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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, **THE PENTAGON**, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

On Arbitrarily Large Postulate Sets For the Propositional Calculus: A Constructive Proof*

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1. Introduction

There are many different-sized postulate sets for the propositional calculus which are independent and complete. For example:

- a) Nicod [3] exhibits the single postulate

$$[p/(q/r)]/([s/(s/s)]/[(t/q)/\{(p/t)/(p/t)\}]),$$

with the Sheffer stroke function as a primitive operation defined by $p/q \equiv_p \sim p \vee \sim q$.

Nicod shows that this postulate is complete.

- b) Rosser [5], with primitive operations \sim and \wedge , shows that the following set of three postulates is sufficient to develop the propositional calculus:

P1. $p \rightarrow p \wedge p$

P2. $p \wedge q \rightarrow p$

P3. $(p \rightarrow q) \rightarrow [\sim(q \wedge r) \rightarrow \sim(r \wedge p)]$.

It may be shown that this set is independent.

- c) Of the famous Russell-Whitehead postulates, the following four are independent and complete [1]:

P1. $p \vee p \rightarrow p$

P2. $q \rightarrow p \vee q$

P3. $p \vee q \rightarrow q \vee p$

P4. $(q \rightarrow r) \rightarrow [(p \vee q) \rightarrow (p \vee r)]$

where \sim and \vee are the primitive terms.

There are other different-sized postulate sets known. Tarski

*A paper presented at the KME Regional Convention at Springfield, Missouri, April 23, 1966, and awarded first place by the Awards Committee.

[6] uses a set of seven postulates; Kleene [2] uses a set of ten; Novikov [4] uses a set of eleven.

2. On arbitrarily large postulate sets.

We pose the obvious question: Does there exist for all n a set of n independent, complete postulates for the propositional calculus? Before proving this and a much stronger result, we proceed to a useful lemma.

In the succeeding work all propositional functions will be written in terms of the two primitive operations negation (\sim) and implication (\rightarrow).

If a function $\alpha(p_1, p_2, \dots, p_k)$ is a tautology in this context then it may be written in the form

$$\Phi_n(\theta(p_1, \dots, p_k) \rightarrow \Psi(p_1, \dots, p_k))$$

where Φ_n is a recursively defined function equivalent to n negations of its argument, and where $\theta(p_1, \dots, p_k)$ and $\Psi(p_1, \dots, p_k)$ consist of the antecedent and consequent, respectively, of the major implication in α .

DEFINITION. Let α be a propositional function. Define $N(\alpha)$ to be the total number of uses of all variables in α .

LEMMA. If α and β are functions of one variable, $\alpha(p) \equiv \phi_i(\theta_1(p) \rightarrow \psi_1(p))$ and $\beta(p) \equiv \phi_j(\theta_2(p) \rightarrow \psi_2(p))$, and

$$\text{if } \frac{N(\theta_1(p))}{N(\psi_1(p))} \neq \frac{N(\theta_2(p))}{N(\psi_2(p))},$$

then there exist no propositional functions λ and γ such that $S_p^\lambda(\alpha(p))$ and $S_p^\gamma(\beta(p))$ are identical.

Proof.

It will suffice to show that under the circumstances described any substitution will preserve the original ratio under the function

N . If $\frac{N(\theta_1)}{N(\psi_1)} = \frac{r}{s}$, then assume that some substitution

$\frac{\lambda}{p}$ is made in α , and suppose that formula λ contains n uses of k variables. Then any substitution made in θ_1 will also be made in ψ_1 ,

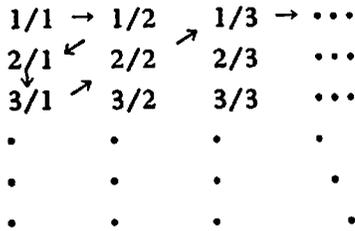
and we now have

$$\frac{N(\theta'_i)}{N(\psi'_i)} = \frac{nr}{ns} = \frac{r}{s}.$$

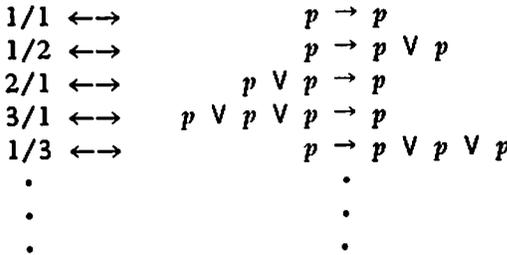
Since all substitutions are thus ratio-preserving as far as the number of variables is concerned, and since the two ratios are initially not equal, they can never under simple substitution be identical.

Q.E.D.

We know that the rationals may be well-ordered by a scheme such as the following:



We now have the sequence 1, 1/2, 2, 3, 1/3, ... in which all rationals appear. If we associate with each rational number a tautology in the following manner:



then by the Lemma above no two of these will ever be identical under substitution, since each has a distinct ratio associated with it. Call the set of these tautologies C. Define C* to be the set whose elements are the quadruple negated elements of C.

THEOREM. For every $n > 2$, we may construct a set of n independent complete postulates for the propositional calculus.

Proof.

Let t_1, t_2, \dots, t_{n-3} be distinct elements of C*.

Consider the set

P1. t_1

P2. t_2

.

.

.

P($n - 3$). t_{n-3}

P($n - 2$). $t_1 \rightarrow (t_2 \rightarrow \dots \rightarrow (t_{n-3} \rightarrow [p \rightarrow pp]) \dots)$

P($n - 1$). $t_1 \rightarrow (t_2 \rightarrow \dots \rightarrow (t_{n-3} \rightarrow [pq \rightarrow p]) \dots)$

P(n). $t_1 \rightarrow (t_2 \rightarrow \dots \rightarrow (t_{n-3} \rightarrow [X] \quad) \dots)$

where X is $(p \rightarrow q) \rightarrow [(\sim qr) \rightarrow (\sim rp)]$.

Completeness. Follows from the completeness of Rosser's set (above) which is obviously derivable from this set.

Independence.

I. Of the t 's. Consider t_i , $1 \leq i \leq (n - 3)$

(1) By substitution

a) No t_j , $j \neq i$, will yield t_i by the lemma;

b) None of the last three will yield t_i by substitution since their major connective is implication while the major connective in t_i is negation.

(2) By modus ponens

a) No t_j , $j \neq i$, will yield t_i by modus ponens since modus ponens may not be applied to a statement whose major connective is not implication.

b) Modus ponens on the last three will yield, without possible further simplification, only the following (with t_i deleted):

$$t_i \rightarrow (t_{i-1} \rightarrow \dots \rightarrow (p \rightarrow pp) \dots)$$

$$t_i \rightarrow (t_{i-1} \rightarrow \dots \rightarrow (pq \rightarrow p) \dots)$$

$$t_i \rightarrow (t_{i-1} \rightarrow \dots \rightarrow (X) \dots)$$

none of which will yield t_i by substitution or by modus ponens.

II. Of P($n - 2$).

We take as a model a 3-valued algebra with operations \sim

and \rightarrow defined by:

p	$\sim p$
0	1
1	2
2	2

\rightarrow	0	1	2
0	0	1	0
1	0	0	0
2	0	1	0

It may be easily verified that all of the postulates except $P(n - 2)$ have the hereditary property in this model of taking only value 0 or 2.

III. Of $P(n - 1)$.

We construct a model similar to the one above, with operations defined by the following tables:

p	$\sim p$
0	1
1	1
2	0

\rightarrow	0	1	2
0	1	1	1
1	0	1	0
2	1	1	1

In this model all postulates except $P(n - 1)$ have the hereditary property of always taking value 1.

IV. Of $P(n)$.

We construct a model similar to the above, with operations defined by:

p	$\sim p$
0	1
1	2
2	3
3	3

\rightarrow	0	1	2	3
0	0	2	2	3
1	1	0	0	1
2	1	0	0	1
3	0	2	2	3

In this model all postulates except $P(n)$ have the hereditary property of always taking on value 0 or 3.

Q.E.D.

THEOREM. For every $n > 3$, there exist an infinite number of independent, complete postulate sets with exactly n elements.

Proof.

Obviously there are an infinite number of ways of choosing a set of $n - 3$ rationals, and it may be seen that the proof of the previous theorem holds regardless of the particular elements of C^* chosen. Therefore, the subset of C^* associated

with any such $(n - 3)$ element subset of the rationals will satisfy the theorem.

Q.E.D.

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Science, particularly mathematics, though it seems less practical and less real than the news contained in the latest radio dispatches, appears to be building the one permanent and stable edifice in an age where all others are either crumbling or being blown to bits.

—E. KASNER and J. NEWMAN

Galois' Theory for the Group of an Equation and the Criterion of Solvability*

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Evariste Galois was a French mathematician who was born in 1811 and lost his life in a silly boyish duel before he had reached the age of twenty-one. The night before the duel he wrote a letter to his friend Aguste Chevalier in which he set forth briefly his discovery of the connection of the theory of groups with the solutions of equations by radicals.

Long before entering their first college analysis course budding mathematicians know that one of the important functions of mathematics is to solve equations. Algebraic equations are classified according to their degree. The general equations of the first and second degree and their solutions were known as early as 1700 B.C.

$$\begin{aligned} 1) \quad ax + b &= 0 & x &= -b/a \\ 2) \quad ax^2 + bx + c &= 0 & x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ 3) \quad ax^3 + bx^2 + cx + d &= 0 \\ 4) \quad ax^4 + bx^3 + cx^2 + dx + e &= 0 \end{aligned}$$

But the solutions for the cubic and quartic were not formulated until the sixteenth century. Note that the solutions are obtained in terms of the coefficients by the use of rational operations and the extraction of roots, and as the degree increases the solution becomes rapidly more difficult. Although sixteenth century mathematicians could not solve general equations of degree higher than four, still they believed that such equations could be solved and eventually would be. It was not until the nineteenth century that solutions of these equations were shown, by means of the theory of groups, to be impossible.

This statement is obviously a very sweeping one, and admit-

*A paper presented at the KME Regional Convention at Springfield, Missouri, April 23, 1966, and awarded second place.

tedly needs qualification. First of all, just what sort of an impossibility is meant here. Whether a problem can or cannot be solved depends upon the conditions imposed upon the solution. The equation, $x + 5 = 3$, can be solved if negative numbers are permitted, but cannot be solved if negative numbers are excluded. Similarly, $2x + 3 = 10$ can be solved if x represents a number of dollars, but cannot be solved if x represents a number of persons since $x = 3\frac{1}{2}$. An algebraic expression may be reducible (that is factorable) or irreducible depending upon the field in which the factoring is to be done. Thus, $x^2 + 1$ is irreducible in the field of real numbers but reducible in the field of complex numbers. In other words, it is meaningless to say that an expression can or cannot be factored without specifying the field. Consequently, mathematicians have learned the importance of indicating the environment in which a statement is true or false or perhaps entirely meaningless.

Therefore, in what sense has it been proved impossible to solve the general equation of degree higher than four? The answer is that it is impossible to solve it by radicals. This means that the unknown cannot be expressed in terms of the coefficients, only, by the use of rational operations and the extraction of roots. Group theory proved to be the important tool that facilitated this discovery.

Essentially a group is a mathematical structure or system and as such it must have elements and an operation. For individual groups this operation may vary widely; and the more common ones are the rational operations, substitutions, and the following of one rotation by another. It is customary, no matter what the operation, to call it multiplication. A group has four definitive qualifications. It must have: 1) closure, 2) associativity, 3) an identity element, and 4) each element must have an inverse.

Also, in order to understand Galois' theory, some information about subgroups is necessary. A subset of the original group, G , is a subgroup if and only if it satisfies the qualifications of a group under the same operation as G . It can be shown that the order of any subgroup is a factor of the order of the given group. For example, a group of order 24 may have subgroups of order 2, 3, 4, 6, 8, and 12 because each of these will divide 24.

A particular kind of subgroup that is important to an understanding of Galois' theory is an invariant subgroup. Now, a subgroup is called invariant if it remains unchanged when all of its

elements are conjugated by all the elements of the original group. To illustrate what is meant by the conjugation of one element by another, examine the group of six substitutions:

$$G = \{(I), (12), (13), (23), (123), (132)\}$$

and a subgroup of it

$$S = \{(I), (12)\}$$

Taking the element (12) multiply it on the right by (123) and on the left by (132) which is the inverse of (123).

$$(132)(12)(123) = (23).$$

The result, (23), is then called the conjugate of (12) by (123).

Particularly important among invariant subgroups is a maximal invariant proper subgroup. It is one which is not contained in a larger invariant proper subgroup. Now if G is a given group and if H is a maximal invariant proper subgroup of G , K a maximal invariant proper subgroup of H , etc., and if the order of G is divided by the order of H , and the order of H divided by the order of K , etc., the numbers obtained are called the composition factors of the group G . If these are all prime numbers, G is called a solvable group.

