THE PENTAGON

Volume IX SPRING, 1950 Number 2

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Kappa Mu Epsilon, national honorary mathematics fraternity, was founded in 1931. The object of the fraternity is four-fold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievement in the study of mathematics in the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

SOLUTIONS OF THE QUADRATIC EQUATION*

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1. Historical Survey

The origin of the solution of the quadratic equation has not been traced definitely to any one person or to any one race. It is a known fact that the geometric solution of the quadratic equation was invented long before the analytic method. Equations of the second degree were solved arithmetically by the Egyptians, geometrically by Euclid and his followers, and algebraically by the Hindus.

Al-Khowarizmi, an Arabian writer of the ninth century, gave rules the validity of which was demonstrated by geometric methods. The use of these rules was followed in Europe until near the end of the sixteenth century. At that time writers began to consider the solution of general equations with literal coefficients. However, even then zero and negative roots were neglected and the complete solution of the quadratic equation was not given until positive, negative, irrational, and complex numbers came into general use around the middle of the seventeenth century.

Before the seventeenth century geometric constructions were devised which employed intersections of straight lines and conic sections, or of straight lines and higher degree curves. Down to the time of Cardan, the geometric methods of the Greeks predominated among the Arabs and people of the Occident.

The algebraic solution of quadratic equations was known to Diophantus, an Alexandrian Greek of the fourth century *A.D.*, and author of a treatise on arithmetic. However, Diophantus rejected negative roots and failed to recognize two roots of a quadratic equation even when both were positive. To the Greeks, the idea of multiple-valued solutions was entirely foreign.

*A thesis presented for Honors in Mathematics at Albion College, first semester 1949-1950. Because of technical difficulties in printing, a brief section treating the origin and development of the mathematical symbolism used in solving quadratic equations has been omitted here.—ED.

The first to observe that a quadratic has two roots and to recognize the existence of negative quantities were the Hindus. The earliest complete method of solution of quadratic equations, together with their applications to practical problems, is found in the writings of Brahmagupta, a Hindu astronomer of the seventh century *A.D.*

There is a great resemblance between the writings of Diophantus and of the Hindu mathematicians. We have reason to believe that Diophantus got his first glimpse of algebraical knowledge from the Hindus during the time of commercial trading between Rome and India by way of Alexandria, while afterwards the Hindus received some of their knowledge from the writings of the gifted Diophantus [32, pp. 204-205].*

2. Early Solutions

The first known solution of the quadratic equation is the one given in the Berlin Papyrus, believed to have been written between 2160 and 1700 *B.C.* The problem may be stated:

> Divide 100 square measures into two squares such that the side of one shall be three fourths the side of the other. That is,
>
> $$x^2 + y^2 = 100, \ y = \tfrac{3}{4}x.$$

It might be noted that this example is one of simultaneous equations, and the solution as given by the Berlin Papyrus is as follows [28, p.443]:

> "Make a square whose side is 1 and another whose side is 3/4. Square 3/4, giving 9/16, add the squares, giving 25/16, the square root of which is 5/4. The square root of 100 is 10. Divide 10 by 5/4, giving 8, and 3/4 of 8 is 6. Then $8^2 + 6^2 = 100$, $6 = 3/4$ of 8, so that the roots of the two implied equations are 6 and 8."

One of the first writers to give a rule for the solution of the quadratic equation was Brahmagupta. In an algebra

*Numbers in brackets refer to the references cited at the end of the paper.

written by him in the early part of the seventh century he gives this rule for solving $ax^2+bx=c$ [21, p. 10]:

> "To the absolute number multiplied by the [coefficient of the] square, add the square of half the [coefficient of the] unknown, the square root of the sum, less half the [coefficient of the] unknown, being divided by the [coefficient of the] square, is the unknown."

If we wish, we might put this in a formula,

$$x = (\sqrt{(\tfrac{1}{2}b)^2 + ac} - \tfrac{1}{2}b)/a.$$

The reader will observe that this is a correct solution for the quadratic equation $ax^2 + bx = c$.

Another Hindu, Sridhara (1020), gave a solution for the quadratic equation $ax^2 + bx = c$ similar to the one above. He has been quoted as saying [28, p. 446]:

> "Multiply both sides of the equation by a number equal to four times the [coefficient of the] square, and add to them a number equal to the square of the original [coefficient of the] unknown quantity. [Then extract the root.]"

That is, given $ax^2 + bx = c$, we multiply by $4a$,

$$4a^2x^2 + 4abx = 4ac.$$

Then adding b^2 to each member,

$$4a^2x^2 + 4abx + b^2 = b^2 + 4ac,$$

whence

$$2ax + b = \sqrt{b^2 + 4ac},$$

$$x = (-b+\sqrt{b^2+4ac})/2a.$$

The negative root being rejected, the purpose of multiplying by $4a$ was to avoid fractions.

The first systematic treatment of algebra was written by an Arab, Mohammed ibn Mussa, Al-Khowarizmi. The word "algebra" is a part of the title of his treatise, *al-jebra W'almuqabula.* Al-Khowarizmi considered the quadratic equations $ax^2 = bx$, $ax^2 = n$, $ax^2 + bx = n$, $ax^2 + n = bx$, and $ax^2 = bx + n$. He gives the method of solution for $ax^2 + bx = n$ (or essentially the same equation $x^2 + px = q$) on the basis of geometry as follows [28, p. 446-447]. He

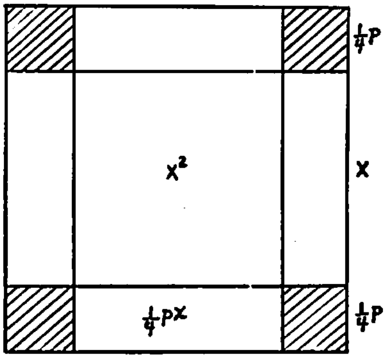

Fig. 1

first constructed the square as shown in Figure 1. Then the unshaded part is $x^2 + px$ and therefore equal to q. In order, however, to make this a square we must add the four shaded squares, each of which is $(\frac{1}{4}p)^2$ and the sum of which is $\frac{1}{4}p^2$. It will be noted that since we now have a square of area $x^2 + px + \frac{1}{4}p^2 = q + \frac{1}{4}p^2$, the length of one side must be $\sqrt{q + \frac{1}{4}p^2}$. But from the diagram one side on the square is $x + \frac{1}{2}p$. Therefore, we can set these two values equal; that is,

$$x + \tfrac{1}{2}p = \sqrt{q + \tfrac{1}{4}p^2}.$$

As an example, consider $x^2 + 10x = 39$. To make the left-hand member a perfect square we must add $\frac{1}{4}p^2$ or 25, so that

$$\begin{aligned} x^2 + 10x + 25 &= 39 + 25, \\ x + 5 &= \sqrt{64}, \\ x &= -5 + 8 = 3. \end{aligned}$$

For the same problem Al-Khowarizmi gives a second method of solution which yields results equivalent to the values obtained from our quadratic formula. In this method he proceeds in much the same manner as he did in the first method. In Figure 2 the unshaded part is $x^2 + px$; now add the square of $\frac{1}{2}p$. This gives $x^2 + px + \frac{1}{4}p^2 = \frac{1}{4}p^2 + q$, whence, $x + \frac{1}{2}p = \sqrt{\frac{1}{4}p^2 + q}$ of which he takes but the positive square root.

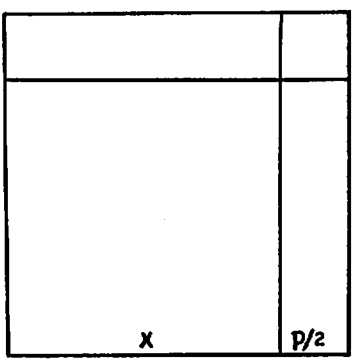

Fig. 1

Let us now turn from the algebra of Al-Khowarizmi to the *Elements* of Euclid. To solve the equation $x^2 + ax = b^2$ Euclid proceeds as follows [14, pp. 79-80]. To the segment $AB = a$ in Figure 3, apply the rectangle DH, of known area equal to b^2, in such a way that CH shall be a square. The

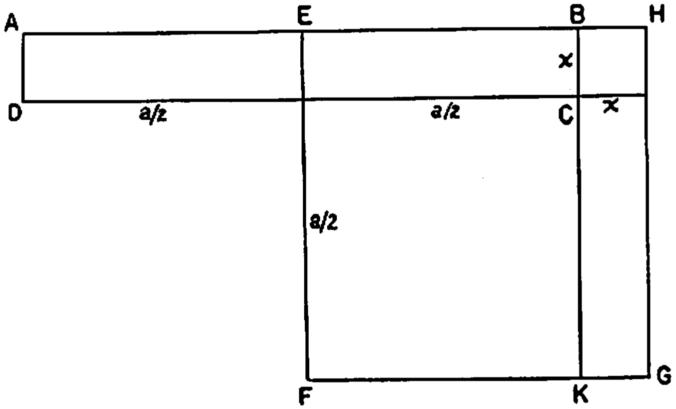

Fig. 3

figure shows that, for $CK = \frac{1}{2}a$, $FH = x^2 + 2x(\frac{1}{2}a) + (\frac{1}{2}a)^2 = b^2 + (\frac{1}{2}a)^2$. But we observe from Figure 3 that $b^2 + (\frac{1}{2}a)^2 = c^2$, where $c = EH = \frac{1}{2}a + x$ and $x = c - \frac{1}{2}a$. The solution obtained by applying areas, in which the square root is always considered positive, is nothing more than a construction of a line representing

$$x = -\tfrac{1}{2}a + \sqrt{b^2 + (\tfrac{1}{2}a)^2} = c - \tfrac{1}{2}a.$$

In the same manner, Euclid solved all equations of the form $x^2 \pm ax \pm b^2 = 0$. He noted that if $\sqrt{b^2 - (\frac{1}{2}a)^2}$ is involved the condition for a solution is $b > \frac{1}{2}a$. Nowhere in Euclid's work are negative quantities considered [14, p. 80].

Marco Aurel (1540), a German residing in Spain, classified the quadratic equations into three classes as did most of the writers of this period. The three classes were [17, p. 58]

$$ax^2 + bx = n,\ ax^2 + n = bx, \text{ and } ax^2 = bx + n.$$

For the first form Aurel gives a method of solution which we illustrate by an example.

Given:	$2x^2 + 12x = 32.$
Divide by the coefficient of x^2:	$x^2 + 6x = 16.$
Take ½ the coefficient of x:	$\frac{1}{2}$ of $6 = 3.$
Square this:	$3^2 = 9.$
Add this to the constant term:	$16 + 9 = 25.$
Then	$x = \sqrt{25} - 3,$
	$x = 5 - 3 = 2.$

Aurel's solution for the second form is similar to this, the addition being replaced by a subtraction. For the third form he proceeds as in the case of form one. Whenever negative quantities occur, the equation is reduced to one with all positive terms by adding the same quantity to both members, and thus it can be classified under one of the three classes above.

Aurel made no attempt to avoid irrational numbers as did the other writers of his time. He also assumed an equation to have two roots if both were positive. One of the problems appearing in his writings was, "Divide a number into two parts such that one multiplied by the other will give 3 [17, p. 60]." His solution is essentially as follows.

Let x = one part and $10 - x$ = the other part. Then $(10-x)x = 3$, whence $x^2 + 3 = 10x$ and $x = 5 \pm \sqrt{22}$.

Menher De Kempton (1565) demonstrated the solution of the quadratic equation and accompanied each example with a very clear geometrical demonstration [17, p. 66]. For example, given

$$4x^2 + 12x + 10 = 50.$$

Subtract 1 from both sides:

$$4x^2 + 12x + 9 = 49.$$

Factor:	$(2x + 3)^2 = 7^2.$
Take the root:	$2x + 3 = 7.$
Thus,	$x = 2.$

This method, in which a certain quantity is added to each member of the equation thereby making them perfect squares, was the first of its kind in this period.

Albert Girard (1629) dealt mostly with cubic equations but the following example of his solution of a quadratic equation shows how primitive was the method he used [17, p. 75].

Given:	$x^2 + 6x = 40.$
Divide by x:	$x + 6 = 40/x.$
Factors of 40:	$2\times20,\ 4\times10,\ 5\times8.$

If one chooses 2 for x, then $2 + 6 = 8$ which is not in accord with the other factor 20. If one chooses 4, then $4 + 6 = 10$ which does correspond to the other factor 10; therefore $x = 4$. Like most of his predecessors, he considered positive roots only.

The first work in which both positive and negative roots were indicated by plus (+) and minus (−) was *Algebra ofte Nieuwe Stel-Regel* (1639) written by Stampioen. He gives the following example [17, p. 83]:

$$x^2 + 4x = 60$$
$$\tfrac{1}{2} \text{ of } 4 = 2,\ 2^2 = 4$$
$$4 + 60 = 64,\ \sqrt{64} = \pm 8$$
$$x = 8 - 2 = 6,\ x = -8 - 2 = -10.$$

In the check of this problem as well as in the usual explanation accompanying the reduction of the equation, Stampioen emphasizes the fact that the square root of 64 is +8 or −8 because +8 or −8 squared gives 64. Furthermore, in order to check the solution he substitutes the roots in the equation to see if they satisfy.

In the works of Vieta (1540-1630), the analytic methods replaced the geometric and his solutions of the quadratic equation were therefore a distinct advance over those of his predecessors [17, p. 89]. For example, to solve

$x^2 + ax + b = 0$, he lets $x = u + z$. Then,

$$u^2 + (2z+a)u + z^2 + az + b = 0.$$

If $2z+a=0$, that is $z = -\tfrac{1}{2}a$, then

$$u = \pm\tfrac{1}{2}\sqrt{a^2 - 4b}$$

and
$$x = u+z = -\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{a^2 - 4b},$$

which will be noted is the quadratic formula when the coefficient of the x^2 term is unity.

Concerning the works of Descartes, D. E. Smith says [27, p. 375]: "The fundamental idea in Descartes mind was not the revolutionizing of geometry so much as it was the elucidating of algebra by means of geometric intuition and concepts, in a word, the graphic treatment of the equation . . ." Descartes gives the following method to solve the equation $x^2 = ax + b$ [17, p. 96]:

"Construct (Fig. 4) the right triangle *NLM* with one side *LM* equal to b, the square root of the known quantity b^2, and the other side *LN*, equal to $\frac{1}{2}a$, that is, to half of the other known quantity which was multiplied by x, which I suppose to be the unknown line. Then prolong *MN*, the hypotenuse of this triangle, to *O*, so that *NO* is equal to *NL*; the whole line *OM* is the required line." From the figure, $x = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}$.

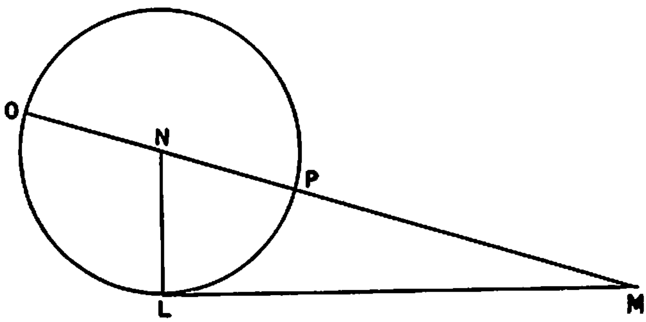

Fig. 4

For the equation $y^2 = -ay + b^2$ Descartes says [17, p. 96]: "But I have $y^2 = -ay + b^2$ where y is the quantity whose value is desired. I construct the same right triangle *NLM*, and on the hypotenuse *MN* lay off *NP* equal to *NL*, and the remainder *PM* is y, the desired root. Thus I have

$$y = -\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2}.$$

"Finally, if I have $x^2 = ax - b^2$, I make *NL* equal to $\frac{1}{2}a$ and *LM* equal to b as before; then, instead of joining the points *M* and *N*, I draw *MQR* parallel to *NL*, and with *N*

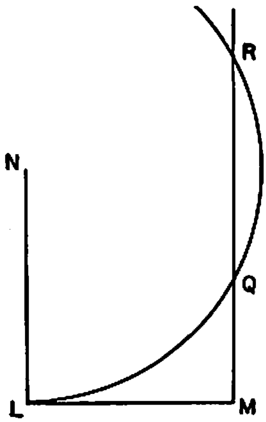

Fig. 5

as center describe a circle through L cutting MQR in the points Q *and* R (Fig. 5); then x, the line sought, is either MQ or MR, for in this case it can be expressed in two ways, namely, $x = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b^2}$, and $x = \frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b^2}$. Since $MR \cdot MQ = (LN)^2$, then if $R = x$, we have $MQ = a - x$, and so $x(a-x) = b^2$ or $x^2 = ax - b^2$." If instead of this $MQ = x$, then $MR = a - x$, and again $x^2 = ax - b^2$.

