

THE PENTAGON

Volume XXVII

Spring, 1968

Number 2

CONTENTS

	<i>Page</i>
National Officers -----	70
Communication Networks Using Matrices <i>By Judy Kaldenberg</i> -----	71
Rotation about an Axis <i>By Ali R. Amir-Moéz</i> -----	82
The Moveable Figures <i>By Andrea Lee Meyer</i> -----	86
The Gamma Function: Or Why $0! = 1$ <i>By Frank Fishell</i> -----	96
Quantum Chess <i>By Douglas A. Engel</i> -----	99
An Interesting Consequence of Rolle's Theorem <i>By R. S. Luthar</i> -----	104
The Mathematical Scrapbook -----	106
The Book Shelf -----	114
The Problem Corner -----	121
Kappa Mu Epsilon News -----	128

National Officers

- FRED W. LOTT - - - - - *President*
University of Northern Iowa, Cedar Falls, Iowa
- GEORGE R. MACH - - - - - *Vice-President*
California State Polytechnic College,
San Luis Obispo, California
- LAURA Z. GREENE - - - - - *Secretary*
Washburn University of Topeka, Topeka, Kansas
- WALTER C. BUTLER - - - - - *Treasurer*
Colorado State University, Fort Collins, Colorado
- EDDIE ROBINSON - - - - - *Historian*
Southwest Missouri State College, Springfield, Missouri
- CARL V. FRONABARGER - - - - - *Past President*
Southwest Missouri State College, Springfield, Missouri
-

Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Communication Networks Using Matrices*

JUDY KALDENBERG

Student, University of Northern Iowa, Cedar Falls

In our complex world of today, many physical problems can be represented and solved using a mathematical model. One such physical problem is that of a communication network, which we will represent and analyze using matrices. A communication network consists of a set of n people, call them $A_1, A_2, A_3, \dots, A_n$, such that between some pairs of persons there exists a communication link. This communication link may be either one-way or two-way. Examples of one-way communication links include sending a messenger, a smoke signal, or a signal light, while two-way communication links may be represented by a telephone, telegraph, or a short-wave radio where both parties can receive and transmit. We are interested here not only in the communication links between single pairs of persons but in those links that could continue from one person to another, involving three, four, or more people. This type of communication network takes place when one person of a group notifies several members of the group of a message, they, in turn, notify several other members, and this procedure is continued until everyone in the group has been contacted.

To indicate communication ability, we shall use the symbol \rightarrow , where $A_1 \rightarrow A_2$ means that A_1 can communicate with A_2 (in that order). It will be convenient to make one restriction; that is, it is false that $A_i \rightarrow A_i$, for any i , meaning that an individual cannot (or need not) communicate with himself. It is possible, however, that both $A_i \rightarrow A_j$ and $A_j \rightarrow A_i$, that is, a two-way self-communication link is possible. We describe such situations by saying symmetry exists between A_i and A_j . When the communication is not two-way, that is $A_i \rightarrow A_j$, or $A_j \rightarrow A_i$, but not both, it is said antimetry exists between A_i and A_j .

One way of representing communication networks is by directed graphs with arrows indicating the direction in which communication is possible. A double arrow indicates communication is possible in both directions. Using four people, two possible communication networks could be represented as:

*A paper presented at the 1967 National Convention of **IME** at Atchison, Kansas, April 7-8.

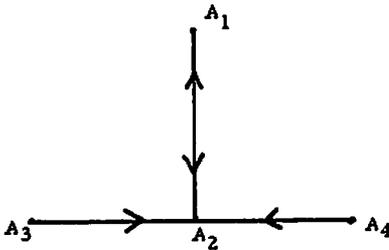


Figure 1

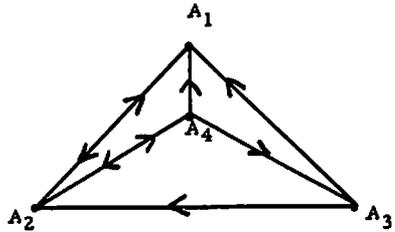


Figure 2

As more people become involved, it becomes more and more difficult to represent the communication network using directed graphs.

It has been suggested that the analysis and presentation of results concerning relations among groups of people, as communication networks, may be greatly improved by using matrix algebra [1, p. 95-116]. It will be necessary to add, subtract, and multiply square matrices. It will be assumed the reader is familiar with adding (or subtracting) matrices by adding (or subtracting) elements in corresponding positions and the usual row by column definition of matrix multiplication.

Using a square matrix, the communication network can be represented as follows:

$$\begin{matrix}
 & A_1 & A_2 & A_3 & \dots & A_j & \dots & A_n \\
 A_1 & x_{11} & x_{12} & x_{13} & \dots & x_{1j} & \dots & x_{1n} \\
 A_2 & x_{21} & \dots & \dots & \dots & x_{2j} & \dots & \dots \\
 \cdot & \cdot & & & & \cdot & & \cdot \\
 \cdot & \cdot & & & & \cdot & & \cdot \\
 \cdot & \cdot & & & & \cdot & & \cdot \\
 \cdot & \cdot & & & & \cdot & & \cdot \\
 A_i & x_{i1} & \dots & \dots & \dots & x_{ij} & \dots & \dots \\
 \cdot & \cdot & & & & \cdot & & \cdot \\
 \cdot & \cdot & & & & \cdot & & \cdot \\
 A_n & x_{n1} & \dots & \dots & \dots & \dots & \dots & x_{nn}
 \end{matrix} = (x_{ij})$$

where the entry x_{ij} has the value 1 if $A_i \rightarrow A_j$, and the value 0 if it is false that $A_i \rightarrow A_j$. The main diagonal terms are all 0 since $A_i \rightarrow A_i$ is impossible.

As an illustration, the directed graph of Figure 2 shows that $A_1 \rightarrow A_2$, $A_2 \rightarrow A_1$, $A_2 \rightarrow A_3$, $A_4 \rightarrow A_2$, $A_3 \rightarrow A_1$, $A_3 \rightarrow A_2$, $A_4 \rightarrow A_1$, and $A_4 \rightarrow A_3$. The matrix X associated with this network is:

$$X = \begin{matrix} & A_1 & A_2 & A_3 & A_4 \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

From the matrix X , we form a new matrix, S , which indicates the times when both A_i can communicate with A_j and A_j can communicate with A_i . The entries of S are such that $s_{ij} = s_{ji} = 1$ if $x_{ij} = x_{ji} = 1$ and otherwise $s_{ij} = s_{ji} = 0$. S will be a symmetric matrix since $s_{ij} = s_{ji}$, for every i and j . For the illustration above

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

From X , we have $A_1 \rightarrow A_2$, $A_2 \rightarrow A_1$, and $A_4 \rightarrow A_3$. Therefore, it can be seen that A_1 can communicate with A_3 by a three-step link. This is called a three-chain from A_1 to A_3 , and may be written as $A_1A_2A_4A_3$. It can be seen that an n -step sequence would have $n + 1$ members.

When the same element appears more than once in an n -chain, the n -chain is said to be redundant. The chains $a \rightarrow b \rightarrow e \rightarrow d \rightarrow b \rightarrow d$ ($abedbd$) and $a \rightarrow c \rightarrow a \rightarrow b \rightarrow d \rightarrow c \rightarrow e$ ($acabdce$) are both redundant since b and d occur twice in the former and the elements a and c both occur twice in the latter.

It can be seen that by squaring the X matrix, the entry $x_{ij}^{(2)}$ will give the number of two-step links from A_i to A_j . If $E = X^2$ and $e_{ij} = x_{ij}^{(2)}$ is the element in the i -th row and j -th column of E , the definition of matrix multiplication implies that $e_{ij} = \sum_{k=1}^n x_{ik}x_{kj}$. For each k in this sum, the product $x_{ik}x_{kj}$ will be 1 if and only if $x_{ik} = 1$ and $x_{kj} = 1$, that is, when there is a two-step link from A_i to A_j through A_k ($A_i \rightarrow A_k \rightarrow A_j$). When at least one of x_{ik} or x_{kj} is 0, such a two-step link does not exist, and the product $x_{ik}x_{kj}$ will be 0. The element e_{ij} of E will be the result of a sum of ones and zeros with the total indicating the number of distinct two-step links from A_i to A_j .

With the help of induction, it is easy to show that in the matrix X^n , the entry $x_{ij}^{(n)} = c$ if and only if there are c distinct n -chains from A_i to A_j [1].

If we square our matrix

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

we get

$$X^2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 1 \end{bmatrix}$$

It is obvious that these numbers in our squared matrix indicate the number of two-step links since, for example, the 1 in the third row and fourth column arises upon multiplying the third row of our X matrix by the fourth column, due to the fact that A_3 can communicate with A_2 and A_2 can communicate with A_4 . Therefore, A_3 can

communicate with A_1 in a two-step link through A_2 , which accounts for the 1 in the squared matrix. It can also be seen that A_4 can communicate with A_1 in two two-step links, and since before, he could communicate directly with A_1 , he now has three different methods which he can use to communicate with A_1 .

The redundant two-step links are those where A_i communicates with himself through some other person A_k ($A_i \rightarrow A_k \rightarrow A_i$). These links can be classified as redundant since the A_i element occurs more than once, at the beginning and at the end. The number of such links is given by the i -th diagonal element of the squared matrix. Therefore

$$R^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is the matrix of redundant two-step links where the diagonal elements are the diagonal elements from X^2 and the other elements are 0.

Redundant three-steps would include redundancies of the type where the first element is repeated, the last element is repeated, the first element equals the last, or where there are two redundancies and the first and last elements are redundant but they do not equal each other. It can be shown that to obtain the matrix of redundant three-steps, first the following matrix is computed where $R^{(2)}$ stands for the matrix of redundant two-steps:

$$XR^{(2)} + R^{(2)}X - S$$

Deleting in this sum, the main diagonal and replacing it by the main diagonal of X^3 , gives the matrix of redundant three-steps. If the main diagonal of $XR^{(2)} + R^{(2)}X - S$ is denoted by $Y^{(3)}$ and the main diagonal of X^3 by $Z^{(3)}$, then the matrix of redundant three-steps, $R^{(3)}$, is given by:

$$R^{(3)} = XR^{(2)} + R^{(2)}X + Z^{(3)} - Y^{(3)} - S.$$

By now, you are probably beginning to wonder what good all of this one-step and two-step communication business is anyhow. We are going to illustrate this by using a hypothetical situation. Let us take two groups of seven men each and let us let group X and group Y declare war on each other. Group X consists of men A, B, C, D, E, F, G and group Y consists of men J, K, L, M, N, O, P. Suppose that they are in a jungle and a victory greatly depends on their ability to keep in communication with each other. We will let the communication networks of X and Y be respectively:

$$X = \begin{matrix} & A & B & C & D & E & F & G \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{matrix} \quad = \quad \begin{matrix} & J & K & L & M & N & O & P \\ \begin{matrix} J \\ K \\ L \\ M \\ N \\ O \\ P \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{matrix}$$

It can be seen that the corresponding men A and J, B and K, etc., of the X and Y matrices, can communicate with others in the same number of ways but in a different order (the corresponding rows have the same number of ones). This arrangement was arbitrary with no special outcome intended. It can also be seen that matrix Y is a symmetric matrix, while X is not, which means that all the men in group Y who can communicate in one direction can communicate back. We will see what effect this has on their further communication ability. The two-step connections among members of Group X and Group Y are then represented respectively by X^2 and Y^2 :

self-communication links as the number of individuals he can communicate with in the original matrix.

From here, we will now go on, and compare their communication abilities on the three-step links by computing the third powers of their respective matrices as follows:

$$X^3 = \begin{matrix} & A & B & C & D & E & F & G \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{matrix} & \left[\begin{array}{cccccc} 3 & 6 & 3 & 7 & 4 & 4 & 5 \\ 4 & 4 & 4 & 4 & 3 & 3 & 3 \\ 7 & 9 & 7 & 6 & 6 & 7 & 6 \\ 5 & 10 & 6 & 7 & 3 & 6 & 7 \\ 6 & 8 & 5 & 8 & 6 & 6 & 7 \\ 4 & 8 & 4 & 6 & 2 & 5 & 5 \\ 6 & 9 & 6 & 5 & 3 & 6 & 7 \end{array} \right] \end{matrix}$$

$$\begin{matrix} & J & K & L & M & N & O & P \\ \begin{matrix} J \\ K \\ L \\ M \\ N \\ O \\ P \end{matrix} & \left[\begin{array}{cccccc} 2 & 4 & 8 & 4 & 4 & 4 & 8 \\ 4 & 2 & 5 & 3 & 3 & 3 & 3 \\ 8 & 5 & 4 & 10 & 10 & 5 & 5 \\ 4 & 3 & 10 & 8 & 9 & 9 & 11 \\ 4 & 3 & 10 & 9 & 8 & 9 & 11 \\ 4 & 3 & 5 & 9 & 9 & 6 & 8 \\ 8 & 3 & 5 & 11 & 11 & 8 & 6 \end{array} \right] \end{matrix} = Y^3$$

Just from looking at the above matrices, there does not really seem to be too great a difference in the communication abilities of these two groups at the three-step level.