With these few facts at hand, it is now possible to explore Galois' discovery concerning the group of an equation. Every equation has a definite group associated with it for a given field. Taking for example, an equation of the third degree, $ax^3 + bx^2 + cx + d = 0$ having three distinct roots, x_1, x_2, x_3 , and some function of these roots, such as, $x_1x_2 + x_3$ in which these x 's can be replaced by each other, there would be six possible substitutions, that is, 3!. Similarly for an equation with four roots, there would be 4! substitutions in the function we choose. And in general for n roots there would be $n!$ possible substitutions. It is important to note that when a substitution is applied to a function it may or may not alter the value of the function. For instance, the substitution (12) applied to $x_1 + x_2$ does not alter its value because addition is commutative; but if (12) is applied to $x_1 - x_2$, it does alter the value since it changes $x_1 - x_2$ to $x_2 - x_1$, and subtraction is not commutative.

Supposing an equation of degree n , having n distinct roots $x_1, x_2, x_3 \dots x_n$, it can be shown that in the function,

$$V_1 = m_1x_1 + m_2x_2 + m_3x_3 + \dots + m_nx_n,$$

which is sometimes called the Galois function, the m 's can be chosen so that every possible substitution ($n!$ in all) of the x 's does alter the value, V , of this function; and therefore, this function can have $n!$ different values. These $n!$ values are represented by $V_1, V_2, V_3, \dots, V_n$; and the expression

$$P(y) = (y - V_1)(y - V_2) \cdots (y - V_n)$$

can be formed where y is a variable. This polynomial may or may not be factorable depending upon the field in which the factoring is to be done. Suppose, for example, that for a given field $P(y)$ is factored so that the part containing V_1 which is not further reducible in that field is $(y - V_1)(y - V_2) = y^2 - (V_1 + V_2)y + V_1V_2$. Note that in this case the only V 's involved are V_1 and V_2 ; and there are just two possible substitutions for these, the identity substitution and that substitution of the x 's which changes these V 's into each other. Now, these substitutions form a group which is called the group of the given equation for the given field.

$$\{I, (V_1 V_2)\} \quad \begin{array}{c|cc} & I & (V_1 V_2) \\ \hline I & I & (V_1 V_2) \\ (V_1 V_2) & (V_1 V_2) & I \end{array}$$

This function remains unaltered by all the substitutions of this group. Similarly, if the irreducible part of $P(y)$ had contained besides the V_1 , also V_2 and V_3 , the group would then consist of all those substitutions which would leave this irreducible part unaltered. In general, then, the group of an equation for a given field is determined by that part of $P(y)$ which is irreducible in the given field and contains V_1 . If this irreducible part is denoted by $G(y)$, then $G(y) = 0$ is called a Galois resolvent.

There are two important characteristic properties of the group of an equation which enable one to find which of the possible substitutions form this group without actually going to the trouble of finding a Galois resolvent. First, it can be proved that if the value of any function of the roots of an equation is in a given field, then, this function must remain unaltered in value by all the substitutions of the group of this equation for that field. Secondly, if the value of a function is not in the field, the group must contain a substitution which does alter the value of the function.

To illustrate these properties one may consider the quadratic equation $x^2 + 3x + 1 = 0$ having two roots, x_1 and x_2 . Since there are only two roots, the only possible substitutions are I and (12) . The group of this equation must contain either both of these or I alone, and that depends on the field which is chosen. With the function of the roots $x_1 - x_2$, the quadratic formula and a little algebra yields that $x_1 - x_2 = \frac{\sqrt{b^2 - 4ac}}{a}$. Since in the given

quadratic equation $a = 1$, $b = 3$, and $c = 1$, $x_1 - x_2 = \sqrt{5}$. Now if the field chosen is the field of rational numbers, then the value of this function is not in the field; and therefore, this group must have a substitution which does alter this function. Obviously (12) must be in the group, and the group for a rational field, therefore, contains both I and (12) . If on the other hand, the field of real numbers is chosen, the value $\sqrt{5}$ is in the field; and therefore, $x_1 - x_2$ must remain unaltered by all the substitutions of the group. Then the group cannot contain (12) because this substitution alters $x_1 - x_2$. Consequently, the group of this equation for the field of real numbers contains only I — not very interesting, but it is a group!

Finally, this information concerning the group of an equation for a given field and how to find it can be used in the Galois criterion of solvability. It should be recalled that a solvable group is one for which the composition factors are all prime numbers. Galois showed that an equation is solvable by radicals if and only if its group, for a field containing its coefficients, is a solvable group.

Since the general quadratic $ax^2 + bx + c = 0$ has two roots, its group, G , for a field containing its coefficients, consists of the substitutions I and (12) . Its only maximal invariant proper subgroup is obviously I , so its only composition factor is $2/1 = 2$.

$$\begin{array}{l} G = I, (12) \\ H = I \end{array} \qquad G/H = 2/1 = 2$$

Since this factor is prime, then according to the Galois criterion it has been proved that every quadratic is solvable by radicals.

The general equation of degree four,

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

has a group of order $4!$ or 24 . A series of maximal invariant proper subgroups:

G: order 24	$24/12 = 2$	} Prime
H: order 12	$12/4 = 3$	
K: order 4	$4/2 = 2$	
L: order 2	$2/1 = 2$	
M: order 1		

consists of orders 12, 4, 2, and 1 respectively; and the composition factors are 2, 3, 2, and 2 — all *prime* numbers!

For the general equation of degree five, G contains $5!$ substitutions, H contains $5!/2$ substitutions, and the only invariant proper subgroup of H has only one element, I . The proof of this statement can be found in *Modern Algebraic Theories* by Leonard Dickson. Therefore, the composition factors are 2 and $5!/2$, but $5!/2$ is *not prime*. Consequently, the general equation of degree five is not solvable by radicals. In fact this is true for the general equation of degree $n = 4$, since the composition factors are 2 and $n!/2$, and the latter is not prime.

The use of groups to determine the solvability of equations is by no means the only application of the wonderful idea of groups. In fact, group theory is fundamental in projective geometry and the theory of relativity. The farsighted speculation of C. J. Keyser in his book, *Mathematical Philosophy*, suggests that the group concept will revolutionize modern thought in history, philosophy, and even psychology.

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Some Applications of Partially Ordered Boolean Matrices

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Introduction. In any mathematical system having a well-defined set, it is possible to define a partial ordering, \cong , on the set. It is, however, difficult in some cases to display this partial ordering in a systematic fashion so that an analysis of the system can be undertaken. We show here a method of displaying the partial order on a set by means of Boolean matrices. In this discussion, we will limit ourselves to sets called lattices according to the following definition:
D1: A lattice L is a partially ordered finite set such that every set consisting of any two elements of L has a least upper bound and a greatest lower bound.

In addition to this, we will need the following definitions:
Consider a set L of n elements with a partial ordering, \cong . Let the set of all a_i for i any positive integer from 1 to n be elements of L .

D2. Let $a_i \cong a_j$. Then by $a_i > a_j$ we shall mean $a_i \cong a_j$ and $a_i \neq a_j$. By $a_i \leq a_j$ we shall mean $a_j \cong a_i$ and then $a_i < a_j$ will mean $a_i \leq a_j$ and $a_i \neq a_j$.

D3: Let $c = a_1 > a_2 > a_3 > \dots > a_n$ be any simply ordered subset of L . Then we call c a chain of L . We say c has length $n - 1$ where n is the number of elements of L in the chain.

D4: Let L be a partially ordered set and a_1, a_2 , elements of L . Then a_1 is said to cover a_2 if $a_1 > a_2$ and if there exists no x in L such that $a_1 > x > a_2$. a_1 is said to order a_2 if $a_1 \cong a_2$.

In view of these definitions we can now display a partially ordered set by two Boolean matrices as defined below.

D5: Let L be a partially ordered set with elements x_i . The n by n Boolean matrix (a_{ij}) , having $a_{ij} = 1$ if x_i covers x_j or $i = j$, and $a_{ij} = 0$ if x_i does not cover x_j and $i \neq j$, is called the cover matrix of L . The matrix will be denoted by C .

D6. Let L be a partially ordered set with elements x_i . The n by n Boolean matrix (a_{ij}) , having $a_{ij} = 1$ if x_i orders x_j and $a_{ij} = 0$ if $x_i \not\cong x_j$, is called the order matrix of L . This matrix will be denoted by M .

Before proving the three main theorems noted here, let us consider an example of the order and cover matrices. Consider the set L of subsets of the set $\{0, 1\}$. Clearly L has four elements, namely, $x_1 = \{0, 1\}$, $x_2 = \{1\}$, $x_3 = \{0\}$, and $x_4 = \emptyset$. If we partially order L by means of set inclusion, then L is capable of matrix treatment defined in D5 and D6. We have the following chains in L : $x_1 > x_2 > x_3$, and $x_1 > x_3 > x_4$. The cover and order matrices are displayed in Figure One.

$$C = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure One

There is an interesting relationship between these cover and order matrices and indeed for those of any partially ordered set. We note in the case considered $C^2 = M$. The rest of this paper will be concerned with finding the smallest n such that $C^n = M$. We remark that in taking powers of matrices Boolean algebra is involved.

An n by n matrix is said to be upper triangular if $a_{ij} = 0$ for $i > j$. Our definitions of C and M imply that each of these is upper triangular. Since the product of any two upper triangular matrices is also upper triangular, it follows that $(C)^t$, for any positive integer, t , is also upper triangular. Thus, if we represent the i th row and j th column of $(C)^t$ by ${}_i a_{ij}$, we have ${}_i a_{ij} = 0$ for $i > j$.

Now Parker [1] has proved for any Boolean matrices X and Y having each of their diagonal elements equal to 1, that whenever an element x_{ij} of X is equal to 1, then the corresponding element $(xy)_{ij}$ of the matrix XY is equal to 1 and the element $(yx)_{ij}$ of YX is equal to 1. A cover matrix C is a Boolean matrix having each of its diagonal elements equal to 1. Therefore, by applying Parker's theorem, we see that $a_{ij} = 1$ in C implies that ${}_i a_{ij} = 1$ in $(C)^t$.

Comparing the matrices C and M for the partially ordered set L , we now note that:

- (1) Both C and M have the element in the i th row and j th column equal to one if $i = j$.
- (2) Both C and M have the element in the i th row and j th column equal to zero if $i > j$.
- (3) If C has an element $a_{ij} = 1$ for $i < j$, then for the cor-

responding element b_{ij} of M , $b_{ij} = 1$.

(4) If M has an element $b_{ij} = 0$ for $i < j$, then for the corresponding element a_{ij} of C , $a_{ij} = 0$.

(5) M has an element $b_{ij} = 1$ for $i < j$ when the corresponding element a_{ij} of C is equal to zero if and only if there exists in the partially ordered set, L , a chain of length p , as follows:

$$x_i > x_{r_1} > x_{r_2} > x_{r_3} > \dots > x_{r_{p-1}} > x_j,$$

where each element covers the one which follows it, and $p \geq 2$.

We show the complete relationship between C and M by means of the following three theorems.

THEOREM 1: Let C be the n by n cover matrix for the partially ordered set L . Let $i < j$ and $a_{ij} = 0$ in C . If ${}_q a_{ij} = 1$ in $(C)^q$, where $q \geq 2$, then there exists some element x_r in L such that $x_i > x_r > x_j$.

Proof. The proof is by induction on q . Since $a_{ik} = 0$ in any cover matrix for $i > k$, then for $q = 2$ we have ${}_2 a_{ij} = a_{i(i+1)} \cdot a_{(i+1)j} + \dots + a_{i(j-1)} \cdot a_{(j-1)j} = 1$. Hence at least one of these terms, say $a_{ir} \cdot a_{rj}$, $i > r > j$, must be 1. Hence both a_{ir} and a_{rj} must be 1 by Boolean multiplication. By D5 (since $r \neq i$ and $r \neq j$), x_i covers x_r and x_r covers x_j . Thus, $x_i > x_r > x_j$.

Now assume the theorem holds for q . Then let $a_{ij} = 0$, $i < j$ in C , and let ${}_{q+1} a_{ij} = 1$ in $(C)^{q+1}$ where $q + 1 \geq 2$. Suppose ${}_q a_{ij} = 1$. Then $q \geq 2$, and by the inductive hypothesis there exists some x_r such that $x_i > x_r > x_j$. On the other hand, if ${}_q a_{ij} = 0$, then ${}_{q+1} a_{ij} = {}_q a_{i(i+1)} \cdot a_{(i+1)j} + \dots + {}_q a_{i(j-1)} \cdot a_{(j-1)j} = 1$. Thus, at least one term, say ${}_q a_{it} \cdot a_{tj} = 1$, $i < t < j$. Hence both ${}_q a_{ij}$ and a_{tj} are 1. Consider a_{it} in C . Suppose $a_{it} = 1$. Then x_i covers x_t and $x_i > x_t$. But since $a_{tj} = 1$, $x_t > x_j$. Thus $x_i > x_t > x_j$. But if $a_{it} = 0$, then $q \geq 2$ and by the induction hypothesis, there exists some x_r such that $x_i > x_r > x_t$. But since $a_{tj} = 1$, $x_t > x_j$. Hence $x_i > x_r > x_j$.

THEOREM 2: Suppose in C we have the following conditions:

- (1) $a_{ij} = 0$ for some i, j such that $i < j$;
- (2) there exists in the partially ordered set L a chain of length p from x_i to x_j ; i.e., there are elements

$$x_{r_1}, x_{r_2}, \dots, x_{r_{p-1}} \text{ of } L$$

$$\text{such that } x_i > x_{r_1} > x_{r_2} > \dots > x_{r_{p-1}} > x_j$$

and each element covers the one which follows it;

- (3) there exists in L no such chain from x_i to x_j of length greater than p .