As for the last method, if MR is tangent to the circle, that is if $b = \frac{1}{2}a$, the roots will be equal; but if $b > \frac{1}{2}a$, the line MR will not meet the circle and the roots will be imaginary. It can be noted that Descartes considered only three types of equations, namely, $x^2 + ax - b^2 = 0$, $x^2 - ax - b^2 = 0$, and $x^2 - ax + b^2 = 0$. He does not consider the type $x^2 + ax + b^2 = 0$ because it has only negative roots.

3. Derivations of the Quadratic Formula

The preceding pages of this article have dealt with the history and some of the early solutions of the quadratic equation. The discussion has been brief and the number of examples has been limited. However, the examples are fairly general and show the important steps in the evolution of a standardized method of solution for the quadratic equation. In the remainder of this paper we shall deal with solutions which are in use at the present time. Some of these are very similar to those given above, but they generally are of a more scientific nature.

One of the more modern methods for the development of the quadratic formula, interesting mostly from the standpoint of theory, is the one that follows. This method uses determinants and is due to Euler (1750) and Bezout (1775), but was improved by Sylvester (1840) and Hesse (1844) [28, p. 450].

Given $x^2 + px + q = 0$, let $x = u + z$. Then

$$x^3 + px^2 + qx = 0,$$
$$x^2 - (u+z)x = 0,$$
$$x^3 - (u+z)x^2 = 0.$$

Now the eliminant of this system of equations is the determinant

$$\begin{vmatrix} 1 & p & q \\ 0 & 1 & -(u+z) \\ 1 & -(u+z) & 0 \end{vmatrix} = 0.$$

Expanding, we get

$$-p(u+z) - (u+z^2) - q = 0,$$

whence, rewriting,

$$u^2 + (2z+p)u + (z^2+pz+q) = 0.$$

Since $x = u + z$, we can choose u or z at our convenience; thus, let $2z+p = 0$, that is, $z = p/2$. Then

$$u^2 + p^4/4 - p^2/2 + q = 0,$$
$$u = \pm (\sqrt{p^2-4q})/2,$$

and

$$x = u + z = -p/2 \pm (\sqrt{p^2-4q})/2.$$

One also can obtain the quadratic formula by solving the quadratic equation in a manner very similar to that used by Ferrari in his solution of the biquadratic equation. Given $x^2 + px + q = 0$, transpose everything but the x^2 term to the right-hand side of the equation and complete the square in y by adding $yx + y^2/4$ to both members. Then

$$x^2 + yx + y^2/4 = -px - q + yx + y^2/4,$$

that is,

$$(x + y/2)^2 = (y-p)x + (y^2/4 - q).$$

Now since y can take any value, and for any value of y the left-hand member will be a perfect square, let us choose $y = p$. Then, extracting roots,

$$(x + p/2) = \pm(\sqrt{p^2-4q})/2,$$

or

$$x = -p/2 \pm (\sqrt{p^2-4q})/2.$$

Let us next turn our attention to the solution of the quadratic equation by a trigonometric method. Given

(1) $$ax^2 + bx + c = 0.$$

Let $x = \sqrt{c/a} \tan \theta$. Substituting in (1),

$$c \tan^2 \theta + b\sqrt{c/a} \tan \theta + c = 0,$$

or $$c \sec^2 \theta + b\sqrt{c/a} \tan \theta = 0.$$

Multiplying by $2 \cos^2 \theta$,

$$2c + b\sqrt{c/a}(2 \sin\theta \cos\theta) = 2c + b\sqrt{c/a} \sin 2\theta = 0,$$

whence

$$\sin 2\theta = -2\sqrt{ac}/b.$$

Using the identity $\tan \theta = (1-\cos 2\theta)/\sin 2\theta$, we get after some simplification,

$$\tan\theta = (-b\sqrt{c} \pm \sqrt{b^2c - 4ac^2})/2c\sqrt{a}.$$

Since $x = \sqrt{c/a} \tan \theta$, we have

$$x = (-b \pm \sqrt{b^2-4ac})/2a.$$

We shall now consider an altogether different method of deriving the quadratic formula. In this method we let the roots of $ax^2 + bx + c = 0$ be p and q. Then

(1) $$p + q = -b/a$$

(2) $$pq = c/a.$$

Squaring (1) and subtracting four times (2) gives

$$p^2 + 2pq + q^2 - 4pq = b^2/a^2 - 4c/a,$$

$$(p-q)^2 = (b^2-4ac)/a^2,$$

(3) $$p-q = \pm(\sqrt{b^2-4ac})/a.$$

Now if we solve (1) for q and substitute this value in (3), we get

$$p = (-\mathrm{b} \pm \sqrt{b^2 - 4ac})/2a.$$

The solution of the quadratic equation by means of completing the square will be given now as a final development of the quadratic formula.

Every function of the form $x^2 + px + q$ can be made a complete square in x by the addition of a constant k. That is,

$$x^2 + px + q + k = (x + m)^2 = x^2 + 2mx + m^2$$

where m is a second constant. Now since we have two expressions which are identical, the coefficients of like powers must be equal; consequently, $p = 2m$, $q + k = m^2$. Hence,

$m = p/2$ and $k = m^2 - q = (p/2)^2 - q$. Thus we see that in order to make $x^2 + px + q = 0$ a perfect square in x we must add the constant $(p/2)^2 - q$ to each member of the equation. This can be extended to the general form $ax^2 + bx + c = 0$ by first dividing each member by a; then $p = b/a$, $q = c/a$, and $k = (b/2a)^2 - c/a = (b^2 - 4ac)/4a^2$. Thus,

$$(x + b/2a)^2 = (b^2 - 4ac)/4a^2,$$
$$x = -b/2a \pm \sqrt{b^2 - 4ac}/2a.$$

4. Methods of Factoring

One of the most common methods of solution of the quadratic equation, if not the most common, is the method of factoring. There are four methods of factoring in common practice: (1) trial method, (2) multiplication by the coefficient of x^2, (3) division by the coefficient of x^2, and (4) decomposition. Of these methods probably the trial method is the one most commonly used. The easiest way of explaining these methods is by example [15]. We shall illustrate each of the four methods of factoring with the equation,

$$18x^2 + 37x - 20 = 0.$$

Under the trial method, we know that, in this example, we must find two numbers which have a product 18, and two other numbers which have a product -20; however, these four numbers must be chosen in such a way that their "cross product" is 37. Now 18 can be factored as 1×18, 2×9, or 3×6. In this method the possibility that 1×18 is the right combination is rather small, so we save it till last; 3×6 has the best chance so we will try it first. In a similar manner 20 can be factored into 1×20, 2×10, and 4×5. Here again 4×5 has the best chance, so we will try it first. Setting down some of the possibilities for the factors of $18x^2 + 37x - 20$, we have

$(3x-5)(6x+4)$, $(3x+4)(6x-5)$, $(3x+5)(6x-4)$, $(3x-4)(6x+5)$, $(3x+2)(6x-10)$, $(3x-2)(6x+10)$, $(3x-10)(6x+2)$, $(3x+10)(6x-2)$, $(2x+4)(9x-5)$, $(2x-4)(9x+5)$, etc.

We now test the 3×6 and the 4×5 in all possible combinations; however, we find that none of these gives the correct cross product 37 that is required by the equation. We proceed with the 3×6 and the 2×10 combinations which again lead us nowhere. Next we try 2×9 and 4×5 (note that we exhaust all possibilities first in which the numbers are closest together.) When we come to the combination $(2x+5)(9x-4)$ we find that this gives the required terms. It now is an easy matter to complete the solution; all we must do is to set each factor equal to zero and solve for x. Doing this we obtain $-5/2$ and $4/9$ as the roots of the equation. In most cases that arise it will be found that the factors are more easily obtained than they were in this example.

To factor the same example by the method of multiplication by the coefficient of x^2, we write

$$18x^2 + 37x - 20 = [18(18x^2) + 18(37x) - 20(18)]/18.$$

Now if we set $z = 18x$, the right member becomes

$$(z^2 + 37z - 360)/18 = (z + 45)(z - 8)/18,$$

or

$$\begin{aligned}(18x + 45)(18x - 8)/18 &= 9(2x + 5)\cdot2(9x - 4)/18\\ &= (2x + 5)(9x - 4).\end{aligned}$$

Much work could have been saved by observing the fact that $(18x)^2$ will always be produced by squaring $18x$ and we could just as well have written $(18x+\quad)(18x-\quad)$ and for the missing terms find two numbers whose sum is 37 and whose product is -360. It might also be noted that we must use the trial method for this process. However, since the coefficient of the second degree term is unity, much laborious work is saved.

In the method of division by the coefficient of x^2, we proceed in a manner very similar to that in the preceding method. Thus,

$$\begin{aligned}18x^2 + 37x - 20 &= 18(x^2 + 37x/18 - 20/18)\\ &= 18(x^2 + 37x/18 - 360/324)\\ &= 18(x + 45/18)(x - 8/18)\\ &= 18(x + 5/2)(x - 4/9)\\ &= 2(x + 5/2)\cdot9(x - 4/9)\\ &= (2x + 5)(9x - 4).\end{aligned}$$

In the method of decomposition we assume a solution of the form $(ax+b)(cx+d) = acx^2 + (bc+ad)x + bd$. If the constant term of the trinomial be multiplied by the coefficient of x^2, the product is $bcad$ which has two factors bc and ad. These are also components of the x term. Thus, the product of the first and third terms is $abcdx^2$ which is to be factored into bcx and adx. In our example, multiply $18x^2$ by -20 giving $-360x^2$. By trial, this may be factored into the two components of the middle term, namely, $45x$ and $-8x$. Thus,

$$\begin{aligned} 18x^2 + 37x - 20 &= 18x^2 + 45x - 8x - 20 \\ &= 9x(2x+5) - 4(2x+5) \\ &= (2x+5)(9x-4). \end{aligned}$$

The four methods of factoring described above assume that the roots of the equation are rational. The following solution by factoring is one which will apply in every case of the quadratic equation, regardless of the nature of the roots [12].

Any given quadratic equation can be changed to the form

$$x^2 - 2px + p^2 - q^2 = 0, \tag{1}$$

where $-2p$ is the coefficient of the first power of the unknown, and the constant term is $p^2 - q^2$. Then the left-hand member of (1) can be factored as the difference of two squares and the solution follows readily. Thus,

$$(x-p)^2 - q^2 = 0,$$
$$(x-p-q)(x-p+q) = 0,$$
$$x_1 = p+q,\ x_2 = p-q. \tag{2}$$

Since every equation of the second degree can be changed to the form (1), p can be found, then q can be determined from $p^2 - q^2$, and the roots are determined from (2). As an example, consider $x^2 - 4x - 12 = 0$. Here $p = 2$ and $p^2 - q^2 = -12$, so that $q = \pm 4$. Substituting in (2), $x_1 = 2 + 4 = 6$, and $x_2 = 2 - 4 = -2$.

If it so happens that the coefficient of the second power of the unknown is not unity, we divide the equation by this value and proceed in the manner outlined above.

5. Graphical Solutions for Real Roots

Another important method of solution, but which in general gives only approximate results, is that of graphing the quadratic equation. If care is taken, a fairly accurate result can be obtained. Let us consider the graphical solution of a specific equation, $x^2 - 3x - 4 = 0$. We begin by setting the left-hand member equal to y,

$$y = x^2 - 3x - 4,$$

and then compute the following table of values:

x	-2	-1	0	1	2	3	4	5
y	6	0	-4	-6	-6	-4	0	6

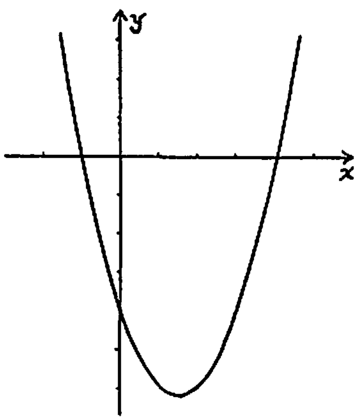

Fig. 6

Plotting these points, we obtain the graph shown in Fig. 6. At the points where the curve crosses the X-axis the value of y is zero. Hence, the values of x at these points are the roots of the given equation. We note in our example that the curve crosses the X-axis at $x = -1$ and $x = 4$, and these are therefore the roots of $x^2 - 3x - 4 = 0$.

The following graphical solution is more convenient than that just described. The quadratic equation $ax^2 + bx + c = 0$ is equivalent to the system of equations,

$$y = x^2, ay + bx + c = 0.$$

Now $ay + bx + c = 0$ represents a straight line and $y = x^2$ represents a parabola with the vertex at the origin. To illustrate the method, let us consider the equation $x^2 - x - 2 = 0$. We construct the following graphs: $y = x^2$, $y - x - 2 = 0$. We observe (Fig. 7) that the straight line cuts the parabola in the points $(-1,1)$ and $(2,4)$. Consequently, the two roots of the given equation are $x_1 = -1$ and $x_2 = 2$. The reader will observe that the graphs of all quadratic equations reduce to a straight line and a parabola, and in every case the parabola is represented by the equation $y = x^2$.

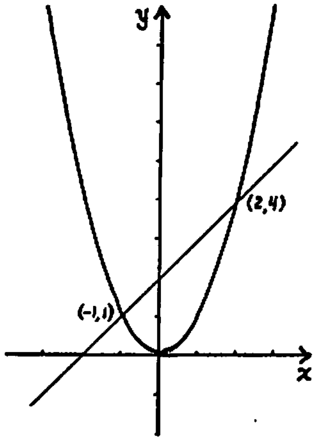

Fig. 7

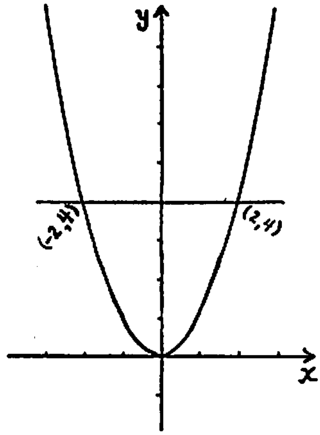

Fig. 8

If the quadratic equation has the incomplete form $ax^2 + c = 0$, we have the system, $y = x^2$, $ay + c = 0$. If, for example, the equation is $3x^2 - 12 = 0$, we construct the graphs of $y = x^2$ and $3y - 12 = 0$. Then from the graph (Fig. 8) it is clear that the roots are 2 and -2.

If the quadratic equation has equal roots, the graph takes on only a slightly different form. For the equation $x^2 - 4x + 4 = 0$, we graph $y = x^2$ and $y - 4x + 4 = 0$. From the graph (Fig. 9) we see that the line is tangent to the parabola at the point where $x = 2$; thus it is said that 2 is a double root of the equation $x^2 - 4x + 4 = 0$.

The case where the quadratic equation has imaginary roots can be detected very easily by the graphical method. Consider $x^2 - 2x + 5 = 0$. From the graph (Fig. 10) we note that the straight line and the parabola do not touch; thus, there is no solution in the real plane.

The parabola $y = x^2$ is not the only conic section that may be used with the straight line for a graphic solution of the quadratic equation. The hyperbola $y = 1/x$ gives a very

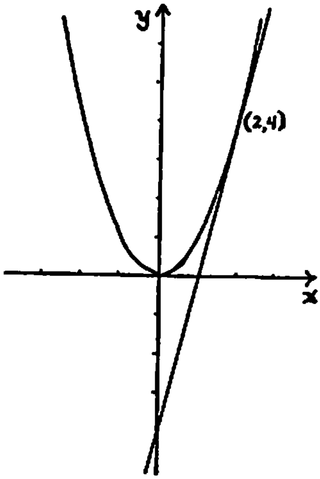

Fig. 9

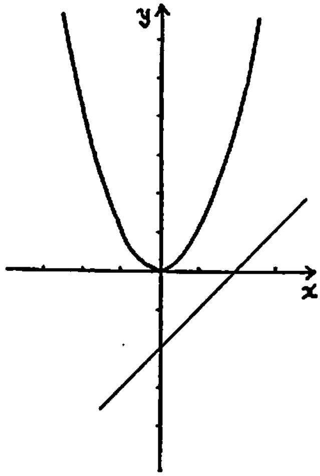

Fig. 10

convenient solution. The equation $ax^2 + bx + c = 0$ is equivalent to the system of equations,

$$y = 1/x,\ ax + b + cy = 0.$$

The intersection of the hyperbola and straight line will give the solution of the given quadratic equation. For example, given $x^2 + 2x - 8 = 0$, we graph $y = 1/x$ and $x + 2 - 8y = 0$. The two curves are shown in Fig. 11; the required x values are -4 and 2. It might be noted that here again if the line is tangent to the hyperbola the roots are equal, and if the two curves do not intersect the roots are imaginary.