Once again we want to compute the non-redundant three-steps. The matrix of redundancies of the three-step communication network was given by:

$$R^{(3)} = XR^{(2)} + R^{(2)}X + Z^{(3)} - Y^{(3)} - S.$$

The calculated redundancies are:

$$R_x^{(3)} = \begin{matrix} & A & B & C & D & E & F & G \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{matrix} & \begin{bmatrix} 3 & 4 & 0 & 3 & 0 & 0 & 3 \\ 0 & 4 & 0 & 0 & 2 & 2 & 0 \\ 4 & 0 & 7 & 3 & 3 & 0 & 3 \\ 3 & 4 & 3 & 7 & 0 & 0 & 4 \\ 0 & 2 & 0 & 3 & 6 & 2 & 3 \\ 0 & 2 & 3 & 3 & 0 & 5 & 0 \\ 3 & 4 & 3 & 0 & 0 & 3 & 7 \end{bmatrix} & \begin{matrix} J & K & L & M & N & O & P \\ \begin{bmatrix} 2 & 4 & 6 & 0 & 0 & 0 & 6 \\ 4 & 2 & 5 & 0 & 0 & 0 & 0 \\ 6 & 5 & 4 & 7 & 7 & 0 & 0 \\ 0 & 0 & 7 & 8 & 7 & 6 & 7 \\ 0 & 0 & 7 & 7 & 8 & 6 & 7 \\ 0 & 0 & 0 & 6 & 6 & 5 & 6 \\ 6 & 0 & 0 & 7 & 7 & 6 & 6 \end{bmatrix} & \end{matrix} = R_y^{(3)}$$

Once again, group Y seems to have a larger share of redundancies, which will eliminate some of their possible communication links.

The matrix of non-redundant three-step communication links would be given, therefore, by deleting the redundancies from the total three-step links, or in other words by:

$$X^3 - R_x^{(3)} = \begin{matrix} & A & B & C & D & E & F & G \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{matrix} & \begin{bmatrix} 0 & 2 & 3 & 4 & 4 & 4 & 2 \\ 4 & 0 & 4 & 4 & 1 & 1 & 3 \\ 3 & 9 & 0 & 3 & 3 & 7 & 3 \\ 2 & 6 & 3 & 0 & 3 & 6 & 3 \\ 6 & 6 & 5 & 5 & 0 & 4 & 4 \\ 4 & 6 & 1 & 3 & 2 & 0 & 5 \\ 3 & 5 & 3 & 5 & 3 & 3 & 0 \end{bmatrix} & \end{matrix}$$

$$\begin{array}{c}
 J \ K \ L \ M \ N \ O \ P \\
 \begin{array}{l}
 J \\
 K \\
 L \\
 M \\
 N \\
 O \\
 P
 \end{array}
 \begin{bmatrix}
 0 & 0 & 2 & 4 & 4 & 4 & 2 \\
 0 & 0 & 0 & 3 & 3 & 3 & 3 \\
 2 & 0 & 0 & 3 & 3 & 5 & 5 \\
 4 & 3 & 3 & 0 & 2 & 3 & 4 \\
 4 & 3 & 3 & 2 & 0 & 3 & 4 \\
 4 & 3 & 5 & 3 & 3 & 0 & 2 \\
 2 & 3 & 5 & 4 & 4 & 2 & 0
 \end{bmatrix}
 \end{array} = Y^3 - R_p^{(3)}$$

It can be seen from the above that the three-step links between *J* and *K*, *K* and *J*, and *K* and *L* are all redundant although there are some two-step links between these individuals. In group *X*, it can be seen that there are no redundant three-step links among any individuals. Any individual can communicate with any other individual, other than himself, in a three-step communication link. It seems, on the basis of the three-step links, that group *X* would have an advantage. To see which group really has the advantage in communication over one, two, and three-step communication links, we would want to add the respective non-redundant communication matrices for the various one, two, and three steps. Therefore, the total number of ways the individuals can communicate with each other is given by the following total matrices:

$$\begin{array}{c}
 A \ B \ C \ D \ E \ F \ G \\
 \begin{array}{l}
 A \\
 B \\
 C \\
 D \\
 E \\
 F \\
 G
 \end{array}
 \begin{bmatrix}
 0 & 5 & 5 & 4 & 6 & 6 & 4 \\
 4 & 0 & 5 & 6 & 2 & 3 & 4 \\
 3 & 13 & 0 & 6 & 4 & 9 & 7 \\
 5 & 9 & 5 & 0 & 5 & 8 & 6 \\
 8 & 10 & 8 & 7 & 0 & 7 & 7 \\
 6 & 8 & 3 & 5 & 4 & 0 & 7 \\
 5 & 8 & 5 & 8 & 5 & 5 & 0
 \end{bmatrix}
 \end{array}
 \begin{array}{c}
 J \ K \ L \ M \ N \ O \ P \\
 \begin{array}{l}
 J \\
 K \\
 L \\
 M \\
 N \\
 O \\
 P
 \end{array}
 \begin{bmatrix}
 0 & 2 & 4 & 6 & 6 & 5 & 3 \\
 2 & 0 & 2 & 4 & 4 & 3 & 4 \\
 4 & 2 & 0 & 5 & 5 & 7 & 8 \\
 6 & 4 & 5 & 0 & 6 & 6 & 7 \\
 6 & 4 & 5 & 6 & 0 & 6 & 7 \\
 5 & 3 & 7 & 6 & 6 & 0 & 5 \\
 3 & 4 & 8 & 7 & 7 & 5 & 0
 \end{bmatrix}
 \end{array} = Y_t$$

From these matrices, it appears that X has the larger range of possible ways for its individual members to communicate. For example, group X has ten communication links which can be carried out in eight or more ways while group Y has only two links of this type. Totaling these matrices gives group X 250 communication links and group Y 210 links. It seems then that the fact that the members of group Y were in symmetric relation to each other did not really benefit the group any, but may have even hindered it in that it caused more redundancies. Therefore, if a victory depends on the ability to communicate, group X should win this war!

LIST OF REFERENCES

- [1] Luce, R. Duncan, and Perry, Albert C.; "A Method of Matrix Analysis of Group Structure," *Psychometrika*; The Psychometric Society; Vol. 14, No. 1; March, 1949.
- [2] Trimble, H. C., and Lott, Fred W. Jr.; *Elementary Analysis — A Modern Approach*; Prentice-Hall, Inc.; 1960.
- [3] Kemeny, John G., Snell, J. Laurie, and Thompson, Gerald L.; *Introduction to Finite Mathematics*; Prentice-Hall, Inc.; 1956-57.
- [4] Mac Duffee, Cyrus Colton; *Vectors and Matrices*; The Mathematical Association of America; 1943.



One may be a mathematician of the first rank without being able to compute. It is possible to be a great computer without having the slightest idea of mathematics.

—NOVALIS

Rotation About an Axis

ALI R. AMIR-MOËZ

Faculty, Texas Technological College, Lubbock

The problem of rotation about an axis in a three dimensional Euclidean space has been solved many ways. But often it is ignored in the classroom because it is a little long. In this note we shall give the detail of obtaining the matrix of this transformation.

1. *Notations:* Vectors will be denoted by Greek letters. The inner product of two vectors will be indicated by (ξ, η) . The expression $||\xi||$ means the norm of ξ . A linear transformation will be denoted by a capital letter.

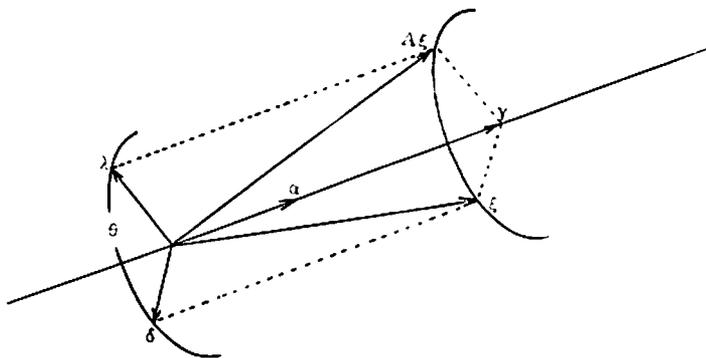
2. *Problem:* Find the matrix of rotation about the line

$$x = at, \quad y = bt, \quad z = ct.$$

Solution: Indeed, at least one of a , b and c is different from zero. One observes that

$$\alpha = \left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}} \right)$$

is a unit vector on the line. Let $\xi = (x, y, z)$ be any vector and let $\gamma = (\xi, \alpha)\alpha$, and $\delta = \xi - \alpha$.



Suppose the rotation is A . Since γ is on the axis of rotation $A\gamma = \gamma$. Let $A\delta = \lambda$. Then

$$(1) \quad \begin{cases} (\lambda, \alpha) = 0 \\ ||\lambda|| = ||\delta|| \\ (\lambda, \delta) = ||\delta||^2 \cos \theta. \end{cases}$$

Thus $A\xi = \gamma + \lambda$.

We observe that

$$\gamma = \left(\frac{a^2x + aby + acz}{a^2 + b^2 + c^2}, \frac{abx + b^2y + bcz}{a^2 + b^2 + c^2}, \frac{acx + bcy + c^2z}{a^2 + b^2 + c^2} \right),$$

and

$$\delta = \left(\frac{(b^2 + c^2)x - aby - acz}{a^2 + b^2 + c^2}, \frac{(a^2 + c^2)y - abx - bcz}{a^2 + b^2 + c^2}, \frac{(a^2 + b^2)z - acx - bcy}{a^2 + b^2 + c^2} \right).$$

Let $\lambda = (t, r, s)$ and $\frac{1}{a^2 + b^2 + c^2} = h$. Then (1) will become

$$\begin{cases} at + br + cs = 0, \\ t^2 + r^2 + s^2 = h^2 \{ [(b^2 + c^2)x - aby - acz]^2 \\ \quad + [(a^2 + c^2)y - abx - bcz]^2 + [(a^2 + b^2)z - acx - bcy]^2 \}, \\ h \{ t [(b^2 + c^2)x - aby - acz] \\ \quad + r [(a^2 + c^2)y - abx - bcz] + s [(a^2 + b^2)z - acx - bcy] \} \\ = h^2 \{ [(b^2 + c^2)x - aby - acz]^2 + [(a^2 + c^2)y - abx - bcz]^2 \\ \quad + [(a^2 + b^2)z - acx - bcy]^2 \} \cos \theta. \end{cases}$$

Now in this set of equations we let $\xi = (1, 0, 0)$.

Then we get

$$(2) \quad \begin{cases} at + br + cs = 0 \\ t^2 + r^2 + s^2 = h(b^2 + c^2) \\ (b^2 + c^2)t - abr - acs = (b^2 + c^2) \cos \theta. \end{cases}$$

From the first and the third equations we get

$$t = h(b^2 + c^2) \cos \theta.$$

Now eliminating t we get

$$(3) \quad \begin{cases} br + cs = -ah(b^2 + c^2) \cos \theta \\ r^2 + s^2 = h(b^2 + c^2) - h^2(b^2 + c^2)^2 \cos^2 \theta. \end{cases}$$

Let us set $h(b^2 + c^2) = k$. Then (3) will become

$$\begin{cases} br + cs = -ak \cos \theta \\ r^2 + s^2 = k - k^2 \cos^2 \theta. \end{cases}$$

Thus we get

$$s = -\frac{b}{c}r - \frac{a}{c}k \cos \theta.$$

Therefore

$$r^2 + \frac{b^2}{c^2}r^2 + \frac{2ab}{c^2}kr \cos \theta + \frac{a^2}{c^2}k^2 \cos^2 \theta - k - k^2 \cos^2 \theta = 0.$$

This equation will become

$$r^2 + (2hab \cos \theta)r + h^2(a^2 + c^2)(b^2 + c^2)\cos^2 \theta - hc^2 = 0.$$

Solving this equation for r we get

$$r = -hab \cos \theta \pm c\sqrt{h} \sin \theta.$$

Indeed, there are two choices. Let us choose

$$r = -abh \cos \theta + c\sqrt{h} \sin \theta.$$

Then

$$s = -ach \cos \theta - b\sqrt{h} \sin \theta.$$

Consequently for $A(1, 0, 0)$ we have

$(h[a^2 + (b^2 + c^2)\cos \theta], abh(1 - \cos \theta) + c\sqrt{h} \sin \theta, ach(1 - \cos \theta) - b\sqrt{h} \sin \theta)$. A similar technique for $A(0, 1, 0)$ gives $(bah(1 - \cos \theta) - c\sqrt{h} \sin \theta, h[b^2 + (c^2 + a^2)\cos \theta], bch(1 - \cos \theta) + a\sqrt{h} \sin \theta)$. Finally for $A(0, 0, 1)$ we obtain $(cah(1 - \cos \theta) + b\sqrt{h} \sin \theta, cbh(1 - \cos \theta) - a\sqrt{h} \sin \theta, h[c^2 + (a^2 + b^2) \cos \theta])$. Thus the matrix of A is:

$$\begin{pmatrix} \frac{a^2 + (b^2 + c^2)\cos \theta}{a^2 + b^2 + c^2} & \frac{ab(1 - \cos \theta)}{a^2 + b^2 + c^2} + \frac{c \sin \theta}{\sqrt{a^2 + b^2 + c^2}} & \frac{ac(1 - \cos \theta)}{a^2 + b^2 + c^2} - \frac{b \sin \theta}{\sqrt{a^2 + b^2 + c^2}} \\ \frac{ba(1 - \cos \theta)}{a^2 + b^2 + c^2} - \frac{a \sin \theta}{\sqrt{a^2 + b^2 + c^2}} & \frac{b^2 + (c^2 + a^2)\cos \theta}{a^2 + b^2 + c^2} & \frac{bc(1 - \cos \theta)}{a^2 + b^2 + c^2} + \frac{a \sin \theta}{\sqrt{a^2 + b^2 + c^2}} \\ \frac{ca(1 - \cos \theta)}{a^2 + b^2 + c^2} + \frac{b \sin \theta}{\sqrt{a^2 + b^2 + c^2}} & \frac{cb(1 - \cos \theta)}{a^2 + b^2 + c^2} - \frac{a \sin \theta}{\sqrt{a^2 + b^2 + c^2}} & \frac{c^2 + (a^2 + b^2)\cos \theta}{a^2 + b^2 + c^2} \end{pmatrix}$$

The reader may verify that this matrix is unitary.

