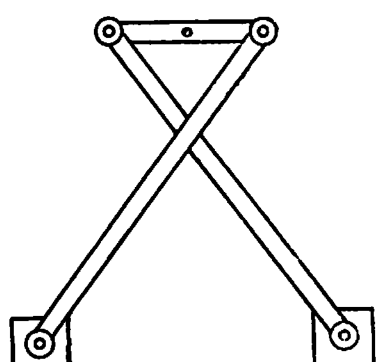


Fig. 4.

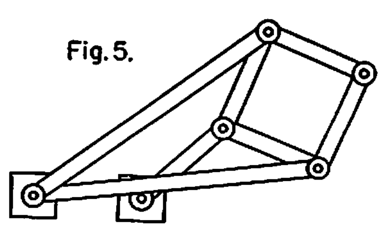
in the figure, parallel to that forming the fixed pivots, it will be found that the tracer will coincide with that line at the points where verticals through the fixed pivots cut it as well as at the mean position, but, as in the case of Robert's parallel motion, it coincides nowhere else, though its deviation is very small as long as it remains between the verticals.

We have failed then with three links, and we must go on to the next case, a five-link motion—for you will observe that we must have an odd number of links if we want an apparatus describing definite curves. Can we solve the problem with five? Well, we can; but this was not the first accurate parallel motion discovered, and we must give the first inventor his due (although he did not find the simplest way) and proceed in a strict chronological order.

In 1864, eighty years after Watt's discovery, the problem was first solved by M. Peaucellier, an officer of Engineers in the French army. His discovery was not at first estimated at its true value, fell almost into oblivion, and was rediscovered by a Russian student named Lipkin, who got a substantial reward from the Russian Government for his supposed originality. However, M. Peaucellier's merit has at last been recognized, and he has been awarded the great mechanical prize of the Institute of France, the "Prix Montyon."

M. Peaucellier's apparatus is shown in Fig. 5. It has, as you see, seven pieces or links. There are first of all two long links of equal length. These are both pivoted at the same fixed point; their other extremities are pivoted to opposite angles of a rhombus composed of four equal shorter links. The portion of the apparatus I have thus far described, considered apart from the fixed base, is a linkage termed a "Peaucellier cell." We then take an *extra* link, and pivot it to

Fig. 5.



a fixed point whose distance from the first fixed point, that to which the cell is pivoted, is the same as the length of the extra link; the other end of the extra link is then pivoted to one of the free angles of the rhombus; the other free angle of the rhombus has a pencil at its pivot. That pencil will accurately describe a straight line.

I must now indulge in a little simple geometry. It is absolutely necessary that I should do so in order that you may understand the principle of our apparatus.

In Fig. 6, *QC* is the extra link pivoted to the fixed point, *Q*, the other pivot on it *C*, describing the circle *OCR*. The straight lines *PM* and *PM'* are supposed to be perpendicular to *MRQOM'*.

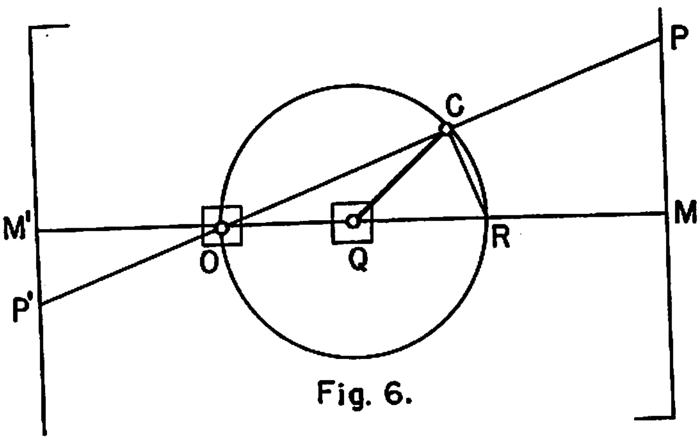


Fig. 6.

Now the angle *OCR*, being the angle in a semicircle, is a right angle. Therefore the triangles *OCR, OMP* are similar. Therefore,

$$OC : OR = OM : OP.$$

Therefore,

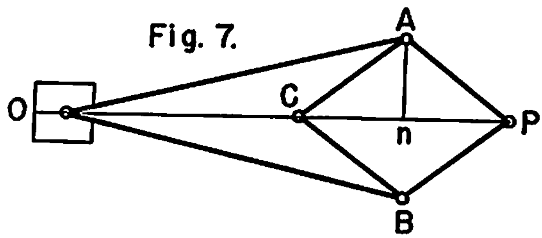
$$OC \cdot OP = OM \cdot OR,$$

wherever *C* may be on the circle. That is, since *OM* and *OR* are both constant, if while *C* moves in a circle *P* moves so that *O, C, P* are always in the same straight line, and so that *OC* · *OP* is always constant; then *P* will describe the straight line *PM* perpendicular to the line *OQ*.

It is also clear that if we take the point P' on the other side of O, and if $OC \cdot OP'$ is constant P' will describe the straight line $P'M'$. This will be seen presently to be important.

Now turning to Fig. 7, which is a skeleton drawing of the Peaucellier cell, we see that from the symmetry of

Fig. 7.



the construction of the cell, O, C, P, all lie in the same straight line, and if the straight line An be drawn perpendicular to CP—it must still be an imaginary one, as we have not proved yet that our apparatus does draw a straight line—Cn is equal to nP. Now

$$OA^2 = On^2 + An^2$$
$$AP^2 = Pn^2 + An^2$$

therefore,

$$\begin{aligned} OA^2 - AP^2 &= On^2 - Pn^2 \\ &= (On - Pn)(On + Pn) \\ &= OC \cdot OP. \end{aligned}$$

Thus since OA and AP are both constant $OC \cdot OP$ is always constant, however far or near C and P may be to O. If then the pivot O be fixed to the point O in Fig. 6, and the pivot C be made to describe the circle in the figure by being pivoted to the end of the extra link, the pivot P will satisfy all the conditions necessary to make it move in a straight line, and if a pencil be fixed at P it will draw a straight line. The distance of the line from the fixed pivots will of course depend on the magnitude of the quantity $OA^2 - OP^2$ which may be varied at pleasure.

I hope you clearly understand the two elements composing the apparatus, the extra link and the cell, and the part each plays, as I now wish to describe to you some modifications of the cell. The extra link will remain the same as before, and it is only the cell which will undergo alteration.

If I take the two linkages in Fig. 8, which are known as the "kite" and the "spear-head," and place one on the

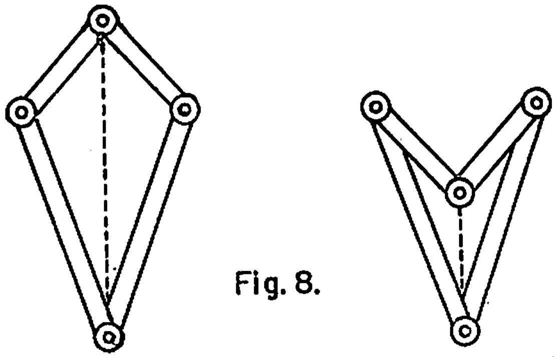


Fig. 8.

other so that the long links of the one coincide with those of the other, and then amalgamate the coincident long links together, we shall get the original cell of Figs. 5 and 7. If then we keep the angles between the long links, or that between the short links, the same in the "kite" and "spear-head," we see that the height of the "kite" multiplied by that of the "spear-head" is constant.

Let us now, instead of amalgamating the long links of the two linkages, amalgamate the short ones. We then get the linkage of Fig. 9; and if the pivot where the short

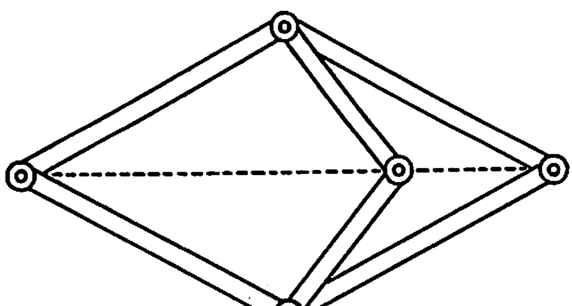


Fig. 9.

links meet is fixed, and one of the other free pivots be made to move in the circle of Fig. 6 by the extra link, the other will describe, not the straight line *PM*, but the straight line *P'M'*. In this form, which is a very compact one, the motion has been applied in a beautiful manner to the air engines which are employed to ventilate the Houses of Parliament. The ease of working and absence of friction and noise is remarkable. The engines were constructed and the Peaucellier apparatus adapted to them by Mr. Prim, the engineer to the Houses, by whose courtesy I have been enabled to see them, and I can assure you that they are well worth a visit.

Another modification of the cell is shown in Fig. 10. If instead of employing a "kite" and "spear-head" of the same dimensions, I take the same "kite" as before, but use a "spear-head" of half the size of the former one, the angles being however kept the same, the product of the heights of the two figures will be half what it was before, but still constant. Now instead of superimposing the links of one figure on the other, it will be seen that in Fig. 10 I fasten the shorter links of each figure together, end to end. Then, as in the former cases, if I fix the pivot at the point where the links are fixed together, I get a cell which may be used, by the employment of an extra link, to describe a straight line. A model employing this form of cell is exhibited in the Loan Collection by the Conservatoire des Arts et Métiers of Paris, and is of exquisite workmanship; the pencil seems to swim along the straight line.

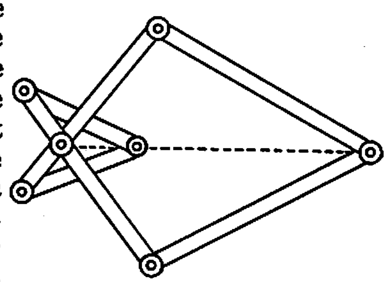


Fig. 10.

Mr. Peaucellier's discovery was introduced into England by Professor Sylvester in a lecture he delivered at the Royal Institution in January, 1874 (5), which excited very great interest and was the commencement of the consideration of the subject of linkages in this country.

In August of the same year Mr. Hart of Woolwich Academy read a paper at the British Asoociation meeting (6), in which he showed that M. Peaucellier's cell could be replaced by an apparatus containing only four links instead of six. The new linkage is arrived at thus.

If to the ordinary Peaucellier cell I add two fresh links of the same length as the long ones I get the double, or rather quadruple cell, for it may be used in four different ways, shown in Fig. 11. Now Mr. Hart found that

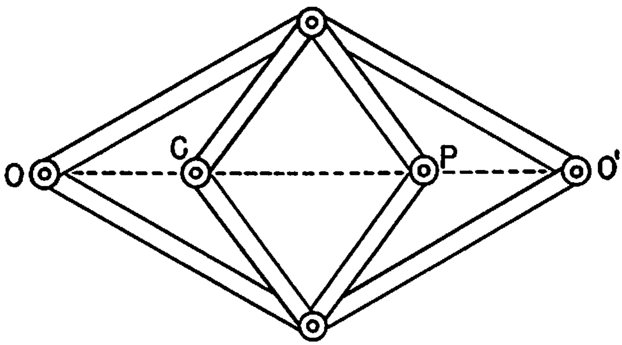


Fig. 11.

if he took an ordinary parallelogrammatic linkwork, in which the adjacent sides are unequal, and crossed the links so as to form what is called a contra-parallelogram, Fig.

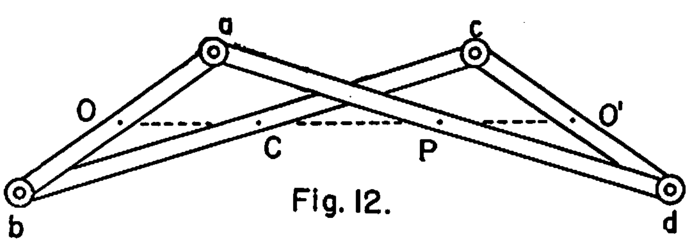


Fig. 12.

12, and then took four points on the four links dividing the distances between the pivots in the same proportion, those four points had exactly the same properties as the four points of the double cell. That the four points always lie

THE CONSTITUTION OF KAPPA MU EPSILON

Article I.—The Fraternity and Its Objectives

Section 1. The name of this organization shall be Kappa Mu Epsilon, Honorary Mathematics Society. Its Greek motto,

'αναπτυσσετε την 'εκτιμησιν του καλλους των μαθηματικων

may be translated, "Unfold the Glory of Mathematics."

Section 2. The object of the society shall be

A. to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program;

B. to help the undergraduate realize the important role that mathematics has played in the development of the western civilization;

C. to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought;

D. to provide a society for the recognition of outstanding achievement in the study of mathematics in the undergraduate level.

Article II.—Membership

Section 1. The active members shall consist of those members who pay their current dues.

Section 2. Qualifications for membership. A member

A. must be or have been a faculty member or a regularly enrolled student of an accredited four-year college;

B. must have completed at least eight semester hours or the equivalent in college mathematics, which *must* include the basic ideas of Cartesian Geometry;

C. must be above the average in his institution in mathematics and in general scholarship.

Section 3. Members of chapters upon moving to the locality of another chapter may become affiliated therewith by presenting credentials of membership in good standing from the secretary of the initiating chapter.

Section 4. During each biennium, a chapter may initiate two honorary members. The qualifications for membership listed in Section 2 do not apply to honorary members. The chapter shall pay the regular initiation fees for such honorary members.

Article III.—Insignia

Section 1. The badge of this organization shall be a pentagon with the center slightly concave; on the upper half shall be a five-leaf rose, and on the lower half the letters K. M. E. The badge may be worn as a pin, key, or pendant.

Section 2. The seal shall be a five-pointed star enclosing a five-leaf rose, and being encircled by the legend "Kappa Mu Epsilon, founded 1931."

Section 3. The crest shall be a shield enclosing the five-pointed star; in the star shall be the rose rho$=$ a sin 5θ, the symbol of pure mathematics. Around the star shall be the symbols for the sciences which apply mathematics: at the upper left a book of knowledge, for students and teachers; in the lower left a shamrock and a slide rule, for the engineers; at the upper right a conventionalized butterfly, for the biological sciences; at the lower right a moon and three stars, for the physical sciences; at the bottom the symbol "s angle n", for the business world. Above the shield proper is the design of the badge of the society, and below it is a streamer upon which is printed the Greek motto.