Then in $(C)^p$, ${}_p a_{ij} = 1$.

Proof: The theorem will be proved by induction on p . If $p = 2$, there exists a chain $x_i > x_{r_1} > x_j$ where x_i covers x_{r_1} and x_{r_1} covers x_j . By definition D5 $a_{ir_1} = 1$ and $a_{r_1j} = 1$, where $i < r_1 < j$. Hence $a_{ir_1} \cdot a_{r_1j} = 1$. Now since $a_{ij} = 0$, ${}_2 a_{ij} = a_{i(i+1)} \cdot a_{(i+1)j} + \dots + a_{i(j-1)i} \cdot a_{(j-1)j} = 1$. Hence the theorem is true for $p = 2$.

Assume the theorem is true for $p = s$. Let $a_{ij} = 0$ and let there exist in L a chain $x_i > x_{r_1} > x_{r_2} > \dots > x_{r_s} > x_j$ of length $s + 1$, where each element covers the one which follows it, such that no chain exists in L from x_i to x_j of length greater than $s + 1$. We shall show that ${}_{s+1} a_{ij} = 1$. Now $x_i > x_{r_1} > \dots > x_{r_{s-1}} > x_{r_s}$ is a longest chain from x_i to x_{r_s} , for otherwise there would exist a chain from x_i to x_j of length greater than $s + 1$. Furthermore, each element covers the one which follows it. Since $s \geq 2$, x_i does not cover x_{r_s} . Hence $a_{ir_s} = 0$. By the induction hypothesis, ${}_s a_{ir_s} = 1$. Now ${}_{s+1} a_{ij} = {}_s a_{ii} \cdot a_{ij} + \dots + {}_s a_{ir_s} \cdot a_{r_sj} + \dots + {}_s a_{nn} \cdot a_{nj}$. But since x_{r_s} covers x_j , $a_{r_sj} = 1$. Hence ${}_{s+1} a_{ij} = 1$.

We can now conclude the following theorem, its proof depending on the results of the two previous theorems:

THEOREM 3: Let C and M be the n by n cover and order matrices, respectively, of a partially ordered set, S . If in S , the maximum length of any chain is p , then $(C)^p = M$.

Proof: Let ${}_p a_{ij}$ and b_{ij} represent corresponding elements in $(C)^p$ and M , respectively. We must show that these elements are equal for all i, j , $1 \leq i \leq n$ and $1 \leq j \leq n$. Clearly ${}_p a_{ij} = b_{ij}$ for $i \geq j$.

Let $i < j$ and let ${}_p a_{ij} = 1$. Consider a_{ij} in C . If $a_{ij} = 1$, then $b_{ij} = 1$ in M . On the other hand if $a_{ij} = 0$, then by Theorem 2, there exists some x_r such that $x_i > x_r > x_j$. Hence $x_i > x_j$ and $b_{ij} = 1$. Then $b_{ij} = 1$ when ${}_p a_{ij} = 1$.

Now let $i < j$ and let ${}_p a_{ij} = 0$. Then $a_{ij} = 0$, for if $a_{ij} = 1$, then by Parker's theorem ${}_p a_{ij} = 1$. Now $b_{ij} \neq 1$, for suppose

$b_{ij} = 1$, then x_i orders x_j by D6. Since $a_{ij} = 0$, there must exist some x_r such that $x_i > x_r > x_j$. Hence there exists at least one chain from x_i to x_j . Let $x_i > x_{r_1} > x_{r_2} > \dots > x_{r_{q-1}} > x_j$ be a longest such chain. Then each element covers the one which follows it. By Theorem 2 ${}_q a_{ij} = 1$. But the maximum length of any chain is p so that $q \leq p$. Hence by Parker's theorem ${}_p a_{ij} = 1$, contradicting our premise that ${}_p a_{ij} = 0$. Hence $b_{ij} = 0$. This concludes the theorem.

Applications of Boolean Matrices. Using these theorems we can now present two applications. This concept of Boolean matrices can be applied directly to sorites. This notion is especially valuable in the interpretation of those sorites with a great number of premises such as those of Lewis Carroll. A simple syllogism such as:

All men are mortal.
John is a man.
Therefore, John is a mortal.

contains the chains $p \rightarrow q$, $q \rightarrow r$, therefore $p \rightarrow r$ or, according to our lattice interpretation, the chain $p > q > r$.

Consider an example of a set of premises from Lewis Carroll:

Babies are illogical.
Nobody is despised who can manage a crocodile.
Illogical persons are despised.

Let us use the notation $b \rightarrow i$ for the first premise, $d \rightarrow -c$ for the second, (where $-c$ means not c), and $i \rightarrow d$ for the third. Since we have four terms here, there is a four by four matrix which can describe this set of premises. We stipulate that for $a \rightarrow b$, then a covers b in the Boolean matrix. The cover matrix for this set of premises is given by:

$$C = \begin{matrix} & b & i & d & -c \\ \begin{matrix} b \\ i \\ d \\ -c \end{matrix} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

In order to see all possible conclusions (orderings) from this set of premises we take the third power of this four by four matrix

since three is the length of the maximum chain given.

$$(C)^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M$$

This matrix displays all possible conclusions and shows that babies cannot manage a crocodile, babies are despised, and illogical persons cannot manage crocodiles.

The moves made by chessmen in a game of chess are also conducive to lattice treatment. If we order an $n \times n$ chessboard by giving each position a number 1 to n^2 , then we can define a legal move from position a to b by saying a covers b in the same sense as in definition D4. Although any of the chessmen are capable of matrix treatment according to this definition, the knight's move offers the most novel application. Its complete move is a vertical or horizontal move for two spaces and then one space to the left or right.

Suppose a knight is in position i of an n by n chessboard and it is desired to move it to position l . The problem to be answered is: How many complete moves will this take?

One considers the chain $i > \dots > l$ whose elements are positions on the chessboard with the ordering $j > k$ meaning the knight can move from j to k in one move. The answer to our question, then, lies in the analysis of the number of elements in the chain $i > \dots > l$; that is, in the length of the chain.

This analysis may be accomplished by using Theorem 3. We must examine a cover matrix having one row and one column for each of the n^2 positions on the board. The resulting n^2 by n^2 cover matrix will have entries $a_{ij} = 1$ if $i = j$ or if the knight can move from position i to j in one complete move.

As a simple example, consider the three by three chessboard with spaces numbered from one to nine as shown in Figure Two.

1	2	3
4	5	6
7	8	9

Figure Two

A Three by Three Ordered Chessboard

A knight in position 1 can move to positions 6 and 8 in one move; a knight in position 2 can move to positions 7 and 9 in one move; etc. Thus $1 > 6, 1 > 8$; $2 > 7, 2 > 9$; etc. The nine by nine cover matrix for the above chessboard for a knight's move is given below where the columns and rows are ordered from 1 to 9 left to right, and top to bottom.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure Three

The Cover Matrix for the Chessboard Given in Figure Two

One move, then, corresponds to the cover matrix raised to the first power. The result of Theorem 3 implies that two moves will correspond to the second power of the cover matrix, three moves will correspond to the third power of the cover matrix, etc.

We make two important observations concerning the cover matrix. First it is always symmetric about its main diagonal since if $a > b$, then certainly $b > a$ by the definition of cover as used in this example. Second, it can be shown (see Parker, 1) that an n by n Boolean matrix, a_{ij} , has the property $(a_{ij})^{n-1} = (a_{ij})^n = a_{ij}^{n+p}$ for any integer p . Interpreting the second of these facts for our example, we find the nine by nine matrix, C , raised to the 8th power will be equal to C raised to $8 + p$. However, by the first observation any of these matrices is symmetrical so that raising it to the fourth power in this case will be sufficient.

The powers of the cover matrix are given in Figure Four and the order matrix in Figure Five. Thus, if one wishes to see if he can move from position i to position j on the chessboard in n moves, he merely notes if there is a one or a zero in position a_{ij} of the matrix C^n . A one implies the move can be made; a zero implies that it cannot be made. We note in our example that the zeros along the fifth row and fifth column imply that *no* moves

can be made from position five on our chessboard.

To answer our original problem one needs to generalize the above method for an n by n chessboard by obtaining the n^2 by n^2 cover and order matrices.

$$C^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad C^3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure Four

Powers of the Cover Matrix for $n = 2$ and 3 for the Chessboard of Figure Two

$$C^4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} = M$$

Figure Five

The Order Matrix for the Chessboard given in Figure Two

FOOTNOTES AND BIBLIOGRAPHY

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Linear Set Equations and Set-Theoretic Matrices

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Introduction. The problem of solving a system of simultaneous non-linear set equations was discussed by Goodman [1]. In this paper we consider special systems of linear set equations. We obtain necessary and sufficient conditions for the homogeneous and constant systems to have solutions. These results, in turn, are used to obtain necessary and sufficient conditions for the existence of an inverse of a set-theoretic matrix. Finally, a slightly more general system of equations is solved by means of the inverse.

Preliminaries. A matrix over a universal set U is a rectangular array of subsets of U . The terms element, dimension, transpose, row vector and column vector have the usual meanings. An $m \times n$ matrix A is denoted by $[A_{ij}]_{(m,n)}$ and A' is the transpose of A . Two matrices are equal if and only if they have the same dimension and corresponding elements are equal.

DEFINITION 1. If $A = [A_{ij}]$ and $B = [B_{ij}]$ have the same dimension, the sum of A and B , written $A + B$, is the matrix $[A_{ij} \cup B_{ij}]$.

DEFINITION 2. If $A = [A_{ik}]_{(m,p)}$ and $B = [B_{kj}]_{(p,n)}$, the product of A and B (in that order), written AB , is the matrix $[C_{ij}]_{(m,n)}$, where $C_{ij} = \bigcup_{k=1}^n (A_{ik} \cap B_{kj})$.

It is easy to verify the following properties:

- (a) matrix addition is commutative and associative;
- (b) matrix multiplication is associative;
- (c) the right and left-hand distributive laws hold;
- (d) $(AB)' = B'A'$ whenever A is conformable to B for multiplication;
- (e) if $\emptyset_n = [\emptyset]_{(n,n)}$, then $A + \emptyset_n = A$ for all $n \times n$ matrices A ;
- (f) if I_U^n is the $n \times n$ matrix with U on the main diagonal and \emptyset elsewhere, then $I_U^m A = AI_U^n = A$ for all $m \times n$

matrices A over U .

Notations. If $A = [A_{ij}]_{(m,n)}$, we will write $A = [A^1, \dots, A^n]$, where each A^k is the column vector $\{A_{1k}, \dots, A_{mk}\}$. We may also write $A = \{A_1, \dots, A_m\}$, where each A_p is the row vector $[A_{p1}, \dots, A_{pn}]$. We define $D(A) = \bigcup_{i=1}^m \bigcup_{j=1}^n A_{ij}$. The constant column vector $[S]_{(n,1)}$ will be denoted by V_S^n .

Linear set equations. The system of equations,

$$(1) \quad \bigcup_{j=1}^n (A_{ij} \cap X_j) = B_i, \quad i = 1, \dots, m$$

is called a system of linear set equations in the n unknowns X_1, \dots, X_n . In matrix form, (1) becomes $AX = B$, where A , X and B are defined in the usual manner. System (1) is said to be a system over U if both A and B are matrices over U . If (1) is a system over U and X is a vector over U which satisfies $AX = B$, then X is called a solution of (1).

Homogeneous system. The system

$$(2) \quad [A^1, \dots, A^n]X = V_\emptyset^m$$

is the homogeneous system. The solution $X = V_\emptyset^n$ is called the trivial solution. A nontrivial solution is a solution $X = \{X_1, \dots, X_n\}$ such that $X_j \neq \emptyset$ for at least one j .

THEOREM 1. Let (2) be a system over U . A necessary and sufficient condition that (2) have a nontrivial solution is that $D(A^i) \neq U$ for at least one i . In this case, $\tilde{X} = \{\tilde{X}_1, \dots, \tilde{X}_n\}$, where $\tilde{X}_i = \complement D(A^i)$, $i = 1, \dots, n$, is a maximal solution. A necessary and sufficient condition that the vector $Y = \{Y_1, \dots, Y_n\}$ be a solution of (2) is that $Y_i \subset \tilde{X}_i$, $i = 1, \dots, n$.

Proof. If (2) has a nontrivial solution $X = \{X_1, \dots, X_n\}$, then $A_{ij} \cap X_j = \emptyset$ for all i and j . Thus $D(A^i) \cap X_i = \emptyset$ for all i . Since X is nontrivial, $X_k \neq \emptyset$ for some k , whence $D(A^k) \neq U$.

We must also have $X_i \subset \complement D(A^i)$ for all i ; this establishes \tilde{X} as a maximal solution. On the other hand, if $D(A^i) \neq U$ for some i , then $\tilde{X} \neq V_\emptyset^n$ from which it follows that \tilde{X} is a nontrivial solution

of (2). The last statement of the theorem is clear.

Constant system. The system,

$$(3) \quad (A_1, \dots, A_m)X = V_s^m,$$

where $S \neq \emptyset$, is called a constant system of linear set equations.