One may observe that (a, b, c) is a set of direction cosines of the line, then $a^2 + b^2 + c^2 = 1$ and the matrix of the rotation will become

$$\begin{pmatrix} a^2 + (b^2 + c^2)\cos \theta & ab(1 - \cos \theta) + c \sin \theta & ac(1 - \cos \theta) - b \sin \theta \\ ba(1 - \cos \theta) - a \sin \theta & b^2 + (c^2 + a^2)\cos \theta & bc(1 - \cos \theta) + a \sin \theta \\ ca(1 - \cos \theta) + b \sin \theta & cb(1 - \cos \theta) - a \sin \theta & c^2 + (a^2 + b^2)\cos \theta \end{pmatrix}$$



The number of grains of sand on the beach at Coney Island is much less than a googol —

10,000,000,000,000,000,000,000,000,
 000,000,000,000,000,000,000,000,000,
 000,000,000,000,000,000,000,000,000,
 000,000,000,000.

—Edward Kasner

The Moveable Figures*

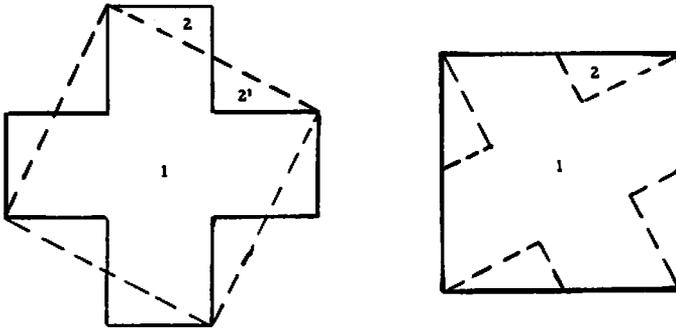
ANDREA LEE MEYER

Student, Mount St. Scholastica College, Atchison

In the field of geometry one often speaks of congruent figures, figures having equal area, and of transforming one figure into another by rearranging its parts. In this paper the writer intends to show the basic relationship between geometric transformations and equidecomposable figures beginning with the definition of the equidecomposable concept.

DEFINITION: Two figures are said to be equidecomposable if it is possible to decompose one of them into a finite number of parts which can be rearranged to form the second figure.

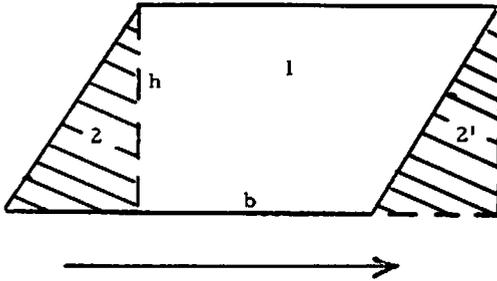
Example:



This cross shaped figure can be decomposed into the square by cutting off pieces numbered "2" and moving them to their new positions, "2." Since we did not change the area of any of the pieces, it can be assumed that areas are preserved in the rearrangement of pieces of equidecomposable figures.

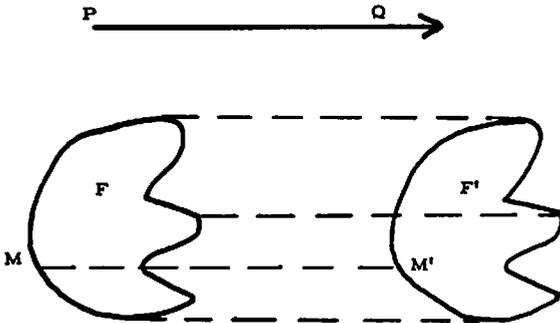
To build upon this "picture definition," more precise formulations of these mathematical operations which rearrange are necessary. The first of these operations, the parallel translation, is easily seen in the example of transforming a parallelogram into a rectangle of equal height, by moving segment "2" to the right, the length of the base.

*A paper presented at the 1967 National Convention of KME at Atchison, Kansas, April 7-8.



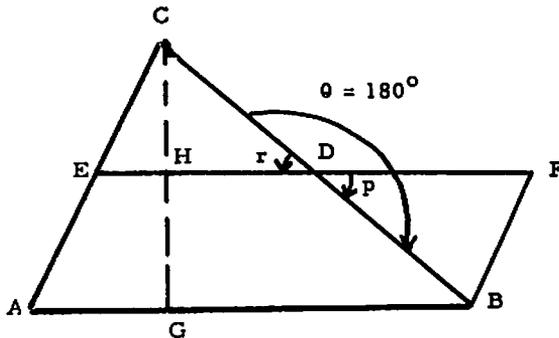
Hence the definition of parallel translation is as follows:

DEFINITION: Let \vec{PQ} be a directed line segment. From an arbitrary point M of figure F , draw the line segment MM' equal and parallel to \vec{PQ} and having the same direction. One then says that the point M' is obtained from M by a parallel translation with respect to \vec{PQ} . Each point in figure F is similarly transformed, such that F' is obtained from F by a parallel translation with respect to \vec{PQ} .



The next mathematical operation is called a rotation and is defined: **DEFINITION:** If the figures F and F' are related by a rotation through an angle θ , then corresponding segments of these figures are equal and make an angle θ with each other.

Next it should be proved that a rotation transformation applied to a general triangle will always yield a parallelogram of the same base and one-half the height of the triangle.



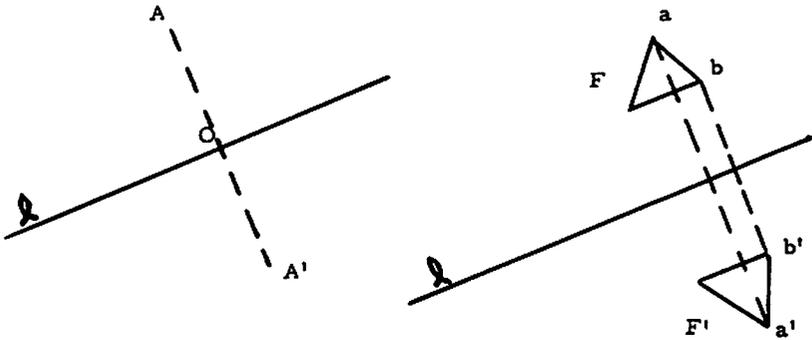
Given $ED \parallel AB$ and $CH = HG$, one need only prove that triangle ECD is congruent to triangle DBF , to prove that any triangle ABC can be transformed into a parallelogram of equal area by a rotation transformation. This transformation is the rotation of triangle ECD about the point D into triangle DBF . Given that the line through the mid-point of one side of a triangle (EH bisects CG in triangle AGC) and parallel to the second side (ED is parallel to AB) goes through the mid-point of the third side, it is shown that $AE = EC$. Similarly one can show $CD = DB$, by the same method applied to triangle GBC .

It is also known that a line which joins the mid-points of two sides of a triangle equals one-half the third side. Since ED joins the mid-points E and D of two sides of the triangle, it equals one-half AB , the third side of the triangle. If ED is extended to twice its length, the triangle formed by drawing BF is congruent to triangle ECD by the side, angle, side theorem of geometry, since $r = p$ by opposite angles of intersecting lines. AE parallel to BF and AB parallel to EF implies that $ABFE$ is a parallelogram. In this case, corresponding parts of our rotated figure are parallel and therefore make an angle of 180° with each other.

The third distinct type of transformation called line symmetry is a reflection in a given segment. It is defined as follows:

DEFINITION: A point A' is said to be the image of a point A by

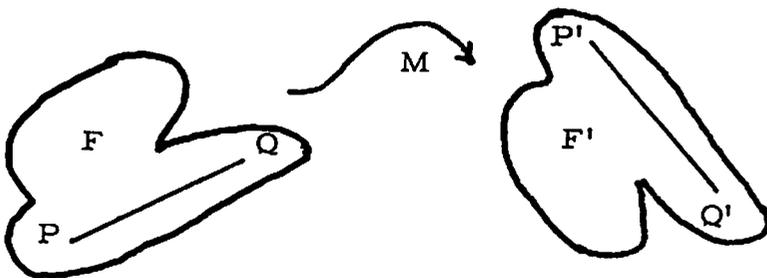
reflection in a line l (called the axis of symmetry) if the segment AA' is perpendicular to the line l and is bisected by the line l .



The line transformation of a figure F is a point by point reflection of the figure in the given line. A familiar example would be an ink blot picture in which the line of symmetry is the fold of the paper.

In general, these geometric transformations all involve the motion of a figure or line segment in the plane. More generally, an arbitrary motion may be represented as follows:

DEFINITION: A figure F is "lifted out" of its plane and transposed as a "rigid body" to a new position F' . This transposition from figure F to figure F' is called a motion.



A motion can also be described as a geometric transformation M which transforms points P into other points P' in such a way that for any two points P and Q , the length of PQ equals the length of $P'Q'$. In other words, distances are preserved under the operation of a general motion. The three transformations defined here—parallel translations, rotations, and reflections in a given line segment—are all motions which fit this general description.

The set of all motions would be even more useful if they were shown to form a group under the multiplicative operation of "followed by." *Closure* holds simply by definition. One motion followed by another is not necessarily a simple motion but certainly fits the general definition of transposition from one position F to another position F' in the plane. The *identity* element of the group is the motion which replaces the figure in the same position.

Each element of the set has a corresponding *inverse* such that a motion "followed by" its inverse is the identity motion. For example: The inverse of a parallel translation would be a parallel translation along the same line segment for the same distance but in the opposite direction. The inverse of a rotation and of line reflection are obtained in a similar fashion.

The *associativity* law is seen to hold when both groupings of the motions $(m_1 \cdot m_2) \cdot (m_3)$ and $(m_1) \cdot (m_2 \cdot m_3)$ transform the same figure A into the same figure A' . The set of all motions, however, does not form a commutative group. For example, a rotation followed by a translation would not yield the same figure (determined by its position) as a translation followed by a rotation.

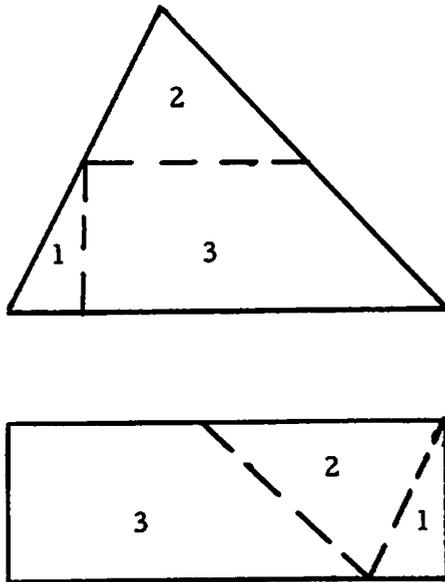
Turning again to the equidecomposable concept, one sees that the set of all motions forming the group are used as operations to show two figures equidecomposable. For example: A triangle is related to the parallelogram of half its height by being equidecomposable with it. Hence the question arises, is this relation an equivalence relation?

Reflexive: Is figure A equidecomposable with itself? The answer is affirmative as can be seen by applying the identity motion to the piece of the whole figure.

Symmetric: If figure A is equidecomposable with figure B , then is figure B equidecomposable with figure A ? The process is one of applying inverse motions on pieces of B congruent to those of A . For example, the cross was shown to be equidecomposable with the

square, the square can easily be shown to be equidecomposable with the cross by applying inverse motions to the segments two prime.

Transitive: If figure A is equidecomposable with figure B, and figure B is equidecomposable with figure C, then is figure A equidecomposable with figure C? Yes, each of the motions used to show A related to B and B related to C is performed in order, on the twice broken figure A to form figure C. For example, our original triangle can be transformed into the rectangle without going through the parallelogram step, as is seen here by the dotted lines.



It has been shown that two figures which are equidecomposable have equal areas simply by nature of the decomposition and subsequent motions used to show equidecomposability. But, is the converse true? Are two polygons having equal areas, necessarily equidecomposable? Although the answer might seem to be "no," the proof of the theorem that two polygons having equal areas are necessarily equidecomposable follows easily from the following three theorems:

THEOREM I. Every triangle is equidecomposable with some rectangle. This example was used in detail for the various types of motions and proved in part for the triangle and the parallelogram.

THEOREM II. Every polygon is equidecomposable with some rectangle. This result follows because every polygon can be decomposed into a finite number of triangles, and as Theorem I states, every triangle is equidecomposable with some rectangle.

THEOREM III. Two rectangles having equal areas are equidecomposable. (The proof of this theorem is tedious involving many constructions, so I will omit it.)

THEOREM: Two polygons which have equal areas are equidecomposable.

Proof: Every polygon is equidecomposable with some rectangle. (Theorem II) The two rectangles will have equal areas because the polygons from which they came had equal areas and areas are preserved under the equivalence relation of equidecomposability. The rectangles will therefore be equidecomposable because they have equal areas. (Theorem III) Thus the two polygons are equidecomposable by being equidecomposable with the same rectangle.

It is now known that given two polygons of equal area, either of them can be subdivided into a finite number of polygonal parts such that, upon rearrangement, these parts form the other polygon. Given this theorem, one might be asked to solve a problem similar to the following: **PROBLEM:** Given a square, one is asked to transform it into three congruent squares, or, similarly, to show one square equidecomposable with three other smaller congruent ones.