Section 4. The colors shall be rose-pink and silver; the flower shall be the wild rose.

Article IV.—Chapters

Section 1. An organized group of at least ten members from an accredited (by the appropriate regional accrediting agency) four-year college may petition for a chapter.

Section 2. A request for the establishment of a chapter shall be referred to the National Council by the National President. This request shall include such information concerning the school and the mathematics faculty as required by the petition. If the Council endorses the request, the National President shall then direct the National Secretary to notify the chapters of this application, giving only the pertinent details. If, within 60 days, one-fourth of the chapters have not filed disapproval with the National Secretary, the National President shall, with the approval of the National Council, grant a charter to the petitioning group.

Section 3. Chapters may be approved by the voting delegates at the biennial convention, as provided in Article VI, Section 1. A three-fourths vote of the voting delegates shall be required to approve all motions to establish a new chapter.

Section 4. Each chapter shall be governed by chapter by-laws which shall not be in variance with the national constitution. Copies of chapter by-laws shall be filed with the National President and the National Secretary.

Section 5. Chapters shall be designated by state and Greek letters. The Greek letter to each chapter shall be the Greek alphabetical order in which the chapter was granted, i.e., Kansas Alpha, Kansas Beta, Kansas Gamma, etc.

Section 6. If a chapter fails to comply with the constitution or receives an indictment from two other chapters for failure to fulfill the purpose of the fraternity, its charter may be withdrawn by a two-thirds vote of the chapters upon a report of the National Council including the charge against the chapter and the chapter's defense.

Section 7. (a) A chapter upon its request presented to the National Council, may become inactive.

(b) A chapter which is delinquent in paying its dues, making required reports, or answering letters, may be suspended by the National Council. Failure to report on any matter where the thirty or sixty days' silence is equivalent to an approval, shall be considered as delinquency in respect to chapter reports, except when the National Council finds the failure to report is unavoidable.

(c) A chapter shall be expelled by the methods of Section 6, Article IV. A chapter which is inactive or suspended shall not be permitted to have membership cards or pins, nor shall it be allowed to participate in any of the benefits of the national fraternity such as the defraying of the expenses of a delegate to the national convention or any other expenditures of money. In any matter requiring approval of two-thirds or three-fourths of the chapters, the number required for approval shall be based on the chapters in good standing; i.e., chapters not inactive or suspended. Upon evidence of willingness to abide by the constitution presented to the National Council and upon payment of a reinstatement fee of ten dollars ($10) and an additional amount to be determined by the National Council, an inactive or suspended chapter may be reinstated.

Article V.—National Officers

Section 1. The national officers shall be President, Vice-President, Secretary, Treasurer, Historian, immediate Past President, and shall be members of Kappa Mu Epsilon. These shall be elected by a simple majority at a national convention. If, for any reason, a national convention is not held, the National Council shall present a nomination for each national officer to the local chapters for their approval or disapproval. Failure to reply within thirty days shall be considered an approval. Each nominee shall be considered elected when two-thirds of the chapters give their approval of his election.

Section 2. The term of office of these officers shall be for two years, except that of the Secretary and Treasurer who may be reelected as often as the chapters deem it advisable. Except for the offices of Secretary and Treasurer, no office shall be filled by the same chapter for more than two successive terms.

Section 3. The duties of the President shall be to preside at the national convention, to provide for the installation of new chapters, to approve all bills before they are paid by the treasurer, and to perform other duties as usually pertain to this office.

Section 4. The duties of the Vice-President shall be to assist the President.

Section 5. The Secretary shall take charge of the records of the society, as provided, and shall perform those duties that usually pertain to this office.

Section 6. The duties of the Treasurer shall be to pay all bills upon the approval of the president, and to perform such duties as usually pertain to this office.

Section 7. The Historian shall be the official historian of Kappa Mu Epsilon. He shall be assistant-editor of the official publication of Kappa Mu Epsilon and shall keep a scrap book of interesting news notes about chapters or members.

Section 8. The immediate Past President shall act as an advisory officer, aiding any and all of the other officers.

Section 9. The above officers shall constitute the National Council. The duties of the National Council shall be to investigate petitioning groups and their institutions as a field of expansion, to fill any vacancies for the unexpired term, and to serve as general executive committee in the interim between conventions. Any four members of the Council shall constitute a quorum for the transaction of business of the fraternity.

Article VI.—Conventions

Section 1. The national convention shall consist of members of the National Council and delegates elected by each chapter. Each chapter may send as many delegates as it desires and each delegate shall be entitled to participate in all discussions, and to vote individually on all motions except on motions to amend the constitution or by-laws, or in the election of national officers, or on motions relating to the establishment of new chapters. On all motions relating to the exceptions mentioned above, each chapter shall be entitled to two votes. These votes shall be cast by two special voting delegates whose names and chapter membership must be certified to, in writing, by the chapter represented.

Section 2. The national convention shall be held every two years, the time and place of which shall be decided by the National Council, and the chapters shall be notified of their action at least six months prior to the date which is set for the convention. The National Organization pays the first-class railroad fare of one delegate from each chapter provided such a sum does not exceed three-fifths of the amount paid by the chapter, during the current biennium, as initiation fees for new members.

Section 3. Biennial reports of all national officers shall be made at each national convention. A written copy of each report must be presented to the National Secretary for permanent filing.

Section 4. The members present at any regularly announced meeting of the convention constitutes a quorum for business purposes providing 30 per cent of the chapters are represented.

Article VII.—Publications

Section 1. The official journal of Kappa Mu Epsilon shall be known as *The Pentagon.*

Section 2. The National Council shall appoint the editor of the official journal. The National Historian shall be assistant editor.

Section 3. The official journal shall contain material which will tend to achieve the objectives of Kappa Mu Epsilon and which will tend to establish fraternal ties between the chapters.

Section 4. The official journal shall be published at least once a year and as more often as the editor and the National Council shall deem advisable.

Section 5. Each initiate shall receive the official journal for a period of two years.

Article VIII.—Amendments

Section 1. This constitution may be amended by two-thirds vote of the national convention provided that a copy of the proposed amendments shall have been sent to every chapter by the National Council at least one month before the convention meets.

Section 2. Amendments may also be proposed at the national convention, and if approved by a two-thirds vote, be submitted to each chapter. Such amendments shall become effective upon ratification by two-thirds of the chapters.

Section 3. Amendments may be proposed at any time by any local chapter or by the National Council to each local chapter for its approval or disapproval. Such amendments become effective upon ratification by two-thirds of the chapters. Failure of any chapter to report within sixty days shall be considered equivalent to approval.

BY LAWS

Article I.—Finance

Section 1. Each new chapter shall pay in advance to the National Treasurer five dollars per capita. Expenses incurred by the national organization incidental to the establishment of a chapter shall be paid by the chapter; however, in any case, this sum shall not be less than thirty-five dollars.

Section 2. Each chapter shall pay the national organization an initiation fee of five dollars for each subsequent member.

Section 3. An auditing committee of three members shall be appointed by the National President to audit the accounts of the National Treasurer prior to the convention.

Article II.—Ritual

Section 1. An official ritual, provided by the National Council, shall be used in all chapter installations and initiations.

Article III.—Chapter Officers

Section 1. The chapter officers shall be President, Vice-President, Recording Secretary, Treasurer, Corresponding Secretary, and Faculty Sponsor.

Section 2. The Corresponding Secretary shall have charge of the official correspondence with the National Society. He shall be a faculty member.

Article IV.—Amendments

Section 1. These by-laws may be amended in the same manner as prescribed for the amendment of the national constitution.

Article V.—Printed Forms

Section 1. Petitions for the establishment of a chapter shall be of the form approved by the National Council.

Section 2. The charter of any chapter shall be of the form approved by the National Council.

Section 3. Permanent record cards of all chapter members shall be sent by each Corresponding Secretary to the National Secretary for permanent filing. These cards shall be provided by the National Secretary and be of the form approved by the National Council.

Section 4. The National Secretary shall, upon the request of the Corresponding Secretary, and upon receipt of the necessary fees and records, issue membership cards of the forms approved by the National Council.

Section 5. All requests for pins or keys must be made on the forms provided for that purpose and must bear the approval of the National Secretary.

in a straight line is seen thus: considering the triangle *abd*, since $aO : Ob = aP : Pd$ therefore OP is parallel to *bd*, and the perpendicular distance between the parallels is to the height of the triangle *abd* as Ob is to *ab*; the same reasoning applies to the straight line CO', and since $ab : Ob = cd : O'd$ and the heights of the triangles *abd*, *cbd*, are clearly the same, therefore the distances of OP and $O'C$ from *bd* are the same, and $OCPO'$ lie in the same straight line.

That the product $OC \cdot OP$ is constant appears at once when it is seen that ObC is half a "spear-head" and OaP half a "kite;" similarly it may be shown that $O'P \cdot O'C$ is constant, as also $OC \cdot CO'$ and $OP \cdot PO'$. Employing then the Hart's cell as we employed Peaucellier's, we get a five-link straight line motion. A model of this is exhibited in the Loan Collection by M. Breguet.

I now wish to call your attention to an extension of Mr. Hart's apparatus, which was discovered simultaneously by Professor Sylvester and myself. In Mr. Hart's apparatus we were only concerned with bars and points on those bars, but in the apparatus I wish to bring before you we have pieces instead of bars. I think it will be more interesting if I lead up to this apparatus by detailing to you its history, especially as I shall thereby be enabled to bring before you another very elegant and very important linkage—the discovery of Professor Sylvester.

When considering the problem presented by the ordinary three-*bar* motion consisting of two radial bars and a traversing bar, it occurred to me—I do not know how or why, it is often very difficult to go back and find whence one's ideas originate—to consider the relation between the curves described by the points on the traversing bar in any given three-bar motion, and those described by the points on a similar three-bar motion, but in which the traversing bar and one of the radial bars had been made to change places. The proposition was no sooner stated than the solution became obvious; the curves were precisely similar. In Fig. 13 let CD and BA be the two radial

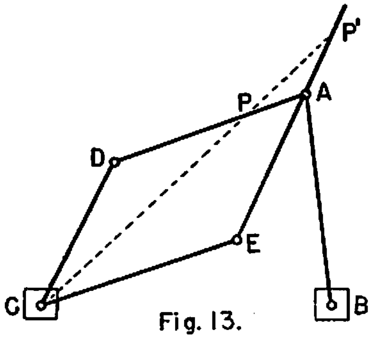
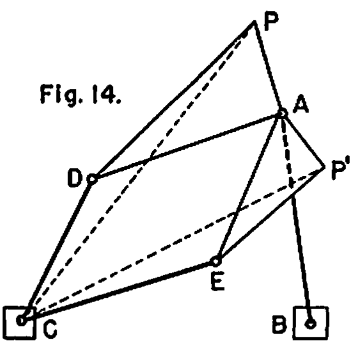


Fig. 13.

bars turning about the fixed centers C and B, and let DA be the traversing bar, and let P be any point on it describing a curve depending on the lengths of AB, BC, CD, and DA. Now add to the three-bar motion the bars CE and EAP', CE being equal to DA, and EA equal to CD. $CDAE$ is then a parallelogram, and if an imaginary line CPP' be drawn, cutting EA produced in P', it will at once be seen that P' is a fixed point on EA produced, and CP' bears always a fixed proportion to CP, viz., $CD : CE$. Thus the curve described by P' is precisely the same as that described by P, only it is larger in the proportion $CE : CD$. Thus if we take away the bars CD and DA, we shall get a three-bar linkwork, describing precisely the same curves, only of different magnitude, as our first three-bar motion described, and this new three-bar linkwork is the same as the old with the radial link CD and the traversing link DA interchanged (7).

On my communicating this result to Professor Sylvester, he at once saw that the property was one not confined to the particular case of points lying on the traversing bar, in fact to three-*bar* motion, but was posessed by three-*piece* motion. In Fig. 14 $CDAB$ is a three-bar motion, as in Fig. 13, but the tracing point or "graph" does not lie on the line joining the joints AD, but is anywhere else on a "piece" on which the joints AD lie. Now, as before, add the bar CE, CE being equal to AD, and the piece AEP', making AE equal to CD, and the triangle AEP' similar to the triangle PDA; so that the angles AEP', ADP

Fig. 14.



are equal, and

$$P'E : EA = AD : DP.$$

It follows easily from this—you can work it out for yourselves without difficulty—that the ratio $P'C : PC$ is constant and the angle PCP' is constant; thus the paths of P and P', or the "grams" described by the "graphs," P and P', are similar, only they are of different sizes, and one is turned through an angle with respect to the other.

Now you will observe that the two proofs I have given are quite independent of the bar AB, which only affects the particular curve described by P and P'. If we get rid of AB, in both cases we shall get in the first figure the ordinary pantagraph, and in the second a beautiful extension of it, called by Professor Sylvester, its inventor, the *Plagiograph* or *Skew Pantagraph.* Like the Pantagraph, it will enlarge or reduce figures, but it will do more, it will turn them through any required angle, for by properly choosing the position of P and P', the ratio of CP to CP' can be made what we please, and also the angle PCP' can be made to have any required value. If the angle PCP' is made equal to 0 or 180°, we get the two forms of the pantagraph now in common use; if it be made to assume successively any value which is a sub-multiple of 360°, we can, by passing the point P each time over the same pattern make the point P' reproduce it round the fixed center C after the fashion of a kaleidoscope. I think you will see from this that the instrument, which has, as far as I know, never been practically constructed, deserves to be put into the hands of the designer. I give here a picture of a little model of a possible form for the instrument furnished by me to the Loan Collection by request of Professor Sylvester (8).

After this discovery of Professor Sylvester, it occurred to him and to me simultaneously—our letters announcing our discovery to each other crossing in the post—that the principle of the plagiograph might be extended to Mr. Hart's contra-parallelogram; and this discovery I shall now proceed to explain to you. I shall, however, be more easily able to do so by approaching it in a different manner to that in which I did when I discovered it.

