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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Biological Growth and Related Mathematical Principles*

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The problems arising from the numerical phenomena of growth are of great importance and illustrate clearly a useful way of applying and adapting general mathematical methods in approaching a specific problem. Many of the methods and results based on growth have more far-reaching applications and implications than presented here.

There are three quantitative aspects of growth in which the zoologist is especially interested. The first, which is the basic form of the growth problem, is concerned with the change over time in some dimension of an animal. As an animal matures and ages, weight, surface area, and length of various organs change in a fairly regular way and this change with time can be thought of as growth in its strictest sense although currently growth has come to mean increase in size of cells, increase in mass of cells, mitosis, migration of cells and other connotations.

Growth is not always positive, for in the aging process, some organs may decrease or even disappear. This is the case with the thymus gland in mammals which is most prominent during puberty and sexual maturity and then slowly atrophies in later life.

The basic measurable aspect of growth from which the others are derived is the change in a single dimension of an animal over time.

The second problem of growth is that of the relative sizes of two dimensions in one animal. If there is a functional relationship between the magnitude of each of two dimensions and time, there must be also some function relating the two dimensions with each other, the time factor being constant. For example:

$$\text{Length} = 4 \times \text{age}$$

$$\text{Width} = 3 \times \text{age}$$

$$\text{Length} = 4/3 \text{ width}$$

Growth relations are never this simple but this serves to dem-

* A paper presented at a Regional Convention of KME at St. Mary's Lake, Michigan on April 28, 1962.

onstrate the point. Time can be eliminated only because it is assumed to be the same for both length and width measurements and this is possible only if both dimensions are measured at the same time on the same individual or on two individuals of the same age.

The third aspect of growth concerns changes of shape and this is the most difficult to attack for shape cannot be defined by single numbers except in some very unusual cases. In only the simplest cases can form be expressed numerically. For example, the change from a circular to ellipsoidal form can be expressed numerically by noting the relative lengths of the major and minor axes of the ellipse. When these are equal, the form is circular and as they become more and more unequal, the ellipse becomes flatter.

The simplest kind of growth is additive change or growth by accretion. A constant amount is added to the organ, regardless of the organ size, in each unit of time.

The relationship between organ sizes and time can be expressed by the following equation:

$$Y_t = Y_0 + Kt$$

Y_0 = initial size of organ

K = additive growth rate

t = elapsed time

Y_t = final size of organ at end of time period

One must assume, when considering the additive growth process, that the new material laid down does not in itself grow but is simply added at a constant rate. This occurs in the growth of such vertebrate organs as feathers, scales, hair, teeth etc. and among the invertebrates such as in the growth of shells of mollusks.

The simple equation above is not entirely suitable unless it applies to relatively short time periods, for animals do not continue to grow throughout their entire lives, for in animals growth is considered to be determinate. Growth definitely slows in later life and may halt completely long before death. Another objection to the simple additive equation above is that if the linear dimensions of an organ grow by an additive process, the area and mass do not and vice versa, for mass varies as the cube of linear dimensions and area as the square. If growth in linear dimensions is expressed by

$$L_t = L_0 + Kt$$

then growth in area must be expressed by

$$A_t = (\sqrt{A_0} + Kt)^2$$

and growth in mass by

$$M_t = (\sqrt[3]{M_0} + Kt)^3.$$

Even these relations are not unfailingly valid for in the growth of hair, the increment is in length, not in the other linear dimensions. Thus, in this case, mass and area increase in the same way as does length. In some cases, mass, rather than length, is subject to a constant increment. Thus,

$$M_t = M_0 + Kt$$

but

$$L_t^3 = L_0^3 + Kt.$$

One can see that each situation must be considered in the light of its particular biology of growth and no general rule can be given.

Most growth is of a nonadditive nature. In many cases, one can assume that changes in size are functions of the size itself, larger size being subject to larger changes. This is the case when the living material added by growth begins itself to grow immediately. Then the increase (or increment) at any moment would depend upon the amount of material already present and the size of the increments would increase continuously from instant to instant. If this pattern continued in a regular way, the size of the organ at any time could be expressed most easily by

$$\log_e X_t = \log_e X_0 + K_p t$$

where

\log_e = natural logarithm

K_p = instantaneous or geometric growth rate

X_0 = value of variable at beginning of growth period in question

X_t = value of variable at end of growth period in question

This can also be expressed as

$$X_t = X_0 e^{K_p t}$$

in which case the constant K_p is referred to as the exponential

growth rate. The logarithmic form is the most useful for it is essentially a line of the type

$$Y = b + at$$

where $\log_e X_t = Y$, $\log_e X_0 = b$ and $K_g = a$.

Again, as in the case of arithmetic growth, the geometric growth rate cannot be expected to remain constant throughout the entire life of the organism.

It is necessary to distinguish between geometric increment and geometric rate of growth for the two are quite different yet often confused.

For example, if an organ grows from 50 to 60 grams in 10 days, it has increased in weight by 20%, its geometric increment. However, contrary to expectation, it has not been growing at 20% per 10 days nor at 2% per day. If the increment was added instantaneously at the end of each day, one would see that the result of a geometric, compound-interest rate of 2% per day would cause 50 grams to increase to nearly 61 grams, not 60 in 10 days. This can be compared to interest of 2% per day compounded daily. However, the above example is discontinuous growth and growth is not discontinuous or compounded daily but is continuous. This is as if the interest was continuously due, was paid instantaneously and immediately became capital and started paying interest itself.

One clear example of this geometric growth is the number of cells in a bacterial colony growing on an unlimited food supply. A single cell divides giving rise to two cells each of which in turn divides into two, the resulting four cells dividing into two, etc. Each new cell that is added to the population itself adds two new cells. Figure 1 shows this sort of growth, the logarithm of the number of cells being plotted against time. The figure shows this relationship is rectilinear which is to be expected of the geometric growth function

$$\log X_t = \log X_0 + K_g t.$$

The slope of the line in Figure 1 is the value of K_g and the point of intersection of the line with the Y-axis is the value of $\log X_0$ (Y-intercept form of a straight line).

This example also is as inadequate as the additive example in representing a true picture of growth. The same objections apply. The last point in Figure 1 demonstrates one of these objections. The number of living cells in the culture has stopped increasing because

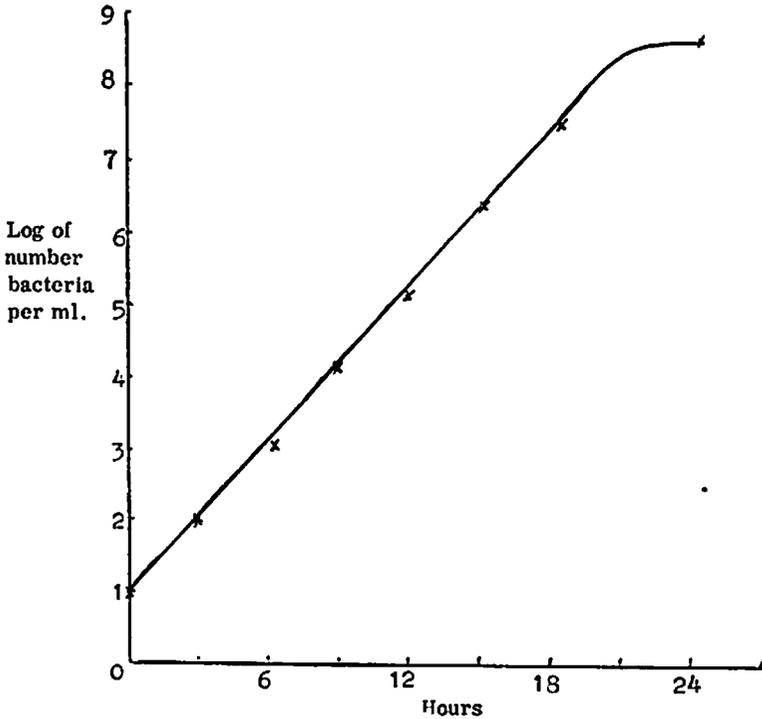


Fig. 1:
Example of almost perfect geometric growth based on actual data.

of a limited food supply. K_p , the instantaneous rate of increase, is not constant in higher organisms as well as in bacterial colonies.

K_p is approximately constant over short periods of time and often for longer periods of time in embryonic or early postnatal life but it is infrequently even roughly constant throughout life. It drops as the organism approaches its ultimate size. Even if growth is continuous throughout life, the rate becomes much less. Constant multiplicative growth leads eventually to such huge increments that a decrease in growth rate is almost inevitable. For example, if an animal weighed 10 pounds at birth, and grew steadily at 1% per day, in 10 years time it would weigh more than 70,000,000,000,000,000 lbs. Yet, most animals grow faster than 1% per day in early life.

K_p can also be negative. This is an indication of a decrease in the size of an organ. If the geometric growth rate were constant throughout life, a growth curve would appear as in Figure 1 excluding the last point. However, because of the lack of constancy of K_p , the logarithmic plot of size against time will not be strictly rectilinear but will level off as the last part of the curve in Figure 1 does.

A curve representing an arithmetic plot of size against time (Figure 2) is concave downward, rising very steeply at first and leveling off gradually until it becomes horizontal or nearly so in adult life. However, if the whole growth curve is considered from the

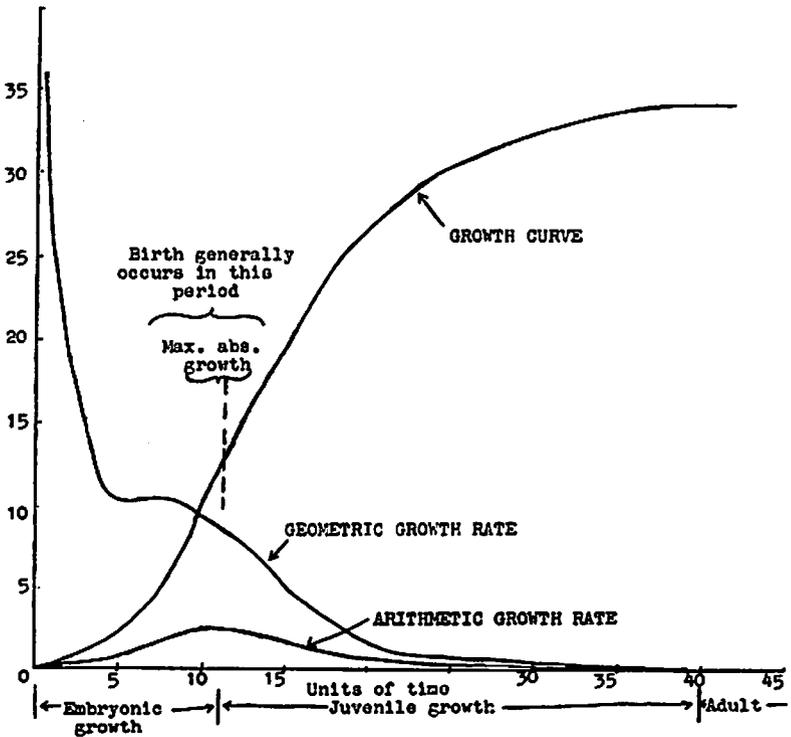


Fig. 2:
Generalized and idealized curves of growth and growth rates. Based on many different sets of data.

fertilized egg through embryonic development and juvenile growth, the early part of the curve will be seen to be concave upward, starting nearly horizontal and curving upward becoming quite steep and finally running into the postnatal curve. The point of inflection where the concavity changes from upward to downward is the point at which the increment per unit time—the arithmetic growth rate—is greatest. Birth usually occurs about this time. Therefore, the earlier part of the curve with increments increasing very rapidly is mainly embryonic while the later part with increments decreasing more slowly is mainly postnatal.

The arithmetic growth rate begins at zero, rises to a maximum at the time of birth and falls off, eventually reaching zero when the animal stops growing. This curve is skewed because the period of accelerating arithmetic growth is generally much shorter than the period of decelerating growth so the peak of the curve is far to the left of the midpoint.

The curve of the geometric growth rate usually follows a different course. The rate is often very high during the earliest stages of embryonic development but it then falls very rapidly and levels out so the rate is relatively constant or fluctuates about a nearly horizontal line through much of the embryonic period. Near the time of greatest arithmetic growth, the graph of the geometric growth rate normally curves downward again and falls off, at first rapidly and then more slowly until it reaches zero when maximum size is attained.

Frequently, two dimensions of organs or an organism grow in such a way that the ratio between their geometric growth rates remains approximately constant over long periods of time. This relationship may be expressed by the equation

$$\frac{K_{gY}}{K_{gX}} = \alpha$$

K_{gY} and K_{gX} are two geometric growth rates and α is their ratio. If α is constant over the time period considered, the relationship between the two dimensions, X and Y , can be expressed by the function

$$Y = bX^\alpha$$

This is called the "allometric equation." X and Y represent the dimensions being considered, α is the ratio of geometric rates and b is another constant specific to the dimensions in question. This con-

stant, b , has a clear mathematical meaning: it is the value of Y when X is unity or the ratio of Y and X when α is unity. However, the biological interpretation of this constant is not so clear. It is related obscurely to the basic size differences between X and Y and is sometimes referred to as the "initial growth constant."

This relationship between Y and X can apply either to dimensions of one organism at a given time or to the same dimension of two organisms of the same age. It is more widely applicable to growth relationships than the geometric growth function itself. Supposedly, this is so because the allometric equation applies when α , the ratio between the two geometric rates, is relatively constant while the geometric growth curves require that the rates themselves be constant.

$K_{\rho X}$ and $K_{\rho Y}$ change markedly during growth and thereby invalidate the geometric growth curves as a descriptive function. However, if the ratio, $K_{\rho X}/K_{\rho Y}$, remains fairly constant, the allometric equation adequately describes all cases where the geometric function applies to both variables. Also, in many cases where the geometric or arithmetic growth curves do not apply because of changing values of K , the allometric equation can be applied.