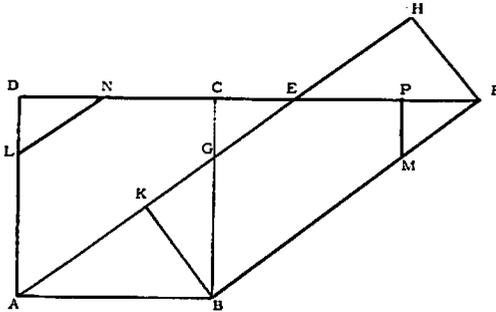
The definition for equidecomposable includes two processes: breaking down the original figure and then rearranging the pieces. Given a figure on which the lines between the pieces are marked, to rearrange the pieces into another figure is a matter somewhat on the order of a jigsaw puzzle in which one uses the geometric motions of the group to transform each piece. However, establishing the lines for breaking the original figure is more complicated and often involves algebra and elaborate construction.

Suppose the original square in our problem has sides of length one. Then its area is also one. If one constructs a rectangle with dimensions x by $3x$ such that $3x^2 = 1$, he will then be able to cut the rectangle into the three squares needed to solve the problem.

$$3x^2 = 1 \quad \Rightarrow \quad x^2 = \frac{1}{3}$$

$$x = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$3x = \sqrt{3}$$



For the construction of the required rectangle of dimensions x by $3x$, mark on the extended segment, DC , the point E such that $DE = \sqrt{2}$ which is also equal to AC , the diagonal of the square. Draw AE , noting that its length equals $\sqrt{3}$ since it is the hypotenuse of a triangle of sides of length 1 and $\sqrt{2}$. Next draw BF through B parallel to AE . The figure $ABFE$ is a parallelogram formed by two sets of intersecting parallel lines. From the geometric theorem stating that any two parallelograms which have equal bases and altitudes, we know that the square, $ABCD$ and the parallelogram, $ABFE$ each have areas equal to one.

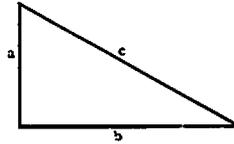
From B and F , draw BK and FH perpendicular to AE extended. The rectangle thus formed $BKHF$, has the same area as the parallelogram $ABFE$, because triangle ABK is congruent to triangle EFH . Thus, since AE was drawn equal to $\sqrt{3}$, $KH = \sqrt{3}$. Therefore, for our rectangle to have an area of one, BK must equal $\sqrt{3}/3$.

Figure $BFHK$, is a rectangle with sides of $\sqrt{3}$ and $\sqrt{3}/3$ which can be broken into three congruent squares of sides $\sqrt{3}/3$, to solve the problem. However, for a complete cutting of the original square, LN must be drawn parallel to AE such that $DL = CG = PM$, thereby making triangle DLN congruent to triangle CGE which is also congruent to triangle PMF . A rectangle of area equal to that of

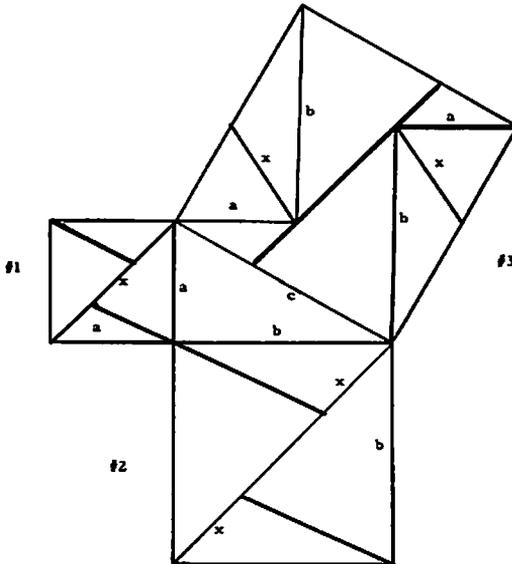
The Pentagon

the original square has been constructed which can now be broken into the three congruent squares needed to solve our problem.

Another interesting illustration of the theorem which states that two polygons having equal areas are equidecomposable is the Pythagorean relationship, $a^2 + b^2 = c^2$, where a , b , and c , correspond to the sides of a right triangle.



The terms a^2 , b^2 , and c^2 imply that squares with sides of length a , b , and c can be built on the sides of the triangle such that a^2 , b^2 , and c^2 represent the area of the squares.



Since it is known that $a^2 + b^2 = c^2$, we know that squares #1 and #2 can be cut and rearranged by motions to form square #3. Conversely, if one could find and prove the proper geometric cutting, this would be a method of proving the Pythagorean Theorem. The lines in squares #1 and #2 can be drawn for any right triangle by simply letting the length of an arbitrary segment x , in #1 equal the length of segment x , in #2.

In summary, two figures are shown to be related by the equivalence relation of equidecomposability by use of motions as mathematical operations. These operations include parallel translations, rotations and line symmetry as geometric transformations and form a group under the multiplicative operation of "followed by." The theorem that two polygons having equal areas are necessarily equidecomposable, was seen to be related to concrete problems and the ancient Pythagorean relationship.

BIBLIOGRAPHY

- *Bolyanskii, V. G. *Equivalent and Equidecomposable Figures*. Trans. from Russian by Alfred K. Henn and Charles E. Watts. "Topics in Mathematics Series." Boston: D. C. Heath and Company, 1963.
- Yaglom, I. M. *Geometric Transformations*. Trans from Russian by Allen Shields. "The Singer Series." New York: Random House, 1962.
- **"A Problem on the Cutting of Squares," *The Mathematics Teacher* XLIX (May, 1956), pp. 332-343.
- Abrahamson, B. *Notes on Plane Co-ordinate Geometry*. London: Butterworths, 1963.
- *These are the main references. The other two were for background.



The Gamma Function: Or Why $0! = 1$

FRANK FISHELL

Faculty, Montcalm Community College

This note uses the gamma function to motivate the definition given to many students that $0! = 1$. Many students and teachers are not entirely clear on why $0! = 1! = 1$, and most of the reasons given are not mathematical.

The gamma function, which arises in many areas of pure and applied mathematics, is defined by the improper integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt. \quad (1)$$

For a proof that this improper integral converges for $x > 0$ and diverges for $x \leq 0$ see [1*] or [2].

Replace x by $(x + 1)$ in (1), so that

$$\Gamma(x + 1) = \int_0^{\infty} t^x e^{-t} dt, \quad (2)$$

which is continuous for $x > 0$. Letting $u = t^x$, $dv = e^{-t} dt$ and integrating (2) by parts, we find

$$\int_0^{\infty} t^x e^{-t} dt = [-t^x e^{-t}]_0^{\infty} + \int_0^{\infty} x t^{x-1} e^{-t} dt.$$

Evaluating the first term on the right we find

$$\begin{aligned} [-t^x e^{-t}]_0^{\infty} &= \lim_{b \rightarrow \infty} [-t^x e^{-t}]_0^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{-b^x}{e^b} - 0 \right] \\ &= 0 \text{ as } b \rightarrow \infty. \end{aligned}$$

Therefore

$$\int_0^{\infty} t^x e^{-t} dt = x \int_0^{\infty} t^{x-1} e^{-t} dt.$$

Combining this result with (1) and (2) we have

$$\Gamma(x + 1) = x \Gamma(x), \quad (3)$$

which expresses the most fundamental property of the gamma

*Numbers enclosed in brackets refer to references at the end of this article.

function. We shall now use (3) as the definition of the gamma function and insist that it hold for all x .

Now we return to (1) and prove that $\Gamma(1) = 1$.

$$\begin{aligned}\Gamma(x) &= \int_0^{\infty} t^{x-1} e^{-t} dt \\ \Gamma(1) &= \int_0^{\infty} t^{1-1} e^{-t} dt \\ &= \int_0^{\infty} e^{-t} dt \\ &= \lim_{b \rightarrow \infty} \left[-e^{-t} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{-e^b} - (-e^0) \right] \\ &= 0 + 1\end{aligned}$$

$$\Gamma(1) = 1.$$

From (3) we find,

$$\Gamma(2) = \Gamma(1 + 1) = 1 \Gamma(1) = 1 = 1!$$

$$\Gamma(3) = \Gamma(2 + 1) = 2 \Gamma(2) = 2 \cdot 1 = 2!$$

$$\Gamma(4) = \Gamma(3 + 1) = 3 \Gamma(3) = 3 \cdot 2 \cdot 1 = 3!$$

$$\Gamma(5) = \Gamma(4 + 1) = 4 \Gamma(4) = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$$

and in general

$$\Gamma(x + 1) = x! \tag{4}$$

For this reason the gamma function is often called the factorial function. Using (4) and the fact that

$$\Gamma(1) = \Gamma(0 + 1), \text{ we see that}$$

$$\Gamma(0 + 1) = 0!$$

and since $\Gamma(1) = 1$, $0! = 1$.

Hence we have a justification for the often quoted definition which is so confusing to many students.

Further study of the gamma function enables one to find the factorial values of negative and rational numbers. For further study of these extensions of the gamma function see [3].

1. Harold K. Crowder and S. W. McCluskey, *Topics in Higher Analysis*, New York: The Macmillan Company, 1964, p. 336.
2. Murray R. Spiegel, *Theory and Problems of Advanced Calculus*, New York: Schaum Publishing Company, 1963, p. 273.
3. Emil Artin, *The Gamma Function*, New York: Holt, Rinehart and Winston, 1964.



Can you supply the missing digits in this multiplication?

$$\begin{array}{r}
 \text{xxx}7 \\
 \text{xxx} \\
 \hline
 \text{xxxx}6 \\
 \text{xx}203 \\
 \text{x}37\text{xx} \\
 \hline
 \text{xxxxxxx}
 \end{array}$$

Quantum Chess

DOUGLAS A. ENGEL

Climax Molybdenum Co., Climax, Colorado

Today a great many chess games are played between computers and men. No computer program has yet been devised that wins every time against mortal man, but there are some that come close. Perhaps man is aided by the fact that he has been playing for over a thousand years. At the same time it is man who has created that diabolic contraption, the computer. So if he begins to lose every time he can only blame himself.

Chess is a highly mathematical game. In fact the moves of several of the pieces can be described by linear equations in x and y . For instance the rook can move any distance orthogonally in two directions. If one thinks of the rook's square as an origin equal to zero, then the moves of the rook are described by the equations $x = n, y = 0$, and $x = 0, y = n$. After the rook is moved the square upon which it resides is a new origin equal to zero and the same set of equations applies to its future movement. One can describe the moves of the bishop by the equation $y = x$, where both directions of x and y are positive. Again the square upon which the bishop stands must be the zero point at all times for this equation to hold. The queen, of course, moves according to both the equations for the rook and the equation for the bishop. The knight moves according to the equation $y = x^2 + 1$, where the only value x can take is 1. Hence the knight can move from the square it stands upon any combination of the two orthogonal directions 1 and 2. A simpler equation could have been given for the knight, but the above one was given in order to show how one could see some of the pieces as being described by a possibly more complex system than is at first apparent.

The highly mathematical moves of the chessmen suggest that one could vary the game to take advantage of an even more purely mathematical structure. One could add constants to the variables and otherwise change the game so that the moves are recognized more for their equations than the word rule for them. This change could result in a game in which one would write a set of equations and then play the game.

A simple version of chess changed by changing the equations of motion of the pieces is played on a 6 by 8 board. The equations

of motion of the pieces and some of the possible moves are shown in Fig. 2. Figure 1 shows a suggested opening setup for beginning the game. The author has named this game quantum chess because only certain integral values are allowed x and y .

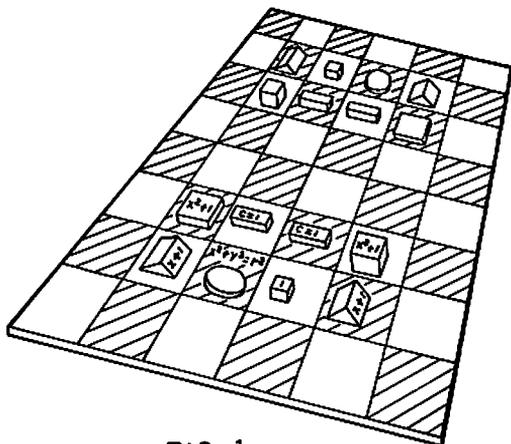


FIG. 1

In Fig. 2 the piece $x^2 + 1$ can move any combination of the distances $(0,1)$, $(1,2)$, $(2,5)$, and $(3, \text{square on border})$. A free path must be open for any of these moves to take place. If a man is captured he can only be captured at the endpoint of the move. For instance, the $x^2 + 1$ piece in Fig. 2 can move to the right 3 squares and down to the border where a dot is placed to indicate this move. The move can take place if either the path down and to the right or to the right and down is open. The direction taken for x or y is, of course, arbitrary, but the x and y directions are always perpendicular. Since the x and y (positive) axes are arbitrary each equation

describes moves that are similar in each of the four quadrants. One of the curves described by the piece $x^2 + 1$ is shown by the curved line in Fig. 2.

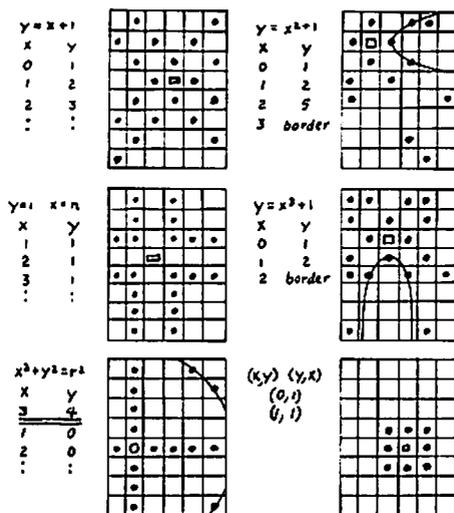


FIG. 2

One of the pieces shown in Fig. 2 obeys the equation for a circle. Since this equation has integral solutions if either x or y is zero the piece can move any orthogonal distance in either direction like a rook. In addition the circle can move any combination of the two perpendicular distances 3 and 4, because 3 and 4 satisfy the equation,

$$3^2 + 4^2 = 5^2$$

The unit piece moves just like the king in chess. The game is won by capturing the unit. There are no pieces equivalent to the pawn as in chess.