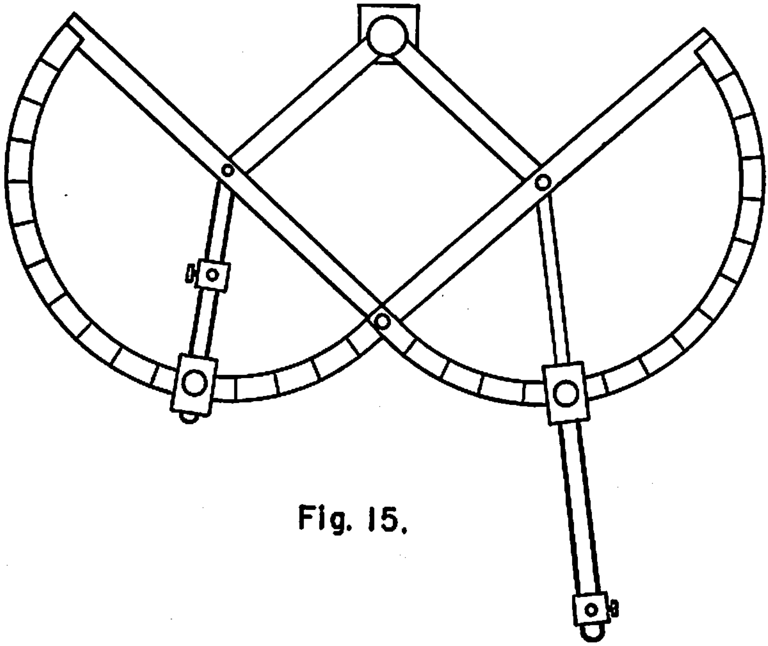


Fig. 15.

If we take the contra-parallelogram of Mr. Hart, and bend the links at the four points which lie on the same straight line, or *foci* as they are sometimes termed, through the same angle, the four points, instead of lying in the same straight line, will lie at the four angular points of a parallelogram of constant angles,—two the angle that the bars are bent through, and the other two their supplements—and of constant area, so that the product of two adjacent sides is constant.

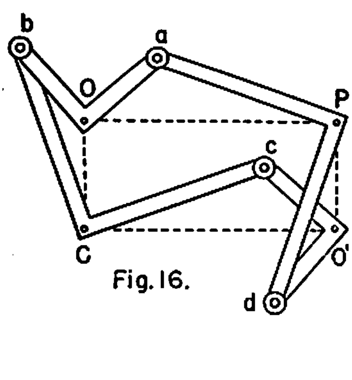


Fig. 16.

In Fig. 16 the lettering is preserved as in Fig. 12, so that the way in which the apparatus is formed may be at once seen. The holes are taken in the middle of the links, and the bending is through a right angle. The four holes *OPO'C* lie at the four corners of a right-angled parallelogram, and the product of any two adjacent sides, as for example $OC \cdot OP$, is constant. It follows that if O be pivoted to the fixed point O in Fig. 6, and C be pivoted to the extremity of the extra link, P will describe a straight line, not PM, but one inclined to PM at an angle the same as the bars are bent through, *i.e.* a right angle. Thus the straight line will be parallel to the line joining the fixed pivots O and Q. This apparatus, which for simplicity I have described as formed of four straight links which are afterwards bent, is of course strictly speaking formed of four plane links, such as those employed in Fig. 1, on which the various points are taken. This explains the name given to it by Professor Sylvester, the "Quadruplane." Its properties are not difficult to investigate, and when I point out to you that in Fig. 16, as in Fig. 12, Ob, bC form half a "spear-head," and Oa, aP half a "kite," you will very soon get to the bottom of it.

I cannot leave this apparatus, in which my name is associated with that of Professor Sylvester, without expressing my deep gratitude for the kind interest which he took in my researches, and my regret that his departure for America to undertake the post of Professor in the new Johns Hopkins University has deprived me of one whose valuable suggestions and encouragement helped me much in my investigations.

Before leaving the Peaucellier cell and its modifications, I must point out another important property they possess besides that of furnishing us with exact rectilinear motion. We have seen that our simplest linkwork enables us to describe a circle of any radius, and if we wished to describe one of ten miles' radius the proper course would be to have a ten-mile link, but as that would be, to say the least, cumbrous, it is satisfactory to know that we can effect our purpose with a much smaller apparatus. When the Peaucellier cell is mounted for the purpose of describing a straight line, as I told you, the distance between the fixed pivots must be the same as the length of the "extra" link. If this distance be not the same we shall not get straight lines described by the pencil, but circles. If the difference be slight the circles described will be of enormous magnitude, decreasing in size as the difference increases. If the distance QO, Fig. 6, be made greater than QC, the convexity of the portion of the circle described by the pencil (for if the circles are large it will of course be only a portion which is described) will be towards O, if less the concavity. To a mathematician, who knows that the inverse of a circle is a circle, this will be clear, but it may not be amiss to give here a short proof of the proposition.

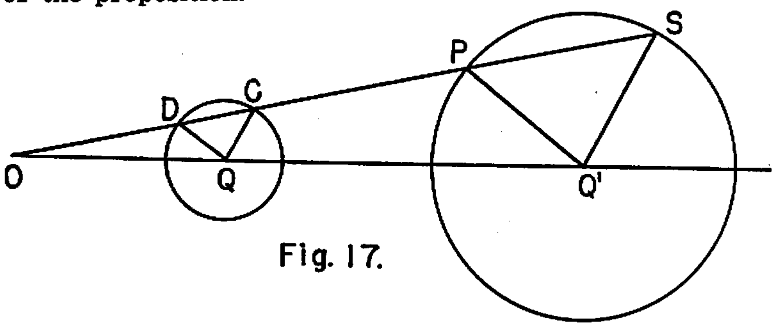


Fig. 17.

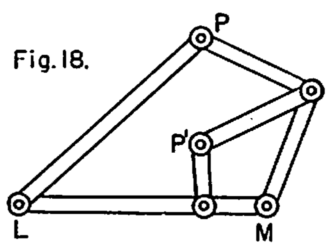
In Fig. 17 let the centers Q, Q' of the two circles be at distances from O proportional to the radii of the circles. If then $ODCPS$ be any straight line through O, DQ will be parallel to PQ', and CQ to SQ', and OD will bear the same proportion to OP that OQ does to OQ'. Now considering the proof we gave in connection with Fig. 7, it will be clear that the product $OD \cdot OC$ is constant, and therefore since OP bears a constant ratio to OD, $OP \cdot OC$ is constant. That is if $OC \cdot OP$ is constant and C describes a circle about Q, P will describe one about Q'. Taking then O, C and P as the O, C and P of the Peaucellier cell in Fig. 7, we see how P comes to describe a circle.

It is hardly necessary for me to state the importance of the Peaucellier compass in the mechanical arts for drawing circles of large radius. Of course the various modifications of the "cell" I have described may all be employed for the purpose. The models exhibited in the Conservatoire by M. Breguet are furnished with sliding pivots for the purpose of varying the distance between O and Q, and thus getting circles of any radius.

My attention was first called to these linkworks by the lecture of Professor Sylvester, to which I have referred. A passage in that lecture in which it was stated that there were probably other forms of seven-link parallel motions besides M. Peaucellier's, then the only one known, led me to investigate the subject, and I succeeded in obtaining some new parallel motions of an entirely different character to that of M. Peaucellier (9). I shall bring two of these to your notice, as the investigation of them will lead us to consider some other linkworks of importance.

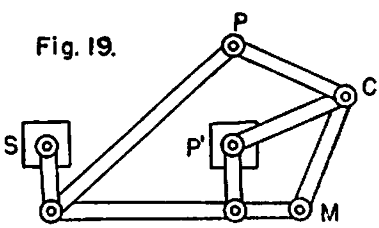
If I take two kites, one twice as big as the other, such that the long links of each are twice the length of the short ones, and make one long link of the small kite lie on a short one of the large, and a short one of the small on a long one of the large, and then amalgamate the coincident links, I shall get the linkage shown in Fig. 18.

Fig. 18.



The important property of this linkage is that, although we can by moving the links about, make the points P and P' approach to or recede from each other, the imaginary line joining them is always perpendicular to that drawn through the pivots on the bottom link LM. It follows that if either of the pivots P or P' be fixed, and the link LM be made to move so as always to remain parallel to a fixed line, the other point will describe a straight line perpendicular to the fixed line. Fig. 19 shows you the parallel motion made by fixing P'. It is unnecessary for me to point out how the parallelism of LM is preserved by adding the link SL, it is obvious from the figure. The straight line which is described by the point P is perpendicular to the line joining the two fixed pivots; we can, however, without increasing

Fig. 19.



the number of links, make a point on the linkwork describe a straight line inclined to the line SP at any angle, or rather we can, by substituting for the straight link PC a plane piece, get a number of points on that piece moving in every direction.

In Fig. 20, for simplicity, only the link CP' and the new piece substituted for the link PC are shown. The new piece is circular and has holes pierced in it all at the same distance—the same as the lengths PC and $P'C$—from C. Now we have seen from Fig. 19 the P moves in a vertical straight line, the distance PC in Fig. 20 being the same as it was in Fig. 19; but from a well-known property of a circle, if H be any one of the holes pierced in the piece, the angle $HP'P$ is constant, thus the straight line HP' is fixed in position, and H moves along it; similarly all the other holes move along in straight lines passing through the fixed pivot P', and we get straight line motion distributed in all directions. This species of motion is called by Professor Sylvester "tram-motion." It is worth noticing that the motion of the circular disc is the same as it would have been if the dotted circle on it rolled inside the large dotted circle; we have, in fact, White's parallel motion reproduced by linkwork. Of course, if we only require motion in one direction, we may cut away all the disc except a portion forming a bent arm containing C, P, and the point which moves in the required direction.

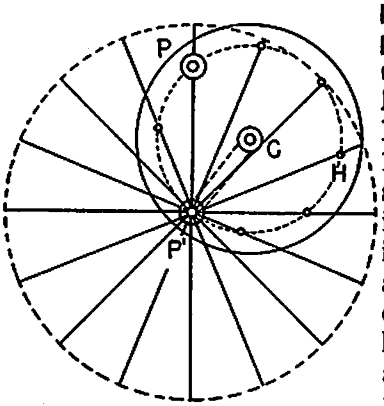


Fig. 20.

The double kite of Fig. 18 may be employed to form some other useful linkworks. It is often necessary to have, not a single point, but a whole piece moving so that all points on it move in straight lines. I may instance the slide rests in lathes, traversing tables, punches, drills, drawbridges, etc. The double kite enables us to produce linkworks having this property. In the linkwork of Fig. 21,

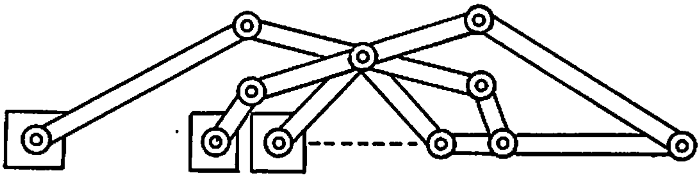


Fig. 21.

the construction of which will be at once appreciated if you understand the double kite, the horizontal link moves to and fro as if sliding in a fixed horizontal straight tube. This form would possibly be useful as a girder for a drawbridge.

In the linkwork of Fig. 22, which is another combination of two double kites, the vertical link moves so that all its points move in horizontal straight lines. There is a modification of this linkwork which will, I think, be found interesting. In the linkage in Fig. 23, which, if the thin links are removed, is a skeleton drawing of Fig. 22, let the dotted links be taken away and the thin ones be inserted; we then get a linkage which has the same property as that in Fig. 22, but it is seen in its new form to be the ordinary double parallel ruler with three added links. Fig. 24 is a figure of a double parallel ruler made on this plan with a slight modification. If the bottom ruler be held horizontal the top moves vertically up and down the board, having no lateral movement.

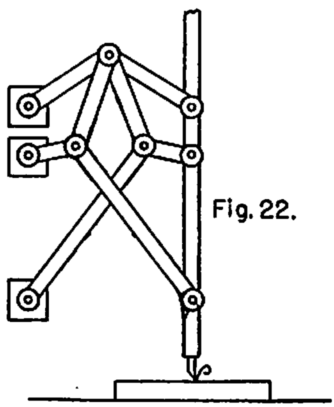


Fig. 22.

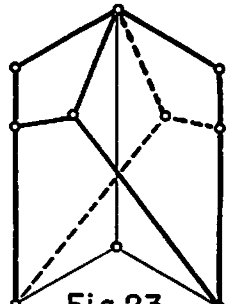


Fig. 23.

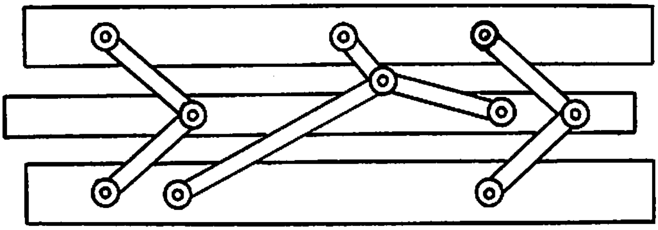


Fig. 24.

While I am upon this sort of movement I may point out an apparatus exhibited in the Loan Collection by Professor Tchebicheff, which bears a strong likeness to a complicated camp-stool, the seat of which has horizontal motion. The motion is not strictly rectiliner; the apparatus being—as will be seen by observing that the thin line in the figure is of invariable length, and a link might therefore be put where it is—a combination of two of the par-

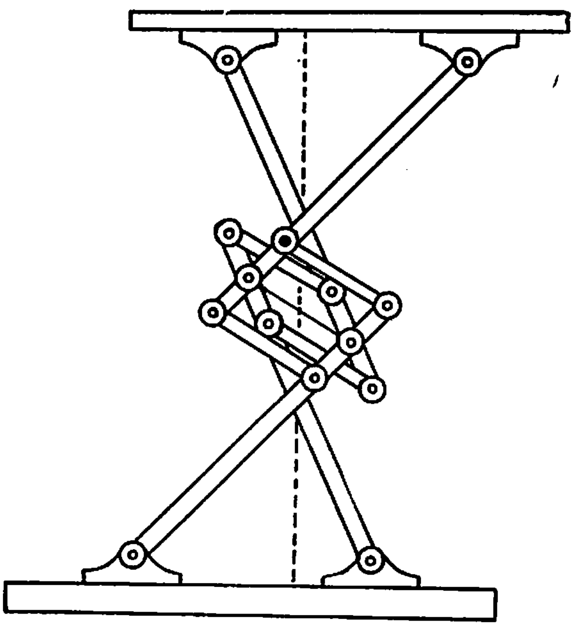


Fig. 25.

allel motions of Professor Tchebicheff given in Fig. 4, with some links added to keep the seat parallel with the base. The variation of the upper plane from a strictly horizontal movement is therefore double that of the tracer in the simple parallel motion.