THEOREM 2. Let (3) be a system over U . A necessary and sufficient condition that (3) have a solution is that $S \subset D(A_i)$, $i = 1, \dots, m$.

Proof. If $S \subset D(A_i)$, $i = 1, \dots, m$, then $X = V_s^n$ is a solution. If a solution X exists, then $A_i X = S$, $i = 1, \dots, m$. This implies $D(A_i) \cap D(X) \supset S$ which yields $S \subset D(A_i)$ for all i .

THEOREM 3. If (3) has a solution, then $\bar{X} = \{\bar{X}_1, \dots, \bar{X}_n\}$, where $\bar{X}_i = S \cup \bigcap D(A^i)$, $i = 1, \dots, n$, is a maximal solution. Furthermore, if $W = \{W_1, \dots, W_n\}$, where $S \cap D(A^i) \subset W_i \subset \bar{X}_i$, $i = 1, \dots, n$, then W is also a solution.

Proof. Assume (3) has a solution. Now $\bar{X} = V_s^n + \tilde{X}$, so that by Theorems 1 and 2, we have $A\bar{X} = AV_s^n + A\tilde{X} = V_s^m + V_\emptyset^m = V_s^m$. Hence \bar{X} is a solution. If $AY = V_s^m$ where $Y = \{Y_1, \dots, Y_n\}$, then for all i and j , we have $A_{ij} \cap Y_j \subset S$. This implies $D(A^i) \cap Y_j \subset S$ for all j . Therefore $Y_j \subset S \cup \bigcap D(A^i)$. This proves the maximality of \bar{X} . The last statement of the theorem may be easily verified by the reader.

Inverses. If A is a matrix of order n over U , A is invertible if and only if there exists a matrix B over U such that $AB = BA = I_\nu^n$. In this case, B is called the inverse of A and we write $B = A^{-1}$.

If C is also over U and $CA = AC = I_\nu^n$, then $C = CI_\nu^n = C(AB) = (CA)B = I_\nu^n B = B$. Therefore inverses, when they exist, are unique. We now show that invertible set-theoretic matrices do indeed exist.

THEOREM 4. Let A be a matrix of order n over U . Then A is invertible if and only if

$$(i). \quad D(A^i) = D(A_i) = U, \quad i = 1, \dots, n.$$

(ii). Every line of A (both rows and columns) is composed of mutually disjoint sets.

In this case, $A^{-1} = A'$.

Proof. If (i) and (ii) hold, it is easily verified that $A'A = AA' = I_U^n$.

Suppose B is a matrix over U such that $AB = BA = I_U^n$. Then

$AB = I_U^n$ implies

$$(4) \quad A_i B^i = U, \quad i = 1, \dots, n.$$

$BA = I_U^n$ implies

$$(5) \quad \{B_i, \dots, B_{i-1}, B_{i+1}, \dots, B_n\} A^i = V_{\emptyset}^{n-1}, \\ i = 1, \dots, n.$$

$$(6) \quad B_i A^i = U,$$

By Theorem 2, (4) and (6) imply

$$(7) \quad D(A_i) = D(A^i) = D(B_i) = D(B^i) = U, \quad i = 1, \dots, n.$$

Theorem 1 and (5) imply

$$(8) \quad A_{ij} \subset C \left(\bigcup_{\substack{k=1 \\ k \neq j}}^n B_{kj} \right).$$

We assert that the rows of A are composed of mutually disjoint sets. Fix i ; if $j \neq p$, then by (8)

$$A_{ij} \cap A_{ip} \subset C \left(\bigcup_{\substack{k=1 \\ k \neq j}}^n B_{ki} \right) \cap C \left(\bigcup_{\substack{k=1 \\ k \neq p}}^n B_{ki} \right) \\ = C D(B^i) \\ = \emptyset.$$

$B'A' = I_U^n$ yields

$$(9) \quad \{(B^1)', \dots, (B^{i-1})', (B^{i+1})', \dots, (B^n)'\} A'_i = V_{\emptyset}^{n-1}, \\ i = 1, \dots, n. \text{ By Theorem 1, we have}$$

$$(10) \quad A_{ji} \subset C \left(\bigcup_{\substack{k=1 \\ k \neq j}}^n B_{ik} \right).$$

To show that the columns of A are composed of mutually disjoint sets, apply (10) in the same manner as we applied (8). Thus A satisfies conditions (i) and (ii) of the theorem. By the first part, A has an inverse A' . Since inverses are unique, $A' = B$.

COROLLARY. Let (1) be a system over U . If A is invertible, there is a unique solution given by $X = A'B$.

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SIXTEENTH BIENNIAL CONVENTION

April 7-8, 1967

The sixteenth biennial convention of Kappa Mu Epsilon will be held on the campus of Mount St. Scholastica College, Atchison, Kansas, on April 7-8, 1967. Students are urged to prepare papers to be considered for presentation at the convention. Papers must be submitted to Professor George R. Mach, National Vice-President, California State Polytechnic College, San Luis Obispo, California, before January 16, 1967. For complete directions with respect to the preparation of such papers, see pages 43-44 of this issue of *The Pentagon*.

I hope that every chapter will be well represented at the convention.

Fred W. Lott
National President

Generalizing the Law of Repeated Trials

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Many college algebra students know the Law of Repeated Trials. This law states that, if the probability of a contingent event is p , then the probability of r successes in n independent trials of the event is

$$\binom{n}{r} p^r [(1 - p)^{n-r}]. \quad (1)$$

Here $\binom{n}{r}$ is the number of combinations of n things taken r at a

time. This result may be stated in slightly different form as follows: Given n independent events each of which has a probability p , then the probability of r successes is given by (1). To generalize this last statement let the probability of the first independent event be p_1 , that of the second independent event be p_2 and so on to the n th independent event with the probability p_n . What then is the probability of r successes? This problem has a pleasing and elegant solution, which is not difficult. Consider first a special case where $n = 3$ and $r = 1$. Let the required probability be $P_{(1)}$. Then,

$$\begin{aligned} P_{(1)} &= p_1(1 - p_2)(1 - p_3) + p_2(1 - p_1)(1 - p_3) \\ &\quad + p_3(1 - p_1)(1 - p_2) \quad (2) \\ &= (p_1 + p_2 + p_3) - 2(p_1p_2 + p_1p_3 + p_2p_3) + 3p_1p_2p_3. \end{aligned}$$

Let S^1 mean $p_1 + p_2 + p_3$; S^2 mean $p_1p_2 + p_1p_3 + p_2p_3$ and S^3 mean $p_1p_2p_3$ then $P_{(1)} = S^1 - 2S^2 + 3S^3$.

To obtain a second solution of this problem, we may proceed as follows: It should be clear that the sum of the products in the right member of (2) can be expressed as $A_1S^1 + A_2S^2 + A_3S^3$, (3) where A_1 , A_2 and A_3 are constants. To determine these constants we return to the problem of repeated trials. In this problem all of the p 's are equal, say $p_1 = p_2 = p_3 = p$. Then (3) becomes the probability of exactly one success in three independent trials of a contingent event with the probability of success p in each trial. By

the Law of Repeated Trials this probability is $\binom{3}{1} p[(1 - p)^2]$ so that we have

$$\binom{3}{1} p [(1 - p)^2] = A_1 S^1 + A_2 S^2 + A_3 S^3 \quad (4)$$

In (4) S^1 has become

$$S^1 = p + p + p = 3p = \binom{3}{1} p. \quad (5)$$

In (4) S^2 has become

$$S^2 = pp + pp + pp = 3p^2 = \binom{3}{2} p^2. \quad (6)$$

In (4) S^3 has become

$$S^3 = ppp = p^3 = \binom{3}{3} p^3. \quad (7)$$

$\binom{3}{1} p [(1 - p)^2]$ can be expanded

$$\binom{3}{1} p [(1 - p)^2] = \binom{3}{1} p \left[\binom{2}{0} - \binom{2}{1} p + \binom{2}{2} p^2 \right]. \quad (8)$$

Substituting the results in (5), (6), (7) and (8) in (4), we have

$$\begin{aligned} \binom{3}{1} p \left[\binom{2}{0} - \binom{2}{1} p + \binom{2}{2} p^2 \right] \\ = A_1 \binom{3}{1} p + A_2 \binom{3}{2} p^2 + A_3 \binom{3}{3} p^3. \quad (9) \end{aligned}$$

Then (9) is an identity in p , and the coefficients of like powers of p in the two members must be equal so that we obtain

$$\binom{3}{1} \binom{2}{0} = A_1 \binom{3}{1}, \text{ and } A_1 = \binom{3}{1} \binom{2}{0} \div \binom{3}{1} = 1.$$

$$- \binom{3}{1} \binom{2}{1} = A_2 \binom{3}{2}, \text{ and } A_2 = - \binom{3}{1} \binom{2}{1} \div \binom{3}{2} = -2.$$

$$\binom{3}{1} \binom{2}{2} = A_3 \binom{3}{3}, \text{ and } A_3 = \binom{3}{1} \binom{2}{2} \div \binom{3}{3} = 3.$$

Now consider the binomial expansion of $S^1(1 + S)^{-2}$ with $S^k = 0$ if $k > 3$, 3 being the number of contingent events in the problem. Using the Binomial Theorem for negative exponents,

$$S^1(1 + S)^{-2} = S^1(1 - 2S + \frac{(-2)(-3)}{2!}S^2) \\ = S^1 - 2S^2 + 3S^3.$$

The value of $P_{\{1\}}$ may, therefore, be expressed symbolically as

$$P_{\{1\}} = S^1(1 + S)^{-2}, \text{ where } S^K = 0 \text{ if } K > 3.$$

Considering the foregoing we might suspect that given n independent, contingent events with the probabilities, p_i , $i = 1, 2, 3, \dots, n$; then the probability $P_{\{r\}}$ of r successes would be given symbolically by

$$P_{\{r\}} = S^r(1 + S)^{-r-1}, \quad (10)$$

where $S^K = 0$ if $K > n$, n being the number of contingent events. Equation (10) may be established by the procedure used in the second solution of our special case with $n = 3$, and $r = 1$. In the general case above, (2) becomes

$$P_{\{r\}} = \Sigma p_a p_b p_c \dots p_m (1 - p_a)(1 - p_b) \dots (1 - p_m). \quad (11)$$

Here in each product there are r factors p_i and $n - r$ factors $(1 - p_j)$, and the sum is formed by adding all the products obtained by choosing the r factors p in every possible way from the given p 's. Then as before $P_{\{r\}}$ may also be expressed as $A_r S^r + A_{r+1} S^{r+1} + \dots + A_n S^n$. Here the S 's are defined as before; that is, S^K means the sum of all possible products of K factors p chosen from the given p 's. Again let all the p 's be equal, and equate the two resulting expressions for $P_{\{r\}}$. We have

$$\binom{n}{r} p^r [1 - p]^{n-r} = \binom{n}{r} p^r \left[\binom{n-r}{0} - \binom{n-r}{1} p + \binom{n-r}{2} p^2 + \dots \right. \\ \left. + (-1)^{n-r} \binom{n-r}{n-r} p^{n-r} \right] = A_r \binom{n}{r} p^r + A_{r+1} \binom{n}{r+1} p^{r+1} \\ + A_{r+2} \binom{n}{r+2} p^{r+2} + \dots + A_n \binom{n}{n} p^n. \quad (12)$$

Equating the coefficients of like powers of p in the last two members of (12), we have

$$\binom{n}{r} \binom{n-r}{0} = A_r \binom{n}{r}, \text{ and } A_r = 1.$$

$$-\binom{n}{r} \binom{n-r}{1} = A_{r+1} \binom{n}{r+1}, \text{ and}$$

$$A_{r+1} = -\binom{n}{r} \binom{n-r}{1} \div \binom{n}{r+1} = -\frac{(r+1)!}{1!r!}.$$

$$\binom{n}{r} \binom{n-r}{2} = A_{r+2} \binom{n}{r+2}, \text{ and}$$

$$A_{r+2} = \binom{n}{r} \binom{n-r}{2} \div \binom{n}{r+2} = \frac{(r+2)!}{2!r!}.$$

• • • • •

$$(-1)^{n-r} \binom{n}{r} \binom{n-r}{n-r} = A_n \binom{n}{n}, \text{ and}$$

$$A_n = (-1)^{n-r} \binom{n}{r} \binom{n-r}{n-r} \div \binom{n}{n} = (-1)^{n-r} \frac{n!}{(n-r)!r!}.$$

Hence $P_{\{r\}} = S^r - \frac{(r+1)!}{1!r!} S^{r+1}$

$$+ \frac{(r+2)!}{2!r!} S^{r+2} + \dots + (-1)^{n-r} \frac{n!}{(n-r)!r!} S^n. \quad (13)$$

But the right member of (13) is the first $(n-r+1)$ terms of the binomial expansion of $S^r(1+S)^{-r-1}$. Hence we see the validity of (10).

