

THE PENTAGON

Volume XXV

Fall, 1965

Number 1

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National Officers

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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Hamilton Quaternions*

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The science of mathematics is ever changing. It seems that development of its fascinating secrets is unlimited. Mathematicians are constantly staggering intellects with concepts reaching far beyond any imaginable ideas. This amazement is especially true for young, hopeful college students of mathematics. We are familiar with the construction of vector multiplication and addition and respective properties for two-dimensional vectors. It seems quite reasonable to ask what happens when vectors are extended to three, four, and even n -dimensions. William Rowan Hamilton, who lived in the early nineteenth century, did ask. In 1852 his lectures on the theory of quaternions were published. He had discovered a new algebra admirably adapted to the description of phenomena in the field of physics and quite generally used today. Within the past century, men, such as Hamilton, Gibbs, Cayley, and Grassmann, have published results of their development of the theory of quaternions. This paper endeavors to give some of these results and to show properties and operations, applications, and extensions of the Hamilton quaternion theory.

A quantity represented by the form $X = a + bi + cj + dk$ is defined to be a quaternion. The letter a and the coefficients of i , j , and k are symbols designating real numbers. Also $i^2 = j^2 = k^2 = -1$. These additional relations also hold: $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. We might note that here ij does not equal ji . The quaternion X can also be represented by the ordered quadruple of real numbers, (a,b,c,d) . Another notation quite extensively used is (A,B) where A and B represent complex numbers, $A = (a,b)$ and $B = (c,d)$. (A,B) can be written as $A + Bj$ which shall now be verified.

$$\begin{aligned}
 (A,B) &= ((a,b), (c,d)) \quad \text{since } A = (a,b) \text{ and } B = (c,d) \\
 &= ((a,b), (0,0)) + ((0,0), (c,d)) \\
 &= (a,b) ((1,0), (0,0)) + (c,d) ((0,0), (1,0)) \quad \text{by definition of addition} \\
 &= A ((1,0), (0,0)) + B ((0,0), (1,0)) \quad \text{by scalar multiplication} \\
 & \quad \quad \quad \text{by substitution}
 \end{aligned}$$

* A paper presented at the 1965 National Convention of KME and awarded first place by the Awards Committee.

$$(A,B) \sim A + Bj \quad \text{where} \quad \begin{aligned} ((1,0), (0,0)) &\sim 1 \\ ((0,0), (1,0)) &\sim j \end{aligned}$$

Thus each quaternion can be expressed as the sum: a complex number A plus the product of a complex number B multiplied by j .

There are four principal notations used for quaternions. These are:

$$\begin{aligned} (A,B) \\ A + Bj \\ a + bi + cj + dk \\ (a,b,c,d) \end{aligned}$$

Let us see how they are related to each other.

We define A and B to be the ordered pair of complex numbers (a,b) and (c,d) respectively. We have shown that (A,B) can be written as $A + Bj$. Substituting the ordered pairs (a,b) for A and (c,d) for B we obtain $(a,b) + (c,d)j$. We recall that complex numbers can be written in the form $a + bi$ and thus we have $(a + bi) + (c + di)j$. Applying the definition $ij = k$ leads us to the common quaternion form $a + bi + cj + dk$. The ordered quadruple (a,b,c,d) consists of the four real coefficients of this four-dimensional vector. These four notations listed above are the ones commonly accepted for a Hamilton quaternion.

Hamilton had great difficulty in securing official recognition for his quaternions. In his hypothesis he equated i^2 , j^2 , and k^2 to negative one which could be tolerated since ideas of imaginary or complex numbers had been developed previously. The relationship between the old concept of a complex number and the Hamilton concept can be shown by the following:

1. $(a,b) = (a,0) + (0,b)$
2. $\quad = (a,0) + (b,0)(0,1)$
3. $(a,b) \sim a + bi$

where

$$\begin{aligned} (a,0) &\sim a \\ (b,0) &\sim b \\ (0,1) &\sim i \end{aligned}$$

Hamilton's complex number (a,b) equals $(a,0) + (0,b)$ by definition of the addition of components of ordered pairs. Step two follows by definition of multiplication of ordered pairs. The equivalence in step 3 follows where the three additional equivalences hold.

However, Hamilton's setting $ij = k$ and $ji = -k$ provoked rousing laughter. Of course, if i and j were ordinary numbers like 2 and 3, it would be foolish to claim $2 \times 3 = 6$ and $3 \times 2 = -6$. But Hamilton's i, j , and k are not ordinary numbers. They are symbols for certain operations, and it does matter in what order the operations are performed.

Knowing, then, what a quaternion is, we can establish some of its properties. The sum of two quaternions, (A,B) and (C,D) , is the new quaternion, $(A + C, B + D)$, whose components are the sum of the components of the two quaternions. Quaternions under this component-wise addition satisfy all the group properties including commutativity. The additive identity quaternion is the zero quaternion. The additive identity for the quaternion, $((a,b), (c,d))$, is $((0,0), (0,0))$. The additive inverse of (A,B) is $(-A,-B)$, that is the negation of each component. The inverse of $((a,b), (c,d))$ is $((-a,-b), (-c,-d))$. These can both be easily verified.

If we denote a complex number by m and denote a quaternion by X , we can define scalar multiplication of quaternions by $mX = ma + mbi + mcj + mdk$. This definition of scalar multiplication of quaternions parallels that defined for scalar multiplication of vectors of two dimensions. But in this new definition for quaternion multiplication the m must denote a complex number instead of a real number and the X must denote a quaternion instead of a complex number. Then this scalar multiplication for quaternions satisfies all the properties of scalar multiplication of vectors, that is, for all m, n that are complex numbers and for all X, Y that are quaternions the following five properties hold.

1. $I \cdot X = X, 0 \cdot X = 0$
2. $m(nX) = (mn)X$
3. $-m(X) = -(mX)$
4. $m(X + Y) = mX + mY$
5. $(m + n)X = mX + nX$

These are familiar from scalar multiplication for vectors. Where

the m, n were real numbers, now they are complex, and where the X, Y were complex numbers, now they are quaternions.

To define the product of two quaternions we must discuss the conjugate of a quaternion. The quaternion $\bar{A} - Bj$ is called the conjugate of the quaternion $A + Bj$, where \bar{A} is the complex conjugate of A . If A, B, C , and D are complex numbers, then we have the following product which is rather long and tedious to remember: $(A + Bj)(C + Dj) = AC - B\bar{D} + (AD + B\bar{C})j$. However, John L. Kelley in *Introduction to Modern Algebra* gives us a rule of thumb to remember this definition. He claims that when two quaternions are multiplied it is assumed that multiplication distributes both on the right and on the left over addition, that multiplication is associative, and that moving j through a complex number changes the number to its conjugate. Let us multiply two quaternions and apply Professor Kelley's rule.

Example of Quaternion Multiplication

$$(A + Bj)(C + Dj) = AC - B\bar{D} + (AD + B\bar{C})j$$

$$\begin{aligned} & \left(\begin{matrix} A & B \\ (5 + 3i) & (1 - 2i)j \end{matrix} \right) \left(\begin{matrix} C & D \\ (3 - i) & (2 - 6i)j \end{matrix} \right) \\ = & \begin{matrix} A & C & B & D \\ (5 + 3i)(3 - i) & + & (1 - 2i)j(2 - 6i)j \\ & & A & D & B & C \\ & & + & (5 + 3i)(2 - 6i)j & + & (1 - 2i)j(3 - i) \end{matrix} \\ = & \begin{matrix} A & C & B & \bar{D} \\ (5 + 3i)(3 - i) & + & (1 - 2i)(2 + 6i)j^2 \\ & & + & \left(\begin{matrix} A & D & B & \bar{C} \\ (5 + 3i)(2 - 6i) & + & (1 - 2i)(3 + i) \end{matrix} \right) j \end{matrix} \\ = & \begin{matrix} A & C & B & \bar{D} \\ (5 + 3i)(3 - i) & - & (1 - 2i)(2 + 6i) \\ & & + & \left(\begin{matrix} A & D & B & \bar{C} \\ (5 + 3i)(2 - 6i) & + & (1 - 2i)(3 + i) \end{matrix} \right) j \end{matrix} \end{aligned}$$

$$\begin{aligned}
 &= (18 + 4i) - (14 + 2i) \\
 &\quad + \left((28 - 24i) + (5 - 5i) \right) j \\
 &= 4 + 2i + (33 - 29i)j \\
 &= E + Fj
 \end{aligned}$$

Moving j through a complex number changes that number to its conjugate. We move j through D and get $B\bar{D}j^2$ and we move j through C and get $B\bar{C}j$. We have used the distributive law to factor out j . When we first defined quaternions, we said that $j^2 = -1$. Therefore, the positive $B\bar{D}j^2$ became negative $B\bar{D}$. We have arrived now at the given definition for a quaternion product $AC - B\bar{D} + (AD + B\bar{C})j$. Simple computation, remembering that $i^2 = -1$, results in the final answer which is a new quaternion. The set of quaternions under both operations of addition and multiplication satisfies every one of the axioms for a field except the commutative law of multiplication.

We have already considered the additive identity and the additive inverses. Now we shall look at the multiplicative identity and the multiplicative inverses. The multiplicative identity for the quaternion $a + bi + cj + dk$ is $1 + 0i + 0j + 0k$.

The multiplicative inverse for the quaternion (A, B) is given to be $\frac{\bar{A} - Bj}{A\bar{A} + B\bar{B}}$; which is expressed in the $a + bi$ notation as

$$Q^{-1} = \frac{a - bi - (c + di)j}{(a + bi)(a - bi) + (c + di)(c - di)}$$

We shall now verify that this quaternion given actually has the property of an inverse, that is, we shall show that the product of the quaternion and the inverse cited actually yields the identity quaternion or $Q \cdot Q^{-1} = I$

$$\text{Quaternion } Q = (a + bi) + (c + di)j$$

$$\text{Multiplicative Inverse } Q^{-1} =$$

$$\frac{a - bi - (c + di)j}{(a + bi)(a - bi) + (c + di)(c - di)} = \frac{\bar{A} - Bj}{A\bar{A} + B\bar{B}}$$

$$\text{Multiplicative Identity } I = 1 + 0i + (0 + 0i)j$$

Show $Q \cdot Q^{-1} = I$

Proof:

$$\begin{aligned} Q \cdot Q^{-1} &= \left[\frac{1}{a^2 + b^2 + c^2 + d^2} \right] \times \\ &\quad \left[\left(a + bi + (c + di)i \right) \left((a - bi - (c + di)j) \right) \right] \\ &= \left[\frac{1}{a^2 + b^2 + c^2 + d^2} \right] \left[(a^2 + b^2) + (c^2 + d^2) \right. \\ &\quad \left. + \left((a + bi)(c + di) - (c + di)(a + bi) \right) i \right] \\ &= 1 + 0i + (0 + 0i)j \\ &= I \end{aligned}$$

$$\therefore \frac{a - bi - (c + di)j}{(a + bi)(a - bi) + (c + di)(c - di)}$$

is the multiplicative inverse of
 $a + bi + (c + di)j$

We notice that the denominator of the inverse quaternion when multiplied yields a real number because we are multiplying complex numbers by their conjugates. We can then treat this denominator as a scalar factor and place it in front of the product of two quaternions one of which is the original quaternion for which the inverse in question was given and the other is the numerator of the given inverse quaternion. Completing the multiplication we find that $Q \cdot Q^{-1} = I$. Therefore, we have proved that the given quaternion when multiplied by the quaternion which we claimed to be its inverse actually yields the identity quaternion.

We have now defined a quaternion. We have investigated addition of two quaternions and scalar multiplication of a quaternion. We have defined the product of two quaternions. Additive and multiplicative identities and inverses have been asserted. Now, we shall look at some extensions of the quaternion concept.

We can have a group of elements known as the quaternion

group where the operation is multiplication on eight symbols with rules of multiplication given in the table.

Quaternion Group

$$\pm 1, \pm i, \pm j, \pm k$$

operation of multiplication

	1	<i>i</i>	<i>j</i>	<i>k</i>	-1	- <i>i</i>	- <i>j</i>	- <i>k</i>
1	1	<i>i</i>	<i>j</i>	<i>k</i>	-1	- <i>i</i>	- <i>j</i>	- <i>k</i>
<i>i</i>	<i>i</i>	-1	<i>k</i>	- <i>j</i>	- <i>i</i>	1	- <i>k</i>	<i>j</i>
<i>j</i>	<i>j</i>	- <i>k</i>	-1	<i>i</i>	- <i>j</i>	<i>k</i>	1	- <i>i</i>
<i>k</i>	<i>k</i>	<i>j</i>	- <i>i</i>	-1	- <i>k</i>	- <i>j</i>	<i>i</i>	1
-1	-1	- <i>i</i>	- <i>j</i>	- <i>k</i>	1	<i>i</i>	<i>j</i>	<i>k</i>
- <i>i</i>	- <i>i</i>	1	- <i>k</i>	<i>j</i>	<i>i</i>	-1	<i>k</i>	- <i>j</i>
- <i>j</i>	- <i>j</i>	<i>k</i>	1	- <i>i</i>	<i>j</i>	- <i>k</i>	-1	<i>i</i>
- <i>k</i>	- <i>k</i>	- <i>j</i>	<i>i</i>	1	<i>k</i>	<i>j</i>	- <i>i</i>	-1

From the table we notice that one is the identity element. This group has a set of elements with a single operation of multiplication for which the closure, associative, identity, and inverse laws hold. The multiplicative commutative law is excluded for we see in the table that $ij \neq ji$.