The allometric equation, as with other growth equations, is applied for purely descriptive purposes. Whether the resulting description is useful and adequate in any given case depends on how well the allometric equation fits the particular data.

The constant α is generally referred to as the constant of allometry. If it equals unity, the two dimensions have the same geometric growth rate and there will be a constant ratio between them no matter what their sizes.

When $\alpha = 1$, $Y = bX$ or $Y/X = b$.

Two dimensions which maintain constant size ratios are said to grow "isometrically," "isomorphically," or "isogonically." However, these simple situations do not frequently occur as is often the case with biology. Dimensions whose ratio is constantly changing are said to grow "allometrically," "allomorphically," or "heterogonically."

One should note that in allometric growth, the ratio of the dimensions, X/Y , is constantly changing while the ratio of rates, $K_{\rho X}/K_{\rho Y}$, is constant.

Deviations from the simple allometric relation may be of two kinds. First, α may change continuously during growth so that one dimension is growing at a greater geometric rate than the other

initially but as time elapses, the rates become more equal and may even reverse their relationship. This happens when one organ or dimension stops growing before the other. Second, there may be an acute change in the constant of allometry, α , at an important period of development in the life cycle such as when moulting occurs in arthropods and a distinct change in the relative growth rates of different structures may occur.

The aspect of growth concerning changes in shape has no standard system of analysis as there is for allometric growth. For one thing, it is not entirely clear what is meant by shape. In a very simple case, it is a change in relative sizes of two dimensions. In this case, shape can be treated by the method of allometric growth. This method, however, describes the relation between two dimensions only while a shape change is often best described in terms of several dimensions simultaneously. In such cases, the use of deformed or transformed coordinates is necessary. This method and others dealing with shape change will not be dealt with here.

All differences between animals, whether morphological, physiological or behavioral, are material that can be used by taxonomists. Differences in patterns of growth are valid materials upon which to base systematic conclusions as are also differences in color or structure. Dimensions of organs or organisms are the main characters of use to the taxonomist. He must, therefore, consider the relationship between age and size in order to make correct biological inferences from results of statistics. For most animals, age classes are difficult to recognize qualitatively and samples are more often than not heterogeneous to an unknown degree. In samples such as these, the only valid method of comparison is based on the age-size relationship itself which is the growth curve. Comparing the growth relationships of different samples from several populations may serve two purposes. First, the growth curve is in itself a physiological character useful as a guide to taxonomic inferences and, second, the growth curve of a certain dimension may be the only true means of comparing the dimension from sample to sample because of the ever-present heterogeneity of the samples.

Although this discussion is far from complete, deals with only one biological aspect and barely scratches the surface of this, it gives a small insight into the interrelationship between biology and mathematics.

Indirect Proof

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Indirect reasoning, as defined by W. Stanley Jevons, is "that which points out what a thing is by showing that it cannot be anything else." [5, p. 81]*. This is like showing a certain country on a map by coloring all that it is not. Lines can be proved equal by showing that it is impossible for them to be unequal; or lines must meet in a certain point because it is impossible for them to meet elsewhere.

When given information suggests two or more reasonable conclusions, each conclusion must be compared with the premises. Any conclusion which violates one or more of the premises must be rejected. After all of the contradictory statements have been rejected, the one that remains is true.

This is the nature of indirect proof. A statement is proved true by showing that all of its contradictions are false. And a contradiction is false if it is contrary to the available data, an axiom, or theorem.

Augustus DeMorgan, in his *Study of Mathematics*, states that "there are many propositions in which the only possible result is one of two things which cannot be true at the same time, and it is more easy to show that one is *not* the truth than that the other *is*." [2, p. 226].

Although many people fail to understand and recognize indirect reasoning as such, it is commonly used in science, law, medicine, and many other areas. There are also a great number of examples from everyday life.

For instance, a man found that the radiator in his office continued to emit heat after it was shut off. He complained of a faulty valve. The reasoning which led him to the conclusion that the fault was in the valve was: the valve is shut off and heat still comes through; the valve is either good or bad; if it were good heat would not come through; therefore, the valve must be faulty.

The FBI used indirect reasoning in solving a kidnap case. They took the victim's description of the house and location where he had been held and determined all of the places where it could not be. The

* The first number in brackets refers to that work in the references at the end of this article and the second number is the page on which the quotation may be found.

information included a prairie landscape and two airliners which passed over the house at certain times of the day. These conditions could apply to many locations, but when all was considered together, only one place did not contradict the facts, and the others were rejected. The FBI checked this house and found their kidnapper.

If a statement is true, its contrapositive must also be true. This statement can be proved by reasoning indirectly. For example: assume the truth of the statement—"a metal is an element". The contrapositive—"a non-element is non-metal"—must also hold true. The only possibilities for a non-metal are that it be metallic or non-metallic.

The statement—"a non-element is a metal"—leads to a contradiction because metal has already been qualified as an element. Therefore, the only non-contradictory possibility is that a non-element be a non-metal.

Let p represent any statement, and q any other statement. Let $\sim p$ and $\sim q$ represent the contradictions of these statements. Below is a table to represent the truth values of various combinations of these two statements. From the table it may be seen that, when a statement is true, its contradiction is false; the conjunction of two true statements is true; the conjunction of two false statements is false; and the conjunction of a true statement with a false statement is false.

p	q	$\sim p$	$\sim q$	p and q	$\sim p$ and $\sim q$	$\sim p$ and q	p and $\sim q$
T	T	F	F	T	F	F	F
T	F	F	T	F	F	F	T
F	T	T	F	F	F	T	F
F	F	T	T	F	T	F	F

Looking back at the previous example, p would be "it is metal" and q would be "it is an element". If both p and q were true, then p and q —"it is metal and it is an element"—is their conjunction and must also be true. But "it is metal and it is not an element" would be the conjunction of a truth with an untruth and would therefore be false.

The form for indirect proof is adaptable to any number of premises and terms. As the number of premises and terms increases,

symbols may be substituted, and the possible alternatives may be listed and all contradictions eliminated, leaving only the true case.

Alternatives may be listed by the process of exhaustive division. For example, take the statement—"all books are English or scientific or both". Let E and S stand for English and scientific respectively, and $\sim E$ and $\sim S$ for the negation of each. All of the possible arrangements of the statements regarding these books are E and S (this book is English and this book is scientific); E and $\sim S$ (this book is English and not scientific); $\sim E$ and S (this book is not English and this book is scientific); and $\sim E$ and $\sim S$ (this book is not English and this book is not scientific). There is no book which does not fit into one of these classifications.

If it can be said that: iron is a metal; and all metals are elements, it can be proved indirectly, that iron is an element. Let F be "it is iron"; M be "it is metal"; and E be "it is an element". The possibilities for iron are: F and M and E ; F and (M and $\sim E$); (F and $\sim M$) and E ; (F and $\sim M$) and $\sim E$. By the truth table, the last two are false because, by hypothesis, (F and $\sim M$) is false. The second is false because (M and $\sim E$) is false.

The only remaining possibility, then is F and M and E which is that which is iron and metal and element all at the same time. Thus, (F and E) is true and the statement is proved.

To determine the nature of a non-element, the pertinent possibilities are: ($\sim E$ and M and F); $\sim E$ and (F and $\sim M$); $\sim E$ and ($\sim F$ and M); $\sim E$ and $\sim F$ and $\sim M$. Again eliminating the contradictions, by the table, the only remaining combination is $\sim E$ and $\sim F$ and $\sim M$ which shows that a non-element is also a non-metal and not iron.

Two of the basic laws of logic which make the above method of indirect proof possible are the Law of Contradiction and the Law of the Excluded Middle. The first law states that a thing cannot both be and not be. Since a metal cannot be an element and a non-element at the same time, all of the above cases which said that the object was metal and non-elemental at the same time violate this law and are untrue.

The Law of the Excluded Middle states that a thing must either be or not be. Thus it is possible to list all of the cases cited because iron must either be an element or not be an element. There is no alternative. By combining the principles of both of these laws, it was possible to eliminate all but the one true case concerning iron, metal, and element.

When more than two alternatives are possible, the conclusion is reached by the method called *exclusion* or *exhaustion*.

This pattern of an indirect method of reasoning begins when all of the possible alternatives are considered. Then all descriptions of the problem according to the premises are listed. Every description which violates the Law of Contradiction may then be discarded, and the remaining terms form the desired, true description.

When only two alternatives are possible, the *reductio ad absurdum* method is used. This is done by assuming the contradiction to a statement is true and showing that this leads to an absurd conclusion. There are three principles upon which this method is based: I. The Excluded Middle (a thing must either be or not be); II. if one of two contradictory statements is true, the other must be false and *vice versa*; III. if a correct reasoning pattern leads to a false conclusion, the premises must be false [1, p. 109].

For example, the proof of the statement "two lines which are perpendicular to the same line are parallel to each other" is most easily proved by the method of *reductio ad absurdum*.

By the law of the Excluded Middle, the lines must be either parallel or not parallel. The two contradictory statements are: "the lines are parallel" and "the lines are not parallel". One or the other of these statements must be true.

If the two lines, A and B, are assumed to be non-parallel, they must meet in some point O. But this is absurd because a previous theorem of Euclid's states that only one line from a point can be perpendicular to the line. Therefore, the lines cannot possibly be non-parallel, and by the third principle, the premises of non-parallelism must be false. The second principle guarantees the truth of the contradictory statement and the lines must therefore be parallel.

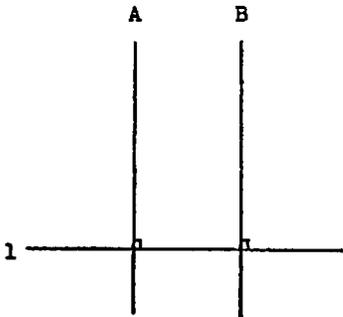


Fig. 1

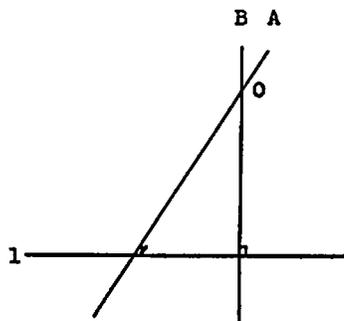


Fig. 2

In many cases, an indirect proof for a statement is much more concise and requires considerably less effort than a direct proof would. Consider again the example of the faulty steam valve. The valve had to be either good or faulty (principle I). If it had been good it would not have allowed steam to pass through. Therefore, it was faulty (principle III). A direct proof would have necessitated removing the valve and examining it.

To prove, indirectly, that "when two lines are cut by a transversal such that the alternate interior angles are equal, the lines are parallel", it is again necessary to assume the contradicting statement, *i.e.* "when the angles are equal, the lines are not parallel.", and show that it again leads to an impossible situation.

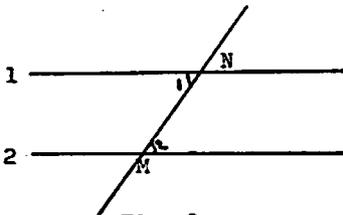


Fig. 3

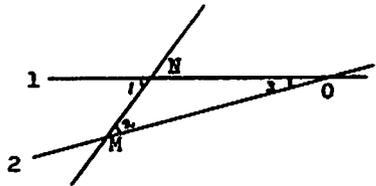


Fig. 4

If lines 1 and 2 were not parallel, they would meet at some point O as in Figure 4, thus forming triangle OMN . But this is absurd because the exterior angles of a triangle must be equal to the sum of the opposite interior angles. That is, angle 1 must be equal to angle 2 plus angle 3 in triangle OMN . Since angle 1 is equal to angle 2 by hypothesis, it could not possibly be equal to angle 2 plus another angle. The true statement is that the lines are parallel.

Even though it is a very useful method of logical reasoning, indirect proof has never been widely accepted for use in a geometry course. Many textbooks mention it and devote two or three paragraphs to it, and use it in a few examples, and then return to direct proofs for the remainder of the book.

Many teachers hesitate to teach indirect proof. They feel that it "beats around the bush" in considering what is not even wanted as a truth. They say that it often seems too absurd to even consider a contradictory statement which seems so obviously false. It is too much like reasoning in a circle". There is a preference for a direct

approach to the proof of a true theorem; it just doesn't seem natural to many of them to *want* something to be untrue.

Other objections to indirect reasoning are that it encourages many students to simply memorize the material; teachers too often don't understand it; it is too complex for students to grasp; students do not know when to apply it; it seems to be dodging the issue; and it requires an inaccurate figure to represent false assumptions [1, p. 103].

Augustus DeMorgan declared that "indirect proof is as logical as direct proof, but not of so simple a kind, hence it is desirable to use direct proof whenever it can be obtained." [1, p. 103].

In the face of these many objections (and there are certainly more than these) it seems a small wonder that indirect proof is even mentioned in a geometry course. But this is part of the difficulty. If it were treated adequately, it would be a positive addition to a geometry. At present it is given too little consideration—95% of the current books are devoted to direct proof [1, p. 122]. It is also often presented too early and this creates a negative attitude among the students.

One reason that indirect proof should be included in a course of geometry, is that many of the things which must be assumed as true at the beginning of the course, can later be proved by indirect reasoning. This gives a more comprehensive understanding of the logical system.

Augustus De Morgan suggests that a pupil experiences some difficulty because he has to assume, as being true, the very thing that he wishes to destroy. One way to get around this difficulty is to state the assumption as a hypothetical case as "would be" instead of "is"; that is, instead of "assume the lines are not parallel," say "what if the lines were not parallel".

Then, instead of trying to see the impossible situation on the figure being used, a new diagram should be introduced, as in the case of the parallel lines and the transversal (Figs. 3 & 4). The students would be forced to not rely on the figure but to use logical reasoning.

The National Council of Teachers of Mathematics Yearbook for year 1930 makes the statement that "it is impossible to develop a syllabus of propositions which can all be proved directly. Euclid used indirect proof eleven times in book I." [1, p. 103].