A further pleasing result is obtained by finding the probability P_r of at least r successes. This is found by summing the probabilities of r successes plus $(r+1)$ successes and so on up to n successes. Thus we have symbolically

$$P_r = S^r(1+S)^{-r-1} + S^{r+1}(1+S)^{-r-2} + \dots + S^n(1+S)^{-n-1}.$$

From probability theory we have

$$P_{r+1} = P_r - P_{\{r\}}$$

Letting $r = 0, 1, 2, \dots :$

$$P_1 = P_0 - P_{\{0\}} = 1 - \frac{S^0}{1+S} = \frac{S}{1+S},$$

$$P_2 = P_1 - P_{11} = \frac{S}{1+S} - \frac{S}{(1+S)^2} = \frac{S^2}{(1+S)^2},$$

$$P_3 = P_2 - P_{121} = \frac{S^2}{(1+S)^2} - \frac{S^2}{(1+S)^3} = \frac{S^3}{(1+S)^3}.$$

In general

$$P_r = \frac{S^r}{(1+S)^r} = S^r(1+S)^{-r},$$

where, of course, it is understood that $S^k = 0$ if $k > n$.



Mathematics is a science. It is the most exact, the most elegant, and the most advanced of the sciences and therefore it has been called the *Queen of Sciences*. Nothing, not even the modern miracles of applied science and technology, gives a better idea of the apparently unlimited capacity of the human mind than higher mathematics.

—H. M. DADOURIAN

The Degeneration of Sequences of Integers by Division*

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1. **Introduction.** If we divide 11 by 2, 3, or 4 we get remainders of 1, 2, and 3, respectively. Similarly, if we divide 59 by 2, 3, 4, or 5 we get the respective remainders of 1, 2, 3, and 4. However, if we divide 59 by 6 also, the next integer in the first sequence, the remainder is 5, the next integer in the second sequence.

In this paper we propose to show how to construct the numbers N_q recursively which when divided by 2, 3, \dots , q give remainders of 1, 2, \dots , $q - 1$, respectively. That is if N_{q-1} is known we shall show how N_q can be determined in all cases.

Problems of a similar nature have been studied for centuries. Sun-Tsu in a Chinese arithmetic, about the first century (see [1]), found all positive integers which have remainders 2, 3, 4 when divided by 3, 5, 7, respectively. In the seventh century Brahmagupta (see [1]) proposed the problem of finding an integer having remainders 5, 4, 3, 2 when divided by 6, 5, 4, 3, respectively; and in the thirteenth century Leonardo Pisano (see [1]) treated the problem of finding an integer which gives the remainders 1, 2, \dots , 9 when divided by 2, 3, \dots , 10, respectively.

The first problem can be dealt with in terms of what is normally called the Chinese remainder theorem (see [2] and [3]), since the given integral divisors in this case are all relatively prime in pairs. In the second and third problems this is not the case and to our knowledge the standard approach has been to deal with each individual problem independently. We propose to develop a system by which this latter class of problems can concisely be dealt with in general.

2. **The Degenerator of a Sequence.** In this section, after

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defining the notation, we present two preliminary lemmas and the main theorem of the paper. In the theorem we consider the three possible cases that occur.

DEFINITION 2.1. For the sequence of positive integers $2, 3, \dots, q$ we define the "degenerator of the sequence," denoted by N_q , to be the smallest positive integer such that

$$(2.1) \quad N_q = (m - 1) \pmod{m}, \quad (m = 2, 3, \dots, q).$$

Notice that the degenerator is defined to be the *least* positive integer satisfying (2.1); it is obvious that the number $(2 \cdot 3 \cdot \dots \cdot q) - 1$ would provide the necessary remainders, but such a number may not be the least such number. However, whether $(2 \cdot 3 \cdot \dots \cdot q) - 1$ is the least or not, the fact that it does provide the desired remainders guarantees the existence of a smallest number due to the well-ordered property of the natural numbers.

THEOREM 2.1. For every sequence of positive integers $2, 3, \dots, q$, the degenerator N_q can be determined whenever N_{q-1} is known and

- (a) $N_q = q(N_{q-1} + 1) - 1$, if q is prime.
- (b) $N_q = p(N_{q-1} + 1) - 1$, if there exists a prime p and a positive integer $t > 1$ such that $q = p^t$.
- (c) $N_q = N_{q-1}$, if q is composite but not equal to a prime raised to an integral power.

The proof of this theorem is straightforward, but first we must establish the following preliminary lemmas.

LEMMA 2.1. For the degenerator N_q of every sequence of positive integers $2, 3, \dots, q$,

$$(2.2) \quad x \mid N_q + 1, \quad (x = 2, 3, \dots, q);$$

moreover, N_q is the smallest positive integer for which (2.2) is true.

Proof. For every given sequence of positive integers $2, 3, \dots, q$ we know that the degenerator N_q exists. By Definition 2.1 we also know that there exist non-negative integers $\gamma_2, \gamma_3, \dots, \gamma_q$ such that

$$N_q = 2\gamma_2 + 1 = 3\gamma_3 + 2 = \dots = q\gamma_q + (q - 1),$$

or

$$N_q + 1 = 2(y_2 + 1) = 3(y_3 + 1) = \cdots = q(y_q + 1),$$

from which it is evident that

$$x \mid N_q + 1, \quad (x = 2, 3, \cdots, q).$$

To see that N_q is the smallest positive integer for which this is true, we assume that there exists a positive integer t smaller than N_q for which

$$x \mid t + 1, \quad (x = 2, 3, \cdots, q).$$

Therefore, there exist positive integers z_2, z_3, \cdots, z_q such that

$$t + 1 = 2z_2 = 3z_3 = \cdots = qz_q,$$

or

$$(2.3) \quad t = 2z_2 - 1 = 3z_3 - 1 = \cdots = qz_q - 1.$$

But from (2.3) we have that

$$t = (m - 1) \pmod{m}, \quad (m = 2, 3, \cdots, q),$$

which contradicts Definition 2.1 since $t < N_q$. Therefore, $t \geq N_q$ and the proof is completed.

LEMMA 2.2. For every prime p and every degenerator N_q , $p \mid N_q + 1$ if and only if $p \leq q$.

Proof. Suppose that there exists a prime $p > q$ such that $p \mid N_q + 1$. Therefore, there exists an integer j such that $N_q + 1 = pj$. By Lemma 2.1 we have that

$$(2.4) \quad x \mid pj, \quad (x = 2, 3, \cdots, q).$$

Since p is prime and $p > q$, (2.4) implies that

$$x \mid j, \quad (x = 2, 3, \cdots, q).$$

But this contradicts Lemma 2.1 since $j < N_q + 1$. Therefore, $p \leq q$.

The converse is immediate, for if $p \leq q$ then by Lemma 2.1 $p \mid N_q + 1$.

Now that we have established the results to prove Theorem 2.1 we can proceed with its proof.

Proof. As was mentioned previously, the proof is completed by considering three cases, the case depending upon the value of the subscript q for N_q . Since the theorem rests upon the assumption

that N_{q-1} 's value is obtainable, we will assume that it is known throughout the proof.

Case 1. We consider first the case when q is prime. By Lemma 2.2 we have that $q \nmid N_{q-1} + 1$. Lemma 2.1 states that N_q is the smallest integer for which

$$(2.5) \quad x \mid N_q + 1, \quad (x = 2, 3, \dots, q).$$

Lemma 2.1 also gives us that N_{q-1} is the smallest integer for which

$$(2.6) \quad x \mid N_{q-1} + 1, \quad (x = 2, 3, \dots, q - 1)$$

It is evident, though, that $q(N_{q-1} + 1)$ does satisfy (2.5). That $q(N_{q-1} + 1)$ is the smallest positive integer to satisfy (2.5) follows from the fact that N_{q-1} is the smallest positive integer to satisfy (2.6) and the fact that q is prime.

Therefore, by Lemma 2.1 when q is prime

$$N_q = q(N_{q-1} + 1) - 1$$

or

$$N_q = qN_{q-1} + (q - 1).$$

Case 2. We now consider the case when q is a composite number for which there exists a prime p and a positive integer $t > 1$ such that $p^t = q$. Again, as in case 1,

$$(2.7) \quad q \nmid N_{q-1} + 1.$$

To see this, assume that (2.7) is false. Then, since $q = p^t$,

$$p^t \mid N_{q-1} + 1.$$

Therefore, there exists an integer k such that

$$(2.8) \quad N_{q-1} + 1 = kp^t.$$

By Lemma 2.1 we have that

$$(2.9) \quad x \mid N_{q-1} + 1, \quad (x = 2, 3, \dots, q - 1).$$

Because p is prime we can conclude from (2.8) and (2.9) that

$$x \mid kp^{t-1}, \quad (x = 2, 3, \dots, q - 1),$$

which contradicts Lemma 2.1 since $kp^{t-1} < N_{q-1} + 1$. Therefore,

$$(2.10) \quad q \nmid N_{q-1} + 1 \text{ or } p^t \nmid N_{q-1} + 1.$$

Observing that $p^{t-1} < p^t = q$ we have by Lemma 2.1 that

$$(2.11) \quad p^{t-1} \mid N_{q-1} + 1.$$

Now, we also know from Lemma 2.1 that $N_{q-1} + 1$ is the smallest positive integer such that

$$(2.12) \quad x \mid N_{q-1} + 1, \quad (x = 2, 3, \dots, q - 1).$$

Therefore, by (2.10), (2.11), and (2.12) we have that $p(N_{q-1} + 1)$ is the smallest integer for which

$$x \mid p(N_{q-1} + 1), \quad (x = 2, 3, \dots, q = p^t).$$

From this fact and Lemma 2.1 we see that when q is composite and equal to a prime p raised to an integral power then

$$N_q = p(N_{q-1} + 1) - 1 = pN_{q-1} + (p - 1).$$

Case 3. Lastly, we consider the remaining case, when q is composite but not equal to any prime raised to an integral power. If we express q as a product of primes as follows:

$$q = p_1 p_2 \cdots p_n,$$

we obtain a set of n prime numbers which we designate as the set P . We know that there exist at least two primes p_i and p_j in P such that $p_i \neq p_j$. Now, take the set P and divide it into two disjoint subsets P_1 and P_2 , $P_1 = \{p_x: p_x \in P; p_x = p_i; i, x \text{ integers}\}$ and $P_2 = \{p_x: p_x \in P; p_x \neq p_i; i, x \text{ integers}\}$. Let t_1 be the integer produced by taking the product of all the elements in P_1 and t_2 the integer formed by repeating this process for the elements in P_2 .

Since both t_1 and t_2 are less than q we have by Lemma 2.1 that

$$(2.13) \quad t_1 \mid N_{q-1} + 1$$

and

$$(2.14) \quad t_2 \mid N_{q-1} + 1.$$

Because t_1 and t_2 are relatively prime, (2.13) and (2.14) imply that

$$t_1 t_2 \mid N_{q-1} + 1.$$

But also, since $t_1 t_2 = q$, we have that

$$q \mid N_{q-1} + 1.$$

Therefore, when q is composite, but not equal to any prime raised to an integral power,

$$N_q = N_{q-1}.$$

This completes the proof of Theorem 2.1 for we have proved that N_q , in each of three cases, can be expressed in terms of N_{q-1} .

3. Application and Discussion. As was pointed out in the introduction of this paper, we proposed to develop a concise means of determining the degenerator of a sequence of integers 2, 3, \dots , q . It is evident that Theorem 2.1 allows us to reach this end but it may not be obvious that this approach is concise due to the recursive nature of the theorem. However, as can be seen by the following example, even if the previous degenerator N_{q-1} is unknown, N_q can quickly be determined.

Example. What is the smallest positive integer which will have remainders 1, 2, \dots , 13 when divided by 2, 3, \dots , 14, respectively?

Recalling that the degenerator N_2 is equal to 1 and assuming that this is the only degenerator known at present we determine N_{14} by applying Theorem 2.1. Observe that 2, 3, 5, 7, 11, 13 are primes and that $4 = 2^2$, $8 = 2^3$, $9 = 3^2$ are composites equal to primes raised to integral powers and finally that 6, 10, 12 are composites which do not fall into one of the above classifications. Therefore, applying our theorem we have that

$$N_{14} = 2 \cdot 3 \cdot 2 \cdot 5 \cdot 7 \cdot 2 \cdot 3 \cdot 11 \cdot 13 - 1$$

or

$$N_{14} = 360359.$$

The only real limitation we have in applying this method is our knowledge of primes and our ability to recognize composites which are equal to primes raised to integral powers. However, with the tables that are available today this limitation need rarely be of concern.

Throughout this paper we have been concerned with the *smallest* positive integer which has the characteristics of the degenerator. It should be pointed out that although this has been our primary concern, Theorem 2.1 does enable us to determine every positive integer in this class. In particular, we have that each integer

$$(N_q + 1)k - 1, \quad (k = 1, 2, \dots)$$

has remainders of $1, 2, \dots, q - 1$ when divided by $2, 3, \dots, q$, respectively. Moreover, every integer with this property is of the above form.

Approaching Theorem 2.1 in another light we see that we can also employ it to obtain integers having remainders of $2, 3, \dots$ less than the divisors, instead of merely one less as developed in this paper. Similarly, Theorem 2.1 enables us to obtain the smallest integer divisible by $2, 3, \dots, q$. In fact, this paper could have easily been developed along any one of the above lines, especially the latter.

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One may be a mathematician of the first rank without being able to compute. It is possible to be a great computer without having the slightest idea of mathematics.