The exclusion of commutativity eliminates the possibility of a quaternion field. However, we do have a "division algebra". In fact, quaternions constitute the only non-commutative division algebra over the field of real numbers.

In 1940, it was shown by a Swiss mathematician, H. H. Hopf, that if it were possible to define a division algebra on vectors of n -dimensions that n must be of the form 2^k where k is an integer. In 1840, Hamilton determined the division algebra for $k = 2$ or 2^2 and developed quaternion algebra for vectors of four dimensions. Cayley developed it for $k = 3$ or eight dimensional vectors.

Grassmann was developing a method of generating still more generalized numbers where $k = n$. Having created n -dimensional vectors of hypercomplex numbers, Grassmann set up rules for combining such numbers.

Quaternions and the theory of the quaternion group and division algebra are applied today in the theory of quantum physics and relativity mechanics. They are used in vector analysis, in the theory of matrices and in the geometry of the straight line, of the plane, of the sphere, and of the cyclic cone.

Much work has been done, startling possibilities have been cited, curiosity has been awakened. The theory of quaternions can and will open doors to future glory and progress if there are mathematicians to meet the challenges. Can we in college hope even to try? Most emphatically, I say we must!

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"It seems that Laplace transforms, like many other things bearing the names of other persons, are inventions of Euler. In any case, Euler used the transforms to solve differential equations when Laplace was -7 years old, and he did it very neatly."

—RALPH AGNEW

Lattice Theory*

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We are living in an age which has well been termed the Atomic Age of mathematics — an age in which order and structure have become focal points of study. For example, in physics, the structure of the atom has become a center of concentrated attention. This study of order and structure has led mathematicians to the fascinating study of LATTICE THEORY — developed greatly by Garrett Birkoff of Harvard University. The purpose of this paper is to investigate exactly what a lattice is and to study several specific examples.

Basic to the concept of lattices is a firm grasp of the meaning of partial ordering in a mathematical system. We know that various relations may hold between the elements of a set. Properties will result which are important in the system. Let us take, for example, the set of real numbers, the relation being less than or equal to. The following properties exist for any x and y in the reals under this relation. The real number system is:

P1) REFLEXIVE

that is, $x \leq x$

P2) ANTI-SYMMETRIC

that is, if $x \leq y$ and $y \leq x$, then $x = y$

P3) TRANSITIVE

that is, if $x \leq y$ and $y \leq z$, then $x \leq z$

Any set which possesses these three properties with respect to some relation (not necessarily \leq) is known as a partially ordered set. If we consider the relation R , then we see the properties in abstract terms as:

P1) Reflexive: xRx

P2) Anti-symmetric: xRy and $yRx \Rightarrow x = y$

P3) Transitive: xRy and $yRz \Rightarrow xRz$

These three properties can pertain to many relations. For

* A paper presented at the 1965 National Convention of EME and awarded second place by the Awards Committee.

example, consider the set of all *positive* integers. Now from the rudiments of elementary arithmetic it is evident that some of these are exactly divisible by others but not all of them. Thus 4 is divisible by 2 but not 3. If a positive integer y is exactly divisible by a positive integer x , we will write $x|y$, read " x divides y ." Notice that this relation of divisibility satisfies our definition for a partially ordered set:

P1) Reflexive

$$x|x$$

P2) Anti-symmetric

$$\text{if } x|y \text{ and } y|x, \text{ then } x = y$$

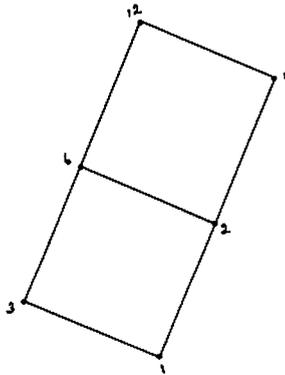
P3) Transitive

$$\text{if } x|y \text{ and } y|z, \text{ then } x|z$$

Therefore, we conclude that the set of all positive integers is a partially ordered set under the divisibility relation.

Observe that one may choose two integers, say 3 and 7, neither one of which may be divisible by the other. This is where the ideas of *partial* ordering comes in.

We can diagram the divisibility relationship of the set of integers $\{1, 2, 3, 4, 6, 12\}$ as follows:



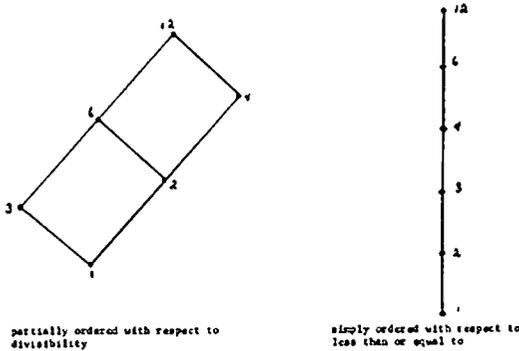
(Figure 1)

Note that if a number x divides a number y , then x is below y in the diagram and connected to it by a rising line. The line need not be unbroken. Thus $2|12$ and is joined to it by a rising line going indirectly to 12 through 4 or 6.

Here also we can have two numbers, neither of which divides the other. For example, 4 and 6. $4 \nmid 6$ and $6 \nmid 4$. We see that they are not connected by a rising line.

Hence, the word *partially* is used to indicate that, given any two elements x and y in the set, it is possible that these elements may not be related; that is, neither xRy nor yRx . If, on the other hand, for any two elements x and y either xRy or yRx and the three properties of partial ordering also hold true, then the set is a simply ordered set.

We see that our previous example with the set $\{1, 2, 3, 4, 6, 12\}$ would be a simply ordered set under the relation "less than or equal to" whereas it was not simply ordered with respect to divisibility.



(Figure 2)

The set is simply ordered with respect to \leq because any x is either less than or equal to a given y or y is less than or equal to x . Whenever a set such as the one given here is simply ordered, it is called a CHAIN.

Before dealing with the concept of a lattice we need to consider one additional thing—the idea of a set being bounded. There are two cases to consider: upper bounds and lower bounds. Consider the following simple example: Given the set of real numbers from 4 to 5 inclusive, it is easy to see that 4 is a lower bound for the set and 5 is an upper bound for the set. But there are others, in fact all numbers less than 4 are lower bounds. Similarly, all numbers greater than 5 are upper bounds. Thus we speak of a greatest lower bound and a least upper bound. If A is a set, then b is the least upper bound of the set A if and only if

- 1) b is an upper bound for A
- 2) there is no other upper bound less than b .

A greatest lower bound is defined similarly. Since these concepts are quite familiar ones in mathematics, we shall not dwell on them in this paper.

The general theory of partially ordered sets is based on a single relation such as \leq or divisibility. That of lattices is also based indirectly on a single relation which will be represented abstractly by the symbol \supset , read “contains,” but more directly on two binary operations represented by the symbols \cup and \cap , read “union” and “intersection.” It is by reason of this analogy that lattice theory is a branch of algebra.

With the concepts of partial ordering and boundedness in mind we can now proceed to define a lattice.

DEFINITION. In simplest terms, a lattice is a partially ordered set P any two of whose elements have a greatest lower bound (*g.l.b.*) which we shall denote $x \cap y$ and a least upper bound (*l.u.b.*) denoted $x \cup y$.

It may be helpful at this point to redefine *g.l.b.* and *l.u.b.* using our notation $x \cap y$ as *g.l.b.* and $x \cup y$ as *l.u.b.* The symbol \supset means “contains under the given relation.”

$g = x \cap y$ (or g is the greatest lower bound of x and y) if and only if:

1. g is a lower bound of x and y (that is $x \supset g$ and $y \supset g$.)
2. if d is any lower bound of x and y then $g \supset d$

Thus the *g.l.b.* contains every other lower bound of x and y .

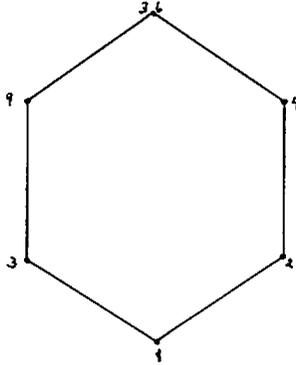
$l = x \cup y$ (or l is the least upper bound of x and y) if and only if:

1. l is an upper bound of x and y (that is, $l \supset x$ and $l \supset y$.)
2. if d is any upper bound of x and y then $d \supset l$.

Thus, any upper bound of x and y contains l .

Now let us see how we can apply our definition to several concrete instances.

Let us consider our divisibility relation once again in connection with the following diagram. Suppose that we are given the set of numbers $\{1, 2, 3, 4, 9, 36\}$.



(Figure 3)

If our relation is defined to be divisibility, then by $x \supset y$ we mean $y|x$. Thus $x \cap y = g$ (*g.l.b.*) is the greatest common divisor of x and y in the given set since $g|x$ and $g|y$, and $x \cup y = l$ (*l.u.b.*) is their least common multiple again in the set. In this example we have a partially ordered set any two of whose elements have a *g.l.b.* and a *l.u.b.* Some examples are:

$$3 \cap 2 = 1 \quad \text{and} \quad 3 \cup 2 = 36$$

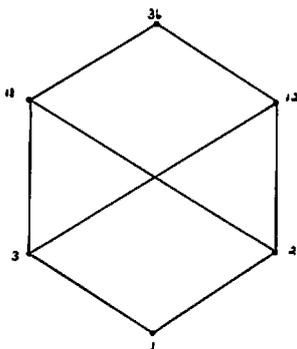
$$2 \cap 9 = 1 \quad \text{and} \quad 2 \cup 9 = 36$$

$3 \cap 2 = 1$ and $2 \cap 9 = 1$ because the greatest common divisor in

the set is one in both cases. $3 \cup 2 = 36$ and $2 \cup 9 = 36$ because the least common multiple in the set is 36.

The set in Figure 3 is a lattice.

In contrast to the above, consider the set with the six elements $\{1, 2, 3, 12, 18, 36\}$ as diagrammed below: (divisibility relation)



(Figure 4)

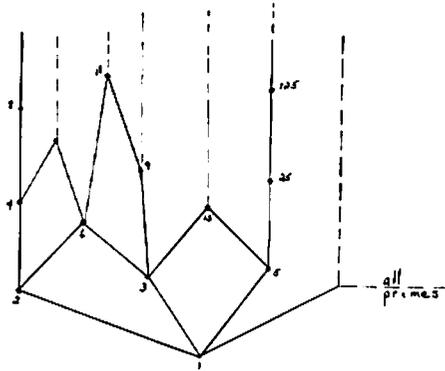
Observe that the intersection of the line segment joining 2 and 18 with the line segment joining 3 and 12 is not a point of our diagram. If it were, the figure would represent a lattice, but as we shall now show, the figure does not represent a lattice. According to our definition of upper bound under the divisibility relation as a common multiple, 12, 18, and 36 are all upper bounds of 2 and 3 since 2 and 3 divide 12, 18, and 36. But the *least* upper bound does not exist. One may object that 12 is certainly the “least” of the three, but recall that our definition of least is *not* in terms of magnitude. If 12 is to be the *l.u.b.* of the set, the following conditions must hold:

1. It must be an upper bound, $12 \supset 3$ and $12 \supset 2$.
This is obviously true since 12 is divisible by both 2 and 3.

2. If d is any upper bound of 2 and 3 then $d \supset 12$ (d is divisible by 12). We have established that 18 is an upper bound but certainly $18 \not\supset 12$ under our divisibility relationship; that is, 12 does not divide 18. Therefore 12 is not the least upper bound.

Neither is 36 the least upper bound. It too fulfills the first requirement; that is, $36 \supset 2$ and $36 \supset 3$. But if d is any upper bound, say 18 or 12, we have neither $18 \supset 36$ nor $12 \supset 36$ (that is, 18 is not divisible by 36, neither is 12 divisible by 36). Therefore this set is not a lattice since there exist two elements, here shown to be 2 and 3, which have no least upper bound.

