THE PENTAGON

A Mathematics Magazine for Students

Volume 59 Number 2 Spring 2000

Contents

Typeset in Scientific Word Printed in the United States of America.

The Pentagon (ISSN 0031-4870) is published semiannually in December and May by Kappa Mu Epsilon. No responsibility is assumed for opinions expressed by individual authors. Papers written by undergraduate mathematics students for undergraduate mathematics students are solicited. Papers written by graduate students or faculty will be considered on a space-available basis. Submissions should be typewritten (double spaced with wide margins) on white paper, standard notation conventions should be respected any special symbols not typed should be carefully inserted by hand in black in. All illustrations must be submitted on separate sheets and drawn in black ink. Computer programs, although best represented by pseudocode in the main text, may be included as an appendix. Graphs, tables or other materials taken form copyrighted works MUST be accompanied by an appropriate release form the copyright holder permitting their further reproduction. Student authors should include the names and addresses of their faculty advisors. Final versions on 3.5 inch disk in Microsoft Word are appreciated. Contributors to *The Problem Corner* or *Kappa Mu Epsilon News* are invited to correspond directly with the appropriate Associate Editor.

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Triangle Tilings on the Hyperbolic Plane

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Presented at the 1999 National Convention and awarded "top four" status by the Awards Committee.

Abstract

How might triangles tile quadrilaterals on the hyperbolic plane? This report summarizes and catalogues the author's research on tilings for which an arbitrary number of triangles may constitute a quadrilateral vertex. Also discussed are the two styles of tilings these triangles may yield: those with infinite possibilities for triangle vertices and those with fixed triangle vertices.

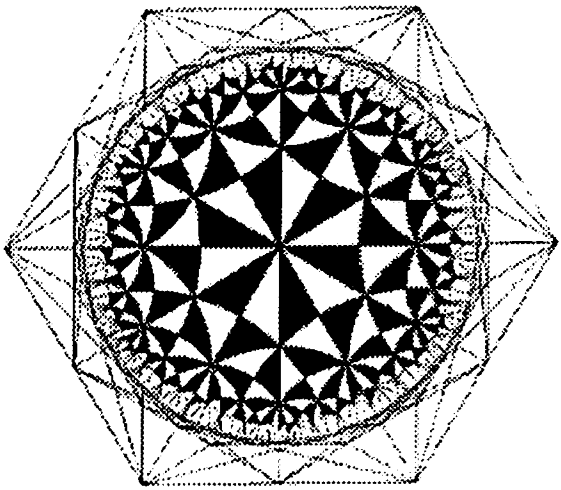
Introduction

This paper provides methods by which one can determine if and how triangles may tile a given quadrilateral in the hyperbolic plane. For each construction, we permit reflections of a single triangle to tile the quadrilateral. We then illustrate a potential method for classifying all cases of free-vertex quadrilaterals.

A Discussion Of The Euclidean And Hyperbolic Planes

Consider a two-dimensional surface, on which we place a quadrilateral. We then reflect this quadrilateral in both horizontal and vertical directions, until the entire plane is covered. This surface is hence called tiled, as a non-overlapping set of quadrilaterals composes it. Now, if we bisect each quadrilateral along its diagonals, our tiles become triangles.

We can easily visualize such a tiling in the Euclidean plane; the quadrilateral angles must sum to 360° and the triangle angles to 180°. We now apply this method of tiling to the hyperbolic plane. The hyperbolic model is depicted either as a unit disc, or the upper half-plane. Lines on the hyperbolic plane are any given diameters that pass through the plane. Below is a unit disc tiled by triangles, as first discovered by HSM Coxeter:



The intersection of two lines in the plane is unique, and the angle of intersection between the two is the same as the Euclidean angle at that point of intersection. Although the lines above appear curved, they are locally straight. Furthermore, although the triangles appear to become smaller as they approach the boundary of the unit disc, all triangles in the hyperbolic plane are congruent if they have the same angle measure.

Research Topics And Main Results

This purpose of this research was to determine triangle tilings of quadrilaterals on the hyperbolic plane. We catalogue these tilings by k, the number of triangles composing each quadrilateral. It was soon discovered that given lower k, both finite and infinite triangle vertex possibilities exist. We then separate triangle tilings into two classifications: those with finite possibilities for a particular tiling style and those of finite possibilities for tilings. The methods used to determine these tilings for lower k are discussed and cataloged. The graphs corresponding to each are illustrated in the appendix. As k increases, we see that no longer may triangle vertices approach infinity.

We call special k the first k yielding no infinite possibilities. A graphical and theoretical proof that special k equals thirteen follows. We then describe methods for determining tilings greater than special k, and classify a few of these cases.

Previous Research In This Area

This research is a continuation of studies by Dawn Haney and Lori McKeough: researchers at the 1997 REU conducted at Rose-Hulman Institute of Technology.

Haney's and McKeough's research satisfied the corner condition which allowed for no more than one triangle to compose each quadrilateral vertex. Similarly, we can extend their discoveries satisfying the corner condition to all triangle tilings greater than special k.

The Hyperbolic Plane And Tilings

Hyperbolic Angles

Although similar to Euclidean geometry, in order to be considered a polygon in the hyperbolic plane, the following requirements for angle sizes exist:

- a. The vertices of a triangle in the hyperbolic plane must sum to less than π, or 180 degrees. The vertices of hyperbolic triangles are denoted by l, m, n where each is the angle size π/l radians, π/m radians, and π/n radians.

  b. The vertices of a quadrilateral in the hyperbolic plane must sum to less than 2π, or 360 degrees. The vertices of hyperbolic quadrilaterals are denoted by s, t, u, v where each is the angle size π/s radians, π/t radians, π/u radians, π/v radians.

For this research, we restrict these polygons so that all l, m, n triangle vertices and s, t, u, v quadrilateral vertices are elements of the integers.

Vertex Sizes For A Quadrilateral

Determining the vertices of a quadrilateral can be done when given the l, m, n vertices of the triangle. When only one triangle exists at a quadrilateral vertex, the corresponding s, t, u, or v is equal to its triangle composition. When more than one triangle exists at a quadrilateral vertex, the vertex is equal to the l, m, or n occurring there divided by the number of triangles, which meets at that particular vertex. Consider a quadrilateral in which two triangle angles of size $\pi/6$ meet at one vertex. $2 \cdot \pi/6 = \pi/3$, so we denote the quadrilateral value as 3.

Finding The Number Of Triangles In A Quadrilateral

Since we only permit one size triangle to be reflected to create a quadrilateral, the simplest method for determining k, the number of triangles within the quadrilateral, is to divide the area of the quadrilateral by the area of the tiling triangle.

The area of a hyperbolic triangle is determined by the formula:

$$At = \pi - \left(\frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n}\right)$$

Similarly, the area of a hyperbolic quadrilateral is determined by the formula:

$$Aq = 2\pi - \left(\frac{\pi}{s} + \frac{\pi}{t} + \frac{\pi}{u} + \frac{\pi}{v}\right)$$

We now define k:

$$k = \frac{Aq}{At} = \left(\frac{2\pi - \left(\frac{\pi}{s} + \frac{\pi}{t} + \frac{\pi}{u} + \frac{\pi}{v}\right)}{\pi - \left(\frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n}\right)}\right)$$

Hence, we can use this equation, knowing the values for the quadrilateral and triangle vertices, to determine that a whole-number k exists.

Lower And Upper Bounds On K

As aforementioned, k is the number of triangles composing a given quadrilateral. Obviously, the lower bound on k is two, for we may reflect a triangle (permitted it is not a right triangle) along any edge and create a quadrilateral.

We now consider a maximum bound of k.

Proposition. *For any given (l, m, n) triangle that successfully tiles a quadrilateral in the hyperbolic plane, $k \leq 60$.*

Proof. Fix l, m, and n. To maximize k we need to maximize the area of the quadrilateral encompassing the triangles. Each integer s, t, u, and v must divide one of l, m, or n. The largest possible quadrilateral is then of the form (b, b, b, b) where be is the largest of l, m, and n. The smallest possible triangle area in the hyperbolic plane is $\pi/42$, for a $(2, 3, 7)$ triangle. Choosing a $(7, 7, 7, 7)$ quadrilateral as suggested above yields $k = 60$. All other triangles in the hyperbolic plane yield an area $\geq \pi/24$, which is the area of a $(2, 3, 8)$ triangle. Here $k \leq 48$. Hence, $2 \leq k \leq 60$.

Two Tiling Types Of Vertices

We classify triangle tilings of quadrilaterals into two types, depending on how the triangles compose the quadrilateral.

1. *Free-Vertices*: A triangle tiling is free if any of the l, m, or n triangle vertices exist only on one or more quadrilateral vertices. See the $k = 4$ tiling of a $(3, 3, 4)$ triangle. Here, the four triangles create a $(3, 2, 3, 2)$ quadrilateral. However, the n value (4) occurs only on a vertex of the quadrilateral. We then determine that since two triangles meet at this vertex, the triangle designation must be some multiple of two, greater than two. Also, since n does not occur on an edge or interior of the quadrilateral, any even $n \geq 4$ will work for a similar tiling.
2. *Fixed Vertices*: A triangle tiling may be considered well-defined, or

fixed, when all three l, m, n angles of the triangle occur either along the edge or interior to the quadrilateral that it tiles. See the $k = 6$ tiling of the $(3, 4, 4)$ triangle. Each vertex of the triangle occurs on an edge of the quadrilateral $(4, 4, 4, 4)$. We needn't consider that the vertices of the quadrilateral can stretch to infinity, and that this is the only $k = 6$ tiling of this particular style.

Note that Haney and McKeough's corner condition implies fixed triangle vertices, for each triangle angle must exist either interior to or on the edge of the quadrilateral.

After studying free and fixed vertices, we find that as k increases, there is a lesser chance that one of the vertices can be free. We now attempt to divide the triangle tilings by this feature.

- **Lower** k contains both free and fixed vertex tilings. Since free vertices exist, infinitely many tilings are possible. However, we can classify these tilings in such a way that all infinite possibilities are accounted for by a set of rules on the l, m, n, vertices.
- **Special** k is the first such k that permits no free vertices of the quadrilateral, and all quadrilaterals tiled by triangles greater than k are fixed. A proof that special $k = 13$ follows.
- **Upper** k contains all tilings greater than special k. These tilings are all fixed vertices.

We now wish to classify all tilings of lower k.

Methods Of Finding Lower k

The three methods below are all applicable for assisting in determining triangle tilings of quadrilaterals for a particular k. The benefits and hindrances for each method follow.

1. **Computer Search for k:** When given a particular pair of triangle and quadrilateral vertices, we then solve to see if k exists for this pair.
2. **Dual Graphs and Circle Diagrams:** A more effective method, creating dual graphs and circle diagrams allow for potential k's to be graphically determined.
3. **Triangulation of given n-gons:** To triangulate an n-gon in the Euclidean plane, we may create a hyperbolic quadrilateral with four of the n vertices of the n-gon.

Computer Search For k

Since k can be determined algebraically, we can create a program which generates all combinations of triangle vertices l, m, n and s, t, u, v and then evaluate k. If k is an integer, then it appears that such a hyperbolic tiling exists. Although this can be done for any given vertices in the hyperbolic plane, we restricted our angles to ≤ 10, but this yielded over 750 potential tilings.

Output failing to yield a tiling

For the massive amount of output, it is obvious many solutions for k are not applicable. Consider a $(2, 3, 8)$ triangle and a $(3, 3, 4, 4)$ quadrilateral. Solving for k:

$$k = \frac{2 - \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4}\right)}{1 - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{8}\right)} = \frac{\frac{1}{3}}{\frac{1}{24}} = 8$$

Yet, we cannot actually create this tiling in the hyperbolic plane.

Output for a successful tiling

Fortunately, a few of the pieces did make successful tilings. One such is $(2, 3, 8)$ triangle on a $(4, 2, 4, 2)$ quadrilateral:

$$k = \frac{2 - \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{2}\right)}{1 - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{8}\right)} = \frac{\frac{1}{2}}{\frac{1}{24}} = 12$$

This tiling is in the $k = 12$ section of the appendix.

Determining hubs as an algorithm refinement

Since not all (l, m, n) triples and (s, t, u, v) quadrilaterals permit a successful tiling, we refine our algorithm to consider hubs. A hub is any point on the edge of or interior to the quadrilateral where the vertices of the triangles meet. Hubs have a value of 1 if they occur on a quadrilateral's edge, 2 if on the interior. The total number of hubs for a particular triangle tiling can be determined by:

$$hb = \left(\frac{K(xb) - cb}{b}\right)$$

Where xb is the number of b angles in the triangle and cb is the number of quadrilateral corners of size b. If more than one triangle composes a corner of the quadrilateral, a fractional hub value will be computed. This value is $1/w$, where w is the s, t, u, or v value at that vertex. When adding these hub values to the algorithm, we can alleviate many of the potential k's which unsuccessfully computed correct tiling styles of triangles in the quadrilateral.

Creating dual graphs

As the above algorithm was successful, but not restrictive enough in determining k, we then attempt to draw the quadrilateral by the k triangles. The *MatLab* program that drew all tilings in the appendix assisted in this task. The program allows for a given (l, m, n) triangle to be created on the hyperbolic plane, and the user then determines which path the reflections of this tile may take on the plane. This program accounts for a few of the triangle tilings of quadrilaterals found, but the necessary step of determining restrictions on free vertices in these diagrams was not considered

We then attempt creating dual graphs for each k value. The dual graphs are essentially all tree graphs of k-vertices, or combinations of cycles and tree graphs for k-vertices. After making the tree/cycle graph, we draw the necessary triangles around the graph, attempting to shape the triangles into a quadrilateral like form.

For extremely small k, this was a nice choice. We could analyze the triangle vertices, and determine sizes and restrictions. As k increased, this option was not satisfactory. When we drew one triangle around a vertex, we had one, two, or three choices of where to reflect the triangle next, depending on the degree of the vertex. Thus, there was up to $\frac{2^k}{4}$ possible triangle tilings for a given k. We attempt a new method.

Creating all possible Tree Diagrams by Computer

A *Mathematica* program created by Mr. Robert M. Dickau, when given a particular k, generates all possible tilings of a polygon with $k+2$ edges. We can then take these drawings and attempt to for a quadrilateral out of this polygon. The sum total of triangle tilings for a given k is equal to the Catalan number corresponding to k. We determine the Catalan number for a given n with the formula:

$$\left(\frac{1}{n+1}\right)\binom{2n}{n}$$

Thus, for a $k = 7$, there are 429 resulting graphs!

We now attempt to refine Dickau's algorithm to eliminate rotational and reflective symmetries for the given polygon, as they result in the same quadrilaterals. This is done by listing the connected vertices as ordered pairs, and creating the $n \times n$ matrix with these pairs. We then create permutations on the columns of an $n \times n$ identity matrix. If we multiply IMI', where I is some permutation of the identity matrix, M is the matrix of the ordered pairs, and I' is the inverse of the permuted identity matrix, we determine a rotated graph of M. We then delete these rotations.

Furthermore, instead of finding a particular polygon's triangulation, we can recursively determine the ordered pairs. Beginning with $k = 2$, we define the ordered pair $(1, 3)$ to be the only possibility of dividing the $(k+2)$ polygon (quadrilateral). Note, $(2, 4)$ works as well, but is a rotation of $(1, 3)$. We then go through and append a vertex to each edge of the polygon between existing vertices. For example, we first place a new vertex between vertices 1 and 2. All points greater than 2 are then incremented by 1, and we append the ordered pair $(1, 3)$ to the set of points. We continue adding vertices between $(2, 3), (3, 4)$, for all edges of the polygon. Finally, the new algorithm eliminates duplicates of sets of ordered pairs.

The aforementioned method was successful for up to $k = 7$ rather quickly, reducing 429 ordered pairs to only 49. Furthermore, cases for $k = 8$ were found rather efficiently, finding only 150 cases as opposed to all 1430. Currently, all cases up to $k = 8$ are found.

Special K Conjecture

Support: We begin by considering that for a large k, we wish to have small triangles. We consider triangles $(2, 3, d), (3, 3, d)$, etc. where $d \longrightarrow \infty$. This allows for some multiple of the d's to exist in the quadrilateral vertices. We then consider the quadrilateral (q, q, q, q) where q is some combination of the d's (perhaps $d/2$ or $d/3$). Since q is approaching infinity as well, the area of the quadrilateral is 2π, as all $1/q$'s $\longrightarrow 0$. We find the smallest possible triangle with the above stipulations is the $(2, 3, d)$ triangle, and we now find k:

$$k = \frac{2 - \left(\frac{1}{q} + \frac{1}{q} + \frac{1}{q} + \frac{1}{q}\right)}{1 - \left(1\frac{1}{2} + \frac{1}{3} + \frac{1}{d}\right)} = \frac{2}{\frac{1}{6}} = 12$$

Hence, the largest k which supports free vertices is $k = 12$, and special $k = 13$.

Graphically, we have found much support for this conjecture. Graphs for $k = 12$ with free vertices do exist, and no found graphs above $k = 12$ permit free vertices.

Upper K

Since upper k contains all fixed vertex quadrilaterals, we can consider the algorithm used by Haney/McKeough to determine tilings greater than special k. The algorithm takes a master tile triangle and then determines a reflected path of the triangle along a given oval. When this path reaches a proposed quadrilateral corner, it then begins its reflected walk in a new direction. The program is designed to allow only one triangle in each of the quadrilateral vertices. A modification of this algorithm permitting more than one triangle to meet in a quadrilateral vertex would allow for classification of upper k.

Further Studies

- Algorithm refinement for the cases of lower k. By creating a more efficient program, we can successfully compute all lower k, thus finding missed elements in $9 \leq k \leq 12$.
- Classifying upper k by working with methods similar to Haney and McKeough's, by eliminating the corner condition from their program, we can classify all upper k with 2, 3, etc. vertices meeting in each quadrilateral corner.

Acknowledgements. The author wishes to thank Professor Allen Broughton (Rose-Hulman Institute of Technology), Ryan Derby-Talbot (Pomona College), Robert Dirks (Wabash College), and Maria Sloughter (Carleton College) for all help and knowledge during the NSF-REU at Rose-Hulman Institute of Technology for the summer of 1998. She further wishes to thank Mr. Robert M. Dickau for the usage and manipulation of his Mathematica program on triangle tilings for any given polygon size. All numeric and Catalan calculations were determined with Mathematica. All hyperbolic tilings were created with MatLab.

References

1. Dickau, Robert M. "Triangle Graphs and Catalan Numbers." http://forum.swarthmore.edu/advanced/robertd/catalan.html
2. Haney, Dawn and McKeough, Lori. "Quadrilaterals Subdivided by Triangles in the Hyperbolic Plane." NSF grant DMS-9619714.