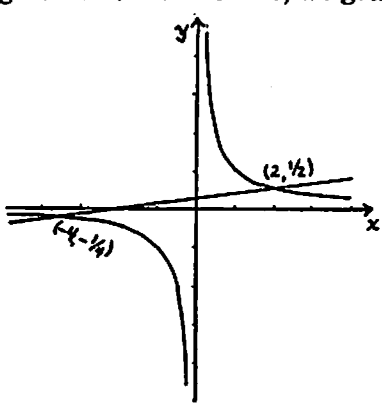

Fig. 11

There is yet another graphical method of solution that uses a straight line and a conic section. In this method we assume that the given quadratic equation can be reduced to the form $x^2 - px + q = 0$, where p and q are real numbers. In Figure 12 the coordinates of the points B and Q are $(0,1)$ and (p,q), respectively. Now draw the circle having BQ as a diameter. Its center is $[p/2, (q+1)/2]$. By the distance formula, the square of the length of BQ is $p^2 + (q-1)^2$. Thus the equation of the circle is

$$[x + p/2]^2 + [y - (q+1)/2]^2 = [p^2 + (q-1)^2]/4.$$

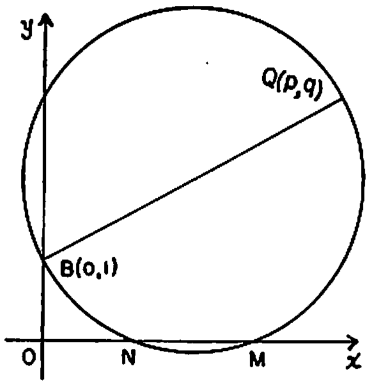

Fig. 12

When $y=0$ this reduces to $x^2-px+q=0$. Hence if the X-axis intersects the circle in two distinct points N and M, their abscissas ON and OM are the two distinct roots of the equation, $x^2 - px + q = 0$. If the circle is tangent to the X-axis the roots are equal, and if the circle does not touch the X-axis the roots are imaginary.

As a final graphical solution we will construct an alignment chart for $x^2 + ax + b = 0$ consisting of two straight lines and a curve between them. The chart will have the property that, if we locate specific values of a and b on the respective straight lines and draw a straight line between these two points, the point in which this line cuts the curve will be the required value for x. The theory of the construction of an alignment diagram is rather complicated and will not be given here; the interested reader is referred to books on nomography.

As specifications for our chart, we will let both a and b range from -10 to 10, and propose that the completed chart shall be 15 inches wide and 20 inches high. First we construct two parallel lines of length 20 inches and 15 inches apart. The same uniform scale is marked off from the

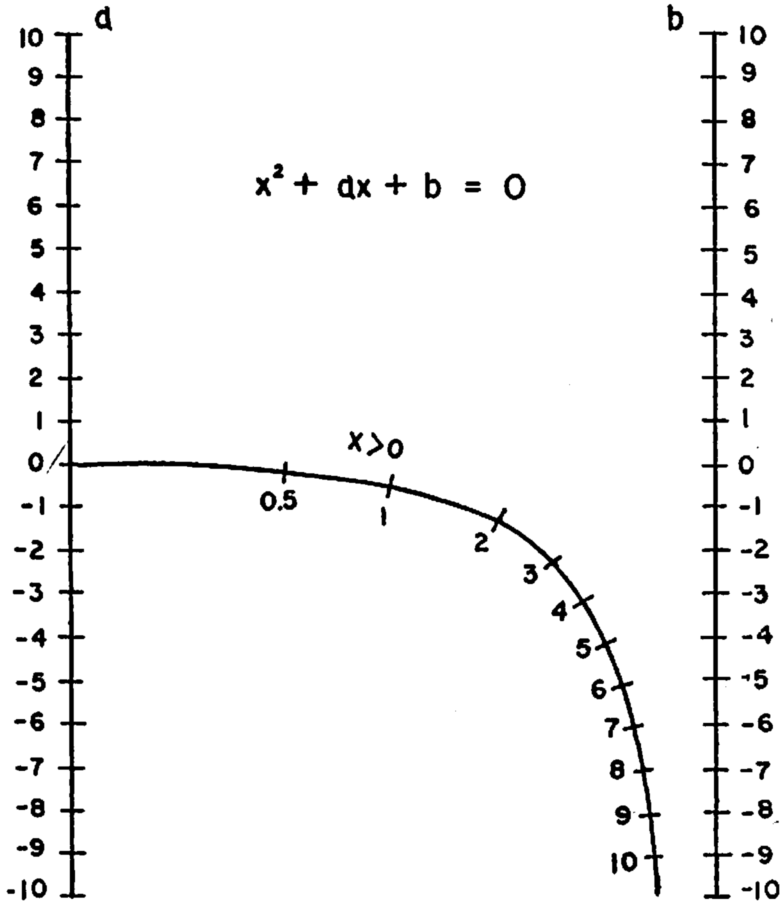

Fig. 13

middle of each of these lines (Fig. 13). The parametric equations of the curve between the two lines are

$$x_1 = 15x/(1+x), \; y_1 = -x^2/(1+x),$$

where the origin of the x_1, y_1 coordinate system is the zero point on the b scale [11, p. 192]. Only one root need be found because of the relation $x_1 + x_2 = -a$. If the equation has no negative roots the nomogram appears to be of no use; however, if we substitute $-x$ for x we will obtain an equation with roots of opposite sign than those of the original equation but of the same numerical value. If the line does not cut the curve the roots are imaginary.

6. Determination of Imaginary Roots

In the preceding discussion we have been concerned primarily with quadratic equations having roots that are real numbers, either rational or irrational. However, this is not the only type of solution that can be obtained; the solution may also take the form of complex numbers. What does one do when he wishes to find the solutions of equations having complex roots? Can the solution be obtained by a graphical method? These questions and many others may be raised about complex roots.

First let us consider a proof that the quadratic formula remains true for complex numbers [9]. We know that any complex number may be expressed in the equivalent forms [18, pp. 170, 183],

$$z = r \cdot e \exp i\theta = r(\cos\theta + i\sin\theta).$$

Also it can be shown that the sum of two complex numbers is graphically the diagonal drawn from the origin of the parallelogram whose sides are the given complex numbers; thus if the sum of three complex numbers is equal to zero, they must form the sides of a triangle. Let us substitute $r \cdot e \exp i\theta$ for z in the equation $az^2 + bz + c = 0$, whence

$$ar^2 \cdot e \exp 2i\theta + br \cdot e \exp i\theta + c = 0.$$

Since this states that the sum of three complex numbers is equal to zero, they will form a triangle. If c be taken horizontal then $br \cdot e \exp i\theta$ will make an angle of θ with c, and $ar^2 \cdot e \exp 2i\theta$ will make an angle 2θ with c or an angle θ with CB (Fig. 14). It follows that triangle OCB is isosceles, whence $ar^2 = c$ and $r = \sqrt{c/a}$. Now $\cos(180° - \theta) = rb/2c$, so that $\cos\theta = -(\sqrt{c/a})(b/2c) = -b/\sqrt{4ac}$. Also,

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1 - b^2/4ac}.$$

Substituting these in the equation $z = r(\cos\theta + i\sin\theta)$, there results

$$z = \sqrt{c/a}[-\sqrt{b^2/4ac} \pm i\sqrt{1 - b^2/4ac}],$$

or

$$z = (-b \pm \sqrt{b^2 - 4ac})/2a.$$

Thus we have proven that the quadratic formula holds for complex numbers.

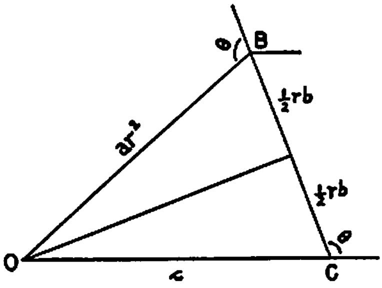

Fig. 14

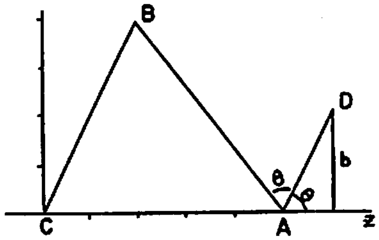

Fig. 15

The preceding proof with a few slight changes can be used to solve graphically for the complex roots of a quadratic equation with real coefficients. If the equation is of the form $az^2 + bz + c = 0$, we divide by a and obtain an equation of the form $z^2 - pz + q = 0$. In Figure 15, CZ is the axis of real numbers, $AB = z^2$, $BC = -pz$, $CA = q$, and $AD = z$; A lies on CZ because q is real. Now triangle ABC is isosceles by the same reasoning that was used in the preceding proof, and this makes possible the graphical determination of the roots z. Possibly an example would help clarify the method. Consider $z^2 - 2z + 5 = 0$. In this case $p = 2$ and $q = 5$. From what has been said, $AC = AB$ so that $q = |z^2|$, or $|z| = \sqrt{q}$. Now construct $AC = AB = 5$, $BC = |-pz| = 2\sqrt{5}$, and bisect angle BAZ (Fig. 15). Since $AB = |z^2| = 5$, construct $AD = |z| = \sqrt{5}$. But AD represents the vector $x = 1 + 2i$; hence the two roots are $z_1 = 1 + 2i$ and $z_2 = 1 - 2i$.

For a second graphical method of finding the complex roots of $az^2 + bz + c = 0$ consider the following [9]. Let

$$z_1 = x_1 + iy_1 = -b/2a + i\sqrt{4ac - b^2}/2a,$$
$$x_2 = x_2 + iy_2 = -b/2a - i\sqrt{4ac - b^2}/2a.$$

The moduli of z_1 and z_2 are each equal to $\sqrt{c/a}$. Hence z_1

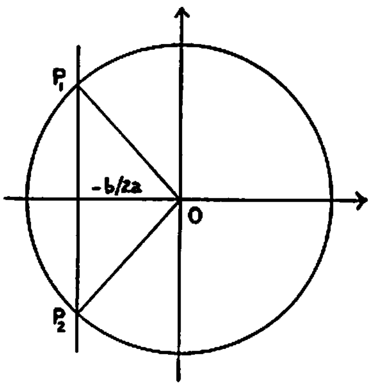

Fig. 16

and z_2 lie on a circle with its center at the origin and radius equal to $\sqrt{c/a}$. Furthermore, the real part of both z_1 and z_2 is $x = -b/2a$. Hence the two complex roots of the equation $az^2 + bz + c = 0$ are represented in both magnitude and direction by the line segments OP_1 and OP_2 (Fig. 16).

Still a third graphical method has been devised [33]. If we consider the quadratic equation to have the form $z^2 + pz + q = 0$ where $p^2 - 4q < 0$, the two roots are conjugate complex numbers of the form

$$z_1 = u + iv = r \cdot e \exp i\theta,\ z_2 = u - iv = r \cdot e \exp(-i\theta),$$

where

$$r = \sqrt{u^2 + v^2},\ \theta = \arctan(u/v).$$

It is known that $z_1 + z_2 = -p$ and $z_1 z_2 = q$; also $e \exp i\theta = \cos\theta + i \sin\theta$. Therefore,

$$p = -(z_1 + z_2) = -r[e \exp i\theta + e \exp(-i\theta)] = -2r\cos\theta,$$

$$q = z_1 z_2 = (r \cdot e \exp i\theta)\ [r \cdot e \exp(-i\theta)] = r^2.$$

Now substituting these values for p and q in the original equation, we obtain

$$z^2 - 2rz\cos\theta + r^2 = 0.$$

If we wish to determine the two complex roots of this equation by the help of a geometric method, we need only construct the radius vector r and the amplitude θ. After we have done this, the imaginary parts of $u + iv$ and $u - iv$ are obtained easily from the graph.

To construct Figure 17 we proceed as follows. With a rectangular coordinate system construct a circle with a diameter $OP = q$ on the X-asis. At the point $E(1,0)$ erect a perpendicular which intersects the circle in points S. With the radius OS construct a circle with its center at O. Through the point $W(-p/2, 0)$ draw a perpendicular which will intersect the latter circle in two distinct points, A and B. Then $z_1 = OA$ and $z_2 = OB$.

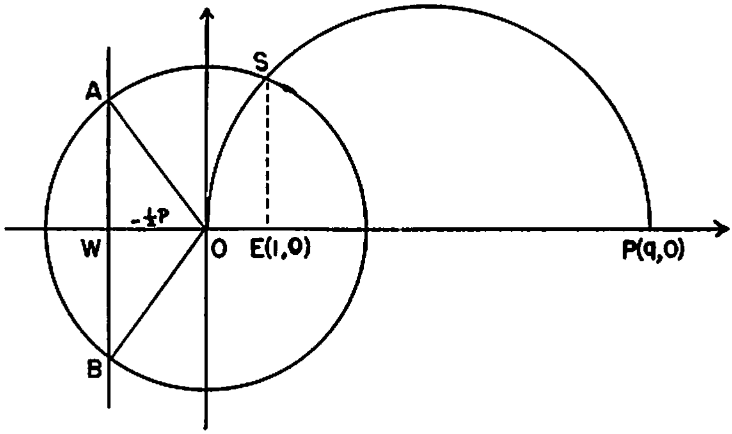

Fig. 17

7. Solutions by Trigonometric Methods

The approximate solutions of all quadratic equations can be obtained by the use of trigonometry [1]. The roots of $ax^2+bx+c=0$ may be written, if $c<0$, as

$$x = \sqrt{-c/a}(-b/2\sqrt{-ac}\pm\sqrt{b^2-4ac}/2\sqrt{-ac}).$$

Choose an angle θ between 0° and 180° such that $\cot\theta = -b/2\sqrt{-ac}$. Making this substitution, we obtain

$$x = \sqrt{-c/a}(\cot\theta \pm \csc\theta),$$

whence

$$x_1 = \sqrt{-c/a}\cot(\tfrac{1}{2}\theta),\ x_2 = -\sqrt{-c/a}\tan(\tfrac{1}{2}\theta).$$

In case $c>0$, we write

$$x = \sqrt{c/a}(-b/2\sqrt{ac} \pm \sqrt{b^2-4ac}/2\sqrt{ac}).$$

Now $-b/2\sqrt{ac}$ may be numerically greater than, equal to, or less than unity. If it is numerically equal to or greater than unity, we let $\csc\theta = -b/2\sqrt{ac}$, where $-90° \leqq \theta \leqq 90°$. The solution then becomes

$$x = \sqrt{c/a}(\csc\theta \pm \cot\theta),$$

whence

$$x_1 = \sqrt{c/a}\cot(\tfrac{1}{2}\theta),\ x_2 = \sqrt{c/a}\tan(\tfrac{1}{2}\theta).$$

If b is numerically less than $2\sqrt{ac}$, b^2-4ac is negative and the values of x are imaginary. To provide for this we rewrite x in the form

$$x = \sqrt{c/a}(-b/2\sqrt{ac} \pm i\sqrt{4ac - b^2}/2\sqrt{ac}).$$

Here we let $\cos\theta = -b/2\sqrt{ac}$ where $0° < \theta < 180°$. Then x has the values

$$x_1 = \sqrt{c/a}(\cos\theta + i\sin\theta),\ x_2 = \sqrt{c/a}(\cos\theta - i\sin\theta).$$

Example 1. $0.184x^2 + 0.0358x - 1.018 = 0$.

Here $c<0$ and $\cot\theta$ is negative.

$\text{Log}\,|\cot\theta| = \text{Log}\,|-b/2\sqrt{-ac}| = 8.6166 - 10$, $\theta = 92°22'$.

$\text{Log}\,x_1 = \text{Log}\,|\sqrt{-c/a}\cot(\tfrac{1}{2}\theta)| = 0.3533$, $x_1 = 2.356$.