The terminology 'annihilated' has been added for a piece that is captured. Strictly speaking annihilation would have to destroy both pieces if any analogy with atomic particles were to hold. Player and antiplayer can begin with the opening setup shown in Fig. 1. Some shapes for pieces are also suggested in Fig. 1. If you have a chessboard you can place a strip of paper over the 2 by 8 portion of squares that is not used for quantum chess. Since there are six basically different pieces in chess you can use these for pieces of quantum chess if you want to. However, it is easy to cut the pieces shown in Fig. 1 out of a square bar of wood. It will also be much less confusing since pieces logically resemble their equations of motion and will be easily remembered if they have the shapes shown.

On the opening moves the $C = 1$ piece ($x = n, y = 1$) cannot move up and down but can move sideways. This rule is added so that a man cannot be captured on the first move. It appears that in the first few moves the man to move first usually has a small advantage. As the game progresses it appears that the advantage disappears, as in chess. The pieces seem to be quite powerful but it can be very difficult to checkmate, even if you have a good advantage. The game develops faster than chess, but rapidly develops complexity so that a player may find himself taking quite a long time to decide on a move. There is also an advantage in quantum chess in learning the moves of the various pieces as equations. This gives the game a distinct educational value and allows an objective approach. One will soon attempt tricky moves from the logical standpoint of solutions to several equations after transforming. It is possible to develop a good deal of respect and facility for cartesian coordinates by merely playing this game, or other versions of it.

ADDING A DIMENSION

The quadruple symmetry of the moves of the various pieces in quantum chess makes it easy to add a dimension of time to the moves. Up and down directions may be considered to belong to the y - axis while sideways directions are the x - axis. On a player's first move all pieces obey the equations $y = x + 1$, ($y = 1, x = 1, 2, \dots$), $y = x^2 + 1$, $y = x^3 + 1$, and the given equation for the circle. On the same player's second move all pieces obey the same equations with x and y interchanged. On all odd moves, first, third, fifth, that a player makes his pieces can move only according to the equations

given above. On all even moves pieces must obey the same equations with x and y interchanged, as stated above for a second move.

Thus time enters into the game using the idea of odd and even parity moves. The time rule alternately rotates the strategy of the game 90° so that one must keep thinking in terms of a change of parity on the next allowable moves. At the same time the power of the pieces diminishes so that the total complexity of the game stays about the same. However, since power has diminished in the pieces it will take longer to play a game with the time rule added. Captures will tend to occur with less frequency through the same number of moves.

Hopefully students of mathematics will take up quantum chess and develop their own versions so that they may enjoy many games while learning valuable mathematics.

References

1. Maurice Kraitchik, *Mathematical Recreations*, (New York: Dover, 1953), pp. 267-323.
2. Joseph S. Madachy, *Mathematics on Vacation*, (New York: Charles Scribner's Sons, 1966), pp. 34-54.
3. Martin Gardner, *Martin Gardner's New Mathematical Diversions from Scientific American*, (New York: Simon and Schuster, 1966), pp. 70-81.



Find the three digits which when substituted for A , B , and C will make this addition correct.

$$\begin{array}{r}
 A A A A \\
 B B B B \\
 C C C C \\
 \hline
 B A A A C
 \end{array}$$

An Interesting Consequence of Rolle's Theorem

R. S. LUTHAR
Faculty, Colby College

If a function $f(x)$ is

- i) continuous in $[a,b]$;
- ii) differentiable in (a,b) ; and
- iii) $\frac{f(a) + f(b)}{a + b} = \frac{f(a) - f(b)}{b - a}$

then, there exists a point, say x_1 of (a,b) , such that

$$f'(x_1) = -\frac{f(x_1)}{x_1}.$$

Proof: Condition (iii) insures that

$$a f(a) = b f(b).$$

Then, if we construct the function

$$F(x) = x f(x), \text{ we have } F(a) = F(b).$$

Thus, $F(x)$ satisfies all the requirements of Rolle's Theorem, and there exists a point x_1 of (a,b) such that

$$F'(x_1) = 0.$$

Consequently, we have

$$x_1 f'(x_1) + f(x_1) = 0, \text{ or } f'(x_1) = -\frac{f(x_1)}{x_1}.$$

Directions for Papers to Be Presented at the Seventeenth Biennial Kappa Mu Epsilon Convention

A significant feature of this convention will be the presentation of papers by student members of KME. The mathematics topic which the student selects should be in his area of interest, and of such a scope that he can give it adequate treatment within the time allotted.

Who may submit papers: Any student KME member may submit a paper for use on the convention program. Papers may be submitted by graduates and undergraduates; however, graduates will not compete with undergraduates.

Subject: The material should be within the scope of the understanding of undergraduates, preferably those who have completed differential and integral calculus. The Selection Committee will naturally favor papers that are within this limitation, and which can be presented with reasonable completeness within the time limit prescribed.

Time limit: The usual time limit is twenty minutes, but this may be changed on the recommendation of the Selection Committee if requested by the student.

Paper: The paper to be presented, together with a description of charts, models, or other visual aids that are to be used in the presentation, should be presented to the Selection Committee. A bibliography of source materials together with a statement that the author of the paper is a member of KME, and his official classification in school, undergraduate or graduate, should accompany his paper.

Date and place due: The papers must be submitted no later than January 25, 1969, to the office of the National Vice-President.

Selection: The Selection Committee will choose about ten to twelve papers for presentation at the convention. All other papers will be listed by title and student's name on the convention program, and will be available as alternates.

George R. Mach
National Vice-President, Kappa Mu Epsilon
Department of Mathematics
California State Polytechnic College
San Luis Obispo, California 93401

The Mathematical Scrapbook

EDITED BY GEORGE R. MACH

Readers are encouraged to submit Scrapbook material to the editor. Material will be used where possible and acknowledgement will be made in THE PENTAGON. All of the Scrapbook material for this issue was submitted by members of the California Gamma Chapter winter 1968 pledge class.

$$= \triangle =$$

Editor's note: The following was submitted by Virginia Traudt.

Many people are familiar with a scheme for squaring numbers between 1 and 100 ending in 5, which goes like this: The last two digits of the product are 25 and the first two are the product of your original tens-digit and the next integer. As an example:

$$(35)^2 = 1225, \quad \text{where } (3)(4) = 12.$$

Actually, there is no reason to limit this scheme to numbers between 1 and 100. Can you provide a proof for the scheme? You might construct one by looking at our example in the following way:

$$\begin{aligned}(35)^2 &= (30 + 5)(40 - 5) \\ &= 1200 + 40(5) - 30(5) - 5(5) \\ &= 1200 + (40 - 30 - 5)(5) \\ &= 1200 + 5(5)\end{aligned}$$

An interesting extension is the product of two numbers symmetrically located about a number ending in 5. For example:

$$(32)(38) = 1216 \quad \text{and} \quad (33)(37) = 1221.$$

Note that $(2)(8) = 16$, and $(3)(7) = 21$. Can you give the scheme and provide a proof for it?

A different extension is the product of any two numbers ending in 5. If the tens-digits are both even or both odd, an example goes like this:

$$(25)(65) = 1625, \quad \text{Note that } 16 = (2)(6) + \frac{2 + 6}{2}$$

If one tens-digit is even and the other is odd, an example goes like this:

$$(25)(75) = 1875, \quad \text{Note that } 18.5 = (2)(7) + \frac{2 + 7}{2}.$$

Can you give the scheme and provide a proof for it? Note how the original square of 35 is just a special case of the more general scheme you have now found?

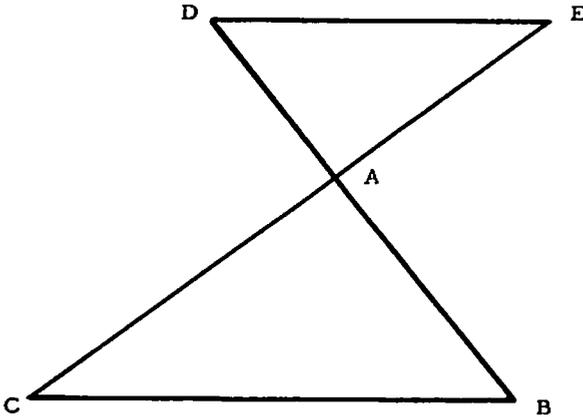
$$= \triangle =$$

Editor's note: The following was submitted by Charlene Matteson.

Arithmetic operations are often long and tedious while graphical solutions can be simple. Graphical computations are never exact, but exact results are not necessary in many practical situations. The single important element in all graphical computations is an arbitrarily selected length of a straight line segment which is accepted as the unit.

Graphical multiplication is based on the property of similar triangles which have a common vertex. When two triangles are similar, their corresponding sides are proportional. Thus, in the following figure, if the triangles ABC and ADE are similar, we have the proportion:

$$\frac{AD}{AE} = \frac{AB}{AC}, \text{ and hence, } (AD)(AC) = (AB)(AE)$$



If AD is our arbitrarily selected unit, that is, $AD = 1$, then $(AB)(AE) = AC$. This shows us the procedure which is employed in graphical multiplication. Lay out two lines intersecting at A . Starting with the vertex A , we mark off a straight line segment AD which is equal to our arbitrarily selected unit. On the same line, we mark off a straight line segment AB which is the first factor and which is expressed in terms of the unit AD . On the other line we mark off the straight line segment AE which is the second factor and which is also expressed in terms of the unit AD . We join the points D and E with the straight line DE . Through the point B we draw the straight line BC parallel to DE . The straight line segment AC is the required product.

= Δ =

Editor's note: The following was submitted by Richard Jacoby.

It is interesting to observe what happens to a function when the independent variable takes on values for which the function is not defined in ordinary analytic geometry. Consider the graph $y^2 = (x - 2)^2(x - 3)$, or $f(x) = \pm (x - 2)(x - 3)^{1/2}$,

Fig. 1.

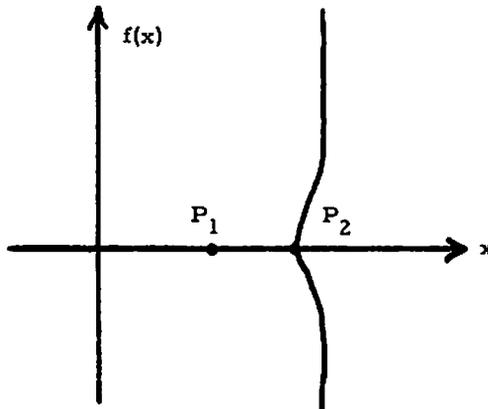


Figure 1

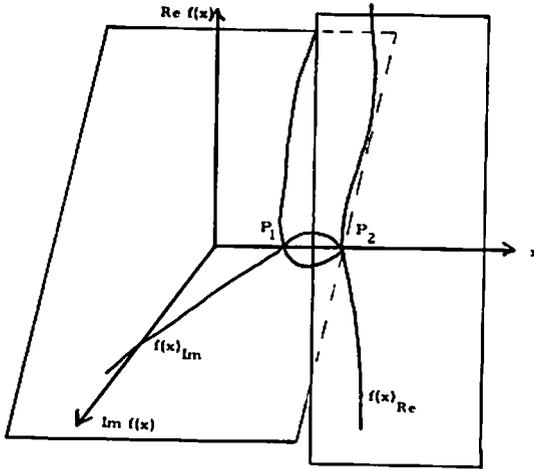


Figure 2

The point $P_1(2,0)$ is a part of the graph but it appears to be a discontinuity. However, if we use any other value of x less than 3, we see that $f(x)$ takes on imaginary values. If we graph these imaginary values of $f(x)$ on an imaginary plane (Fig. 2) we see that P_1 is a point of continuity as is $P_2(3,0)$ in the imaginary graph. (Portions of the real and imaginary planes have been sketched in to aid in visualizing the graphs). The equation is

$$f(x)_{im} = \pm (x - 2)(3 - x)^{1/2}.$$

Then, considering a complex graph (imaginary and real parts) we see that the graph in Fig. 1 is just a special case where the imaginary part is zero. You may also note the slope at P_2 is infinite for both the imaginary and real graphs.

Another interesting example is the parabola $y^2 = x$, or $f(x) = \pm x^{1/2}$, Fig. 3. For positive values of x we see the parabola we would expect, but for negative values of x we get a parabola in the imaginary plane with equation $f(x)_{im} = \pm (-x)^{1/2}$.

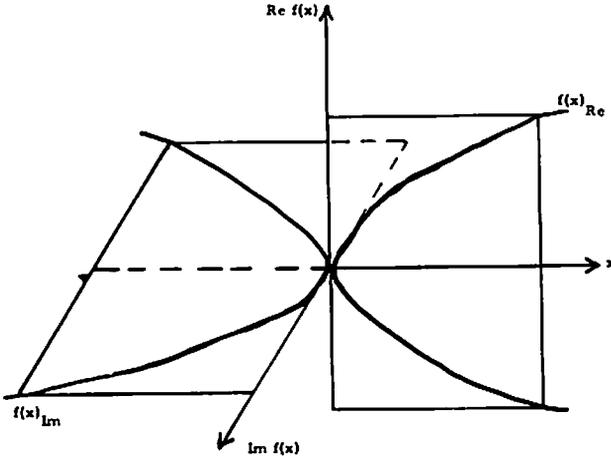


Figure 3

At the origin, the slopes of the real and imaginary parts of $f(x)$ are infinite, each in its own plane. The origin is also the point of continuity for the two parts of the graph.