W. Stanley Jevons was a strong supporter of indirect methods

of proof. He said: "Some philosophers . . . have held that the indirect method of proof has a certain inferiority to the direct method. . . . But there are many truths which we can prove only indirectly. . . . A number is prime . . . the side and diagonal of a square are incommensurable. . . . I believe that nearly half our logical conclusions rest upon its employment." [5, p. 104].

A better understanding of indirect proof leads to a better understanding of an independent, consistent logical system.

Many theorems are too difficult to prove directly and are listed among the assumptions for the system. Many of these could be proved later by indirect reasoning and the system would be more independent.

It is necessary for an axiom system to be consistent for an indirect proof to work. In an inconsistent system, both of the contradicting statements may be true or false at the same time. Conversely, if two contradictory statements can both be proved or disproved, the system is not consistent. Indirect proof is an important part of logic. As a method of analysis, it assumes the contradiction and proceeds forward to a conflict with the data. It can be a powerful tool, but people must learn to accept it and to use it properly.

REFERENCES

- [1] The National Council of Teachers of Mathematics, fifth year-book, *The Teaching of Geometry*, Teachers College, Columbia University, 1930.
- [2] DeMorgan, Augustus, *The Study of Mathematics*, The Open Court Publishing Company, Chicago, 1898.
- [3] Smith, Folland R., *Beginners Geometry*, the MacMillan Company Limited, London, 1925.
- [4] Jevons, W. Stanley, *Elementary Lessons in Logic*, the MacMillan Company Limited, London, 1957.
- [5] Jevons, W. Stanley, *The Principles of Science*, the MacMillan Company Limited, London, 1920.
- [6] Shute, William G., William W. Shirk, George F. Porter, *Plane and Solid Geometry*, American Book Company, New York, 1957.
- [7] Weichons, A. M., W. R. Krickenberg, Helen R. Pearson, *Plane Geometry*, Ginn and Company, New York, 1958.

A Look at Einstein's Theory of Relativity*

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The name, Albert Einstein, calls to mind the great intellectual effort and genius which overturned the most traditional notions of physics and culminated in the establishment of the relativity of the notions of space and time, the inertia of energy, and an interpretation of gravitational forces which is in some ways purely geometrical. In his later years, he attempted to develop a unified-field theory relating gravity, electro-dynamics, and atomic structure. Though unsuccessful here, his personal achievements rank with Newton's. In 1905, he presented his special theory of relativity. It is my intention to discuss the special theory using his book, *Relativity, The Special and General Theory*¹ as source.

The special theory of relativity deals essentially with the apparent effects upon a body in motion in a relative sense.

Within the theory of relativity, and for that matter within reality itself, every description of an event in space involves the use of a rigid body (body of reference) to which that event must be referred. Of course, in order to have a precise location of an event in space, the time must be given also. A four-magnitude system of coordinates, a Cartesian system, is adopted by the special theory. In order to get the concept of a reference body well in mind, let us use this room as one. We can obtain a complete description of where a certain fly is, by saying that the fly is six feet from the floor, three feet from the east wall and ten feet from the north wall at such and such a time.

The concept of motion in space has to be replaced by the idea of motion relative to a particular body of reference. There is no such thing as an independently existing trajectory or path-curve, but only a trajectory relative to a particular body of reference or co-ordinate system. If we have a train passing beneath a bridge and an observer within a car and one on the ground under the bridge, we can easily observe this. At a certain instant, a stone is dropped from the bridge. To the observed on the ground, the stone describes a straight line,

* A paper presented at a Regional Convention of KME at St. Mary's Lake, Michigan on April 28, 1962.

¹ Einstein, Albert E., *Relativity, The Special and General Theory*. Crown Publishers Inc., New York, 1961.

but to an observer on the train, watching the stone fall near the window, the stone describes a portion of a parabola as it seems to drift backwards along the train. Thus, when we say motion, we must say motion with respect to a specified reference system.

In addition to this, we need to know that every reference body has its own particular time. Unless we are told the reference body to which the statement of time refers, there is no meaning in a statement of the time or an event.

The general principle of relativity in the restricted sense can be expressed abstractly in this manner: "If, relative to K , K' is a uniformly moving co-ordinate system devoid of rotation, then natural phenomena run their course with respect to K' according to exactly the same general laws as with respect to K ."² If we could find the relationship between the variables x , y , z , and t in both systems, the general laws of nature would be co-variant or would vary within each system in the same manner. A simple example of the relationships that the principle would bring about is this: if an object were moving uniformly in a straight line in the system K , it would also be moving uniformly in a straight line in K' .

If an object possessed motion within a moving system in the direction of the movement of the system with respect to another, classical physics assumed that the motion of the object with respect to the latter system would be the sum of the velocity of the object with respect to the first system and the velocity of that system with respect to the other. Classical physics would assume that the motion, with respect to the ground, of a man walking inside a train in the direction of its motion would be the sum of his walking speed and the speed of the train.

If this concept is true, then the speed of the impact of light on two co-ordinate systems in motion with respect to each other, would not be a constant, but would vary as the systems moved toward and away from the source. However, there is conclusive proof, the Michaelson-Morley experiment, that no matter at what speed a body is moving away from or towards a source, the speed of impact of light is a constant, 186,000 miles per second.

At this point it seems that we may have reached an impasse where the principle of relativity clashes head on with the law of the propagation of light. But here is where Einstein overthrew two basic hypotheses of classical physics.

² *Ibid.*, page 13.

According to Einstein, classical physics has handed down two mistaken hypotheses. They are these: "(1). The time interval between two events is independent of the condition of motion of the body of reference. (2). The space interval (distance) between two points of a rigid body is independent of the condition of motion of the body of reference."³

If we drop these hypotheses, we have no dilemma concerning the propagation of light. What we need is a relationship between the co-ordinates of two systems in motion with respect to each other, such that c , the speed of light, simultaneously will be equal to x/t and x'/t' . The man of the hour in this case was H. A. Lorentz, who formulated the Lorentz transformation, the key to the mathematics of relativity. If K' is in motion with respect to K at velocity v ,

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} & z' &= z \\ y' &= y & t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}$$

Let us test these equations. We know that c equals the transmission velocity of light along x .

Thus if we let x equal ct , where ct is the distance light travels in time t .

$$\begin{aligned} x' &= \frac{ct - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t(c - v)}{\sqrt{1 - \frac{v^2}{c^2}}} \\ t' &= \frac{t - \frac{vct}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t\left(\frac{c - v}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t}{c} \cdot \frac{(c - v)}{\sqrt{1 - \frac{v^2}{c^2}}} \\ \frac{x'}{t'} &= \frac{t}{t/c} = c \end{aligned}$$

$$\therefore x' = ct'$$

³ *Ibid.*, page 3.

Thus, the velocity of the transmission of light relative to reference body K' is equal to c . But this should not be surprising since these equations were developed in order to fit into both the principle of relativity and the law of the propagation of light.

Merely by inspecting these simple formulas, one can see that c seems to be a limiting value in the Lorentz transformations. By simple application of these formulas, one can show that a rod is shorter when in motion in the direction of its length than when it is stationary. It is also easily shown that a time lapse within an object in motion is longer than an equal time lapse when it is stationary.

If a person could develop a rocket which would move him fast enough he could live for centuries and would still be in the prime of life. For back on earth time would go by just as it always had, but all his life processes would be slowed down as his speed approached the speed of light.

Applying these formulas again, we can obtain the velocity of an object moving within K' with respect to K , if K' is moving with respect to K . Or more simply we can obtain the velocities of two moving objects with respect to each other if we know their velocities with respect to one co-ordinate system.

v = the velocity of K' with respect to K .

w = the velocity of an object in motion with respect to K'
(moving along the $x-x'$ axis)

$$x' = wt'$$

$$\frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{w \left(t - \frac{vx}{c^2} \right)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x - vt = wt - \frac{wvx}{c^2}$$

$$x + \frac{wvx}{c^2} = wt + vt$$

$$x = \left(\frac{w + v}{1 + \frac{wv}{c^2}} \right) t$$

Thus the required velocity is
$$W = \frac{w + v}{1 + \frac{wv}{c^2}}$$

In application, we can now not only see why old physics' answer for the rate of speed of two objects approaching each other at velocities known with respect to another system was incorrect; but also what the correct velocity would be. For example, if there were two rockets, each approaching us from opposite directions with one half the speed of light, the velocity with respect to each other is not the speed of light, but four-fifths the speed of light because

$$W = \frac{\frac{1}{2}c + \frac{1}{2}c}{1 + \frac{\frac{1}{2}c \cdot \frac{1}{2}c}{c^2}} = \frac{c}{5/4} = \frac{4}{5}c$$

Most work with the Special Theory of Relativity has been done in the fields of electrodynamics and optics. One way in which it has helped has been in simplifying the derivation of laws and it has considerably reduced the number of independent hypotheses forming the basis of the theory.

In mechanical physics, the modification required by the theory affects only the laws for very rapid motions amounting to a large fraction of the speed of light. In accordance with the theory of relativity the kinetic energy of a material point of mass m is no longer

given by $\frac{mv^2}{2}$ but rather by $\sqrt{\frac{mc^2}{1 - \frac{v^2}{c^2}}}$.

This expression approaches infinity as the velocity v approaches the velocity of light c . The velocity therefore must always remain less than c , no matter how much energy is added.

One of the most important developments discovered is that the law of the conservation of the mass of a system becomes identical with the law of the conservation of energy. Finally we find that the energy a certain mass m possesses is equal to mc^2 , the familiar formula for the energy given off in nuclear reactions.

Experience has shown that the theory of relativity enables us to predict the effects produced on light reaching us from the fixed stars. Most experiments have involved the study of Beta-rays, electrons of high velocity and low inertia. Further, much experimenta-

tion on the laws of electromagnetism supports the theory. No studies of large scale mass have been feasible in which the velocities attained are near enough to the speed of light for the effects to be noticeable.

However, at the present time, plans, are afoot to test portions of the theory using artificial satellites. Most likely, because of the accuracy of atomic clocks, the effects of velocity upon time could be measured.

The general theory of relativity which was published in 1915 is, as one might expect, rather complicated and involved; however, I must say that it has met with much success, in that, several effects of the theory can be observed, whereas, the special theory was primarily hypothetical work.

But no matter how much conclusive evidence supports the theory of relativity, let us not forget that just as this theory improves and supersedes some of our old laws of physics, there may be someone who comes along just like Einstein to astound the world with a theory which will make the theory of relativity seem relatively inaccurate, at best.

BIBLIOGRAPHY

Einstein, Albert E., *Relativity, The Special and General Theory*.
Crown Publishers Inc., New York, 1961.



Angling may be said to be so like the mathematics that it can never be fully learnt.

—IZAAB WALTON

A Relationship Between Determinants and Progressions

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Any determinant of the second order whose elements, reading across successive rows, are the terms of an arithmetic progression will have the value of the progression's common difference squared multiplied by minus two.

Proof: Let $a, a + d, a + 2d, a + 3d, \dots$ be any arithmetic progression whose first term is a and whose common difference is d . Then

$$\begin{aligned} \begin{vmatrix} a & a + d \\ a + 2d & a + 3d \end{vmatrix} &= a(a + 3d) - (a + 2d)(a + d) \\ &= (a^2 + 3ad) - (a^2 + 3ad + 2d^2) \\ &= -2d^2 \end{aligned}$$

Any determinant of the third order and above whose elements, reading across successive rows, are the terms of an arithmetic progression will have zero as its value.

Proof: Let $a, a + d, a + 2d, \dots$ be any arithmetic progression whose first term is a and whose common difference is d . Let the determinant be of n th order with $n \geq 3$. Then the determinant can be written as

$$\begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-1)d \\ a+nd & a+(n+1)d & a+(n+2)d & \cdots & a+(2n-1)d \\ a+2nd & a+(2n+1)d & a+(2n+2)d & \cdots & a+(3n-1)d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a+(n-1)nd & a+(n^2-n+1)d & a+(n^2-n+2)d & \cdots & a+(n^2-1)d \end{vmatrix}$$

We can multiply the elements of the first column by -1 and add to those in the last column without altering the value of the determinant to obtain

$$\begin{vmatrix} a & a+d & a+2d & \cdots & (n-1)d \\ a+nd & a+(n+1)d & a+(n+2)d & \cdots & (n-1)d \\ a+2nd & a+(2n+1)d & a+(2n+2)d & \cdots & (n-1)d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a+(n-1)nd & a+(n^2-n+1)d & a+(n^2-n+2)d & \cdots & (n-1)d \end{vmatrix}$$

Again multiply the elements of the first column by -1 but now add to the elements of the second column.

$$\begin{vmatrix} a & d & a+2d & \cdots & (n-1)d \\ a+nd & d & a+(n+2)d & \cdots & (n-1)d \\ a+2nd & d & a+(2n+2)d & \cdots & (n-1)d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a+(n-1)nd & d & a+(n^2-n+2)d & \cdots & (n-1)d \end{vmatrix}$$

Removing the common factor of $n-1$ in the last column, we have

$$\begin{vmatrix} a & d & a+2d & \cdots & d \\ a+nd & d & a+(n+2)d & \cdots & d \\ a+2nd & d & a+(2n+2)d & \cdots & d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a+(n-1)nd & d & a+(n^2-n+2)d & \cdots & d \end{vmatrix} = (n-1) \cdot 0 = 0$$

since the elements of two columns are identical.

Any determinant of any order whose elements, reading across successive rows, are the terms of any geometric progression will have zero as its value.