Appendix A

Catalogue of Divisible quadrilaterals with free vertices:

K	l,m,n	Stipulations	s,t,u,v
2	$(l,2m,2n)$	$l,m \geq 4, n \geq 3$	(l,m,l,n)
3	$(2,3m,2n)$	$- = -$	$(2,n,2n,m)$
3	$(3,2m,2n)$	$m,n \geq 3$	$(2m,n,m,2n)$
4	$(2,3m,3n)$	$m,n \geq 2$	$(2m,m,2n,n)$
4	$(3,3,2n)$	$n \geq 2$	$(3,n,3,n)$
4	$(2,2m,2n)$	$m \geq 4, n \geq 2$	(m,n,m,n)
4	$(2,4m,2n)$	$m,n \geq 2$	$(2n,n,2n,m)$
4	$(2,4,2n)$	$n \geq 3$	$(2,n,n,2)$
4	$(3,3,3n)$	$n \geq 2$	$(3,3n,3,n)$
5	$(2,5,2n)$	$n \geq 2$	$(2n,2,n,n)$
5	$(2,4,6n)$	$- = -$	$(4,2n,3n,2)$
6	$(2,3,4n)$	$n \geq 2$	$(2,2n,2,n)$
6	$(2,4,3n)$	$n \geq 2$	$(4,4,n,n)$
6	$(2,3,2n)$	$n \geq 3$	$(2,n,2,n)$
6	$(2,6,2n)$	$n \geq 2$	$(2n,2n,n,n)$
6	$(2,5,6n)$	$- = -$	$(5,6n,3n,2n)$
6	$(2,4,6n)$	$- = -$	$(4,2n,6n,3n)$
7	$(2,4,12n)$	$- = -$	$(2,4n,3,3n)$
7	$(2,3,10n)$	$- = -$	$(2,2n,3,5n)$
8	$(2,4,2n)$	$n \geq 3$	$(3n,3n,3n,3n)$
8	$(2,3,4n)$	$n \geq 2$	$(3,n,3,n)$
8	$(2,4,3n)$	$n \geq 2$	$(3n,n,3n,n)$
8	$(2,4,4n)$	$n \geq 2$	$(4n,2n,4n,n)$
8	$(2,3,2n)$	$n \geq 6$	$(n,3,3n,3)$
9	$(2,3,15n)$	$- = -$	$(3n,n,5n,2)$
9	$(2,3,12n)$	$- = -$	$(2,4n,6n,3n)$
10	$(2,3,20n)$	$- = -$	$(3,6n,15n,10n)$
12	$(2,3,8n)$	$- = -$	$(4n,2n,4n,2n)$
12	$(2,3,4n)$	$n \geq 2$	$(2n,n,2n,n)$
12	$(2,3,12n)$	$- = -$	$(2n,12n,4n,6n)$

Note: table is incomplete for $k \geq 9$.

Appendix B

Catalogue of divisible tilings for quadrilaterals with fixed vertices

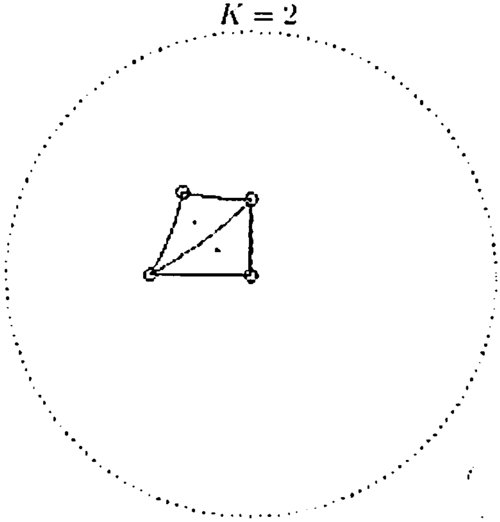
k	l, m, n	s, t, u, v
6	$(3,4,4)$	$(4,4,4,4)$
7	$(3,4,4)$	$(3,3,4,4)$
7	$(3,3,5)$	$(3,3,5,5)$
10	$(2,4,5)$	$(2,4,2,4)$
10	$(2,4,5)$	$(2,2,4,4)$
10	$(3,3,4)$	$(3,3,3,3)$
12	$(3,3,4)$	$(4,4,4,4)$
12	$(2,5,5)$	$(5,5,5,5)$
12	$(2,4,6)$	$(4,4,4,4)$
12	$(2,4,6)$	$(3,3,6,6)$
12	$(2,3,8)$	$(2,2,4,4)$

Note: table is incomplete for $k \geq 9$.

Appendix C

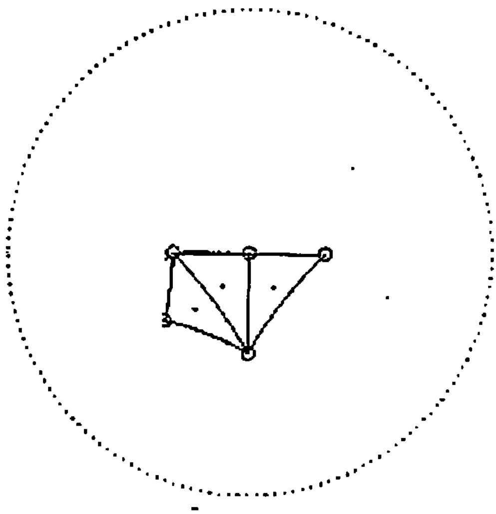
An Example of a Tiling For Each K (2 through 12)

$K = 2$



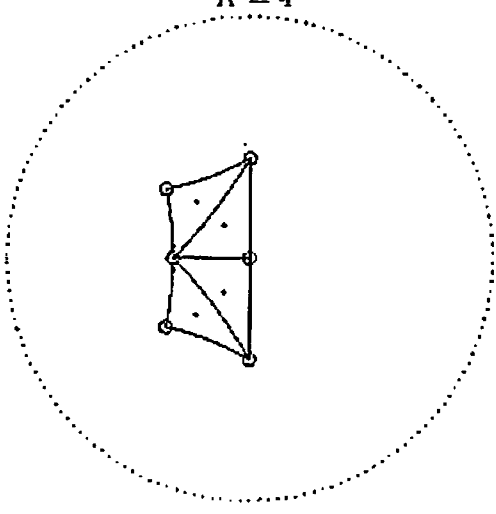
A $(2,6,4)$ triangle with a $(2,3,2,2)$ quadrilateral.

$K = 3$



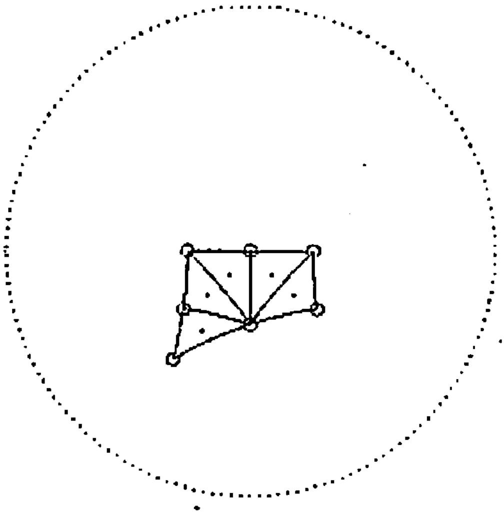
A $(2, 4, 6)$ triangle with a $(2, 2, 4, 2)$ quadrilateral.

$K = 4$



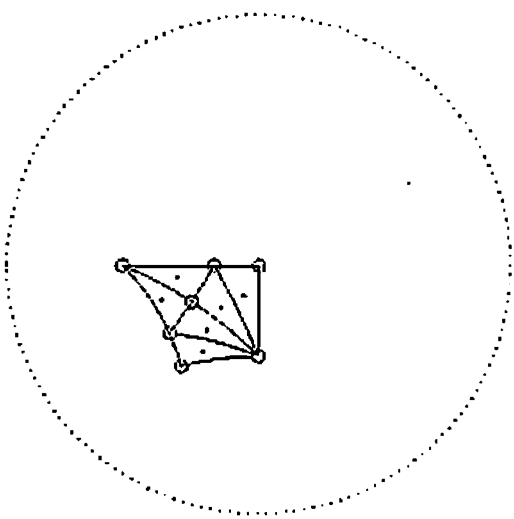
A $(2, 4, 6)$ triangle with a $(2, 3, 3, 2)$ quadrilateral.

$K = 5$



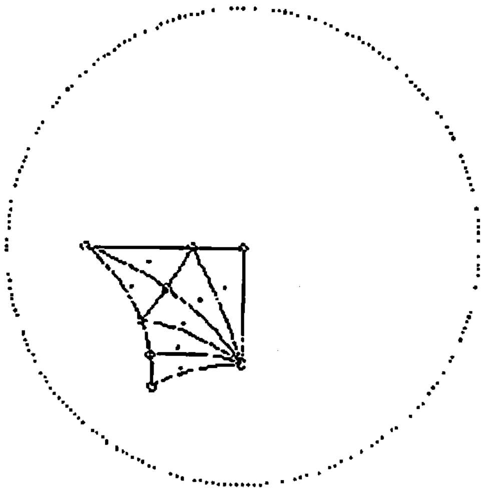
A (2, 4, 5) triangle with a (4, 2, 2, 2) quadrilateral.

$K = 6$



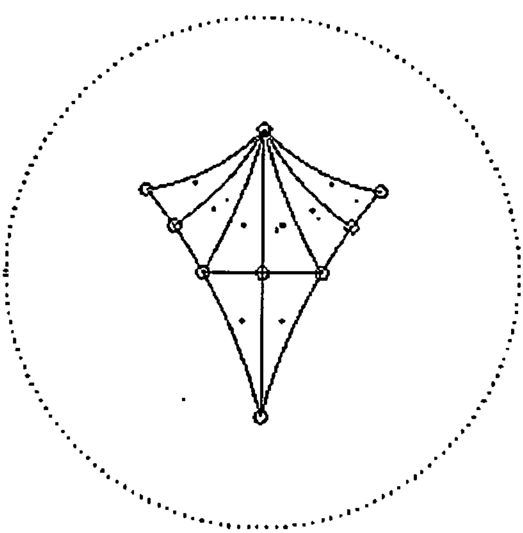
A (2, 3, 8) triangle with a (2, 4, 2, 2) quadrilateral.

$K = 7$



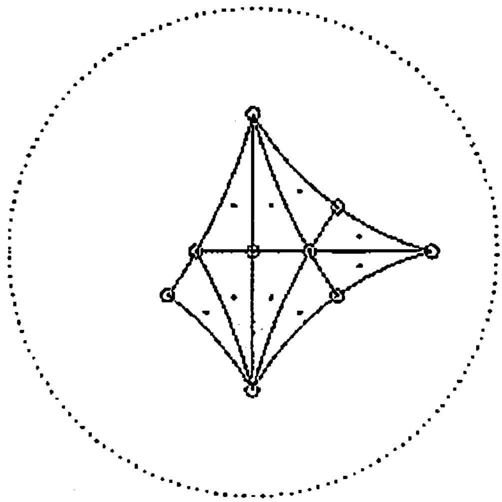
A $(2,3,10)$ triangle with a $(2,2,3,5)$ quadrilateral.

$K = 8$



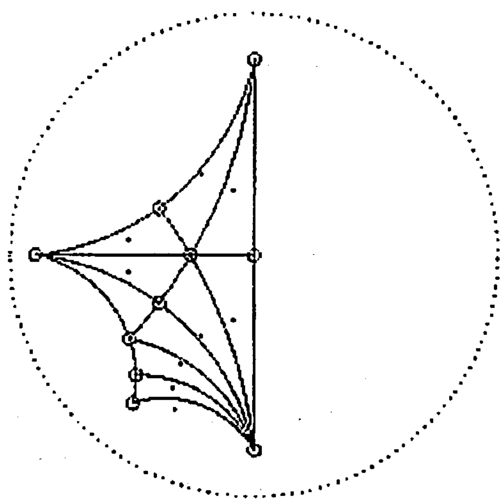
A $(2,3,12)$ triangle with a $(6,3,2,3)$ quadrilateral.

$K = 9$

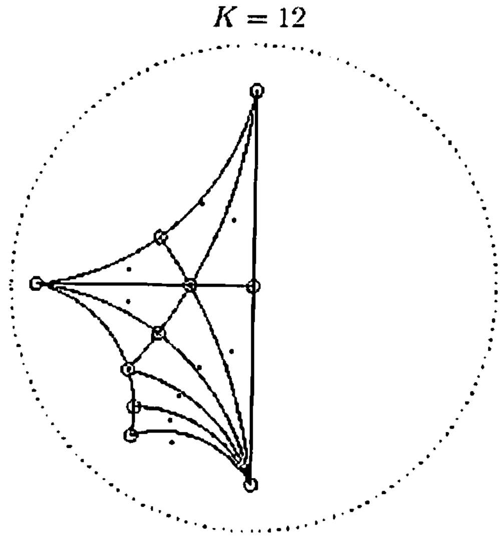


A $(2, 3, 12)$ triangle with a $(2, 4, 6, 3)$ quadrilateral.

$K = 10$



A $(2, 3, 30)$ triangle with a $(3, 6, 15, 10)$ quadrilateral.



A (2, 3, 12) triangle with a (2, 12, 4, 6) quadrilateral

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In the Spirit of Klein: Making Connections Between the Mobius and Lorentz Groups

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Presented at the 1999 National Convention

Introduction

As mathematicians of the 19^{th} century realized, the classical geometries-Euclidean, hyperbolic, and elliptic-lack a coherent unified structure when viewed *axiomatically*. The unification was achieved through Felix Klein's Erlanger Programme of 1872, in which a radical departure was proposed. Klein elevated the transformation group of a given set to primary status, defining geometrical objects as entities which are preserved under selected subgroups of the transformation group and terming them *invariants* (Greenberg, 1993).

Klein's unified approach will be utilized in this paper as it applies to the classical linear groups. By taking an n-dimensional inner product space E, and then looking at the orthogonal group $O(E)$ of all linear transformations from E to E which preserve the inner product, one can establish a correspondence between selected subgroups of $O(E)$ and the geometrical objects which appear as invariants under these subgroups. Furthermore, inner products of an indefinite signature are considered under the orthogonal groups. It is in this way the classical geometries are viewed as invariants in the spirit of Klein. The central theme of this paper highlights the case in which a connection exists between two specific transformation groups, the classical group of Mobius transformations and the Lorentz group of Einstein's Special Relativity.

The Geometry of the Programme

Before examining applications of Klein's geometrical approach, we must first define the guiding principles of it. We first look at an n-dimensional vector space and an associated inner product. Now we are interested in *nondegenerate* inner products. These inner products have the following property in a vector space V:

$$\text{If } \forall \vec{x} \in V, \langle \vec{x}, \vec{y} \rangle = 0, \text{ then } \vec{y} = 0.$$

Nondegenerate inner products can further be classified as positive definite or indefinite. We define these inner products by first considering an arbitrary vector $\vec{v}$ and then examining the inner product with itself. If we have a positive definite inner product, we obtain $\langle \vec{v}, \vec{v} \rangle \geq 0$ for all $\vec{v} \in V$ with equality only when $\vec{v} = 0$. The indefinite inner product is more complex. When taking the inner product of an arbitrary vector $\vec{v}$ with itself, three possible cases exist: $\langle \vec{v}, \vec{v} \rangle > 0, \langle \vec{v}, \vec{v} \rangle = 0, \langle \vec{v}, \vec{v} \rangle < 0$. Physicists denote these cases as timelike, null, and spacelike, respectively, reflecting the physical significance each has in Einstein's Special Relativity Theory. The significance of the distinction between the positive definite and indefinite cases is this: a positive definite inner product will provide a vector space E with a Euclidean structure while non-Euclidean geometries arise from indefinite metrics that are restricted to suitable subsets of the parent vector space. We now have the blueprint for the Kleinian approach to geometry and can apply the concepts discussed to meaningful examples.

Let us first consider a typical map of geometrical significance, a transformation $\sigma : R^2 \to R^2$, where σ is the reflection about a line m. We observe three things about σ: (i) the only invariant set under σ is m itself, (ii) $\sigma \neq Id$, but (iii) $\sigma^2 = Id$. Klein's approach elevates these observations to the status of axioms. Given a two-dimensional space E and a group G of linear isomorphisms $\sigma : E \to E$, we may define a "line" in E as the *invariant* set under an involution, i.e. under a map $\sigma \in G$ such that $\sigma \neq Id$ and $\sigma^2 = Id$.

The proceeding example may be generalized to higher dimensions. Let E be an n-dimensional positive definite inner product space (thus $\langle \vec{v}, \vec{v} \rangle > 0 \forall \vec{v} \neq \overrightarrow{0}$). Let $\vec{v} \in E, \vec{v} \neq \overrightarrow{0}$, define a transformation $\tau\vec{v}$ of E by the following formula:

$$\tau\vec{v}(\vec{x}) = \overrightarrow{x} - 2\frac{\langle \vec{x}, \vec{v} \rangle}{\langle \vec{v}, \vec{v} \rangle}\vec{v}; \vec{x} \in E$$

Now this transformation $\tau_{\vec{v}}$ may be seen to represent a reflection through the $(n-1)$-dimensional hyperplane that is orthogonal to the vector $\vec{v}$ and passes through the origin. Thus the vector $\vec{v}$ itself is mapped under the transformation by $\tau\vec{v}(\vec{v}) = -\vec{v}$. The transformation is illustrated in Figure 1 letting $E = R^3$ with the standard inner (or dot) product.

The invariant set of this transformation is the plane orthogonal to $\vec{v}$, once again demonstrating the spirit of Klein's approach to geometry. Klein would define an affine hyperplane in the n-dimensional space as an invariant set under an involution ($\sigma^2 = Id$ but $\sigma \neq Id$) as we have seen in our transformation in R^2 earlier.

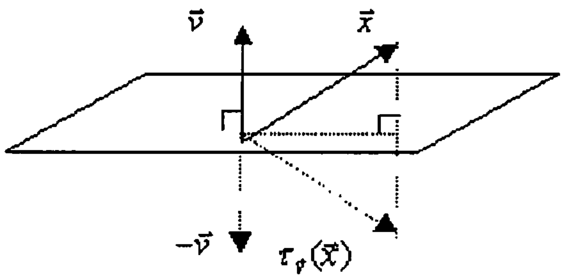

Figure 1

Orthogonal Transformations

Let us now look at the finite dimensional vector space E more closely. We may associate with E an inner product of arbitrary signature. The inner product is written in the form $\langle \vec{x}, \vec{y} \rangle$ and has two important classifications: (i) the positive definite case, with signature $(+, +, \ldots, +)$, and (ii) the indefinite case, with signature $(-, -, \ldots, -, +, +, \ldots +)$. Given a nondegenerate inner product space $(E, \langle , \rangle)$, the automorphism group $O(E)$, also called the orthogonal group, is defined to be the group of all nonsingular linear maps $\sigma : E \to E$ which preserve the inner product. The orthogonal transformations in $O(E)$ also preserve angles and distances since these geometric notions are derived from the inner product. Since the orthogonal group consists of linear transformations, a choice of basis B for E permits the identification of each $\sigma \in O(E)$ with the matrix $[\sigma]_B$ that represents σ with respect to B. The following important result follows:

Theorem. *Let E be a vector space with associated inner product. Then an orthogonal transformation $\sigma \in O(E)$ has determinant 1 or -1* (Iversen, 1992, p. 4).

In Euclidean three-dimensional space ($E = R^3$), this result allows us to distinguish two important categories of orthogonal transformations. For an arbitrary line L through the origin, let L^* denote the subspace of E consisting of all vectors perpendicular to L. According to a proposition of Euler, an orthogonal transformation σ with determinant 1 is a rotation, where a line L is fixed by σ, and the transformation induced by σ in the plane L^* is a rotation. In contrast, an orthogonal transformation σ with determinant -1 has the form $\pi \circ \theta$, where π is a rotation as in the previous case (with a line L as the axis), and θ is reflection in the plane L^* (Iversen, 1992, p. 14). Again, this way of "classifying" the group of orthogonal transformations of R^3 exhibits the incorporation of a Kleinian approach to transformations of R^3 as the rotations and reflections preserve inner products.