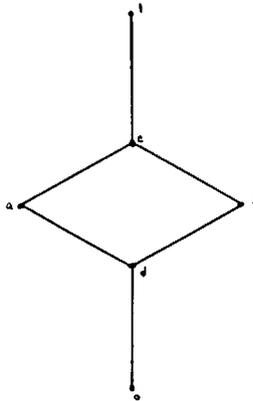
It is interesting to observe that in order to be a lattice, a partially ordered set need not have both a greatest and a least element. For example, here we have a set which is infinite: (divisibility relation)



We can see the reason for considering this example as a lattice if we realize that every pair of integers has a greatest common divisor and a least common multiple. However, this lattice is not a complete one, which is defined as follows: Partly ordered sets in which EVERY SUBSET has a *g.l.b.* and a *l.u.b.* are called complete lattices. In this illustration not every subset (here the entire set) has a *l.u.b.* though every pair of elements does have a *l.u.b.* The entire set does not have a *l.u.b.* because there does not exist a number which contains every x and y (which can be divided by every x and y) unless 0 is included in the set. This interesting distinction was first introduced by Birkhoff. It should be pointed out though that this idea can be applied only to infinite sets, for in order to be a lattice at all every finite set must have a greatest and least element. But the converse does not hold.

One must continually be reminded not to confuse greatest and least in terms of magnitude with *g.l.b.* and *l.u.b.* under a specific given relation.

Another surprising thing is that a subset of elements of a given lattice may themselves be a lattice but not a sublattice of the given lattice. (In group theory, one has the same idea with the idea of a "subgroup" meaning a subset of elements which themselves satisfy the requirements for a group.) Consider the following diagram:



(Figure 6)

If all elements are considered, d is the greatest lower bound of a and b , and c is the least upper bound of a and b , or $a \cap b = d$ and $a \cup b = c$. But if we consider only the elements $0, a, b, 1$, then 1 is the *l.u.b.* and 0 is the *g.l.b.* since c and d are not being considered in this subset. Thus, $a \cap b = 0$ and $a \cup b = 1$.

So this subset is a lattice in itself because it does fulfill the requirements but it is not a SUBlattice because the \cap and \cup of a and b do not give identical results.

This concept of a lattice, which is a fairly recent development in mathematics, sheds light on the study of Boolean algebra, group theory, projective geometry, point set theory and even logic and probability.

Thus we have concluded that a lattice is a partially ordered set any two of whose elements have a *g.l.b.* and a *l.u.b.* The subject of lattices is indeed a fascinating one and can be pursued much further. Even though it is based on extremely simple and general postulates, lattice theory is destined to play a major role in the future of mathematics.

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In general, the examples used in this paper have been adapted from Lieber's book.



"... if I were again beginning my studies, I would follow the advice of Plato and start with mathematics."

—GALILEO

A Theme With Variations*

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Mathematics is an experimental science. The theorems, proofs, and ideas a mathematician produces are seldom reached in the condensed and logical form in which they are published; rather, they are obtained by analogies, accidents, playing, experimentation, hunches, heuristic arguments — and sometimes logic. That this is really so can be verified by listening to the conversation between two equally (and highly) competent mathematicians who work in the same field. The information needed to convince one of them of the truth of a theorem is often only a tiny fraction of the actual proof and frequently this information is not part of a proof at all.

In this article I will try to show how one might arrive at the theorem about the arithmetic and geometric mean from a very simple beginning. Part of this article actually represents my own fumbblings (in a more condensed and logical form — of course) in obtaining proofs for this theorem.

If x is real, then $x^2 \geq 0$. This is our starting point. Replace x by any combination of reals and the square is still non-negative $0 \leq (x y)^2 = x^2 y^2$. (Product of two positive numbers is positive. Not very profound.) $(x^2)^2 = x^4 \geq 0$. Nothing new. $0 \leq (x + y)^2 = x^2 + 2xy + y^2$ Looks more complicated. Transpose: $-2xy \leq x^2 + y^2$. For $x, y \geq 0$, this is trivial since the left hand side is negative; the right hand side is positive. Start again. Assume x and y positive, $0 \leq (x - y)^2 = x^2 - 2xy + y^2$ or $0 \leq 2xy \leq x^2 + y^2$. Left \leq trivial, the right \leq not quite so trivial. $x^2 + y^2$ suggests Pythagoras, fix d , $d^2 = x^2 + y^2$. $xy \leq d^2/2 = (d/\sqrt{2})^2$. Or: All rectangles with given diagonal, d is \leq the area of the rectangle (square) whose two sides are $d/\sqrt{2}$. Reminds me of the conjecture: the square is the largest rectangle with given perimeter, $x + y + x + y = 4s$. That is, if $x + y = 2s =$ fixed, then $xy = \text{area} \leq s^2 = \left(\frac{x + y}{2}\right)^2$. What does this say? $xy \leq (x^2 + 2xy + y^2)/4$, $4xy \leq x^2 + 2xy + y^2$, $0 \leq x^2 - 2xy + y^2$, Ha! a perfect square, $0 \leq (x - y)^2$,

* A paper presented at the banquet of the 1965 National Convention of **KME**.

Oh! THEOREM. Of all rectangles with given perimeter this square is the largest. Proof: Trivial.

Let's look at that inequality again. $xy \leq \left(\frac{x+y}{2}\right)^2$. Here $\frac{x+y}{2}$ is famous, the average of x and y , let us isolate it,

$$\sqrt{xy} \leq (x+y)/2.$$

What is \sqrt{xy} ? It, too, appears several places in mathematics, $G = \sqrt{xy} =$ geometric mean, $A = (x+y)/2 =$ arithmetic mean.

THEOREM $G \leq A$.

$G = A$ if and only if $x = y$, why? Can we generalize? Among all parallelepipeds with given surface area (or sum of edges) is the cube the largest?

Given: $2xy + 2yz + 2zx = S$. Show $xyz =$ maximum for $x = y = z$. Or: given $4x + 4y + 4z = 4L$ show $xyz =$ max for $x = y = z = L/3$, that is, $xyz \leq \left(\frac{x+y+z}{3}\right)^3$. This looks like a more reasonable generalizaion. Conjecture:

$$\sqrt[3]{xyz} \leq (x+y+z)/3.$$

We probably can go further than this.

Conjecture: $G = \sqrt[n]{x_1 x_2 \cdots x_n} \leq (x_1 + x_2 + \cdots + x_n)/n = A$ with $=$ only when $x_1 = x_2 = \cdots = x_n$.

Supporting evidence: Any choice of n and x_1, x_2, \cdots, x_n . Try it yourself.

Attempts at proofs. Induction on n is the first thing we think of. Try it and see how difficult it is! In this connection we show a

proof due to Cauchy. $n = 2$: $\sqrt{x_1 x_2} \leq \frac{x_1 + x_2}{2}$, Done.

$$n = 4: \sqrt[4]{x_1 x_2 x_3 x_4} = \sqrt{\sqrt{x_1 x_2} \sqrt{x_3 x_4}} \leq \frac{\sqrt{x_1 x_2} + \sqrt{x_3 x_4}}{2}$$

$$\leq \frac{\frac{x_1 + x_2}{2} + \frac{x_3 + x_4}{2}}{2} = \frac{x_1 + x_2 + x_3 + x_4}{4}$$

See the pattern?

$$\begin{aligned}
 n = 8: \sqrt[8]{x_1 x_2 \cdots x_8} &= \sqrt[4]{\sqrt{x_1 \cdots x_4} \sqrt{x_5 \cdots x_8}} \leq \\
 &= \frac{\sqrt[4]{x_1 \cdots x_4} + \sqrt[4]{x_5 \cdots x_8}}{2} \\
 &\leq \frac{(x_1 + \cdots + x_4)/4 + (x_5 + \cdots + x_8)/4}{2}.
 \end{aligned}$$

It is now obvious (Is it to you?) that we can prove the theorem for $n = 2^m$ by induction on m . How can we "interpolate" between the powers of 2? Example: $G = \sqrt[11]{x_1 \cdots x_{11}} \leq (x_1 + \cdots + x_{11})/11 = A$? Instead of 11 x 's let us get $16 = 2^4$, the 11 we got and 5 more. Which 5? Well, G or A are natural ones to try.

$$\begin{aligned}
 &\sqrt[16]{x_1 \cdots x_{11} \cdot GGGGG} \\
 &\leq (x_1 + \cdots + x_{11} + G + G + G + G + G)/16 \\
 &= (11A + 5G)/16. \text{ No! we should have used } A\text{'s.}
 \end{aligned}$$

$$\begin{aligned}
 &\sqrt[16]{x_1 \cdots x_{11} A^5} \\
 &\leq (x_1 + \cdots + x_{11} + 5A)/16 = (11A + 5A)/16 = A \\
 x_1 \cdots x_{11} A^5 &\leq A^{16} \text{ or } x_1 \cdots x_{11} \leq A, \text{ etc.}
 \end{aligned}$$

Why won't it work using the G 's? Or does it???

This proof is elegant and artificial. Try something else. If all x 's are multiplied by the same factor kx_1, \dots, kx_n , then G and A are replaced by kG and kA . Can this be used? Let $k = 1/G$.

THEOREM: If the product of $x_1 \cdots x_n = 1$, then the sum $x_1 + \cdots + x_n \geq n$. If $x_1 = x_2 = \cdots = x_n = 1$, then the matter is easy. If $n = 2$, $x_1 > 1$ then $x_2 = \frac{1}{x_1} < 1$. What is the minimum

of $x_1 + \frac{1}{x_1}$? Use calculus. If $n \geq 3$, we may not be able to pair them off so nicely. But try $x_1 \cdots x_n = 1$, $x_n = 1/x_1 \cdots x_{n-1}$. Consider: $A(x_1, \dots, x_{n-1}) = x_1 + \cdots + x_{n-1} + 1/x_1 \cdots x_{n-1}$. $\frac{\delta A}{\delta x_i} = 1 - 1/x_i x_1 \cdots x_{n-1}$, $i = 1, \dots, n-1$. Set $\delta A / \delta x_i = 0$ to find minimum, if it exists! Then $x_i = 1/x_1 \cdots x_n$, $i = 1, \dots, n-1$, that is, they are all equal which gives $x = x_1 = \cdots = x_{n-1}$ and $x_n = 1/x^{n-1}$. Find minimum of $x + \cdots + x + 1/x^{n-1} = (n-1)x + 1/x^{n-1}$ by calculus. That is proof number two. Multiplying all x 's by k led us to a proof, what about adding?

Replace x_i by $x_i + a$, $i = 1, \dots, n$. This increases A to $A + a$ and G to $\sqrt[n]{(x_1 + a) \cdots (x_n + a)}$, which looks like an algebraic mess. If, however, we do not tamper with all the x 's but only a few, maybe the algebra will improve. Make one x smaller, another larger to retain A , what happens to G ? Replace x_1 by $x_1 + h$ and x_2 by $x_2 - h$. A does not change and G becomes

$$\sqrt[n]{(x_1 + h)(x_2 - h)x_3 \cdots x_n}$$

$$\sqrt[n]{x_1 \cdots x_n} \stackrel{?}{<} \sqrt[n]{(x_1 + h)(x_2 - h)x_3 \cdots x_n}$$

or

$$x_1 x_2 \stackrel{?}{<} (x_1 + h)(x_2 - h)$$

$$= x_1 x_2 + h(x_2 - x_1) - h^2? \text{ or } 0 \stackrel{?}{<} h[x_2 - x_1 - h]? \text{ For}$$

$x_1 < x_2$ and $h < x_2 - x_1$ we have $0 < h[x_2 - x_1 - h]$, that is, the G has increased. Thus we should increase the "small" x 's and decrease the "large" ones. This easily leads us to the following construction. Let $0 < x_1 \leq x_2 \leq \dots \leq x_n$. Assume $x_1 < x_n$ so that $x_1 < A < x_n$ (Why?) Set $h = \min(A - x_1, x_n - A)$ and replace x_1 by $x_1 + h$, x_n by $x_n - h$. Then A is unaltered while G increased. Either x_1 or x_n or both is moved to A . Repeat at most n times, until all x 's are moved to A , the G 's increase ending up at A , so the original G is less than A . That is proof number three.

Whenever we run into a product with many factors it is worthwhile to use logarithms to convert it to a sum and have a look at what we get. Thus,

$$\sqrt[n]{x_1 \cdots x_n} \leq (x_1 + \cdots + x_n)/n$$

leads to

$$(\log x_1 + \cdots + \log x_n)/n \leq \log (x_1 + \cdots + x_n)/n \text{ and } G \leq A$$

may be expressed as

THEOREM: The arithmetic mean of the logarithms is \leq the logarithms of the arithmetic mean.

The geometric mean vanished from the picture!

What other functions than $\log x$ has this property, $y = ax + b$ for example? Is $\frac{1}{n} \sum (a x_i + b) \leq a \left(\frac{\sum x_i}{n} \right) + b$? Easy computation

shows we always have \geq . What about $y = x^2$

$$\frac{x_1^2 + \cdots + x_n^2}{n} \geq \left(\frac{x_1 + \cdots + x_n}{n} \right)^2 ?$$

For $n = 2$ we actually get \geq , not \leq . Experimenting with different choices of n and the x 's shows \geq . Same result for the third

power but $\frac{\sqrt{x_1} + \sqrt{x_2}}{2} \leq \sqrt{\frac{x_1 + x_2}{2}}$

Further experimenting shows \leq when the graph is convex and \geq when the graph is concave. When should we have $=$ for all x 's?