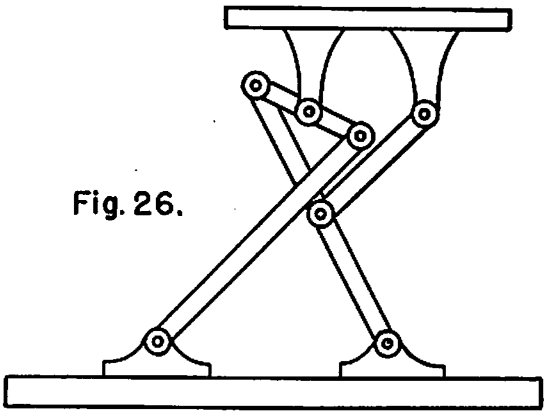


Fig. 26.

Fig. 26 shows how a similar apparatus of much simpler construction, employing the Tchebicheff approximate parallel motion man be made. The lengths of the links forming the parallel motion have been given before (Fig. 4). The distance between the pivots on the moving seat is half that between the fixed pivots, and the length of the remaining link is one-half that of the radial links.

An *exact* motion of the same description is shown in Fig. 27. *O, C, O′, P* are the four *foci* of the quadriplane shown in the figure in which the links are bent through a right angle, so that $OC \cdot OP$ is constant, and *COP* a right angle. The focus *O* is pivoted to a fixed point, and *C* is made by means of the extra link *QC* to move in a circle of which the radius *QC* is equal to the pivot distance *OQ*. *P* consequently moves in a straight line parallel to *OQ*, the five moving pieces thus far described constituting the Sylvester-Kempe parallel motion. To this are added the moving seat and the remaining link *RO′*, the pivot distances of which, *PR* and *RO′*, are equal to *OQ*. The seat in consequence always remains parallel to *QO*, and as *P* moves

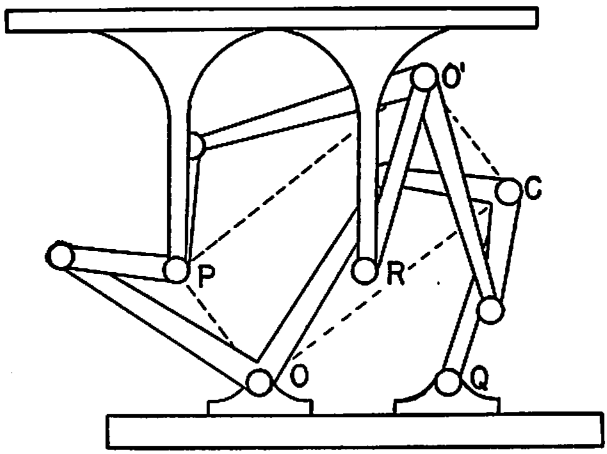


Fig. 27.

accurately in a horizontal straight line, every point on it will do so also. This apparatus might be used with advantage where a very smoothly-working traversing table is required.

I now come to the second of the parallel motions I said I would show you. If I take a kite and pivot the blunt end to the fixed base, and make the sharp end move up and down in a straight line, passing through the fixed pivot, the short links will rotate about the fixed pivot with equal velocities in opposite directions; and, conversely, if the

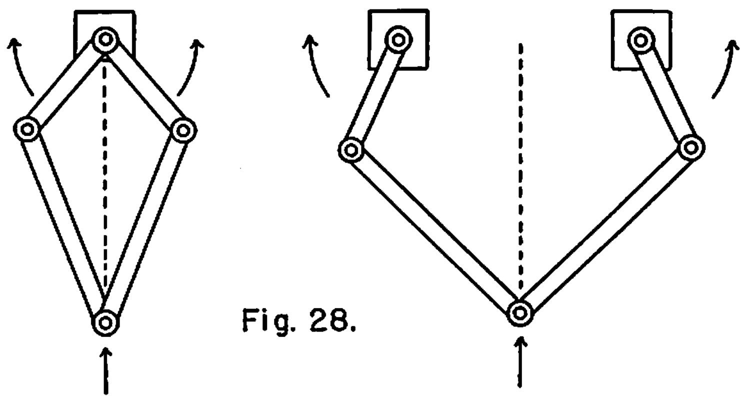


Fig. 28.

links rotate with equal velocity in opposite directions, the path of the sharp end will be a straight line, and the same will hold good if instead of the short links being pivoted to the same point they are pivoted to different ones.

To find a linkwork which should make two links rotate with equal velocities in opposite directions was one of the first problems I set myself to solve. There was no difficulty in making two links rotate with equal velocities in the same direction,— the ordinary parallelogrammatic linkwork employed in locomotive engines, composed of the

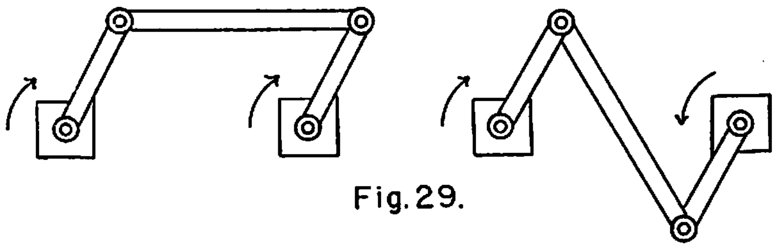


Fig. 29.

engine, the two cranks, and the connecting rod, furnished that; and there was none in making two links rotate in opposite directions with *varying* velocity; the contra-parallelogram gave that; but the required linkwork had to be discovered. After some trouble I succeeded in obtaining it by a combination of a large and small contra-parallelogram put together just as the two kites were in the linkage of Fig. 18. One contra-parallelogram is made twice as large as the other, and the long links of each are twice as long as the short (10).

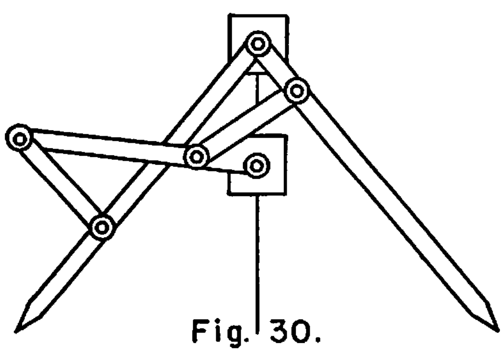


Fig. 30.

The linkworks in Figs. 30 and 31, will, by considering the thin line drawn through the fixed pivots in each as a link, be seen to be formed by fixing different links of the same six-link linkage composed of two contra-parallelograms as just stated. The pointed links rotate with equal velocity in opposite directions, and thus, as shown in Fig. 28, at once give parallel motions. They can of course,

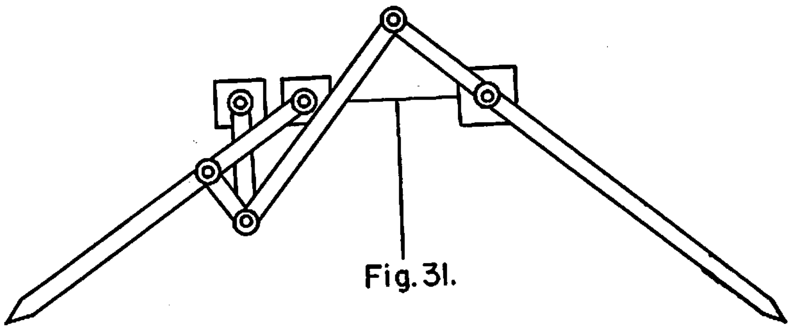


Fig. 31.

however, be usefully employed for the mere purpose of reversing angular velocity (11).

An extension of the linkage employed in these two last figures gives us an apparatus of considerable interest. If I take another linkage contra-parallelogram of half the size of the smaller one and fit it to the smaller exactly as I fitted the smaller to the larger, I get the eight-linkage of Fig. 32. It has, you see, four pointed links radiating from a center at equal angles; if I open out the two extreme ones to any desired angle, you will see that the two intermediate ones will exactly *trisect the angle.* Thus the power we have had to call into operation in order to effect Euclid's first Postulate—linkages—enables us to solve a problem which has no "geometrical" solution. I could of course go on extending my linkage and get others which would divide an angle into any number of equal parts. It is obvious that these same linkages can also be employed as linkworks for doubling, trebling, etc., angular velocity (12).

Another form of "Isoklinostat"—for so the apparatus

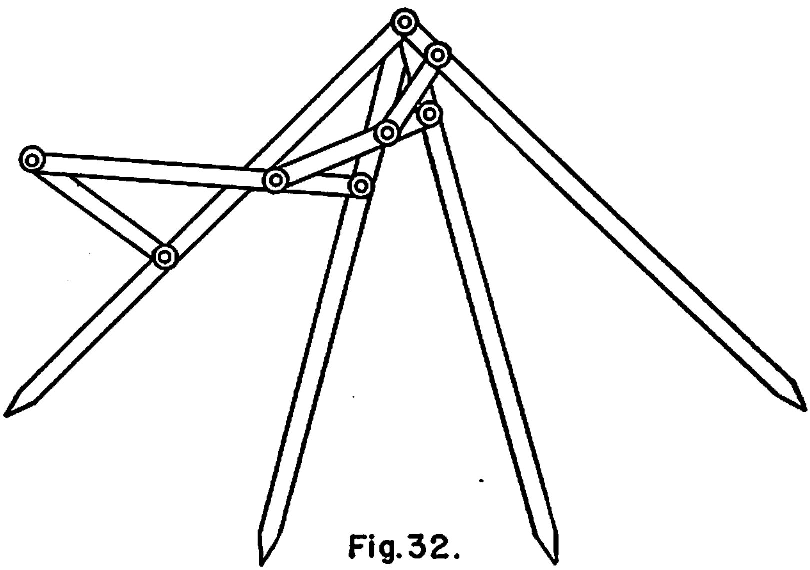


Fig. 32.

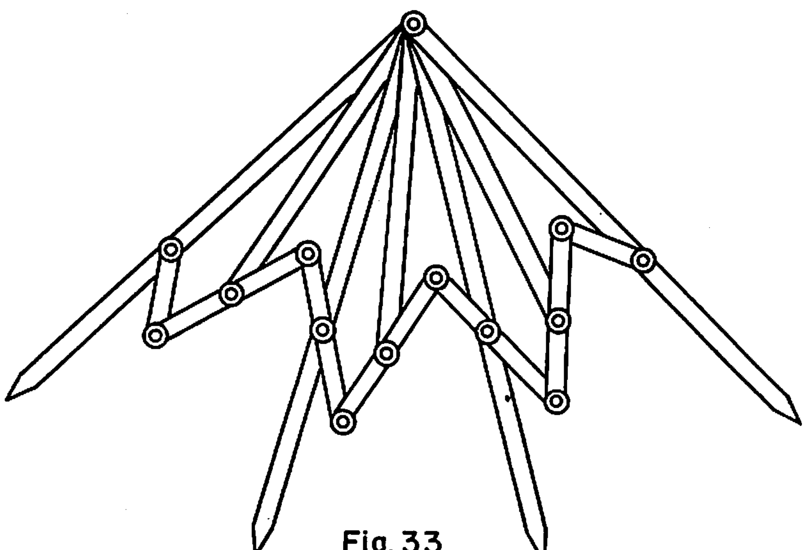


Fig. 33.

is termed by Professor Sylvester—was discovered by him. The construction is apparent from Fig. 33. It has the great advantage of being composed of links having only two pivot distances bearing any proportion to each other, but it has a larger number of links than the other, and as the opening out of the links is limited, it cannot be employed for multiplying angular motion.

Subsequently to the publication of the paper which contained an account of these linkworks of mine of which I have been speaking, I pointed out in a paper read before the Royal Society (13) that the parallel motions given there were, as well as those of M. Peaucellier and Mr. Hart, all particular cases of linkworks of a very general character, all of which depended on the employment of a linkage composed of two similar figures. I have not sufficient time, and I think the subject would not be sufficiently inviting on account of its mathematical character, to dwell on it here, so I will leave those in whom an interest in the question has been excited to consider the original paper.

At this point the problem of the production of straight-line motion now stands, and I think you will be of opinion that we hardly, for practical purposes, want to go much farther into the theoretical part of the question. The results that have been obtained must now be left to the mechanician to deal with, if they are of any practical value. I have, as far as what I have undertaken to bring before you today is concerned, come to the end of my tether. I have shown you that we *can* describe a straight line, and *how* we can, and the consideration of the problem has led us to investigate some important pieces of apparatus. But I hope that this is not all. I hope that I have shown you (and your attention makes that hope a belief) that this new field of investigation is one possessing great interest and importance. Mathematicians have no doubt done much more than I have been able to show you today (14), but the unexplored fields are still vast, and the earnest investigator can hardly fail to make new discoveries. I hope therefore that you whose duty it is to extend the domain of science will not let the subject drop with the close of my lecture.

NOTES

(1) The hole through which the pencil passes can be made to describe a circle independently of any surface (see the latter part of Note 8), but when we wish to describe a circle on a given plane surface that surface must of course be assumed to be plane.

(2) "But" (it is carefully added) "not a graduated one." By the use of a ruler with only two graduations, an angle can, as is well known, be readily trisected, thus—Let RST by the angle, and let PP' be the points where the graduations cut the edge of the ruler. Let $2RS = PP'$. Draw RU parallel and RV perpendicular to ST. Then if we fit the ruler to the figure $RSTUV$ so that the edge PP' passes through S, P lies on RU and P' on RV, PP' trisects the angle RST. For if Q be the middle point of PP', and RQ be joined, the angle TSP = the angle QRP = the angle QRP = half the angle RQS, that is half the angle RSQ.