These results can now be applied to more non-traditional vector spaces. Let us consider a vector space V over R with dimension $n+1$. The vector space V may be provided with a nondegenerate inner product of signature type $(-n, +1)$; simply choose a basis B for V, and define, for each $\vec{v}, \vec{w} \in V, \langle \vec{v}, \vec{w} \rangle = -v_1 w_1 - v_2 w_2 - \cdots - v_n w_n + v_{n+1} w_{n+1}$. Again, the orthogonal group $O(V)$ is of primary interest, particularly its actions on the hyperboloid (or pseudosphere). The *pseudosphere* consists of all vectors $\vec{v} \in V$ such that $\langle \vec{v}, \vec{v} \rangle = 1$. In the case where V has dimension 3, the pseudosphere appears as in Figure 2.

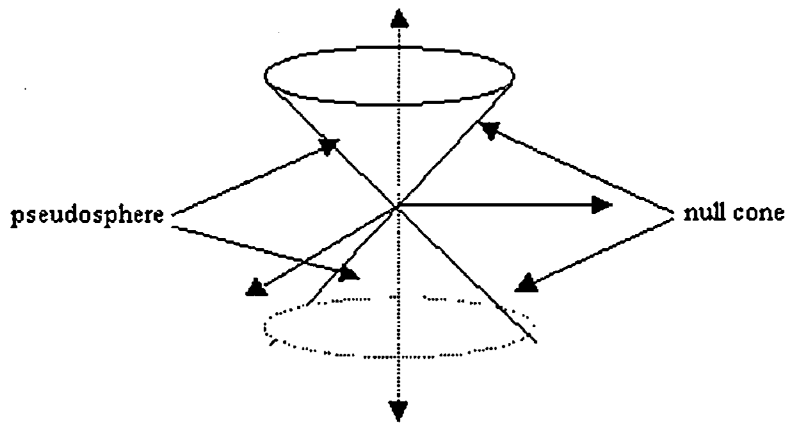

Figure 2

The following result is basic.

Theorem. *The hyperboloid associated with V (denoted $S(V)$) has two connected components. Two points $\vec{x}, \vec{y} \in S(V)$ are in the same connected component if and only if* $\langle \vec{x}, \vec{y} \rangle 0$. (Iversen, 1992, p. 24).

We can now define a Lorentz transformation belonging to the Lorentz group of Einstein's Special Relativity. Using the vector space V from above, a Lorentz transformation is an orthogonal transformation $\sigma \in O(V)$ that preserves the connected components of the pseudosphere. The Lorentz group will be denoted $Lor(V)$. Using the theorem above, a Lorentz transformation σ satisfies $(\langle \vec{x}, \sigma(\vec{x}) \rangle) > 0$ for all . By considering the analogy with rotations of the unit sphere in Euclidean space, we see that we may conceive of a Lorentz transformation as a type of "rotation" of the pseudosphere. A basic example of a Lorentz transformation is provided by reflection along a spacelike vector $\vec{v} \in V$ with $\langle \vec{v}, \vec{v} \rangle = -1$:

$$\tau_{\vec{v}} = +2\langle \vec{x}, \vec{v} \rangle \vec{v}$$

We see that $\langle \tau_{\vec{v}}(\vec{x}), \vec{x}\rangle = \langle \vec{x}, \vec{x}\rangle + 2\langle \vec{x}, \vec{v}\rangle\langle \vec{v}, \vec{x}\rangle = 1 + 2\langle \vec{x}, \vec{v}\rangle^2 0$ for all $\vec{x} \in S(V)$ in accordance with the above-mentioned inequality. It may be shown that any Lorentz transformation σ of an $(n + 1)$-dimensional vector space with an inner product of signature $(-n, 1)$ may be written as the product of at most $(n + 1)$ reflections $\tau_{\vec{v}}$ of this type (Iversen 1992, p. 27). A Lorentz transformation is called even if its determinant is $+1$; the set of all even Lorentz transformations constitute the special Lorentz group $Lor^+(V)$, a subgroup of $Lor(V)$ of index 2 (Iversen, 1992).

There is physical significance to these terms. In Special Relativity, a Lorentz transformation relates space-time observations between two observers (or frames). Since the components of $S(V)$ are preserved, the Lorentz transformations preserve the same sense of time -"future" and "past"- for all observers. The condition $\det(\sigma) = +1$ ensures that two observers related by σ enjoy the same sense with respect to spatial orientation.

Another specific transformation group, the classical group of Mobius transformations, is pertinent to our investigation. To begin, let us consider a Euclidean n-space E; this is an n-dimensional real vector space E provided with a positive definite, nondegenerate inner product. Let C be a sphere of radius $r > 0$ with center $\vec{c}$. We then have a transformation σ defined by

$$\sigma(\vec{x}) = \vec{c} + r^2 \frac{\vec{x} - \vec{c}}{\|\vec{x} - \vec{c}\|}; \quad \vec{x} \in E - \{\vec{c}\}$$

The transformation σ is called *inversion in the sphere C*. The form of the equation defining σ shows that $\sigma(\vec{x})$ lies somewhere on the ray which emanates from $\vec{c}$ and passes through $\vec{x}$. If $\vec{x}$ lies in the interior of C, then $\sigma(\vec{x})$ will lie in the exterior, and vice versa. Moreover, the sphere C appears as the invariant set of σ; $C = \{\vec{x} \in E \mid \sigma(\vec{x}) = \vec{x}\}$ (see Figure 3).

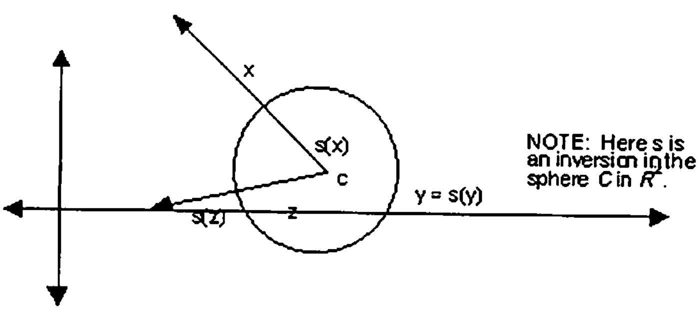

Figure 3

Next, consider an affine hyperplane H in E that passes through the point $\vec{x} \in E$. If $\vec{n}$ is a unit normal to H, the *Euclidean reflection* through H may be represented by the formula

$$\lambda(\vec{y}) = \vec{y} - 2\langle \vec{y} - \vec{x}, \vec{n} \rangle \vec{n}; \quad \vec{y} \in E$$

Observe, for example, that if $\vec{y} \in H$, then $\vec{y} - \vec{x}$ is parallel to H and hence $\langle \vec{y} - \vec{x}, \vec{n} \rangle = 0$; thus $\lambda(\vec{y}) = \vec{y}$ for all $\vec{y} \in H$, showing H is an invariant set under λ.

The Mobius group provides a unification of inversions through spheres and reflections through affine hyperplanes. This is achieved through considering the one-point compactification $E^* = E \cup \{\infty\}$, where the point at infinity is added to the original Euclidean space E. We see that the formula for an inversion σ through a sphere C may be extended to an involution of E^*, also denoted by σ, which interchanges the center $\vec{c}$ of C and ∞. Similarly, the reflection λ through the hyperplane H may be extended to an involution λ on E^* that preserves (or fixes) the point at infinity.

We now can introduce the word "sphere" to refer to either a Euclidean sphere in E or a subset of E^* of the form $H \cup \{\infty\}$ where H is an affine hyperplane in E. Now the two types of transformations of E^* discussed above may both be called *inversions in "spheres"* in E^*. We can define the subgroup of all homeomorphisms of E^* generated by inversions in "spheres" as the Mobius group, $Mob(E)$.

The Connection between the Mobius and Lorentz Groups

As stated in the introduction, we can establish a connection between the Mobius groups and the Lorentz groups previously discussed. We require two preliminary definitions.

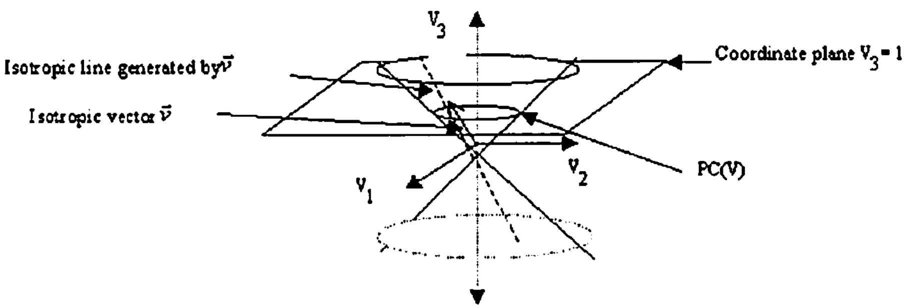

Figure 4

First, consider once again our vector space V with indefinite inner product of signature $(-, ..., -, +)$. The isotropic cone (or null cone to mathematical physicists) $C(V)$ is the set of all vectors $\vec{x}$ such that $\langle \vec{x}, \vec{x} \rangle = 0$. An isotropic line is merely any line spanned by an isotropic vector. Following the ideas of classical projective geometry, the space whose "points" consist of isotropic lines in V is called the projectivised cone $PC(V)$. $PC(V)$ may be visualized by considering the intersection of the isotropic cone with the hyperplane $\vec{v}_n = 1$. This visual model is shown in Figure 4 above for the case $dim(V) = 3$.

Secondly, we must define a new space $E \oplus R^2$ that contains a copy of Euclidean space E embedded in an ambient space of indefinite signature. Let $(E, <>)$ be Euclidean n-space, and denote by $E \oplus R^2$ the direct sum of E with R^2. We will associate with this space an indefinite inner product of signature $(-(n+1), +1)$ defined as follows:

$$\langle (\vec{x}, a, b), (\vec{y}, c, d) \rangle = -\langle \vec{x}, \vec{y} \rangle + \frac{1}{2}(ad + bc);\ \vec{x}, \vec{y} \in E \text{ and } a, b, c, d \in R$$

Observe that $\left\langle (\vec{0}, 1, 0), (\vec{0}, 1, 0) \right\rangle = \left\langle (\vec{0}, 0, 1), (\vec{0}, 0, 1) \right\rangle = 0$, so the a-axis (spanned by $\left(\vec{0}, 1, 0\right)$) and b-axis (spanned by $\left(\vec{0}, 0, 1\right)$) in R^2 are both isotropic. Also, $\left\langle (\vec{0}, 1, 1), (\vec{0}, 1, 1) \right\rangle = +1$ and $\left\langle (\vec{0}, -1, 1), (\vec{0}, -1, 1) \right\rangle = -1$. Thus, a "timelike" axis in $E \oplus R^2$ is spanned by the vector $(\vec{0}, 1, 1)$ while $(\vec{0}, -1, 1)$ spans the spacelike axis. Let us visualize the projectivised cone $PC\left(E \oplus R^2\right)$ by intersecting the isotropic cone with the hyperplane $b = 1$ in (E, a, b)-coordinates. This is shown in Figure 5 where we depict E as one-dimensional.

Let us consider the transformation $f : E \longrightarrow PC\left(E \oplus R^2\right)$ defined by $f(\vec{e}) = (\vec{e}, \langle \vec{e}, \vec{e} \rangle, 1), \vec{e} \in E$. Observe that $\langle f(\vec{e}), f(\vec{e}) \rangle = -\langle \vec{e}, \vec{e} \rangle + \langle \vec{e}, \vec{e} \rangle = 0$ for all $\vec{e} \in E$, so the image of E under f is contained in $PC\left(E \oplus R^2\right)$ as claimed. Moreover, if $(\vec{e}, a, b)$ is any isotropic vector with $b \neq 0$, then the point $\left(\frac{1}{b}\vec{e}, \frac{a}{b}, 1\right)$ in $PC\left(E \oplus R^2\right)$ represents the null direction given by $(\vec{e}, a, b)$; since $f\left(\frac{1}{b}\vec{e}\right) = \left(\frac{1}{b}\vec{e}, \frac{\langle \vec{e}, \vec{e} \rangle}{b^2}, 1\right) = \left(\frac{1}{b}\vec{e}, \frac{a}{b}, 1\right)$, the transformation f identifies E with the complement of the point of $PC\left(E \oplus R^2\right)$ that represents the line through $\left(\vec{0}, 1, 0\right)$ The map f may be extended to a bijection $f : E^* \longrightarrow PC\left(E \oplus R^2\right)$ where $f(\infty)$ is the line through the point $\left(\vec{0}, 1, 0\right)$.

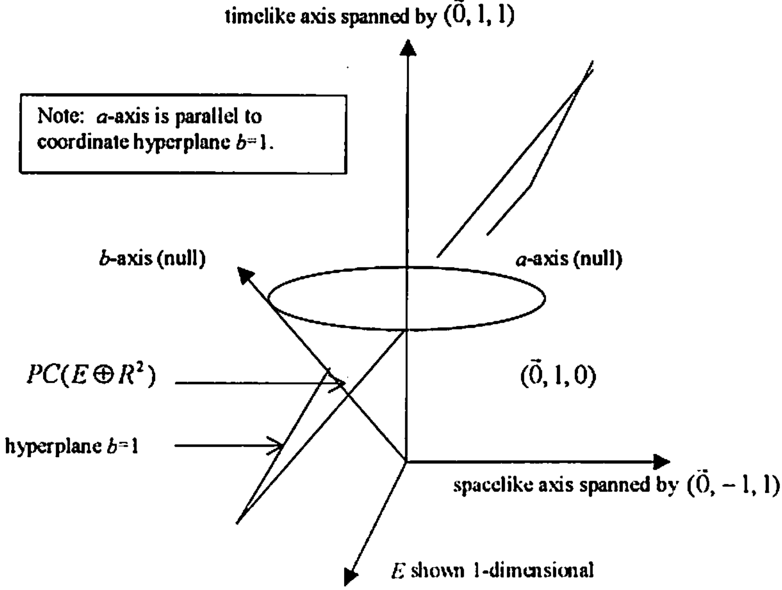

Figure 5

The connection between the Mobius group and the Lorentz group now appears through the following theorem.

Theorem. The action of the Lorentz group $Lor\left(E \oplus R^2\right)$ on the projectivised cone $E \oplus R^2$ and the homeomorphism $f : E^* \longrightarrow PC\left(E \oplus R^2\right)$ identifies the Mobius group $Mob(E)$ with the Lorentz group $Lor\left(E \oplus R^2\right)$.

Sketch of Proof. We consider the maps

$$E^* \xrightarrow{f} PC\left(E \oplus R^2\right) \overset{Lor(E\oplus R^2)}{\longrightarrow} PC\left(E \oplus R^2\right)$$

$$\vec{e} \longrightarrow (\vec{e}, \langle \vec{e}, \vec{e} \rangle, 1) \overset{Lor(E\oplus R^2)}{\longrightarrow} \tau\left(\vec{e}, \langle \vec{e}, \vec{e} \rangle, 1\right)$$

Consider a Euclidean sphere C in E^* with center $\vec{c}$ and radius $r > 0$. Let $\sigma : E^* \longrightarrow E^*$ be inversion through C:

$$\sigma(\vec{e}) = \vec{c} + r^2 \frac{\vec{e} - \vec{c}}{\|\vec{e} - \vec{c}\|^2}; \quad \vec{e} \in E$$

Form the vector $\vec{C} = \left(\vec{c}, \langle \vec{c}, \vec{c} \rangle - r^2, 1\right)$ in $\left(E \oplus R^2\right)$. Direct computation shows $\left\langle \vec{C}, \vec{C} \right\rangle = -r^2$, so $\vec{C}$ is spacelike. Now let $\tau_{\vec{C}} : PC\left(E \oplus R^2\right) \longrightarrow PC\left(E \oplus R^2\right)$ be the Lorentz "reflection" along $\vec{C}$: $\tau_{\vec{C}}\left(\vec{e}, \langle \vec{e}, \vec{e} \rangle, 1\right) = \left(\vec{e}, \langle \vec{e}, \vec{e} \rangle, 1\right) + 2\left(\left\langle \left(\vec{e}, \langle \vec{e}, \vec{e} \rangle, 1\right), \frac{\vec{C}}{r} \right\rangle\right) \frac{\vec{C}}{r}$. Now direct but tedious computation shows that $f(\sigma(\vec{e})) = \frac{r^2}{\|\vec{e}-\vec{c}\|^2} \tau_{\vec{C}}\left(f(\vec{e})\right) \, \forall \vec{e} \in E$. Thus the following diagram commutes:

$$\begin{array}{ccc} E^* & \xrightarrow{f} & PC\left(E \oplus R^2\right) \\ \downarrow_{\sigma} & & \downarrow_{\tau_{\vec{C}}} \\ E^* & \xrightarrow{f} & PC\left(E \oplus R^2\right) \end{array}$$

So each inversion in E^* is directly identified with a Lorentz "reflection" $\tau_{\vec{C}}$ in $PC\left(E \oplus R^2\right)$. A similar proof may be given to associate each reflection λ through an affine hyperplane in $E*$ with a Lorentz reflection τ_N acting on $PC\left(E \oplus R^2\right)$. This completes the proof.

The surprising connection between the Mobius groups and the Lorentz groups is now recognized.

Conclusion

The power of Klein's Erlanger Programme has been demonstrated in this project. The approach taken in this paper reflects a fusion of Klein's Erlanger Programme with standard techniques from the theory of linear (vector) spaces. Since our studies in linear spaces include inner products, Klein's group-theoretic approach to geometry leads one to consider the important classical transformation groups. A great deal of algebraic machinery is now available for use in the study of primitive geometric and topological notions. We have seen a connection arise between the classical Mobius groups (from geometry and complex analysis) and the Lorentz groups from Einstein's Special Relativity. In recent years, Sir Roger Penrose has utilized this connection in his Spinor/Twistor Program, in which he hopes to provide a unified foundation for Quantum Mechanics and General Relativity (Penrose and Rindler, 1984).

Paradoxically, Klein's Erlanger Programme never became a dominant force in the teaching of geometry; this is true of both secondary and university level education. However, Klein's central idea-studying invariants under a group of transformations-has become central to most of modern mathematics. (Witness such definitions as "Topology is the study of invariants under homeomorphisms.") The legacy left by Klein opens the door to many beautiful connections between mathematics and the world. As mathematicians continue research in the "spirit" of Klein's work, perhaps more intriguing connections between geometry and linear algebra will be developed.

Acknowledgements. I would like to thank Dr. Kevin Easley for serving as my advisor on this project. His dedication to the Kleinian "spirit" and to my project is truly appreciated. I also would also like to thank my academic advisor, Dr. Susan LaGrassa, for her time in proofreading this paper. I would also like to acknowledge Dr. Mary Sue Beersman, sponsor of the Kappa Mu Epsilon Mo Eta Chapter at Truman State University, for her dedication to KME and for instilling in me the confidence to submit this paper.

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Starting a KME Chapter

Complete information on starting a chapter of KME may be obtained from the National President. Some information is given below.

An organized group of at least ten members may petition through a faculty member for a chapter. These members may be either faculty or students; students must meet certain coursework and g.p.a. requirements.