$\text{Log}\,(-x_2) = \text{Log}\,|\sqrt{-c/a}\tan(\tfrac{1}{2}\theta)| = 0.3897$, $x_2 = -2.452$.

Example 2. $x^2 - 5x + 6 = 0$.

Here $c>0$,

$\text{Log}\,|\csc\theta| = \text{Log}\,|-b/2\sqrt{ac}| = 0.0089$, $\theta = 78°28'$.

$\text{Log}\,x_1 = \text{Log}\,|\sqrt{c/a}\cot(\tfrac{1}{2}\theta)| = 0.4771$, $x_1 = 3$.

$\text{Log}\,x_2 = \text{Log}\,|\sqrt{c/a}\tan(\tfrac{1}{2}\theta)| = 0.3010$, $x_2 = 2$.

The above are not the only substitutions that can be used to determine a trigonometric solution. The following substitutions will work just as well. We write the roots of $ax^2+bx+c=0$ in the form,

$$x = (-b/2a)(1 \pm \sqrt{1 - 4ac/b^2}),$$

and shall restrict our attention to the case $b^2-4ac>0$. If $c>0$, let $\sin^2\theta = 4ac/b^2$. Then $x = (-b/2a)(1 \pm \cos\theta)$, or

$$x_1 = (-b/a)\sin^2(\tfrac{1}{2}\theta),\ x_2 = (-b/a)\cos^2(\tfrac{1}{2}\theta).$$

If $c<0$, let $\tan^2\theta = -4ac/b^2$. Then $x = (-b/2a)(1 \pm \sec\theta)$ and

$$x_1 = (b/a)\sec\theta\sin^2(\tfrac{1}{2}\theta),\ x_2 = (-b/a)\sec\theta\cos^2(\tfrac{1}{2}\theta).$$

As a final trigonometric method of solution consider $x^2-px+q=0$, where again we restrict our attention to the case $p^2-4q>0$. Let $x_1 = p\cos^2\theta$, $x_2 = p\sin^2\theta$. Then

$$x_1 + x_2 = p(\sin^2\theta + \cos^2\theta) = p,$$

$$x_1x_2 = p^2\sin^2\theta\cos^2\theta = (p^2\sin^2 2\theta)/4.$$

But $x_1x_2 = q$. Therefore,

$$\sin 2\theta = 2\sqrt{q/p}.$$

Example. $x^2 - 93.7062x + 1984.74 = 0.$

Here $\sin 2\theta = 2\sqrt{1984{\cdot}74}/93.7062$, $\theta = 35°58.9'$.

Thus, $x_1 = p \cos^2\theta = 61.361$, $x_2 = p \sin^2\theta = 32.345$.

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THE PYTHAGOREAN DOCTRINE

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The article on "Pythagoras: Mathematician and Philosopher" in the Spring issue of this journal[1] discussed philosophy and Pythagoras as a philosopher, made frequent references to the "Pythagorean Number Philosophy," mentioned philosophy of mathematics, and seemed to regret the fact that little has been written on the influence of mathematics on philosophy.

The present article is concerned, principally, with revealing the mathematical nature of Pythagorean philosophy. It is also concerned with doing justice to Pythagoras. The thesis to be advanced here is that the Pythagorean Doctrine, in its arithmo-geometric aspects, was an attempt, perhaps the first, to bring mathematics to the help of philosophy in the study of nature. It is pertinent to insert here that the word mathematics was first used by Pythagoras. It is derived from *mathema,* that which is learnt, a lesson.

Consider then, briefly, the views of the cosmogonists preceding Pythagoras. Anaximander of Miletus (611-547 B.C·) held[2] that from an endless, boundless, and shapeless mass, the *apeiron,* which is subject to a circular movement, a flat disc, the earth, is formed at the center, then rings of water, air, and fire are thrown out, spreading away from the center in ever thinning layers, but not without limit for an infinite mass cannot rotate. The universe thus formed, however, is not stable; the celestial fire devours and dissipates the center and the outflung layers and thus, in the course of time, everything returns to the original state. But there is an end to this period of dissipation and the same causes that once formed the universe reform it. There is thus an endless succession of worlds and the only

[1]R. H. Moorman, "Pythagoras: Mathematician and Philosopher," *The Pentagon,* Vol. 8, pp. 79-84 (Spring, 1949).
[2]Paul Tannery, *Pour l'Histoire de la science hellene,* 2nd ed. Paris, 1930.

thing that remains immortal and imperishable is the circular movement.

Anaximenes of Miletus, a younger contemporary of Anaximander and, by report, a pupil and friend of his, added the elaboration that the boundless air, subject to an eternal movement, is the source of everything. Expanding under the influence of heat, or contracting under that of cold, it has formed all the phases of existence.

In contrast with this we find the Pythagorean Doctrine proceeding more cautiously, from within the cosmos, and trying to build up a theory by abstraction, logical construction, and generalization. The geometric point is defined as "unity in position" and, conversely, the unit of number as "a point without position."[3] This identification of the point and the unit of number led immediately to the well-known practice of representing numbers by figures (*schematographein*). For if a number is a plurality of units, and a unit is a point, than a number may be represented by an aggregate of points, arranged into such a figure as the nature of the number might suggest. In this practice a method was developed for the successive generation of numbers of a given type; this was the use of gnomons.[4] Thus Proclus, Diogenes Laertius,[5] and Plutarch[6] attribute to Pythagoras the method of forming successive square numbers by the addition of equilateral gnomons to unity (Fig. 1), while Lucian and Aristotle mention the formation, by the Pythagoreans, of the triangular and the oblong numbers by the addition of the corresponding gnomons (Fig. 2).

That this practice was not a fruitless pastime is shown by the many arithmetic discoveries they made by its use. Thus, from the fact that the gnomons of the squares are the odd numbers after unity, in succession, and the resulting figures are the next higher squares, they deduced, by successive additions,[7]

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[3]Proclus, *Commentary on Euclid's Elements I*, ed. Friedlein, 1873, p. 95.

[4]In geometry, a gnomon is the figure which, added to any figure, preserves the original shape.

[5]*Diogenes Laertius*, *ed.* Hubner, Leipsic, 1831.

[6]Plutarch, *Opera Moralia*, Symposium 6, ed. D. Wyttenbach. Oxford, Claredon Press, 1795-1821.

[7]Nicomachus of Gerasa, *Introduction to Arithmetic*, tr. by M. L. D'Ooge, with studies in Greek arthmetic by F. E. Robbins and L. C. Karpinski. New York, Macmillan, 1926.

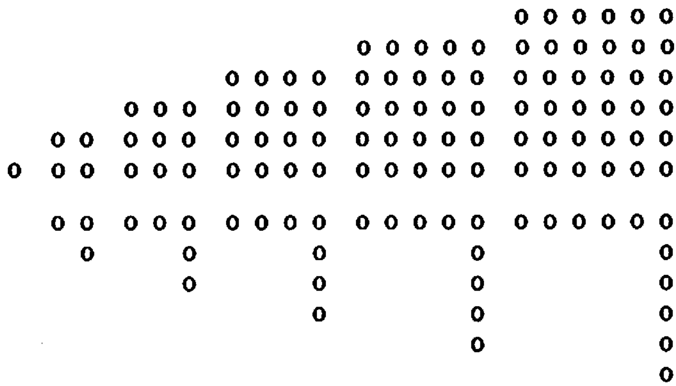

Fig. 1. Square numbers and their gnomons.

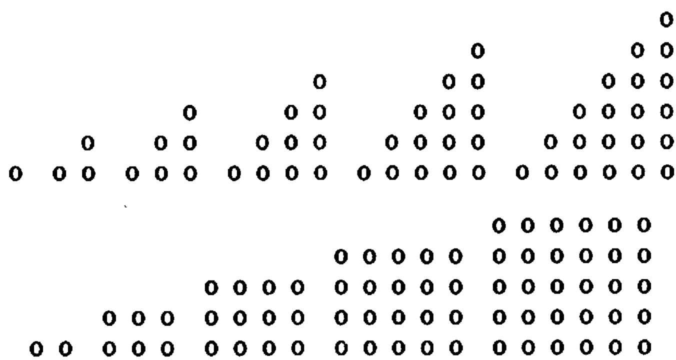

Fig. 2. Triangular and oblong numbers.

$$1 + 3 + 5 + \ldots + (2n-1) = n^2,$$

and from the fact that the gnomon $(2n+1)$ added to the square n^2 produces the next higher square, $n^2 + (2n+1) = (n+1)^2$ they found[8] that if $(2n+1) = k^2$, where k is any odd number, then the numbers k, $\frac{1}{2}(k^2-1)$, $\frac{1}{2}(k^2+1)$ are the sides and hypotenuse of a right triangle or, in present-day terms, they are Pythagorean numbers. Proclus[9] says:

> But there are delivered certain methods of finding triangles of this kind (sc. right-angled triangles

[8]G. J. Allman, *Encyclopaedia Britannica*, Werner Edition, 1900, Vol. XX, p. 141.
[9]Proclus, *op. cit.*, p. 428

> whose sides can be expressed by whole numbers) one of which they refer to Plato, but the other to Pythagoras, as originating from odd numbers. For Pythagoras places a given odd number as the lesser of the sides about the right angle, and when he has taken the square erected upon it, and diminished it by unity, he places half the remainder as the greater of the sides about the right angle; and when he has added unity to this he gets the hypotenuse . . . . But the Platonic method originates from even numbers. For when he has taken a given even number he places it as one of the sides about the right angle, and when he has divided this into half, and squared the half, by adding unity to this square he gets the hypotenuse, but by subtracting unity from the square he forms the remaining side about the right angle.

The reader will find that if two successive gnomons have a sum which is an even square, that is, if $(2n-1) + (2n+1) = m^2$, m even, then, since $(n-1)^2 + (2n-1) + (2n+1) = (n+1)^2$, it follows that the numbers $(n-1)$, m, and $(n+1)$ provide Plato's solution.

These are only a few of the mathematical discoveries of Pythagoras and his school. A complete list cannot be given here; it is enough to say that the first two books of Euclid's *Elements* (on triangles, rectangles, and areas), part of the third (on circles), part of the fourth (constructions of polygons), the seventh (theory of proportions), part of the sixth (applications of proportions), the bulk of the thirteenth (on the regular solids), and the discovery of the irrationals are now known to be due to Pythagoras and his school.[10] These form the bulk of their contribution to the subject matter of mathematics.

But of at least equal importance to these must be counted certain methodological contributions they made to mathematics. One of these is the method of definition and logical proof essentially as we use it today. This finds its greatest expression in the many and varied uses made of

[10] G. J. Allman, *Greek Geometry from Thales to Euclid*. Dublin, Longmans, 1889.

the theory of proportions as a mathematical tool. This theory, begun by Pythagoras and brought to its highest development by his successor, Archytas of Tarentum,[11] became the chief tool and most striking characteristic of all later Greek mathematics. It was skillfully used by Euclid, Aristarchus, Archimedes, Apollonius, Ptolemy, and many others; it survived through all later periods and reappeared as late as the days of Galileo (*Two New Sciences*, 1638) and Newton (*Principia*, 1687).

Latent in this theory lies Pythagoras' deep insight of the oneness of magnitude. Lengths, numbers, all magnitudes in fact can, as magnitudes, be represented by the same symbols. This was a profound abstraction and a broad generalization. In the theory of proportions lines represent numbers, and numbers lines. It is then a theory applicable both to geometry and arithmetic; it effects a fusion of the two. "In this respect," says Allman, "Pythagoras is comparable to Descartes to whom is due the combination of Algebra and Geometry."

We have seen, in the practice of *schematographein,* the early attempts to represent numbers by points. In the theory of proportions this develops into the conception of the line as a series of juxtaposed points. It was easy to pass then to the conception of the plane as a series of juxtaposed lines, and the solid as a series of juxtaposed planes. We are told by Diogenes Laertius[12] that

> Pythagoras taught that the principle of all things is the monad, or unit; arising from this monad is the infinite dyad . . . from the monad and the infinite dyad arise numbers; from the numbers points; from points lines, from lines planes, from these solid figures and from these sensible bodies . . .

One may see in this the attempt to supply to the study of geometry, and physical bodies, the necessary abstract background for their study by means of numbers. The Cantor-Dedekind Axiom, which postulates a one-to-one cor-

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[11] F. Cajori. *A History of Mathematics.* New York, The Macmillan Company, 1938, p. 20.
[12] Diogenes Laertius, *op. cit.*: *De Vit. Pyth.*

respondence between the points of a line and the real numbers, is indispensable in present-day geometry. The corresponding Pythagorean Axiom seems to have assumed (not explicitly of course) a one-to-one correspondence between the points of a line and the rational numbers. And they seem to have looked upon the physical particle as analogous to the geometric point. This would make the last sentence in the quotation from Diogenes Laertius (above) comprehensible. And since, as we have seen, points and numbers were identified, it would also lend meaning to Aristotle's statement[13] that

> The Pythagoreans seem to have looked upon number as the principle and, so to speak, the matter of which beings consist,

and to Philolaus' assertion[14] that

> Number is perfect and omnipotent, and the principle and guide of divine and human life.

For, these utterances of later followers and commentators express but crudely and sensuously the tenets of the school and must therefore be taken with a certain degree of freedom of interpretation. The qualifying phrase, "so to speak," in the quotation from Aristotle should be noticed.

It is instructive now to survey briefly the theories of the cosmogonists after Pythagoras; they are the group of philosophers now known as "The Atomists."

> Anaxagoras of Clazomenae (500-428) held that a chaotic mass existed from the beginning and contained within it, in infinitesimally small fragments, the seeds of things.
>
> Leucippus (circa 480 B.C.), pupil and friend of Zeno of Elea believed in an infinity of atoms (*atoma* = indivisibles) as the ultimate constituents of things.
>
> Democritus (circa 460 B.C.), pupil and friend of Philolaus and Leucippus, elaborated the theory of the latter and applied it to geometry.

[13] A. Seth, *Encyclopaedia Britannica*, Werner Edition, 1900, Vol. XX, p. 138.

[14] Philolaus of Thebes (496-396), the Pythagorean who gave the first written exposition of the Pythagorean doctrine.

> Epicurus (341-270) adopted the views of Democritus and systematized them into a broader philosophy.
>
> Lucretius (94-55) gave the best written exposition of the Epicurean philosophy.

The difference between these views and the earlier cosmogonies is noticeable. From chaotic, boundless masses of amorphous substance the emphasis passes to considerations of the structure of matter from an infinity of indivisible particles, arranged according to law, and with due regard to form, size, internal properties, and numerical relations. The atomic theory of the *De Rerum Natura* of Lucretius[15] is a far cry indeed from the vague conceptions of the older cosmogonists. But, whence came it?

We have seen that the Pythagoreans had conceived of sensible bodies as consisting of particles that corresponded in a one-to-one way with the geometric points. From this it would follow that, since points are the indivisibles of space in the Pythagorean Doctrine, particles would be the indivisibles of matter. Atomism seems to have been forecast by Pythagoreanism. Further, it has been pointed out that Leucippus and Democritus, the founders of the Atomism, had had contacts with and received instruction from men acquainted with the Pythagorean Doctrine. For Zeno is well-known as the propounder of the famous paradoxes against the Pythagorean Doctrine, and Philolaus was, after Archytas, the chief exponent of Pythagoreanism. It displays, of course, the fundamental error involved in the Pythagorean Axiom as to the correspondence of material points, geometric points, and the rational numbers. Consider, for example, the following dilemma to which Democritus was led by his adherence to the indivisibles. In the letter to Eratosthenes prefixed to *The Method of Archimedes*[16] we find the following statement:

> This is a reason why, in the case of the theorem the proof of which Eudoxus was the first to discover, namely, that the cone is a third part of the cylinder,

[15] Lucretius, *On the Nature of Things*, tr. by H. A. J. Munro. Lonon, 1932.

[16] T. L. Heath, *The Method of Archimedes*. Cambridge, University Press, 1912.

> and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it.

And Plutarch[17] presents Democritus as saying:

> If a cone were cut by planes parallel to its base, what must we think of the surfaces of the sections, that they are equal or unequal? For, if they are unequal, they will show the cone to be irregular, as having many indentations like steps, and unevennesses; and if they are equal, the sections will be equal and the cone will appear to have the property of a cylinder, namely, to be composed of equal and not unequal circles, which is very absurd.