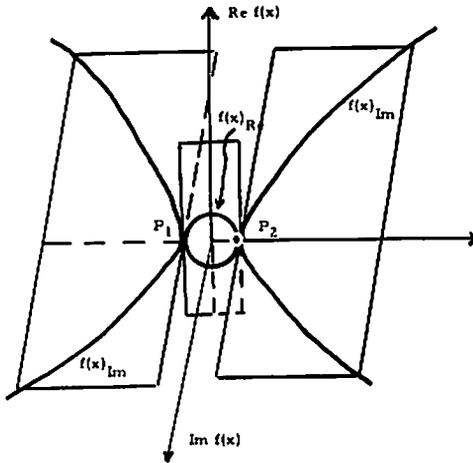


Figure 4

A third interesting function to investigate is the circle $x^2 + y^2 = 1$, or $f(x) = \pm (1 - x^2)^{1/2}$. For values where $|x| > 1$, $f(x)$ is imaginary and has the equation $f(x)_{im} = \pm (x^2 - 1)^{1/2}$, which is the hyperbola, Fig. 4.

The points $P_1(-1,0)$ and $P_2(1,0)$ are points of continuity for the real and imaginary graphs. P_1 and P_2 are also points where the slopes of the real and imaginary graphs go to infinity. Again, the circle is a special case of the complex graph where the imaginary part is zero.

It is interesting to note that a function which has a real hyperbola has an imaginary ellipse as part of its graph and that a function which has a real ellipse has an imaginary hyperbola as part of its graph. Interested students might investigate three dimensional cases such as spheres or paraboloids.

= Δ =

Editor's note: The following was submitted by Ronald Jones.

Choose any three-digit number whose first and third digits are not the same. Reverse the order of the digits. Upon subtracting the smaller from the larger of the two numbers, you will obtain a third three-digit number, call it X . (Note: If your subtraction should yield a two-digit number, put a zero in the hundreds place as a place holder). Upon reversing the order of the digits of X , and then adding this new number to X , you will *always* obtain the same sum. Try it.

Now, can you prove it? Start by letting your original number be $(100a + 10b + c)$. Then $X = 99(a - c)$, and you are on your way. You will need the "reversal" of X and that is a little tricky to get.

= Δ =

Editor's note: The following quotation was submitted by Richard Grialou.

"Pure mathematics consists entirely of such asseverations as that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is of which it is supposed to be true . . . If our hypothesis is about *anything* and not about some one or more particular things, then our deductions constitute mathematics. Thus mathematics may be defined

as the subject in which we never know what we are talking about, nor whether what we are saying is true."

—Bertrand Russell

= Δ =

Editor's note: The following was submitted by William Henry.

Suppose we place a tightly fitting band around the equator of the earth. If we add a 12 inch section to this band so as to increase its length by that amount, how far above the earth would the band now be if we raise it an equal distance above the surface at all points? On first thought, it would seem that the distance would be unnoticeable. Using either calculus or geometry we can see that:

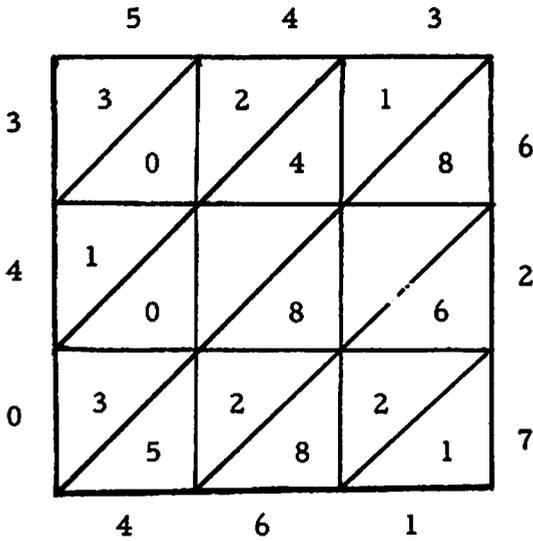
$$\begin{array}{ll}
 2 \pi r = c & 2 \pi r = c \\
 2 \pi dr = dc & 2 \pi (r + r_1) = c + 12 \\
 dr = dc/2\pi & 2 \pi r_1 = 12 \\
 dr = 12/2\pi & r_1 = 12/2\pi \\
 \approx 1.91 \text{ inches} & \approx 1.91 \text{ inches}
 \end{array}$$

We see that the band moves up a startling distance of almost 2 inches all the way around. If we were to take a softball and perform the same operation, by approximately how much would the radius increase? Note that the change in the radius is not dependent upon the original circumference or radius.

= Δ =

MEDIEVAL MULTIPLICATION: Multiplication hasn't always been as easy as it is today. Many complicated methods were devised and discarded before modern positional arithmetic was discovered. During the 12th century a method which probably originated in India, but in Europe was known as the *gelosia* method, was in use. *Gelosia* is from the Italian, meaning "window grating."

Suppose you want to multiply 543 by 627. First a 3 by 3 diagram is drawn and each cell is divided into an upper and a lower portion. Then 543 is written across the top and 627 is written down the right side.



Begin with the upper right cell: $6 \times 3 = 18$. Put the 1 in the upper portion and the 8 in the lower portion of this cell. Next put the product of 4×6 in the upper middle cell and fill in the other cells in a like manner. Next add the diagonals, beginning with the lower right one which is 1, carrying any digits as necessary to the next diagonal. The next diagonal is $6 + 2 + 8 = 16$. The 6 is recorded and the 1 is carried to the next diagonal, etc. The completed diagram gives the desired answer, $543 \times 627 = 340461$.

The Book Shelf

EDITED BY JOHN C. BIDDLE

This department of *The Pentagon* brings to the attention of its readers published books (both old and new) which are of a common nature to all students of mathematics. Preference will be given to those books written in English or to English translations. Books to be reviewed should be sent to Dr. John C. Biddle, Mathematics Department, Central Michigan University, Mt. Pleasant, Michigan 48858.

Famous Problems of Mathematics, Heinrich Tietze, Graylock Press, New York, 1965, xvi + 367 pp., \$10.00.

This book is a translation of the German second edition which is titled *Gelöste und Ungelöste Mathematische Probleme aus alter und neuer Zeit*. The book has fourteen chapters concerning a wide selection of classical and modern problems of mathematics. Some of the problems discussed are completely solved such as construction of n -gons and trisection of the angle. A considerable part of the book concerns problems which are unsolved such as the twin prime conjecture, the four color problem and Fermat's last theorem. Each problem is developed historically with extensive biographical information about the mathematicians who worked on the problem.

In brief, the chapter titles are: Prime Numbers and Prime Twins, Traveling on Surfaces, Trisection of an Angle, On Neighboring Domains, Squaring the Circle, Three Dimensions, Prime Numbers Again, Counting and Calculating, The Regular Polygon of 17 Sides, Solution of Algebraic Equations by Means of Root Extraction, The Four Color Problem, Infinity in Mathematics, Fermat's Last Problem, and Space Curvature. Each chapter has drawings, many in several colors, and photographs of mathematicians when appropriate. There is also an addendum of notes giving more historical and mathematical details, with some note sections being several pages long. The end of the book contains a detailed bibliography by chapter and a supplementary list by mathematical interests and a good index.

The book is completely intelligible to any student with knowledge of the basic facts of trigonometry. Most of the book can be read by persons with interest in mathematics but with less background, yet it also contains material unfamiliar to mathematicians. The reader's interest is aroused enough to demand more information than the author presents about each problem. For those who want more information the supplementary lists provide more than enough. The combination of mathematical and biographical information makes the

book even more interesting; the breadth of coverage is astonishing for the size of the book.

Some of the chapter headings are self-explanatory, others need elaboration. Chapter II is on geodesics of different surfaces and does not involve differential calculus. In discussing angle trisection in Chapter III, the author does not attempt to prove the impossibility using ruler and compass but does discuss close approximations. The chapter on neighboring domains begins with the old problem of joining houses by roads so that no pair of roads cross each other. Leading from this problem the author ends discussing topological problems. The second chapter on primes discusses distribution of primes and the notes do include some integration involving $Li(x)$. The notes on the chapter concerning the 17-gon contain a trigonometric discussion of the construction problem. The chapter on infinity deals with denumerability and shows that the continuum is not denumerable. Chapter XIII contains a proof of the necessary and sufficient conditions for Pythagorean triples and a proof of the nonsolvability of the equation $x^4 + y^4 = z^4$. The historical discussion of Fermat in this chapter is particularly nice. The last chapter has a brief discussion of curvature concepts prior to Einstein.

The weaknesses of the book are few. The writing style is awkward and seems to suffer from translation, since the English sentences tend to have a German sentence structure. The many color plates contained in the book are frequently out of place, for example, plates on geodesics split the table dealing with $Li(x)$ and the number of primes less than x on pages 16 and 17 while the discussion of $Li(x)$ comes on page 151. Also plate XIII follows page 208, two chapters ahead of the discussion of it on page 236. Occasionally discussions end too much unresolved, confusing the reader rather than increasing his interest.

In summary, the book definitely belongs in any school or public library and would serve as an interesting and valuable supplement to anyone's personal library. Although not suitable as a textbook in any standard course, it could be used as a reference in all high school courses and some college courses such as history of mathematics, number theory and mathematics for teachers. It surely would provide valuable reading for all levels of persons with mathematical interests or persons who are puzzle solvers.

—James K. Bidwell
Central Michigan University

Modern Fundamentals of Mathematics, Henry Sharp, Jr., Prentice-Hall, Inc., Englewood Cliffs New Jersey, 1968, pp. 390 + ix.

This text is another in the set of books designed to fill the gap between secondary school mathematics and an integrated course in analytic geometry and calculus. That the gap exists is proven inductively by the large cardinal number of this set. Although it will be used primarily in colleges as an "algebra-trig" text, Sharp's book could also be used as a senior year text in secondary schools.

The text covers the essentials of functions, circular functions, polynomial functions, and exponential functions. It has a definite flavor of mathematical structure. As the author says in the preface:

Topics have been selected to provide reasonably complete coverage at the pre-calculus level. They have been organized, however, in an effort to emphasize the structure of mathematics, recognition of which is an important step toward "mathematical maturity."

The author has chosen to use consistently definition, example and theorem-proof to develop the material. The use and importance of theorem-proof is increased as the book progresses. The terminology and symbolism is "modern" and is properly utilized. The material presented is, generally speaking, standard for such a book. It is the mathematical flavor which makes the text a good one, and consequently, in my opinion, different from many similar "algebra-trig" texts.

The chapter titles are, in order: Sets and Axiomatic Structure, Real Number System, Relations and Functions, Linear Functions, Circular Functions, Quadratic Functions, Counting and Mathematical Induction, Composite Functions, Polynomials and Exponential Functions, Linear Systems, and Probability.

The first two chapters concentrate on set theory, axioms of a group and properties of real number system. An appendix is given for logic in Chapter One. Integers (also called *whole numbers* in the text) and rational numbers are well developed. The completeness axiom is also stressed.

The chapter on relations and functions is well handled and is standard material. Linear functions are presented using good set notation. Analytic geometry of the line is covered. Quadratic functions are given little space and compete in the same chapter with quadratic

trigonometric equations and a development of complex numbers which seems to be misplaced in this location.

Circular functions are introduced using angles of rotation and the general wrapping function is omitted. Periodicity and inverse functions are carefully developed. Triangle problems are given relatively little space, which is good in this reviewer's opinion.

Beginning with the chapter on induction, the book has a more rigorous approach and increases in difficulty, emphasizing reading of proofs, etc. The induction axiom is different from the common form. It states that if M is a set of integers such that (1) $t \in M$ and (2) if $k \in M$, then $(k + 1) \in M$. Then $W_t \subset M$, where $W_t = \{x \mid x \text{ is an integer and } x \geq t\}$. This axiom is used to show that statements have an inductive truth set; that is, the set of natural numbers is a subset of the truth set. Induction proof is well illustrated and is used strongly throughout the rest of the book.

The chapters on composite functions, polynomials and exponential functions are good but heavily mathematical. For example, the generalized sine function is developed using composite functions. Slope of tangents to polynomial functions are developed using

$$\frac{f(r + h) - f(r)}{h}$$

and Newton's iterative method for approximating roots is introduced. A standard development of logarithms as inverse functions of exponential functions is included.

The book concludes with brief chapters on linear systems and probability. The Gauss-Jordan method and Cramer's rule are developed for general solution of systems. The probability chapter is only fourteen pages and contains only essential material.

The problem sections are adequate and non-trivial material is introduced occasionally. Answers to odd-numbered problems are included. There are brief but adequate tables and an index.

The text is clearly an adequate pre-calculus text and provides an opportunity for an interested student of average ability or better to grow in mathematical maturity. This maturity is something rarely accomplished until after the standard calculus course. The reviewer believes that good knowledge of this text would allow a much better calculus course than is presently possible with the frequent "algebra-trig" rehash of advanced secondary material. Assuming a partial

ordering of quality in the set of books similar to this one, the text reviewed surely is an upper bound of the set.

—James K. Bidwell
Central Michigan University

Graphics: Analysis and Conceptual Design, (2nd ed.), A. S. Levens. John Wiley and Sons, Inc., New York, 1968, 771 pp., illustrated and annotated.