The financial obligation of new chapters to the national organization includes the cost of the chapter's charter and crest (approximately $50) and the expenses of the installing officer. The individual membership fee to the national organization is $20 per member and is paid just once, at that individual's initiation. Much of the $20 is returned to the new members in the form of membership certificates and cards, keypin jewelry, a two-year subscription to the society's journal, etc. Local chapters are allowed to collect semester or yearly dues as well.

The petition itself, which is the formal application for the establishment of a chapter, requests information about the petitioning group, the academic qualifications of the eligible petitioning students, the mathematics faculty, mathematics course offering and other facts about the institution. It also requests evidence of faculty and administrative approval and support of the petition. Petitions are subject to approval by the National Council and ratification by the current chapters.

Graphical Encryption Technique

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Brief Summary and Description of Research

Graphical encryption means taking a plain document and encoding it to a graphical image.

Ideally, an encryption should be able to be decrypted back to its original state without a single error. A process that will do this desired encryption is a combination of mathematics and abstract visualization. What follows is one way this may be done.

In order to encrypt a text document to a picture, one needs to see what kinds of parameters they are bound to before deciding on a proper algorithm. In most computers today, graphics can have up to 16,777,216 different colors and could be any size (see Appendix A). Knowing there are a limited number of colors, this must be taken into consideration when constructing our algorithm.

The layout of a computer graphic is a graph. Similar to the graph of a math equation on the Cartesian coordinate system, one can specify the coordinates for plotting each point. What is required are a vertical location value (X coordinate), a horizontal location value (Y coordinate) with respect to the origin and a color value. In order to create the desired output, it will be necessary to have a three-dimension numerical output (e.g. X coordinate, Y coordinate, color).

To determine how to achieve this format, one needs to figure out how to make all the letters, punctuation marks, and numbers (text) represent a uniform set of numbers. Fortunately, that process has already been done. Computers use a built-in table that is called the ASCII (American Standard Code for Information Interchange) table (see Table 1 in Appendix B). Encryption algorithms have used a simple letter-to-number encoding for years, so this method will be employed.

The proposed algorithm so far converts letters to numbers, performs an operation, and computes three numbers as an output that represent the X-coordinate, the Y-coordinate, and the color. The result looks promising, until you look at the decryption because when going from the picture to the text, one does not know in what order the points went onto the graph. For example, the X and Y values for consecutive letters may send each point to opposite corners of the graph. To solve this problem a text ordering system will need to be adopted. To keep it simple, the text that comes first will have the lower position number and then use that pattern in with the algorithm as another input variable. This will also prevent duplicate inputs and preserve the uniqueness of the outputs.

As it stands, two inputs and three outputs are needed to make the required mathematical function. In addition, this function must have an inverse to provide for the proper decryption. An easy way to develop the inverse of a mathematical function is to use matrices. If matrices are used, the input is represented as a 1 by 2 matrix, the transformation matrix as a 2 by 3 matrix to result in a 1 by 3 matrix to achieve the three-dimension output required. Unfortunately, that method will not work because a 2 by 3 matrix does not have an inverse since it is not square. To have a square matrix, say 3 by 3 to insure our three-dimension output, a 1 by 3 input matrix will be needed for the matrix multiplication to compute properly. Using two characters at a time from the text instead of one will complete the input matrix. This will also decrease the number of points that are graphed for the picture. This reduced size of the picture will make its transmission across the network faster. In the event that an interceptor would try to break the code, it would be theoretically more difficult by having more letter-letter-position combinations than letter-position combinations. The two letters create 676 (262) possibilities for just the alphabet alone, not taking in account for numbers and punctuation marks.

Discussion of Possible Interpretations of Research

Knowing the type of algorithm necessary to make this encryption work in theory, different variations were attempted. Altering the algorithm by increasing both the input and the output to six or even nine turned out to do nothing more or nothing better than the original algorithm did. Just having two sets of input points be turned into two sets of output points and increased the transformation matrix by repeating the numbers provided the exact results the original algorithm did, just twice as fast (and twice as confusing). Unless the output is changed from three dimensions, the input will have to be three dimensions or the transformation matrix will not have an inverse. A possible alternative way might be to have an X coordinate, a Y coordinate, a red value, a green value, and a blue value (see Appendix A). With that as a target output, one could use four consecutive characters with a position as the input. This worked, however, with matrix multiplication the larger the matrix meant more numbers will be involved.

If more numbers are multiplied and added to each other, the results become larger. Since the value for the red, green, and blue variables is limited to 256 each, this method seems extremely inefficient.

A transformation matrix for any method will need to contain all positive numbers. Any negative numbers could leave the door open to a negative number in the output. Any negative number being sent to the computer as a coordinate for a picture either triggers the program to end, resulting in an unfinished picture, or crashes the program altogether since a typical image fort uses only positive number for pixel coordinates.

Further investigation of this process shows us exactly how many different encryptions are actually taking place. Directly, there are three: the letter-to-number association, the transformation matrix, and the number-to-color association. Indirectly, other possibilities include white-space filler on the picture or a graphical compression.

One goal of this particular encryption is to deceive third party interceptors into believing the picture being electronically sent is just a picture and not a text document. This goal will never be 100% met since some crucial encryption information will have to be embedded in the header of the file for proper decryption. This information, although not graphically viewable, will allow the computer to distinguish the output graphic file from "normal" graphic files. Further research on graphic headers will have to be done to resolve this problem.

Even though deceiving third party interceptors is one of the main goals of this project, it is obvious a graphical encryption is not the only way this can be achieved. Conversions to sound or video are also possible and those options may open the door for more embedded encryptions therefore making the resultant algorithm even more difficult to crack.

An added bonus to this algorithm is that it may allow an ordinary graphic file to decrypt as if it was an encrypted message to see what hidden messages are there. This may just be the beginning of a new pseudoscience with endless numbers of possibilities.

Hypothesized Variations for Computer Applications

When attempting to create an application that will actually perform this encryption, one may encounter many unforeseen technicalities. A technicality may be encountered if not forcing the person encrypting the document to transmit it immediately is a creation goal. An encryption method that allows one to not be forced to transmit immediately is a public-key method like the Rivest Shamir Adleman (RSA) algorithm (see Appendix C).

By looking at the Graphical Encryption Technique algorithm, there are three encryption keys required to create the graphic: letter-to-number, a transformation matrix, and number-to-color. The person decrypting the graphic will need this information to decode the message. Although sending this information along with the graphic is risky, it will ensure that the person will have the correct information and will be able to decrypt it. However, an interceptor may obtain the graphic file and also decrypt it, so it may be wise to use the RSA algorithm and have it embedded internally to encode the necessary information. The only other thing necessary would be a process to ensure proper encryption keys are used.

A method to ensure that proper encryption keys are used is to create a random number generator and have the program use the original date stamp of the computer being used as the seed of the random number generator. The program then takes nine consecutive numbers from the function to create the transformation matrix. Since the numbers are generated "randomly" and do not guarantee a non-singular matrix the program should automatically test the matrix and see if it is invertible. If not, simply repeat the process over until an invertible matrix is obtained. After the transformation matrix is created, use the RSA algorithm to encrypt the seed used to obtain the transformation matrix and embed it in the header file.

To decide on how to associate colors with the calculated numbers, it is possible to have a mathematical function that takes the calculated color number and get three distinct values that are less than or equal to 256 (see Appendix A). Sending the necessary decryption information along with the message is to eliminate the need for prior contact with an individual using this encryption technique. The receiver of the message can publish their numbers for encryption in a directory, and the sender can "address" the letter to them by using the specified numbers in the encryption process.

To increase the security of the algorithm, multiple encryptions may be done during any step of the algorithm. Currently, ways of accomplishing this and notifying the receiver of these actions to obtain the proper original message are being researched. In addition to increasing security, research is being conducted to find ways to include graphics in documents are also being researched.

Sample Encryption

Original Text File Contents

```
The dog ran fast
```

Using the ASCII table values, it will be coded to the numbers in Figure 1.

T	h	e	□	d	o	g	□	r	a	n	□	f	a	s	t
84	104	101	32	100	111	103	32	114	97	110	32	102	97	115	116
1		2		3		4		5		6		7		8	

Figure 1

Place each number in the correct order in the 1 by 3 matrix for second encryption (see Figure 1):

$[84, 104, 1][101, 32, 2][100, 111, 3][103, 32, 4]$
$[114, 97, 5][110, 32, 6][102, 97, 7][115, 116, 8]$

The first two numbers are ordered as they appear in the document and the third number denotes the position in the document. The value 32 (blank space) on the end of the final matrix creates an even number of numbers for encryption.

Next choose a 3 by 3 transformation matrix to encrypt these 1 by 3 matrices. For this sample use the transformation matrix of:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

By performing the matrix multiplication as such:

$$\begin{bmatrix} 84 & 104 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 398 & 275 & 461 \end{bmatrix}$$

and

$$\begin{bmatrix} 101 & 32 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 201 & 240 & 369 \end{bmatrix}$$

produces a list of transformed numbers that looks like this:

$[398, 275, 461][201, 240, 369][439, 320, 525][207, 250, 377]$
$[415, 340, 541][218, 270, 400][407, 322, 507][479, 370, 585]$

Analysis of the output indicates this encryption will require an X field that ranges from 0 to 479, an Y field that ranges from 0 to 370, and nine different colors. Thus, in this algorithm, is out of the possible 178,080 different coordinate points in the plotting field, only nine are in use. Notice on the next page that the "graphic" may be difficult to see at first glance because of the dominance of white since only 0.005% for filler to be used while programming a graphic.

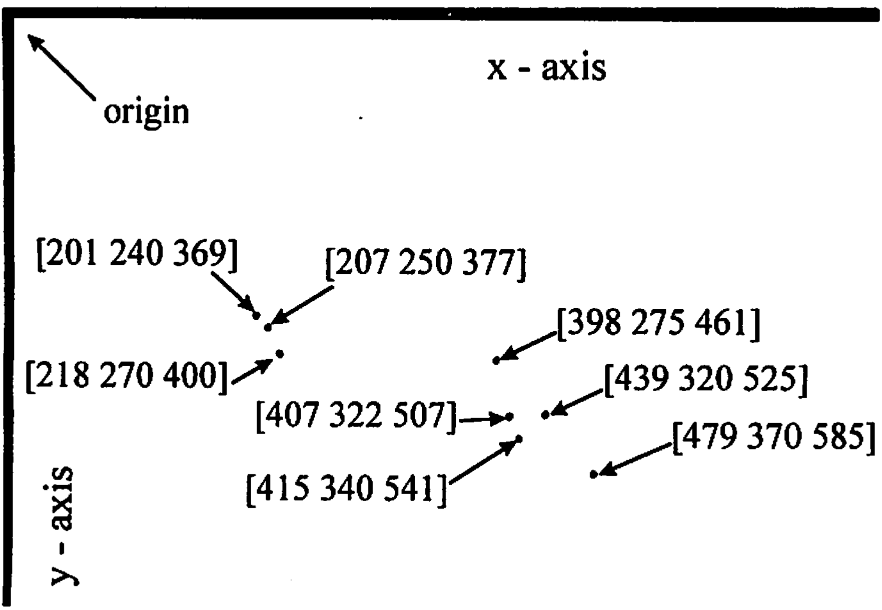
Corresponding Graphical Image for Sample

Assigning random colors to the sample encryption may generate a table that looks like this:

Table 1

211 = Red	369 = Blue	377 = Yellow
400 = Green	461 = Brown	507 = Black
525 = Purple	541 = Orange	585 = Gray

Resulting in an image that looks like this:



Careful inspection of the printout reveals all nine of the colors listed above. Please note that each pixel in the picture has been magnified by a factor of nine to allow for easier viewing. Additionally, due to the grayscale reproduction of this image, the colors may not be distinguishable.

Appendix A

Each of the colors are represented by a red, green, and blue variable with each of the variables having a memory value of eight bits (one byte) of memory. In the binary number system, eight consecutive true bits (1's instead of 0's, 11111111) is converted to 255 in the decimal number system. Using eight consecutive false bits (0's instead of 1's, 00000000) as an additional number, one can get 256 possible values for each variable that represents a color. To get the 16,777,216 different color values, multiply the total possible value from each element together (or just cube 256).

Appendix B

American Standard Code for Information Interchange (ASCII)

	0	1	2	3	4	5	6	7	8	9
0	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	HT
1	LF	VT	FF	CR	SO	SI	DLE	DC1	DC2	DC3
2	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS
3	RS	US	□	!	"	#	$	%	&	'
4	(	)	*	+	,	-	.	/	0	1
5	2	3	4	5	6	7	8	9	:	;
6	<	=	>	?	@	A	B	C	D	E
7	F	G	H	I	J	K	L	M	N	O
8	P	Q	R	S	T	U	V	W	X	Y
9	Z	[	\	]	^		`	a	b	c
10	d	e	f	g	h	i	j	k	l	m
11	n	o	p	q	r	s	t	u	v	w
12	x	y	z	{	|	}	.	DEL		

Codes 0 through 31 and 126 and 127 represent nonprintable control characters

Note. From "Structured and Object-Oriented Techniques" by Andrew C. Staugaard, Jr., 1997, 744-745. Copyright 1997 by Prentice-Hall, Inc.

Acknowledgements. I would like to thank my mentor Mr. Jim Kaus for all the help and support he has given me through the duration of this project. With out the help from Mr. Kaus I would have had difficulty on much of the theory behind certain areas of mathematics and would not have understood other encryption standards as well as I do to help build my own. I would also like to thank Mr. Brent Rickenbach for the time he has volunteered towards the computer program aspect of this project. With out Mr. Rickenbach, the reality of writing a computer program to make this idea come to life would have remained a seemingly unachievable feat.

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`eagle.eku.edu/faculty/pjcostello/kme/`

Postmodernism in Mathematics: The Loss of Certainty as Seen in a Discussion of Kurt Gödel's Incompleteness and Consistency Theorems

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Presented at the 1999 National Convention

Introduction

In 1931 a twenty-five year old German mathematician named Kurt Gödel, who attended the University of Vienna, published a relatively short paper titled "On Formally Undecided Propositions of Principia Mathematica". The Principia Mathematica mentioned in the title is the three-volume treatise by Alfred Whitehead and Bertrand Russell on mathematical logic and the foundations of mathematics ([4], p. 3). Gödel's paper dealt with a central problem in mathematics, which is proving the consistency of a large class of deductive systems. Before his paper, it was assumed unquestionably that each sector of mathematical thought can be supplied with a set of axioms "sufficient for developing systematically the endless totality of true propositions about the given area of inquiry ([4], p. 6). Also, within this axiomatic method, the presumption was that no inconsistencies can arise within it ([3], p. 73). By inconsistencies, it is meant that no two propositions can be derived from the set of axioms in which one is the negation of the other. Gödel's paper showed that this assumption is implausible. He presented mathematicians with the conclusion that the axiomatic method has certain inherent limitations, and thus, the axioms are incomplete. He also proved that it is impossible to establish the internal logical consistency of a large class of deductive systems ([4], p. 6). Before discussing the essence of Gödel's reasoning behind his proof, a background on the formalist philosophy of mathematics as founded by Kant and developed further by David Hilbert is needed. Also, a more in depth discussion on the form of a proof of consistency would be advantageous.

Formalism

Formalism is the view that mathematics is concerned with formal symbolic systems. Mathematics is regarded as "a collection of such abstract developments, in which the terms are mere symbols; the ultimate base of mathematics does not lie in logic but only in a collection of prelogical marks or symbols and in a set of operations with these marks" ([1], p. 633). Kant holds that in mathematics, even though the theorems follow from the axioms according to principals of logic, the axioms and theorems are not themselves principles of logic ([3], p. 72). David Hilbert adapted Kants philosophy, and expressed the idea that if mathematics contains only ideal symbols which are the descriptions of concrete objects of a certain kind and the logical relations between the descriptions, then no inconsistencies can arise within it. Thus, the establishment of the consistency of the various branches of mathematics becomes an important and necessary part of the formalist program. Without a proof of consistency, the whole formalist study is senseless ([1],p. 633).

In 1920 Hilbert and his collaborators, Bernays, Ackermann, and von Neumann started work on the formalist program. Previous consistency proofs based upon "interpretations and models usually merely shift the question of consistency from one domain of mathematics to another" ([1], p. 634). In general, the proofs are only relative. Hilbert conceived of a new, direct approach to the consistency problem.

Hilbert's Form of an Absolute Proof of Consistency (His Program)

The first step in the construction of an absolute proof is the complete formalization of a deductive system. This means draining the expressions occurring within the system of all meaning. The signs are to be combined and manipulated by a set of precisely stated rules ([4], p. 26). Hilbert's form of an absolute proof of consistency deals with finite mathematical systems. He sought to develop a method that would yield demonstrations of consistency "as much beyond genuine logical doubt as the use of finite models for establishing the consistency of certain sets of postulates. By an analysis of a finite number of structural features of expressions in completely formalized system of signs (called a 'calculus') an absolute proof can be constructed" ([4], p. 33).

There are four steps in the formalization process:

1. A complete catalogue is prepared of the elementary signs to be used in the calculus. These are its vocabulary.

2. The "Formation Rules" are laid down. They declare which of the combinations of the signs in the vocabulary are accepted as "formulas" (in effect, as sentences). The rules may be viewed as constituting the grammar of the system.

3. The "Transformation Rules" are stated. They describe the precise structure of formulas from which other formulas of given structure are derivable. These rules are in effect the rules of inference.

4. Finally certain formulas are selected as axioms (or as "primitive formulas"), which serve as the foundation for the entire system[1].

For the sake of brevity the actual list of signs along with the various rules and axioms will not be listed.

The purpose in the proof is to show that the set of axioms is not contradictory or in other words to prove absolutely that it is impossible by using the Transformation Rules to derive from the axioms a formula S together with its formal negation $\sim$ S ([4], p. 50). If this does occur, then one can conclude that any formula constructed from the Formation Rules would be deducible from the axioms. Putting this statement in theorem form we have: "If the calculus is not consistent, every formula is a theorem." This theorem has a contrapositive: "If **not** every formula is a theorem (i.e., if there is at least one formula that is not derivable from the axioms), then the calculus is consistent" ([4], p. 51). The problem in the proof, therefore, is to illustrate that there is at least one formula that cannot be derived from the axioms. If this can be done with a system of mathematics, then one has an absolute proof of consistency.

Hilbert's method codifies only a fragment of formal logic. In major systems of mathematics such as all of geometry and abstract algebra, a one-to-one correspondence can be established between these systems and arithmetic (i.e., these systems are reducible to arithmetic). "The reducibility to arithmetic of physical and mathematical theories which contain ideal notions raises the question of the consistency of arithmetic itself" ([3], p. 75). The main question to be solved comes to be, is Hilbert's formal apparatus and vocabulary (a finitistic method) powerful enough to prove the whole of arithmetic consistent and not merely a fragment? Many attempts to construct a proof were unsuccessful. Gödel's paper showed that operating within the strict limits of Hilbert's original program, one such proof is impossible ([4], p. 58). The motivation behind Gödel's proof and the machinery needed to understand his reasoning have been presented. Now a brief discussion of the essence of his proof can be done.

The Essence of Gödel's Proof

Gödel Numbering

Gödel's proof was achieved through an ingenious form of mapping. He showed that it is possible to assign a unique number to each elementary sign, each formula (or sequence of signs), and each proof (or finite

[1] The complete list of these steps along with the actual complete catalogue of the signs, the Formation Rules, Transformation Rules, and axioms are found on pages 46-51 of Nagel and Newman's book on Gödel's Proof.

sequence of formulas) in a formalized calculus. These numbers are tags called a Gödel number. The table below displays the signs, the number associated with each, and the meaning of each sign.