It appears then that Democritus was led to assert that the volume of a cone is one third that of a cylinder having the same base and equal height, but was puzzled by the dilemma he mentions and could not prove the theorem.

The root of the difficulty lay, of course, in the conception of matter as composed of indivisible and juxtaposed particles. This difficulty does not arise today (the irregularities and unevennesses are smoothed out) because the Cantor-Dedekind Axiom has replaced the Pythagorean Axiom. Eudoxus proved the theorem, and others similar to it, by ignoring the question of the structure of matter, that is, whether it is discrete or continuous, and using instead the well-known Eudoxian Axiom (erroneously attributed by some to Archimedes) which asserts that:[18]

> Of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with it and with one another.

This lemma marked the beginning of a new trend in Greek mathematics. With its help Eudoxus created a new

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[17]Plutarch, *De Communibus Notitiis*, Vol. IV, ed. by Didot, Paris, p. 1321.

[18]T. L. Heath, *The Works of Archimedes: On the Sphere and Cylinder*. Cambridge, University Press, 1897.

number system (essentially as given in the fifth book of Euclid) and a new method (the method of exhaustions) for handling problems of the types that balked Democritus. The "indivisibles" were banished from mathematics.

"Jacobi states that at various times he had tried to persuade a young man to begin research in mathematics, but this young man always excused himself on the ground that he did not yet know enough. In answer to this statement Jacobi asked this man the following question: Suppose your family would wish you to marry, would you then also reply that you did not see how you could marry now, as you had not yet become acquainted with *all* the young ladies?"

—G. A. MILLER.

CURIOSUM

$$(A+\sqrt{B})+(A-\sqrt{B})+(A+i\sqrt{B})+(A-i\sqrt{B})=4A$$
$$(A+\sqrt{B})^2+(A-\sqrt{B})^2+(A+i\sqrt{B})^2+(A-i\sqrt{B})^2=4A^2$$
$$(A+\sqrt{B})^3+(A-\sqrt{B})^3+(A+i\sqrt{B})^3+(A-i\sqrt{B})^3=4A^3$$

—DR. ALFRED MOESSNER.[1]

[1]Dr. Moessner requests that persons interested in diophantics and problems of theoretical numbers correspond with him at the following address: in (13a) Gunzenhausen, Altes Schulhaus, Amerikanische Zone, Germany-Bayern.

AN INSOLUBLE EXPONENTIAL CODE

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Virtually since the beginning of man's existence on this old sphere of ours, he has found it necessary to communicate in some manner with his neighbor. Often in fields of commerce, diplomacy, or war, this communication requires a certain degree of secrecy to prevent its falling into the hands of unwanted would-be sharers of the information. The universal answer to this problem has been found to lie in the use of some type of cipher or code. Thus, a coded message which may fall into the enemy's hands is useless unless it can be converted to its original form. The development of "unbreakable codes" in time of war has become almost as important as life itself, while at the same time an equally important possession is a means of cracking any or all enemy ciphers that are intercepted.

Early efforts at cryptography were led by the ancient Greeks who frequently employed simple substitution of numbers for letters. A representative example of this type of substitution is based on the following square array.

	1	2	3	4	5
1	*A*	*F*	*L*	*Q*	*V*
2	*B*	*G*	*M*	*R*	*W*
3	*C*	*H*	*N*	*S*	*X*
4	*D*	*I*	*O*	*T*	*Y*
5	*E*	*K*	*P*	*U*	*Z*

The use of this diagram involves reading first vertically, then horizontally. For example, *R* becomes the intersection of horizontal row 2 and vertical row 4, or 24 in coded form. It will be seen that only 25 letters are permitted in such an arrangement as above. This is easily adjusted to the conventional alphabet by arbitrarily omitting a letter such as *J*. Many different codes result from a permutation of the letters.

An analysis of the above typical code reveals several factors upon which might be based an efficient method for cracking it. First and foremost is the quite obvious fact that, regardless of the transposed arrangement of corresponding numbers or letters in the particular diagram, the same number always represents one, and only one, letter. Conversely, the same letter is always represented by one, and only one, number. Again, in cracking any coded message of considerable length, the odds are high in favor of the message having a definite related percentage of letter content. It has been found from extensive study of characteristic languages that each has a pronounced letter frequency into which messages chosen at random from that language will fall. For example, in English the single letter frequency is in the order, *ETOANIRSHDLCWUMFYGPB-VKXQJZ*. Similar orders of frequency are available for two-, three-, and four-letter combinations, double letters, initial and final letters, one-, two-, three-, and four-letter words, and numerous other combinations. We see, then, that this universally accepted practice of "one number for one letter" code selection is subject to easy solution by the experienced, well-equipped cryptanalyst.

It is possible to use certain mathematical processes to create a code not having the defects common to conventional substitution codes. Thus, a chosen exponent n may be applied to numbers x to produce, correct to two significant figures, other numbers x^n the sum of whose two significant digits is some constant k. For example, if $n = 0.85$ and $k = 5$, we derive the following table:

x	1.49	2.66	3.93	5.26	6.64
x^n	1.4	2.3	3.2	4.1	5.0

Now k, the sum of the digits of x^n, may be used as one of the digits in the number which indicates a particular letter in the kth row or kth column of our diagram.

Although the process of raising each number to a power appears to be somewhat cumbersome, the wide range of accuracy allowed permits the use of the ten-inch log log slide rule so familiar to engineering students. The selection of a value for the exponent n may be based on any convenient scheme. However, if the exponent is allowed to vary periodically, a still greater array of different numbers is made possible; in fact, the same number over two periods can yield two different letters. It is suggested that n might be based on the time of transmission of the message. The first digit in n might represent the day of the week and the second digit the forenoon or afternoon of the particular day. For example, an n of 0.51 would be used on Thursday morning, 0.52 on Thursday afternoon, 0.61 on Friday morning, etc.

Now, summing up our proposed scheme, let us receive a sample message and carry it through the different processes to completion. The message, sent on Sunday afternoon, reads as follows:

(16.4-2.315)-(210.0-0.0252)-(1020-4.56)-
(16,100-488.0)-(0.073-5400)-(0.0018-488.0)-
(0.04-0.0026)-(1020-0.00025)-(0.0006-0.0093)-
(0.00039-0.0026)-(710-2.315)-(5400-0.00039)-
(0.036-0.0093)-(0.091-0.234).

On Sunday afternoon, $n = 0.12$, which yields the following table:

x	x^n	k		
16.4	1.4	5		
2.315	1.1	2	52	*K*
210.0	1.9	1		
.0252	.64	1	11	*A*
1020	2.3	5		
4.56	1.2	3	53	*P*
16,100	3.2	5		
488.0	2.1	3	53	*P*
.073	.73	1		
5400	2.8	1	11	*A*
.0018	.47	2		
488.0	2.1	3	23	*M*
.04	.68	5		
.0026	.49	4	54	*U*

1020	2.3	5		
.00025	.37	1	51	*E*
.0006	.41	5		
.0093	.57	3	53	*P*
.00039	.39	3		
.0026	.49	4	34	*S*
710.0	2.2	4		
2.315	1.1	2	42	*I*
5400	2.8	1		
.00039	.39	3	13	*L*
.036	.67	4		
.0093	.57	3	43	*O*
.091	.75	3		
.234	.84	3	33	*N*

Our message has a very good chance of remaining unsolved by any unsuspecting cryptographer because we have so jumbled the identical numbers used that they have no bearing upon one another. In fact, although several combinations (such as 2.315 in both *K* and *I*) are repeated, they are unrelated insofar as a final decoded solution is concerned. Again, two-number groups representing the same letter have no similarity; for example,

P = (1020-4.56) = (16;100-488.0) = (0.0006-0.0093).

So, we see, the exponential code has at least two advantages over the ordinary substitution code: (1) The same letter can be represented by several different sets of numbers. (2) The same number can contribute to several unrelated letters.

It is to be conceded that the enemy cryptographer would eventually discover certain repetitions of number combinations if we were forced to send a long message. By this time, however, the clock would have brought forth a new half day, and with that new period of time a new exponent, and, once again, a just-as-bewildered "expert."

MYSTICAL SIGNIFICANCE OF NUMBERS

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Certain numbers were employed over and over again by ancient peoples in mystic and symbolic ways in their religious ceremonies. Many divergent and geographically separate races had similar uses for numbers. They all applied abstract numbers to concrete objects and phenomena in an effort to explain happenings that mystified them. This universal use of symbolistic numbers developed in three main ways.

The basic elements in number mysticism came from nature and happenings in the physical world. In the *Rig-Veda* of India the gods were grouped in *three* classes, the gods of heaven, air, and earth; the Egyptian god was of three personalities, morning, noonday, and setting sun; many races used three with the cycle of man's life—birth, life, and death. *Four* had special significance from nature's four winds and four corners of the world, thus yielding a god who saw four ways.

Number mysticism also developed from the Babylonian culture which influenced the culture and religion in the pre-Christian world, Old Testament, New Testament, and thus Christian peoples. It was through Babylonian astronomy and astrology that the number *seven* was developed into the most used of all the mystical numbers. The Babylonians based their astrology on seven moving celestial bodies (the sun, moon, and five planets visible to the naked eye), and their navigation on the seven stars of the Pleiades and other constellations. The four phases of the moon take 29½ days for completion and seven is the integer closest to the length of one phase. There are, therefore, four weeks of seven days in a month.

The third area for the development of mystic numbers was the Greek philosophy that everything in the universe—beauty, order, music, art—have their origins in numbers. Pythagoras delved into the make-up of the universe, and by experiments in the structure of things observed relationships between numbers and happenings in the universe. In his search for the true philosophy, Pythagoras traced the origin of all things to numbers. For example, intervals of an octave, a fifth, and a fourth could be produced by strings of equal length stretched in a 1/2, 2/3, and 3/4 proportion by different weights. His conclusion: Harmony depends on musical proportion; it is nothing but a mysterious numerical relation. Where there is number there is also harmony.

Other Greek philosophers gave various numbers certain characteristics. *One* was considered the essence of things; it is an absolute number, hence the origin of all numbers and so of all things. *Four* was the most perfect number; in some mystic way it was conceived to correspond to the human soul. This connotation for four could have come from the idea of the composition in man of the four basic elements (fire, water, air, and earth), and the idea that four is represented geometrically by a solid. (In the Pythagorean philosophy, one represented a point, two represented a line, three represented a triangle, and four, placed above the triangle in space, represented a tetrahedron, the first of the regular solids.)

Hebrew use of mystic numbers in the Old and New Testaments, and, consequently, in the Christian world up to today evolved from these three main sources. It cannot be said that all numbers used in the Bible have hidden meanings, but some are definitely mystic and symbolistic. These are *seven, twelve, ten,* and *three.* There is an arithmetical as well as historical basis for the constant use of these particular numbers. *Three* is a complete number consisting of a beginning, middle, and end; it is the simplest group of units. *Seven* is a double group with a central point. *Twelve* is four times the group of three; it is also the first number divisible by four numbers. *Ten* is the basis of decimals and comes from the ten fingers of man with five denoting half the complete ten, and nine falling just short of completeness.

Seven, the most widely used number, appears in many of the ancient religions as well as the Bible. It is a symmetric unit composed of not only a central point, but also of two balanced groups of threes which add greatly to its harmonic symbolism of completeness. The Chinese emperor ruled seven provinces, he prayed to seven chief spirits, seven days after his death he was placed in his coffin, and he was buried in the seventh month. In the Old Testament the number seven occurs frequently in connection with religious ceremonies and signs. ". . . A candlestick all of gold, with a bowl upon the top of it and his seven lamps thereon and seven pipes to the seven lamps . . ." (Zechariah 4:2). In this verse seven seems to be used to express the completeness and divineness of light, and possibly knowledge. Seven was transferred by the people from their religion to daily life. Job had many blessings with seven sons and three daughters. In *Proverbs* 6:16-35, there are listed seven traits of man that "the Lord hates and these are an abomination unto Him." The New Testament mentions seven many times in such groups as seven days, seven parables, and seven petitions of the Lord's Prayer. Besides signifying light, omniscience, and forethought, in some uses seven expressed completeness in denoting evil. *Luke* 8:2 relates how seven devils, signifying total sin and complete wickedness, were cast out of Mary Magdalene. Today, seven is used in Christian theology and liturgies as the symbol of perfection and completion in such cases as seven Sundays in Lent, seven sacraments, seven words of the cross, etc.

Three occurs next to seven in frequency, its arithmetic absoluteness lending to its popularity. The Christian doctrine of Trinity follows from three being absolute and the universe being divided into three parts. With three persons in the Godhead, three becomes an especially symbolistic and mystic number, reaching its peak in the resurrection of Jesus in three days.

The definite number *ten* developed from man's fingers. It represents oneness and its use in the Ten Commandments as well as the Tenth in the Law of the Tithe is more significant than mystic. Man has ten fingers which are constantly with him—good reminders of these rulings which he is to follow.

Twelve is the sum of five (half of ten) and seven (the complete number) which adds to its Biblical popularity. Jesus had twelve apostles and there were twelve tribes of Israel. In *Revelation,* the heavenly city has twelve gates which gives each of the twelve tribes a part in heaven.

The mysticism of numbers still fascinates man, and the best challenge left by mystical numbers is found in *Revelation* 13:18. St. John says, "Let him that hath understanding count the number of the beast: for it is the number of a man; and his number is six hundred three-score and six." This beast with the number 666 is the symbol of the Antichrist. Peter Bungus, in *Numerorum Mysteria,* worked diligently and with great satisfaction to reduce this number to the name of the "unholy" Martin Luther. Letting a=1, b=2, . . . , k=10, l=11, . . . , s=90, t=100, u=200, etc., and using a Latinized spelling, Bungus wrote

(30)	(1)	(80)	(100)	(9)	(40)	(20)	(200)	(100)	(5)	(80)	(1)
M	A	R	T	I	N	L	U	T	E	R	A

In response, Michael Stifel, a friend of Luther and a German mathematician, exercised equal ingenuity and showed that the number referred to Pope Leo X. And the game of beasting has continued unto this day. During World War II it was pointed out that the number of the beast referred to no one else but Hitler. Letting A=100, B=101, . . , Z=125, we find $H+I+T+L+E+R = 107+108+119+111+104+117 = 666$.

"A Bell Laboratories mathematician took a long look at the cross-section diagram of an early carrier current field and then showed, by means of a single theorem from trigonometry, that it could carry twice as many messages as the engineers had figured."

—GERARD PIEL.

25. RATIONAL-SIDED TRIANGLES.

In the diary of Lewis Carroll there appears the following note dated Dec. 19, 1897: "Sat up last night till 4 P.M. (sic) over a tempting problem sent me from New York: to find three equal rational-sided right triangles. I found two whose sides are 20, 21, 29 and 12, 35, 37 but could not find *three*." This is but one of many interesting problems that have been proposed concerning pythagorean triplets, that is, rational numbers satisfying the relation $x^2+y^2=z^2$. The ancient Egyptians were familiar with the fact that $3^2+4^2=5^2$, and a recently deciphered cuneiform tablet of the ancient Babylonians revealed a table of fifteen triplets, the largest being (13,500; 12,709; 18,541). Pythagoras, Plato, Euclid, and Diophantos gave rules for determining triplets, and the complete solution of $x^2 + y^2 = z^2$ in rational numbers is now known. The problem has been generalized in many ways: an additional condition is introduced as in Lewis Carroll's problem above, or the restriction to right triangles is replaced by another restriction such as the requirement that the area be rational.

N. Allison, *Mathematical Snack Bar.* New York, Chemical Publishing Co., 1936 (pp. 20-27), 116-131).

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26. THE "FIFTEEN" PUZZLE.

One of the most popular puzzles ever invented is the Fifteen Puzzle which made its appearance about 1878. The puzzle consists of a 4×4 matrix of numbers 1, 2, 3, . . . , 16 mounted on small blocks. The blocks are arranged at random and number 16 is removed. The puzzle consists of moving the remaining blocks about so that the numbers assume their natural order. The puzzle attracted tremendous interest and huge prizes were announced for the solutions of certain initial arrangements. Interest in the puzzle subsided when mathematicians analyzed the various arrangements and showed that some were impossible of solution. The puzzle still is sold in large numbers, a modern version being attractively made of plastic material.

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27. SQUARING THE CIRCLE.