Proof: Let a, ar, ar^2, \dots be any geometric progression whose first term is a and whose common ratio is r . Let the determinant be

of the n th order. We can remove a common factor of r^n from one row, say the third row, to obtain

$$\begin{vmatrix} a & ar & ar^2 & \dots & ar^{n-1} \\ ar^n & ar^{n+1} & ar^{n+2} & \dots & ar^{2n-1} \\ ar^{2n} & ar^{2n+1} & ar^{2n+2} & \dots & ar^{3n-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ ar^{n^2-n} & ar^{n^2-n+1} & ar^{n^2-n+2} & \dots & ar^{n^2-1} \end{vmatrix}$$

$$= r^n \begin{vmatrix} a & ar & ar^2 & \dots & ar^{n-1} \\ ar^n & ar^{n+1} & ar^{n+2} & \dots & ar^{2n-1} \\ ar^n & ar^{n+1} & ar^{n+2} & \dots & ar^{2n-1} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ ar^{n^2-n} & ar^{n^2-n+1} & ar^{n^2-n+2} & \dots & ar^{n^2-1} \end{vmatrix}$$

$$= 0$$

since the elements in two rows, the second and third, are equal.

EDITOR'S QUERY: Is there some connection between the methods used in the above article on determinants and the solution to problem E 1511 proposed in *The American Mathematical Monthly*, Vol. 69, No. 4, April 1962, page 311?

Probability in Genetics

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Probability concepts are becoming more important in a wide variety of fields. Increasing amounts of study are now being done in all the science fields, physical, biological, and social. In this report, I will attempt to delve into a part of the biological and social science field, in particular some aspects of probability in genetics.

We shall suppose some familiarity with certain basic principles in the mathematics of probability. We will use capital letters to symbolize events that may occur and write $P(A)$ for the probability that event A occurs. The compound event "either A or B occurs" is denoted with symbol $A \cup B$ while "both A and B occurs" is represented by $A \cap B$ (e.g., see [1, p. 52]*). We will need properties of probabilities such as:

(1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [1, p. 67]

(2) If A_1, A_2, \dots, A_m are mutually exclusive events, that is, no two of them may occur simultaneously, then

$$P(A_1 \cup A_2 \cup \dots \cup A_m) = P(A_1) + P(A_2) + \dots + P(A_m) \quad [1, p. 73]$$

(3) The conditional probability that A occurs given that B has occurred is given by

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \quad [1, p. 76]$$

(4) If A and B are independent events, then

$$P(A \cap B) = P(A) \cdot P(B)$$

and

$$P(A | B) = P(A)$$

[1, p. 102]

(5) If E_1, E_2, \dots, E_m are mutually exclusive events and at least one of them must occur, then for any event A ,

$$P(A) = P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + \dots + P(E_m) \cdot P(A|E_m) \quad [1, p. 91]$$

* The first number in brackets refers to the References at the end of this article; the second number denotes the page in that reference.

We shall first consider only a single gene which can have one of two forms, recessive, r , or dominate, D . This single gene, when combined with another single gene, will form a combination of either rr , rD , or DD . Since genes are supposed to occur in pairs in each individual, we could classify individuals of each generation in one of these types. Each individual would obtain one gene from the male species and one from the female species. Let us give symbols to the probability of these three types occurring. Let x_n equal the probability of rr occurring in the n th generation, y_n equal the probability of rD occurring in the n th generation, and z_n equal the probability of DD occurring in the n th generation. Since these events are mutually exclusive, (2), and at least one of them must occur, we can easily see that [1, p. 124]

$$x_n + y_n + z_n = 1.$$

First, let us look at the probability of an individual from the $(n + 1)$ st generation having a pair of genes with a certain characteristic type rr . Clearly, one can see that this individual must receive a recessive gene from both parents. This can be done in one of two ways from each parent. He can receive a recessive gene from the rr type with probability 1, or from the rD type with probability $\frac{1}{2}$. The probability of the event A of receiving a recessive gene from either parent is, by (5),

$$\begin{aligned} P(A) &= P(E_1) \cdot P(A|E_1) + P(E_2) \cdot P(A|E_2) + P(E_3) P(A|E_3) \\ &= x_n \cdot 1 + y_n \cdot \frac{1}{2} + z_n \cdot 0 \\ &= x_n + \frac{1}{2}y_n \end{aligned}$$

Likewise, the probability of the event B of receiving a recessive gene from the other parent would be the same. Since these two events would be independent, the probability of receiving a recessive gene from both parents and thus obtaining the desired rr type can be written, by (4), as

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B) \\ &= (x_n + \frac{1}{2}y_n)^2 \end{aligned}$$

Similarly, the probability of an individual from the $(n + 1)$ st generation having a certain characteristic of the type DD would be $(z_n + \frac{1}{2}y_n)^2$.

Now let us look at the probability of an individual from the $(n + 1)$ st generation having a characteristic of the type rD . He can

obtain these genes in one of two ways, getting the recessive gene from the male and the dominant gene from the female or vice versa. Let E be the event that he gets the recessive gene from the male and the dominant gene from the female, and F the event that he gets the recessive gene from the female and the dominant gene from the male. We then want to find $P(E \cup F)$. In analyzing event E , we know that the male parent can have a recessive gene in one of two types, rr or rD . From the previous work, we can see that the probability of this occurring is $x_n + \frac{1}{2}y_n$. In the same manner, we can find that the probability of the female parent having a dominant gene is $z_n + \frac{1}{2}y_n$. From this we obtain

$$P(E) = (x_n + \frac{1}{2}y_n)(z_n + \frac{1}{2}y_n).$$

We can also readily see that $P(E) = P(F)$. Therefore, by (1)

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) \\ &= 2(x_n + \frac{1}{2}y_n)(z_n + \frac{1}{2}y_n) \end{aligned}$$

since events E and F cannot occur at the same time.

Gregor Johann Mendel, in his experiments concerning inheritance, found that the DD type will usually be found $\frac{1}{4}$ of the time, the rr type will also be found $\frac{1}{4}$ of the time, and the rD type will be found $\frac{1}{2}$ of the time [2, p. 32]. By substituting these numbers for x_n , y_n , and z_n in the equations that we obtained, we can see that the probability of an individual in the $(n + 1)$ st generation having a characteristic of one of the three types is the same as that for an individual in the n th generation. This is true for individuals in the $(n + m)$ th generation.

Thus far we have considered genes which remain either dominant or recessive throughout each generation. At any time, a gene may change from a dominant characteristic to a recessive one or vice versa. This is called mutation of a gene and is something that does not occur very often [2, p. 9]. Since recessive mutations occur more often, we will consider them first. Let us say that D mutates to r with probability p . Going back to our previous problem of finding an rr type in the $(n + 1)$ st generation, we now see that a recessive gene can be received from either parent in one of four ways. He can receive a recessive gene directly from the rr type or the rD type or he can receive a dominant gene from the DD type or the rD type and then it will mutate to a recessive gene. Let us call these four events E , F , G , and H respectively.

We will then let A be the event that he gets a recessive gene from the male parent and B the event that he gets it from the female parent. We can see that

$$\begin{aligned} P(A) &= P(B) = P(E \cup F \cup G \cup H) \\ &= P(E) + P(F) + P(G) + P(H) \end{aligned}$$

since no two of the four events can occur at the same time. We already know that $P(E) = x_n$ and $P(F) = \frac{1}{2}y_n$; we must now find $P(G)$ and $P(H)$. We can see that a dominant gene can be received from the DD type with probability 1 and we know that it will mutate to a recessive gene with probability p . Similarly, a dominant gene can be received from the rD type with probability $\frac{1}{2}$ and it will mutate with probability p . Therefore,

$$P(G) = pz_n \text{ and } P(H) = \frac{1}{2}py_n.$$

Now we want to find $P(A \cap B)$. Since A and B are independent events,

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B) \\ &= [P(E) + P(F) + P(G) + P(H)]^2 \\ &= (x_n + \frac{1}{2}y_n + pz_n + \frac{1}{2}py_n)^2. \end{aligned}$$

We have supposed that a dominant gene may mutate to a recessive gene with probability p but that a recessive gene does not mutate. If, now, it is also possible for a recessive gene to mutate to a dominant one with probability q , then the probability of events E and F will need to be modified to meet the requirement that the r gene received from an rr or an rD parent does not mutate. Thus

$$P(E) = (1 - q)x_n \text{ and } P(F) = (1 - q)\frac{1}{2}y_n$$

where $1 - q$ is the probability that an r gene does not mutate, and the probability of an rr type in the $(n + 1)$ st generation now becomes

$$P(A \cap B) = [(1 - q)(x_n + \frac{1}{2}y_n) + p(z_n + \frac{1}{2}y_n)]^2$$

From $x_n + y_n + z_n = 1$, we see that $z_n + \frac{1}{2}y_n = 1 - (x_n + \frac{1}{2}y_n)$. Making this substitution and simplifying, we obtain

$$P(A \cap B) = [(x_n + \frac{1}{2}y_n)(1 - p - q) + p]^2.$$

In like manner, we can find that the probability of a DD type

occurring in the $(n + 1)$ st generation is

$$[(z_n + \frac{1}{2}y_n)(1 - p - q) + q]^2.$$

Now let us find the probability of obtaining an individual from the $(n + 1)$ st generation with a characteristic of the rD type. We can again see that the recessive gene can be obtained from the male or female parent. It can be received from either parent with probability $(x_n + \frac{1}{2}y_n)(1 - p - q) + p$. The dominant gene can likewise be transferred with probability $(z_n + \frac{1}{2}y_n)(1 - p - q) + q$. We will then find that the rD type characteristic in the $(n + 1)$ st generation will occur with probability

$$2[(x_n + \frac{1}{2}y_n)(1 - p - q) + p][(z_n + \frac{1}{2}y_n)(1 - p - q) + q].$$

If there are no mutations of this sort, then $p = 0$ and $q = 0$. In this case these results reduce to those established earlier.

Again we consider the probability of an individual in the $(n + 1)$ st generation having and rr type characteristic, but now let us suppose that all the recessive genes in the male or female parent are not capable of being transferred to their offspring [1, p. 126]. We will suppose that a certain percentage t of recessive genes are not capable of being transferred from the male or female parent. Let A be the event that the recessive gene comes from the rr type and B the event that it comes from the rD type. Also let C be the event that the recessive gene is transferrable in the rr type and let G be the event that the recessive gene is transferrable in the rD type. Then we want to find $P(A|C)$ and $P(B|G)$. We already know, (3), that

$$P(A|C) = \frac{P(A \cap C)}{P(C)} \quad \text{and} \quad P(B|G) = \frac{P(B \cap G)}{P(G)}.$$

First of all, we can see that the probability of the recessive gene coming from the rr type and having t percentage not transferrable is

$$P(A \cap C) = x_n - tx_n$$

and the probability of its coming from the rD type and having t percentage not transferrable is

$$P(B \cap G) = \frac{1}{2}(y_n - ty_n).$$

Also $P(C) = 1 - tx_n$ and $P(G) = 1 - ty_n$, thus we obtain

$$P(A|C) = \frac{x_n - tx_n}{1 - tx_n} \quad \text{and} \quad P(B|G) = \frac{\frac{1}{2}(y_n - ty_n)}{1 - ty_n}.$$

Since the recessive gene must come from the rr type or the rD type, we see that the probability of getting a transferrable recessive gene from one parent is

$$\frac{x_n - tx_n}{1 - tx_n} + \frac{1/2(y_n - ty_n)}{1 - ty_n}.$$

Of course, if the individual is to have the rr type characteristic, he must get a recessive gene from both parents. Since these two events would be independent of each other, the probability of the individual having this type would just be the above number squared. You can see that if all recessive genes in the parents are transferrable, ($t = 0$), then the above results reduce to those established earlier.

We will now discuss the use of the binomial theorem in determining probabilities in genetics. If $(a + b)^n$ is expanded by the use of the binomial theorem [1, p. 149], we can find the coefficient of each term in the expansion. Suppose, for instance, that you wanted to find the probability that a family of five children would have 3 girls and 2 boys. If we let a stand for the probability that a child is a boy and b for the probability that a child is a girl, we can expand $(a + b)^5$ and then by picking out the correct term in the expansion and substituting for a and b , one can find the required probability. In this case you would want the term Ha^2b^3 , where H is the correct coefficient in the expansion, since you want the case when there are 3 girls and 2 boys. Upon expansion, you will obtain $10a^2b^3$. We will then say that the probability of having a boy at birth is $1/2$ and that of having a girl is also $1/2$. When $1/2$ is substituted for a and for b , you obtain $5/16$ which is the probability of having three girls and two boys.

Now let us suppose that both parents had the rD type characteristic and we wanted to find the probability that in a family of six, three of them would have the rr type characteristic. According to Mendel's laws of inheritance, one out of every four would be of the rr type. If we let a stand for the probability that a person is not of the rr type and b stand for the probability that he is, then we can solve the original problem. We will expand $(a + b)^6$ and pick out the right term; in this case, Ha^3b^3 , since we want three of them to be of the rr type. By setting a equal to $3/4$ and b equal to $1/4$, we then obtain a probability of $135/1024$. The binomial theorem can be used in solving many other problems in genetics and those concerning inheritance [2, p. 84].

Although we have touched on only a few of the ways that probability theories and the laws of probability can be used in genetics, you can see that these and many other principles can be used by doctors, hospital personnel, and many other people. Most people do not realize that it is possible to predict with a certain degree of accuracy the characteristics that a person will inherit from his parents. Of course, these predictions do not always hold true, but still it gives the people connected with this field a basis for explaining and predicting certain phenomena which take place.

REFERENCES

- [1] Goldberg, Samuel, *Probability, An Introduction*, Prentice-Hall, Inc., 1960.
- [2] Snyder, Laurence H. and David, Paul R., *The Principles of Heredity*, D. C. Heath and Company, 1957.

Fourteenth Biennial Convention

April 8-9, 1963

The fourteenth biennial convention of Kappa Mu Epsilon will be held on the campus of Illinois State Normal University, Normal, Illinois on April 8-9, 1963. Students are urged to prepare papers to be considered for presentation at the convention. Papers must be submitted to Professor Harold E. Tinnappel, National Vice-President, Bowling Green State University, Bowling Green, Ohio, before February 1, 1963. For complete directions with respect to the preparation of such papers, see pages 114-115 of the Spring issue of THE PENTAGON.