Constant Signs	Gödel Number	Meaning
$\sim$	1	not
$\vee$	2	or
$\supset$	3	ifthen
$\exists$	4	There is an
$=$	5	equals
0	6	zero
s	7	the immediate successor of
(	8	punctuation mark
)	9	punctuation mark
,	10	punctuation mark

Numerical Variable	Gödel Number	A Possible Substitution Instance
x	11	0
y	13	s0
z	17	y

Numerical values are associated with prime numbers greater than 10.

Sentential Variable	Gödel Number	A Possible Substitution Instance
p	11^2	$0 = 0$
q	13^2	$(\exists x)\,(x = sy)$
r	17^2	$p \supset q$

Sentential variables are associated with the squares of prime numbers greater than 10.

Predicate Variable	Gödel Number	A Possible Substitution Instance
P	11^3	Prime
Q	13^3	Composite
R	17^3	Greater than

Predicate variables are associated with the cubes of prime numbers greater than ten[2].

Suppose we have the formula, '$(\exists x)\,(x = sy)$'. Translated in meta-

[2] The entire notation and arrangement of reasoning concerning Gödel's proof used in this paper is consistent with Nagel and Newman's book on Gödel's proof (pp 68-97). The notation in this book and its arrangement is easy to follow if one does not have much experience with advanced mathematics.

mathematical terms this says, "There is an x such that x is the immediate successor of y", and says in effect, that every number has an immediate successor. The numbers associated with its ten constituent elementary signs are, respectively, 8, 4, 11, 9, 8, 11, 5, 7, 13, 9. Rather than have ten numbers represent a formula, it would be more desirable to have a single number represent the formula. This is done by assigning the formula the unique number that is the product of the first ten primes in order of magnitude, each prime being raised to a power equal to the Gödel number of the corresponding elementary sign. Thus, the formula is associated with the number $2^8 \times 3^4 \times 5^{11} \times 7^9 \times 11^8 \times 13^{11} \times 17^5 \times 19^7 \times 23^{13} \times 29^9$; let us refer to this number as m. In a similar fashion, a unique number, the product of as many primes as there are signs (each prime being raised to a power equal to the Gödel number of the corresponding sign), can be assigned to every finite sequence of elementary signs and, in particular, to every formula.

Now consider a sequence of formulas that may occur in some proof:

$$(\exists x)\,(x = \mathrm{s}y)$$

$$(\exists x)\,(x = \mathrm{s}0)$$

The second formula reads: "There is an x such that x is the immediate successor of 0"; it is derivable from the first by substituting the numeral '0' for the numerical variable 'y'. The Gödel number of m has already been assigned to the first formula, and suppose that n is the Gödel number assigned to the second formula. As before, instead of having two numbers to represent a sequence, it is convenient to have a single number as a tag for the sequence. Using the same rules as before, we will agree to associate with it the number which is the product of the first two primes in order of magnitude, each prime being raised to a power equal to the Gödel number of the corresponding formula in the sequence. If we call this number k, we can write $k = 2^m \times 3^n$. By applying this compact procedure we can obtain a number for each sequence of formulas. In conclusion, every expression in the system, whether an elementary sign, a sequence of signs (i.e., a formula), or a sequence of sequences (i.e., a proof), can be assigned a unique Gödel number. Thus, the formal calculus has been completely "arithmetized". The method is essentially a set of directions for setting up a one-to-one correspondence between the expressions in the calculus and a certain subset of the integers. Given an expression, the Gödel number uniquely corresponding to it can be calculated. Working in reverse, given a number it can be determined whether it is a Gödel number, and, if it is, the expression it represents can be exactly analyzed and retrieved.

Every meta-mathematical statement (or a sentence form of a mathematical relation) is represented by a unique formula within arithmetic; and the relations of logical dependence between the meta-mathematical statements are fully reflected in the numerical relations of dependence between their corresponding arithmetical formulas. The mapping Gödel uses reduces the relations between formulas and the relations between proofs into an investigation of the arithmetical properties and relations of certain integers.

Demonstrable Formulas

Focusing attention to the meta-mathematical statement: "The sequence of formulas with Gödel number x is a proof of the formula with Gödel number z". This statement is represented by a definite formula in the arithmetical calculus, which expresses a purely arithmetical relation between x and z. (To understand the complexity of this relation recall the example in which the Gödel number $k = 2^m \times 3^n$ was assigned to the proof whose conclusion was the Gödel number n. Here there is a definite relation between k, the Gödel number of the proof, and n, the Gödel number of the conclusion.) We write this relation between x and z as the formula '$Dem(x, z)$'. This means, "The sequence of formulas with Gödel number x is a proof (or demonstration) of the formula with Gödel number z". Thus, it can be said that the formula is demonstrable. To establish the truth or falsity of the meta-mathematical statement under discussion, we must ask whether the relation 'Dem' holds between the two numbers. Conversely, we can establish that the arithmetical relation holds between a pair of numbers by showing that the meta-mathematical statement represented by this relation between the numbers is true. Similarly, the meta-mathematical statement "the sequence of formulas with the Gödel number x is not a proof for the formula with the Gödel number z" is represented by a definite formula in the formalized arithmetical system. This formula is the formal contradictory of '$Dem(x, z)$', namely, '$\sim Dem(x, z)$'.

Thus far a mere "taste" of the foundation constructed for Gödel's proof has been given. We can now move to the heart of the argument Gödel presents. Because of the complexity and the length of his proof, only a general explanation will be given.

The Main Argument

Gödel showed, using methods that have been described in the previous paragraphs of this paper, (i) how to construct an arithmetical formula G that represents the meta-mathematical statement: "the formula G is not demonstrable". In Gödel's argument, the formula G is associated with the number h, and is constructed so that it corresponds to the statement: "The formula with the associated with the number h is not demonstrable". But (ii) Gödel also showed that G is demonstrable if, and only if, its formal negation $\sim$ G is demonstrable. However, from Hilbert's form of an absolute proof of consistency, if a formula and its own formal negation are both formally demonstrable, the arithmetical calculus is not consistent. Accordingly, if the calculus is consistent, neither G nor $\sim$ G is formally derivable from the axioms of arithmetic. Therefore, if arithmetic is consistent, G is a formally undecidable formula. Gödel then proved (iii) that, though G is not formally demonstrable, it is nevertheless a true arithmetical formula. (iv) Since G is both true and formally undecidable, the axioms of arithmetic are incomplete. In other words, we cannot deduce all arithmetical truths from the axioms. More encompassing, Gödel established that arithmetic is essentially incomplete: even if additional axioms were assumed so that the true formula G could be derived from the augmented set, another true but formally undecided formula could be constructed (i.e., by adding a recursively definable class of axioms, there exist undecidable arithmetical propositions) ([2], p. 193). This is Gödel's Incompleteness Theorem. (v) Next, Gödel described how to construct an arithmetical formula A that represents the meta-mathematical statement: "Arithmetic is consistent"; and he proved that the formula '$A \subset G$'(if A, then G) is formally demonstrable. Finally, he showed that the formula A is not demonstrable since it was previously proven that G was not demonstrable ([4], p. 86). From this it follows that the consistency of arithmetic cannot be established by an argument that can be represented in the formal arithmetical calculus (i.e., the question of consistency is undecidable). This is Gödel's Consistency Theorem.

Conclusion

The conclusions in Gödel's paper are monumental for the field of formalized mathematics. When formalizing any mathematical system and establishing a one-to-one correspondence with a subset of the integers, there are true arithmetical statements that cannot be derived from the set of arithmetical axioms. This is the fundamental limitation in the axiomatic method, for the axioms are essentially incomplete. Also, the consistency of arithmetic cannot be proved using arithmetic ([5], p. 110)! Even if one could prove the consistency of arithmetic using another system, then occurs the problem of proving the consistency of that more powerful system. It is a never-ending undecidable paradigm on consistency. Thus, the formalist approach to mathematics leaves many unanswerable questions. It appears that Gödel believes that only a philosophy of the Platonic type can supply us with an adequate perception of mathematics ([4], p. 99). The philosophy is that mathematics is discovered and is not just a system of symbols related to one another by a created set of rules, as the formalist would perceive mathematics. If mathematics is as the formalist perceives and a consistency proof could have been achieved, then mathematics would be a science untouched by the postmodernist views of uncertainty. However, this paper describes that such a proof is impossible. Therefore, the very foundations of mathematics are lacking in certainty. The once thought "perfect" science is, as all of man's sciences, imperfect.

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From the Pages of...

"A mathematician wishing to write a paper needs to prove a particular point. He works continuously for 48 hours and finally proves it. In his paper he casually refers to this point with the remark, "It is easily seen that . . ."

Another mathematician wishing to write a paper needs to prove a particular point. He also works without pause for 48 hours but is unable to prove it. He wishes to write the paper anyway so he casually refers to this point with the remark, "It is obvious that · · ·."

-*The Pentagon*, Spring 1956 (pp.110-111)

Using Mathematics to Search for the Fourth Dimension

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The purpose of this paper is to try to use mathematics to analyze aspects of the fourth-dimension, even though we cannot possibly visualize it due to the limits placed on us by our three-dimensional world. Some might say that it is ridiculous to talk about any dimension higher than the third-dimension, but right now, physicists are talking and debating about the concept of string theory. These small particles may be the key to formulating a universal law that would be fundamental to all physics. Interestingly, these strings are theorized to exist as 10- and 26-dimensional objects, but this idea of multidimensionality is nothing new to the field of mathematics, queen of the sciences. Linear algebra has long dealt with the idea of n-dimensional equations, and calculus is not restricted by our three- dimensional world. You can easily add another independent variable into a function, and some even explain phenomena in this world. There is even some talk among physicists that the fourth dimension might just be gravity, and the fifth is light. Then at 10- and 26- dimensions are strings. So the message I am trying to convey in this introductory paragraph is that it is not ridiculous to talk about the fourth-dimension, because physicists are even talking about higher dimensions! This paper will demonstrate the use of mathematics to analyze some aspects of the fourth-dimension, even though we cannot visualize it.

While in Calculus IV, my class began studying functions of several variables. Two independent variables would determine a third dependent one, which was a three-dimensional graph in R^3. Surprisingly, there were equations that had three independent variables which determined a fourth dependent variable. This would be described as being the fourth-dimension in R^4. The text book stated that it could not be graphed or visualized, but that calculus was not limited to our three-dimensional space. I became curious about this and wanted to learn more about the fourth-dimension. It seemed that calculus could prove its existence.

Some believed that the fourth-dimension was time, which can be a fourth-dimension mathematically. Yet, time can be added to the second-dimension to get a third-dimension mathematically, but it is not *the* third-dimension. I wanted to look at the fourth-dimension as being based on four axes. Another point that I have heard about time is that watches slow down in airplanes compared to stationary ones on the ground. I decided to show this mathematically by using the time dilation equation, which is derived from Einstein's General Theory of Relativity. It would calculate the ratio of one tick of a clock on the ground per ticks in the moving clock. The time dilation factor (slowing down of a clock) is calculated as:

$$\frac{\text{Time between ticks of a moving clock}}{\text{Time between ticks of a stationary clock}} = \frac{1}{\left(1 - \frac{V_r^2}{c^2}\right)^{1/2}}$$

where V_r is the relative velocity, and c is the speed of light in a vacuum.

An airplane with a velocity of $239\frac{m}{s}$, compared to a stationary clock on the ground, would have a time dilation factor of:

$$\tau_d = \frac{1}{\left(1 - \frac{V_r^2}{c^2}\right)^{1/2}} = \frac{1}{\left[1 - \frac{\left(239\frac{m}{s}\right)^2}{\left(3.00\times 10^8\frac{m}{s}\right)^2}\right]^{1/2}} = 1 + \frac{3}{10^{12}}$$

Of course, this is negligible to us, but as velocity increases, what happens to the time dilation factor? To determine this, I took the limit of the time dilation factor. The domain of the function includes all values of the velocity from 0 to c. A negative relative velocity does not exist and no velocity can exceed c, for then you would be taking the square root of a negative number. This also proves that no velocity can exceed c, which is in agreement with Einstein's General Theory of Relativity.

$$\lim_{V_r \to c}\left[\frac{1}{\left(1 - \frac{V_r^2}{c^2}\right)^{1/2}}\right] = \frac{1}{0} = \infty$$

$$\lim_{V_r \to 0}\left[\frac{1}{\left(1 - \frac{V_r^2}{c^2}\right)^{1/2}}\right] = 1$$

which means both clocks have the same time.

Let t_{mc} = ticks of the moving clock
t_{sc} = ticks of the stationary clock
τ_d = time dilation factor

$$\boxed{\frac{t_{mc}}{t_{sc}} = \tau_d}$$

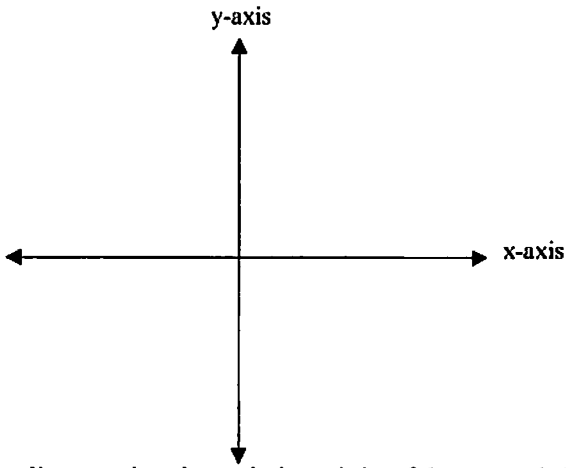
To the stationary clock, a clock moving at a velocity of c appears to have actually stopped, for $\tau_d = \frac{t_{mc}}{t_{sc}} = \infty$, which means that there are an infinite number of ticks for the moving clock within a single tick for the stationary clock. Therefore, time slows down as velocity approaches c, and stops where velocity is c. Due to this, it is proven that there is no absolute time throughout space. Time can be a fourth-dimension mathematically, but not *the* fourth-dimension. Time seems to be dependent on velocity. Calculus adds axes for each higher dimension. By extending this, the fourth-dimension is based on the x-axis, y-axis, z-axis, and a fourth axis. Therefore, the fourth-dimension is another axis that will be called the w-axis.

If we start with examining the zeroth-dimension, we can visualize it as being a single point (e.g. C , like a sphere or circle with $r = 0$). Now to find the first-dimension, imagine a perpendicular line through the paper at that point. You now have the x-axis, or first-dimension, which is a line. Now redraw the first-dimension as a line on the paper.

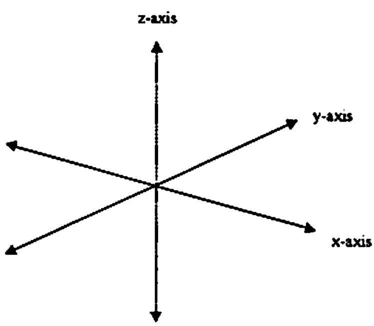
e.g. ◀ – – – – – – – – – – – – – – – – – – ▶

At a point in the line (◀ – – – – – – – – • – – – – – – – – – – – ▶), imagine a line passing through that point perpendicular to that line and the paper. Now you have the y-axis, which added to the x-axis, makes an xy plane. This is the second-dimension. Now redraw the second-dimension onto the paper.

e.g.



Now imagine a line passing through the origin of the second-dimension and is perpendicular to both axes. This would be the z-axis, and you would now have the third-dimension. Now redraw this onto the paper.



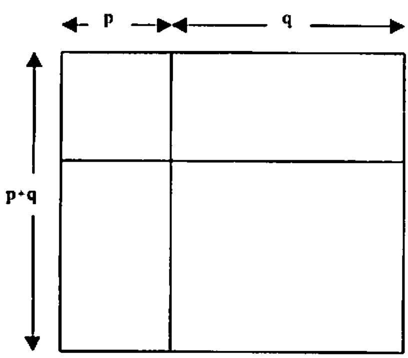
Now imagine a line through the origin coming and going through the paper, and is a normal line to the paper. This can be called the w-axis, and is the axis that determines the fourth-dimension. The cross-section of it would be a three-dimensional object, for a slice would be the image on the paper having three axes. This is evident in calculus, such as in the second dimension where $y = f(x)$, in the third-dimension where $z = f(x, y)$, and in the fourth-dimension it can be continued by writing $w = f(x, y, z)$. In the zeroth-dimension, only a point exists, like a sphere with $r = 0$. In the first-dimension, only length and points exist, and can be thought of as a circle with $\sqrt{r = 4}$ or just a straight line. In the second-dimension, only an object with length and width exists. Here a circle exists with other polygons. In the third-dimension, length, width, and depth exist, and cubes and spheres can exist. Objects of the fourth-dimension cannot be visualized, but their cross-sections can be observed in our third-dimension. The equation for a circle is $y^2 = C - x^2$, where C is a constant and the $r = C^{1/2}$. The equation for a sphere is $z^2 = C - x^2 - y^2$. This pattern can be continued to see what a hypersphere equation would by in the fourth-dimension. The equation of the hypersphere is $w^2 = C - x^2 - y^2 - z^2$. Also, since the last two dimensions (2-D and 3-D) had the radius, or distance from the center to the edge, determined by $C^{1/2}$, then the distance from the center of the hypersphere to its edge should also be $C^{1/2}$. This measurement can also be in feet, meters, etc., for going from the first-dimension to the second-dimension and to the third-dimension, all could have the length measured in feet, meters, etc.

Even though an object in the fourth-dimension cannot be visualized or graphed, a cross-sectional area of the object can be graphed and visualized. The book *Flatland*, by Edwin Abbott Abbott, showed how a square in the second-dimension, which is an xy plane, would be perceived by a being in the first dimension, which is a straight line. As the square would pass through the first-dimension, it would appear as a line and then disappear when it left. If a circle from the third-dimension were to pass through the second-dimension, it would appear at first as a point, and then would grow into a circle with the radius increasing, until the radius was equal to its radius in the third dimension. Then the radius would decrease until it was a point and then disappear. Thus, extending this information, we can propose that we can perceive a fourth-dimensional object in our third-dimensional world as its cross-section in our world. A sphere in the fourth-dimension was called a hypersphere in *Flatland*. Even though we cannot visualize the fourth-dimensional hypersphere, its cross-section, or shape of appearance in the third-dimension, can be graphed and visualized. The equation for a hypersphere is $w^2 = C - x^2 - y^2 - z^2$, where C is a constant.

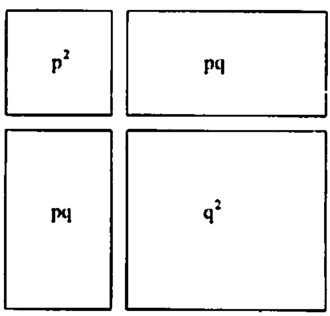
If we set $w^2 = 0$ to remove the plotted points on the w-axis, then $0 = C - x^2 - y^2 - z^2$, and we can solve for z^2. Therefore, $z^2 = C - x^2 - y^2$, which is the equation of a sphere. What this tells us is that even though we cannot visualize a hypersphere in the fourth-dimension, if it either passes through or enters our dimension, first it appears as a point, and then it will appear as a sphere. The larger the hypersphere passing through our dimension, the large r the radius of the sphere appearing in our dimension, because if a sphere were to pass through the second-dimension, it would appear at first as a point that would grow larger as a circle until it had the same r as the sphere. Also, the radius of a hypersphere would still be $C^{1/2}$. The diameter of a sphere appearing in the second-dimension would be equal to the diameter of the largest circle, which is its cross-section. Therefore, the diameter of the cross-section of a hypersphere in our three-dimensional world would be equal to the diameter in the fourth-dimension. Although units of length seem to continue up into each dimension, cubic feet is only existent in our three-dimensional world. In the fourth-dimension, volume as we would call it, would be expressed in feet raised to the fourth power. It would not really be volume, but it describes what the w-axis is, and is not possible to see in its entirety, but can be seen in cross-section as it enters our world.