One of the three "classical geometrical problems of antiquity" was that of squaring the circle, that is, of constructing a square having an area equal to that of a given circle. The Greek mathematicians noted that this problem was solved if a line can be constructed of length equal to the circumference of the circle. If the only tools permitted are the compass and the straight-edge, the problem is impossible. The proof of this was not completed until 1882 when Lindemann proved that π was transcendental. If tools other than the compass and straight-edge are permitted, the problem is solvable. Regardless of the fact that mathematicians have disposed of circle-squaring with finality, would-be-solvers continue to attempt the impossible.

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"In the pure mathematics we contemplate absolute truths, which existed in the Divine Mind before the morning stars were together, and which will continue to exist there, when the last of their radiant host shall have fallen from Heaven."

—E. T. BELL.

THE PROBLEM CORNER

EDITED BY JUDSON W. FOUST
Central Michigan College of Education

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solutions should not demand any tools beyond the calculus. Although new problems are preferred, old problems of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before October 1, 1950. The best solutions submitted by students will be published in the Fall 1950 number of THE PENTAGON. Credit will be given for all correct solutions received. Address all communications to Dr. Judson Foust, Central Michigan College of Education, Mt. Pleasant, Michigan.

PROBLEMS PROPOSED

A solution to Problem 8 has not been received. The student submitting the best solution of this problem will be given a one-year subscription to THE PENTAGON.

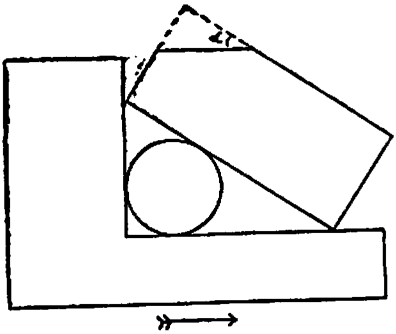
25. *Proposed by Frank Hawthorne, Hofstra College, Hempstead, New York.*

Triangle ABC has medians AD and CF meeting at H; E is the midpoint of AC; ED meets CF in G. Show that the area of triangle DGH is 1/24 of the area of triangle ABC.

26. *Proposed by Stanley Ogilvy, New York, N.Y.*

Four mothers, each with one daughter, went out to buy ribbons. Each mother bought twice as many yards as her daughter. Each person bought as many yards of ribbon as the number of cents paid per yard. No stores sold any ribbon in fractions of cents per yard. Mrs. Jones spent 76¢ more than Mrs. White. Nora bought three yards less than Mrs. Brown. Gladys bought two yards more than Hilda, who spent 48¢ less than Mrs. Smith. What is the name of Mary's mother?

27. *Proposed by Cleon C. Richtmeyer, Central Michigan College of Education, Mount Pleasant, Michigan.*



In order to saw a rectangular block at an angle, the block is held on the moving table against a cylinder as indicated in the figure. If the block is m inches long, find a formula for the diameter of the cylinder necessary to cut the block at an angle α.

28. *Proposed by Norman Anning, University of Michigan, Ann Arbor, Michigan.*

If 70 per cent have lost an eye, 75 per cent have lost an ear, 80 per cent an arm, 85 per cent a leg, what per cent, at least, must have lost all four?

29. *Proposed by Cleon C. Richtmeyer, Central Michigan College of Education, Mount Pleasant, Michigan.*

An interesting variation of a familiar problem in the mathematics of finance may be stated as follows: A father sets aside $4,000 for his son when he starts to college, to provide him with a fixed monthly income while he is in school. He is to receive equal payments at the end of each of the nine months of each of the four school years. If the $4,000 is invested at 3% compounded monthly one month before he is to receive the first payment, how large will the monthly payment be?

30. *Proposed by John K. Osborn, Central Michigan College of Education, Mount Pleasant, Michigan.*

Find the length of the largest runner of carpet two feet wide that can be placed diagonally in a room 24 feet by 30 feet so that each of the four corners of the runner touch a wall of the room.

SOLUTIONS

1. *Selected from the second Stanford University Mathematics Examination, April 19, 1947.*

To number the pages of a bulky volume the printer used 1890 digits. How many pages has the volume?

Solution by Robert E. Doyle, Iona College, New Rochelle, N.Y.

Pages 1-9 inclusive require 9 digits, pages 10-99 require 180 digits, and pages 100-999 require 2700 digits. Therefore there must be more than 99 pages and less than 999 pages. Now $180+9=189$, $1890-189=1701$, $1701\div 3=567$, and $567+99=666$. There were 666 pages in the volume.

Also solved by Victor Delisle, Iona College, and Otto C. Juelich, Forest Hills, N.Y.

2. *Selected from the second Stanford University Mathematics Examination, April 19, 1947.*

Among grandfather's papers a bill was found: 72 turkeys $-67.9-. The first and last digits of the number that obviously represented the total price of these fowls are replaced here by blanks, for they have faded and are now illegible. What are the two faded digits and what was the price of one turkey?

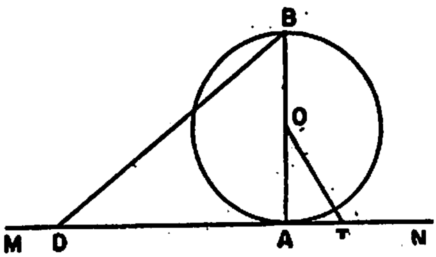
Solution of John Messera, Hofstra College, Hempstead, N.Y.

Since the total number of cents must be divisible by 72, it must also be divisible by 9 and divisible by 8. In order for the total number of cents be divisible by 8, the number formed by the last three digits must be divisible by 8. From this we see that 2 is the last digit because $792\div 8=99$. In order that the number of cents be divisible by 9, the sum of the digits must be divisible by 9. Upon adding the digits known we obtain $6+7+9+2=24$. Therefore in order to make the sum of the digits divisible by 9, we see that it is necessary to add 3 to 24. Thus the total bill in cents is 36792 or in dollars $367.92, and the price of one turkey is $367.92 $\div 72=$ $5.11.

Also solved by Otto C. Juelich, Forest Hill, N.Y., Victor Delisle, Iona College, and Robert E. Doyle, Iona College.

5. *Proposed by the Problem Corner Editor.*

The following approximate construction of π was presented in 1685 by Kochansky, a Polish mathematician. Let O be the center of a circle with a radius of one unit. Let A be the point of tangency of a tangent MN. At O construct an angle AOT equal to 30°, T being the intersection of the side of the angle with the tangent. From T, on MN, in the direction of A, lay off TD equal to 3 units. Then, if B is the other end of the diameter through A, BD is approximately equal to π. Find to the fifth significant figure the error in this approximation.



Solution by Otto C. Juelich, Forest Hill, N.Y.

$AT = \text{Tan}30° = \sqrt{3}/3$, $DA = 3-\sqrt{3}/3$, $BA = 2$

$BD^2 = BA^2 + DA^2 = 2^2 + (3-\sqrt{3}/3)^2 = (40-6\sqrt{3})/3.$

Error $= \pi - BD = 3.141592653 - 3.141533338$

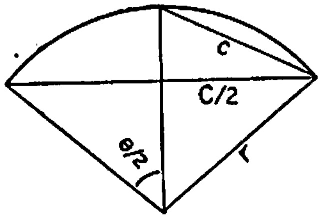
$= 0.000059315.$

Also solved by Victor Delisle, Iona College, and Robert E. Doyle, Iona College.

7. *Proposed by the Problem Corner Editor.*

Reference is made in *There is Fun in Geometry* by Kasper to Huyghen's approximation to the length of an arc. The Problem Corner Editor has sought without success to locate more definite reference to this. Show that approximately $L = (8c-C)/3$ in which L is the length of the arc, c is the chord of half the arc, and C is the chord of the arc.

Solution by Otto C. Juelich, Forest Hill, N.Y.



From the figure,

$c = 2r \sin(\frac{1}{4}\theta)$,

$C = 2r \sin(\frac{1}{2}\theta)$.

Let $\Phi = \frac{1}{4}\theta$. Then

(1) $(8c-C)/3 = (16r \sin \Phi - 2r \sin 2\Phi)/3$.

Substituting Φ and 2Φ for x in the sine series,

$$\sin x = x - x^3/3! + x^5/5! - x^7/7! + \ldots,$$

which converges for all x, and the results in turn in (1), we obtain

$$\begin{aligned}(8c-C)/3 &= 4r\Phi - 16r\Phi^5/5! + 80r\Phi^7/7! \ldots \\ &= r\theta - r\theta^5/7680 + \ldots\end{aligned}$$

a convergent series since it is the difference of two series that converge for all values. Since it is an alternating series, it follows that $(8c-C)/3$ differs from $L = r\theta$ by less than $r\theta^5/7680$.

19. *Proposed by Dr. C. B. Read, University of Wichita, Wichita, Kansas.*

A problem frequently found in algebra books a generation ago was: At what time after a specified hour will the hour and minute hands of a clock be together? The modern electric clock often has an hour, a minute, and a second hand; at what time after twelve o'clock will the three hands again be together?

Solution by Earl T. Boone, Wayne University, Detroit, Michigan.

Obviously if the three hands are to be together it is first required to have two of the hands together and then to determine whether or not the third hand coincides. If all the times when the hour and minute hands are together are calculated the criterion for determining whether or not the second hand is in the same position will be that the number of minutes and the number of seconds after the given hour must be the same. The hour and minute hands coincide every 1-1/11 hours or every 1 hour 5 minutes 27-3/11 seconds. Therefore the coincidences after 12 o'clock are 1:5:27-3/11, 2:10:54-6/11, 3:16:21-9/11, 4:21:49-1/11, 5:27:16-4/11, 6:32:43-7/11, 7:38:10-10/11, 8:43:38-2/11, 9:49:5-5/11, 10:54:32-8/11, and 12:0:0. Obviously at no time is the second hand within 5 seconds of coinciding with the other two. Therefore the time after twelve o'clock when all three hands are together is the following twelve o'clock.

Also solved by Otto C. Juelich, Forest Hill, N.Y., and the proposer.

20. *Proposed by the Problem Corner Editor. (A problem of historic interest due to John Bernoulli.)*

Find the numerical value of i^i.

Solution by Erwin Deal, Nebraska Wesleyan University, Lincoln, Nebraska.

In trigonometric form, $i = \cos(\frac{1}{2}\pi) + i \sin(\frac{1}{2}\pi)$. Hence, by DeMoivre's theorem,

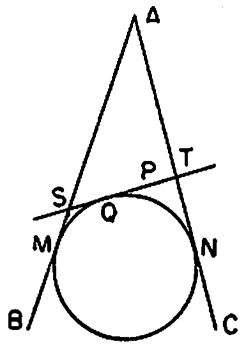
$$i^i = \cos(\tfrac{1}{2}i\pi) + i \sin(\tfrac{1}{2}i\pi) = \cosh(\tfrac{1}{2}\pi) - \sinh(\tfrac{1}{2}\pi)$$
$$= 2.5089 - 2.3011 = 0.2078.$$

Also solved by Roy E. Crane, Morristown, N.J., Charles Gillilard, Washington, D.C., Hugh Morris, Hofstra College, Clifford E. Harralson, Southwest Missouri State College, and Otto C. Juelich, Forest Hill, N.Y. Mr. Deal generalized the problem to that of finding the numerical value of $(a+bi)\exp(c+di)$.

21. *Proposed by the Problem Corner Editor. (From Taylor and Bartoo,* Introduction to College Geometry, *The Macmillan Company, 1949. Exercise 2, page 43.)*

Through a given point within a given angle to draw a line which will form with the given angle a triangle having a given perimeter.

Solution by William Douglas, Courtenay, British Columbia.

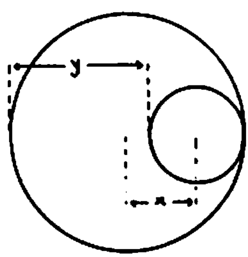


Denote the angle by BAC, the point by P, and let the given perimeter be K. On AB and AC lay off $AM = AN = \frac{1}{2}K$. Describe a circle tangent to AB and AC at M and N, respectively. Through P draw a line tangent to the circle at Q and cutting AB in S and AC in T. Then STA is the required triangle for $QS = SM$ and $QT = TN$ and the perimeter of triangle AST $= AM + AN = K$.

Note. It would appear obvious that the construction is not possible if P falls within or beyond the circle, and that there are two solutions in the figure above since there are two distinct tangents from P.

23. *Proposed by the Problem Corner Editor.* (*From Christman,* Shop Mathematics, *The Macmillan Company, 1926, page 26.*)

Find y if x is 0.312 inches.



Solution by Lotta Stallman, Paterson, N.J.

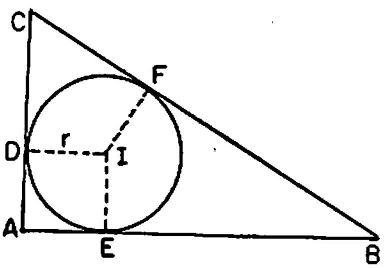
Let r = radius of the small circle and R = the radius of the large circle. Then $x+r=R$ and $x-r=y-R$. Solving simultaneously, $y=2x=0.624$ inches.

Also solved by Erwin Deal, Nebraska Wesleyan University, William Douglas, Courtenay, British Columbia, Ralph Nyberg, Normal, Illinois, Robert Reichert, Hofstra College, Earl T. Boone, Wayne University, and H. P. Thompson, North Hollywood, California.

24. *Proposed by the Problem Corner Editor. (From Christman,* Shop Mathematics. *The Macmillan Company, 1926, page 87.)*

Prove that in any right triangle the sum of the two legs is equal to the sum of the hypotenuse and the diameter of the inscribed circle.

Solution by Lotta Stallman, Patterson, N.J.



Designate the center of the inscribed circle as I, the radius r, and the points of tangency D, E, and F. Points A, B, and C are the intersections of tangents to the circle so that $AE = AD$, $CD = FC$, and $EB = BF$. All the angles in $AEID$ are right angles, therefore $AD = DI = AE = EI = r$. But $AB+AC = (AE+BE) + (AD+DC) = (AE+AD) + (BE+DC) = 2r+(BF+FC) = 2r+BC$.

Also solved by Earl T. Boone, Wayne University, William Douglas, Courtenay, British Columbia, Otto C. Juelich, Forest Hill, N.Y., and H. P. Thompson, North Hollywood, California.

THE BOOK SHELF

EDITED BY CARL V. FRONABARGER
Southwest Missouri State College

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be considered and preference will be given to books written in English. When space permits, older books of proven value will be described. Please send books for review to Professor Carl V. Fronabarger, Southwest Missouri State College, Springfield, Missouri.

Number Theory and Its History. By Oystein Ore. McGraw-Hill Book Company (330 West 42nd St., New York 18, N.Y.), 1948. 10 + 370 pages. $4.50.

Guided largely by his experience in teaching number theory to undergraduate students, Professor Ore has written a book teeming with many thought-provoking topics which can be understood without much mathematical experience and background. A perusal of its contents will yield the student an invaluable development of his understanding of the basic fundamentals related to our number system and arithmetic operations.

The blending of historical facts into the presentation is a rich characteristic of this book. It is further characterized by clear, concise definitions and very readable proofs of theorems. Another feature of considerable merit is the inclusion of bibliographies after most chapters.

The author's organization of topics is very well done. The inclusion of numerous problems, with an adequate number of illustrative examples preceding the selections, will offer a definite challenge. The topics—Counting and Recording of Numbers, Euclid's Algorism, Prime Numbers, Aliquot Parts, Indeterminate and Diophantine Problems, Congruences, Theory of Decimal Expansions, Classical Construction Problems, and several portions or whole chapters dealing with the well-known theorems attributed to Euler, Fermat, Wilson and other mathematicians of distinction—can scarcely fail to augment the student's appreciation of the subject.

This book should provide, along with the standard undergraduate course in the theory of equations, an adequate background in algebra for students entering graduate school, and it will remain an important source of reference for number theory topics.

—J. HAROLD SKELTON.

Mathematics Our Great Heritage. By William L. Schaaf. Harper and Brothers (49 East 33rd St., New York 16, N.Y.), 1948. 11 + 291 pages. $3.50.

This book should be accessible to laymen through public libraries, and should also be in every high school and college library. It can be listed as one of the fine books to motivate interest and study in mathematics. The few gems of mathematical thought are well chosen and exemplify the objectives selected by the author. The reading of this book should help the reader realize that mathematics has been most effective in developing cultural and scientific leadership, and it should inspire one to go to the source material and continue further study in the subject. Professor Schaaf has given us a notable piece of literature. More books like this should be written by competent authors.