Assume $f(x)$ is convex in the interval $a \leq x \leq b$ and $a \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq b$. Why is

$$(*) \quad \frac{1}{n}[f(x_1) + \cdots + f(x_n)] \leq f\left(\frac{x_1 + \cdots + x_n}{n}\right) = f(A) ?$$

x_1, \cdots, x_n and A are carried from the x -axis to the y -axis via the graph of $f(x)$. If, instead of $f(x)$, we use a straight line, $y = ax + b$, through $(A, f(A))$ lying above $f(x)$, (here we use the convexity) then we have equality in $(*)$, and $f(x_i) \leq ax_i + b$, giving a smaller arithmetic average which proves the inequality. This is the fourth proof. Example: $f(x) = \sin x$, $0 \leq x \leq \pi$,

$$n = 2 \text{ gives } \sin \frac{x_1 + x_2}{2} \cos \frac{x_1 - x_2}{2} = \frac{1}{2}(\sin x_1 + \sin x_2) \leq$$

$\sin \frac{x_1 + x_2}{2}$. How many new, simple and interesting inequalities

can you now make?

The definite integral is an analogue of the sum. There is no nice analogue of the product however. But writing $G \leq A$ as $\frac{1}{n} \sum \log x_i \leq \log \frac{1}{n} \sum x_n$ we may generalize to

THEOREM:

$$\frac{1}{b-a} \int_a^b \log x \, dx \leq \log \frac{1}{b-a} \int_a^b x \, dx$$

(Continued on page 61.)

The Parallel Postulates of Non-Euclidean Geometry*

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The study of geometry is an old and sacred science, originating in Ancient Egypt. It is one of the most basic because it studies the world around us, which was the first thing that man investigated. But the early geometry of straight lines on a plane surface is not the only geometry. There are several others which are of particular interest to the modern mathematician. I would like to investigate two of these geometries, specifically the non-Euclidean geometries, Hyperbolic and Elliptical. During the course of this paper we shall see their relation to Euclidean geometry which gives them the name non-Euclidean.

In the Fourth Century B.C. Euclid organized the early study of geometry into his thirteen-volume *Elements*. To him and other mathematicians, geometry seemed a very logical science built upon the concept of a plane surface. It was intended as a description of nature, as it seemed to these men. It looked like the shortest distance between two points was a straight line and so Euclid assumed it. He took a total of five self-evident truths which were the basis of his geometry. Since they were the very foundation of the system, there was no way to prove them. He exercised the mathematician's right to set up the postulates which define his system.

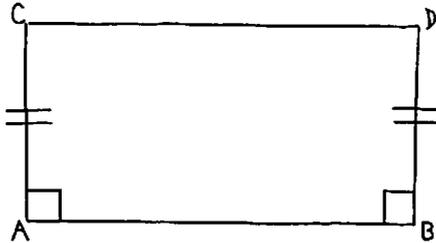
The first five postulates of Euclidean geometry are:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line.
3. To draw a circle with any center and any distance as radius.
4. That all right angles are equal to one another.
5. That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which are the angles less than two right angles.

* A paper presented at the 1965 National Convention of KME and awarded third place by the Awards Committee.

We are more used to the fifth postulate stated as Playfair phrased it: *Through a point not on a line, one and only one line can be drawn parallel to the given line.*

It had long been suggested that this fifth postulate of parallel lines be a theorem because of its complicated nature and reliance on the previous four. Many mathematicians, including the Greeks and Arabians, tried to prove this last postulate and no one could adequately do it. Around 1700, a Jesuit priest by the name of Giovanni Saccheri became interested in proving Euclid's fifth postulate. Saccheri taught theology and philosophy in Turin and Pavia, Italy. Under the influence of Father Tomasso Ceva, the study of the fifth postulate became a lifetime pursuit for Saccheri. He used the method of indirect proof and the quadrilateral in figure 1 in an attempt to find a contradiction which would prove the uniqueness of parallel lines.



(Figure 1)

Remember from elementary geometry that if two lines AC and BD of equal length are perpendicular to the same line AB , the line CD connecting the endpoints will form equal, in fact, right angles at C and D . Saccheri called the angles at C and D the summit angles and came up with three possibilities:

1. The summit angles are right angles.
2. The summit angles are obtuse angles.
3. The summit angles are acute angles.

By assuming each of these in turn, Saccheri came to the corresponding relation between the lines AB and CD . Besides the type of angle,

his proof depended upon the fact that AC and BD were equal and perpendicular to AB . The first hypothesis, that the angles are right, resulted in equal lines which, of course, led to Euclid's parallel postulate. The second hypothesis, that of the obtuse angle, when CD will be less than AB , turned out to be a contradiction, and was thereby eliminated. But when Saccheri came to the acute angle supposition, he could not find the desired contradiction. He had to trust to intuition in his last proof saying that the "hypothesis to the acute angle is absolutely false because repugnant to the nature of the straight line."

Now, from our own background, we can tell that Saccheri was probably not satisfied with this conclusion. But he did not go any further into the results of his efforts. He had started out to free Euclid from all contradiction so he was not looking for anything new. Besides not expecting to find a flaw in Euclid, some other factors held Saccheri back. The Kantian philosophy at that time denied the possibility of a non-Euclidean geometry, because it held that space was not subject to experiments. The mathematicians were strongly influenced by this philosophy and could not see that geometry was a theoretical science which could accept the convenient postulates and reject the others.

About fifty years after Saccheri, three men working with the acute angle hypothesis finally developed a geometry different from Euclid's. The greatest of these men was Karl Friedrich Gauss. He adopted Saccheri's alternative of the acute angle, leaving the rest of Euclid's postulates alone. The result was a geometry just as rational and valid as Euclid's. But Gauss did not publish his work right away. Perhaps he did not want to face the heated criticism of those who still held to the 2000-year-old habit of thought. Nicholas Lobachevsky, a Russian, and John Bolyai, a Hungarian, were not the prominent mathematicians that Gauss was, but at about the same time, they developed a geometry identical to that of Gauss. Further, they had the courage to publish their findings. Unfortunately, their work was almost completely ignored, except by those who saw fit to remove Lobachevsky from his post as rector of the University of Kazan. When Gauss died, his work on non-Euclidean geometry was found among his papers and published posthumously. Only then did people take a second look at the works of Lobachevsky and Bolyai.

In his proof of the parallel postulate, Saccheri was able to discard the obtuse angle hypothesis because he used Euclid's infinite

lines. In 1854, Georg F. Riemann, a German, took another look at the definition of the line, and by bounding it, failed to find the contradiction that Saccheri had found, which allowed him to establish a "spherical" or "elliptical" geometry. Thus, Saccheri's obtuse angle hypothesis led to the postulate that there are no parallels to a line through a point outside the line.

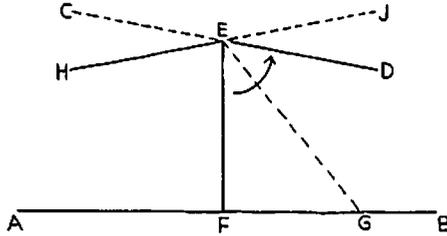
Before we examine the particulars of the parallel postulates of Hyperbolic and Elliptical geometries, we must see their logic. Only one of Euclid's postulates, the fifth (the parallel postulate), was contradicted in each case. A new fifth postulate was used in conjunction with the other four to prove all the necessary theorems. The reason why non-Euclidean geometry is valid is the fact that the new postulate did not bring about a contradiction anywhere in the system.

Furthermore, it is not necessary to apply a geometry to a particular figure for it to be accepted. The visual aids which allow us to examine the properties of the new parallel postulates are only helps in our understanding.

Upon first analysis, the geometry of Euclid seems to be the one which would describe the physical world around us. But upon further investigation, we see that Euclid's system does not work in all cases. As we look down a road, the parallel lines on each side seem to converge. Also the earth upon which we live is of an ellipsoidal shape and certainly there are no straight lines on it. A straight line had always been defined as the shortest distance between two points. On a plane surface, this is obvious. But the shortest distance between two points on a sphere is the length of the arc of a great circle which is called a geodesic, which on a plane surface would be the same as a straight line.

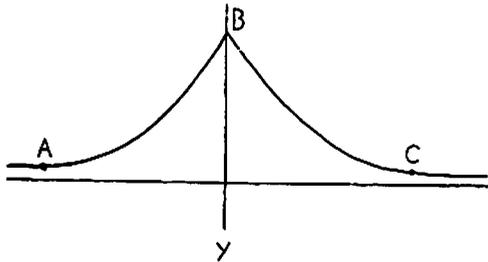
HYPERBOLIC GEOMETRY

Hyperbolic geometry, because the angle of parallelism is acute, maintains that two parallels can be drawn through a point if the direction of parallelism is also considered. Figure 2 shows two lines CD and HJ parallel to another, AB , through the same point E . By drawing EG from CD to AB and then letting G move to the right along AB past B , EG will reach CD and continue as a limit where CD is parallel to AB . If EG goes past ED it will no longer be parallel, but will be simply a non-intersecting line. The direction of parallelism in this case is from E to D .



(Figure 2)

Perhaps this postulate will become clearer if we see that it is applied to figures on a pseudo-sphere. A pseudo-sphere is the solid of revolution obtained by rotation of the tractrix ABC in Figure 3 around the x -axis.

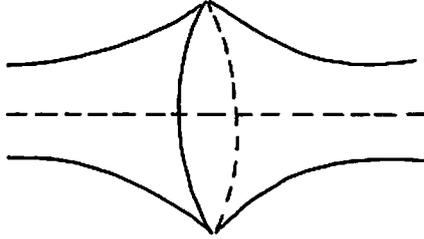


(Figure 3)

The equation for the tractrix is:

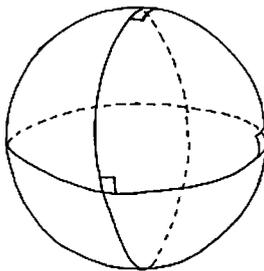
$$x = -\sqrt{1^2 - y^2} + 1 \cdot \log \frac{1 + \sqrt{1^2 - y^2}}{y}$$

When this is done, we get a solid such as this:



(Figure 4)

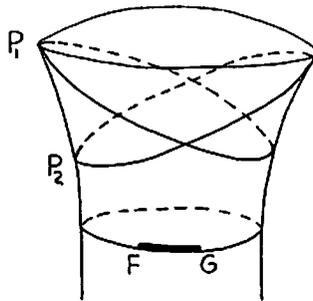
The use of this figure is very logical. Remember that the sum of the measures of the angles of a plane triangle must equal π as a result of Euclid's parallel postulate. On a sphere, the sum of the angles of a triangle is greater than π by an amount known as the spherical excess. If two great circles which are perpendicular at a pole are used as sides subtended by the equator, as shown in figure 5, we have a triangle with three mutually perpendicular sides. The area of any sphere can be divided into eight such orthogonal triangles.



(Figure 5)

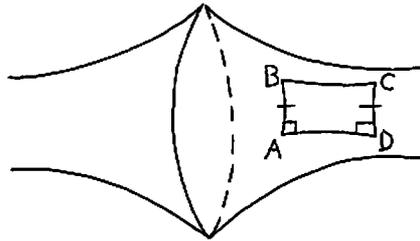
The interior angles of each triangle then equal $3\pi/2$. $3\pi/2 - \pi = \pi/2$ is the spherical excess of the triangle. Multiply $\pi/2$ times the eight possible congruent triangles on the sphere, $8(\pi/2) = 4\pi$, and we see a relationship between the surface area of the sphere $4\pi r^2$, and the spherical excess of the sphere, 4π . Thus we have $A = r^2E$, where A is now the area of the triangle and E is the excess of the triangle. It can be proved that the excess is additive just like the area. When R is the radius of the sphere, then the excess of a spherical triangle equals A/R^2 . The parallel postulate of Hyperbolic geometry implies that the sum of the angles of a triangle is less than π , which would imply a negative excess or in that case a deficiency shown by the equation $E = A/R^2$, where A is still the positive area function of the triangle. R^2 can only be negative if $R = iS$ where $i = \sqrt{-1}$ and S is a real positive number. This gives the figure an imaginary radius. Since the pseudo-sphere is a sphere of imaginary radius, it is used to illustrate Hyperbolic geometry. An interesting note at this point is the way a plane corresponds to this formula. In a plane triangle, the excess equals 0, which implies that $A/R^2 = 0$. The area is still positive and R^2 must go to infinity as E tends to zero. From this, we can say that the plane is a sphere of infinite radius. From this one formula, we can see the relationship between the three geometries, Elliptic, Hyperbolic, and Euclidean, on the basis of their angle of parallelism.