This solution is of course not a "geometrical" one in the sense I have indicated, because a graduated ruler and the fitting process are employed. But does Euclid confine himself to his three Postulates of construction? Does he not use a graduated ruler and this fitting process? Is not the side AB of the triangle ABC in Book I. Proposition 4, graduated at A and B, and are we not told to take it up and fit it on to DE?

It seems difficult to see why Euclid employed the second Postulate—that which requires "that a terminated straight line may be produced to any length in a straight line,"—or rather, why he did not put it among the propositions in the First Book as a problem. It is by no means difficult by a rigid adherence to Euclid's methods to find a point outside a terminated straight line which is in the same straight line with it, and to prove it to be so, without the employment of the second Postulate. That point can then, by the first Postulate, be joined to the extremity of the given straight line which is thus produced, and the process can be continued indefinitely, since by the third Postulate circles can be drawn with any center and radius.

(3) It is important to notice that the fixed base to which the pivots are attached is really one more link in the system. It would on that account be perhaps more scientific, in a general consideration of the subject, to commence by calling any combination of pieces (whether those pieces be cranks, beams, connecting-rods, or anything else) jointed or pivoted together, a "*linkage.*" When the motion of the links is confined to one plane or to a number of parallel planes, the system is called a "*plane linkage.*" (It will be seen that this lecture is confined to plane linkages; a few remarks about solid linkages will be found at the end of the note.) The motion of the links among themselves in a linkage may be deter-

minate or not. When the motion is determinate the number of links must be even, and the linkage is said to be "*complete.*" When the motion is not determinate the linkage is said to have 1, 2, 3, etc. degrees of freedom, according to the amount of liberty the links possess in their relative motion. These linkages may be termed "*primary,*" "*secondary,*" etc. linkages. Thus if we take the linkage composed of four links with two pivots on each, the motion of each link as regards the others is determinate, and the linkage is a "*complete linkage.*" If one link be jointed in the middle the linkage has one degree of liberty and is a "*primary linkage.*" So by making fresh joints "*secondary*" or "*tertiary,*" etc. linkages may be formed. These primary, etc. linkages may be formed in various other ways, but the example given will illustrate the reason for the nomenclature. When one link of a linkage is a fixed base the structure is called a "*linkwork.*" Linkworks, like linkages, may be "*primary,*" "*secondary,*" etc. A "*complete linkwork,*" *i.e.* one in which the motion of every point on the moving part of the structure is definite, is called a "*link-motion.*" The various "grams" described by these link-motions are very difficult to deal with. I have shown, in a paper in the *Proceedings of the London Mathematical Society*, 1876, that a link-motion can be found to describe any given algebraic curve, but the converse problem, "Given the link-motion, what is the curve?" is one towards the solution of which but little way has been made; and the "tricircular trinodal sextics," which are the "grams" of the simple three-piece motion, are still under the consideration of some of our most eminent mathematicians.

Taking them in their greatest generality, the theoretically simplest form of link-motion is not the flat circle-producing link, but a solid link pivoted to a fixed center, and capable of motion in all directions about the center, so that all points on it describe spheres in space; and the most general form a number of such links pivoted together, forming a structure the various points on which describe surfaces. If two simple solid links, turning about two fixed centers, are pivoted together at a common point, that point will describe a circle independently of any plane surface, the other points on the links describing portions of spheres. The form of pivot which would have to be adopted in solid linkages would be the ball-and-socket joint, so that the links could not only move about round the fixed center, but rotate about any imaginary axis through that center. It is obvious that it would be impossible to construct any joint which would give the links perfect freedom of motion, as the fixed center about which any link turned must be fastened to a fixed base in some way, and whatever means were adopted would interfere with the link in some portion of its path. This is not so in plane link-motions. The subject of solid linkages has been but little considered. Hooke's joint may be mentioned as an example of a solid link-motion. (See also Note 11.)

(4) I have been more than once asked to try and get rid of the objectionable term "parallel motion." I do not know how it came to be employed, and it certainly does not express what is intended. The apparatus does not give "parallel motion," but approximate "rectilinear motion." The expression, however, has now become crystallized, and I for one cannot undertake to find a solvent.

(5) See the *Proceedings of the Royal Institution,* 1874.

(6) This paper is printed *in extenso* in the *Cambridge Messenger of Mathematics,* 1875, vol. iv., pp. 82-116, and contains much valuable matter about the mathematical part of the subject.

(7) The interchange of a radial and traversing bar converts Watt's Parallel Motion into the Grasshopper Parallel Motion. The same change shows us that the curves traced by the linkwork formed by fixing one bar of a "kite" are the same as those traced by the linkwork formed by fixing one bar of a contra-parallelogram. This is interesting as showing that there is really only one case in which the sextic curve, the "gram" of three-bar motion, breaks up into a circle and a quartic.

(8) For a full account of this and the piece of apparatus next described, see *Nature,* vol. xii., pp. 168 and 214.

(9) See the *Messenger of Mathematics,* "On Some New Linkages, 1875, vol. iv., p. 121.

(10) A reference to the paper referred to in the last note will show that it is not necessary that the small kites and contra-parallelograms should be half the size of the large ones, or that the long links should be double the short; the particular lengths are chosen for ease of description in lecturing.

(11) By an arrangement of Hooke's joints, pure solid linkages, we can make two axes rotate with equal velocities in contrary directions (See Willis's *Principles of Mechanism,* 2nd Ed. sec. 516, p. 456), and therefore produce an exact parallel motion.

(12) The "kite" and the "contra-parallelogram" are subject to the inconvenience (mathematically very important) of having "dead points." These can be, however, readily got rid of by employing pins and gabs in the manner pointed out by Professor Reuleaux. (See Reuleaux's *Kinematics of Machinery,* translated by Professor Kennedy, Macmillan, pp. 290-294.)

(13) *Proceedings of the Royal Society,* No. 163, 1875, "On a General Method of Obtaining Exact Rectilinear Motion by Linkwork." I take this opportunity of pointing out that the results there arrived at may be greatly extended from the following simple consideration.

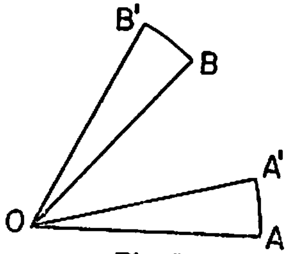


Fig. 34.

If the straight link OB makes any angle D with the straight link OA, and if instead of employing the straight links we employ the pieces $A'OA$, $B'OB$, and if the angle $A'OA$ equals the angle $B'OB$, then the angle $B'OA'$ equals D. The recognition of this very obvious fact will enable us to derive the Sylvester-Kempe parallel motion from that of Mr. Hart.

(14) In addition to the authorities already mentioned, the following may be referred to by those who desire to know more about the mathematical part of the subject of "Linkages." "*Sur les Systèmes de Tiges Articulées,*" par M. V. Liguine, in the *Nouvelles Annales*, December, 1875, pp. 520-560.

Two papers "*On Three-bar Motion,*" by Professor Cayley and Mr. S. Roberts, in the *Proceedings of the London Mathematical Society*, 1876, vol. vii., pp. 14 and 136. Other short papers in the *Proceedings of the London Mathematical Society*, vols. v., vi., vii., and the *Messenger of Mathematics*, vols. iv. and v.

TOPICS FOR CHAPTER PROGRAMS—XIII

37. NIM.

The game of Nim is usually played with matches. Any number of matches is divided into any number of piles, equal or unequal. Two players move alternately, each in his turn choosing any one pile from which he draws one or more matches. The player who succeeds in picking up the last match is the winner. The game is particularly interesting because it has a complete mathematical analysis. In fact, a mechanical "Nim-player" has been constructed.

W. W. R. Ball and H. S. M. Coxeter, *Recreations in Mathematics.* New York, The Macmillan Company, 1947.

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38. PERFECT NUMBERS.

"A perfect number is one which equals the sum of its aliquot parts, i.e., divisors which are less than the number itself. Thus 6 is a perfect number, being equal to the sum of 1, 2, and 3. Continuously for twenty centuries a widespread interest has been taken in the purely numerical questions which arose in the study of these remarkable numbers."

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39. THE GOLDEN SECTION.

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"It might not be absurd to contend that the maintenance of a high standard of mathematical attainment among the scientific men of a country is an object of almost national concern."

—H. J. STEPHEN SMITH

THE PROBLEM CORNER

EDITED BY JUDSON W. FOUST

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solutions should not demand any tools beyond the calculus. Although new problems are preferred, old problems of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before October 1, 1952. The best solutions submitted by students will be published in the Fall 1952 number of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Dr. Judson Foust, Central Michigan College of Education, Mt. Pleasant, Michigan.

PROBLEMS PROPOSED

46. *Proposed by Harold Larsen, Albion College, Albion, Michigan.*

Find all five-digit numbers N such that the cube root of N is exactly equal to the sum of the digits of N.

47. *From the American Mathematical Monthly, October 1951.*

Derive a formula for the sum of the first n terms of a progression in which the first term is a, each even placed term is obtained from its preceding term by multiplying by the constant u, and each odd placed term (after the first) is obtained from its preceding term by multiplying by the constant v.

48. *Proposed by James P. Bradford, Laurel, Mississippi.*

A man cashed a check and found he had received twice the amount of the check plus $3.50 or he received in dollars what the check read in cents and he received in cents what the check read in dollars. What was the actual value of the check?

49. *Proposed by the Problem Corner Editor.*

Three men—Arthur, Bernard and Charles—with their wives—Ann, Barbara and Cynthia—made some purchases.

When their shopping was finished each found that the average cost in dollars of the articles he or she had purchased was equal to the number of his or her purchases. Arthur had bought 23 more articles than Barbara and Bernard had bought 11 more than Ann. Each husband spent $63 more than his wife. Who is the husband of whom?

50. *Proposed by the Problem Corner Editor.*

A roll of paper tape is 6 inches in diameter and has a center core of 1 inch diameter. How many feet of tape is contained in the roll if the tape is 1/250 inch in thickness?

UNSOLVED PROBLEMS

Solutions to problems 34 and 41 have not been received.

SOLUTIONS

33. *Selected from the tenth annual William Lowell Putnam Mathematical Competition, March 25, 1950.*

In each of n houses on a straight street are one or more boys. At what point should all the boys meet so that the sum of the distances they walk is as small as possible?

Solved by George Ladner, Washburn University, Topeka, Kansas.

EDITOR'S NOTE: For lack of space, the detailed proof by Mr. Ladner is omitted. He assumes that the houses are all on the same side of the street and that there is a fixed "exit point" from each house. Counting the boys in order from house to house, we may let x_i be the distance of the ith boy from some arbitrary origin. If P be the meeting point, then the total distance walked is $\Sigma|x_i-P|$. Now this sum is a minimum when P is the median of the x's. Two cases arise depending on the distribution of the boys: the median may fall at a particular house, or it may be an arbitrary point between two houses.

37. *Proposed by H. D. Larsen, Albion College, Albion, Michigan.* (From *Journal de Mathematiques Elementaires.*)

If a, b, c are three numbers in arithmetic progression, b, c, d three numbers in geometric progression, c, d, e three numbers in harmonic progression, show that a, c, e are in harmonic progression.

Solution by Sharon Murnick, Hofstra College, Hempstead, L.I., New York.

From the data, the means of the progressions are $b = \frac{1}{2}(a+c)$, $c = \sqrt{(bd)}$, and $d = 2ce/(c+e)$. Therefore,

$$c^2 = \tfrac{1}{2}(a+c) \times 2ce/(c+e)$$
$$c^2 = c(ae+ce)/(c+e)$$
$$c^2+ce = ae+ce$$
$$c^2 = ae.$$

Therefore c is the geometric mean between a and e.

Also solved by Walter Old, Hofstra College, Hempstead, L.I., New York; N. Bun Furlong, Southern Methodist University, Dallas, Texas; Henry Beersman, Southwest Missouri State College, Springfield, Missouri.

42. *Proposed by the Pentagon Editor.* (This problem appeared in an early number of *The American Mathematical Monthly.*)

A teacher looks at his watch on leaving school at noon. When he comes back the two hands have changed places. Find both times.

Solution by Bill Northrip, Southwest Missouri State College, Springfield, Missouri.

Let x equal the number of minute spaces the minute hand has traveled after 12:00 when the teacher leaves. Then $x/12$ equals the number of minute spaces the hour hand has traveled beyond 12:00. Since the minute hand moves 12 times as far as the hour hand, when the hands have changed places the following equation may be written:

$$60+x/12-x = 12(x-x/12).$$

Solving, we get $x = 5\ 5/143$. Therefore the teacher left at 5 5/143 minutes after 12:00 and returned at 60/143 minutes after 1:00.

Also solved by Sharon Murnich, Hofstra College, Hempstead, L.I., New York; and Harvey Fiala, St. Johns University, Collegeville, Minnesota.

43. *Proposed by the Problem Corner Editor.*

How many cards must you be dealt from a shuffled deck of cards before you can be sure of holding (a) a royal flush? (b) four of a kind? (c) a full house?

Solution by Henry Beersman, Southwest Missouri State College, Springfield, Missouri.

(a) A royal flush consists of the ten, Jack, Queen, King, and Ace of one of the suits. Therefore, to be sure that you hold a royal flush you need to be dealt 49 cards since the last four cards in the deck might be one of the required cards from each suit.

(b) To be sure of holding four of a kind you must be dealt 40 cards since the first 39 cards could be three of each kind and the fortieth card the one needed to make four of a kind.

(c) To be sure of holding a full house, which is three of one kind and two of another, you would need to be dealt 27 cards since the first 26 might consist of pairs and the next card would be needed to make three of one kind.

Also solved by Harvey Fiala, St. John's University, Collegeville, Minnesota.

44. *Proposed by the Problem Corner Editor.*

How many parallelograms are formed when six parallel lines cross eight parallel lines?

Solution by Sharon Murnick, Hofstra College, Hempstead, L.I., New York.