I hope that every chapter will be well represented at the convention.

CARL V. FRONABARGER
National President

Representation of Real Numbers by a Pencil of Lines*

FAE LOUISE SEAY

Student, Southwest Missouri State College

In this paper I will show how the field of real numbers may be represented as a pencil of lines. This will involve showing that the pencil is an ordered field and demonstrating that an isomorphism exists between the two fields.

The pencil will be defined as having its vertex at point P . (See Figure I) Take any two lines of the pencil which are mutually perpendicular and label one of these n . Then construct line m , parallel to n , at the point O on the line perpendicular to n . Now the lines in the pencil can be defined such that L_R is the line which is determined by the point P and a point of the line m at the end point of the directed line segment OR . The line L_R will be equal to the line L_S if and only if $\overline{OR} = \overline{OS}$.

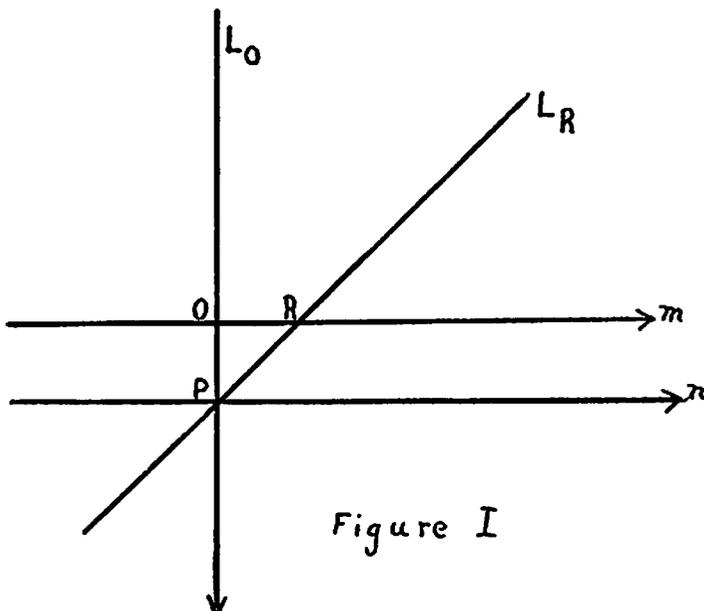


Figure I

* A paper presented at a Regional Convention of KME at Mount St. Scholastic College, Atchison, Kansas on April 7, 1962.

The definition of an ordered field requires two binary operations on the pencil such that the properties of a field will hold for these operations.

The operation of "addition" will be defined such that

$$L_R + L_S = L_T \leftrightarrow \overline{OR} + \overline{OS} = \overline{OT}$$

In Figure II it is easy to see that this operation will be performed by the Euclidean method of adding line segments.

"Multiplication" is next defined such that

$$L_R \cdot L_S = L_T \leftrightarrow \frac{\overline{OR} \cdot \overline{OS}}{\overline{OP}} = \overline{OT}$$

This operation (shown in Figure III) is performed by striking an arc of a circle with center at O and radius \overline{OR} , in the clockwise direction until it intersects line L_o at the point N . Through N a line is constructed parallel to L_S and it intersects line m at the point T . By

use of similar triangles it is easily seen that $\frac{\overline{OS}}{\overline{OP}} = \frac{\overline{OT}}{\overline{ON}}$. This expression, since $\overline{OR} = \overline{ON}$, reduces to the form $\frac{\overline{OR} \cdot \overline{OS}}{\overline{OP}} = \overline{OT}$ which

is the same as that of the definition, therefore, the line that is determined by points P and T is the product line.

Both addition and multiplication are closed operations in this set since no matter what segment is determined on line m by either "addition" or "multiplication" a line can be constructed connecting the end point of that segment with P and thus forming another line in the pencil. This is a good time to note that the line n must be excluded from this set of lines since it will at no time intersect with the line m and; therefore, it is not in the set of lines in the pencil that can be added or multiplied.

It will now be shown that the properties of an ordered field hold for this set of lines.

$L_R + L_S = L_T$ implies that $\overline{OT} = \overline{OR} + \overline{OS}$. Now let $L_S + L_R = L_{T'}$ which implies that $\overline{OT'} = \overline{OS} + \overline{OR}$. But it is known that $\overline{OR} + \overline{OS} = \overline{OS} + \overline{OR}$ and it follows that $\overline{OT} = \overline{OT'}$ which implies that the point T coincides with T' . Therefore $L_R + L_S = L_S + L_R$ and the addition of lines of the pencil is a commutative operation.

Since the addition of line segments is associative it can be

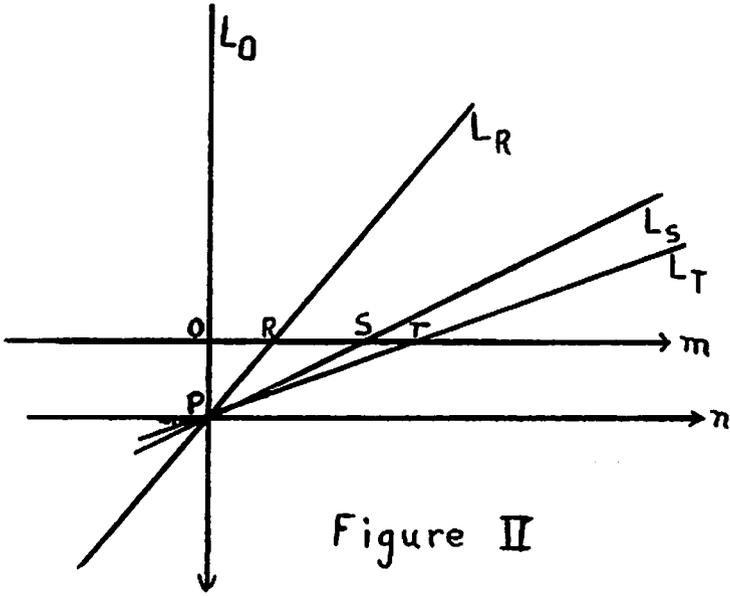


Figure II

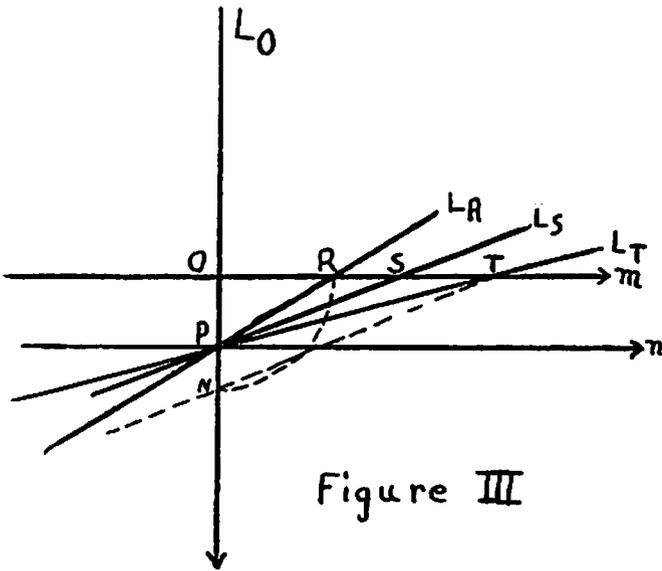


Figure III

easily established that the addition of the lines of the pencil is associative by a proof similar to that of the commutative law.

It is now readily seen that the zero element of the set is the line L_0 which by definition is determined by the null segment \overline{OO} . It is seen (again using Figure I) that adding $L_0 + L_R = L_T$ implies that $\overline{OO} + \overline{OR} = \overline{OT}$. It is immediately seen that $\overline{OR} = \overline{OT}$ and therefore

$$L_0 + L_R = L_R$$

As shown in Figure IV, we define $-L_R$ to be the reflection of L_R in L_0 . We shall now show that $-L_R$ plays the role of an additive inverse of L_R . By the definition of addition, $L_R + (-L_R) = L_T$ implies $\overline{OR} + \overline{RO} = \overline{OT}$. But it is known that the addition of inverse segments results in the null segment \overline{OO} thus implying that O coincides with T . Therefore, $L_R + (-L_R) = L_0$.

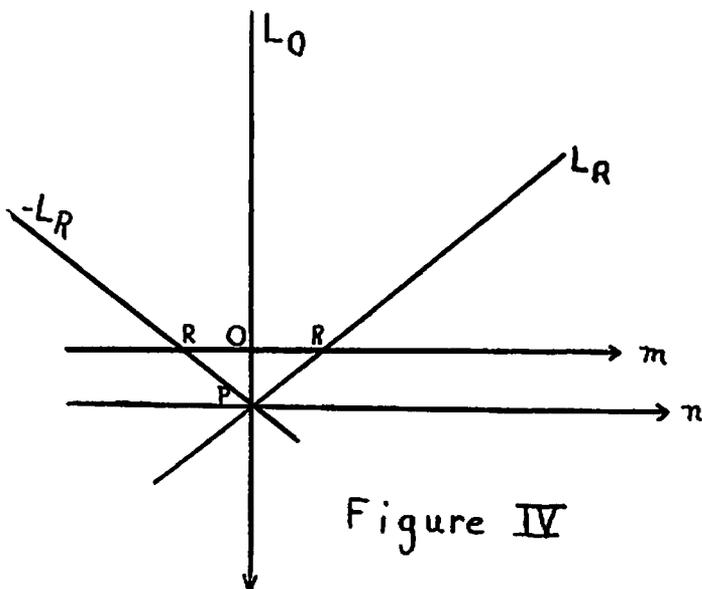


Figure IV

It will now be shown that the associative law for multiplication holds for this pencil of lines.

$$\begin{aligned} (L_R \cdot L_S) \cdot L_T &= L_R \cdot (L_S \cdot L_T) \\ L_Q \cdot L_T &= L_R \cdot L_V \\ L_X &= L_Y \end{aligned}$$

$$L_R \cdot L_S = L_Q \rightarrow \frac{\overline{OR} \cdot \overline{OS}}{\overline{OP}} = \overline{OQ}$$

$$L_Q \cdot L_T = L_X \rightarrow \frac{\overline{OQ} \cdot \overline{OT}}{\overline{OP}} = \overline{OX}$$

$$\overline{OX} = \frac{\overline{OR} \cdot \overline{OS} \cdot \overline{OT}}{\overline{OP} \cdot \overline{OP}}$$

$$L_S \cdot L_T = L_V \rightarrow \frac{\overline{OS} \cdot \overline{OT}}{\overline{OP}} = \overline{OV}$$

$$L_R \cdot L_V = L_Y \rightarrow \frac{\overline{OR} \cdot \overline{OV}}{\overline{OP}} = \overline{OY}$$

$$\overline{OY} = \frac{\overline{OR} \cdot \overline{OS} \cdot \overline{OT}}{\overline{OP} \cdot \overline{OP}}$$

$\overline{OX} = \overline{OY}$ implies by the definition of equal lines that $L_X = L_Y$ and thus, since the steps are reversible, the associative law of multiplication holds.

The distributive law will now be shown in a similar manner.

$$\begin{aligned} L_R (L_S + L_T) &= L_R \cdot L_S + L_R \cdot L_T \\ L_R \cdot L_Q &= L_V + L_W \\ L_X &= L_Y \end{aligned}$$

$$L_S + L_T = L_Q \rightarrow \overline{OS} + \overline{OT} = \overline{OQ}$$

$$L_R \cdot L_Q = L_X \rightarrow \frac{\overline{OR} \cdot \overline{OQ}}{\overline{OP}} = \overline{OX}$$

$$\overline{OX} = \frac{\overline{OR} \cdot \overline{OS} + \overline{OR} \cdot \overline{OT}}{\overline{OP}}$$

$$L_R \cdot L_S = L_V \rightarrow \frac{\overline{OR} \cdot \overline{OS}}{\overline{OP}} = \overline{OV}$$

$$L_R \cdot L_T = L_W \rightarrow \frac{\overline{OR} \cdot \overline{OT}}{\overline{OP}} = \overline{OW}$$

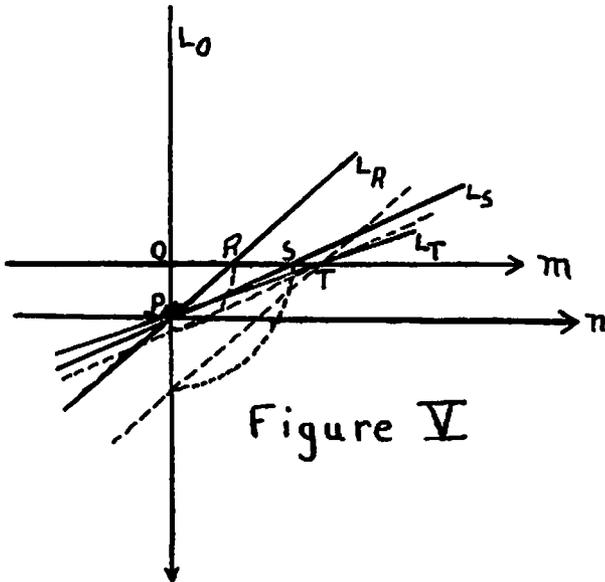
$$L_V + L_W = L_Y \rightarrow \overline{OV} + \overline{OW} = \overline{OY}$$

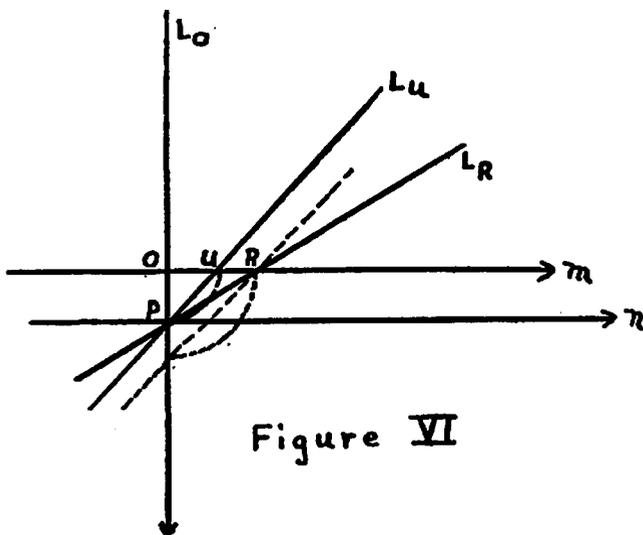
$$\overline{OY} = \frac{\overline{OR} \cdot \overline{OS} + \overline{OR} \cdot \overline{OT}}{\overline{OP}}$$

Since the steps are reversible the distributive law holds.