Figure 6 shows two parallel lines on a pseudo-sphere. The lines P_1 and P_2 are parallel to FG . We can see that the extended lines will not intersect FG but only tend toward it.



(Figure 6)

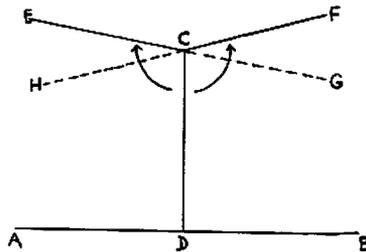
A Saccheri quadrilateral on a pseudo-sphere has the characteristics of $ABCD$ in figure 7. The side BC is greater than AD and the summit angles B and C are acute.



(Figure 7)

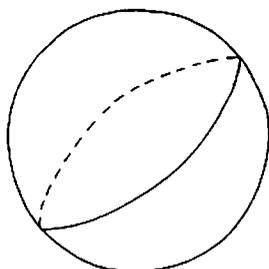
ELLIPTIC GEOMETRY

The basis of Riemann's Elliptic geometry is the obtuse angle of parallelism. In figure 8 CD is perpendicular to AB and both ECD and DCF are the angles of parallelism. If these lines are extended negatively, they will intersect the line AB and thus are not parallel.



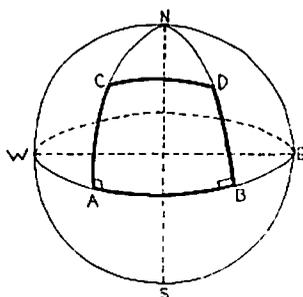
(Figure 8)

This type of non-Euclidean geometry can be illustrated on a sphere because the geodesics or straight lines on a sphere are the great circles.



(Figure 9)

Two properties which they have of themselves are especially interesting. First, straight lines on a sphere are finite, because they begin and end at the same point. Secondly, all the great circles intersect. Therefore, there are not two lines which fulfill the definition of parallelism, that is, two parallel lines are everywhere equidistant and never meet.



(Figure 10)

A quadrilateral on a sphere as shown in figure 10 has unequal sides with CD less than AB and the sides are arcs of great circles. When the sides are drawn as a part of the circles, as shown, we can see that they intersect. The angles of a quadrilateral and a triangle are obtuse, so that the angles at C and D in the quadrilateral are each greater than $\pi/2$.

It is interesting to know that these new frames of reference

lead to many diverse areas. Since the development of the Hyperbolic and Elliptic geometries, mathematicians have come upon the fourth dimension, finite geometries, and the study of curvature or differential geometry. As recently as thirty to forty years ago, Einstein developed his theory of relativity, which relies on different frames of reference. He said that if a Cartesian coordinate system or plane surface, made up of heat conducting metal rods, was heated unequally, the curvatures formed would require a new type of geometry. If it were of positive curvature, some of the postulates of Elliptic geometry would apply, while if it were negative, the Hyperbolic system would contain it. There is even the possibility that there would be a combination of negative and positive curves which would need a more complex set of postulates to define it.

These developments and those of the past 250 years have been caused by the change in the philosophy of mathematicians. Once men realized that postulates are simply man-made assumptions and not "self evident truths," the way was free for new areas of thought and theories. The non-Euclidean geometries have made history in the study of mathematics, which can lead to future progress more startling than that of the unconscious discovery of Saccheri, and the simultaneous findings of Gauss, Lobachevsky and Bolyai. But, if this is to be possible, we must remember never to become too fixed in our attitudes and always to look for the new in mathematics.

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The Problem Corner

EDITED BY F. MAX STEIN

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before March 1, 1966. The best solutions submitted by students will be published in the Spring 1966 issue of *The Pentagon*, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to Professor Howard Frisinger, Colorado State University, Fort Collins, Colorado, the new Problem Corner Editor.

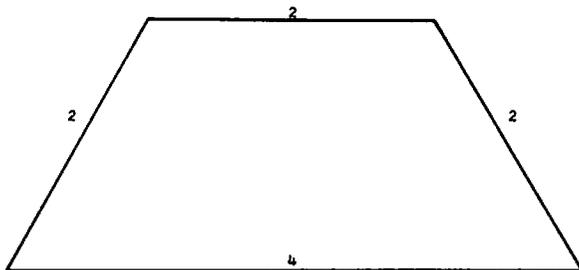
PROPOSED PROBLEMS

186. *Proposed by Fred W. Lott, State College of Iowa, Cedar Falls, Iowa.*

Prove that the square of an integer ends in 6 if and only if the ten's digit of the square is odd.

187. *Proposed by Ervin R. Deal, Colorado State University, Fort Collins, Colorado.*

Break the isosceles trapezoid up into four congruent parts.



188. *Proposed by T. L. Zimmerman, Kansas State Teachers College, Emporia, Kansas.*

Prove that \sqrt{p} is an irrational number if p is a prime other than 1.

189. *Proposed by Howard Frisinger, Colorado State University, Fort Collins, Colorado.*

Consider the table:

$$\begin{array}{rcl} 1 & = & 0 + 1 \\ 2 + 3 + 4 & = & 1 + 8 \\ 5 + 6 + 7 + 8 + 9 & = & 8 + 27 \\ 10 + 11 + 12 + 13 + 14 + 15 + 16 & = & 27 + 64 \end{array}$$

Express the general law suggested by this table and prove it.

190. *Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.*

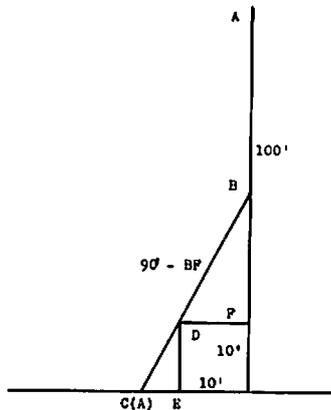
Find three right triangles such that the lengths of all sides are integral and the lengths of the legs are consecutive integers.

SOLUTIONS

181. *Proposed by George W. Norton, III, Marietta College, Marietta, Ohio.*

Suppose a shack, 10 feet by 10 feet, stands next to a tree 100 feet tall. If the tree breaks at B, the top A falls down (rotating about point B) and meets the ground at C. This fallen part BC just touches the shack at D. How high from the ground is point B?

Solution by LeRoy Simmons, Washburn University, Topeka, Kansas.



Since $(CE + 10)^2 + (BF + 10)^2 = (90 - BF)^2$ or
 (1) $(CE)^2 + 20(CE) - 7900 + 200(BF) = 0$, and

$$\frac{CE}{10} = \frac{10}{BF} \text{ or } CE = \frac{100}{BF},$$

then substituting for CE in (1) we obtain:

$$2(BF)^3 - 79(BF)^2 + 20(BF) + 100 = 0.$$

Using Budan's Theorem the two real roots of this equation are found to be $1 < BF < 2$ and $39 < BF < 40$. Using Newton's method the roots are found to be 1.281 and 39.212. Therefore, the distance from point B to the ground can be either 11.281 ft. or 49.212 ft.

Also solved by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio; Jerry L. Lewis, Drake University, Des Moines, Iowa, and the proposer.

182. *Proposed by J. Frederick Leetch, Bowling Green State University, Bowling Green, Ohio.*

If x is irrational, what is the nature of $x + h$ and $x - h$?

Solution by Frank Gutekunst, LaSalle College, Philadelphia, Pennsylvania.

Let h be a rational number. By definition we can write $h = r/s$, r and s integers. Suppose that the sum is rational. Then it may be written as the quotient of two integers. That is, $x + h = x + r/s = p/q$, p and q integers. Multiplying through by sq we get

$$sqx + qr = sp \quad \text{or} \quad x = (sp - qr)/sq.$$

But since the product and difference of integers is an integer, we have that x , an irrational, is written as the quotient of two integers, a direct contradiction. Therefore, our assumption must be wrong and $x + h$ must be irrational for h rational.

The same argument applies for $x - h$; i.e., $x - h$ is likewise irrational if h is rational.

In the case that h is irrational and $h = -x$, then $x + h = 0$, a rational, (and $x - h = x - (-x) = 2x$, an irrational number. *Ed.*) Likewise $h = x$ implies that $x - h = 0$, a rational, (and $x + h = 2x$, an irrational number. *Ed.*)

For the final case, let h be irrational where $h \neq -x + m/n$, m and n integers. Again assume that $x + h$ is rational; i.e.,

$x + h = p/q$, p and q integers. Subtracting x from both members gives $h = -x + p/q$, which violates our restriction. Therefore, when h is irrational $x + h$ must be irrational for $h \neq -x + m/n$. For $h = -x + m/n$, $x + h = m/n$, a rational number. (Also $x - h = x - (-x + m/n) = 2x - m/n$, an irrational number. *Ed.*)

The same argument as in the preceding case applies for $x - h$. That is, $x - h$ is irrational for $h \neq x + m/n$. (Also $x - h$ is rational and $x + h$ is irrational for $h = x + m/n$. *Ed.*)

183. *Proposed by the Editor.*

Sammy Sophomore couldn't perform the integration $\int \frac{dx}{x}$, so he multiplied the numerator and denominator of the integrand by x . He then integrated by parts as follows:

$$\int \frac{dx}{x} = \int \frac{xdx}{x^2} = -\frac{x}{x} + \int \frac{dx}{x}.$$

He then concluded that $-1 = 0$. Find the fallacy in his reasoning (if there is one).

Solution by LeRoy Simmons, Washburn University, Topeka, Kansas.

Going one step further helps to explain his fallacy:

$$\ln x + c_1 = \int \frac{dx}{x} = \int \frac{xdx}{x^2} = -\frac{x}{x} + \int \frac{dx}{x} = -1 + \ln x + c_2.$$

By saying that $-1 = 0$ Sammy is assuming that $c_1 = c_2$.

Also solved by Frank Guetkunst, LaSalle College, Philadelphia, Pennsylvania.

Mr. James F. Ramaley calls attention to a similar problem F21, p. 62, *Mathematics Magazine*, January, 1964 (V. 37, No. 1) and to his comment on that problem, p. 360, *Mathematics Magazine*, November, 1964. He also gives a reference to similar problems on p. 65 of E. A. Maxwell's book *Fallacies in Mathematics*.

184. *Proposed by Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.*

Let P be any point on an ellipse with semi-major axis a and semi-minor axis b . The circle with center P and radius b intersects the line containing the major axis in two points. Let A denote the

point of intersection which is farthest from the center of the ellipse. The circle with center P and radius a intersects the line containing the minor axis in two points. Let B denote the point of intersection which is farthest from the center of the ellipse. Prove that the points P , A , and B are collinear.

Solution by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.

From the conditions of this problem, the equation of the ellipse can be written as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the small circle as $(X - x)^2 + (Y - y)^2 = b^2$, and the large circle as $(X - x)^2 + (Y - y)^2 = a^2$. The coordinates of A are $(x + \sqrt{b^2 - y^2}, 0)$, and the coordinates of B are $(0, y + \sqrt{a^2 - x^2})$ when the coordinates of P are (x, y) . If the three points are collinear, then the slope of line BP should equal the slope of line PA . The slope of BP is

$$\frac{y + \sqrt{a^2 - x^2} - y}{0 - x} = \frac{\sqrt{a^2 - x^2}}{-x},$$

and the slope of PA is

$$\frac{y - 0}{x - x - \sqrt{b^2 - y^2}} = \frac{y}{-\sqrt{b^2 - y^2}}.$$

Equating these slopes we get

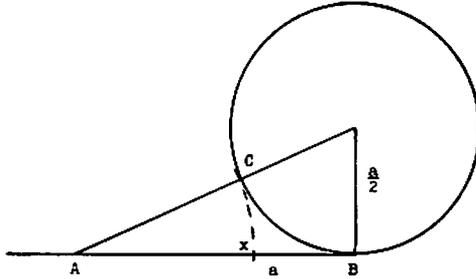
$$\frac{\sqrt{a^2 - x^2}}{-x} = \frac{y}{-\sqrt{b^2 - y^2}} \text{ or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Since this satisfies the original equation for x and y , then the slopes are equal. Therefore, the three points are collinear.

Also solved by Frank Gutekunst, LaSalle College, Philadelphia, Pennsylvania; George Arthur Murr, III, LaSalle College, Philadelphia, Pennsylvania, and LeRoy Simmons, Washburn University, Topeka, Kansas.

185. *Proposed by Howard Frisinger, Colorado State University, Fort Collins, Colorado.*

Show that $\frac{Ax}{a} = \frac{\sqrt{5} - 1}{2}$ in the following figure.