Some people still visualize the engineer as one who is engaged in precise measurements, endless mathematical calculations, and limited to the realm of science. The author in this second edition of *Graphics* logically dispels this fantasy. Engineering as a trade takes on the stature of design, particularly creative effort in design. As quoted in this text, "Scientists explore what has been discovered and the engineers create what has never been."

The author, assisted by professional publishing skills, has created an avante-garde reference text that has youthful appeal and mature reasoning leading one through the open door of creativity.

With a blend of today's mundane and glamor problems that contain insight for tomorrow, Professor Levens has expertly organized a course of study into three areas: (1) orthogonal projection; (2) graphical mathematics; (3) design. All three areas are correlated in the conceptual approach to the study of creative engineering graphics.

Using the system of descriptive geometry and orthogonal projection, after a concise lesson in free-hand sketching, Part I emphasizes problems and their solutions by graphical representation of angles, skew lines, development, intersections, and vectors. An analysis of each situation is given with the problem to encourage further study and reinforce learning.

The mathematics of industry is vividly and *graphically* demonstrated in Part II. This section has particular appeal to anyone with a desire to "see" arithmetical solutions of work-a-day problems displayed in other than a numerical digit form. This section of the work could well serve as a reference text for a contemporary approach to problem-solving for analytical mathematics (algebra—even complex numbers—geometry, calculus, statistics, and nomography). The graphical solution of problems in electrical circuits will be important to students and instructors in the area of electricity-electronics.

In the area of design (Part III) the author has brought together the fundamental concepts of graphics as a tool for creative work. Concern is expressed for precision in workmanship, in knowledge of hardware, in exacting measurement; but the inspiration is for creativity in our work. The ultimate goal, the designer, is one who recognizes and uses all aspects of our society in his creative efforts. Several suggestive problems from industry as stimulative examples are detailed in this section. They include an instant coffee dispenser, calligraphic pen, automatic instrumentation (Osomometer), and a wheeled toy.

The appendices are well arranged and for those mathematically oriented, appendices D, E, and F are concerned with calculations for geometric, algebraic, and calculus solutions of graphical problems.

Physically, the book is attractively crafted, graphically excellent, and the nearly square format gives a balanced page arrangement for the illustrations and descriptive literature. The weight (almost two kilos) may cause some irritations. As this is a second edition one may assume that errors of printing and problems have been corrected.

—Virgil W. Davis
Central Michigan University

Teaching General Mathematics, Max A. Sobel, Prentice-Hall, Teachers' Mathematics Reference Series, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1967, 93 pages. \$3.95

This is the sixth and latest book in the Prentice-Hall Teachers' Mathematics Reference Series. The title is a little misleading in that the content is directed toward materials to be used with the slow learner.

Chapter 1 discusses the slow learner and some of his problems while Chapter 2 comments on the progress that has been made in production of materials and curriculum for the slow learner. Several topics that seem to show promise for further research are also indicated in this chapter. Among these are ". . . extended research on such topics as the effectiveness of mathematics teachers who have been specially prepared to teach low achievers, . . ."

The remainder of the book is devoted to different areas that may be useful as topics for discussion or units of study by the students. These topics range from number systems through geometry to probability with the final chapter a "bag of tricks."

Several parts of these latter chapters pose the questions for the students or give the teacher ideas about how to introduce the topic but little else, thereby leaving it to the teacher's imagination as to how to go on from there. This procedure is very apparent in Chapter 6 "Explorations in Probability," where the author discusses the "birthday problem" and conditional probability. Both of these areas would be extremely difficult if not impossible for the teacher to explain in language that is understandable to the slow learner.

In general, however, the author makes several references to having the students "discover by doing" which keeps them busy and at the same time gives them the needed practice in the basic fundamentals without making this practice boring.

I would recommend that this volume become a part of the personal library of every teacher who plans to teach the slow learner.

JOHN C. BIDDLE
Central Michigan University
Mt. Pleasant, Michigan



Consider a triangle (T), its inscribed circle (C_1) and its circumscribed circle (C_2). The sides of (T) are chords of (C_2) and tangent to (C_1). Selecting any point on (C_2), lay out two chords, each tangent to (C_1). Is it possible that a third chord of (C_2) might complete another triangle *and* be tangent to (C_1)? What happens when (T) is an equilateral triangle? Are there any other interesting special cases? Can any generalizations be made? Readers are invited to submit any interesting results to the editor.

The Problem Corner

EDITED BY H. HOWARD FRISINGER

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before October 1, 1968. The best solutions submitted by students will be published in the Fall 1968 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor Robert L. Poe, Department of Mathematics, Texas Technological College, Lubbock, Texas 79409, the new Problem Corner Editor.

PROPOSED PROBLEMS

211. *Proposed by J. F. Leetch, Bowling Green State University, Bowling Green, Ohio.*

Prove that in the Fibonacci sequence 1, 1, 2, 3, 5, ... every 5th term is divisible by 5 and that these are the only terms having this property.

212. *Proposed by Charles W. Trigg, San Diego, California.*

There is only one three-digit number which is six times the sum of the fourth powers of its digits. Find this number.

213. *Proposed by R. S. Luthar, The University of Wisconsin, Waukesha, Wisconsin.*

If n is an odd integer with at least two distinct factors, prove that:

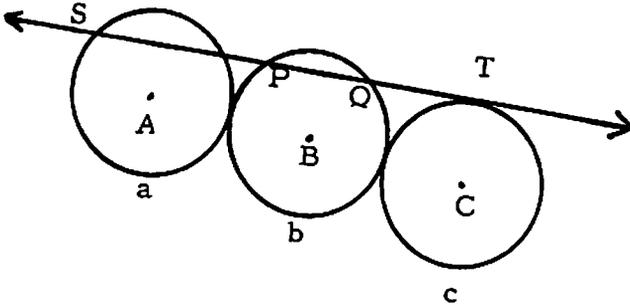
$$\log_n \geq (k - 1) \log_3 + \log_5$$

where k is the number of distinct prime factors of n .

214. *Proposed by J. F. Leetch, Bowling Green State University, Bowling Green, Ohio.*

Join consecutively the points $(1,0)$, $(\frac{1}{2},(\frac{1}{2})^2)$, $(\frac{1}{3},0)$, $(\frac{1}{4},(\frac{1}{4})^2)$, ..., $(\frac{1}{2n},(\frac{1}{2n})^2)$, $(\frac{1}{2n+1},0)$, ... with line segments, and include $(0,0)$ in the resulting graph. Does this graph have length?

215. *Proposed by Leigh Jones, State University of New York at Albany, Albany, New York.*



Circles a , b , c with respective centers A , B , C and radius one are such that a and c are tangent to b . Points S , A , B , C are collinear.

Line \overleftrightarrow{ST} is tangent to c at T , and intersects circle b at P and Q . How long is \overline{PQ} ?

SOLUTIONS

206. *Proposed by Raymond Huck, Marietta College, Marietta, Ohio.*

Show that $\tan^2 18^\circ + \tan^2 36^\circ + \tan^2 54^\circ + \tan^2 72^\circ = 12$.

Solution by Don N. Page, William Jewell College, Liberty, Missouri.

If $\alpha_1 = 36^\circ$ and $\alpha_2 = 72^\circ$, $\tan 5\alpha_1 = \tan 180^\circ$ and $\tan 5\alpha_2 = \tan 360^\circ = 0$. Since $\tan n\alpha = \frac{\tan(n-1)\alpha + \tan \alpha}{1 - \tan(n-1)\alpha \tan \alpha}$, $\tan 5\alpha = \frac{5 \tan \alpha - 10 \tan^3 \alpha + \tan 5\alpha}{1 - 10 \tan^2 \alpha + 5 \tan^4 \alpha}$. Thus $5 \tan \alpha - 10 \tan^3 \alpha + \tan^5 \alpha = 0$ for $\tan 36^\circ$ and $\tan 72^\circ$, so $\tan^2 36^\circ = \frac{10 - \sqrt{100 - 20}}{2} = 5 - 2\sqrt{5}$ and $\tan^2 72^\circ = \frac{10 + \sqrt{100 - 20}}{2}$

$$\begin{aligned}
 &= 5 + 2\sqrt{5}. \text{ Then since } 18^\circ = 90^\circ - 72^\circ \text{ and } 54^\circ = \\
 &90^\circ - 36^\circ, \tan^2 18^\circ = \cot^2 72^\circ = \frac{1}{\tan^2 72^\circ} = \frac{1}{5 + 2\sqrt{5}} \\
 &= 1 - \frac{2}{5}\sqrt{5}, \text{ and } \tan^2 54^\circ = \frac{1}{\tan^2 36^\circ} = \frac{1}{5 - 2\sqrt{5}} = 1 + \frac{2}{5}\sqrt{5}. \\
 &\text{Therefore, } \tan^2 18^\circ + \tan^2 36^\circ + \tan^2 54^\circ + \tan^2 72^\circ \\
 &= 1 - \frac{2}{5}\sqrt{5} + 5 - 2\sqrt{5} + 1 + \frac{2}{5}\sqrt{5} + 5 + 2\sqrt{5} = 12.
 \end{aligned}$$

Also solved by Gregory Holdon, Indiana University of Pennsylvania, Indiana, Pennsylvania.

207. Proposed by Charles W. Trigg, San Diego, California.

There is only one three-digit number which is equal to twice the sum of the squares of its digits. Find this number.

Solution by William R. MacHose, Grove City College, Grove City, Pennsylvania.

The 3 digit number is of the form $100x + 10y + z$, where $1 \leq x \leq 9$, $0 \leq y \leq 9$, and $0 \leq z \leq 9$. The number is given equal to $2(x^2 + y^2 + z^2)$, indicating that the number is even, therefore, we now know that z must be even.

$$\text{Given: } 100x + 10y + z = 2(x^2 + y^2 + z^2) \quad (1)$$

$$\text{therefore, } 2x^2 + (-100)x + (2y^2 - 10y + 2z^2 - z) = 0.$$

Solving for x in the quadratic formula,

$$x = \frac{100 \pm \sqrt{10000 - 4(2)(2y^2 - 10y + 2z^2 - z)}}{2(2)}$$

$$x = 25 - \sqrt{625 + (-y^2 + 5y) + (-z^2 + z/2)} \quad (2)$$

The minus sign before the radical is needed because $x \leq 9$. For convenience let $f(y) = -y^2 + 5y$ and $f(z) = -z^2 + z/2$.

y	$f(y)$	z	$f(z)$
0	0	0	0
1	4	2	-3
2	6	4	-14
3	6	6	-33
4	4	8	-60
5	0		
6	-6		
7	-14		
8	-24		
9	-36		

From these tables we can readily see

$$\text{that } -96 \leq f(y) + f(z) \leq 6. \quad (3)$$

It follows that:

$$529 \leq 625 + f(y) + f(z) \leq 631. \quad (4)$$

We also know that since $1 \leq x \leq 9$ and from equation (2), then $16 \leq \sqrt{625 + f(y) + f(z)} \leq 24$; squaring this we have

$$256 \leq 625 + f(y) + f(z) \leq 576 \quad (5)$$

Combining the conditions in equations (4) and (5) we see that:

$$529 \leq 625 + f(y) + f(z) \leq 576.$$

Now since $625 + f(y) + f(z)$ is a perfect square between the values of 529 and 576, then $625 + f(y) + f(z)$ must equal either $23^2 = 529$ or $24^2 = 576$. For 576 to be our choice, $f(y) + f(z)$ must equal -49 which cannot occur, according to the tables above.

For 529 to be the choice, then $f(y) + f(z)$ must equal -96 which it does for $y = 9$ and $z = 8$. If $y = 9$ and $z = 8$, then by equation (2) x must equal 2. Putting these values into equation (1) we find our number to be 298.

Also solved by Criss Cosner, La Mesa, California, and Don N. Page, William Jewell College, Liberty, Missouri.

208. Proposed by Steven R. Conrad, Flushing, New York.

Find all values of x for which the expression $4^x + 4^8 + 4^{11}$ is a perfect square.

Solution by Burl V. Bachman, State University of New York at Albany, Albany, New York.

There must exist integral values of x and y such that

$$4^8(4^{x-8} + 1 + 4^3) = y^2.$$

The fact that $4^8 = (2^8)^2$ and y^2 are perfect squares demands that $4^{x-8} + 1 + 4^3$ also be a perfect square, since $4^{x-8} + 1 + 4^3$

$$= \frac{y^2}{4^8} = \left(\frac{y}{2^8}\right)^2. \text{ Hence we may set } 4^{x-8} + 1 + 4^3 = 4^{x-8}$$

$$+ 1 + 64 = 4^{x-8} + 65 = z^2, \text{ where } z = \frac{y}{2^8}. \text{ In other words,}$$

$$z^2 = 65 + 4^{x-8}. \quad (1)$$

Since $x < 8$ is impossible for equation (1) to yield an integral solution, we turn to $x \geq 8$. For $x = 8$, we have $z^2 = 65 + 4^0 = 65 + 1 = 66$, which is no solution. For $x = 9$, we have $z^2 = 65 + 4^1 = 65 + 4 = 69$, which is no solution, either.

However, for $x = 10$, we have $z^2 = 65 + 4^2 = 65 + 16 = 81 = 9^2$. Hence $x = 10$, $z = 9$ is a solution of equation (1).

Similarly, we now pursue $x \geq 11$. For $x = 11$, we have $z^2 = 65 + 4^3 = 65 + 64 = 129$, and $x = 11$ yields no solution. For $x = 12$, we have $z^2 = 65 + 4^4 = 65 + 256 = 321$, and $x = 12$ is no solution either.