It will next be shown that multiplication is commutative (See Figure V). Let $L_R \cdot L_S = L_T$ and $L_S \cdot L_R = L_{T'}$. This implies that $\overline{OT} = \frac{\overline{OR} \cdot \overline{OS}}{\overline{OP}}$ and $\overline{OT'} = \frac{\overline{OS} \cdot \overline{OR}}{\overline{OP}}$ and since multiplication of segments is commutative $\overline{OT} = \overline{OT'}$, T coincides with T', and $L_R \cdot L_S = L_S \cdot L_R$.

Referring to Figure VI it will now be shown that the identity element is the line L_U , where $\overline{OU} = \overline{OP}$. $L_R \cdot L_U = L_T$ implies that $\frac{\overline{OR} \cdot \overline{OU}}{\overline{OP}} = \overline{OT}$. Since $\overline{OP} = \overline{OU}$, it follows that $\overline{OR} = \overline{OT}$ which





implies that T coincides with R and $L_R \cdot L_U = L_R$.

The multiplicative inverse for any $L_R \neq L_0$ is defined such that

$$L_R \cdot L_{R'} = L_U \leftrightarrow \overline{OR'} = \frac{\overline{OP} \cdot \overline{OU}}{\overline{OR}}$$

It is then seen in Figure VII by the use of similar triangles that the inverse of any non-zero line L_R is its reflection in the line L_U .

It has now been shown that the pencil of lines is a field and the only thing left to show is that it is an ordered field. This requires that the pencil contain a non-empty subset which is closed with respect to addition and multiplication. Let L_+ be the set of lines in the pencil which lie between line n and L_0 going from n to L_0 in the counterclockwise direction. It is seen that there is closure with respect to the two operations since when any two lines of L_+ are added or multiplied they result in another line which is an element of L_+ . The trichotomy law must also hold before the field can be ordered. This law states that every element of the field must be either equal to the zero element, an element of the subset L_+ , or its additive inverse must be an element of L_+ . It is obvious that this is so.

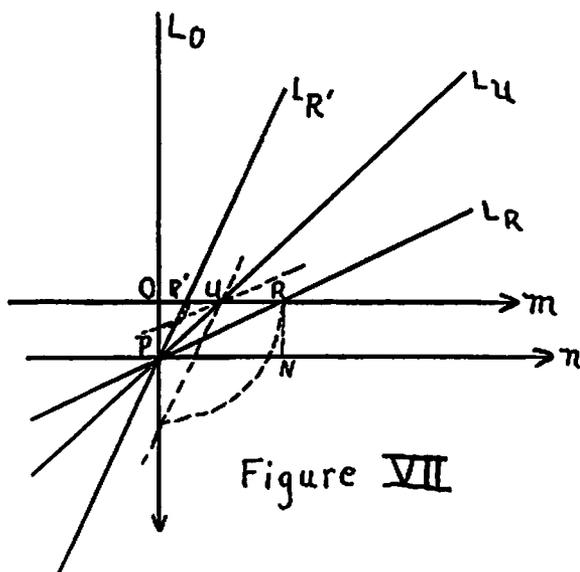


Figure VII

Now that the pencil has been shown to be an ordered field it is left to be shown that an isomorphism exists between the pencil and the real numbers. This requires that there exist a one-to-one mapping between the two fields and that addition and multiplication be preserved.

Let \overline{OP} now be equal to one unit length. Then, on the line m , units will be marked off as on the continuum used in elementary algebra.

It is obvious that there is a one-to-one mapping between any line L_R and a point R on the line m and that there is a one-to-one mapping between any point R and the segment \overline{OR} . It is next seen that there is a one-to-one mapping between the segment \overline{OR} and the real number r . Since mappings are transitive, there is a one-to-one mapping between the lines and the real numbers.

The preservation of the two operations is easily seen since $L_R + L_S \leftrightarrow \overline{OR} + \overline{OS} \leftrightarrow r + s$, and in multiplication, $L_R \cdot L_S \leftrightarrow \frac{\overline{OR} \cdot \overline{OS}}{\overline{OP}} \leftrightarrow r \cdot s$. Since \overline{OP} has been designated as one unit it can be seen that $L_R \cdot L_S \leftrightarrow \overline{OR} \cdot \overline{OS} \leftrightarrow r \cdot s$.

It has been shown that the pencil of lines is an ordered field and that it is isomorphic to the real numbers, thus implying that the pencil can be used to represent our system of real numbers.

Though it is interesting to see that there is a geometric method of addition and multiplication, I am sure that the majority will go on using the standard way of addition and multiplication rather than changing to complicated constructions such as these.

A Statement on the Peace Corps

FROM

R. SARGENT SHRIVER, JR., DIRECTOR

The United States is sending some of its most outstanding young men and women as Peace Corps Volunteers to the developing nations. As teachers, engineers, nurses, coaches and surveyors, and in community development work, these Volunteers are providing leadership and knowledge to people throughout the world.

Fraternities and sororities have prided themselves on their ability to attract and develop leadership. Responsibility, too, has come with this leadership.

Let me suggest that an even greater responsibility and challenge awaits you now. The chance to serve overseas, and thus to continue the work of more than 4,000 Peace Corps Volunteers now in the field, offers a rare fulfillment and experience. Inform yourself about the Peace Corps and how you may become a part of it after college. Contact the Peace Corps Liaison Officer on your campus, or write directly to PEACE CORPS, College and University Division, Washington 25, D.C.

EDITOR'S NOTE: Perhaps some members of Kappa Mu Epsilon are now teaching or working in some other capacity as Peace Corps Volunteers in Asia, Africa, or Latin America. If any readers know of such, please inform the Editor of the KME News Section. The Editor of THE PENTAGON would be happy to publish letters or articles on the experiences of our members serving in another culture.

159. *Proposed by the Editor.*

Given a circle O of radius r and a point P outside the circle. With compasses construct the inverse of P . That is, find P' so that $OP \cdot OP' = r^2$.

160. *Proposed by Perry Smith, Albion College, Albion, Michigan.*

State a rule by means of which any repeating decimal can be written as the sum of two rational numbers.

SOLUTIONS

151. *Proposed by Sam Sesskin, Hofstra College, Hempstead, New York.*

The algebra teacher restricts the use of logarithms to $\log 3$ and $\log 2$. Find $\log 5$ accurate to the nearest thousandths.

Solution by Perry Smith, Albion College, Albion, Michigan.

Find the ratio of two integral powers of 2 and of 3 that will approximate 5. For example $2^{15}/3^8 = 4.9944$, thus $\log 4.9944 = 15 \log 2 - 8 \log 3$. This gives $\log 5$ with the required accuracy.

Also solved by Gerald Christman, Kansas State College of Pittsburg; Richard Hannes, Upsala College, East Orange, New Jersey; John W. Hardtla, University of Southern Mississippi, Hattiesburg; Phil Huneke, Pomona College, Claremont, California; Blaine E. Myers, Westminster College, New Wilmington, Penn.; Roger Richards, Westminster College, New Wilmington, Penn.; Steve Wingert, Southern Methodist University, Dallas, Texas; Clinton Lee Wood, Colorado State University, Ft. Collins.

Editor's note: Christman and Hannes computed $\log 5$ as follows:

$$\log 5 = 2 \log 2 + 2 \left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5(9)^5} + \dots \right)$$

Using only two terms of this series will provide the accuracy desired.

152. *Proposed by C. W. Trigg, Los Angeles City College.*

Find a set of three digit numbers, each of which is a permutation of the same three digits, which when divided by the sum of the digits yields two pairs of consecutive integers.

Solution by Phil Huneke, Pomona College, Claremont, California.

Let the desired digits be a , b and c . We then desire

$$(100a + 10b + c)/(a + b + c) = m(\text{an integer})$$

and

$$(100a + 10c + b)/(a + b + c) = m + 1$$

or, eliminating m ,

$$9c - 9b = a + b + c.$$

Therefore 9 divides $a + b + c$. Now

$$a + b + c \leq 9 + 8 + 7 = 24,$$

Thus we have

$$\text{Case 1: } a + b + c = 9$$

$$\text{Case 2: } a + b + c = 18$$

Case 1: If $a + b + c = 9 = 9c - 9b$ then $c - b = 1$.

Similarly for another permutation of a , b , and c we get the difference of another pair of a , b and c to also be 1. Therefore a , b , c are consecutive integers and $a + b + c = 9$, thus $a, b, c, = 2, 3, 4$ and a desired set is: 234, 243, 432, 423.

Case 2: If $a + b + c = 18$ then $9c - 9b = 18$ or $c - b = 2$.

Similarly for another permutation of a , b and c we get the difference of another pair of a , b and c to also equal 2.

Therefore $a + b + c = x + (x + 2) + (x + 4) = 18$, or $x = 4$. Thus $a, b, c = 4, 6, 8$ and a desired set is: 468, 486, 846, 864. These two sets are the only solution to the problem.

Also solved by John W. Hardtla, University of Southern Mississippi, Hattiesburg; Joe Heffelfinger, Anderson College, Anderson, Indiana; Roger Richards, Westminster College, New Wilmington, Penn.; Karen L. Smith, State University of New York, Albany.

153. *Proposed by Mary Sworske, Mount Mary College, Milwaukee, Wisconsin. (From The Theory of Numbers by Burton Jones)*

A woman with a basket of eggs was knocked down by a bicycle. In presenting the bill to the rider's father, she said she did not know how many eggs she had, but when she counted them two at a time she had one egg left, similarly when she counted them three, four, five, and six at a time; but in sevens there were not any left. What is the smallest number of eggs she could have had?

Solved by George T. Hawkins, University of South Carolina, Columbia.

Let n represent the number of eggs; then $n - 1$ is divisible by 2, 3, 4, 5, and 6, and therefore is divisible by their l.c.m. which is 60. Thus $n = 60x + 1$, but n is divisible by 7, therefore we can write $n = 7y$ or $7y = n = 60x + 1$. Now when 60 is divided by 7 the remainder is 4, thus we can examine $4x + 1$ for divisibility by 7. Inspection yields $x = 5$. Thus $n = 60x + 1 = 301$.

Also solved by John W. Hardtla, University of Southern Mississippi, Hattiesburg; Joe Heffelfinger, Anderson College, Anderson, Indiana; Phil Huneke, Pomona College, Claremont, California; Robert T. Kurosaka, State University of New York, Albany; Norman Nielsen, Pomona College, Claremont, California; Ro Patrick, State University of New York, Albany; Roger Richards, Westminster College, New Wilmington, Penn.; Sam Sesskin, Hofstra College, Hempstead, New York; Karen L. Smith, State University of New York, Albany; Perry Smith, Albion College, Albion, Michigan; Richard M. Turkel, Hofstra College, Hempstead, New York; Steve Wingert, Southern Methodist University, Dallas, Texas; Clinton L. Wood, Colorado State University, Ft. Collins.

154. *Proposed by the Editor (From The American Mathematical Monthly).*

Find a number $abcde$ so that 3 times that number gives $abcde1$.

Solution by Pauline Lockett, Texas Woman's University, Denton.

$$3(abcde) = abcde1$$

$$300,000 + 30,000a + 3,000b + 300c + 30d + 3e =$$

$$100,000a + 10,000b + 1,000c + 100d + 10e + 1$$

$$\text{or } 10,000a + 1,000b + 100c + 10d + e = 42,857$$

thus $abcde = 42857$ and the desired number $abcde = 142,857$.

Also solved by George T. Hawkins, University of South Carolina, Columbia; John W. Hardtla, University of Southern Mississippi, Hattiesburg; Joe Heffelfinger, Anderson College, Anderson, Indiana; Phil Huneke, Pomona College, Claremont, California; Robert T. Kurosaka, State University of New York, Albany; Norman Nielsen, Pomona College, Claremont, California; Ro Patrick, State University of New York, Albany; Roger Richards, Westminster College, New Wilmington, Penn.; Sam Sesskin, Hofstra College, Hempstead, New

York; Karen L. Smith, State University of New York, Albany; Perry Smith, Albion College, Albion, Michigan; Rickey M. Turkel, Hofstra College, Hempstead, New York; Steve Wingert, Southern Methodist University, Dallas, Texas; Clinton L. Wood, Colorado State University, Ft. Collins; Thomas Watson, Tennessee Polytechnic Institute, Cookeville, Tennessee.

155. *Proposed by Sam Sesskin, Hofstra College, Hempstead, New York.*

In the set of natural numbers, suppose $N = 2n + 1$, show that the sum S , of all products whose divisors total N is $2(1^2 + 2^2 + \dots + n^2)$. In other words show

$$\sum_{i=1}^n [(2n + 1)i - i^2] = 2 \sum_{i=1}^n i^2$$

Solution by John W. Hardtla, University of Southern Mississippi, Hattiesburg.

$$\sum_{i=1}^n [(2n + 1)i - i^2] = (2n + 1) \sum_{i=1}^n i - \sum_{i=1}^n i^2$$

Since

$$\sum_{i=1}^n i = n(n + 1)/2$$

and

$$\sum_{i=1}^n i^2 = n(n + 1)(2n + 1)/6$$

we get

$$\begin{aligned} \sum_{i=1}^n [(2n + 1)i - i^2] &= n(n + 1)(2n + 1)/2 - n(n + 1)(2n + 1)/6 \\ &= 2n(n + 1)(2n + 1)/6 \\ &= 2 \sum_{i=1}^n i^2 \end{aligned}$$

Also solved by Phil Huneke, Pomona College, Claremont, California; Norman Nielsen, Pomona College, Claremont, California; Perry Smith, Albion College, Albion, Michigan.