Turning to algebra, a square could be expressed as the square of a binomial. The binomial is the length, represented by $(p + q)$. The square of this binomial,

$$\begin{aligned}(p+q)^2 &= (p+q)(p+q)\\ &= p(p+q) + q(p+q)\\ &= p^2 + pq + qp + q^2\\ &= p^2 + 2qp + q^2\end{aligned}$$



Banchoff, pg. 24



Banchoff, pg. 24

To get a cube, cube the binomial

$$\begin{aligned}(p+q)^3 &= (p+q)(p+q)^2\\ &= p(p+q)^2 + qp(p+q)^2\\ &= p(p^2+2pq+q^2) + q(p^2+2pq+q^2)\\ &= (p^3+2p^2q+pq^2) + (qp^2+2pq^2+q^3)\\ &= p^3+3p^2q+3pq^2+q^3\end{aligned}$$

This tells us that a cube with length $(p+q)$, has eight pieces, found by adding up the coefficients. There will be one cube of length p, one cube of length q, three rectangular prisms with a square base of length p and a height of q, and three rectangular prisms with a square base of length q and a height of p.

A hypercube can be expressed as raising a binomial to the fourth power.

$$(p+q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$$

Therefore, the hypercube will have 16 pieces. The dimension of a piece is found by adding the exponents. There will be a hypercube of length p, a hypercube of length q, 4 four-dimensional prisms with a cubical base of p and height q, 6 four-dimensional objects with two lengths of p and two lengths of q at each vertex, and 4 four-dimensional prisms with a cubical base of q and height p. The above is stated by Thomas F. Banchoff, in his book *Beyond the Third Dimension*, allows perceiving the hypercube as a binomial to the fourth power. He also states that coefficients of binomial expansion can be expressed by each row of Pascal's triangle:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

Pascal's Triangle

Now lets look at the components of the hypersphere by noticing the change from the zeroth-dimension onwards. A chart will organize the data.

Object	**Dimension**	**Vertices**	**Edges**	**Faces**	**Solids**
Point	0-D	1	0	0	0
Segment	1-D	2	1	0	0
Square	2-D	4	4	1	0
Cube	3-D	8	12	6	1
Hypercube	4-D	16	32	24	8
Hyper-hypercube	5-D	32	80	80	40

Vertices $= 2^n$, where n is the dimensional #.
Edges $= n \times 2^{n-1}$, where n is the dimensional #.

Also, if you look at a square with length ℓ, you can calculate the diagonal as $\ell(2)^{1/2}$. A cube's longest diagonal would be $\ell(3)^{1/2}$. Therefore, the longest diagonal of a hypercube or beyond can be found by the equation $n^{1/2}$, when ℓ is one. This can be done when you keep finding the distance between the origin and $(1, 1)$ for 2-D, $(1, 1, 1)$ for 3-D, $(1, 1, 1, 1)$ for 4-D, etc. So, a diagonal of a cube of $\ell = 1$ projected into the fourth-dimension would be $4^{1/2}$.

The next step in trying to understand the fourth dimension is using coordinate geometry. To simplify the process of finding coordinates for a square, cube, and hypercube, let $\ell = 1$, or use the unit square. Beginning in the second-dimension, or xy plane, with the square, the origin is $(0, 0)$. Two other points may be found with one on each axis. They are represented by $(x, 0)$ and $(0, y)$. The fourth point will be represented by (x, y). Knowing that it is a unit square, the values will be $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. To extend this to the third-dimension, all possible combinations of (x, y) are used. The first is the origin at $(0, 0, 0)$. The three axes give $(x, 0, 0) = (1, 0, 0)$, $(0, y, 0) = (0, 1, 0)$, and $(0, 0, z) = (0, 0, 1)$. Then there is $(x, y, 0) = (1, 1, 0)$, $(x, 0, z) = (1, 0, 1)$, $(0, y, z) = (0, 1, 1)$, and $(x, y, z) = (1, 1, 1)$. The fourth- dimension hypercube has 16 vertices, determined from the chart of components of shapes in different dimensions. The form will be (x, y, z, w). All the coordinates of the hypercube will then be:

$$\begin{array}{cccc} (0,0,0,0) & (1,0,0,0) & (1,1,0,0) & (0,1,0,0) \\ (0,0,1,0) & (1,0,1,0) & (1,1,1,0) & (0,1,1,0) \\ (0,0,1,1) & (1,0,1,1) & (1,1,1,1) & (0,1,1,1) \\ (0,0,0,1) & (1,0,0,1) & (1,1,0,1) & (0,1,0,1) \end{array}$$

The maximal diagonal $[(0, 0, 0, 0)$ to $(1, 1, 1, 1)]$ is also proven here to be $4^{1/2}$ as mentioned earlier, using the distance formula.

Returning back to the string theory mentioned earlier, some physicists believe that gravity is the fourth-dimension. What we perceive as gravity in our third-dimension could really be the fourth-dimension entering ours. Gravity attracts equally in all directions. This is like a sphere. A hypersphere would appear in our three-dimensional world similar to the effects of gravity. Black holes are also believed by some physicists to be connected to white holes, or quasars, even though they are separated by great distances, and there is no connection observed between them. This could be explained by higher dimensions. In *Flatland* and *Sphereland*, a hand of a human in the third-dimension can pass through the second-dimension and appear as separate circles not connected, but being the same and one united object. The cross-section of the fingers would appear as such a phenomena to the two-dimensional beings. This can be turned into an analogy between the third- and fourth-dimensions. A black hole and a quasar may be united only in the fourth-dimension. Matter pulled into the black hole passes through the fourth-dimension and re-enters our third-dimension by way of the quasar. In conclusion, even though the fourth-dimension cannot be visualized or graphed, the existence can be analyzed by mathematics. We can also hypothesize about the form the fourth dimension takes in our world.

Acknowledgements. I would like to thank Dr. Skoner, Ms. Miko, Mr. Deskevich, Dr. Harris, and Father Oliver Hebert, T.O.R., for all their help in this project. I also would like to acknowledge the help I received in writing this paper from Banchoff's *Beyond the Third Dimension*, which showed me many new aspects to analyze the fourth dimension.

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1. Atkins, K. R., *Physics*, John Wiley & Sons, Inc., New York, 1965
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3. Bradley, Gerald L., and Karl J. Smith, *Calculus*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
4. Novikov, Igor D., and Valery P. Frolov, *Physics of Black Holes*, Kluwer Academic Publishers, Boston, 1989.
5. Rucker, Rudy, *The Fourth Dimension: A Guided Tour of the Higher Universes*, Houghton Mifflin Company,Boston, 1984.

Cumulative Subject Index

The Cumulative Subject Index for *The Pentagon* is up and running! Check it out at `www.cst.cmich.edu/org/kme/`, the national KME homepage, or directly at `www.cst.cmich.edu/org/kme/indpent.htm`.

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The Problem Corner

Edited by Kenneth M. Wilke

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following problems should be submitted on separate sheets before July 1, 2000. Solutions received after the publication deadline will be considered also until the time when copy is prepared for publication. The solutions will be published in the Fall 2000 issue of *The Pentagon*, with credit being given to the student solutions. Affirmation of student status and school should be included with solutions. Address all communications to Kenneth M. Wilke, Department of Mathematics, 275 Morgan Hall, Washburn University, Topeka, Kansas 66621 (e-mail: xxwilke@acc.wuacc.edu).

PROBLEMS 535-539

Problem 535. Proposed by the editor.

Determine the smallest value of $|18^p - 7^q|$ where p and q are positive integers and $||$ denotes absolute value.

Problem 536. Proposed by the editor.

Let P(n) denote the product of the divisors of n (including 1 and n) Find, with proof, the smallest integer n such that $P(n) = n^8$ where:

(a) n is an integer;

(b) n is a perfect square; and

(c) n is a perfect cube.

Problem 537. Proposed by the editor.

Let r, s, t, u and v be integers such that both their sum and the sum of their squares are divisible by an odd prime p. Prove that p also divides the quantity $r^5 + s^5 + t^5 + u^5 + v^5 - 5rstuv$.

Problem 538. Proposed by the editor.

An eccentric gardener with a mathematical penchant has a group of gardens which have the following common properties: each garden has a triangular shape such that the area of the garden is twice the perimeter; each side is an integral number of feet; and in each garden two sides are consecutive integers. How many gardens does the eccentric gardener have and what are the dimensions of each garden?

Problem 539. Proposed by the Albert White, St. Bonaventure University, St. Bonaventure, New York.

For points x, y, z, let $[x, y, z]$ denote the area of the triangle formed by the pointsx, y and z. Let a, b and c be the vertices of a right triangle. Find the point x such that $[a, b, x]^2 + [a, c, x]^2 + [b, c, x]^2$ is a minimum. For the purposes of this problem assume that all points lie in the same plane.

Please help your editor by submitting problem proposals.

SOLUTIONS 525-529

Editor's comment. The editor wishes to acknowledge that Carl Libis, Antioch College, Yellow Spring, Ohio, was inadvertently omitted from the list of solvers for problem 509. Mr Libis submitted a solution for parts (a) and(b) of problem 509. The editor apologizes for any inconvenience this has caused.

Problem 525. Proposed by Pat Costello, Eastern Kentucky University, Richmond Kentucky.

Determine the last two digits of the number
$N = 19^{19^9} + 29^{29^9} + 39^{39^9} + 49^{49^9} + 59^{59^9} + 69^{69^9} + 79^{79^9} + 89^{89^9} + 99^{99^9}$.

Solution by Clayton Dodge, Orono, Maine.

Observe that $49^2 \equiv 99^2 \equiv 1(mod\ 100)$. Also $9^2 \equiv 59^2 \equiv 81$, $19^2 \equiv 69^2 \equiv 61, 29^2 \equiv 79^2 \equiv 41$ and $39^2 \equiv 89^2 \equiv 21$ all modulo 100. Furthermore, $21^2 \equiv 41, 41^2 \equiv 81, 61^2 \equiv 21$ and $81^2 \equiv 61$ all modulo 100. Note also that $61 \cdot 41 \equiv 21 \cdot 81 \equiv 1(mod\ 100)$. It follows that $9^{10} \equiv 19^{10} \equiv 29^{10} \equiv \ldots = 99^{10} \equiv 1(mod\ 100)$. Hence each of the exponents $19^9, 29^9, \ldots, 99^9$ can be reduced modulo 10. Since $9^2 = 81$, it follows that if $n \equiv 9(mod\ 10)$, then odd powers of n are congruent to 9 modulo 10 and even powers of n are congruent to 1 modulo 10, so we only need to consider the values of 199,292, . . (mod 100). Since we have $9 \cdot 89 \equiv 19 \cdot 79 \equiv 29 \cdot 69 \equiv 39 \cdot 59 \equiv 29^2 \equiv 1(mod\ 100)$, then $19^9 \equiv 79, 29^2 \equiv 69 \ldots (mod\ 100)$. Finally we see that the given sum becomes $79 + 69 + 59 + 49 + 39 + 29 + 19 + 9 + 99 \equiv 51(mod\ 100)$ so the given sum ends in 51.

Also solved by: Charles Ashbacher, Hiawatha, Iowa; Tom Elsner, Kettering University, Flint, Michigan; Carl Libis, Antioch College, Yellow Spring, Ohio; Jeff Stevens, student, Eastern Kentucky University, Richmond, Kentucky; and the proposer.

Problem 526. Proposed by Bryan Dawson, Union University, Jackson, Tennessee.

Given ΔABC and its image $\Delta A'B'C'$ under an unknown glide reflection, give a compass-and-straightedge construction that determines both the line of reflection and the vector of translation parallel to that line that constitute the unknown glide reflection.

Solution by Clayton Dodge, Orono, Maine.

Triangles ABC and $A'B'C'$ must be distinct, oppositely congruent, and not related by a single reflection, but rather by a product of three reflections. Let m be the perpendicular bisector of segment AA'. Reflect triangle ABC in line m; i.e. perform the reflection $\sigma_m(ABC)$. Now A maps to A' and we let B_1 and C_1 be the images of B and C respectively. Thus $\sigma_m(ABC) = A'B_1C_1$. Let n be the perpendicular bisector of $B'B_1$. Since n passes through point A', then $\sigma_n(A'B_1C_1) = A'B'C_2$ for some point C_2. Then line $A'B'$, which we denote by p, is the perpendicular bisector of $C'C_2$, and $\sigma_p(A'B'C_2) = A'B'C'$. Therefore $\sigma_p\sigma_n\sigma_m(ABC) = A'B'C'$. Assuming that the glide reflection does not reduce to just a reflection, then the lines m, n, and p do not form a pencil; i.e. they are not all parallel nor do they pass through a common point. Then line n must intersect at least one of lines m and p. We assume that lines $m$and n intersect. If not, then lines n and p intersect and the construction is similar to the following.

Let lines m and n intersect at point D. The product $\sigma_n\sigma_m$ is a rotation about point D through twice the angle from line m to line n. We rotate lines m and n about D into m' and n' maintaining the angle between them, and with line n' perpendicular to line p. Then $\sigma_n\sigma_m = \sigma_{n'}\sigma_{m'}$. Let lines n' and p meet at point E. Rotate n' and p about E into n'' and p', maintaining the angle between them, and such that n'' is perpendicular to m'. Then $\sigma_p\sigma_n\sigma_m = \sigma_{p'}\sigma_{n''}\sigma_{m'}$. Then n'' is the mirror of the reflection and the translation is through twice the vector from line m' to line n'. [Since the copying of angles is a ruler and compass construction, the construction described also can be performed using a ruler and compass.]

Also solved by the proposer.

1. C.W. Dodge, Euclidean Geometry and Transformations, Addison-Wesley Publishing Co., Reading, MA (1972), Theorem 13.13, p.65; Theorems 14.9 and 14.10, page 69; Theorem 16.4, pp 74-75.

Problem 527. Proposed by the Carol Collins, Drury College, Springfield, Missouri.

Prove that in the expansion of $(x^2+x+1)^n$, the coefficient of the x term is n and the coefficient of the x^2 term is $\frac{n(n+1)}{2}$ for all integers $n > 1$.

Solution by Russ Euler and Jawad Sadek (jointly), Northwest Missouri State University, Maryville, Missouri.

Let $f(x) = (x^2+x+1)^n$. By Taylor's Theorem, the coefficients of x and x^2 will be given by $f'(0)$ and $\frac{f''(0)}{2}$; respectively. Then since $f'(x) = n(x^2+x+1)^{n-1}$and $f''(x) = n(x^2+x+1)^{n-2}[2(x^2+x+1)+(n-1)(2x+1)^2]$, it follows that $f'(0) = n$ and $\frac{f''(0)}{2} = \frac{n(n+1)}{2}$.

Also solved by: Charles Ashbacher, Hiawatha, Iowa; Amy Bittinger, Western Maryland College, Westminster, Maryland; James K. Bryan, student, California State University, Fresno, California; Tom Elsner, Kettering University, Flint, Michigan; Carl Libis, Antioch College, Yellow Springs, Ohio and the proposer.

Problem 528. Proposed by the editor.

Consider a paired number $p(n)$ to be formed by concatenating the same number twice; e.g. $p(1234) = 12341234$. What is the smallest integer n for which $p(n)$ is a perfect square? What is the next smallest integer nn for which $p(nn)$ is a perfect square and nn has more digits than n does? smallest square twin?

Since no solutions have been received for this problem, it will remain open for another issue.

Problem 529. Proposed by Bryan Dawson, Union University, Jackson,Tennessee.

Let BC be a fixed line segment, ℓ a line parallel to BC, and A an arbitrary point on ℓ. Describe (with proof) the path followed by the orthocenter of $\triangle ABC$ as A moves along ℓ.

Solution by Albert White, St. Bonaventure University, St. Bonaventure, New York.

In triangle ABC, let vertex B be the point $(0,0)$ and vertex C be the point $(c,0)$. Let the line ℓ be the line $y = b$. Then let vertex A be the point (a,b) since A lies on line ℓ. Since the orthocenter is the point of intersection of the altitudes, it must lie on each of the lines $x = a$, $y = \frac{a(c-x)}{b}$ and $y = \frac{(c-a)x}{b}$. The intersection of these three lines is the point $(a, \frac{a(c-a)}{b})$. Hence the path of the orthocenter is given by the equation $y = \frac{x(c-x)}{b}$ which is a parabola.

Also solved by: Brian Albright, student, Emporia State University, Emporia, Kansas; Clayton Dodge, Orono, Maine; and the proposer.

Thank You, Referees!

The current and previous editors wish to thank the following individuals who refereed papers submitted to *The Pentagon* during the last two years.

Charles Allen
Drury College
Springfield Missouri

V. Sagar Bakhshi
Virginia State University
Petersburg, Virginia

Gerald Bergum
South Dakota State University
Brookings, South Dakota

Duane M. Broline
Eastern Illinois University
Charleston, Illinois

Patrick Cassens
Missouri Southern State College
Joplin, Missouri

Jeffrey Clark
Elon College
Elon College, North Carolina

Deborah Denvir
Marshall University
Huntington, West Virginia

Vincent Dimicelli
Oral Roberts University
Tulsa, Oklahoma

Timothy Flood
Pittsburg State University
Pittsburg, Kansas

John Fuelberth
Wayne State College
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Chenglie Hu
Fort Hays State University
Hays, Kansas

Lyndell Kerley
East Tennessee State University
Johnson City, Tennessee

Yaping Liu
Pittsburg State University
Pittsburg, Kansas

William R. Livingston
Missouri Southern State College
Joplin, Missouri

Rhonda McKee
Central Missouri State University
Warrensburg, Missouri

Mohammad Riazi-Kermani
Fort Hays State University
Hays, Kansas

Andrew Rockett
C.W. Post Campus of Long Island University
Brookville, New York

Michael Seyfried
Shippenburg University
Shippenburg, Pennsylvania

Thomas J. Sharp
State University of West Georgia
Carrollton, Georgia

Don Tosh
Evangel College
Springfield, Missouri

Lisa Townsley-Kulich
Benedictine University
Lisle, Illinois

Glyn Wooldridge
Liberty University
Lynchburg, Virginia

Betsy Yanik
Emporia State University
Emporia, Kansas

We also wish to thank the many other individuals who volunteered to serve as referees but were not used during the past two years. Referee interest forms will again be sent by mail in the near future so that interested faculty may volunteer. If you wish to volunteer as a referee, feel free to contact the editor (see page 2) to receive a referee interest form.

Kappa Mu Epsilon News

Edited by Don Tosh, Historian

News of chapter activities and other noteworthy KME events should be sent to Don Tosh, Historian, Kappa Mu Epsilon, Mathematics Department, Evangel College, 1111 N. Glenstone, Springfield, MO 65802, or to toshd@evangel.edu.