The six parallel lines will form $C(6,2)$ pairs of opposite sides and the eight parallel lines will form $C(8,2)$ pairs of opposite sides. Therefore, $C(6,2) \times C(8,2)$ or 420 parallelograms are formed.

Also solved by Robert J. Wood, Trinity University, San Antonio, Texas; and Harvey Fiala, St. John's University, Collegeville, Minnesota.

45. *Proposed by Norman Anning, University of Michigan, Ann Arbor.* (From *Logical Nonsense*, by Lewis Carroll.)

Two travelers spend from 3 o'clock until 9 in walking along a level road, up a hill, and then home again; their pace on the level being 4 miles an hour, up hill 3 and down hill 6. Fnd the distance walked and (within half an hour) the time of reaching the top of the hill.

Solution by Robert R. Phelps, Los Angeles City College, California.

Let x equal the level distance one way and y equal the distance on the hill one way. Then the total time for the trip is $2(x/4)+y/3+y/6 = 6$ hours. Solving, we find $x+y = 12$, or the total distance is 24 miles. The time to reach the top of the hill is $t = x/4+y/3$. Solving for t in terms of x by substitution, we have $t = (48-x)/12$. It is obvious that $0<x<12$, so that $3<t<4$, and the time of reaching the top is 6:30 (within half an hour).

Also solved by Sharon Murnick, Hofstra College, Hempstead, L.I., New York; Walter Old, Hofstra College; Henry Beersman, Southwest Missouri State College, Springfield, Missouri; and Richard C. Hill, Pomona College, Claremont, California.

Note on Problem 35.

Herr Alfred Moessner adds three more statements of problems without numbers that yield answers with numbers.

(a) A cube-shaped granite stone has just as many cubic decimeters of volume as its surface has of square decimeters in its area. How long is an edge of this granite stone? How large is its volume? How large is its area?

(b) Harold has in his purse just as many dollars as Alfred has cents. If one multiplies the number which represents how many dollars Harold has by itself, he will get the number which represents the combined money wealth of Harold and Alfred expressed in cents. How much money has Harold? How much has Alfred?

(c) Mr. Larsen is asked what is the number of his automobile. He answers, "The digits of my auto number form an arithmetical progression and the cube of the end digit is equal to the sum of the cubes of the preceding digits." What is the number of Mr. Larsen's auto?

A prison consists of 36 cells arranged like the squares of a chessboard. There are doors between all adjoining cells. A prisoner in one of the corner cells is told that he can have his freedom if he can get into the diagonally opposite corner cell by passing through each of the cells once and only once. Can the prisoner gain his freedom?

—SCHO. SCI. AND MATH.

P L E A S E !

If you are changing your address, please notify us. Otherwise, leave instructions and postage with your postmaster for forwarding your copy.—THE PENTAGON, 310 Burr Oak St., Albion, Michigan.

He compounded a simple—an amazingly simple—which he learned out of a book of simples.

—STEPHEN LEACOCK

$=\nabla=$

"No horse has two tails. Every horse has one more tail than no horse; therefore every horse has three tails."

—BOON

$=\nabla=$

If three equal circles are drawn through a point, the circle through their intersections is equal to each of them.

$=\nabla=$

$$\frac{17^3+7^3}{17^3+10^3} = \frac{17+7}{17+10}$$

—NORMAN ANNING

$=\nabla=$

The difference between simple interest and simple discount varies directly as the square of the time.

$=\nabla=$

WHY WORRY ABOUT METHOD?

Solve $(5-3x)(7-2x) = (11-6x)(3-x)$.

Answer: $5-3x+7-2x = 11-6x+3-x$

$$12-5x = 14-7x$$

$$x = 1$$

—MATH. GAZETTE

$=\nabla=$

Prime number "formulas":

$$x^2+x+17$$
$$x^2+x+41$$
$$2x^2+29.$$

$=\nabla=$

"If all that the Greeks knew about the quadratic equation were known by one student, but if this student had no further knowledge about this subject, he would fail to make a passing grade on an ordinary set of examination questions now given to freshman students in our universities."

—G. A. MILLER

=∇=

The following cryptarithm has two solutions which differ only in the interchange of *O* and *I*.

$$\begin{array}{r} NEWTON \\ KLEIN \\ \hline KEPLER \end{array}$$

—AM. MATH. MONTHLY

=∇=

To multiply two 2-digit numbers with complementary unit digits:

Multiply together those multiples of 10 which are respectively just less than the smaller number and just larger than the greater number. Add to this the product of the differences from the first of these of the two given numbers.

Example: 27×83.

a) $20 \times 90 = 1800$

b) $(27-20) \times (83-20) = 7 \times 63 = 441$.

c) $1800 + 441 = 2241$.

—MATH. GAZETTE

=∇=

A butcher wears a size 16-32 shirt and is 6 ft. tall. What does he weigh? Ans. Meat.

=∇=

The series derived from the harmonic series by striking out those terms containing the digit 9 is a convergent series. More generally, the series derived from the harmonic series by striking out those terms containing the digit 9 at least *a* times converges.

—AM. MATH. MONTH.

=∇=

"Up by five o'clock, and after my journal put in order, to my office about my business, which I am resolved to follow, for every day I see what ground I get by it.

By and by comes Mr. Cooper, mate of the Royall Charles, of whom I intend to learn mathematiques, and do begin with him to-day, he being a very able man, and no great matter, I suppose, will content him. After an hour's being with him at arithmetique (my first attempt being to learn the multiplication-table); then we parted till tomorrow."—*The Diary of Samuel Pepys*, July 4, 1662.

=∇=

$$\begin{array}{rcl} V & = & 1+1/2+1/3+1/4+1/5+1/6+\ \ldots \\ 2V/2 & = & \quad 2/2 \qquad +2/4 \qquad +2/6+\ \ldots \\ \hline 0 & = & 1-1/2+1/3-1/4+1/5-1/6+\ \ldots\ = \ln 2. \end{array}$$

Therefore, $1 = 2$.

—NORMAN ANNING

=∇=

A rectangular patch of ground 1 mile by 3/4 mile consists of a rectangle of grass 1 mile by 1/2 mile, and a rectangle of gravel 1 mile by 1/4 mile. A horseman who can do 12 mph on grass and 8 mph on gravel is to cross it from one corner to the diagonally opposite corner; find the shortest time in which he can do it.

—BOON

=∇=

Once a bright young lady called Lillian
Summed the NUMBERS from one to a billion
But it gave her the "fidgits"
To add up the DIGITS.
If you can help her, she'll thank you a million.

—LEO MOSER

(The correct sum is 40,500,000,001. Cf. *Scripta Mathematica*, Vol. 16, p. 126.)

=∇=

"The life-work of the mathematician is richly compensated; but the compensations are not material—they are spiritual. One of them is the joy of life-long contact and intimate association with the eager minds of the young. Another is life-long companionship with men devoted to science and other fields of scholarship. Another is the privilege of long summer vacations affording special opportunities for study, research, writing, and travel. The mathematician's subject is an honored one, and his life is a life of perpetual contact with fundamental thought. He knows that his science is the science of eternal verities and that its service is essential alike to the prosperous conduct of ordinary human affairs, to the advancement of science, and to the support and progress of civilization. And, though he can not gain material wealth, his work, if he be a man of genius, may bring him fame—'the lofty lucre of renown.' "—C. J. Keyser, *Scientific Monthly*, Vol. 17, pp. 489-97.

=∇=

MAGIC SQUARE OF PRIMES

A	43	1	67
B	61	37	13
C	7	73	31
	D	*E*	*F*

All the numbers are prime. In each horizontal row, and each diagonal, the sum is 111. Further, $\Sigma A^2 = \Sigma C^2$ and $\Sigma D^2 = \Sigma F^2$.

—Alfred Moessner

=∇=

"When in 1814 the allied armies crossed the Rhine, all the students of the Ecole Polytechnic asked to be permitted to enter the army; but the Emperor Napoleon refused to grant this request in the following words: 'We have not yet reached that extreme when it is necessary to kill the hen that lays the golden eggs.' "

—I. J. Schwatt

=∇=

π = *3.14159 26535 89793 23846 26433 83279*

Now I know a spell unfailing
An artful charm, for tasks availing,
Intricate results entailing.
Not in too exacting mood,
(Poetry is pretty good),
Try the talisman.—Let be
Adverse ingenuity!

—Nature (Oct. 26, 1905)

Art is an expression of the world order, and is, therefore, orderly, organic, subject to mathematical analysis.

—CLAUDE BRAGDON

=∇=

NEGATIVE DIGITS

Because our number system is based on the decimal scale, it employs ten digits, 0, 1, 2, . . ., 9. It is interesting to note that these ten digits need not be positive; in fact, the only requirement is that they be consecutive positive or negative integers including 0. In particular, all numbers may be expressed by the digits −4, −3, −2, −1, 0, 1, 2, 3, 4, and 5. For convenience, we shall write the positive digits in ordinary type, the negative digits in italic type. Thus, $28 = 3\mathit{2}$, $387 = 4\mathit{1}3$, $3196 = 320\mathit{4}$, etc.

The use of negative digits simplifies many of the operations of arithmetic.[1] Of course, it is necessary to obey the algebraic rules of signs for operating with signed numbers, but there is compensation in the fact that the number of primary multiplication combinations is reduced from 100 to 36. Moreover, multiplication (and hence division) is simplified by replacing the large multipliers 6, 7, 8, 9 by the smaller multipliers $\mathit{4}$, $\mathit{3}$, $\mathit{2}$, $\mathit{1}$ respectively.

The following examples illustrate the convenience in the use of negative digits.

$$(97)^2 = (10\mathit{3})^2 = 10{,}\mathit{6}09 = 9{,}409.$$

$$\begin{aligned}(839)\times(4{,}718) &= (1{,}\mathit{2}4\mathit{1})\times(5\mathit{3}2\mathit{2})\\ &= 4{,}\mathit{1}6\mathit{2}{,}402\\ &= 3{,}958{,}402.\end{aligned}$$

$$\begin{array}{r} 243 \\ 20\mathit{4}\overline{)5\mathit{2}\mathit{4}3\mathit{2}} \\ 40\mathit{8} \\ \hline 1\mathit{2}43 \\ 1\mathit{2}\mathit{2}4 \\ \hline 6\mathit{1}\mathit{2} \\ 6\mathit{1}\mathit{2} \\ \hline \end{array} \qquad\qquad \begin{array}{r} 5\mathit{3}2\mathit{2} \\ 1\mathit{2}4\mathit{1} \\ \hline \mathit{5}3\mathit{2}2 \\ 19\mathit{2}8\mathit{8} \\ \mathit{1}06\mathit{4}4 \\ 5\mathit{3}2\mathit{2} \\ \hline 4\mathit{1}6\mathit{2}402 \end{array}$$

$$\begin{aligned}(47{,}628)\div(196) &= (5\mathit{2}\mathit{4}3\mathit{2})\div(20\mathit{4})\\ &= 243.\end{aligned}$$

[1] Nature, Aug. 3, 1893, p. 316.

THE BOOK SHELF

EDITED BY CARL V. FRONABARGER
Southwest Missouri State College

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Carl V. Fronabarger, Southwest Missouri State College, Springfield, Missouri.

Mathematics, Queen and Servant of Science. By Eric Temple Bell. McGraw-Hill Book Company. (330 West 42nd Street, New York 18, N.Y.), 1951. 20+437 pages. $5.00.

How many times has a student asked himself, or his teacher, "Where can I find further information on this topic?" Or perhaps a teacher, wishing to stimulate a student, might suggest, "Why not read further on this topic?" This book by Professor Bell will very often prove to be the reference to be used. The easily read, narrative style with humorous touches makes for inviting reading. At the same time, because of the ideas presented, it is challenging reading. The material is arranged so that the book can be read consecutively or a chapter or a part of a chapter can be read by itself.

A student planning to teach mathematics in high school should read at least the first chapter. The discussion on the classical proof of Euclid's theorem on primes will be ample reward for the time spent. And of equal value in a later chapter is the discussion of postulates in general, and of Euclid's parallel postulate in particular. The geometries of Riemann and Lobachevski are used to illustrate the discussion. There are many sections which can be read by high school students to gain a glimpse of mathematics beyond that possible to be taught in class.

For college students many sections will be interesting and informative, while other sections will be challenging indeed. The cliche, "Cannot see the forest for the trees," applies with especial force to mathematics and mathematicians. Professor Bell introduces the reader to many branches of mathematics. Even though a student may have studied a particular subject, the discussion given in the book will be valuable. For example, even after a student has had some work in calculus, a reading of the several chapters on "calculuses" will give a new view point. But particularly valuable will be the reading of material in fields unknown to the student, and the gaining of an overall picture of mathematics in its entirety.

In addition to this material, the book stresses the mathematical spirit and the historical steps in the development of certain mahtematical concepts, as well as the power of mathematics when applied to certain "practical" problems.

—E. E. INGALLS

Meaningful Mathematics. By H. S. Kaltenborn. Prentice Hall, Inc. (70 Fifth Avenue, New York), 1951. +371 pages. $4.80.

Dr. Kaltenborn begins this treatise by discussing the nature of mathematics in three illuminating sections. The "nature" of the various topics is carried through every chapter of the book. We have the same clear discussion of the nature of the number system, functions and graphic representation, the algebraic processes, geometric processes, etc. The lengthy explanation of each topic is an important feature of the book.

The plan of the book is to give a brief integrated discussion of a large number of topics of elementary mathematics, including brief discussions of algebraic geometry, non-Euclidean geometry, vectors, differential and integral calculus, and complex numbers. Trigonometry and conic sections are briefly discussed.

The advanced topics, while treated briefly, are discussed clearly, giving to the person with limited mathematical background an understanding and appreciation of more advanced mathematics. His explanations show the use of mathematics in the world in which we live. The remark that "Calculus is an indispensable tool in all branches of science" is a key to the point of view of this book.

However, there is sufficient development of real mathematical topics to make it a valuable text book for a survey course. It is also an excellent treatise for popular reading.

The practical nature of the book is illustrated in such chapters as "Consumers Mathematics," "Functions and Graphical Representation," and "Logarithms." The book also presents a discussion of the classical problems of geometry as well as the three famous "unsoluble" problems. Here we have a wholesome blend of the practical and the pure mathematics. Any one completing the study of this book will have an introduction to the basic principles of mathematics through calculus.