—C. N. MILLS.

How to Solve It. By G. Polya. Princeton University Press (Princeton, New Jersey), 1948. 15 + 224 pages. $3.00.

This is not a mathematics book; it is a book about mathematics for students and teachers of mathematics. It does not present lists of exercises and problems to be solved, but has to do with general methods of solving problems. In the words of the author, "Mathematics has two faces; it is the rigorous science of Euclid but it is also something else." Textbooks tend to present mathematics as a systematic deductive science after the Euclidean pattern; this book deals with the "something else," the experimental, inductive approach to a problem which leads us to the solution. The Euclidean pattern is not an effective approach to the solution of a problem; rather, it is a vehicle for reporting the solution. This book is devoted to an exposition of methods and techniques for discovering solutions.

According to the author, solving problems is a practical skill to be learned by observing and imitating what other people do when solving problems. But solving the problem at hand, particularly in a classroom situation, may be a relatively unimportant outcome of the student's activity. As the author points out, if a student fails to get acquainted with a particular geometric fact, he may not have missed much that will be of value to him later on but if he fails to get acquainted with the idea of geometric proof in his study of geometry, "He lacks a true standard with which to compare alleged evidence of all sorts aimed at him in modern life."

As the author suggests, it is an old philosophical dream to find unfailing rules applicable to all sorts of problems, but it can never be more than a dream. We can, however, study procedures which are typically useful in solving problems. This is what the author has attempted to provide. The results of his efforts will be of interest to all teachers and students of mathematics. His book should be of particular value to the young, inexperienced teacher. Good students and successful teachers of mathematics will find that they are already using many of the procedures described, but even they will find much of value in this book.

—CLAUDE H. BROWN.

The Psychology of Invention in the Mathematical Field. By Jacques Hadamard. Princeton University Press (Princeton, New Jersey), 1945. 13 + 143 pages. $2.50.

How many times have you asked yourself, "What makes a mathematician and how does he tick?" The answers to these and related questions are given by the author in a clear, logical exposition of his and other's introspections and studies on the methods of mathematical discovery. This fascinating little book should be of interest not only to those who will do graduate work in mathematics, but also to prospective teachers of secondary school mathematics and students of a philosophical or psychological bent, perhaps also to others because of the recent interest in "mechanical brains."

In sections I-V the author develops the thesis that mathematical invention is accomplished by repetitions of the cycle of operations of *preparation, incubation,* and *illumination* (inspiration), two consecutive cycles being properly tied together by "relay-results." Preparation consists of a period of (conscious) intense and controlled thought, during which ideas (hooked atoms) are activated and projected in certain fairly well-defined (but not too narrowly confined) directions. These (usually) apparently fruitless endeavors are followed by a period in which the individual's attention is elsewhere (i.e., he "sleeps" on it), so that incubation occurs. During incubation, some of the above "atoms" collide (subject, perhaps, to the laws of chance) with other "atoms" (including stationary ones) and form "molecules" (combinations of ideas) of various kinds, some of which are likely to be important because of the "fire control apparatus" used in preparation. The unconscious then selects those "molecules" which affect most deeply the person's emotional sensibility, resulting in illumination. Then ensues the conscious work of *verifying* and *precising* these inspirations (to obtain a "relay-result"), a process of "shoring up" the excavations already made, without which further digging is impossible.

The remainder of the book consists of a discussion of some items auxiliary to the main thesis. Synthesis of a whole argument is accomplished by means of a mental image, usually visual or kinetic in nature, the actual steps in the proof being somewhere in the "fringe-consciousness," the top layer of the unconscious, so that they can be "peeled off" easily. For most mathematicians, words (in the usual sense) are not a necessary adjunct to thought. "Common sense" and space intuition, incidentally, do not necessarily lead to correct results. Some authorities classify mathematicians either as "intuitive" or "logical," but the author believes all are intuitive (some more than others) in the sense that an initial intuition (discovery) is followed by logic (enunciation). The author points out that most research leads in the direction of scientific beauty and this kind usually proves most fruitful and (eventually) practically useful. One of the appendices contains a letter from Professor Einstein about his own methods of thought.

—THOMAS H. SOUTHARD.

A Concise History of Mathematics. By Dirk J. Struik. Dover Publications, Inc. (1780 Broadway, New York 19, N.Y.), 1948. Vol. I: 18 + 123 pages. Vol II: 6 + 175 pages. $1.50 per volume.

Here, as the name indicates, is a concise history of mathematics. Published in two small, pocket-size volumes, the work gives a brief account of the development of mathematics from early times through the nineteenth century. No attempt is made to include all that is known about mathematics, either ancient or modern, but important trends and developments are indicated, with a surprising amount of detail for such a short work. In the first volume the author mentions the probable state of mathematics in the Stone Age, traces its development in the ancient Orient, the Grecian period, and introduces the beginnings in Western Europe. In the second volume he takes up the contributions of individuals to the development of the subject in the seventeenth, eighteenth, and nineteenth centuries. Because the history is brief, a quantity of material has been reduced to a small space, but nevertheless the outstanding mathematicians and their works are treated in such a way as to give the account a more human aspect than would have been the case with a topical approach.

While he holds strictly to his subject, Dr. Struik frequently inserts comments on the political situation and general history of the times which shed much light and give some interesting sidelights on the mathematics itself. Some knowledge of the history of ancient, medieval, and early modern times will help in understanding these comments, but is not necessary to an understanding of the work itself. The reader with a general knowledge of high school mathematics, or even less, will find no difficulty in following the mathematics included in the first volume, but for a good understanding of the second volume he should be acquainted with the subject matter of mathematics beyond the high school level. For those who wish to pursue the history of mathematics further and in more detail, lists of appropriate works on the subject are given at the end of each chapter. This is a commendable feature of both volumes. The work is illustrated with a number of facsimile pages from old writings, as well as with pictures of a number of men who contributed to the development of mathematics.

The author seems not to have written his book with any particular group in mind. In some places his language is not simple; he chooses some big words where more simple terms might do, and some of his allusions are somewhat obscure. However, the narrative, brief and direct as it is, gives a good account of the main trend of mathematics over the years. Many students who may hesitate to read a fuller treatment because its size is forbidding may well select this work to gain a knowledge of what the author calls "a vast adventure in ideas."

—S. B. MURRAY.

You Can't Win. By Ernest E. Blanche. Public Affairs Press (2153 Florida Ave., Washington 8, D.C.), 1949. 155 pages. $2.00.

The contents of this little book are well described by the subtitle "Facts and Fallacies about Gambling." The author, who is at present Chief Statistician for the Logistic Division of the Army General Staff, has spent much time during the past twenty years studying games of chance and the "techniques of gambling and the foibles of gamblers." He here presents many of the results of his investigations, with special emphasis on the odds against the average better.

The plan of the book is indicated by the chapter headings. A brief introduction lists fourteen reasons why "you can't win." Then the first two chapters on "The Gambling Trend" and "Wagers and Wagering" are concerned with the definition of gambling and a short discussion of systems for attempting to beat gambling games. The later chapters take up in turn the various games or gambling devices: "Dice Games, Playing Cards, Poker and Three-Card Monte, Betting on the Horses, The Numbers Racket, Lotteries and

Pools, Pin-Ball and Slot Machines, Roulette, Carnival Games, Bingo, The Chain-Letter Racket, The Pyramid Club Scheme, Children's Gambling Games, Confidence Games." A typical chapter will give a brief history of the game or device, describe the method of play, discuss and list odds for and against winning, describe "gimmicks and gadgets" used by professionals and sharpers to swindle the unwary player, and finally point out that "you can't win."

The book is very readable, due to simple language and structure, and contains interesting information. A student of mathematics will certainly wish that the author had included a discussion of elementary probability such as that given in his article on "The Mathematics of Gambling" in *School Science and Mathematics*, March 1946, and had indicated more clearly how odds are computed. (Others may think that there is too much mathematics.) It is not clear to the reviewer just what the author expects of the reader in the way of mathematical background — probably none. However, on page 37 he finds the probability of obtaining 6 at least once in n tosses of a die. This involves solving an exponential equation, the use of logarithms and inequalities, and might cause difficulty for some students of college mathematics.

The author sometimes fails to make his point that "you can't win." For example, in chapter 5, he tells the anecdote of the statesman whose "poker was so good that" while he was in the army "he kept his company continually out of funds." This story is followed on the next page by the statement that "it is difficult to understand how anyone can conclude that poker is a game of skill rather than of chance."

Anyone interested in gambling for any reason, whether as a participant, or as a student of psychology, sociology, or mathematics should get something from this book. However, in most cases, he would probably wish to read farther. For these readers, a rather long list of references is given, including a few on mathematics. The date given in the reference to the author's article on "The Mathematics of Gambling" is incorrect. —PAUL EBERHART.

Come, come, let us circle the square, and that will do us good.
—BOSWELL.

=∇=

Perpendiculars from the incenter to the sides of the 3-4-5 right triangle divide it into areas which are numerically 1, 2, and 3.

=∇=

It is often related that DeMoivre, always interested in number series, had foretold that each day he should need 15 minutes more sleep than on the preceding day, and that his death would occur when the total reached 24 hours.

=∇=

Kenneth is making the acquaintance of decimals and has been asked to divide ten by three. He proceeds:

"Three into ten, three decimal point."

"Right. Go on."

"Carry one—three into ten, three—another three, and another. Why, they are all threes!"

"Yes, indeed."

"Well, how long do they go on?"

"As long as you like."

"What, for a million years?"

"Yes, if you like."

"No, not for a million years."

"Why not?"

"Well, you see, by that time they will know much more about mathematics than we do and they'll soon put a stop to that."

—MATH. GAZETTE.

=∇=

3.14159265358979323846264338327 9

See, I have a rhyme assisting
My feeble brain its tasks sometime resisting,
Efforts laborious can by its witchery
Grow easier, so hidden here are
The decimals all of circle's periphery.

—L. R. STOKELBACH.

A goldsmith charged 2% commission when purchasing some gold from A (meaning that A received only 98% of the value of the gold), and 2% when he sold the same gold to B (meaning that B paid 102% of the value of the gold). But the goldsmith made an extra $25 in the deal by cheating, as he bought with a "pound" weight which actually weighed 17 oz., and sold with a "pound" weight which only weighed 15 oz. How much did A get for his gold? (Ans. $183.51)

—SCH. SCI. AND MATH.

=∇=

Leibnitz was the last of the universals.

—DE QUINCY.

=∇=

"With a series of equations, [A. Brothman, a mathematician who practices as a chemical engineer in New York City,] showed how the Buna-S process [for making synthetic rubber] could be put on a continuous instead of a batch basis. At a cost of $15,000 for extra piping, the capacity of the plant was stepped up forty percent over original design estimates."

—GERARD PIEL.

=∇=

To square numbers between 25 and 75 we may use the identity,

$$N^2 = [25 + (N - 50)]\times 100 + (N - 50)^2.$$

Thus, we determine $N - 50$, add 25, annex two zeros, and then add $(N - 50)^2$. To illustrate, for 56^2 we find in order $56-50=6$, $6+25=31$, $31\times 100=3100$, $3100+6^2=3136$.

=∇=

The following cryptarithm has a unique solution. The x's indicate missing digits, not necessarily equal.

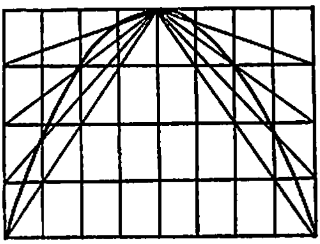
```
xxxx)x55xx5x(x5x
     xx5xx
     -----
      xxxxx
      xxxxx
      -----
        xxxx
        xxxx
        ----
```

—MATH. GAZETTE.

A certain youth was asked his age
By one who seemed to be a sage;
To whom the youth made this reply,
Sir, if you wish your skill to try,
Eight times my age increased by four
A perfect square, nor less nor more;
Its triple square plus nine must be
Another square as you will see.
He tried but sure it posed him quite,
His answer being far from right.
You skilled in science I implore
This mystic number to explore.

—THE SCIENTIFIC JOURNAL, 1818.

=∇=



Here is an easy geometrical construction of a parabolic arch having given width and height. Divide the width into $2n$ equal parts, the height into n equal parts, and draw radial lines as indicated in the figure. The intersections of the radial and vertical lines are points on the desired parabola.

=∇=

The harmonic series

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \ldots$$

is the reciprocal of the continued product

$$(1-1/2)\ (1-1/3)\ (1-1/5)\ (1-1/7)\ \ldots$$

in which only the primes enter.

—EULER.

=∇=

At the time of the French Revolution, delegates of the Convention inquired of the illustrious Lagrange what subject he would be willing to profess for the benefit of the community. Lagrange answered meekly, "I will lecture on Arithmetic."

—NATURE.

Why is it that we entertain the belief that for every purpose odd numbers are the most effectual?

—PLINY THE ELDER.

=Δ=

The radical sign was once an *r*, but it has become worn down by constant use.

=∇=

The Quiz Kids flunked the following problem. A baker sent a boy to deliver an order for 9 doughnuts. The baker placed the doughnuts in a box, and wrote IX on the cover to indicate the number of doughnuts inside. On the way the boy ate 3 of the doughnuts, and then decided it might be good policy to change the number on the box accordingly. His pencil had no eraser. How did he change the number?

=∇=

HORIZONTAL:

1. Insects.
2. Four.
3. Annoy.
4. No teacher has this kind of life.

VERTICAL:

1. Dogs do this.
2. So do suckers.
3. A large one is a mouthful.
4. A little more than one bit, but less than two-bits.

1	2	3	4
2			
3			
4			

=∇=

Sir William Rowan Hamilton (1805-1865), the discoverer of quaternions (1852), was an infant prodigy, competing with Zerah Colburn as a child. He was a linguist of remarkable powers, being able, at thirteen years of age, to boast that he knew as many languages as he had lived years. When only sixteen he found an error in Laplace's *Mécanique céleste.*

—DE MORGAN.

"The advance and perfecting of mathematics are closely joined to the prosperity of the nation."

—NAPOLEON.

$=\nabla=$

The following item appears in Jacques Bernoulli's *Opera* (1744): "Titius gave his friend, Sempronius, a triangular field of which the sides, in perticas, were 50, 50, and 80 in exchange for a field of which the sides were 50, 50, and 60. I call this a fair exchange." Can you find other such pairs of Bernoullian triangles?

$=\nabla=$

"He was an arithmetician rather than a mathematician. None of the humor, the music or the mysticism of higher mathematics ever entered his head. Men might vary in height or weight or color, just as 6 is different from 8, but there was little other difference."

—JOHN STEINBECK, *The Moon Is Down.*

$=\nabla=$

"It is a perfectly erroneous truism, repeated by all copybooks and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of operations which we can perform without thinking about them."

—A. N. WHITEHEAD.

$=\nabla=$

Prime numbers in arithmetical progression. Note that the difference in each case is a multiple of all the primes preceding the first term.

3, 5, 7.
3, 11, 19.
5, 11, 17, 23, 29.
5, 17, 29, 41, 53.
7, 157, 307, 457, 607, 757, 907.

$=\nabla=$

Nothing is so difficult but that it may be found by seeking. —TERENCE.

KAPPA MU EPSILON NEWS

EDITED BY CLEON C. RICHTMEYER, *Historian*

At their annual banquet on January 13, twenty-three new members were initiated into **California Alpha.** Dr. Aubrey J. Kempner spoke on the subject, "What is True in Mathematics?"

—+—

Illinois Beta holds its discussion meetings in conjunction with the Mathematics Club, with the KME business meeting following. The chapter has approved fourteen nominations for prospective new members.

—+—

Iowa Alpha held a Homecoming Breakfast on October 22 at the home of Dr. H. Van Engen, national president of KME.

—+—

Coffee and doughnuts are served after each program meeting of **Kansas Alpha.** The annual dinner held in February featured reports of the NCTM meeting at Wichita.

—+—

Jeri Sullivan was the official delegate of **Kansas Gamma** at the Wichita NCTM meeting. Mary Alice Weir represented the chapter at the Regional N.F.C.C.S. meeting in Omaha. Miss Weir, who is vice-president of the Kansas Gamma chapter, has been awarded a scholarship to the Institutum Divi Thomae in Cincinnati where she will do medical research. Plans are being made for a joint meeting of the four Kansas chapters in connection with the spring meeting of the Kansas section of the M.A.A. in Pittsburg.