Solved by Jerry L. Lewis, Drake University, Des Moines, Iowa.
From the diagram and using the Pythagorean Theorem,

$$AD = \sqrt{a^2 + \frac{a^2}{4}} = \sqrt{5 \frac{a^2}{4}}, \text{ and}$$

(1)

$$AC = Ax = \sqrt{\frac{5a^2}{4}} - \frac{a}{2} = \frac{a\sqrt{5} - a}{2} = \frac{a(\sqrt{5} - 1)}{2}.$$

Furthermore, by dividing both sides of (1) by a we have the desired relation:

$$\frac{Ax}{a} = \frac{\sqrt{5} - 1}{2}.$$

Also solved by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio; Fred Homburg, Manchester College, North Manchester, Indiana; Frank Gutekunst, LaSalle College, Philadelphia, Pennsylvania; Thomas A. Jones, Colorado State University, Fort Collins, Colorado; George Arthur Murr III, LaSalle College, Philadelphia, Pennsylvania; Patricia Robaugh, Duquesne University, Pittsburgh, Pennsylvania; LeRoy Simmons, Washburn University, Topeka, Kansas; T. L. Zimmerman, Kansas State Teachers College, Emporia, Kansas.

Installation of New Chapters

EDITED BY SISTER HELEN SULLIVAN

PENNSYLVANIA ZETA CHAPTER

Indiana State College, Indiana, Pennsylvania

Pennsylvania Zeta Chapter was installed on May 6, 1965, by Professor Paul E. Brown, Chairman of the Department of Mathematics and faculty sponsor of Pennsylvania Alpha Chapter of Westminster College, New Wilmington, Pennsylvania. The installation was held after a banquet attended by the charter members, members of the mathematics faculty of the college and their spouses, and guests from the mathematics staff of Westminster College.

Fourteen faculty members and thirty-one students are charter members of the chapter. They are: Joseph Angelo, Ida Z. Arms, Linda Atty, Edwin Bailey, Carol Bunce, John Busovicki, Joyce A. Collins, Blaine Crooks, Ruth Dixon, William Ealy, Dennis Edwards, Diane Finley, James Flango, Louise Fucile, Donald J. Funk, Raymond D. Gibson, Lynne Heidenreich, Elizabeth Houk, Adrienne Kapisak, A. Marie Klapak, William F. Long, Sr., James Maple, Dale E. Markel, Doyle McBride, Lynne McCormick, James E. McKinley, Carl P. Oakes, Mildred Reigh, Margaret Reitz, Joan Reznar, Dale M. Shafer, Robert Sheraw, Kathryn Sirich, Jean Smith, William R. Smith, Patricia Spondike, Patricia Springman, Nancy Templeton, Michael Thornton, Diane Tullius, Robert C. Vowels, Michael Werner, Samuel Wieand, Robert R. Woods, and Florence D. Zampogna.

The officers of the chapter are:

President	Dale E. Markel
Vice-President	Samuel Wieand
Secretary	Lynne McCormick
Treasurer	Michael Thornton
Faculty Sponsor	William R. Smith
Corresponding Secretary	Ida Z. Arms

ARKANSAS ALPHA CHAPTER

Arkansas State College, State College, Arkansas

Arkansas Alpha Chapter was installed on May 21, 1965, by

Dr. Carl V. Fronabarger, Past President of Kappa Mu Epsilon. The ceremony took place in Room 204 of the Reng Center and was followed by a banquet for members and guests in the State Room. Dr. Fronabarger gave an interesting talk on the history and purposes of Kappa Mu Epsilon.

Charter members are: Trumann Baker, Ann Ball, John Basinger, Tom Bishop, Larry Brandon, Kay Campbell, Glenna Clarida, Ronald Clark, Linda Deck, Patricia Duncan, Jerry Elphingstone, Shirley Fowler, Taylor Francis, Connie Henry, Sharon Henson, Nancy House, Nancy Howell, John Kent, Janet Lieblong, Paul Madden, Joyce Mann, Joe Miller, Lonnie Minton, Paul Mugge, Doyne Null, John Rousey, Louise Sharp, Nancy Sigler, Joy Stephens, and Tom R. Trevathan.

The new chapter's officers are:

President	Ronald Clark
Vice-President	Nancy House
Secretary	Kay Campbell
Treasurer	Glenna Clarida
Faculty Sponsor	Tom R. Trevathan
Corresponding Secretary	Nancy Sigler

The mathematics club at Arkansas State College was formed in 1963. Since then the organization has held monthly meetings with discussion of such topics as Hilbert spaces, topological spaces, Boolean algebra, mathematical paradoxes, history and construction of the slide rule, and vector operations.

Arkansas State College, State College, Arkansas, developed from one of the four state agricultural schools established in 1909 by an act of the Arkansas General Assembly. Present undergraduate enrollment is approximately 3500.

ALABAMA EPSILON CHAPTER

Huntingdon College, Montgomery, Alabama

The Alabama Epsilon Chapter was installed on April 15, 1965, by Dr. Robert E. Wheeler, head of the department of mathematics at Howard College, Birmingham. Professor Joseph Faulkner, also a faculty member of Howard College, assisted in the installation.

(Continued on page 45.)

The Mathematical Scrapbook

EDITED BY GEORGE R. MACH

Editor's Note: For about ten years the Mathematical Scrapbook has been edited by Jerome M. Sachs, Illinois Gamma Chapter, Chicago Teachers College. On behalf of the many *Pentagon* readers, special thanks are extended to the retiring editor for a job well done.

Commencing with this issue the new Mathematical Scrapbook editor is George R. Mach, California Gamma Chapter, California State Polytechnic College. Your new Scrapbook editor became a student member of Iowa Alpha Chapter in 1948. He is now Associate Professor of Mathematics at California State Polytechnic College, San Luis Obispo campus, and is faculty advisor and corresponding secretary of California Gamma Chapter.

Readers are invited to correspond with the Scrapbook editor. Student, faculty, and chapter contributions to the Scrapbook will be solicited in the future.

= Δ =

Many algebraic fallacies can be exposed by exhibiting a division by zero. Here is an interesting "proof" that $\log(-1) = 0$ and it does not involve a division by zero.

$$\begin{aligned}\log(-1) &= \frac{1}{2}[2 \log(-1)] \\ &= \frac{1}{2} \log(-1)^2 \\ &= \frac{1}{2} \log(1) \\ &= 0.\end{aligned}$$

Wherein lies the fallacy? If $\log(-1)$ isn't zero (and it isn't), then what is it? [Hint: Write (-1) as a complex number, put it in exponential form, and you will be on the track.]

= Δ =

A race car circles a one mile track once at 30 miles per hour. Assuming that he can immediately accelerate to any desired speed, how fast should the driver go on the second lap to average 60 miles per hour for the two laps? What if he wants to average 90 miles per hour for the two laps? What about any speed (s) for the two laps?

= Δ =

The reciprocal of a prime (p) is, of course, a rational number and so can be written as a repeating decimal. Let (k) be the number of digits in the repeating portion. Note that for some primes $(k) = (p - 1)$.

$$\frac{1}{p} = \frac{1}{2} = 0.50\bar{0} \dots \quad k = 1, p - 1 = 1$$

$$\frac{1}{p} = \frac{1}{7} = 0.\overline{142857} \dots \quad k = 6, p - 1 = 6$$

$$\frac{1}{p} = \frac{1}{19} = 0.\overline{052631578947368421} \dots \quad k = 18, p - 1 = 18$$

Note for other primes that (k) divides ($p - 1$).

$$\frac{1}{p} = \frac{1}{5} = 0.20\bar{0} \dots \quad k = 1, p - 1 = 4$$

$$\frac{1}{p} = \frac{1}{11} = 0.09\overline{09} \dots \quad k = 2, p - 1 = 10$$

$$\frac{1}{p} = \frac{1}{13} = 0.\overline{076923} \dots \quad k = 6, p - 1 = 12$$

Can you find any primes for which (k) does not divide ($p - 1$)?
Can you prove that (k) always divides ($p - 1$)?

=△=

The Mathematics Dictionary defines a magic square as "a square array of integers such that the sum of the numbers in each row, each column, and each diagonal are all the same." Various rules are found for constructing magic squares of odd order and of even order.

Interesting squares of any order may be constructed using any numbers such as negatives, rationals, irrationals, imaginaries, numbers in a system with any base, etc., and seemingly having no rules of construction. The following squares are examples:

8	3	6	4
9	4	7	5
7	2	5	3
11	6	9	7

2	11	3
0	3	1
1	10	2

These squares are not "magic" in the usual sense but have the property that any set of numbers, including one from each row and one from each column, will have the same sum. The sum in the 4×4 is 24. If the numbers in the 3×3 are in the base 4 system, the sum is 13.

It may take a little while to find the pattern of construction and to see that these squares are really addition tables. As an addition table, the 4×4 would have its columns headed by 6, 1, 4, 2 and its rows headed by 2, 3, 1, 5 with the entries in the square being the indicated sums. Note that the sum of these eight numbers is 24. Do you see why any set including one from each row and one from each column sums 24? Can you now make your own square of any order with your own chosen sum? As an addition table, what should be the column and row headings of the 3×3 square above?



(Continued from page 42.)

Charter members are: Marlin H. Anderson, Jr., Jimmy Bright, Beppe LeCroy Gordon, Mary Jane Jeffards, Rebecca Jean Jones, Rex C. Jones, Lester Lee, Freida Little, Marilyn Schneider, John A. Tindall, Larry Vinson, Richard G. Vinson, Kaye Wilkenson, Camille Woodward, Sandra Yawn.

The officers for 1965-66 are:

President	Beppe LeCroy Gordon
Vice-President	Larry Vinson
Secretary	Marilyn Schneider
Treasurer	Camille Woodward
Historian	Freida Little
Faculty Sponsor	Dr. Richard G. Vinson
Corresponding Secretary	Rex C. Jones

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of *The Pentagon*. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Harold E. Tinnappel, Bowling Green State University, Bowling Green, Ohio.

Mathematical Bafflers, compiled and edited by Angela Dunn, with woodcut illustrations by Ed Kysar, McGraw-Hill Book Company, New York, 1964, 217 pages, \$6.50.

"A collection of the best puzzles from the famous 'Problematical Recreations' series of Litton Industries, together with dozens of provocative posers created especially for this volume."

This 217 page book includes over 150 selected problems presented in seven chapters, entitled: 1. Say It With Letters; 2. Axioms, Angles and Arcs; 3. Solving in Integers; 4. The Data Seekers; 5. Minding your P's and Q's; 6. Now You See It; and 7. Permutations, Partitions and Primes.

This is a very attractive book with one brief, well-stated problem on a page, each problem having a catchy title, and each illustrated with one of Kysar's drawings. Answers are also given for each problem. In some cases the author passes along comments which readers have sent her about problems which had appeared in "Problematical Recreations." These comments add greatly to the interest.

Problems are included which can be attempted, and perhaps solved, by the least sophisticated. There are also problems to frustrate those who consider themselves quite expert. Even after working a problem, the expert may still feel somewhat frustrated when he later reads the author's nicer solution.

The problems are not difficult in the sense of requiring knowledge of advanced mathematics. However, a wide variety of topics and techniques is involved: modular arithmetic; Diophantine equations (including some more complicated than linear); compound interest; limit processes; theorems of geometry; formulas from trigonometry; inequalities; progressions; number bases; number theory; logic and deduction; relations between coefficients and roots of polynomial equations; permutations and combinations; analytic geome-

try. In addition, symbols such as $\sqrt{4}$, 4^4 , $4!$, $.4$, 23 , and $\square (4)$ are found.

Mathematical Bafflers is recommended as an answer for the Math Club or K.M.E. chapter looking for a small prize to present in a mathematics contest. It is just the thing for a mathematician with a Christmas list, or for a mathematician on a Christmas list. It is one book which a mathematician can have in his library which he, his family and friends might all read and enjoy.

—Emmet C. Stopher
State University College

Mathematical Induction, Bevan K. Youse, Emory University Series, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1964, 55 pp., \$2.95.

This book is a brief but very extensive exposition of an important tool of mathematics—Mathematical Induction. The development of this powerful technique is undertaken in such a fashion as to be interesting to the beginner as well as to the advanced student. Beginning with sets, functions and sequences, the author proceeds to the various principles of Mathematical Induction. Normally only the first principle is seen in an introductory work. The well-ordering property is discussed and used in the proof of the first principle.

In addition to the above mentioned features of this book there are three other important (if not the most important) aspects. 1. The explanation and use of the sigma notation and inductive definitions. 2. The numerous statements of varying variety and difficulty proved in detail. These examples go well beyond the type normally found in such courses as College Algebra. 3. The last chapter contains 72 exercises some of which have previously been used as examples in the earlier portion of the book.

This book is well written and easily read. In the opinion of the reviewer this book should prove to be a valuable supplementary or reference text for high school, college or special institute courses.

—Lloyd L. Koontz Jr.
Eastern Illinois University

Mathematical Discovery, Volume II, George Polya, John Wiley and Sons, Inc., New York, 1965, xxii + 191 pp., \$5.50.