The problems selected are relatively simple as compared with those found in standard textbooks. Many of the exercises, especially in the beginning, require the development of a short essay on some particular topic. Consequently, the book may appeal to those not interested in a rigid study of mathematics.

—J. E. DOTTERER

The Education of T. C. Mits. By Hugh C. and Lillian R. Lieber. W. W. Norton and Company, Inc. (101 Fifth Avenue, New York 3, New York), 1944. 230 pages. $3.50.

The authors have done a very successful job of explaining what mathematics—and in particular pure mathematics—attempts to do; what its objectives are, and how these objectives are met; and yet this is accomplished in a very simple and interesting manner.

The mechanical arrangement of the material is both interesting and helpful in reading, with each line being devoted to a single phrase and most pages being accompanied by a cartoon or diagram bearing on the subject under discussion. The book is full of humor and wit, yet very accurately depicts the fundamental concepts of mathematics. T. C. Mits (*T*he *C*elebrated *M*an *i*n *t*he *S*treet) is a fictitious character who experiences all of the fascinating events of the book and emerges much the wiser for having done so.

The book represents a very complete and accurate answer to the frequently occuring question, "What's the good of all this mathematics?" The author explains how consistency is far more important and abstractions more efficient, to the mathematician, than the immediate application of his theory to some other area of science; although this often occurs from time to time after the theory has been developed.

Two divisions of the material are made. One division is devoted to the "Old" or traditional in mathematics; the other to the "New" or modern developments. There is an abundance of material in each section for mathematics clubs as well as for individual reading. It is the type of book that every college freshman interested in mathematics should read, and then re-read as a senior—he may be amazed at what he has learned in four short years.

—J. D. Haggard

Mathematics For The Million. By Lancelot Hogben. W. W. Norton and Co., Inc. (70 Fifth Avenue, New York), 1937. 12+647 pages. $5.95.

Mr. Hogben has described, largely with a historical development point of view, a number of branches of mathematics. They are Algebra, Geometry, Trigonometry, Analytic Geometry, Calculus, Statistics, and Arithmetic. His professed aim in writing the book is to put at the disposal of everybody a book on elementary mathematics so written that an average person could read and understand it.

Mr. Hogben has not produced a textbook to be used in the classroom. But he has a collection of very carefully expressed thoughts on some branches of mathematics. He does have at the end of each subject a number of exercises for the reader to work out. Further, it is an excellent source for the historical development of each subject and could be so used by teachers.

One feature of the book is the remarkable effort Mr. Hogben has made to guide the reader into proper habits in the process of problem solving. He apparently feels that the solving of problems by algebraic means is an art few students learn easily. At any rate he has made a special effort to teach the proper translation of English into algebraic symbols.

Squeezed in here and there are comments which enliven the text and keep the interest of the reader from lagging. Quite obviously Mr. Hogben is not a bookworm, as is evidenced by his acid remarks concerning various political systems. His approach to the various subjects is very direct. He states fundamental ideas inherent in each of these subjects and then gradually probes deeper and deeper into them. Further, in the introduction to a topic, he attempts to present to the reader the reasons for the invention of the particular branch of mathematics. Thus, in his opening description of logarithms he is painstaking in his efforts to demonstrate the superiority of logarithms in the evaluation of involved multiplication processes over the actual arithmetical procedures.

It might almost be said that Mr. Hogben has taken formal subjects and treated them in such a way as to make his comments virtually recreational reading.

—HAROLD W. ZEOLI

"A difficulty raiseth the Spirits of a great man: he hath a mind to wrestle with it and give it a fall. A man's mind must be very low if his difficulty doth not make part of his pleasure."

—NEWTON

THE CONSTITUTION OF KAPPA MU EPSILON

Article I.—The Fraternity and Its Objectives

Section 1. The name of this organization shall be Kappa Mu Epsilon, Honorary Mathematics Society. Its Greek motto,

'αναπτυσσετε την 'εκτιμησιν του καλλουζ των μαθηματικων

may be translated, "Unfold the Glory of Mathematics."

Section 2. The object of the society shall be

A. to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program;

B. to help the undergraduate realize the important role that mathematics has played in the development of the western civilization;

C. to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought;

D. to provide a society for the recognition of outstanding achievement in the study of mathematics in the undergraduate level.

Article II.—Membership

Section 1. The active members shall consist of those members who pay their current dues.

Section 2. Qualifications for membership. A member

A. must be or have been a faculty member or a regularly enrolled student of an accredited four-year college;

B. must have completed at least eight semester hours or the equivalent in college mathematics, which *must* include the basic ideas of Cartesian Geometry;

C. must be above the average in his institution in mathematics and in general scholarship.

Section 3. Members of chapters upon moving to the locality of another chapter may become affiliated therewith by presenting credentials of membership in good standing from the secretary of the initiating chapter.

Section 4. During each biennium, a chapter may initiate two honorary members. The qualifications for membership listed in Section 2 do not apply to honorary members. The chapter shall pay the regular initiation fees for such honorary members.

Article III.—Insignia

Section 1. The badge of this organization shall be a pentagon with the center slightly concave; on the upper half shall be a five-leaf rose, and on the lower half the letters K. M. E. The badge may be worn as a pin, key, or pendant.

Section 2. The seal shall be a five-pointed star enclosing a five-leaf rose, and being encircled by the legend "Kappa Mu Epsilon, founded 1931."

Section 3. The crest shall be a shield enclosing the five-pointed star; in the star shall be the rose $\rho = a \sin 5\theta$, the symbol of pure mathematics. Around the star shall be the symbols for the sciences which apply mathematics: at the upper left a book of knowledge, for students and teachers; in the lower left a shamrock and a slide rule, for the engineers; at the upper right a conventionalized butterfly, for the biological sciences; at the lower right a moon and three stars, for the physical sciences; at the bottom the symbol "s angle n", for the business world. Above the shield proper is the design of the badge of the society, and below it is a streamer upon which is printed the Greek motto.

Section 4. The colors shall be rose-pink and silver; the flower shall be the wild rose.

Article IV.—Chapters

Section 1. An organized group of at least ten members from an accredited (by the appropriate regional accrediting agency) four-year college may petition for a chapter.

Section 2. A request for the establishment of a chapter shall be referred to the National Council by the National President. This request shall include such information concerning the school and the mathematics faculty as required by the petition. If the Council endorses the request, the National President shall then direct the National Secretary to notify the chapters of this application, giving only the pertinent details. If, within 60 days, one-fourth of the chapters have not filed disapproval with the National Secretary, the National President shall, with the approval of the National Council, grant a charter to the petitioning group.

Section 3. Chapters may be approved by the voting delegates at the biennial convention, as provided in Article VI, Section 1. A three-fourths vote of the voting delegates shall be required to approve all motions to establish a new chapter.

Section 4. Each chapter shall be governed by chapter by-laws which shall not be in variance with the national constitution. Copies of chapter by-laws shall be filed with the National President and the National Secretary.

Section 5. Chapters shall be designated by state and Greek letters. The Greek letter to each chapter shall be the Greek alphabetical order in which the chapter was granted, i.e., Kansas Alpha, Kansas Beta, Kansas Gamma, etc.

Section 6. If a chapter fails to comply with the constitution or receives an indictment from two other chapters for failure to fulfill the purpose of the fraternity, its charter may be withdrawn by a two-thirds vote of the chapters upon a report of the National Council including the charge against the chapter and the chapter's defense.

Section 7. (a) A chapter upon its request presented to the National Council, may become inactive.

(b) A chapter which is delinquent in paying its dues, making required reports, or answering letters, may be suspended by the National Council. Failure to report on any matter where the thirty or sixty days' silence is equivalent to an approval, shall be considered as delinquency in respect to chapter reports, except when the National Council finds the failure to report is unavoidable.

(c) A chapter shall be expelled by the methods of Section 6, Article IV. A chapter which is inactive or suspended shall not be permitted to have membership cards or pins, nor shall it be allowed to participate in any of the benefits of the national fraternity such as the defraying of the expenses of a delegate to the national convention or any other expenditures of money. In any matter requiring approval of two-thirds or three-fourths of the chapters, the number required for approval shall be based on the chapters in good standing; i.e., chapters not inactive or suspended. Upon evidence of willingness to abide by the constitution presented to the National Council and upon payment of a reinstatement fee of ten dollars ($10) and an additional amount to be determined by the National Council, an inactive or suspended chapter may be reinstated.

Article V.—National Officers

Section 1. The national officers shall be President, Vice-President, Secretary, Treasurer, Historian, immediate Past President, and shall be members of Kappa Mu Epsilon. These shall be elected by a simple majority at a national convention. If, for any reason, a national convention is not held, the National Council shall present a nomination for each national officer to the local chapters for their approval or disapproval. Failure to reply within thirty days shall be considered an approval. Each nominee shall be considered elected when two-thirds of the chapters give their approval of his election.

Section 2. The term of office of these officers shall be for two years, except that of the Secretary and Treasurer who may be reelected as often as the chapters deem it advisable. Except for the offices of Secretary and Treasurer, no office shall be filled by the same chapter for more than two successive terms.

Section 3. The duties of the President shall be to preside at the national convention, to provide for the installation of new chapters, to approve all bills before they are paid by the treasurer, and to perform other duties as usually pertain to this office.

Section 4. The duties of the Vice-President shall be to assist the President.

Section 5. The Secretary shall take charge of the records of the society, as provided, and shall perform those duties that usually pertain to this office.

Section 6. The duties of the Treasurer shall be to pay all bills upon the approval of the president, and to perform such duties as usually pertain to this office.

Section 7. The Historian shall be the official historian of Kappa Mu Epsilon. He shall be assistant-editor of the official publication of Kappa Mu Epsilon and shall keep a scrap book of interesting news notes about chapters or members.

Section 8. The immediate Past President shall act as an advisory officer, aiding any and all of the other officers.

Section 9. The above officers shall constitute the National Council. The duties of the National Council shall be to investigate petitioning groups and their institutions as a field of expansion, to fill any vacancies for the unexpired term, and to serve as general executive committee in the interim between conventions. Any four members of the Council shall constitute a quorum for the transaction of business of the fraternity.

Article VI.—Conventions

Section 1. The national convention shall consist of members of the National Council and delegates elected by each chapter. Each chapter may send as many delegates as it desires and each delegate shall be entitled to participate in all discussions, and to vote individually on all motions except on motions to amend the constitution or by-laws, or in the election of national officers, or on motions relating to the establishment of new chapters. On all motions relating to the exceptions mentioned above, each chapter shall be entitled to two votes. These votes shall be cast by two special voting delegates whose names and chapter membership must be certified to, in writing, by the chapter represented.

Section 2. The national convention shall be held every two years, the time and place of which shall be decided by the National Council, and the chapters shall be notified of their action at least six months prior to the date which is set for the convention. The National Organization pays the first-class railroad fare of one delegate from each chapter provided such a sum does not exceed three-fifths of the amount paid by the chapter, during the current biennium, as initiation fees for new members.

Section 3. Biennial reports of all national officers shall be made at each national convention. A written copy of each report must be presented to the National Secretary for permanent filing.

Section 4. The members present at any regularly announced meeting of the convention constitutes a quorum for business purposes providing 30 per cent of the chapters are represented.

Article VII.—Publications

Section 1. The official journal of Kappa Mu Epsilon shall be known as *The Pentagon*.

Section 2. The National Council shall appoint the editor of the official journal. The National Historian shall be assistant editor.

Section 3. The official journal shall contain material which will tend to achieve the objectives of Kappa Mu Epsilon and which will tend to establish fraternal ties between the chapters.

Section 4. The official journal shall be published at least once a year and as more often as the editor and the National Council shall deem advisable.

Section 5. Each initiate shall receive the official journal for a period of two years.

Article VIII.—Amendments

Section 1. This constitution may be amended by two-thirds vote of the national convention provided that a copy of the proposed amendments shall have been sent to every chapter by the National Council at least one month before the convention meets.

Section 2. Amendments may also be proposed at the national convention, and if approved by a two-thirds vote, be submitted to each chapter. Such amendments shall become effective upon ratification by two-thirds of the chapters.

Section 3. Amendments may be proposed at any time by any local chapter or by the National Council to each local chapter for its approval or disapproval. Such amendments become effective upon ratification by two-thirds of the chapters. Failure of any chapter to report within sixty days shall be considered equivalent to approval.

BY LAWS

Article I.—Finance

Section 1. Each new chapter shall pay in advance to the National Treasurer five dollars per capita. Expenses incurred by the national organization incidental to the establishment of a chapter shall be paid by the chapter; however, in any case, this sum shall not be less than thirty-five dollars.

Section 2. Each chapter shall pay the national organization an initiation fee of five dollars for each subsequent member.

Section 3. An auditing committee of three members shall be appointed by the National President to audit the accounts of the National Treasurer prior to the convention.

Article II.—Ritual

Section 1. An official ritual, provided by the National Council, shall be used in all chapter installations and initiations.

Article III.—Chapter Officers

Section 1. The chapter officers shall be President, Vice-President, Recording Secretary, Treasurer, Corresponding Secretary, and Faculty Sponsor.

Section 2. The Corresponding Secretary shall have charge of the official correspondence with the National Society. He shall be a faculty member.

Article IV.—Amendments

Section 1. These by-laws may be amended in the same manner as prescribed for the amendment of the national constitution.

Article V.—Printed Forms

Section 1. Petitions for the establishment of a chapter shall be of the form approved by the National Council.

Section 2. The charter of any chapter shall be of the form approved by the National Council.

Section 3. Permanent record cards of all chapter members shall be sent by each Corresponding Secretary to the National Secretary for permanent filing. These cards shall be provided by the National Secretary and be of the form approved by the National Council.

Section 4. The National Secretary shall, upon the request of the Corresponding Secretary, and upon receipt of the necessary fees and records, issue membership cards of the forms approved by the National Council.

Section 5. All requests for pins or keys must be made on the forms provided for that purpose and must bear the approval of the National Secretary.