The Mathematical Scrapbook

EDITED BY J. M. SACHS

In mathematics, a certain surprising thing happens again and again. Someone poses a simple question, a question so simple that it seems no useful result can come from answering it. And yet it turns out that the answer opens the door to all kinds of interesting developments, and gives great power to the person who understands it.

—W. W. SAWYER

= Δ =

The mathematician has reached the highest rung on the ladder of human thought.

—HAVELOCK ELLIS

= Δ =

The fall seems a particularly appropriate time for this football version of an old favorite. This was sent in by a friend whose alma mater is having such a de-emphasized season that he must get his football fun with such efforts as this. Coach X was delighted to have 75 candidates turn out for football. His joy was tempered however when he found that some of them had never played the game before. The experienced players were asked to raise their hands if:

1. The experience was with a defensive unit.
2. The experience was with an offensive backfield.
3. The experience was with an offensive line.

No one of the experienced players raised his hand to all three questions. Some however had two of the three experiences. Half of group 1 were also in group 2. Half of all the experienced players were in group 2. Group 3 contained three times as many players as the intersection of groups 1 and 2. Only one fourth of the players in group 3 were also in group 1. The number of players who were in group 2 only was twice the number in group 1 only. How many of the 75 candidates were inexperienced?

= Δ =

In investigating the binary quadratic form $x^2 + y^2$, what can be said about the primes represented by this form from a discussion of the various cases possible with x and y odd or even? It is obvious that if x and y are both odd, the only prime which can be represented is 2. What happens if both are even? One even and one odd? How does this apply if the form is $x^2 + Ny^2$ where N is a positive integer?

The preceding item reminds the editor of the prejudice or superstition of a colleague who, while refusing to take attendance in his classes, agreeably turns in a report of total number of absences each month. This integer is, of course, completely a product of his imagination. He insists that it can be as outlandish and unlikely as he chooses as long as it is positive and odd. An even integer would seem suspicious he asserts. The following quotation from Pliny the Elder seems to indicate that his attitude is not a new one. "Why is it that we entertain the belief that for every purpose odd numbers are the most effectual?"

$$= \Delta =$$

The subject of representation of integers by quadratic forms has many interesting aspects which can be investigated at several levels of sophistication. Consider the pattern one gets by asking what integers can be by the form $1^2 + n^2$. The first of these would be 2. Suppose we consider a lattice for the positive integers with those which can be represented by $1^2 + n^2$ filled in.

$$\begin{array}{ccccccc}
 _ & \underline{2} & _ & _ & \underline{5} & _ & _ & _ & \underline{10} \\
 _ & _ & _ & _ & _ & _ & \underline{17} & _ & _ \\
 _ & _ & _ & _ & _ & \underline{26} & _ & _ & _ \\
 & & & & \dots & & & &
 \end{array}$$

There seems to be an obvious pattern in the gaps between entries. Does it persist? Can you prove that it persists? Consider a similar lattice for $2^2 + n^2$.

$$\begin{array}{ccccccc}
 _ & _ & _ & _ & \underline{5} & _ & _ & \underline{8} & _ \\
 _ & _ & \underline{13} & _ & _ & _ & _ & _ & \underline{20} \\
 _ & _ & _ & _ & _ & _ & _ & \underline{29} & _ \\
 & & & & \dots & & & &
 \end{array}$$

What generalizations can you see from these two examples? Can you prove them? Obvious example: If N is represented by

$1^2 + n^2$ then $N + 3$ can be represented by $2^2 + n^2$, $N + 8$ by $3^2 + n^2$, $N + 15$ by $4^2 + n^2$, etc. Can you apply these ideas to $1^2 + 2n^2$, $2^2 + 2n^2$, etc. Can you put these lattices together and try to find out what integers can be represented by such forms.

= Δ =

The current attempts to present geometry without the artificial separation of the geometry of the plane from the geometry of three dimensions gets support from the following remark of Felix Klein made early in this century. "It has long been the custom in the schools as well as the university, first to study geometry of the plane and then, entirely separated from it, the geometry of space. On this account, space geometry is unfortunately often slighted, and the noble faculty of space perception, which we possess originally, is stunted."

= Δ =

Round numbers are always false.

—SAMUEL JOHNSON

This succinct remark caused your editor to ask some of his students what round numbers are and what faith they put in the above statement. A number of answers, serious and otherwise, were received, the following among them.

The round numbers, interpreted to mean digits, are obviously 0,2,3,5,6,8,9 since these cannot be symbolized in our notation by combinations of straight line segments but must have parts which are curved, that is round. The non-round digits are thus 1,4,7. Argument that the round numbers are false: The word false is not interpreted in the dictionary sense but is simply a name or synonym for the set of numbers called round. This set can be considered ordered and also using its largest member as a modulus is closed in the following curious fashion. The sums and products of consecutive members are always round. Examples, $2 + 3 = 5$, $2 \cdot 3 = 6$, $5 + 6 = 2 \pmod{9}$, $5 \cdot 6 = 3 \pmod{9}$, etc. It is this curious closure which is meant by the word "false", i.e. belonging to this set. What do you think Johnson meant? Please send interpretations to your editor for publication in future issues. How ingenious can you be in interpreting this and other curious remarks mathematically?

= Δ =

If the nature of proof cannot be described or formulated in detail how can anyone learn it? It is learned, to use an oversimplified analogy, in the same manner as a child learns to identify colors

namely by observing someone else identify green things, blue things, etc. and then by imitating what he has observed.

—IVAN NIVEN

= Δ =

. . . . the mathematical method of proof served as a prototype for Plato's dialectics and for Aristotle's logic.

—B. L. VAN DER WAERDEN

= Δ =

I am sure that all of the readers of *The Pentagon* have heard and solved the nine coin problem but let me restate it as a starting point for problems related to it. You have nine coins, one of them underweight and a beam balance which permits you weigh one coin against another or one group of coins against another. In two weighings find the underweight coin. As you will all recall, this can be done by weighing two groups of three coins against one another.

Now suppose that we have an unspecified number of coins and two are underweight. Can you develop a strategy so that no matter which of the possible cases develops, the number of required weighings is the same and that this is the minimum number which will apply to all cases?

Examples:

1. Two lightweight coins out of three. One weighing is needed. Weigh any one against another. If they weigh the same, they are the two light ones. If they are different in weight the lighter one of the pair weighed and the one not weighed are the light ones.
2. Two lightweight coins in four. If you weigh two against two can you argue that in one case this single weighing will settle the question while in the other, it will not so that this approach is not within the conditions set that all cases require the same number of weighings?

If you weigh one coin against another then if they weigh the same, they are both or neither light and that pair against the remaining pair will settle the issue. If they do not weigh the same, pick the lighter and weigh it against one of the remaining pair to settle the issue. Are you convinced that two weighings will settle this within the conditions?

Can you develop a strategy for two lightweight coins in five? How about three lightweights?

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Harold E. Tinnappel, Bowling Green State University, Bowling Green, Ohio.

College Algebra, Charles H. Lehmann, John Wiley and Sons, Inc., (440 Fourth Avenue South), New York 16, 1962, vi + 432 pp., \$5.95.

Here is a refreshingly new text free from the extremes of modernism and the staleness of tradition. There are, of course, the many stereotyped practice problems, and classical ones such as the lot to be fenced along a river, but there are new ones, and new versions of old ones, such as, for example, "Show by mathematical induction that the rule for synthetic division holds in general", and "A man draws 3 coins at random from a bag containing 8 dimes, 4 quarters, and 3 half-dollars. Find the value of his expectation."

The book is well constructed and well written. The fine balance between small and large print, and the use of italics instead of bold-face letters, on non-glare paper make reading physically easy. The orderly and unusually clear display of examples and equations, the exceptionally complete and clear explanations, and the smooth intuitive transitions from one topic to the next make reading mentally easy. The author does a good job in each chapter without straining the student's mind at any given level of his development, but continuing to stretch it a little. For instance, he presents the axioms and definition of a group on page 182, and shows the roots of unity to be a group. This is in the chapter on complex numbers, a very good chapter, where he introduces the student to ideas of vectors in a plane and gives a section on "Functions of a Complex Variable."

This text contains all the topics to be expected in a college algebra text. It contains more. In the second chapter there are discussions of a number field and necessary and sufficient conditions. Most of the chapters are typical of those in college algebra books, but some chapters have unusual additions. In the chapter on variation there is a good section on curve tracing where horizontal and vertical asymptotes are introduced. In the chapter on Progressions there is a

very clear discussion of the limiting process used in obtaining the formula for the sum of a convergent geometric series. In the chapter on Probability there is a short treatment of the binomial distribution, cumulative distributions and probability curves. The 37-page chapter on Determinants is superior. The arrangement of topics is truly progressive. The display and clarity of explanations motivates quick and complete learning. The use of the * to indicate column or row being altered in writing equal determinants is a great help in following solved examples. The exercises in this chapter especially have been preceded by more careful thought and planning than many such lists. There are plenty of good exercises throughout the book.

It is true that a student has to dig out mathematics by himself if he expects to learn it, but Lehmann in *College Algebra* communicates mathematics to the freshman so that the digging is a rewarding pleasure. Even the calculus student would enjoy reading *College Algebra*.

—NORMAN E. DODSON
Wittenberg University

Studies in Modern Analysis, R. C. Buck, Editor, MAA Studies in Mathematics, Volume I, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1962, 182 pp., \$4.00.

As explained in the introduction written by the editor, this volume contains expository articles by four authors.

In the first article, E. J. McShane discusses a Theory of Limits beginning with the simplest concept of limit and extending it to its most general form used in present-day analysis.

In the second, M. H. Stone begins with the linear approximation that was introduced by Weierstrass and Approximation Theorem.

In the third, Edgar R. Lorch discusses The Spectral Theorem which deals with eigenvectors, eigenvalues and linear transformations in complex space of finite or infinite dimensions. An understanding of this article requires a thorough grounding in certain aspects of the most modern analysis.

In the fourth article, Casper Goffman explains the foundations of functional analysis. It is, as the title indicates, Preliminaries to Functional Analysis.

This volume, and others like it to follow, will be of special interest to two classes of readers. First, the expert in the particular field will enjoy a concise, unified discussion of the material just as

an expert musician would enjoy a familiar piece of music when it is well-played. Second, the beginning graduate student who has a sufficient background in vocabulary and training will profit by the summarizing discussions. Such a reader could easily obtain more benefit from certain articles than from others he might have difficulty in reading.

For the general reader, this volume is not much help. If he is to understand the vocabulary he must put in many hours of supplementary study to acquire a reading knowledge. At the end of this time he has already become familiar with the material discussed since he must have studied it in greater detail than it is presented here.

For either the expert or amateur, this is not a book for speed reading. To a great extent this situation is true of any worth while text. For example, there are a few minor misprints in the first of the four articles; hence the reader must read into the discussion what ought to be said rather than what is written.

—CECIL G. PHIPPS
Tennessee Polytechnic Institute

Mathematical Statistics, Samuel S. Wilks, John Wiley and Sons, Inc. (440 Fourth Avenue South), New York 16, 1962, xvi + 644 pp., \$15.00.

This book is intended for study on a graduate level. There is no hedging by the author. He frankly states that the treatment given to topics requires a "good undergraduate background in mathematics." The treatment is rigorous. Background for it should include undergraduate courses in both the methodology and theory of statistics and a course in matrix algebra.

While no mention is made of the number of semesters of work involved, it would seem that there is enough material for at least two semesters of substantial work.

Some authors feel that a compromise must be made between the two aspects of statistics, mastering of operational procedures and demonstrating the mathematical analysis that validates these procedures. The attitude of the author of this text is to the contrary. He states that in a book such as this, it would be "most unwise to attempt to deal with both aspects of each topic with equal emphasis" and that while statistical methodology and the underlying theory are of equal importance, experience indicates that the combination of the two aspects would better be confined to "research papers, monographs, and books restricted to specific topics."

The book is, indeed, a comprehensive treatment of the mathematical analysis of statistics on a graduate level.

—NURA D. TURNER
State University of New York
College of Albany

Numerical Analysis, Second Edition, by Zdenek Kopal, John Wiley and Sons, Inc. Printed by the offset process in Great Britain by William Clowes and Sons, Limited, London and Beccles, 1961, xvi + 594 pp., \$12.00.

Here is a book of more than 1600 formulas, with some examples illustrating their uses, and a good deal of theoretical discussion of the bases for them.

The first edition of this book appeared in 1955; in this 2nd edition misprints have been corrected and Chapter IX and some other material in the appendixes is new.

The approximation theorem of Weierstrass provides "protection" for the operations in the text; the proof of the theorem is omitted, but the explanatory remark is made (page 19) that "roughly speaking, any continuous function over a finite interval may be treated very much like a polynomial." Thus, the theme of the book is the algebraic approximation theory of a single real variable, using power polynomials as the basis for the development of the theme.

The book is written for the person who is familiar with classical numerical analysis; it is a necessary reference book for any complete mathematical library, and in particular for actuarial and computing laboratory libraries. For those who use "cookbook" formulas for approximation theory, this book will tell them why the recipes work. The actual amount of numerical examples or exercises is very small, in proportion to the theoretical discussions.

Excellent bibliographical-historical comments follow each chapter, after Chapter I, as also do problems. The reader is expected to practice interpolation using logarithms or other existing tables (page 83); the problems in this book aim to stimulate critical thinking as opposed to "mechanical practice." Thus problem VII. 15 (page 438) concerns results established in an MIT Bachelor's Thesis of 1950.

