The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before April 15, 2022. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring 2022 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 881-889

Problem 881. *Proposed by Mathew Cropper, Eastern Kentucky University, Richmond, KY.*

Find a formula (possibly recursive) for the number of integers with n digits that contain exactly one 47 in the integer.

Problem 882. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*

Find all functions f : $\mathbb{R} \rightarrow \mathbb{R}$ that satisfy $f(x^4 + y) = f(x) + f(y^4)$ for all $x,y \in \mathbb{R}$.

Problem 883. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*

If $a,b,c \in \mathbb{C}$ such that $|a^8 + 1| \leq 1$; $|b^{10} + 1| \leq 1$; $|c^{12} + 1| \leq 1$; $|a^4 + 1| \leq 1$; $|b^5 + 1| \leq 1$; $|c^{16} + 1| \leq 1$, then $|a + b + c| + 3 \geq |a + b| + |b + c| + |c + a|$.

Problem 884. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Romania.*

If $a,b,c > 0$ and $a^4 + b^4 + c^4 = 3$, then
$$\frac{(a^2+b^2)^6}{3a^8+10a^4b^4+3b^8} + \frac{(b^2+c^2)^6}{3b^8+10b^4c^4+3c^8} + \frac{(c^2+a^2)^6}{3c^8+10c^4a^4+3a^8} \leq 12.$$

Problem 885. *Proposed by Dorin Marghidanu, Colegiul National 'A. I. Cuze', Corabia, Romania.*

If $a, b, x, y > 0$ and $n \in \mathbb{N}^*$ prove that

$$\frac{(x+y)^n}{2^{n-1}} \le \frac{(ax+by)^n+(bx+ay)^n}{(a+b)^n} \le x^n + y^n.$$

Problem 886. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Prove that for any $a \in (1,2)$ and any integer $n \ge 1$, there exist $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}$ such that

$$(a-1)|\varepsilon_0 + \varepsilon_0 a + \cdots + \varepsilon_n a^n| < 1.$$

Problem 887. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Prove that in any triangle ABC with semiperimeter s, inradius r and usual notations, the following is true

$$\frac{a^{m+1}}{(s-b)^m} + \frac{b^{m+1}}{(s-c)^m} + \frac{c^{m+1}}{(s-a)^m} \ge 3 * 2^{m+1} * \sqrt{3} * r.$$

Problem 888. *Proposed by D.M. Bătinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Let the positive real sequence $(a_n)_{n\ge 1}$ be such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n \sqrt[n]{(n!)^2}} = a \in \mathbb{R}_+^*$ Compute

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{(2n-1)!!}} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right).$$

Problem 889. *Proposed by Seán Stewart, Bomaderry, NSW, Australia.*

If $h_n = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$, then evaluate the following two limits

(i) $\lim\limits_{n\to\infty} (\log(2) - h_n)n$,

(ii) $\lim\limits_{n\to\infty} \left(h_n h_{n+1} - log^2(2)\right)n$

Problem 890. *Proposed by Robert Stanton, St. Johns University, Jamaica, NY.*

For digits, a, b, c, d, let $abcd$ represent the ordinary decimal representation $10^3a + 10^2b + 10c + d$. Prove that there is a unique positive integer $n = aabb$ that is a perfect square.