KAPPA MU EPSILON NEWS

EDITED BY LAURA Z. GREENE, *Historian*

Dick Hill and Jon Matthews, two of the six juniors elected to Phi Beta Kappa at Pomona College, are members of **California Alpha.** Jon Matthews was also awarded the Phi Beta Kappa scholarship.

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Iowa Alpha served as hosts at two tables at the banquet of the National Council of Teachers of Mathematics in Des Moines, April 17, 18, 19. The Chapter provided table decorations and favors.

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Professor C. B. Tucker, National President of Kappa Mu Epsilon, addressed the Initiation dinner of **Kansas Delta** at Washburn University on November 7, 1951.

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Eighteen students at Mount St. Scholastica College qualified for proficiency examinations in mathematics during the first semester. *An Introduction to the Philosophy of Natural and Mathematical Sciences,* by Sister Helen Sullivan, O.S.B., faculty sponsor for **Kansas Gamma** will be ready for distribution in March, 1952.

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Mississippi Gamma will sponsor a special booth on the "Campus Midway" for Career Day in an effort to interest future students in mathematics and to acquaint them with its uses.

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Norman Hoover, alumnus of **Missouri Beta,** is teaching mathematics at the University of South Dakota. Prof. Loren W. Akers is on leave this year for advanced study at the University of Michigan. Keith Stumpff is doing graduate work in mathematics at the University of Missouri.

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Homer E. Scace member of **Nebraska Alpha** and director of governmental affairs for the Empire State Chamber of Commerce, completed work for the Ph.D. degree at Columbia University in June, 1951.

The organization of a Speakers Bureau for local high schools as a service activity was undertaken by **New Jersey Beta.** The response from the high schools has been quite favorable. Chapter members speak to the mathematics clubs of the local high schools on topics which the high school students choose.

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The annual initiation banquet of **New York Alpha** was held March 7. The speaker was Dr. L. W. Cohen, Associate Secretary of the American Mathematical Society and Professor of Mathematics at Queens College. The chapter is continuing to sponsor mathematics help sections.

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In February Professor Marshall H. Stone, University of Chicago, spent three days in Bowling Green. He gave two evening lectures on *Theories of Integration* and one evening lecture on *Revising the Mathematics Curriculum.* Professor Stone was initiated as an honorary member of **Ohio Alpha.**

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Ohio Gamma reports that the film *The Origin of Mathematics* (400 ft., black and white, sound, 16 mm. Brandon Films, Inc., 1600 Broadway, New York 19, N.Y.) proved to be very interesting.

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Professor Ray Carpenter, sponsor of **Oklahoma Alpha,** entertained members, pledges, and their guests at dinner preceding the formal intiation on November 29. One of the eleven who were initiated into the chapter was L. P. Woods, Jr., son of one of the founders of the National Organization.

PROGRAM TOPICS, FALL SEMESTER, 1951-52

Alabama Beta, Alabama State Teachers College
Tricks With Numbers, by Mr. Thomas O. Williams

California Alpha, Pomona College
How Much Ya Wanna Bet? (Probability), by Dr. C. G. Jaeger
Number Mysticism, by Barbara Jobbins
The Normal Curve, by Dick Hill
Number Continuum, by Joe Seewerker
Mapping, by Ernest Kimme

Illinois Beta, Eastern Illinois State College
Progressions, by Dr. L. A. Ringenberg
Mathematics Problems, Solutions and Problems Proposed by Members of K.M.E.
Short Methods of Computation, by Herbert Wills
Machine Calculations, by Dr. Davis
The Littlest Infinity, by Dr. L. A. Ringenberg

Illinois Delta, College of St. Francis
Application of Symbolic Logic to Mathematics, by Sr. M. Claudia
Methods of Mathematics, by Pauline Yatsko
Luca Paciole and Bookkeeping, by Patricia Jensen
Tesselated Polygons, by Patricia Jensen and Pauline Yatsko
The Humanistic Bearings of Mathematics, by Alice Del Favero
A Critical Analysis of Keyser's "Introduction to Mathematical Philosophy," by Bernadine Arseneau

Iowa Alpha, Iowa State Teachers College
Boolean Algebra, by Margaret Green
Systems of Numbers, by Dorothy Johnson
Contrapositives, by Marion Rigdon
Cubic Equations, by Darlene Nelson
The Polar Planimeter, by Hubert White

Iowa Beta, Drake University
History of Iowa Beta, by Mary Ann Beaver
History and Use of the Slide Rule, by Jack Hannaford
Linear Determinants, by Jerry Brockett
Primitive Number Systems, by Jim Knight

Kansas Alpha, Kansas State Teachers College, Pittsburg
Teaching of Mathematics in Israel, by Mr. Michael Palti, visiting educator from Israel
Report on N.C.T.M. Christmas meeting at Stillwater, Oklahoma by members attending—James McKimson, Gerald Kyser, Louis Juertsch, Dr. J. D. Haggard, and Helen Kreigsman

Kansas Beta, Kansas State Teachers College, Emporia
Purposes and Objectives of Kappa Mu Epsilon, by Dr. O. J. Peterson
Egyptian Mathematics, by Lawrence Huntley

Aircraft Radar, by George Gallagher
The Mathematics in Model Airplane Building, by David Cropp

Kansas Gamma, Mount St. Scholastica College
The Types of Training Offered by the College Disciplines, by Ann Robben, Jill Sullivan, and Ellen Dreiling
Contributions of Mathematics to the Student's Life, by Margaret McBride and Kathleen Feldhousen
Short-cuts in Mathematics for the Applied Sciences, by Prof. George Baumgartner, Mary Ellen Kuhlman, and Selma Winn
Mathematical Twenty Questions, by Sister Helen Sullivan, O.S.B., Josephine Dever, Rita Moran, Lucy Bradley, and K. Cleary
Mathematics and the Fine Arts, by Kathleen Feldhousen, Jacque Vader, Viola Moeder, and Bernadine Law

Kansas Delta, Washburn University
Problems from the Pentagon, by Nancy Martin, Richard Fisher
Theory of Probability, by George Ladner
Trends in Mathematics, by Professor Charles B. Tucker, National President
History of Kappa Mu Epsilon, by Doris Baker

Louisiana Beta, Southwestern Louisiana Institute
Diophantine Equations, by Prof. Scholz
Physical Application of Mathematics, by Percy Smith
Magic Squares, by Nelda Gilbert
Binary Systems, by Molly Foreman
Everiste Galois, by Wilhelmina Foss

Michigan Beta, Central Michigan College
What is Number?, by Dana R. Sudborough
Externing, by Courtney Carr and Donald McPhee, Student teachers

Mississippi Gamma, Mississippi Southern College
Vocations Open to Mathematics Majors, by Floyd O'Neal
How My Conceptions of Practice Teaching Have Changed Since I Became a Practice Teacher, by Eddie Miley
The Slide Rule

Missouri Alpha, Southwest Missouri State
Elementary Curve Tracing with Sign Lines and Tangent Carriers, by Mr. L. T. Schiflett
Euler's Formula for Polyhedra, by Henry Beersman
Determinants, by Ralph Siddens
Computation with the Abacus, by Norma L. Jones
Paper Folding, by Paul Long
Sun with Mathematics and the Wheatstone Bridge, by Dr. L. V. Whitney

Missouri Beta, Central Missouri State College
Some Practical Applications of Higher Mathematics, by James Green

The Discovery Method of Teaching Mathematics, by Edna May Schapp
Mathematics and Music, by Zetia Zulanf
Vectors, by George Hutton
The Slide Rule, by Charles Edwards

Missouri Epsilon, Central College
Determinants, by George Koonce
Chess, by David Morrison
Inter-Planetary Rockets, by Hector McDonald

Nebraska Alpha, Nebraska State Teachers College
Mathematical Wrinkles and Riddles, by Warren Emery

New Jersey Beta, New Jersey State Teachers College
Mathematics in Television, by Dr. Virgil S. Mallory
Certain Interpolation Procoedures from the Calculus of Finite Differences, by William Koellner
Einstein's Theory of Relativity, by John Loughlin

New York Alpha, Hofstra College
Geometric Inversion, by Marjorie Liers
Fourth Dimensional Ramblings, by Robert Blasch
Moebus Strips, by John Prussen
Fermagoric Triangles, by Louis Bronzo
Graphic Solutions of Equations having Complete Roots, by Richard Lamm
The Role of a Mathematician in Industry, by Dr. Alfred M. Peiser of Hydrocarbon Research Inc.
Platonic Solids, by Mary Pawelko
How to Tell Time, by William R. Newman
Physics and Electronics, by Walter Tolles

Ohio Alpha, Bowling Green State University
The Philosophy of Mathematics, by Dr. Tom Tuttle
A Mathematical Quiz, by William Elderbrock
Outline of a Problem in Cremona Transformations, by Dr. F. C. Ogg
Theories of Integration, by Dr. M. H. Stone of the University of Chicago

Ohio Beta, College of Wooster
Vocational panel

Ohio Gamma, Baldwin-Wallace College
Origin of Mathematics, a film
Micro-wave Phenomena, by Thomas Budiscak of Ohio Bell
Powdered Metallurgy, by Francis Lowey, metallurgist
Art and Mathematics, by Dr. S. F. Greenwood, Department of Art

Oklahoma Alpha, Northwestern State College
Extracting Roots by Arithmetic, by Ray Carpenter
Solving Higher Equations, by Jim Reeves
Trisecting an Angle, by Robert Tinney

Why it Is Impossible to Trisect an Angle, by Mike Reagan
A Problem of Finance, by Monte York
Squaring a Circle, by Ray Carpenter

South Carolina Alpha, Coker College
Sailors, Coconuts, and Monkeys (Diophantine Equations), by F. W. Saunders
Application of Probability in Card Games, by Bill Johnson
Some Methods of Solving Equations of Higher Degree, by Ann Merck
How Series Are Used in Computing Log Tables, by F. W. Saunders

Tennessee Alpha, Tennessee Polytechnic Institute
The Symmetries of the Equilateral Triangle, by Professor G. C. Holt
Some Aspects of Analytic Projective Geometry, by Professor B. F. Mullins
Elliptic Integrals, by President James L. Comer

Texas Alpha, Texas Technological College
Probability and Games of Chance, by Prof. E. R. Heineman
A Showing of Astronomical Slides with Explanations, by Prof. R. S. Underwood
Loci of the Notable Points of Variable Triangles, by Prof. Gordon Fuller

Texas Epsilon, North Texas State College
A Diophantine Equation, by James C. Bradford
Some Cryptic Divisions, by Laroy R. Carry
Pythagorean Number Triples, by Robert Barham
A Visit to the Mathematics Laboratory Room of the N.T.S.C. Laboratory School, Conducted by Dr. J. V. Cooke

Wisconsin Alpha, Mount Mary College
Report of the Wisconsin Mathematics Council Meeting, by Sister Mary Felice
Summary given of Miss John's lecture on Arithmetic in 1951, by Sister Mary Felice

CHAPTERS OF KAPPA MU EPSILON

ALABAMA ALPHA, Athens College, Athens.
ALABAMA BETA, Alabama State Teachers College, Florence.
ALABAMA GAMMA, Alabama College, Montevallo.
CALIFORNIA ALPHA, Pomona College, Claremont.
COLORADO ALPHA, Colorado A & M College, Fort Collins.
ILLINOIS ALPHA, Illinois State Normal University, Normal.
ILLINOIS BETA, Eastern Illinois State College, Charleston.
ILLINOIS GAMMA, Chicago Teachers College, Chicago.
ILLINOIS DELTA, College of St. Francis, Joliet.
INDIANA ALPHA, Manchester College, North Manchester.
IOWA ALPHA, Iowa State Teachers College, Cedar Falls.
IOWA BETA, Drake University, Des Moines.
KANSAS ALPHA, Kansas State Teachers College, Pittsburg.
KANSAS BETA, Kansas State Teachers College, Emporia.
KANSAS GAMMA, Mount St. Scholastica College, Atchison.
KANSAS DELTA, Washburn Municipal University, Topeka.
LOUISIANA ALPHA, State University, Baton Rouge. (Inactive)
LOUISIANA BETA, Southwestern Louisiana Institute, Lafayette.
MICHIGAN ALPHA, Albion College, Albion.
MICHIGAN BETA, Central Michigan College, Mount Pleasant.
MICHIGAN GAMMA, Wayne University, Detroit.
MISSISSIPPI ALPHA, State College for Women, Columbus.
MISSISSIPPI BETA, Mississippi State College, State College.
MISSISSIPPI GAMMA, Mississippi Southern College, Hattiesburg.
MISOURI ALPHA, Southwest Missouri State College, Springfield.
MISSOURI BETA, Central Missouri State College, Warrensburg.
MISSOURI GAMMA, William Jewell College, Liberty.
MISSOURI DELTA, University of Kansas City, Kansas City.
MISSOURI EPSILON, Central College, Fayette.
NEBRASKA ALPHA, Nebraska State Teachers College, Wayne.
NEW JERSEY ALPHA, Upsala College, East Orange.
NEW JERSEY BETA, New Jersey State Teachers College, Montclair.
NEW MEXICO ALPHA, University of New Mexico, Albuquerque.
NEW YORK ALPHA, Hofstra College, Hempstead.
NORTH CAROLINA ALPHA, Wake Forest College, Wake Forest.
OHIO ALPHA, Bowling Green State University, Bowling Green.
OHIO BETA, College of Wooster, Wooster.
OHIO GAMMA, Baldwin-Wallace College, Berea.
OKLAHOMA ALPHA, Northeastern State College, Tahlequah.
PENNSYLVANIA ALPHA, Westminster College, New Wilmington.
SOUTH CAROLINA ALPHA, Coker College, Hartsville.
TENNESSEE ALPHA, Tennessee Polytechnic Institute, Cookeville.
TEXAS ALPHA, Texas Technological College, Lubbock.
TEXAS BETA, Southern Methodist University, Dallas.
TEXAS GAMMA, Texas State College for Women, Denton.
TEXAS DELTA, Texas Christian University, Fort Worth.
TEXAS EPSILON, North Texas State College, Denton.
WISCONSIN ALPHA, Mount Mary College, Milwaukee.