INSTALLATION OF NEW CHAPTERS

Texas Lambda

Trinity University, San Antonio

The Texas Lambda chapter of Kappa Mu Epsilon was installed on Monday, November 22, 1999, at a 4:00 p.m. reception in the Gold Room of the Chapman Memorial Center on the campus of Trinity University in San Antonio. Donna K. Hafner, CO Delta, Director of the South Central Region of the society, served as installing officer. Benjamin Passty, student President of the Trinity University Math Society, served as the conductor.

Preceding the installation ceremony, Professor Hafner presented a history of Kappa Mu Epsilon, discussed the traditions of the society, and presented the chapter with a permanent Chapter Registry. Following the ceremony, Professor Donald Bailey, Dean of Science, offered congratulatory remarks and spoke briefly about some mathematics of historical importance in the development of our world.

TX Lambda chapter has twenty-eight charter members. Student members are: Brian Cervantes, Ming Chiu, Kristan Doerfler, Amy Gass, Wendy Hendricks, Jessica Hilton, Fernanda Iturbe, Rick Kimple, Farica Paul, Ellen Phifer, Leslie Ramirez, Claire Reynolds, Cristina Rodriguez, Rebecca Rosser, Scott Schaefer, Rusty Schultz, Maria Tafolla, and Lam Yu. Faculty members are: Donald Bailey, Scott Chapman, Richard Cooper, Saber Elaydi, Roberto Hasfura, Kenneth Hummel, Phoebe Judson, Jeff Lawson, Darwin Peek, and Diane Saphire.

The officers installed during the ceremony are: Maria Tafolla, President; Jessica Hilton, Vice President; Lam Yu, Recording Secretary; Rick Kimpel, Treasurer; Prof. Saber Elaydi, Faculty Sponsor and Corresponding Secretary.

Chapter News

AL Zeta Chapter President—Elizabeth White

Birmingham Southern College, Birmingham 7 actives, 11 associates

In November, 11 new members were initiated. The speaker for the occasion was Annie Laurie McCullogh, a BSC graduate and KME member who talked about career opportunities in mathematics. Plans for the year include assisting with a mathematics fair at a nearby elementary school. Other fall 1999 officers: Cody Morris, vice president; Jennifer Clem, secretary/treasurer; Mary Jane Turner, corresponding secretary; Shirley Branan, faculty sponsor.

AR Alpha Chapter President—Michael Mott}
Arkansas State University, State University 8 actives, 5 associates

Other fall 1999 officers: Laura Firestone, Secretary; Jacob Hamilton, Treasurer; William Paulsen, Corresponding Secretary.

CO Gamma Chapter President—Bethany Lyles
Fort Lewis College, Durango 20 actives

An initiation ceremony was held on 10/20/99. Seven new members were inducted. President Bethany Lyles gave a report on her summer research fellowship at Miami University in Ohio. Other fall 1999 officers: Jason Winokur, vice president; Claire Knuckles, secretary/ treasurer; Richard Gibbs, corresponding secretary; Deborah Berrier, faculty sponsor.

CO Delta Chapter President—Natalie Todd
Mesa State College, Grand Junction 20 actives, 12 associates

Twenty-six initiates, members, and guests attended the reception, initiation ceremony, and business meeting held on December 5, 1999. Eleven students and one faculty member were initiated. Student members voted to increase annual chapter dues beginning in the year 2000 to pay for pink and silver honor cords for graduating members. Other fall 1999 officers: Valerie Coniff, vice president; Richard Hasenauer, secretary; Sylvia Myhre, treasurer; Donna Hafner, corresponding secretary; Kenneth Davis, faculty sponsor.

GA Alpha Chapter President—Nancy Boyette
State University of West Georgia, Carrollton 25 actives

On November 11, 1999, we held our Fall Social at a local Mexican restaurant and a fine time was had by all who attended. We also sponsored our annual Food and Clothing Drive for the Needy-with the proceeds being donated to the Salvation Army. Other fall 1999 officers: Tatiana Mack, vice president; Amy Smith, secretary; Blake Smith, treasurer; Joe Sharp, corresponding secretary; Joe Sharp and Mark Faucette, faculty sponsors.

IL Theta Chapter President—Mike Nielson
Benedictine University, Lisle 20 actives

We took a field trip to a local college consortium and had a special fall speaker in November. We also took a field trip to Navy Pier in Chicago to observe mathematics at play. Other fall 1999 officers: Tracii Friedman, vice president; Lisa Townsley Kulich, corresponding secretary.

IN Delta Chapter President—Katherine Zimmer
University of Evansville, Evansville 45 actives

KME student members offered free math tutoring for the entire university community. Other fall 1999 officers: Richard Kribs, vice president; Melia Aldridge, secretary; Mohammad Azarian, treasurer/corresponding secretary/faculty sponsor.

IA Alpha Chapter President—Gary Spieler
University of Northern Iowa, Cedar Falls 45 actives

Student member Allysen Edwards presented her paper "The Rademacher Orthogonal Sequence" at our first fall meeting in September. Marcus Bishop presented his paper "Mathematics and Music" at the second in October while Barb Meyers presented "Sierpinski" at the third in November. Student member Julie Binneboese addressed the fall initiation banquet with "Game Theory". In addition, we initiated seven new student members at our December banquet. The major event of the semester was the homecoming coffee held at professor (emeritus) Carl Wehner's residence in October. Other fall 1999 officers: Allysen Edwards, vice president; Kamilla Guseynova, secretary; Douglas Stockel, treasurer; Mark Ecker, corresponding secretary/faculty sponsor.

IA Gamma Chapter President—Ken Smith
Morningside College, Sioux City 14 actives

Other fall 1999 officers: Jason Hargens, vice president; Lori Gibson, secretary; Mary Curry, treasurer; Doug Swan, corresponding secretary/faculty sponsor.

IA Delta Chapter President—Paul Seberger
Wartburg College, Waverly 39 actives, 3 associates

The September meeting involved planning KME activities for the year including Homecoming activities, KME t-shirt design, and agreement to hold some meetings jointly with the physics club, PsiPhi. A math puzzler and refreshments completed the activities for this meeting. The October business meeting was very brief followed by a Road Rally planned and conducted by Drs. Mariah and Brian Birgen. In November, student member Scott Kahler shared his internship experiences with an environmental engineering firm in Denver, Colorado. A pizza party study-break during final exam week was the December meeting. Other fall 1999 officers: Robyn Brent, vice president; Janelle Young, secretary; Daniel Bock, treasurer; August Waltmann, corresponding secretary; Mariah Birgen, faculty sponsor.

KS Alpha Chapter President—Carrie Denton

Pittsburg State University, Pittsburg 25 actives, 5 associates

The first meeting for the fall was held October 5. At this meeting, the plans for the year were discussed, the group picture for the yearbook was taken, and Dr. Woodburn presented "Coloring Pascal's Triangle." Refreshments were provided following the meeting. The second meeting was held on November 9 and included the initiation ceremony. Following the meeting, the members went out for pizza to welcome the new members. The final meeting of the semester was the Christmas Party. Dr. Davis and his wife hosted the party at their house on December 8. Faculty members provided "goodies" for everyone to enjoy and a variety of games were played by the members. Other fall 1999 officers: Jennifer Malan, vice president; Jennifer Laswell, secretary; Melissa Hess, treasurer; Tim Flood, corresponding secretary; Yaping Liu, faculty sponsor.

KS Gamma Chapter President—Lance Hoover

Benedictine College, Atchison 7 actives, 19 associates

Two new faculty members graced our presence this fall. Barbara Kushan, S.C.L., took the full time position vacated by Linda Herndon, OSB, when Herndon took a leave from the College to pursue graduate studies in Educational Technology at the University of Wisconsin. A local secondary teacher, Donna Roe, took on the teaching of one course while part-time faculty member in our department, Vern Ostdiek, took a fall sabbatical. On September 8, a record crowd enjoyed the fall picnic on the deck of the home of faculty member Richard Farrell. In October, sophomore student Linda Heyd discussed "Function Iteration" at a group gathering in Westerman Hall. Her presentation was based on her directed study completed during the semester last spring under faculty moderator Jo Ann Fellin, OSB. In late November Richard Delaware from UMKC spoke to the group on "Kissing Numbers." Kansas Gamma ended the semester with the traditional Wassail party on December 5 at Marywood, home of Sister Jo Ann. Other fall 1999 officers: Curtis Sander, vice president; Jana Rupp, associate vice president; Jo Ann Fellin, corresponding secretary/faculty sponsor.

KS Delta Chapter President—Laurie Payeur

Washburn University, Topeka 24 actives

During the Fall, 99 semester the chapter members met with Mathematica, the Washburn Mathematics club, for four noon luncheon meetings. At three of these meetings there were speakers; Dr. Ron Wasserstein - Probability and the Kansas Lottery, Dr. Kevin Charlwood - Bridging the Gap from n=1 to n=2 (on solving Cubic Equations), and Dr. David Surowski (Kansas State University) - Ramsey Theory and Monotone Sequences. Other fall 1999 officers: Stephanie Adelhardt, vice president; Melissa Mikkelsen, secretary/treasurer; Allan Riveland, corresponding secretary; Ron Wasserstein and Donna LaLonde, faculty sponsors.

KS Epsilon Chapter President—Adam North
Fort Hays State University, Hays 27 actives

We held a meeting to orient new students, and also had regular meetings. We helped organize the Math Relays, a Western KS annual event. We also had guest speakers, and Halloween and Christmas parties. Other fall 1999 officers: Emily McDonald, vice president; Wendy Scott, secretary/treasurer; Cary Estes, historian; Chenglie Hu, corresponding secretary; Greg Force, Lanee Young, faculty sponsors.

KS Zeta Chapter President—Ashley Helfrich
Southwestern College, Winfield 9 actives

Other fall 1999 officers: Kristin Kraemer, vice president; Deana Pennington, secretary/treasurer; Mehri Arfaei, corresponding secretary; Reza Sarhangi, faculty sponsor.

KY Alpha Chapter President—Shannon Purvis
Eastern Kentucky University, Richmond 21 actives

The semester began with floppy disk sales (together with the ACM chapter) to students in the computer literacy class and the Mathematica class. At the September meeting, we had the election of new officers and discussed plans for the year. At the October meeting, Dr. Ray Tennant gave a talk on "Ramanujan: The Man Who Loved Numbers." In addition to speaking on Ramanujan and his mathematics, Dr. Tennant shared some of his personal experiences while in India over the summer. The November meeting was a joint meeting with the local chapter of ACM. The meeting contained a panel discussion on graduate schools. In December we had our White Elephant Gift Exchange at the Christmas party. During the fun, we played charades and Pictionary. The Boings team won the charades game. As a service project at Christmas time, members collected toys for children ages 3-5, put them in stockings, and took them to the Salvation Army. Other fall 1999 officers: Katy Fritz, vice president; Jennie Campbell, secretary; Kensaku Umeda, treasurer; Pat Costello, corresponding secretary.

KY Beta Chapter President—Velma Birdwell
Cumberland College, Williamsburg 42 actives

On September 7 the chapter officers helped to host an ice cream party for the freshmen math and physics majors. Along with the Mathematics and Physics Club and Sigma Pi Sigma, the chapter had a picnic at Briar Creek Park on September 21. Dr. Jonathan Ramey gave a special presentation about Sir Isaac Newton on November 17. On November 19, KY Beta held a special induction for two new members, both seniors. On the last day of classes, December 10, the entire department, including the Math and Physics Club, the Kentucky Beta chapter, and Sigma Pi Sigma had a Christmas party with about 45 people in attendance. The highlight of the evening was to see professors with pies in their faces, in an effort to raise money to buy Christmas presents for a needy child. Other fall 1999 officers: Simeon Hodges, vice president; Amanda Kidd, secretary; Melanie Maxson, treasurer; Jonathan Ramey, corresponding secretary; John Hymo, faculty sponsor.

LA Gamma Chapter President—Ross Chiquet
Northwestern State University, Natchitoches 22 actives, 1 associate

We meet twice a month for fellowship and to plan future activities. Other fall 1999 officers: Matt Creighton, vice president; Kristen Russell, secretary; Sarah Rankin, treasurer; Leigh Ann Myers, corresponding secretary; Lisa Galminas, faculty sponsor.

MD Alpha Chapter President—Kristen Balster
College of Notre Dame of Maryland, Baltimore 7 actives, 6 associates

On December 2 we held an induction ceremony for four new initiates. At out luncheon meeting Jane Orcutt gave a presentation describing her Internship Experience at the Space Telescope Science Institute. Other fall 1999 officers: Francesca Palek, vice president; Jane Orcutt, secretary; Joan Crawford, treasurer; Sister Marie Augustine Dowling, corresponding secretary; Joseph DiRienzi, faculty sponsor.

MD Delta Chapter President—Heather Barr
Frostburg State University, Frostburg 25 actives

A busy November saw two programs: a viewing of the video "The Proof", and a talk by Dr. Frank Barnet entitled "Dublin Ireland, and the Discovery of the Quaternions." A 50-50 raffle was held as a fund-raiser in December. Other fall 1999 officers: Eric Moore, vice president; Erin Resh, secretary; Rachel Duncan, treasurer; Edward White, corresponding secretary; John Jones, faculty sponsor.

MS Alpha Chapter President—Gordona Bauhan
Mississippi University for Women, Columbus 11 actives, 3 associates

On Sept. 28 we had our monthly meeting. On Oct. 14 we held our initiation. In November we had a monthly meeting, held a Bake Sale, and sponsored a talk: "Historical Problems in Trigonometry", by Ms.

Kathy McGarvey from the Mississippi School for Mathematics and Science. Other fall 1999 officers: Chris Sansing, vice president; Jennifer Kimble, secretary; Kent Smith, treasurer; Shaochen Yang, corresponding secretary; Beate Zimmer, faculty sponsor.

MS Epsilon Chapter President—Eric Carpenter
Delta State University, Cleveland 14 actives

Other fall 1999 officers: Audrey Stewart, vice president; Sallie Simpson Bodiford, secretary/ treasurer; Paula Norris, corresponding secretary; Rose Strahan, faculty sponsor.

MO Alpha Chapter President—Sam Blizard
Southwest Missouri State University, Springfield 22 actives, 7 associates

MO Alpha began the Fall Semester by hosting a departmental picnic. During the remainder of the semester we held monthly meetings with featured speakers, including a representative from the placement office, a faculty member, and two students. Other fall 1999 officers: Rachel Netzer, vice president; Erin Stewart, secretary; Sheri Puestow, treasurer; John Kubicek, corresponding secretary/faculty sponsor.

MO Beta Chapter President—Darin Tessier
Central Missouri State University, Warrensburg 25 actives, 5 associates

MO Beta initiated four full and three associate members in the fall. Trisha White was presented the Claude H. Brown Mathematics Achievement Award for Top Freshman of 1998-99. Graduate student Evan Maki spoke to KME at the September meeting about his undergraduate research done while he was an exchange student in Sweden. A representative from the Office of Career Services spoke at the October meeting. Dr. McKee demonstrated "Tips and Tricks for the Graphing Calculator" at the November meeting. The annual book sale was held in November and, as another way to raise funds, the members helped clean the stadium after a football game in October. The highlight of the semester may have been the student vs. faculty volleyball game held in December. The students took an early lead, but the faculty came back to tie at 2 games each. Everyone enjoyed the Christmas party held after the game. Other fall 1999 officers: Beth Hilbish, vice president; Becky Stafford, secretary; Jeff Callaway, treasurer; Beth Usher, historian; Rhonda McKee, corresponding secretary; Steve Shattuck, Phoebe Ho, and Larry Dilley, faculty sponsors.

MO Gamma Chapter President—Josh Stephenson
William Jewell College, Liberty 11 actives

MO Gamma participated in a Homecoming event with a display and welcome area for alums, who were members of the chapter at Jewell. Plans were also made to attend the Spring 2000 convention. Other fall 1999 officers: Laura Cline, vice president; Becky Huyser, secretary; Joseph Mathis, treasurer/corresponding secretary/faculty sponsor.

MO Epsilon Chapter President—April Smith
Central Methodist College, Fayette 5 actives

Other fall 1999 officers: Ryan Pharr, vice president; Sarah Moulder, secretary; William McIntosh, corresponding secretary; Linda Lembke & William McIntosh, faculty sponsors.

MO Theta Chapter President—Amanda Wachsmuth
Evangel University, Springfield 6 actives, 3 associates

Monthly Meetings were held. We had a party in November at the home of Dr. Tosh. Other fall 1999 officers: John Elliott, vice president; Don Tosh, corresponding secretary/faculty sponsor.

MO Iota Chapter President—Doug Osborne
Missouri Southern State College, Joplin 20 actives

Programs for regular monthly meetings were presented by students in the majors' Probability Course. Chapter members assisted the Math Department with a reception for all math, math ed, and compu-math majors. Members also worked the concession stands at home football games as a money making activity. A Christmas Party and while elephant gift exchange at the home of Dr. Chip Curtis wrapped up a successful semester. Other fall 1999 officers: Christin Mathis, vice president; Dondi Mitchell, secretary; Ted Walker, treasurer; Mary Elick, corresponding secretary; Charles Curtis, faculty sponsor.

MO Kappa Chapter President—Nathan Ratchford
Drury University, Springfield 13 actives, 5 associates

There was a high turn out for the first social event of the semester held at Dr. Pam Reich's house. The winner of the annual math contest this year was Kristi Smith for the Calculus II and above division and Kim Kocian for the Calculus I and below division. Prize money was awarded to the winners at a pizza party held for all contestants. The chapter participated in the Annual Exploration in Mathematics and the Physical Sciences, which is a recruitment workshop designed for high school students. Sub sandwiches were served to the chapter at the preliminary undergraduate research talks given by Kristi Smith and Nathan Ratchford. The Math Club has also been running a tutoring service for both the day school and the continuing education division (Drury Evening College). The semester ended with a Christmas party at Dr. Carol Collins' house. Other fall 1999 officers: Kristen Hannah, vice president; Kristi Smith, secretary; Luis Lee, treasurer;

Charles Allen, corresponding secretary; Pam Reich, faculty sponsor.

MO Lambda Chapter President—Shawna Smith
Missouri Western State College, St. Joseph 34 actives, 6 associates

The Missouri Lambda Chapter initiated six new members on October 17. Dr. Donald Vestal was the speaker for the program. He gave an interesting talk on the history of the Four Color Problem. Other fall 1999 officers: Shane Taylor, vice president; Charisa Greenfield, secretary; Byron Robiboux, treasurer; John Atkinson, corresponding secretary; Jerry Wilkerson, faculty sponsor.

NE Alpha Chapter President—Jeff Thoene
Wayne State College, Wayne 25 actives

Other fall 1999 officers: Russ Volk, vice president; Brandi Hall, secretary; Brian Kesting, treasurer; John Fuelberth, corresponding secretary; Jim Paige, faculty sponsor.

NE Beta Chapter President—Mike Sullivan
University of Nebraska at Kearney, Kearney 20 actives, 4 associates

A KME raffle to raise money for a scholarship netted $325. Other fall 1999 officers: Tisha Maas, vice president; Denise Anderson, secretary; Rich Arch, treasurer; Steve Bean, corresponding secretary; Richard Barlow, faculty sponsor.

NE Gamma Chapter President—Andy Boell
Chadron State College, Chadron 14 actives, 5 associates

Other fall 1999 officers: Shaun Daugherty, vice president; Craig Bruner, secretary; Kendra Pedersen, treasurer; Robert Stack, corresponding secretary; Don Anderson and Brent Rickenbach, faculty sponsors.