Readers of Volume I will be delighted to continue their study

of problem solving in this second part of Polya's two-volume approach to the topic. As in Volume I, model solutions of problems together with practice examples are explored in detail. However, the major point of view in Volume II is a general analysis of the ways and means of discovery and is thus far more philosophical and expository than the first part. Thus there is an excellent chapter entitled "The Working of the Mind," that presents a fine psychological discussion of a problem solver's mental experience as he attempts to solve a problem.

As stated in his first volume, the author's major concern in writing these texts is "to give opportunity for creative work on an appropriate level" to current and prospective high school mathematics teacher. His aim is thus to improve the preparation of high school teachers. It appears that he successfully meets this aim—those who read and study his works cannot help but improve their teaching of mathematics through this detailed study of problem solving. In addition, the author included an excellent chapter entitled "On Learning, Teaching, and Learning Teaching," part of which appeared earlier in an issue of the *American Mathematical Monthly*. In this chapter he not only presents his point of view on the process of learning, but goes on to explore the art of teaching as well as to comment on teacher training. Polya's "Ten Commandments for Teachers" is certainly as fine a set of principles for teachers of mathematics to follow as can be found in any text or course on methods of teaching mathematics.

Polya believes that the best way to become a problem solver is to solve problems! His text abounds with problems, some clever and some routine, that are explored in every detail in order to provide the reader with a means of "understanding, learning, and teaching problem solving." The teacher of mathematics, high school or college level, could do little else but improve his teaching skills by reading both parts of this excellent treatment of an all-important topic.

—Max A. Sobel
Montclair State College

Lectures in Abstract Algebra, Volume III, Theory of Fields and Galois Theory, Nathan Jacobson, D. Van Nostrand, Inc., New York, 1964, 323 pp., \$9.75.

In 1951 Professor Jacobson published the first volume (*Basic Concepts*) of his proposed three-volume treatise on abstract algebra.

That 217-page introduction to groups, rings, ideals, integral domains, fields, and lattices, was followed in 1953 by his 280-page text on *Linear Algebra*. Eleven years later his scholarly third volume appeared on the mathematical scene, and is certainly the grand climax of a project which has spanned two decades.

Volume III is not for the beginner. The best preparation for the reading of this work would be a thorough understanding of most of Volume I and much of Volume II, or equivalently, nine semester hours of course work in algebra. References to specific pages in the earlier volumes are to be found in every chapter of Volume III. The Introduction is not a simple review of basic concepts; in his introductory treatment of extensions of homomorphisms, algebras, and tensor products, the author assumes that the reader has had some previous contact with these ideas, and that he knows about prime fields, quotient fields, algebraic and transcendental extensions of fields, vector spaces, dimensionality, linear transformations, etc.

In Chapter I (Finite Dimensional Extension Fields) the author defines the Galois group of an extension field, and then derives results such as the fundamental theorem of Galois theory, basic theorems on algebraic extensions, and the principal results on finite fields as applications of Galois theory. In Chapter II (Galois Theory of Equations), he gives the classical application of Galois theory to the question of solvability by radicals of a polynomial equation. In Chapter III (Abelian Extensions), a study is made of Kummer extensions and abelian p -extensions.

Chapter IV (Structure Theory of Fields) brings us back to the mainstream that we were following in the first chapter. Here the author generalizes the classical Galois theory so that it applies to infinite dimensional algebraic extensions. Chapter V (Valuation Theory) is of interest, not only because valuations are needed in studying the arithmetic of algebraic fields, but because valuations enable one to apply methods of analysis to arithmetic questions arising in algebra. Both archimedean and non-archimedean valuations are considered, and important properties of p -adic fields are developed. In Chapter VI (Artin-Schreier Theory), the author takes up the theory of formally real fields. This final chapter contains useful results such as Tarski's algorithm for testing the solvability in a real closed field of a finite system of polynomial equations and inequalities in several variables.

The book is devoid of examples, except for three examples of

splitting fields and four examples of real valuations of fields. If the reader looks for a "change of pace" from theorem-proving in the 225 exercises, he will be disappointed; for all but four or five of them consist of more theorems to prove, many of which are extremely challenging.

This well-written work gives the student a comprehensive background in field theory. The author presents the classical results of the earlier algebraists, modified and extended to conform with current thinking and usage. Then he leads the reader to the frontiers of recent developments in the theory by giving results of his own research and that of his contemporaries. The graduate student would find enough material in the book to keep him busy in a year course; and he would discover that chapters I, II and IV (or I, IV and V) could provide the background for a good one-semester course. There is a dearth of abstract algebra textbooks at this level, and Professor Jacobson's welcome addition will undoubtedly become a classic.

—Violet Hachmeister Larney
State University of New York
at Albany

An Introduction to the Fundamental Concepts of Analysis, William E. Hartnett, John Wiley and Sons, Inc., New York, 1964, 154 pp., \$6.95.

The author's stated objective is "The book was written to serve a need—to provide an elementary treatment of the ideas of Analysis . . . Analysis is viewed as the fusion of an algebraic system (usually a commutative field) and a suitable topology for the field. In this book all of the important concepts introduced use the idea of a convergent sequence."

In achieving these objectives, the author used a well-motivated "trial and error" approach. The original motivation stems from the desire to find "nice" sequences (e. g., Cauchy) and "nice" functions (e. g., functions with "nice" pictures). This leads to a careful and meaningful development of the real numbers, continuity, and then differentiability. Further geometric motivations lead to the concept of an integral via step functions.

The book has many refreshing features in the exposition, as well as in the mathematical development.

The personality of the author is revealed as one with deep mathematical insights and substantial pedagogical talent, including

smooth, clear and entertaining exposition. The author communicates directly with the reader—challenging him to find the solution to a problem, as in a good mystery. He thus neatly leads him into an understanding of the basic concepts of analysis and of the creative rôle of the mathematician.

Free use of good clear diagrams and graphs throughout serve well in motivating the intuitive development. Numerous illustrative examples and problems of real interest are included. The problems are clearly stated and deal with basic concepts. They serve to encourage the reader to become more “involved” in the basic ideas which are being developed, and to help him acquire some technical facility.

The novel mathematical approach stems from the fact that sequences are used as a model as well as a tool in the development of some of the basic concepts of analysis. Also, facts ordinarily taken for granted are explicitly and effectively stated; e.g., (1) the difference between a constant function and a constant, (2) the continuity of the sin and cos at zero imply their continuity everywhere, and (3) the rôle of the identity function in the development of polynomials.

A major strength of the mathematical development is that the basic concepts developed are readily extendable to more general situations. For example, in any space in which one can define the notion of a neighborhood, sequences with limits and continuous functions can also be defined.

Only a minimal background is required—high school algebra, trigonometry, and some graphing technique would seem to suffice.

This book could serve well as a text for secondary teachers or mathematics majors in courses in Introductory Analysis or Intermediate Calculus, or for a Seminar in the Foundations of Analysis. In addition it is good entertaining reading for any qualified student interested in college mathematics.

—Gloria Olive
Anderson College



“In every department of physical science there is only so much science, properly so-called, as there is mathematics.”

—IMMANUEL KANT

Kappa Mu Epsilon News

EDITED BY J. D. HAGGARD, HISTORIAN

The Fifteenth Biennial Convention of Kappa Mu Epsilon was held April 25, 26, and 27, 1965, with Colorado Alpha at Colorado State University, Ft. Collins, as host chapter. Thirty-six chapters were represented with a total individual registration of 139.

MONDAY, APRIL 26, 1965

The meetings were held in the University Student Union. National President, Loyal Ollmann, of New York Alpha presided. President William E. Morgan of Colorado State University gave the welcome address and Vice-President Harold E. Tinnappel of Ohio Alpha responded for the Society. The following chapters, approved for membership since the last national convention, were welcomed:

Arkansas Alpha, Arkansas State College, State College
Alabama Epsilon, Huntingdon College, Montgomery
California Delta, California State Polytechnic, Pomona
Maryland Alpha, College of Notre Dame of Maryland, Baltimore

Oklahoma Beta, University of Tulsa, Tulsa
Pennsylvania Delta, Marywood College, Scranton
Pennsylvania Epsilon, Kutztown State College, Kutztown
Pennsylvania Zeta, Indiana State College, Indiana
Tennessee Gamma, Union University, Jackson
Wisconsin Beta, Wisconsin State College, River Falls

Petitions for new chapters at Morningside College, Sioux Falls, Iowa, and Western Maryland College, Westminster, were presented and approved.

Professor Harold E. Tinnappel presided during the presentation of the following papers:

1. *The Exploding Population*, Bradford Roth, California Beta, Occidental College.
2. *Construction of Conics*, Adolf Pohlis, Illinois Gamma, Chicago Teachers College.
3. *Rings of Prime Order*, Clyde Martin, Kansas Beta, Kansas State Teachers College.
4. *Hamilton Quaternions*, Joan Carlow, Kansas Gamma, Mount St. Scholastica College.

After lunch in the Student Union, the faculty members and students met separately in two "Let's Exchange Ideas" discussion sections. The entire convention reconvened at 2:30 p.m. and after reports from the two sections, the following student papers were presented:

5. *Lattice Theory*, Mary Koob, Kansas Gamma, Mount St. Scholastica College.
6. *Incorporation of Some Mathematical Ideas Through Application to Electrical Circuits*, Bill Chauncey and Jerry Ridenhour, Missouri Beta, Central Missouri State College.
7. *An Introduction to Bipolar Coordinates and Applications to Ovals of Descartes*, Dale Schoenefeld, Nebraska Alpha, Wayne State College.
8. *Computer Operations in Base N*, Donna M. Guyle, New York Gamma, Oswego College of Education.

Following a choice of several area tours which were available to conventionaires, the banquet was served in the Student Center Ballroom with Professor M. Leslie Madison, Colorado Alpha, as Master of Ceremonies. Professor Franklin A. Graybill gave the invocation. Professor Arne Magnus of the University of Colorado was the guest speaker. His topic was "Arithmetic and Geometric Means."

TUESDAY, APRIL 27, 1965

The program began at 8:30 a.m. with the following student papers:

9. *Contributions to the Founding of the Theory of Transfinite Numbers*, Maureen O'Grady, New York Epsilon, Ladycliff College.
10. *The Parallel Postulates of Non-Euclidean Geometry*, Mary Irene Solon, Kansas Gamma, Mount St. Scholastica College.
11. *Concerning Functional Conjugates*, Allen R. Grissom, Tennessee Alpha, Tennessee Polytechnic Institute.

The following papers were listed by title:

1. *Projective Geometry and Desargues Theorem*, Suzanne Dulle, Kansas Gamma, Mount St. Scholastica College.
2. *Networks in Topology*, Jo Ingle, Kansas Gamma, Mount St. Scholastica College.

3. *Euclid's Algorithm*, Mary Noonan, Kansas Gamma, Mount St. Scholastica College.

At the second general business session, reports of the national officers were read as well as the report of the auditing committee and resolutions committee. No invitations for the next biennial convention were given at this time.

Professor Carl V. Fronabarger reported for the nominating committee. There was one nomination from the floor and the following list of national officers was elected for 1965-67.

President	Dr. Loyal F. Ollmann Hofstra College
Vice-President	Dr. Fred W. Lott, Jr. State College of Iowa
Secretary	Professor Laura Greene Washburn University of Topeka
Treasurer	Professor Walter C. Butler Colorado State University
Historian	Dr. J. D. Haggard Kansas State College

Dr. Fred W. Lott, Jr., Iowa Alpha, chairman of the awards committee made the following awards to the students listed below for papers presented during the convention.

First Place	Joan Carlow, Kansas Gamma
Second Place	Mary Koob, Kansas Gamma
Third Place	Mary Irene Solon, Kansas Gamma

Sister Helen Sullivan, Kansas Gamma, reported for the resolutions committee. The following resolutions were adopted:

Whereas this Fifteenth Biennial Convention in this scenic state of Colorado has been a very enjoyable and profitable conference, be it resolved that we express our appreciation:

1. To the host chapter, Colorado Alpha, and to Colorado State University for their hospitality, the use of their comfortable facilities, the efficient organization of all preliminary details and for all the other factors (too numerous to list) that contribute so markedly to the success of meetings such as this.

2. To each of our national officers whose unceasing efforts and continual inspiration are responsible for the growth of our society both in prestige and in membership. To Professor Leslie Madison and the mathematics staff, to Mr. Burritt Tomlinson of the Student Union, to Professor Kenneth Whitcomb whose housing arrangements left nothing to be desired, to the faculty sponsor Professor Floyd L. Leidal and the chapter for all the splendid planning that made our stay so very pleasant. To all the unnamed and unknown contributors who promoted the smooth functioning of this conference. To Professor Harold E. Tinnappel, vice-president, for his work in organizing the program of student papers and for assisting the students in their presentation.
3. To Professor Fred W. Lott, Jr., editor of *The Pentagon*, who has so satisfactorily maintained the quality of our magazine.
4. To the eleven students who prepared and presented excellent papers at these sessions, to the three students whose papers were listed by title as well as to all the other students who by their presence contributed both ideas and scholarly attitudes to the convention.
5. To all here present for the warm spirit of fellowship and courtesy that makes these meetings so memorable, and again to Colorado Alpha for making this stay in the Rockies so very pleasurable.