—+—

Miss Laura Greene, **Kansas Delta,** was in charge of arrangements for the KME luncheon held in connection with the N.C.T.M. meeting in Wichita on December 30. Dr. O. J. Peterson, **Kansas Beta** presided, and Kenneth Lake, Kansas Delta senior, addressed the group on the topic, "An Elementary Discussion of Fundamental Concepts in Modern Algebra." Twenty-three members and two guests were present, representing six chapters.

Edward Czarnecki, Treasurer of **Michigan Beta,** was nominated by the college student body for listing in *Who's Who in American Colleges and Universities.*

—+—

Mississippi Beta is planning a spring initiation banquet, with an invited speaker.

—+—

Missouri Alpha has extended an invitation to the fraternity to hold its next biennial convention on the campus of Southwest Missouri State College. There are at present twenty-eight active members in the chapter.

—+—

An "Open House" meeting was held in January by **Missouri Beta,** to which everyone interested was invited, whether or not he was a member. The chapter purchased a $5.00 Christmas Seal Bond from the National Tuberculosis Association.

—+—

Nebraska Alpha will assist in plans for the meeting of the Nebraska section of the N.C.T.M. to be held in April. KME and Lambda Delta Lambda will hold a joint banquet for which the speaker will be furnished by KME.

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On November 16, **New Jersey Beta** held a joint meeting with **New Jersey Alpha** on the Montclair campus. Dr. Howard Fehr of Columbia was the speaker. The October 12 initiation was held at the home of Dr. Virgil S. Mallory.

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At the semi-annual initiation and banquet on January 11, eleven new members were initiated into **New Mexico Alpha.**

—+—

The national officers of KME met on the campus of **New York Alpha** during the Christmas holidays. The chapter was represented by five faculty and five student members at the KME luncheon held in New York City at the time of the M.A.A. meeting. A total of twenty-seven people attended this luncheon, held in Butler Hall, Columbia University. In addition to New York Alpha, Iowa

Alpha, Michigan Alpha, Michigan Beta, New Jersey Alpha, and New Jersey Beta were also represented.

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South Carolina Alpha built the programs for the first semester around the theme, "Modern Higher Plane Geometry." It was felt that this procedure accomplished more than a series of unrelated papers on a variety of subjects.

—+—

Tennessee Alpha has by far the highest scholastic requirements of any organization on the campus of T.P.I.

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PROGRAM TOPICS, FALL SEMESTER, 1949-50

California Alpha, Pomona College
Method of Casting out Nines, by Don Benson
Sir Isaac Newton, by Carolyn Grove
Linkages, by John Dienes
Basic Problems of Number Theory, by Mr. Ralph Vernon
Conic Sections, Illustrated with Plastic Models, by Dr. C. G. Jaeger and Mr. Charles Halberg
Navigation, by Gilbert Madden
Continuous Fractions, by Don Benson

Illinois Beta, Eastern Illinois State College
Approximations to Pi, by Donald Fraembs
Problems of Antiquity, by several speakers
Paradoxes and Fallacies, by several speakers
Pythagorean Triples, by Dr. L. A. Ringenberg

Illinois Delta, College of St. Francis
The New Year, by Mary Lou Hodor
We Claim Lewis Carroll Too, by Sister M. Claudia
The Geometry of Life, by Nan Hutchings
Journal Reports, by Lois Gilgen and Bernadine Arseneau
Problem Proposed, by Alice Del Favero
Movie, *Years of Progress*, by Sister M. Rita Clare
Isaac Newton, by Margaret Ann Dreska

Iowa Alpha, Iowa State Teachers College
Eisenstein's Irreducible Criteria, by Don Edwards
Magic Squares, by Sam Weigert
The Theory and Uses of the Planimeter, by Don Richardson
The Problem of Appolonius, by Eddie Sage

Iowa Beta, Drake University
Highlights of The National Convention, by Lewis Workman
Seven Come Eleven, by Waid Davidson
Euclidean Algorism, by Enid Allbaugh

Boolean Algebra, by Jack Matsui

Kansas Alpha, State Teachers College, Pittsburg

Contributions of Surveying to Mathematics, by Lawrence Fields

Development of Computing Devices and Machines, by Robert Green

Curious Numbers, by Ronald Lehman

Lengths of Circular Arcs by Drafting Methods, by Charles Crane

Probability and Its Uses, by Prof. J. A. G. Shirk

Determination of Pi, by Robert Sommerfield

Proofs of The Pythagorean Theorem, by Robert Thomas

Logarithms of Complex Numbers, by Dr. R. G. Smith

Kansas Beta, State Teachers College, Emporia

Problem Dealing with Simple Harmonic Motion, by Roger Ruth and Virginia Reed

Program of Thought Problems, by Bill Varvel and Saul Straud

Kansas Gamma, Mount St. Scholastica College

Status of Mathematics with the Greek Philosophers, by Sister Helen Sullivan

Mathematics in The Patristic Period, by Noreen Hurter

Mathematics in The Scholastic Period, by Mary Alice Weir

Contemporary Mathematical Philosophers, by Jeanne Cullivan

Kansas Delta, Washburn Municipal University

Theory of Numbers, by Dr. S. Chowla

Problems in Algebra, panel discussion by Mr. Hugo Rolfs, Howard Sperry, William Powell and Kenneth Lake.

The Trisection of an Angle, panel discussion by Mr. Norman Hoover, Margery Gamble, Edna Metzenthin, and Nancy Martin

Mathematical Needs of The Psychology Major, by Robert S. Hage

Michigan Alpha, Albion College

Carl Friedrich Gauss, by Edwin Kehe

Life of Abel, by Deane Floria

New Methods for Evaluating Determinants, by Prof. Paul Cox

Repeating Decimals, by Philip McKean

Some Uses of Mathematics in Chemistry, by David Harmer

Michigan Beta, Central Michigan College

Some of the Mathematics Used in Music, by Margaret King

Report on The Christmas Luncheon in New York City, by Dr. C. C. Richtmeyer

Mathematics in Photography, by Roger Ewing

Michigan Gamma, Wayne University

Newton's Method of Solving Equations, by Dr. K. W. Folley

Foundations of Mathematics, by Dr. R. Ackoff

The Differential Analyzer, by Dr. A. W. Jacobson

Mississippi Gamma, Mississippi Southern College

Finding the Bend Points of Quadratic Equations by Inspection of Roots, by Arthur McCary

Mathematical Probabilities in Various Forms of Gambling, by Virginia Felder

Beginnings of Modern Algebra, by Harold Leone

Short Method of Integration for Volumes of Certain Solids, by E. C. Stanford

Alpha and Beta Functions, by John Jones, Jr.

Missouri Alpha, Southwest Missouri State College

Some Geometric Interpretations of the Binomial Theorem, $n = 2$, $n = 3$, *and Approximations of Roots by Geometry,* by Earl Bilyeu

Diophantine Problems, by J. H. Skelton

Puzzle Problems, by James Jakobsen and Joe Guida

Statistical Research, by Mar J. Robinette

Works of Some Well-Known Greek Mathematicians: Archimedes, by Jessie Belveal; *Appolonius,* by Evelyn Ruark; *Euclid,* by Ernest Fontheim; *Pythagoras,* by George Nash; *Eudoxus,* by Patricia Maddus; *Thales,* by Earl Phillips

Missouri Beta, Central Missouri State College

Denumerable and Non-Denumerable Infinities, by Sammy Vaughn

The Euclidean Algorithm, by Wayne Vanderlinden

Types of Discontinuities, by Philip Burford

Nomography, by Rex Wyrick

Complex Numbers and Their Geometric Interpretation, by Kathryn Lou Baker

Puzzles Based on Binary and Ternary Number Systems, by Martin Rowland

Short Cuts in Multiplication, by Peggy June Taylor

The Fourth Dimension, by Barbara Wurth

An Oblique Coordinate System, by Mrs. Marcia Jackson

Missouri Gamma, William Jewell College

The History of Social Mathematics, by Thomas Henry

The Mathematical Deviation Used in Chemistry, by Donald Williams

Mathematics and Reality, by Bob Fitzwater

The Calendar, by Peggy Beecher

Missouri Epsilon, Central College

Newton and Leibniz, by Paul Calvert

Hamilton and Quaternions, by Mark Barton

The Kinetic Theory of Gasses, by Merle Cartwright

An Introduction to Modern Geometry, by Dr. Floyd Helton

Cantor's Theory of Infinity, by Niels C. Nielson

A Brief History of the Development of The Engineering Profession, by Clifton Denny

History of The Number System, by David Bouldin

Gauss, Prince of Mathematics, by Norman Drissell

Nebraska Alpha, State Teachers College, Wayne

Sir Isaac Newton, by Howard Prouse

Blaise Pascal, by Neil Sandahl

New Jersey Beta, State Teachers College, Montclair

Trisection of a General Angle by Means of Curves, by Audrey Jensen

Appreciation of Elementary Mathematics, by Dr. Howard Fehr

New Mexico Alpha, University of New Mexico
Lagrangian Mechanics, by Norman Riebe
Colds and The Anti-Histamines, by Dean Roy Bowers

New York Alpha, Hofstra College
Mathematics in the Schools of England, by Mr. Charlesworth
The Four Color Problem, by Dr. Ollman
Verb Tense Determined Symbolically, by Mr. Beller
Complex Roots of Polynomials Determined Graphically, by Prof. Howard Fehr
Non-Euclidean Geometry, by Perry Watts

Ohio Alpha, Bowling Green State University
The Color Problem, Mr. Harold Tinnappel
Dimensional Analysis, by Norman Fleck, Ned Krugh, Arthur Miller, and Harry Ling
Mathematical Physics, by Dr. Donald Bowman

Ohio Beta, College of Wooster
Symbolic Logic, by Prof. Wilford Bower
Centrifugal Force in Industry, by Mr. Walter H. Craig
Science and Philosophy of Mathematics, by Robert Miller and George Smolensky
Mathematics of Sound, by Dr. E. Unnewehr

Oklahoma Alpha, Northeastern State College
To Prove that "e" is a Real Number and has a Limit, by Harry Henson and Doyle Sanders
Recovering Oil through Water Pressure, by Doyle Reich

South Carolina Alpha, Coker College
Properties of Cyclic Quadrilaterals, by Shirley Jenkins
The Use of Analysis in Construction Problems, by Frank Saunders
The Theorems of Ceva and Menelaus, by Betty Reaves
Some Properties of Orthogonal Circles, by Frank Saunders

Tennessee Alpha, Tennessee Polytechnic Institute
History of K.M.E., Miquel Jorge Garcia
The Constitution of K.M.E., by B. A. Limpert and H. Baxter Norman.
Mathematical Quiz With Prizes, by M. T. Morgan, W. A. Brown, and F. Joel Witt
Unusual Facts and Figures, by Elbert H. Gilbreath
Chances of Winning, by Charles A. Swallows

Texas Alpha, Texas Technological College
Highlights of Astronomy, by Dr. R. S. Underwood
Probability in Games of Chance, by Prof. E. R. Heineman

Wisconsin Alpha, Mount Mary College
Non-Euclidean Geometry, by Joan Daley
The Fourth Dimension, by Mary Kilkelly
History of The Calculus, by Mary Hunt
Relativity, by Kathleen Hanley
Number Systems, by Wanda Kropp
The Slide Rule, by Janet Haig

THE EIGHTH BIENNIAL CONVENTION OF KAPPA MU EPSILON

The National Council of Kappa Mu Epsilon has accepted the invitation of the Missouri Alpha chapter to hold the Eighth Biennial Convention at Springfield, Missouri. The dates have been set for Friday, April 27, and Saturday, April 28, 1951.

Those members of Kappa Mu Epsilon who attended the Seventh Biennial Convention held at Topeka, Kansas, will recall that a profitable time, and a good time, was had by all who attended. It is the desire of the National Council that the Eighth Biennial Convention meet the standards of excellence set by previous conventions of Kappa Mu Epsilon. There is no reason why it should not be an excellent demonstration of the kind of work that can be done by our fraternity. Such chapters as Kansas Delta and Illinois Alpha have shown how to organize a convention and the national officers have now had some recent experience in organizing a program. These two factors and an outstanding chapter of Kappa Mu Epsilon as host chapter should insure another in the series of conventions for which Kappa Mu Epsilon is famous.

Local chapter officers should give considerable thought to the type of paper their chapter will offer for the convention program. It is easy to offer papers which lack "understandability" at conventions such as those sponsored by Kappa Mu Epsilon. Papers which are suitable for small groups, such as more commonly occur in local chapters, are not always suitable for large groups. In small groups students will feel at ease to ask questions if the paper is not understood. In large groups they hesitate to do so, particularly, if they know so few of the members of the group. Furthermore, papers delivered at local meetings usually have more time alloted than it is possible to allot at conventions. The time handicap is a very severe one when presenting a paper which is to be understood by juniors and seniors.

In all probability there will be many more papers submitted than can be given at the convention. In order to enable the program committee to make a wise choice, fifty word abstracts should accompany each title submitted for a place on the program. The abstract should include a statement as to the kind of visual aid, if any, that will be used by the student in order to clarify the main idea of the paper. These abstracts should be ready by January or February of 1951.

The convention is one means of stimulating interest in your local organization. The possibility of giving a paper at the convention should create an additional interest in the local chapter as well as the national organization. It is hoped that the local officers will give this interest creating activity much consideration. It usually takes some thought and effort on the part of local officers to get members of their chapter to develop an outstanding paper suitable for presentation at a Kappa Mu Epsilon Convention.

—HENRY VAN ENGEN, President.

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310 Burr Oak St.
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CHAPTERS OF KAPPA MU EPSILON

ALABAMA ALPHA, Athens College, Athens.
ALABAMA BETA, Alabama State Teachers College, Florence.
ALABAMA GAMMA, Alabama College, Montevallo.
CALIFORNIA ALPHA, Pomona College, Claremont.
COLORADO ALPHA, Colorado A & M College, Fort Collins.
ILLINOIS ALPHA, Illinois State Normal University, Normal.
ILLINOIS BETA, Eastern Illinois State College, Charleston.
ILLINOIS GAMMA, Chicago Teachers College, Chicago.
ILLINOIS DELTA, College of St. Francis, Joliet.
IOWA ALPHA, Iowa State Teachers College, Cedar Falls.
IOWA BETA, Drake University, Des Moines.
KANSAS ALPHA, Kansas State Teachers College, Pittsburg.
KANSAS BETA, Kansas State Teachers College, Emporia.
KANSAS GAMMA, Mount St. Scholastica College, Atchison.
KANSAS DELTA, Washburn Municipal University, Topeka.
MICHIGAN ALPHA, Albion College, Albion.
MICHIGAN BETA, Central Michigan College, Mount Pleasant.
MICHIGAN GAMMA, Wayne University, Detroit.
MISSISSIPPI ALPHA, State College for Women, Columbus.
MISSISSIPPI BETA, Mississippi State College, State College.
MISSISSIPPI GAMMA, Mississippi Southern College, Hattiesburg.
MISSOURI ALPHA, Southwest Missouri State College, Springfield.
MISSOURI BETA, Central Missouri State College, Warrensburg.
MISSOURI GAMMA, William Jewell College, Liberty.
MISSOURI DELTA, University of Kansas City, Kansas City.
MISSOURI EPSILON, Central College, Fayette.
NEBRASKA ALPHA, Nebraska State Teachers College, Wayne.
NEW JERSEY ALPHA, Upsala College, East Orange.
NEW JERSEY BETA, New Jersey State Teachers College, Montclair.
NEW MEXICO ALPHA, University of New Mexico, Albuquerque.
NEW YORK ALPHA, Hofstra College, Hempstead.
OHIO ALPHA, Bowling Green State University, Bowling Green.
OHIO BETA, College of Wooster, Wooster.
OHIO GAMMA, Baldwin-Wallace College, Berea.
OKLAHOMA ALPHA, Northeastern State College, Tahlequah.
SOUTH CAROLINA ALPHA, Coker College, Hartsville.
TENNESSEE ALPHA, Tennessee Polytechnic Institute, Cookeville.
TEXAS ALPHA, Texas Technological College, Lubbock.
TEXAS BETA, Southern Methodist University, Dallas.
TEXAS GAMMA, Texas State College for Women, Denton.
TEXAS DELTA, Texas Christian University, Fort Worth.
WISCONSIN ALPHA, Mount Mary College, Milwaukee.