—NOVALIS

The Mathematical Scrapbook

EDITED BY GEORGE R. MACH

Readers are encouraged to submit Scrapbook material to the editor. Material will be used where possible and acknowledgment will be made in *The Pentagon*

= Δ =

Dice are always constructed so that the spots on opposite faces total seven. Even with this restriction a die can be made in more than one essentially different way. With no restrictions at all, in how many essentially different ways can the faces of a cube be numbered with the digits 1 to 6? You might be surprised that there are 30 ways. In how many essentially different ways can the faces of a regular tetrahedron be numbered with the digits 1 to 4? It is even more surprising that there are only 2 ways.

A general formula for the number of ways (W) of numbering all regular polyhedra is given by

$$W = \frac{F!}{(F)(e)},$$

where F is the number of faces and e is the number of edges of each face. You can see the development of this formula by considering a cube. At first glance there might seem to be $(6)(5)(4)(3)(2)(1) = 720$ ways. But since any one of the faces might be numbered first (placed down), we must divide by 6. Then for each of these possibilities we may view the cube from four sides since the base has four edges and hence we must divide by 4 to get the essentially different ways.

= Δ =

Pythagorean triangles, like our familiar 3-4-5 and 5-12-13 ones, have been of interest for many years. The early Greeks knew the following formulas to yield integer sides, A - B - C , of a right triangle:

$$A = m^2 - n^2$$

$$B = 2mn$$

$$C = m^2 + n^2,$$

where m and n are arbitrary integers, called generators. If m and n

are consecutive integers, the sides B and C are always consecutive integers. Can you prove this?

If m and n are unlike parity (odd-even) and with no common factor, then A , B , and C will have no common factors and constitute what is called a primitive Pythagorean triangle. In every primitive Pythagorean triangle, one of the sides is divisible by 3 and one by 5. The product AB is divisible by 12 and the product ABC is divisible by 60. Can you prove these properties?

$$= \Delta =$$

An interesting and very useful extension of the technique of integration by parts can frequently be effected by inserting a wisely chosen arbitrary constant in the proper place. As a review, consider the usual

$$\int u \, dv = uv - \int v \, du .$$

In calculating v from dv , we always omit adding an arbitrary constant to v , preferring to have one grand constant of integration at the end. Adding an arbitrary constant to the v is permissible but it's usually not done.

Consider this example:

$$I = \int (x) \arctan x \, dx$$

$$u = \arctan x \qquad dv = x \, dx$$

$$du = \frac{1}{1 + x^2} \, dx \qquad v = \frac{x^2}{2}$$

$$I = \frac{x^2}{2} \arctan x - \int \frac{x^2}{2} \frac{dx}{1 + x^2}$$

This latter integral can be solved but it requires long division first. Wouldn't it be nice if that x^2 in the numerator were $(1 + x^2)$? By adding an arbitrary constant of $\frac{1}{2}$ to v the numerator will be $(1 + x^2)$.

Preferred Solution:

$$u = \arctan x \qquad dv = x \, dx$$

$$du = \frac{1}{1 + x^2} \, dx \qquad v = \frac{x^2}{2} + \frac{1}{2} = \frac{x^2 + 1}{2}$$

$$\begin{aligned}
 I &= \frac{(x^2 + 1)}{2} \arctan x - \int \frac{dx}{2} \\
 &= \frac{(x^2 + 1)}{2} \arctan x - \frac{x}{2} + C.
 \end{aligned}$$

It can easily be verified that both methods yield the same result. Here's another example:

$$\begin{aligned}
 I &= \int \frac{\ln x}{(x + 1)^2} dx \\
 u &= \ln x & dv &= \frac{dx}{(x + 1)^2} \\
 du &= \frac{1}{x} dx & v &= \frac{-1}{x + 1} + 1 = \frac{x}{x + 1} \\
 I &= \frac{x \ln x}{x + 1} - \int \frac{dx}{x + 1} \\
 &= \frac{x \ln x}{x + 1} - \ln(x + 1) + C.
 \end{aligned}$$

See how we avoided integration by partial fractions $\left(\int \frac{-dx}{x(x + 1)} \right)$

by adding 1 to v and creating an x in the numerator of v ?

$$= \Delta =$$

Editor's note: The following was submitted by R. S. Luthar, Waterville, Maine.

A specialized solution of Fermat's last theorem,

$$x^n + y^n = z^n,$$

is found if we consider the set $S = \{0, 1, 2, 3, 4, 5\}$ with multiplication and addition defined on it as ordinary multiplication and addition reduced modulo 6. It can now be verified that

$$3^n + 1^n = 4^n, \quad n \in S \text{ and } n \neq 0.$$

Directions for Papers to Be Presented at the Sixteenth Biennial Kappa Mu Epsilon Convention

ATCHISON, KANSAS
April 7-8, 1967

A significant feature of this convention will be the presentation of papers by student members of KME. The topic on mathematics which the student selects should be in his area of interest and of such a scope that he can give it adequate treatment within the time allotted. By this time the preparation of his paper should be well underway, and he should take advantage of all opportunities available to present his paper before groups interested in mathematics.

Who may submit papers: Any member may submit a paper for use on the convention program. Papers may be submitted by graduates and undergraduates; however, undergraduates will not compete against graduates. Awards will be granted for the best papers presented by undergraduates. Special awards may be given for the best papers presented by graduates, if a sufficient number are presented.

Subject: The material should be within the scope of the understanding of undergraduates, preferably the undergraduate who has completed differential and integral calculus. The Selection Committee will naturally favor papers that are within this limitation and which can be presented with reasonable completeness within the time limit prescribed.

Time Limit: The usual time limit is twenty minutes but this may be changed on the recommendation of the Selection Committee.

Paper: The paper to be presented together with a description of charts, models or other visual aids that are to be used in the presentation of the paper should be submitted to the Selection Committee. A carbon copy of the complete paper may be submitted, or in lieu of the complete paper an outline (sufficient in detail to give the committee a clear idea of the content, methods, and scope of the paper) may be submitted before the January 16th deadline to be followed by the complete paper

before February 15, 1967. A bibliography of source materials together with the statement that the author of the paper is a member of KME and his official classification in school, undergraduate or graduate, should accompany his paper.

Date and Place Due: The papers must be submitted no later than January 16, 1967, to the office of the National Vice-President.

Selection: The Selection Committee will choose about ten to twelve papers for presentation at the convention. All other papers will be listed by title and student's name on the convention program. The authors of the papers selected for presentation will be notified as soon as possible after the selection is made.

Prizes:

1. The author of each paper presented will be given a two-year extension of his subscription of *The Pentagon*.
2. Authors of the two or three best papers presented by undergraduates, according to the judgment of a committee composed of faculty and students will be awarded copies of suitable mathematics books.
3. If a sufficient number of papers submitted by graduate students are chosen for presentation, then one or more similar prizes will be awarded for the best paper or papers from this group.

George R. Mach
National Vice-President
California State Polytechnic College
San Luis Obispo, California 93402



Mathematics in its widest signification is the development of all types of formal, necessary, deductive reasoning.

—A. N. WHITEHEAD

Installation of New Chapter

EDITED BY SISTER HELEN SULLIVAN
NEW YORK ZETA CHAPTER
Colgate University. Hamilton, New York

New York Zeta Chapter was installed on May 16, 1966, by Professor Emmet C. Stopher of State University College, Oswego. The installation was held at Merrill House, followed by dinner and a meeting at which Professor Stopher gave an informal talk on Kappa Mu Epsilon.

Charter members are Edwin Downie, Barry Fernbach, Stephen Garypic, Brian Gerber, Ira Haspel, Edward Macias, William Mastrocola, Jay Menitove, Carl Munshower, Donald Oakleaf, Malcolm Pownall, James Reynolds, Munir Saltoun, George Schwartz, and James Wardwell.

The officers of the chapter are:

President ----- Jay Menitove
Vice-President ----- James Reynolds
Secretary ----- Ira Haspel
Treasurer ----- Ira Haspel
Faculty Sponsor ----- James Wardwell
Corresponding Secretary ----- Malcolm Pownall

Guests at the installation ceremony and dinner included President and Mrs. Vincent Barnett, Dean James A. Storing, and other interested faculty and students.



No process of sound reasoning can establish a result not contained in the premises.

—J. W. MELLOR

The Problem Corner

EDITED BY H. HOWARD FRISINGER

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before March 1, 1967. The best solutions submitted by students will be published in the Spring, 1967, issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor H. Howard Frisinger, Department of Mathematics and Statistics, Colorado State University, Fort Collins, Colorado.

PROPOSED PROBLEMS

196. *Proposed by William K. Sjoquist, University of California at Berkeley, Berkeley, California.*

If $y = uv$ where u and v are functions of x , prove that the n th derivative of y with respect to x is given by

$$y^{(n)} = uv^{(n)} + nu'v^{(n-1)} + n(n-1)u''v^{(n-2)}/2! + n(n-1)(n-2)u'''v^{(n-3)}/3! + \dots + u^{(n)}v$$

197. *Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.*

Let us denote a set of sequences $\{X_{m,n}\}$ by

$$\{X_{1,n}\} = (1, 1, 2, 3, 5, \dots) \text{ where } X_{1,1} = 1$$

$$X_{1,2} = 1$$

$$X_{1,3} = 2$$

•

•

•

$$X_{1,k} = X_{1,k-1} + X_{1,k-2}$$

$$\{X_{2,n}\} = (1, 3, 4, 7, 11, \dots) \text{ where } X_{2,1} = 1$$

$$X_{2,2} = 3$$

$$X_{2,3} = 4$$

•

•

•

$$X_{2,k} = X_{2,k-1} + X_{2,k-2}$$

$\{X_{3,n}\} = (1, 4, 5, 9, 14, \dots)$ for similar definitions of $X_{3,n}$

•
•
•

$\{X_{1,n}\} = (1, i + 1, i + 2, 2i + 3, \dots)$ for $i > 1$, and where the X 's are defined by the same recurrence relation as before.

Now express the n th term of the i th sequence in terms of the n th term of the first sequence, where the first sequence is actually the Fibonacci sequence.

198. *Proposed by the Editor.*

For what values of n is $(11 \times 14^n) + 1$ prime?

199. *Proposed by R. S. Luthar, Colby College, Waterville, Maine.*

Let ΔABC be a right triangle with right angle at A . Construct regular n -gons on AB , AC , and BC with respective areas α , β , γ . Prove $\alpha + \beta = \gamma$.

200. *Proposed by E. R. Deal, Colorado State University, Fort Collins, Colorado.*

"Are those your children I hear playing in the garden?" asked the visitor.

"There are really four families of children," replied the host. "Mine is the largest, my brother's family is smaller, my sister's is smaller still, and my cousin's is the smallest of all. They are playing drop the handkerchief," he went on. "They prefer baseball but there are not enough children to make two teams." "Curiously enough," he mused, "the product of the members in the four groups is my house number, which you saw when you came in."

How many children were there in each of the four families?

SOLUTIONS

191. *Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.*

Express the square root of any positive integer as an infinite continued fraction.

Solution by William Mikesell, Indiana University, Indiana, Pennsylvania.

Take any positive integer N . Let $N = r^2 + s$, where r is the

largest perfect square contained in N without exceeding N .

Then: $\sqrt{N} = r + \sqrt{N} - r$

Multiply $(\sqrt{N} - r)$ by $\frac{\sqrt{N} + r}{\sqrt{N} + r} = 1$.

$$r + (\sqrt{N} - r) \frac{\sqrt{N} + r}{\sqrt{N} + r} = r + \frac{N - r^2}{\sqrt{N} + r}$$

Set $\sqrt{N} + r = 2r + (\sqrt{N} - r)$ and again multiply $\sqrt{N} - r$ by

$$\frac{\sqrt{N} + r}{\sqrt{N} + r}.$$

Doing this and setting $N - r^2 = s$ we get $r + \frac{s}{2r + \frac{s}{\sqrt{N} + r}}$

We may set $(\sqrt{N} + r) = 2r + \sqrt{N} - r$ again and multiply $\sqrt{N} - r$ by $\frac{\sqrt{N} + r}{\sqrt{N} + r}$. This process can be continued as often as the solver pleases.

Also solved by Layne Watson, Evansville College, Evansville, Indiana.

192. Proposed by LeRoy Simmons, Washburn University, Topeka, Kansas.

Find a positive integer X such that $ax + b(x + 1)$ will be equal to all integers greater than or equal to 110, but will not equal 109; where $a, b \in \{0, 1, 2, 3, \dots\}$.

Solution by Patricia Robaugh, Duquesne University, Pittsburgh, Pennsylvania.

Observation: $X = 11$ is solution.