REPORT OF THE NATIONAL PRESIDENT

I want to express my sincere appreciation to the faculty and student delegates who so willingly and ably performed the assignments given to them; to the members of the local chapter of Kappa Mu Epsilon who have worked these many months to make our stay here so pleasant; to the mathematics staff under Professor Madison and the administrative officers of Colorado State University for the generosity and hospitality given in truly "Western" style.

It was a pleasure to work in close harmony with your national officers whose reports have just been heard—particularly with your Secretary, Miss Greene, and your Treasurer, Mr. Butler. Dr. Fronbarger has been of much help in outlining my job as your president. Finally, I want to give my personal thanks to our retiring Editor of

The Pentagon, Dr. Fred W. Lott, Jr., and to Dr. Wilbur Waggoner, the Business Manager of *The Pentagon*. Their efforts enabled us to carry on what I consider to be the greatest asset of our Society—the publication of a mathematics magazine for students.

Looking in the future, it is my hope that we will continue to grow as reported by our Secretary—and I believe that we will. I am now corresponding with sixteen prospective new chapters who have indicated a desire to join Kappa Mu Epsilon.

It is also my hope that many more student papers will be written and presented at sectional meetings and at our next convention. I urge all delegates to encourage students in the writing of papers reflecting some original work.

One of the responsibilities of your Executive Committee will be to investigate the advantages and disadvantages of becoming affiliated with the national organization of Honor Societies. Our findings will be reported back to the individual chapters before action can be finalized.

By action of the Executive Committee in our last session, we have considered the following:

1. We have invited Dr. Helen Kriegsman of Kansas State College of Pittsburg, Pittsburg, Kansas, to be the new Editor of *The Pentagon*. We have not had a positive acceptance, but all of you who know Dr. Kriegsman will be happy if she accepts.
2. We have considered the possibility of an increase in initiation fees. The \$5.00 fee which pays for *The Pentagon*, for regional meetings, for expenses of national officers, for travel expenses of delegates to the convention, and many more items, has not been adequate to enable us to assist some chapters as much as we wish in order that they may send delegates to the conventions. This fee has been stable since 1953.

Finally, I would like to tell all of you how much I have enjoyed working with the people in Kappa Mu Epsilon. I've found many new friends and think you are all a fine group of people. Thank you very much.

Loyal F. Ollmann

REPORT OF THE NATIONAL VICE-PRESIDENT

The principal function of the vice-president is to make arrangements for the program of student papers. A preliminary announcement describing the procedure for submitting papers for the Fifteenth Biennial Convention appearing in the Spring, 1964, issue of *The Pentagon* was followed by a second invitation appearing in the Fall, 1964, issue. In addition, a reminder of this notice was sent to the corresponding secretary of each chapter early in December, 1964.

Fourteen papers were submitted by students from nine different chapters. Sister M. Denis, head of the Mathematics Department of St. Bonaventure's High School, Paterson, New Jersey; Professor Marion P. Emerson, head of the Mathematics Department of Kansas State Teachers College, Emporia, Kansas; and the vice-president served on the Student Paper Selection Committee. This committee selected eleven papers to be presented at the convention and listed the remaining three papers by title on the program.

The vice-president strongly recommends that the valuable experience a student obtains in preparing a paper and in presenting this paper before an audience be one which should be shared by a greater number of members of Kappa Mu Epsilon. The interest at the level of the local chapter will be stimulated and the number of excellent articles written by student authors will appear with greater frequency in *The Pentagon*.

Harold Tinnappel

REPORT OF THE NATIONAL SECRETARY

The total membership of Kappa Mu Epsilon is now 20,977 in sixty-nine chapters. Before the close of the semester, four chapters will be installed and by your vote at this convention, two others will be installed in the near future. These chapters will bring our total to seventy-five active chapters in twenty-eight states.

All orders for jewelry, invitations, and supplies should be sent to the Secretary. All checks should be made payable to the Treasurer, but sent to the Secretary. To all of you who carefully report your initiation, our thanks.

Laura Z. Greene

REPORT OF THE EDITOR OF THE PENTAGON

The manuscripts for the Spring, 1965, issue of *The Pentagon* are now in the hands of the printer. In fact, when I return to Iowa from this convention, I anticipate finding the galley proofs waiting and to begin the task of assembling text and figures into a page dummy. You should receive your copy sometime in May. When this is completed, there will have been four issues of *The Pentagon* published this biennium, consisting of approximately 266 pages. In addition to chapter news, book reviews, the Problem Corner, and the Mathematical Scrapbook, there have been nineteen articles of which fifteen were written by student authors.

The Pentagon is the result of the voluntary contributions of many people. I would like to express the appreciation of the entire Kappa Mu Epsilon organization to all those whose time-consuming and effective work have contributed to producing the journal.

For the past biennium, our National Historian, J. D. Haggard of Kansas State College of Pittsburg, has edited the KME News section. Jerome Sachs, Chicago Teachers College, is the editor of the Mathematical Scrapbook. The Book Shelf is edited by Harold Tinnappel, Bowling Green State University, our National Vice-President. The Problem Corner editor is F. Max Stein, Colorado State University, and Sister Helen Sullivan, Mount St. Scholastica College, edits the reports of the Installation of Chapters. Substantial contributions to producing *The Pentagon* are made by the Business Manager, Wilbur Waggoner of Central Michigan University, and the National Secretary, Laura Greene of Washburn University. I would like to commend Irwin Campbell, manager of the Central Michigan University Press where *The Pentagon* is printed, and his staff for their excellent work. Finally, we are indebted to all those persons who have written the articles published, contributed to the Problem Corner, or written reviews for The Book Shelf.

The Pentagon has been published by Kappa Mu Epsilon since 1941; we are now in our 24th year. For many years, several of the early issues have been out of print. We are now working on arrangements for these early volumes to be reprinted and there will be an announcement in *The Pentagon* when they are available. This will be an opportunity for any individual or for your school library to complete the entire set of *Pentagons*.

Let me close by pointing out that *The Pentagon* can exist only if people will write for it. I urge students and faculty to send in your

best papers and to contribute solutions to The Problem Corner. It takes a large amount of effort and careful attention to minute details to prepare a good manuscript for publication, but the rewards of seeing your work in print are great.

Fred W. Lott, Jr.

REPORT OF THE BUSINESS MANAGER
OF THE PENTAGON

This is the fourth report I have given concerning the activities, policies, and duties of the business manager of *The Pentagon* to a biennial convention of Kappa Mu Epsilon. Perhaps because I teach statistics, the previous reports have primarily been concerned with numerical facts concerning our official journal. In the process of mailing over 11,300 *Pentagons* since I last reported to this convention, some interesting observations can be made about these mailings. For example, these *Pentagons* went to all of the fifty states except Idaho. Our official journal was sent to Argentina, Taiwan, Holland, New Zealand, Venezuela, Canada, British West Indies, Tunisia, Syrian Arab Republic, England, Hong Kong, and Germany. More of these eleven thousand plus *Pentagons* went to Illinois than any other state. Missouri, Kansas, and New York, in that order, followed Illinois in the number of *Pentagons* mailed to that state. Over one-fourth of the *Pentagons* mailed went to the above four states.

I feel that some comments about policies of the business manager are appropriate. *The Pentagon* is mailed in the latter part of May and December of each year. This is to enable subscribers of *The Pentagon*, which is mailed to home addresses, to receive their copies of our journal while on vacation from their respective schools. For students who are initiated during the summer or winter this would mean a considerable time lapse between initiation and receipt of their first *Pentagon*. For this reason I order from the printer two or three hundred more copies than are required for the subscribers on file at the time of publication. These extra magazines then are mailed to initiates as soon as cards are received from the national secretary. When the number of *Pentagons* for a given issue that are in the hands of the business manager reaches approximately fifty, new initiates then must wait until the next printing to receive their first copy of our journal. The fifty remaining copies of each issue are put into reserve so that requests for back issues of *The Pentagon* may be filled. The inside front cover of each issue of *The Pentagon*

lists those back copies which are available from the business manager.

It is the policy of the National Council of Kappa Mu Epsilon that the library of each college or university which has an active chapter shall receive a complimentary copy of the official journal. Each student who has spoken to this fifteenth biennial convention will automatically have his subscription to *The Pentagon* extended two years. Complimentary copies of *The Pentagon* are also sent to institutions of higher education which express an interest in chartering a chapter of Kappa Mu Epsilon. Five complimentary copies are sent to each author of an article which appears in the current issue.

One duty of the Business Manager is to remove the subscription cards from the files for each subscriber whose magazine was undeliverable because of an incorrect address. The inside front cover of *The Pentagon* carries a statement that copies lost because of failure to notify the Business Manager of a change of address cannot be replaced. Over six per cent of the total income of the office of the Business Manager was expended in the past two years to pay return postage on some five hundred journals. This is not only costly in terms of time and money for this office, but also each returned *Pentagon* represents a magazine someone paid for but did not receive. I would ask that each of you in attendance at this convention please stress to the members of your chapter the necessity of making sure the business manager has a current address for each subscriber.

Our editor, Dr. Lott, and the associate editors do an outstanding job of publishing a fine issue each time. It is my privilege to serve you by aiding in the distribution of *The Pentagon* to its many readers.

Wilbur J. Waggoner

REPORT OF THE NATIONAL HISTORIAN

Two years ago we succeeded Professor Frank Gentry as national historian. At that time he transferred the historical file to our office. An examination of the file reveals that it is fairly complete. There is a complete set of *The Pentagon* and a file on each chapter, active or not, of Kappa Mu Epsilon. The chapter files vary a great deal in their contents from very complete to very little.

We have maintained the practices recently revived by Profes-

sor Gentry, of soliciting news items annually from each of the chapters. If every chapter responded, we would likely be required to radically edit the news items before including them in *The Pentagon*. However, to date we have been able to print the material provided by the secretary essentially as it is submitted since only about one-half of the chapters provide us with material for inclusion in the journal. We would encourage more chapters to return the questionnaire we mail to you, not only that we may print the news items about the local chapters but these become a part of the permanent file of each chapter.

I would like to express appreciation to the National Secretary, Laura Greene, for her cooperation and assistance throughout the biennium.

J. D. Haggard



(Continued from page 24.)

Is this theorem true? Can you prove it, and generalize to other functions than $\log x$? Is

$$\left[\begin{matrix} a_1 & a_2 & \dots & a_n \\ x_1 & x_2 & \dots & x_n \end{matrix} \right]^{1/(a_1 + a_2 + \dots + a_n)}$$

$$\cong (a_1x_1 + a_2x_2 + \dots + a_nx_n)/(a_1 + a_2 + \dots + a_n)$$

true?

FINANCIAL REPORT OF THE NATIONAL TREASURER**April 1, 1963 to April 20, 1965**

Cash on hand April 1, 1963		\$ 8,789.01	
Receipts			
Initiates (2364 at \$5.00)	\$11,820.00		
Miscellaneous (Supplies, Jewelry, Installations, etc.)	<u>2,579.95</u>		
Total Receipts from Chapters		\$14,399.95	
Miscellaneous Receipts			
Interest on Bonds	187.92		
Balfour Company (Commissions)	34.80		
Pentagon (Surplus)	157.74		
So. Carolina State (Escrow)	50.00		
Protest Charge	<u>7.00</u>		
Total Miscellaneous Receipts		<u>437.46</u>	
Total Receipts			<u>14,837.41</u>
Total Receipts Plus Cash on Hand			<u>\$23,626.42</u>
Expenditures			
National Convention, 1963			
Paid to Chapter Delegates	\$ 2,555.04		
Officers Expenses	602.96		
Miscellaneous (Speaker, Prizes, etc.)	100.50		
Host Chapter	<u>500.94</u>		
Total National Convention		3,759.44	
Balfour Company (Memberships, Certifi- cates, Stationery, etc.)		4,267.36	
Pentagon (Printing, mailing of 4 issues)		5,641.96	
Installation Expense		64.20	
National Office Expense		614.09	
2 Regional Conventions		<u>200.00</u>	
Total Expense			14,547.05
Cash Balance on Hand April 20, 1965			<u>9,079.37</u>
Total Expenditures Plus Cash on Hand			<u>\$23,626.42</u>
Bonds on Hand			
April 20, 1965	3,000.00		
Savings Account + 277.54 int.	<u>3,243.94</u>		
Total Assets as of April 20, 1965		6,243.94	
Total Assets 1963			15,323.31
Net Gain for Period			<u>14,755.41</u>
			\$567.90

Walter C. Butler



Fifteenth Biennial KME Convention, Colorado State University, Fort Collins, April 25-27, 1965