Now, for $x = 13$, we have $z^2 = 65 + 4^5 = 65 + 1024 = 1089 = (33)^2$. Therefore, $x = 13$, $z = 33$ is another satisfactory solution of equation (1).

By analyzing the sequence $\{i^2\}_{i=0}^{\infty} = 0, 1, 4, 9, \dots, 64, 81, \dots, 1024, 1089, \dots$, we see that the n^{th} and $(n+1)^{\text{th}}$ terms of the sequence differ from each other by two more than the n^{th} and $(n-1)^{\text{th}}$ terms. Since $(33)^2 - (32)^2 = 1089 - 1024 = 65$, we, therefore, know that the difference between any two adjacent terms of the sequence $\{i^2\}_{i=0}^{\infty}$ beyond $n = 33$ will be greater than 65.

Hence, since both sides of the equation $z^2 - 65 = 4^{x-8}$

$= (2^{8-x})^2$ must be perfect squares, equation (1) will yield no more integral solutions other than $x = 10$ and $x = 13$.

The perfect squares created by $x = 10$ and $x = 13$ are:

$$\text{a) } 4^{13} + 4^8 + 4^{11} = 4^8(4^5 + 1 + 4^3) = (2^8)^2(1089) = \\ (256)^2(33)^2 = (256 \times 33)^2 = (8448)^2$$

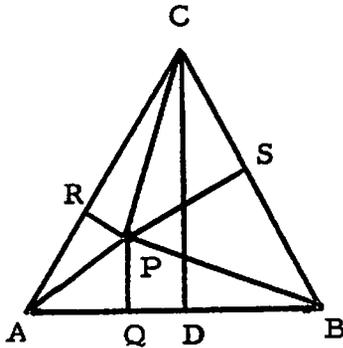
$$\text{b) } 4^{10} + 4^8 + 4^{11} = 4^8(4^2 + 1 + 4^3) = (2^8)^2(81) = \\ (256)^2(9)^2 = (256 \times 9)^2 = (2304)^2.$$

Also solved by Don N. Page, William Jewell College, Liberty, Missouri.

209. *Proposed by Thomas F. Cleary III, State University of New York at Albany, Albany, New York.*

Given: An arbitrary (equilateral) triangle and an arbitrary point P in the interior of the triangle.

Prove: The sum of the lengths of the perpendiculars from point P to each of the sides of the triangle equals the length of an altitude of the triangle.



Solution by Don N. Page, William Jewell College, Liberty, Missouri.

Let $\triangle ABC$ be the equilateral triangle with the arbitrary point P within it and $PQ \perp AB$, $PR \perp AC$, $PS \perp BC$, and $CD \perp AB$. Since $\triangle ABC$ is equilateral, $\overline{AB} = \overline{AC} = \overline{BC}$. By the formula for the area of a triangle, $\triangle APB = \frac{1}{2}\overline{AB} \overline{PQ}$, $\triangle APC = \frac{1}{2}\overline{AC} \overline{PR}$

$= \frac{1}{2} \overline{AB} \overline{PR}$, and $\triangle BPC = \frac{1}{2} \overline{BC} \overline{PS} = \frac{1}{2} \overline{AB} \overline{PS}$. $\triangle ABC = \triangle APB + \triangle APC + \triangle BPC = \frac{1}{2} \overline{AB} \overline{PQ} + \frac{1}{2} \overline{AB} \overline{PR} + \frac{1}{2} \overline{AB} \overline{PS} = \frac{1}{2} \overline{AB} (\overline{PQ} + \overline{PR} + \overline{PS})$. But also, $\triangle ABC = \frac{1}{2} \overline{AB} \overline{CD}$. Therefore $\frac{1}{2} \overline{AB} (\overline{PQ} + \overline{PR} + \overline{PS}) = \frac{1}{2} \overline{AB} \overline{CD}$, and so $\overline{PQ} + \overline{PR} + \overline{PS} = \overline{CD}$, proving that the sum of the lengths of the perpendiculars from point P to each of the sides of $\triangle ABC$ equals the length of an altitude of $\triangle ABC$.

210. *Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.*

The Fibonacci sequence $\{F_n\}$ is defined by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Similarly, the Lucas sequence $\{L_n\}$ is defined by $L_1 = 1, L_2 = 3, L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$.

Then for a given positive integer k , find

$$\lim_{n \rightarrow \infty} F_{n+k} / L_n .$$

Solution by Don N. Page, William Jewell College, Liberty, Missouri.

We may let $F_n = ac^n + bd^n$ and solve for a, b, c , and d by using the defining conditions $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1}$

+ F_{n-2} , obtaining $F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n \sqrt{5}}$. Likewise we

may set $L_n = df^n + eg^n$ and solve for d, e, f , and g , obtaining

$$L_n = \frac{(1 + \sqrt{5})^n + (1 - \sqrt{5})^n}{2^n} .$$
 Then

$$\frac{F_{n+k}}{L_n} = \frac{(1 + \sqrt{5})^{n+k} - (1 - \sqrt{5})^{n+k}}{2^k \sqrt{5} [(1 + \sqrt{5})^n + (1 - \sqrt{5})^n]} = \frac{(1 + \sqrt{5})^k - \left(\frac{1 - \sqrt{5}}{1 + \sqrt{5}}\right)^n (1 - \sqrt{5})^k}{2^k \sqrt{5} \left[1 - \left(\frac{1 - \sqrt{5}}{1 + \sqrt{5}}\right)^n\right]} ,$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{F_{n+k}}{L_n} = \frac{(1 + \sqrt{5})^k - 0(1 - \sqrt{5})^k}{2^k \sqrt{5} [1 - 0]} = \frac{(1 + \sqrt{5})^k}{2^k \sqrt{5}} .$$

Kappa Mu Epsilon News

EDITED BY EDDIE W. ROBINSON, *Historian*

Twenty-five years ago:

Several chapters reported projects of making surgical dressings for the American Red Cross. Chapter rolls were greatly diminished by members being called into the Army. Missouri Beta reported that the size of the chapter was reduced to two members. No new chapters were installed in 1943.

Twenty years ago:

Colorado Alpha and California Alpha were installed. Missouri Alpha initiated twenty-three members, the largest class ever to be initiated by this chapter. Kansas Gamma presented a radio program telling of the achievements of five great women mathematicians. The Society consisted of thirty-seven chapters.

Ten years ago:

Kansas Beta hosted the 1958 Regional Convention for a six-state area. California Gamma was the only chapter installed in 1958. The Society consisted of fifty-one chapters.

Alabama Beta, Florence State College, Florence

Twenty-one new members were initiated at a banquet in April, 1967. The speaker was Mr. John Blackwell, 1963 chapter president. Six members attended the National Convention in Atchison, Kansas. Dr. Elizabeth T. Wooldridge was the installing officer for South Carolina Beta Chapter at South Carolina State College, Orangeburg. Assisting in the installation were Pamela Sims, Mary Darby, Benjamin Fouts, Ronald Williams, Barbara Wright. At the 1967 homecoming, a coffee hour was attended by fifty-five alumni members and guests from sixteen different years of initiation.

California Gamma, California State Polytechnic College, San Luis Obispo

The chapter by-laws were revised to conform to the revisions in the national constitution. A pledge system has been instituted which provides for a pledging period of one quarter and our first pledge class of fifteen members is now at work. Monthly meetings usually feature student speakers.

Illinois Alpha, Illinois State University, Normal

The homecoming float of Illinois Alpha received fourth place in the organizational float competition. More than thirty new members have been initiated during the past year, making Kappa Mu Epsilon one of the larger honorary societies on the campus.

Illinois Beta, Eastern Illinois University, Charleston

Sixty-two members were initiated in the past year, bringing the total membership to 564.

Illinois Delta, College of St. Francis, Joliet

Two new undergraduates and one faculty member have been initiated during the past year.

Kansas Alpha, Kansas State College of Pittsburg, Pittsburg

Syrilda Hughes Miller was the recipient of the Robert Miller Mendenhall award as the outstanding senior mathematics major for 1966-67 and was presented a KME pin.

Chapter activities this year have included a departmental picnic sponsored by KME for all students interested in mathematics. Seven students were initiated during the fall semester of 1967. At this meeting, Calvin Mein summarized the National Convention held April, 1967, for the benefit of students unable to attend.

Other programs featured a talk about mathematics as taught in India by Dr. J. D. Haggard, past national historian of KME, and student papers by Charles McGuire and Homer Watson.

Kansas Gamma, Mount St. Scholastica College, Atchison

This fall Kansas Gamma has twenty-three returning members and sixteen pledges. The theme for the year's discussions and activities is "KME, a Mathematical Group that Functions." Meetings are held semi-monthly and two expository papers, prepared by students, are presented at each meeting.

Special events of the year included: a mathematics movie, "Curves of Constant Width;" a field trip to the government storage caves in Atchison; a lecture by Dr. Fred Van Vleck, Kansas University, who spoke on "Applications of Matrices to the Social Sciences," and a tea honoring the speaker; the High School Invitational Mathematics Tournament; and the Regional Convention at Tahlequah, Oklahoma.

Chapter activities included a formal initiation of new members, a chili supper honoring the new pledges, the Christmas Wassail Bowl Ceremony, a pledge party for the members, and the farewell banquet.

Sister Helen Sullivan, faculty moderator, was the guest speaker for the banquet and initiation ceremonies of Pennsylvania Epsilon at Kutztown State College on November 2, 1967.

Kansas Epsilon, Fort Hays Kansas State College, Hays

Ten members were initiated in December, 1967. Plans are being made to attend the Regional Convention in Oklahoma in April.

Maryland Alpha, College of Notre Dame of Maryland, Baltimore

The theme for the fall meetings was "How, Why and What of Statistics." Maryland Alpha and Beta held a joint meeting in January. Spring meetings will consist of student papers and five members will be initiated on May 7.

Michigan Alpha, Albion College, Albion

Fall activities included a talk by Dr. George Francis of Michigan University on the subject of generalized derivatives and a discussion and demonstration of the IBM 1401 computer which was recently installed on the campus.

Mississippi Beta, Mississippi State University, State College

Mississippi Beta was installed as an active chapter in May, 1967, and has since initiated seventy-six members. Officers are the following:

President:	David Speights
Vice President:	James Hace
Secretary:	Janice West
Treasurer:	Deborah Mager
Corresponding Secretary:	Miss Claris Armstrong
Faculty Sponsor:	Dr. Virginia Rohde

Mississippi Gamma, University of Southern Mississippi, Hattiesburg

Meetings usually consist of films and papers. The May meeting will be a steak cookout plus election of officers.

Missouri Alpha, Southwest Missouri State College, Springfield

During the fall semester, Missouri Alpha has featured at its monthly meetings, programs given by students, mathematics faculty

members and a physics faculty member. A Christmas party was held in December, the highlight of which was the decorating of a tree with geometrical ornaments designed and constructed by guests at the party.

Songbooks have been constructed and distributed and the practice of singing the national song at each meeting was introduced. An initiation banquet was held for members and new initiates in November. The speakers for the evening were the department chairman, Dr. L. T. Shiflett, and the national historian, Eddie W. Robinson, who spoke on the history of KME and of Missouri Alpha Chapter. Plans are now being made for the regional meeting and for our annual spring picnic, at which is presented the freshman award and the merit award.

Missouri Beta, Central Missouri State College, Warrensburg

The chapter has formed a mathematics club for those students interested in mathematics. Faculty members present programs at the monthly mathematics club meetings. A money-making project of selling candles at Christmas time netted approximately \$75. Missouri Beta will host a banquet for Kappa Mu Epsilon-Sigma Zeta in the spring. Two papers are being prepared for presentation at the regional convention.

Nebraska Beta, Kearney State College, Kearney

On April 7-8, 1967, eight students and two faculty members from our chapter attended the sixteenth biennial convention of Kappa Mu Epsilon at Atchison, Kansas. This year we have initiated thirty-six new members, making our active membership over fifty. Our chapter is again sponsoring a Visiting Lecturer in Mathematics this spring and the Mathematics Booster Hour two nights a week to help students in the lower level mathematics courses. We have had our annual Christmas party and are looking forward to our spring banquet and the regional KME convention at Tahlequah, Oklahoma.

Nebraska Gamma, Chadron State College, Chadron

Five new members were initiated for the fall term and ten were taken in for the spring semester. As a service to the college and as a fund-raising project for the chapter, members taught a ten-week slide rule class. We had a float entered in the homecoming parade and the new initiates will put on a humorous skit for the Blue Key Review.

KME has for several years been in charge of grading and recording test scores at the annual Inter-high school Scholastic Contest. It will do so again this year.

Ohio Gamma, Baldwin-Wallace College, Berea

Our chapter is completing its first year of theme programming. This year's theme has been "Computer Emphasis" and has been very successful. It is hoped that the substantially increased attendance of meetings will continue under the new administration, to be elected in the near future.

Oklahoma Alpha, Northeastern State College, Tahlequah

The mathematics club and KME officers alternate in responsibility for programs. Student programs are drawing larger attendance than faculty programs. Several KME members are entering programming work in Tulsa, Oklahoma.

Oklahoma Alpha will host the Regional Convention on April 20.

Wisconsin Alpha, Mount Mary College, Milwaukee

October: Demonstration with an electric calculator by Mr. Chuck Wendt.

November: Talk entitled, "Degrees of Infinity" by Sue O'Connor.

December: Students made mathematical ornaments for Christmas tree, using colored straws and ribbons.

January: Talk entitled, "What's the Odds" by Mary Ellen Nabor.

Talk entitled, "Dynamite Symmetry" by Rosemary Wieczorek.

We are now making plans for our annual mathematics contest for high school students and also plans for attending the regional KME convention, which will be held at Rosary College, River Forest, Illinois.