The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before March 15, 2019. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring 2019 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 820-828

Problem 820. *Proposed by the editor*.

Find a 4-digit positive integer $N = abcd$ which is divisible by 11 and $N/11 = b^2 + c^2 + d^2$.

Problem 821. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a,,b,c \in \mathbb{R}$ then $4\sum_{cyclic} a\,|\,b(1-b^2)\,| \le \sum_{cyclic} a(1+b^2)^2$

Problem 822. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that in any acute-angled ΔABC you have

$$2\sum_{cyclic} \tan^3 A \ge \sum_{cyclic} \sqrt{\frac{\tan^6 A + \tan^6 B}{2}} + 3(\tan A + \tan B + \tan C)$$

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Problem 823. *Proposed by Pedro H.O. Pantoja, University of Campina Grande, Brazil.*

Let x,y,z be positive real numbers. Prove that

$$\frac{1}{xy+yz+zx} \le \frac{3x}{(y+2z)^3} + \frac{3y}{(z+2x)^3} + \frac{3z}{(x+2y)^3} \le \frac{x^3y + y^3z + z^3x}{(3xyz)^2}$$

Problem 824 *Proposed by Pedro H.O. Pantoja, University of Campina Grande, Brazil.*

Find all positive integers a, b, c where a and b are prime numbers with $a \neq 0 \pmod{c}$ such that $51a + 7ab + bc^2 = abc^2$.

Problem 825. *Proposed by Ovidiu Furdui and Alina Sintamarian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*.

Let $k \geq 0$ be an integer. Calculate

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\dots\right)-\frac{1}{n+k}\right]$$

Problem 826. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Let F_n and L_n be the nth Fibonacci and Lucas numbers defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ and by $L_1 = 1$, $L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

Let k be a positive integer and $F(k) = \begin{pmatrix} F_k^2 & F_{k+1}^2 \\ F_{k+1}^2 & F_k^2 \end{pmatrix}\begin{pmatrix} L_{k+1} & L_k \\ L_k & L_{k+1} \end{pmatrix}$.

Evaluate $\prod_{k=1}^{n} F(k)$ as a multiple of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Problem 827. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Let (a_n) be a sequence of positive real numbers such that $\lim_{n\to\infty}\frac{a_n}{n!} = a > 0$. Find

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{a_{n+1}}}-\frac{n^2}{\sqrt[n]{a_n}}\right)$$

Problem 828. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Determine all injective functions $f: \mathbb{R} \to \mathbb{R}$ with $f(0) \neq 1/b$ and

$$f(f(x)y^3) + ax^9y^9 = bf(x^3)f(y^3) \text{ for all } x,y \in \mathbb{R} \text{ where } a>0,\ b>0.$$