The style of writing is "Americanized early 20th century British with a continental flavor". The appearance of the print varies but in general is average.

After an interesting historical-motivational introductory chapter (17 pages), we have the "working chapters" as follows:

Chapter II: (68 pages) polynomial interpolation formulas including those of Lagrange, Newton, Hermit, Gauss, Stirling, Bessel, Everett, and Steffensen. This chapter is fundamental to the rest of the book, since the following chapters deal with the differentiation and integration (and application of these techniques) of these formulas and some of their variations.

Chapter III: (31 pages) approximate computation of derivatives of functions, using derivatives of interpolation formulas. On page 108, occurs a numerical example of computing $\cos x$ from a table of $\sin x$ by numerical differentiation. An example also is given to curve fitting.

Chapter IV: the longest chapter (120 pages); Taylor series method, the Picard method and other methods (related to the formulas of Chapter II) of approximation to solutions of ordinary differential equations are discussed.

Chapter V: (58 pages): Some of the formulas of earlier chapters are extended to more recondite uses with regard to some more involved differential equations with boundary conditions. The "ordinate" approach is used, except in the last section (8 pages) where finite differences are used. One "algebraizes" the equation and its conditions.

Chapter VI: (47 pages): Some special classical methods for solving boundary value problems using variational, iterative, and other techniques. The Raleigh-Ritz method (variational), the Schwarz (iterative), the "collocation" (interpolation) method, and the least-squares method, are treated.

Chapter VII: (92 pages): This chapter serves to bring the ("misnomered") subject of Mechanical Quadratures to the attention of the modern computer. Gaussian formulas are considered from the algebraic, geometric, and analytic points of view; various classical polynomials (Legendre, Laguerre, Hermite) are used; the appropriate appendix provides tabulated abscissae and weight coefficients. A few remarks are made on "Monte Carlo" methods.

Chapter VIII: (45 pages): Herein the equivalence of various integral equations and boundary-value problems is discussed. "Algebraization" of the solution of the integral equation is related to similar treatment of differential equations and boundary-value problems in earlier chapters.

Chapters IX: a new chapter (36 pages): Some earlier concepts are extended to a more general (operational calculus) formal method of derivation of various earlier formulas. Techniques called: "Padé rational approximations" are developed. Appendix II of Padé tables is related to this chapter.

An apparent omission occurs on page 494, line 9 from the top: the sentence probably should read: "... will contain even as well as [odd powers] of . . ." (the words in brackets are missing).

—WILLIAM A. SMALL
Tennessee Polytechnic Institute

Sequential Decoding, John M. Wozencraft and Barney Reiffen, The Technology Press of The Massachusetts Institute of Technology, and John Wiley and Sons, Inc. (440 Fourth Avenue) New York 16, New York, 1961, 74 pp., \$3.75.

This Technology Press Research Monograph gives a brief but excellent discussion on encoding and decoding, and the machinery which is used. To transmit messages or solve problems by machine necessitates putting them into some type of code and, after transmission, decoding to get the messages.

Sequential decoding is basically probabalistic as compared with ordinary decoding which is algebraic in nature. The discussion has been applied to the binary symmetric channel with references to other channels in the last chapter.

The scope of the book is given in the following six chapter headings: Coding and Communication, Block Codes, Sequential Decoding, Convolutional Encoding, Simulation, Extensions and Applications. The Appendix—Bounds on Sums of Random Variables—and a page of references finish the book.

The book gives a most rigorous mathematical treatment of the subject. It will serve as an excellent reference for those working in this field.

—RAYMOND CARPENTER
Northeastern State College
Tahlequah, Oklahoma

Progress in Operations Research, Volume I, Edited by Russell L. Ackoff, John Wiley and Sons, Inc., (440 Park Avenue South), New York 16, 505 pp., 1961, \$11.50.

This is the initial volume of a review series proposed by its Publications Committee and approved by the Council of the Opera-

tions Research Society of America, and is intended as the first of a series of books which will review from time to time the progress of a comparatively new field of endeavor called "Operations Research".

This first volume, according to the editor, is supposed to emphasize the area of technical progress both in the development of modeling techniques and in ways of using these techniques to solve problems.

The first and last chapters are broader in scope than the intervening technical ones. In the first, the meaning, scope, and methodology of Operations Research is discussed in an effort to identify the requirements for continued progress. The last chapter traces the growth of this new science from a purely military services tradition, which it has become since the beginnings of World War II, to its present status as a business service and discusses the opportunities which are opened up to Operations Research and the chances of their meeting the challenge.

The intervening chapters are more technical and are written by experts in each phase of the work. They include a very clear and detailed account of the purpose, structure, and methods of decision and value theory, inventory theory, linear and dynamic programming, the dynamics of Operational systems such as the Markov and Queuing processes, sequencing theory, replacement theory, the theory and application of simulation in Operations Research, and military gaming. In each of these chapters the author has showed what has been done, explains the structure and aim of the theory in detail, points out the weaknesses that have been found, and suggests new approaches to give the science a firmer foundation and to assure its continued progress.

Operations Research is a field about which many teachers of mathematics and even mathematicians have heard, but into which they have not gotten to any depth. This is somewhat the same as engineering or physics or chemistry wherein one could make the same statement. The difference, however, is that the fields in which people in Operations Research work are much different from those in any of these sciences. This book enables one to get a basic understanding of the meaning of Operations Research and then extends this to specific fields of interest within Operations Research. The outstanding bibliography for each chapter makes this book of tremendous value for anyone who wishes to get into some of the prob-

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Installation of New Chapters

EDITED BY SISTER HELEN SULLIVAN

NEBRASKA GAMMA CHAPTER

Nebraska State Teachers College, Chadron, Nebraska

The Nebraska Gamma Chapter was installed on the campus of Nebraska State Teachers College, Chadron, Nebraska, on May 19, 1962. Professor Walter C. Butler, of Colorado State University in Fort Collins was the installing officer. Professor Butler is National Treasurer of Kappa Mu Epsilon.

Thirty-nine charter members participated in the ceremony at which husbands or wives and other guests were present.

Officers for the year 1962-1963 are:

President	Stephen Jones
Vice President	Joyce Ritchey
Recording Secretary	Judith Dunbar
Treasurer	Jerry Terry
Historian	James Moravek
Correspondence Secretary	Prof. Lenora Briggs
Faculty Sponsor	Prof. Eugene M. Hughes



It would appear that Deductive and Demonstrative Sciences are all, without exception, Inductive Sciences: that their evidence is that of experience, but that they are also, in virtue of the peculiar character of one indispensable portion of the general formulae according to which their inductions are made, Hypothetical Sciences. Their conclusions are true only upon certain suppositions, which are, or ought to be, approximations to the truth, but are seldom, if ever, exactly true; and to this hypothetical character is to be ascribed the peculiar certainty, which is supposed to be inherent in demonstration.

—JOHN STUART MILL

Kappa Mu Epsilon News

EDITED BY FRANK C. GENTRY, HISTORIAN

REGIONAL CONFERENCE, Mount St. Scholastica College, Atchison, Kansas.

The Kansas-Missouri-Nebraska Regional Conference of Kappa Mu Epsilon was held at Mount St. Scholastica College on April 7, 1962. All chapters but two in the three-state area were represented. There were 28 faculty members and 107 students present from the ten chapters as well as 2 faculty members and 11 students, who were guests from St. Benedict's College, Atchison.

Professor Lester J. Heider, S. J., of Marquette University was guest speaker at the luncheon. His subject was "The Number System". Student papers were presented as follows:

"Representation of Rational Numbers by a Pencil of Lines",
Fae Louise Seay, Missouri Alpha.

"A Real Number", William Livingston, Kansas Alpha.

"Postulational Methods", Margaret Leary, Kansas Gamma.

"The Algebra of Sets in the Light of Field Properties", Martha
Heidlage, Kansas Gamma.

"Two Solutions of Diophantine Equations", Bob Bailey, Mis-
souri Gamma.

"Patterns in Mathematics", Frances Barry, Kansas Gamma

"Topology", Barbara Kuhn, Nebraska Alpha.

The following awards were made: "Excellent" Ratings to Fae Louise Seay and Margaret Leary; "Superior" Rating to Frances Barry and "Good" Ratings to the other four speakers.

REGIONAL CONFERENCE, St. Mary's Lake, Michigan

The Second Regional Conference of the Kappa Mu Epsilon chapters in Illinois, Indiana, Michigan, Ohio and Wisconsin was held at the Michigan Education Association Camp, St. Mary's Lake, Battle Creek, Michigan, on April 27 and 28, 1962. Michigan Alpha, Albion College, and Michigan Beta, Central Michigan University, were the host chapters. Fifteen chapters of the region were represented by 57 faculty members and students.

Professor Phillip S. Jones, of the University of Michigan, was guest speaker at the banquet. His subject was "Mathematics and the Arts".

The following student papers were presented:

"Mathematical Induction", Gloria Sergen, Illinois Delta.

"Biological Growth and Related Mathematical Principles",
Norma Cooney, Ohio Gamma.

"Conformal Mapping", Eugene Pringle, Indiana Beta.

"Transfinite Numbers", Marilyn Vanden Burgt, Wisconsin
Alpha.

"Invariance of Circle Product", David Rine, Illinois Alpha.

"Solution of Cubics and Quartics", Sally Littleton, Michigan
Alpha.

"A Look at the Special Theory of Relativity", Ken Benedict,
Ohio Gamma.

"Historical Presentation on the Role of the Axiom of Choice in
the Foundations of Mathematics", Alwyn Randall, Indi-
ana Beta.

Each of the students preparing a paper for the Conference was
given a year's subscription to THE PENTAGON.

Alabama Delta, Howard College, Birmingham.

Our chapter initiated nine new members in March, 1962.

California Beta, Occidental College, Los Angeles.

A campaign to revitalize the chapter was started last spring by
re-writing the By-Laws and encouraging participation of associate
members. Bi-weekly meetings with student speakers are held. Our
requirements for membership are: One A or two B's in upper division
mathematics courses and an overall grade point average of 2.75 on a
4 point basis. An associate member must be taking an upper division
mathematics course and must have a 3.00 grade point average in two
mathematics courses.

Florida Alpha, Stetson University, DeLand.

Our chapter has 18 new members and 18 new associate mem-
bers. Last year we had an interesting meeting at the University Plan-
etarium and made one field trip to visit the Martin Marietta Co. at
Orlando.

Indiana Beta, Butler University, Indianapolis.

Last year we initiated seven new student members and two
faculty members. Dr. Crull gave a paper on "Foundations of Mathe-
matical Thought" at the initiation banquet.

Iowa Beta, Drake University, Des Moines.

Most of our programs last year consisted of "Projects by Pledges". We had from one to three such papers at each regular meeting.

Kansas Epsilon, Fort Hays Kansas State College, Fort Hays.

We initiated 18 new members in February and sent a delegation to the Regional Conference at Atchison.

Nebraska Alpha, Nebraska State Teachers College, Wayne.

Our chapter and the chapter of Lambda Delta Lambda, honorary science society, began selecting an outstanding professor in the Science and Mathematics Departments at W.S.T.C. in 1961-62. Mr. Irwin C. Brandt, Associate Professor of Chemistry was the first recipient of the award. We initiated 14 new student members and one faculty member last year. Three of our 1962 graduates have graduate assistantships this year.

Nebraska Beta, Nebraska State Teachers College, Kearney.

Dr. James Douglas, of Rice Institute, Houston, Texas, gave a series of five lectures on mathematics on our campus last spring.

New York Alpha, Hofstra College, Hempstead.

One of our members received a National Science Foundation Fellowship and another an Atomic Energy Commission Fellowship last year. Six chapter members participated in the Putman Competition.

Ohio Delta, Wittenberg University, Springfield.

We initiated 14 new members last spring. Our programs included the following papers by visitors from other universities:

"Prime Number Theory", Dr. Blau, Antioch College.

"Theory of Sets", Dr. Phillip Dwinger, Purdue University.

"Game Theory", Dr. Robert Wilson, Ohio Wesleyan University.

Ohio Gamma, Baldwin-Wallace College, Berea.

Mr. Kemper from the Cleveland Electric Illuminating Company spoke to our chapter on the future work of the computer and the programming process. Mr. Blair Kopfstein, of the Parma Research Center spoke to the chapter on the use of mathematics in research.

Pennsylvania Gamma, Waynesburg College, Waynesburg.

Our most interesting meeting last year was a trip to the Westinghouse Research Laboratory at Pittsburg to visit and hear about their digital computer.

Texas Beta, Southern Methodist University, Dallas.

We initiated 30 new members at a banquet in April. Dr. Robert E. Stoltz, of the Department of Psychology, was our guest speaker.

Wisconsin Alpha, Mount Mary College, Milwaukee.

Dr. Karl Menger, of the Illinois Institute of Technology, was a visiting lecturer on our campus last spring. We held our annual mathematics contest for high school students. About 120 participated. We sent a delegation to the Regional Conference at St. Mary's Lake, Michigan.

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lems in greater depth. As with many surveys, special readers will find that their interests may have been left out or at least minimized. The tremendous work that has been done in the field of marketing is touched on very lightly, but is of great interest to teachers and mathematicians and to the public at large. This is a book for the professional worker in the field and for the person who wants to get an idea of what some of the problems are. It is excellent for the teacher, because it enables him to find problems in fields that are virtually unknown to the average mathematician and teacher of mathematics. Problems such as what is the optimum allocation of men and machines to the various orders that are available can now be handled from the Operations Research point of view. Fascinating applications in the field of machine replacement theory are discussed succinctly. Various ways of considering the inventory problem as a mathematical system should prove of great interest to those who may not even have thought of possibilities in this field. Systems of simultaneous linear equations in which the solutions are restricted to integers should prove of interest to many. These are just a few insights into what lies ahead for the fortunate reader of this book.

—SISTER MARY FELICE
Mount Mary College
and
WILLIAM A. GOLOMSKI
Milwaukee, Wisconsin