Where $a + b = 9$, the following occurs:

$$\begin{aligned} 9(11) + 0(11 + 1) &= 99 \\ 8(11) + 1(11 + 1) &= 100 \\ 7(11) + 2(11 + 1) &= 101 \\ 6(11) + 3(11 + 1) &= 102 \\ 5(11) + 4(11 + 1) &= 103 \\ 4(11) + 5(11 + 1) &= 104 \\ 3(11) + 6(11 + 1) &= 105 \\ 2(11) + 7(11 + 1) &= 106 \end{aligned}$$

$$1(11) + 8(11 + 1) = 107$$

$$0(11) + 9(11 + 1) = 108$$

Where $a + b = 10$, the following occurs:

$$10(11) + 0(11 + 1) = 110$$

$$9(11) + 1(11 + 1) = 111$$

$$8(11) + 2(11 + 1) = 112$$

$$7(11) + 3(11 + 1) = 113$$

$$6(11) + 4(11 + 1) = 114$$

$$5(11) + 5(11 + 1) = 115$$

$$4(11) + 6(11 + 1) = 116$$

$$3(11) + 7(11 + 1) = 117$$

$$2(11) + 8(11 + 1) = 118$$

$$1(11) + 9(11 + 1) = 119$$

$$0(11) + 10(11 + 1) = 120$$

Where $a + b = 11$, the following occurs:

$$11(11) + 0(11 + 1) = 121$$

$$10(11) + 1(11 + 1) = 122$$

$$9(11) + 2(11 + 1) = 122$$

$$8(11) + 3(11 + 1) = 124$$

$$7(11) + 4(11 + 1) = 125$$

$$6(11) + 5(11 + 1) = 126$$

$$5(11) + 6(11 + 1) = 127$$

$$4(11) + 7(11 + 1) = 128$$

$$3(11) + 8(11 + 1) = 129$$

$$2(11) + 9(11 + 1) = 130$$

$$1(11) + 10(11 + 1) = 131$$

$$0(11) + 11(11 + 1) = 132$$

Integers through infinity can be found in this manner but 109 cannot be represented.

Also solved by Layne Watson, Evansville College, Evansville, Indiana.

193. *Proposed by Patricia Robaugh, Duquesne University, Pittsburgh, Pennsylvania.*

Prove that the Egyptian method of multiplication gives correct results in all cases.

Solution by Wilma Yates, Stetson University, Leland, Florida.

Multiply $x \cdot y$, where $x = 2^k + 2^l + \dots + 2^n$ and y is also

an integer. Multiplication is performed by defining x as above in one column and then choosing numbers in a second column corresponding to each addend. When the numbers chosen from the second column are added together, they should give the proper product as follows.

$$\begin{array}{r}
 2^0 = 1 \quad y \\
 2^1 = 2 \quad 2y \\
 2^2 = 4 \quad 4y \\
 \cdot \quad \cdot \\
 \cdot \quad \cdot \\
 \cdot \quad \cdot \\
 \cdot \quad \cdot
 \end{array}$$

$$\begin{aligned}
 \text{Thus the product becomes } xy &= 2^k y + 2^l y + \cdots + 2^n y \\
 &= (2^k + 2^l + \cdots + 2^n) y \\
 &= xy.
 \end{aligned}$$

Also solved by Ivan F. Arnold, Southwest Missouri State College, Springfield, Missouri; Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio; Philip Haverstick, William Jewell College, Liberty, Missouri; Tom S. Johnson, Drake University, Des Moines, Iowa; William Mikesell, Indiana University, Indiana, Pennsylvania; Gail E. Norfolk, California, State Polytechnic College, San Luis Obispo, California; Don Scarpero, University of Missouri at Rolla, Rolla, Missouri; William K. Sjoquist, University of California at Berkeley, Berkeley, California; Layne Watson, Evansville College, Evansville, Indiana.

194. *Proposed by E. R. Deal, Colorado State University, Fort Collins, Colorado.*

Fill in the missing digits

$$\begin{array}{r}
 8 \\
 \hline
 xxx \overline{) xxxxxx} \\
 xxx 5 \\
 \hline
 xxx x \\
 9 xx \\
 \hline
 x x x \\
 x x x \\
 \hline
 x x x
 \end{array}$$

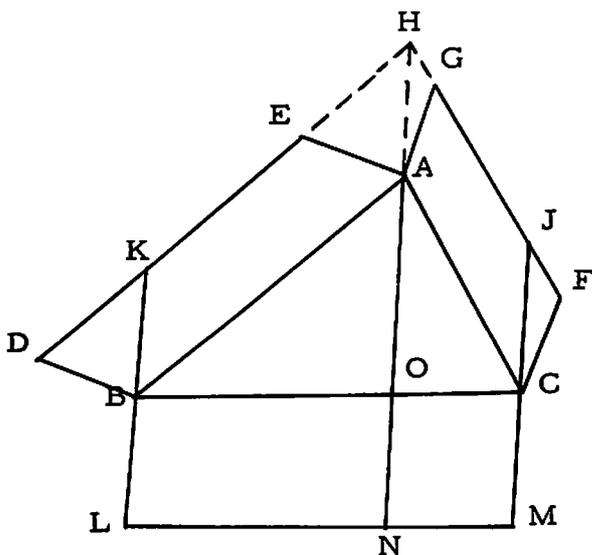
Solution by Larry McFarling, Anderson College, Anderson, Indiana.

$$\begin{array}{r}
 988 \\
 115 \overline{) 113620} \\
 \underline{1035} \\
 1012 \\
 \underline{920} \\
 920
 \end{array}$$

Also solved by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio; Charles Loeffler, State University College, Oswego, New York; Patricia Robaugh, Duquesne University, Pittsburgh, Penn; Wilma L. Yates, Stetson University, DeLand, Florida; William K. Sjoquist, University of California at Berkeley, Berkeley, California.

195. *Proposed by the Editor.*

Given any $\triangle ABC$, any two parallelograms $DBAE$ on AB and $ACFG$ on AC . Let DE and FG meet in H and draw BL and CM equal and parallel to AH . Prove area $ABDE + \text{area } ACFG = \text{area } BCML$.



Solution by Layne Watson, Evansville College, Evansville, Indiana.

- | | |
|--|--|
| 1. Extend BL to meet DE at K , AH to meet LM at N , and CM to meet FG at J . | Through two points there can be one, and only one, straight line. |
| 2. AH is parallel and equal to CM and BL . | Given |
| 3. BL is parallel and equal to CM . | Two lines parallel and equal to a third line are parallel and equal to each other. |
| 4. $BCML$ is a parallelogram. | If two sides of a quadrilateral are equal and parallel, it is a parallelogram. |
| 5. BC is parallel LM , ON is parallel and equal to BL and CM . | Opposite sides of a parallelogram are parallel and equal. Given. Segments of parallels included between parallels are equal. |
| 6. $OCMN$ and $BONL$ are parallelograms. | Reason 4. |
| 7. $ACFG$ and $ABDE$ are parallelograms. | Given. |
| 8. AC parallel FG , AH parallel CJ , AB parallel DE , and BK parallel AH . | Definition of a parallelogram. Given. |
| 9. $ACJH$ and $ABKH$ are parallelograms. | Statement 8, definition of parallelogram. |
| 10. Area $ACFG =$ area $ACJH$ and area $ABDE =$ area $ABKH$. | Parallelograms having equal bases and equal altitudes are equal in area. |
| 11. $ON = BL = CM = AH$. | Statement 5. Given. Quantities equal to the same quantity are equal to each other. |
| 12. Area $OCMN =$ area $ACJH$ and area $BONL =$ area $ABKH$. | Reason 10. |
| 13. Area $BCML =$ area $OCMN +$ area $BONL$ | Whole is equal to the sum of its parts. |

14. Area $BCML = \text{area } ACFG + \text{area } ABDE$ A quantity may be substituted for its equal.

Also solved by Ivan F. Arnold, Southwest Missouri State College, Springfield, Missouri; William D. Edwards, Eastern Illinois University, Charleston, Illinois; Philip Haverstick, William Jewell College, Liberty, Missouri; R. S. Luthar, Colby College, Waterville, Maine; Gail E. Norfolk, California State Polytechnic College, San Luis Obispo, California; Wilma L. Yates, Stetson University, DeLand, Florida.



Note two numbers, each containing all nine digits, and whose sum and difference also each contain all nine digits.

$$\begin{array}{r}
 371294568 \\
 + 216397845 \\
 \hline
 587692413
 \end{array}
 \qquad
 \begin{array}{r}
 371294568 \\
 - 216397845 \\
 \hline
 154896723
 \end{array}$$

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of *The Pentagon*. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor James P. Burling, State University College, Oswego, New York.

The Treasury of Mathematics. Edited by Henrietta O. Midonick. Philosophical Library, Inc., New York, 1965, 820 pp., \$15.00.

This book is truly a treasury of mathematics. It contains a series of fifty-four articles pertaining to the development of mathematics from the very early stages to the fundamental knowledge of calculus, mathematical logic and philosophy.

These articles are translations from ancient manuscripts dating back to the time of Appollonius, the Babylonian mathematical tablets, the Bakhshali manuscripts written on leaves of birch-bark, the works of Chu Chi-Chieh and Li Yeh — to mention just a few.

Each of the articles is preceded by a biographical or historical essay which gives an introductory picture of the article to be discussed.

In many instances reproductions of pages from the original manuscripts have been used to develop a more concrete conception of the work to be discussed, for example — *A Method of Division* by Rabbi Immanuel, *The Geometry of Rene Descartes*, *Maya Numeration*, *The Whetstone of Witte*.

Mathematicians and those interested in the development and history of mathematics should have access to this "Treasury" because of its far-reaching exploration into the past which is a source of instruction for us in the gradual development of mathematics as found in the mathematical and scientific ideas of the present.

Sister Edmund Marie
Ladycliff College

Functions, Limits, and Continuity, Paulo Rinenboim, John Wiley & Sons, Inc., New York, 1964, 133 pp., \$5.95.

One must thoroughly understand the basic concepts of function, limit, and continuity if he is to seriously study mathematical

analysis. To enable the mathematics student to attain such an understanding by reducing the task of studying these basic ideas to an extent as to make it attractive and by eliminating most of the applications usually taught in calculus courses in order to deliberately focus attention upon these essential principles to analysis is a praiseworthy accomplishment. The achievement of this goal is attempted by the author of this book in a direct and compact manner. Most of the applications usually taught in elementary calculus are eliminated in the book, which aids in directing attention upon the ideas of function, limit, and continuity. However, due to the brief treatment given several topics in the text the attractiveness of the task of studying these basics of mathematical analysis may be reduced.

Chapter 1, two pages in length, presents two concepts, sets and correspondences. The next two chapters are a concise, intuitive construction of the real numbers via Cantor's method. A construction of the set of complex numbers is omitted. Chapters 4 and 5 of the book are not as brief in development as the first three chapters and very nicely discuss bounded sets, accumulation points, and sequences of the set of real numbers. Cauchy convergence criterion, monotone sequences, divergence of sequences, limit superior, limit inferior, and the Bolzano-Weierstrass Theorem, which are treated quite rigorously in terms of neighborhoods, are outstanding features of these chapters. Single-valued functions of a real variable are adequately presented in Chapter 6. The content of this chapter is complete in that it includes graphs of functions, polynomial functions, rational functions, composites of functions, inverse functions, and monotone functions over intervals of the reals. Limits and continuity of functions are the topics of Chapters 7 and 8. Epsilon-delta definitions are finally evolved in these sections. The use of epsilon and delta had been avoided in the preceding material. Right-hand limit, left-hand limit, the Intermediate-value Theorem, and the Weierstrass Maximum-Minimum Theorem are the noteworthy items of Chapters 7 and 8. The book ends with a good, elementary discussion of uniform continuity and applications of uniform continuity. The last chapter presents coverings for intervals of the real line, proves the Heine-Borel Theorem for the reals, discusses compactness for sets of real numbers, and then obtains the global property of uniform continuity for real functions of a single variable.

For a book which is an elementary introduction to mathematical analysis there are some noticeable deficiencies. The ideas of subset, intersection, union, Cartesian product sets, and relation,

which are usually considered fundamental in the development of the real number system and essential in a discussion concerning neighborhoods, coverings, and functions, are omitted. Connectedness, the well-ordering principle, and the axiom of choice are used but not mentioned. Zero is denoted as a positive integer. Countable sets, uncountable sets, and cardinal number are touched upon in Appendix B but unfortunately are not portions of the text. Various types of proof are to be found in the book but no discussion of the logic involved is given. The exercises are stated precisely but demand mathematical maturity; that is, considerable guidance is needed if they are to be solved by beginning students.

On the other hand the book has some fine qualities. It has an ample supply of exercises. Examples are given which illustrate the definitions and theorems of the text. Also, some examples are presented which do not satisfy the conditions of these definitions and theorems. The book stresses the need for an operation to be well-defined. The elimination of engineering and physical applications that require a knowledge of science and may obstruct recognition of mathematical principles by computations directly presents the theoretical aspects of function, limit, and continuity. The conciseness with which the material is written provides a challenge for the student. Common sense and intuition are used to guide the reader through a logical sequence of topics designed to reveal the precision and thoroughness required in mathematics.

In summary it should be stated that the book is not designed for self-study but is a teachable textbook for a first course in analysis.

—Robert L. Poe

Texas Technological College

First Course in Functional Analysis, Casper Goffman and George Pedrick, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965, 282 pp., \$12.00.

This well organized text covers the basic material in real analysis and is directed toward the beginning courses of graduate study. The content is relevant and follows the view stated in the preface, that "analysis itself is basic and that the abstract theories to which it leads are primarily of interest as tools which may be used in treating problems in analysis."