NM Alpha Chapter President—William Tierney
University of New Mexico, Albuquerque 90 actives, 21 associates

See our web pages at www.math.unm.edu/~kme/. Other fall 1999 officers: Jennifer Gill, vice president; Tony Malerich, secretary/treasurer; Archie Gibson, corresponding secretary/faculty sponsor.

NY Alpha Chapter President—Patricia Scavuzzo
Hofstra University, Hempstead 7 actives, 4 associates

We held a volleyball game and had a holiday party. Other fall 1999 officers: Rosemary Escobar, vice president; Kimberly Bleier, secretary; Vincent Perniciaro, treasurer; Aileen Michaels, corresponding secretary/faculty sponsor.

NY Kappa Chapter President—Kimberly Farrell

Pace University, New York 6 actives, 4 associates

Other fall 1999 officers: Svetlana Kolomeyskaye, vice president; Monica Mitrofanoff, secretary; Tim Zihharev, treasurer; Geraldine Taiani, corresponding secretary.

NY Lambda Chapter President—Rénèé des Etages

C.W. Post Campus of Long Island University, Brookville 22 actives

Other fall 1999 officers: Stephanie Anne Calzetta, vice president; Suzann Weaver, secretary; Steven McKinnon, treasurer; Andrew Rockett, corresponding secretary; John Stevenson, faculty sponsor.

NC Gamma Chapter President—Brooklyne Tester

Elon College, Elon College 16 actives

Other fall 1999 officers: Hilary Shannon, vice president; Jessica Pollard, secretary; Brian Neiberline, treasurer; Skip Allis, corresponding secretary/faculty sponsor.

OH Gamma Chapter President—Anila Xhunga

Baldwin-Wallace College, Berea 19 actives

Duke Hutchings gave a talk about his experiences in Miami U's summer mathematics program.

Gordon Wade of Bowling Green State University and Chris Cox of Clemson University gave talks regarding their mathematics programs. The chapter participated in the Putnam Exam. Paul Monsky of Brandeis University gave a talk on number theory and fractals. Other fall 1999 officers: Duke Hutchings, vice president; Jeff Smith, secretary; Corina Moise, treasurer; David Calvis, corresponding secretary; Chungsim Han and David Calvis, faculty sponsors.

OK Alpha Chapter President—Aaron Lee

Northeastern State University, Tahlequah 32 actives, 2 associates

Our fall initiation ceremonies brought eleven students and one faculty member into our chapter. Our fall speaker was Captain Ron Thrasher, Commander of the Criminal Investigation Division of the Stillwater Police Department. His talk was "Crunching the Numbers: Math in Modern Policing." The KME annual book sale was held in October. At NSU, the Northeastern State Government Association sponsors a free annual Halloween Carnival. KME's booth was the "KME Pumpkin Patch." We floated plastic pumpkins on water and the children fished for the pumpkins with meter stick fishing poles. The number on the bottom of the pumpkin determined how many candies the child received. Who says a "1" doesn't mean 1 candy for each of the child's fingers? Our Christmas party in December was great-games and pizza and Christmas treats. Other fall 1999 officers: Rhonda Cook, vice president; Chris Burba, secretary; Gregg Eddings, treasurer; Joan Bell, corresponding secretary/faculty sponsor.

OK Gamma Chapter President—Kory Hicks
Southwestern Oklahoma State University, Weatherford 23 actives, 7 associates

Other fall 1999 officers: Christy Koger, vice president; Shelly Davenport, secretary; Linda Coley, treasurer; Wayne Hayes, corresponding secretary; Gerry East, faculty sponsor.

PA Alpha Chapter President—Ryan Vacarro
Westminster College, New Wilmington 8 actives

Other fall 1999 officers: Larry Mumford, vice president; Sara Lieb, secretary; Eric Bass, treasurer; Warren Hickman, corresponding secretary; Carolyn Cuff, faculty sponsor.

PA Iota Chapter President—Thomas Ruffner
Shippensburg University, Shippensburg 18 actives

During the fall semester, KME co-sponsored a weekly seminar series. A variety of talks were presented ranging form careers, to computer science topics, to interesting math related questions. More of the same is planned for the spring. In addition, a year end pizza party was held. A good time was had by all. Other fall 1999 officers: Jaymie Kenny, vice president; Lisa Seniuk, secretary; Mark Reidenbaugh, treasurer/historian; Mike Seyfried, corresponding secretary; Cheryl Olsen, faculty sponsor.

PA Kappa Chapter Presidents—Linda Bruce and Lindsay Janka
Holy Family College, Philadelphia 6 actives

On September 15 the chapter hosted a Welcome for freshman math majors followed by a Math Careers Night Presentation, which was meant to answer the proverbial question: What can I do with a degree in math besides teach? The speakers, who were all graduates of Holy Family and KME members, were: Vince Frascatore '90 (Senior Analyst - Smith-Kline Beecham), Jacqueline Fallon '96 (Actuarial Analyst at Provident Mutual Life) and Lisa Esposito '97 (Systems Engineer at Lockheed-Martin). On October 19 the chapter hosted its third annual High School Math Competition. Eleven local high schools participated. A trophy went to the top school and the top student. Each participant received a certificate. On November 16 Dr. Jane Scanlon, Professor Emeritus of Mathematics at Rutgers University, New Brunswick Campus, gave the inaugural Fall Math Lecture. Dr. Scanlon spoke to the math majors and KME members on her research linking the study of differential equations to cardiology. On December 8, at the college's Christmas Rose program, the Calculus III class presented 'Twas the Night Before Calculus, a parody of 'Twas the

Night Before Christmas. Throughout the Fall Semester, the chapter posted a Problem of the Week on the Math Bulletin Board. Anyone submitting a correct answer to any of the ten problems of the week was eligible for the December 10 drawing for a $20 gift certificate to Franklin Mills Mall. The chapter also published several issues of a chapter newsletter which was distributed to all students in the 200, 300 and 400 level math courses. Other fall 1999 officers: Shannon Marczely, secretary; Sister Benedykta Mazur, treasurer; Sister Marcella Louise Wallowicz, corresponding secretary/faculty sponsor.

PA Mu Chapter President—Kate Wallace
Saint Francis College, Loretto 21 actives

PA Mu co-sponsored the Sixth Annual Science/Mathematics Day on November 19 where 385 high school students and teachers from 25 high schools came to campus for a variety of presentations, a science bowl competition, and several other challenging activities. KME members picked trash along the two-mile stretch of adopted highway. Several members also participated in a week-long summer mathematics camp for middle school girls, where 45 girls participated in enrichment activities and applications of mathematics. KME members participated as group leaders, mentors, presenters, and role models. Other fall 1999 officers: Rebecca Espenlaub, vice president; Kourosh Barati-Sedeh, secretary; Brian Quinn, treasurer; Pete Skoner, corresponding secretary; Amy Miko, faculty sponsor.

PA Omicron Chapter President—Erin Hepinger
University of Pittsburgh at Johnstown, Johnstown 28 actives

In conjunction with the UPJ Math Club and Student Chapter of MAA we had a Welcome Back pizza party in September, a video showing of Donald Duck in Mathmagicland in November, and a student/faculty Christmas party in December. Other fall 1999 officers: Marie Hufford, vice president; Christy Lynch, secretary; Michelle Vincenzini, treasurer; Nina Girard, corresponding secretary/faculty sponsor.

PA Pi Chapter President—Melissa Persing
Slippery Rock University of Pennsylvania, Slippery Rock 18 actives

Meetings were held to complete the constitution for our local chapter. Other fall 1999 officers: Crystal Hogue, vice president; Carrie Birckbichler, secretary; Nishan Jayaratnam, treasurer; Elise Grabner, corresponding secretary; Robert Vallin, faculty sponsor.

SC Gamma Chapter President—Kelly Clardy
Winthrop University, Rock Hill 8 actives

Other fall 1999 officers: Kourtnee Barnett, vice president; Stephanie

Boswell, treasurer; Frank Pullano, corresponding secretary.

SC Delta Chapter President—Angie Sears
Erskine College, Due West 15 actives, 12 associates

We had 4 meetings during which we worked logic puzzles, took a hiking trip with the physics club, and listened to Walt Patterson from Lander University speak to a joint meeting of the Lander Math Club and Erskine's KME Chapter. At the end of the semester, the chapter sent its president on a shopping trip for Christmas presents for a needy child. Other fall 1999 officers: Tad Whitesides, vice president; Minh Le, secretary/treasurer; Ann Bowe corresponding secretary; Susan Patterson, faculty sponsor.

TN Beta Chapter President—B.J. Smith
East Tennessee State University, Johnson City 17 actives, 9 associates

Eight students were inducted into Tennessee Beta, and officers were elected for the year. We also had a social at the firehouse restaurant. Other fall 1999 officers: Mark Taylor, vice president; Susan Hosler, secretary; Tabitha Taylor, treasurer; Lyndell Kerley, corresponding secretary.

TN Gamma Chapter President—Lindsey Crain
Union University, Jackson 17 actives

TN Gamma started the year with a back-to-school pizza party. The social was held on October 26 in the MacAfee Commons Area on campus. On November 18, student member Jamie Mosley gave a talk on his summer internship, where he used his mathematical and statistical training in the business world. The annual KME Christmas party was held on December 4 at the home of Dr. Lunsford. The organization sponsored two needy children for the annual Carl Perkins Christmas Program held on December 9. Other fall 1999 officers: Cathie Scarbrough, vice president; Melissa Culpepper, secretary; Sarah Shaub, treasurer; Bryan Dawson, corresponding secretary; Matt Lunsford, faculty sponsor.

TN Epsilon Chapter President—Jennifer Dowdy
Bethel College, McKenzie 7 actives, 3 associates

TN Epsilon inducted one new member at the installation ceremony held in November. In addition to monthly meetings, the chapter participated in a fund-raising event with Gamma Beta Phi Honor Society in October. Plans are being made for a social event in the spring and possible attendance at the convention by one or more members. Other fall 1999 officers: Belinda Thompson, vice president; Christina Hill, secretary/treasurer; Russell Holder, corresponding secretary; David Lankford, faculty sponsor.

TX Kappa Chapter President—Sheri Asbury
University of Mary Hardin-Baylor, Belton 12 actives, 5 associates

An induction ceremony was held on Nov. 19. Other fall 1999 officers: Robin Stokes, vice president; Kimberly Klentzman, secretary/treasurer; Peter Chen, corresponding secretary; Maxwell Hart, faculty sponsor.

VA Gamma Chapter President—Sarah Lemon
Liberty University, Lynchburg 19 actives, 7 associates

We had four former students in to speak to the chapter. Two were actuaries (a married couple), one was a mathematics teacher in a local high school, and the other was a graduate student who is a research assistant while working on his doctorate in mathematics. One of the actuaries was a recent graduate; the other has been in the profession several years. Each has passed some of the actuarial exams. All of these programs were very helpful to the students in determining what careers they might like. Other fall 1999 officers: Sean Burtt, vice president; Bobbi Heim, secretary; Curtis Hartman, treasurer; Glyn Wooldridge, corresponding secretary; Sandra Rumore, faculty sponsor.

WI Gamma Chapter President—Elizabeth Whitney
University of Wisconsin-Eau Claire, Eau Claire 25 actives, 10 associates

Other fall 1999 officers: Julie Loasching, vice president; Kia Xiong, secretary; Terry Svihovec, treasurer; Marc Goulet, corresponding secretary/faculty sponsor.

Oops

In the installation report for the North Carolina Delta chapter (reported in the Fall 1999 issue of *The Pentagon*), the names of the founding group were inadvertently left off. Here they are (only a few months tardy):

Students: Phill Carroll, Mindy Cox, Misty Dills, Jess Frey, Brian Fulton, Amanda Hall, Melinda Harvey, Adrienne Hill, Michelle Holland, Jen Kleinrichert, Jason Lowder, Burton Martin, Rebecca Smith, and Julie Weavil.

Faculty: Jeff Butera, Lisa Carnell, Rob Harger, Nelson Page, and Shirley Robertson.

Announcement of the Thirty-Third Biennial Convention of Kappa Mu Epsilon

The Thirty-Third Biennial Convention of Kappa Mu Epsilon will be hosted by the Kansas Beta chapter located at Washburn University in Topeka, Kansas. The convention will take place April 19-21, 2001. Each attending chapter will receive the usual travel expense reimbursement from the national funds as described in Article VI, Section 2, of the Kappa Mu Epsilon Constitution.

A significant feature of our national convention will be the presentation of papers by student members of Kappa Mu Epsilon. The mathematical topic selected by each student speaker should be of interest to the author and of such scope that it can be given adequate treatment in a timed oral presentation. Student talks to be judged at the convention will be chosen prior to the convention by the Selection Committee on the basis of submitted written papers. At the convention, the Awards Committee will judge the selected talks on both content and presentation. The rankings of both the Selection and Awards Committees will determine the top four papers.

Who may submit a paper?

Any undergraduate or graduate student member of Kappa Mu Epsilon may submit a paper for consideration as a talk at the national convention. A paper may be co-authored. If selected for presentation at the convention, the paper must be presented by one (or more) of the authors.

Presentation topics

Papers submitted for presentation at the convention should discuss material understandable by undergraduates who have completed only differential and integral calculus. The Selection Committee will favor papers that satisfy this criterion and which can be presented with reasonable completeness within the time allotted. Papers may be original research by the student(s) or exposition of interesting but not widely known results.

Presentation time limits

Papers presented at the convention should take between 15 minutes and 25 minutes. Papers should be designed with these time limits in mind.

How to prepare a paper

The paper should be written in the standard form of a term paper. It should be written much as it will be presented. A long paper (such as an honors thesis) must not be submitted with the idea that it will be shortened at presentation time. Appropriate references and a bibliography are expected. Any special visual aids that the host chapter will need to provide (such as a computer and overhead projection system) should be clearly indicated at the end of the paper.

The first page of the paper must be a "cover sheet" giving the following information: (1) title, (2) author or authors (these names should not appear elsewhere in the paper), (3) student status (undergraduate or graduate), (4) permanent and school addresses, phone numbers, and e-mail addresses, (5) name of the local KME chapter and school, (6) signed statement giving approval for consideration of the paper for publication in *The Pentagon* (or a statement about submission for publication elsewhere), and (7) a signed statement of the chapter's Corresponding Secretary attesting to the author's membership in Kappa Mu Epsilon.

How to submit a paper

Five copies of the paper, with a description of any charts, models, or other visual aids that will be used during the presentation, must be submitted. The cover sheet need only be attached to one of the five copies. The five copies of the paper are due by February 15, 2000. They should be sent to:

Dr. Robert Bailey, KME President-Elect
Department of Mathematics
Niagara University, NY 14109-2044

Selection of papers for presentation

A Selection Committee will review all papers submitted by undergraduate students and will choose approximately fifteen papers for presentation and judging at the convention. Graduate students and undergraduate students whose papers are not selected for judging will be offered the opportunity to present their papers at a parallel session of talks during the convention. The President-Elect will notify all authors of the status of their papers after the Selection Committee has completed its deliberations.

Criteria used by the Selection and Awards Committees

The papers submitted to the Selection Committee should be of appropriate length so that the associated oral presentation could reasonably be given in 15 to 25 minutes.

Judging criteria used by both the Selection Committee and Awards Committee will include: (1) originality of topic and appropriateness for student audience, (2) clarity and organization of materials, (3) depth, significance, and correctness of content, and (4) credit given to literature sources and faculty directors as appropriate.

In addition to the above criteria, the Awards Committee will judge the oral presentation of the paper on: (1) overall style and effectiveness of presentation, (2) presenter's apparent understanding of content, (3) effective use of technology and/or appropriate audio-visual aids, and (4) adherence to any time constraints imposed.

Prizes

All authors of papers presented at the convention will be given two-year extensions of their subscription to *The Pentagon*. Authors of the four best papers presented by undergraduates, as decided by the Selection and Awards Committees, will each receive a cash prize of $100.

Publication

All papers submitted to the convention are generally considered as submitted for publication in *The Pentagon*. Unless published elsewhere, prize-winning papers will be published in *The Pentagon* after any necessary revisions have been completed (see page 2 of *The Pentagon* for further information). All other papers will be considered for publication. The Editor of *The Pentagon* will schedule a brief meeting with each author during the convention to review his or her manuscript.

Kappa Mu Epsilon National Officers

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Kappa Mu Epsilon, Mathematics Honor Society, was founded in 1931. The object of the Society is fivefold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics due to its demands for logical and rigorous modes of thought; to provide a Society for the recognition of outstanding achievement in the study of mathematics at the undergraduate level; and to disseminate the knowledge of mathematics and familiarize the members with the advances being made in mathematics. The official journal of the Society, *The Pentagon*, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the Chapters.

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 April 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Southwest Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
MS Beta	Mississippi State University, Mississippi State	14 Dec 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
NM Alpha	University of New Mexico, Albuquerque	28 March 1935
IL Beta	Eastern Illinois University, Charleston	11 April 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 April 1937
OH Alpha	Bowling Green State University, Bowling Green	24 April 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	Central Missouri State University, Warrensburg	10 June 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
TX Beta	Southern Methodist University, Dallas	15 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 June 1941
NY Alpha	Hofstra University, Hempstead	4 April 1942
MI Beta	Central Michigan University, Mount Pleasant	25 April 1942
NJ Beta	Montclair State University, Upper Montclair	21 April 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 March 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 June 1947
CO Alpha	Colorado State University, Fort Collins	16 May 1948
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 April 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960

OH Epsilon	Marietta College, Marietta	29 Oct 1960
MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
IL Epsilon	North Park College, Chicago	22 May 1963
OK Beta	University of Tulsa, Tulsa	3 May 1964
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 April 1965
AL Epsilon	Huntingdon College, Montgomery	15 April 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
AR Alpha	Arkansas State University, State University	21 May 1965
TN Gamma	Union University, Jackson	24 May 1965
WI Beta	University of Wisconsin—River Falls, River Falls	25 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	Western Maryland College, Westminster	30 May 1965
IL Zeta	Rosary College, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 March 1971
KY Alpha	Eastern Kentucky University, Richmond	27 March 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 April 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 April 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State College, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sept 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis College, Loretto	14 Sept 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983

MO Kappa	Drury College, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 March 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 April 1986
TX Iota	McMurry University, Abilene	25 April 1987
PA Nu	Ursinus College, Collegeville	28 April 1987
VA Gamma	Liberty University, Lynchburg	30 April 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 April 1990
CO Delta	Mesa State College, Grand Junction	27 April 1990
NC Gamma	Elon College, Elon College	3 May 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 April 1991
SD Alpha	Northern State University, Aberdeen	3 May 1992
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 March 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 April 1997
MI Delta	Hillsdale College, Hillsdale	30 April 1997
MI Epsilon	Kettering University, Flint	28 March 1998
KS Zeta	Southwestern College, Winfield	14 April 1998
TN Epsilon	Bethel College, McKenzie	16 April 1998
MO Mu	Harris-Stowe College, St. Louis	25 April 1998
GA Beta	Georgia College and State University, Milledgeville	25 April 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
NY Xi	Buffalo State College	12 May 1998
NY Xi	Buffalo State College, Buffalo	May 12, 1998
NC Delta	High Point University, High Point	March 24, 1999
PA Pi	Slippery Rock University, Slippery Rock	April 19, 1999
TX Lambda	Trinity University	November 22, 1999

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