A chapter on metric space which includes the Arzelà-Ascoli theorem, Stone-Weierstrass theorems and semi-continuity, and one on Banach spaces which includes the Hahn-Banach theorem, the

uniform boundedness principle, weak convergence, the Riesz representation theorem and the closed graph theorem, precede the chapter on measure and integration and the classical L_p spaces. The development here is fairly standard, a discussion of Lebesgue measure leading to measurable functions and convergence, then summable functions and the Fatou-Lebesgue theorems and so to absolute continuity and the Radon-Nikodym theorem. The Fubini theorem is covered in the exercises while the sections of the L_p spaces include a discussion of Fourier series. There follows a chapter on Hilbert space which includes the Müntz-Szász theorem and reproducing kernels, then one on topological vector spaces, covering the Tychonoff theorem, FK spaces, ordered vector spaces and Kothe spaces, with a final chapter on Banach algebras.

The presentation is very clear and concise, and the student will appreciate the extensive collection of exercises at the end of each chapter—they can be tackled without the usual prejudice from a section just concluded in the text though there is a cross reference to provide clues as required.

In contrast to many introductory books on functional analysis, the authors do not develop the theory of linear operators nor deal with spectral analysis. They have produced a compact text which will provide a sound foundation in modern analysis.

—T. Robertson
Occidental College

Lectures on Modern Mathematics, vol. I. Edited by T. L. Saaty. John Wiley & Sons, New York, 1963, 175 pp., \$5.75.

The reading of this book is highly recommended for every serious student of mathematics, advanced undergraduate, graduate, and postgraduate. The book consists of a series of six expository lectures given at George Washington University by six eminent mathematicians and sponsored jointly by the University and the Office of Naval Research. The lectures have done a masterful job of discussing recent results and interesting unsolved problems in six current and productive areas of mathematical research. The specialist will find the work interesting, and the non-specialist will find a source of inspiration for further study. All six lectures are supplemented by extensive lists of references.

Paul Halmos takes "A Glimpse into Hilbert Space." More specifically, he discusses fundamental concepts, results, and unanswered questions in his area of interest of linear operators on

Hilbert Spaces. The technical aspects of a specialized subject are made plausible to a non-specialist by good organization and good examples.

Laurent Schwartz discusses "Some Applications of the Theory of Distributions." He defines distributions and related properties and applies the theory to the solutions of partial differential equations. Ideas are presented but additional reading is needed to complement the lecture.

A. S. Householder lectures on "Numerical Analysis." The development of this new subject is outlined and the work and problems of the numerical analyst are discussed. Error analysis, the nature of the problems, and methods of attack are illustrated with theoretical examples.

Samuel Eilenberg arouses an interest in "Algebraic Topology" by introducing algebraic structure (category) into the study of topological spaces and their continuous mappings, giving a number of examples and applications to topology and other areas.

Irving Kaplansky discusses the development of "Lie Algebras," their classical connection with groups, and the classification and representation of simple algebras.

"Representation of Finite Groups" is presented by Richard Brauer. In giving a survey of the theory, he poses many unanswered questions which might arouse interest in further study.

C. J. Pipes

Southern Methodist University

Calculus and Analytic Geometry, Vols. I and II, Melcher P. Fobes and Ruth B. Smyth, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963, Vol. I - 660 pp., \$8.50; Vol. II - 450 pp., \$6.95.

The preface to Melcher P. Fobes and Ruth B. Smyth's two volume text on Calculus and Analytic Geometry indicates that the authors have given a great deal of thought to the teaching of these subjects.

The approach used reflects the presence of the teacher as you read the text material as homely words and phrases such as "fly-swatter principle," "swapped," "fussy," "all over" the denominator, etc., continually infiltrate the material. While there is some virtue in bringing the discussions down to earth it does seem to encourage

a student to do likewise, thus tending to cause him to persist in the imprecise manner of speaking about mathematical ideas that he has already acquired. I have always found that he needs very little encouragement in this direction.

In this connection, one should ideally expect the material of the course and the approach to its understanding to take on greater sophistication as things proceed. However, there is considerable inconsistency in this in the text. In particular, one proceeds in this manner in developing the notion of finding the area under a curve, continually refining the student's notions right through the idea of upper and lower Riemann sums. This is accomplished by the end of Chapter 13, that is by page 325, Volume I.

However, the authors later lapse into such expressions as "taking the limits as the slicing grows thinner," found on page 589. At this state, the student should be addressed as if he had attained greater maturity. Until Chapter 15, Volume I, is reached, the material is presented as though the student has virtually no trigonometric concepts in spite of the fact that the preface says that the authors presuppose "a semester's worth of analytic trigonometry." Thus radian measure is not broached until page 355, Volume I. The fundamental defect in this organization of the material is that it cannot be tolerated when the course is taught in a modern curriculum which is scientific or engineering oriented.

The text gives answers to the odd-numbered problems. The index is adequate but has some gaps—I could not find differentiation of functions defined implicitly indexed in any manner. The figures are well done but one can find exceptions to this also: The figure 23.2 on page 588, Volume I, is faulty and figure 12.2, page 276, Volume II is neither artistic nor very illuminating. I suppose it would be rather precarious having the man stand on the edge of an *overhanging* cliff, as the text implies. The use of the unconventional symbolism π^r to represent " π radians" is hardly necessary. The organization of the text into two volumes has its disadvantages as it tends to dictate a division of the material in a particular way. One is forced to wait until Volume II for polar coordinates, for example. Problems seem generally to be extensive enough and well graduated.

One can never make a truly valid criticism of a text until he has taught from it. In this sense, what I find in the text is entirely from a reading and scanning of it with the eye of one who has taught this material for about twenty years in an engineering and science

oriented institution. It is inevitable that I should see it, then, in a particular light—not without a certain prejudice. My feeling is that it would be particularly adaptable to a liberal arts curriculum in an institution in which majors other than in science and engineering predominate. In this "*locus operandi*" I feel that it will work out very satisfactorily. It seems to be very teachable, especially to a clientele who themselves wish to become teachers.

—C. A. Johnson

University of Missouri at Rolla

Guidebook to Departments in the Mathematical Sciences in the United States and Canada, Edited by Raoul Hailpern, Mathematical Association of America, State University of New York at Buffalo, Buffalo, New York, 1965, 59 pp., Paperbound, \$0.50.

The *Guidebook* provides, for the prospective student, information about departments of mathematics in four-year colleges and universities in the United States and Canada. Data is summarized on location, size, staff, library facilities, course offerings and number of degrees granted. Additional information concerning the number of graduate students and financial support available for the advanced student is given for departments in institutions conferring the Ph.D. degree in a mathematical science.



The earliest English law defining length is said to be the law of the year 1324 which read, "Three barley corns, round and dry, placed end to end, make an inch."

Kappa Mu Epsilon News

EDITED BY J. D. HAGGARD, HISTORIAN

ARKANSAS — IOWA — KANSAS — MISSOURI —
NEBRASKA — OKLAHOMA

REGIONAL CONVENTION

April 23, 1966

Southwest Missouri State College
Springfield

Papers presented during the conference include:

"Transformations in Differential Equations" by Jerry Ridenhour of Missouri Beta.

"Galois' Theory for the Group of an Equation and the Criterion for Solvability" by Leora Ernst of Kansas Gamma.

"On Arbitrarily Large Postulate Sets for the Propositional Calculus" by John W. Bridges of Missouri Alpha.

"Dr. I. Q." by Joseph Walton of Oklahoma Alpha.

"Where Is In and Where Is Out" by Paul Mugge of Arkansas Alpha.

"Concerning Prime Numbers" by Ronald R. Brown of Missouri Gamma.

"Mathematics of Cards" by Bernita Meyers of Kansas Gamma.

"Conic Sections with Circles as Focal Points" by Thomas M. Potts of Kansas Alpha.

Dr. Paul E. Long of the University of Arkansas was the guest speaker at the banquet where he discussed the topic "The Mathematical Shelf: Purpose and Style."

There were 15 chapters represented at the conference with a total attendance of 101 members.

WISCONSIN — MICHIGAN — OHIO —
INDIANA — ILLINOIS

REGIONAL CONFERENCE

March 18-19, 1966

Mount Mary College

Milwaukee, Wisconsin

Papers presented during the conference include:

"Vector Shorthand" by Duane Larson of Wisconsin Beta.

"Theory of a Complex Variable" by Bill Hibbard.

"Infinity in Mathematics" by Sandra Mertes of Wisconsin Alpha.

"Paul Bunyan Versus the Conveyor Belt" by Karen Johnson, Delores Peirick, and Mary Ann Raczka of Wisconsin Alpha.

Dr. Arthur Bernhart of the University of Oklahoma was the guest speaker at the banquet and spoke on the topic "The Five Color Problem."

There were four chapters represented with a total attendance of sixty students and faculty members.

CHAPTER NEWS

Arkansas Alpha, Arkansas State College, Jonesboro

Arkansas Alpha Chapter of Kappa Mu Epsilon held a spring initiation and installation of officers on May 6, 1966. Twenty-five were initiated at that time, bringing the total active membership to fifty-six. At the banquet which followed, Dean B. Ellis, Professor Emeritus of Arkansas State College, was presented an honorary membership. Guest speaker at the banquet was Dr. Glen Haddock, Academic Dean of Arkansas College, Batesville, Arkansas. His topic was, "Does the Pendulum Really Swing?"

During the school year the chapter has heard talks presented by guests, faculty members, and students on such topics as logic, transfinite numbers, data processing, and Gaussian integers.

The chapter also helped sponsor two picnics for the entire science department. In March we assisted our science faculty in conducting the Northeast Arkansas Science Fair, which is held on our campus each year.

Colorado Alpha, Colorado State University, Ft. Collins

The annual initiation ceremony was conducted in November, bringing the total number of initiates for the year to twenty-eight. In January we elected our new officers. Throughout the year we have enjoyed student and professor talks on such topics as the importance of Kappa Mu Epsilon, the history of probability, probability in statistics, logic, and recreational mathematics. Marilyn Brown was honored at the Associated Women Students Honors Night as the most outstanding woman member of Kappa Mu Epsilon and all other female members were also recognized. The chapter

members also received recognition at the All School Honors Night.

California Gamma, California State Polytechnic College, San Luis Obispo

California Gamma held monthly meetings with speakers being students, faculty, industrial representatives, and visiting professors. We initiated thirty-eight new members at banquets held in the fall and spring quarters. In April we assisted our departmental faculty in hosting its fourteenth annual mathematics contest, which attracted over 500 high school students to our campus.

Illinois Epsilon, North Park College, Chicago

Eighteen new members were initiated at the annual installation banquet on May 25, 1966. The guest speaker for the occasion was Dr. Ralph Shively of Lake Forest College, who spoke on the topic, "Some Unsolved Problems in Mathematics."

Other programs for the year include: "Computers," given by Professor William Herrin of the physics department at North Park College, and "Relativity," given by Professor John Baumgart of North Park College.

Indiana Alpha, Manchester College, North Manchester

Our programs have emphasized the history of mathematics. Outside speakers have been Professor Lipsich, Head of the Mathematics Department at the University of Cincinnati and Professor Retzer from Illinois State University.

Indiana Beta, Butler University, Indianapolis

The Indiana Beta Chapter of Kappa Mu Epsilon at Butler University of Indianapolis reorganized and initiated thirty-seven new members during 1965-66. A wiener roast was held in the spring. Officers for 1966-67 are President, Cal Jared II, Vice-President, Melvin Piepho, Secretary, Pat Gordon, and Treasurer, Paul Davis.

Indiana Delta, Evansville College, Evansville

Mrs. Janet K. Markham has joined the Eli Lilly Company as an associate pharmacologist. Mrs. Markham was a mathematics teacher at Creston Junior High School in Indianapolis, Indiana, before joining Eli Lilly. She was active in Kappa Mu Epsilon while a student at Evansville College.

Louisiana Beta, University of Southwestern Louisiana, Lafayette

The freshman and senior award contest was conducted in April, 1966. Eugene Garcia of Milton, Louisiana, won the freshman award, a CRC Mathematics handbook, and John C. Peck, Lafayette, Louisiana, won the senior award, which was a copy of *Men of Mathematics* by Bell.

New York Gamma, State University College, Oswego

On April 11, the chapter initiated thirty-five new members, by far the largest group ever.

New York Epsilon, Ladycliff College, Highland Falls

New York Epsilon conducted a symposium on March 10 and 12, 1966. The topic under discussion was "Probability and Statistics with Regard to Elementary and High School Students and its Applications."

During March, Kappa Mu Epsilon members attended lectures by Professor Leonard Gilman and Mr. St. John of the New York Telephone Company. On April 25, 1966, Colonel Charles P. Nicholas, U.S.M.A. at West Point was the guest speaker. His topic was "An Adventure in Mathematics."

Seven new members were initiated on May 11, 1966.

Virginia Beta, Radford College, Radford

Dr. Herta Taussig Freitag, Professor of Mathematics at Hollins College, spoke to Virginia Beta at their installation banquet for new members on "Mathematics and Art." Later during the year, Mr. William R. Battle, Vice-President and Actuary of the Shenandoah Life Insurance Company, spoke on "The Function of An Actuary," and various other aspects of the profession. Other monthly meetings have covered the topics of modern mathematics, algorithms, concept of maps, fun in mathematics, and Pascal.