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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

Some Properties of Square Matrices With Integers As Elements

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The purpose of this paper is to develop some properties of square matrices with integers as elements. As the reader will discover, these properties are similar to corresponding properties of integers.

A matrix is a set of elements which are arranged in rows and columns. When a matrix has n rows and n columns, it is called a square or an n by n matrix. The number n is called the order of the matrix. A matrix is represented by enclosing the elements in parentheses. Where no confusion can arise, a matrix is denoted by a capital letter.

If S_n is the set of all n by n matrices whose elements are integers, then S_n is closed under addition and multiplication. That is, if A and B are n by n matrices with integer elements, then $A + B$ and AB are n by n matrices with integer elements.

For illustrative purposes, let

$$A = \begin{pmatrix} 10 & 0 \\ -1 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 3 \\ 1 & 4 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 8 & 3 \\ 0 & 6 \end{pmatrix} \text{ and } AB = \begin{pmatrix} -20 & 30 \\ 4 & 5 \end{pmatrix}.$$

Since A and B are both contained in the set S_2 , it is seen that the matrices $A + B$ and AB are also in S_2 . It should be noted that $AB = BA$ is not an identity since commutativity as a law fails to hold for matrix multiplication. However, the matrix BA is contained in S_n whenever B and A are in S_n .

A matrix of order n whose elements are all zero is called the zero matrix and written O_n , or simply O when the order of the matrix is implied. The unit matrix of order n is the matrix which has ones for all of its principal diagonal elements and zeros elsewhere. The unit matrix is written I_n , or simply I when the order of the matrix is apparent. That is, O and I in the set S_2 are

$$O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ respectively.}$$

A matrix A is called non-singular if there exists a matrix B such that $AB = I = BA$. When the matrix B exists it is called the inverse of A and written A^{-1} . If there is no inverse, then A is called singular. For each matrix A in S_n , there exists a matrix, $-A$, in S_n such that $A + (-A) = O$. This matrix $-A$ may be called an additive inverse. However, throughout this discussion an inverse matrix is understood to be a multiplicative inverse.

If the determinant, $|U|$, of a matrix U is 1 or -1 , then the matrix is called unimodular. A square matrix A is non-singular if and only if $|A| \neq 0$. Hence, a unimodular matrix is always non-singular and is the only matrix in S_n which has an inverse in S_n .

If three matrices A , B , and C exist in S_n such that $A = BC$, then C is called a right divisor of A , and B is called a left divisor of A .

If a matrix D is a right divisor of two matrices A and B (i.e., $A = A_1D$ and $B = B_1D$ where A_1 and B_1 are in S_n .) and if every right divisor of A and B is a right divisor of D (i.e., if $A = A_2C$ and $B = B_2C$, then there exists a matrix C_2 in S_n such that $D = C_2C$.), then D is called a greatest common right divisor, g.c.r.d., of A and B and is written (A, B) . The double parentheses on the right indicate a greatest common divisor is a greatest common right divisor.

If a matrix D is a left divisor of two matrices A and B , and if every left divisor of A and B is a left divisor of D , then D is called a greatest common left divisor, g.c.l.d., of A and B and is written $((A, B))$.

Two matrices A and B are called left associates if there exists a unimodular matrix U such that $A = UB$. A and B are called right associates if $A = BU$. A unimodular matrix is both a left and a right divisor of every matrix A . Since U is unimodular, U^{-1} exists and is in S_n . Let

$$U^{-1}A = A_1 \quad \text{and} \quad AU^{-1} = A_2,$$

then

$$A = UA_1 \quad \text{and} \quad A = A_2U.$$

From the above equations, it is seen that U is both a left and a right divisor.

If a matrix B can be obtained from a matrix A by a finite number of elementary row transformations, then B is a left associate of A . If B can be obtained from A by a finite number of elementary column transformations, then B is a right associate of A . Elementary row operations in this paper are defined as: the interchange of two rows, the multiplication of the elements of a row by -1 , and the replacement of the i th row by the sum of the i th row and k times the j th row where k is an integer. The elementary column operations are: the interchange of two columns, the multiplication of a column by -1 , and the replacement of the i th column by the sum of the i th column and k times the j th column where k is an integer.

Theorems 1 and 2 are given in MacDuffee¹, and they are summarized as follows to add continuity to the discussion of greatest common divisors.

THEOREM 1. Every pair of matrices A and B in S_n has a g.c.r.d., D , expressible in the form $D = MA + NB$ where D , M , and N are in S_n .

Consider the matrix

$$H = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$$

of order $2n$. MacDuffee has shown that there exists a unimodular matrix W of order $2n$ such that WH consists entirely of zeros except for the elements of the n by n block in the upper left corner. That is,

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

where the $2n$ unimodular matrix W has been partitioned into four n by n matrices which multiply as elements of a matrix. Hence,

$$(1) \quad W_{11}A + W_{12}B = D.$$

The matrix W is unimodular and has an inverse V which can likewise be partitioned into four n by n matrices. That is, $V = W^{-1}$ and

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \cdot \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \cdot \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \cdot \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

¹Cyrus MacDuffee, *Vectors and Matrices*. The Mathematical Association of America, 1943. Chap. V.

$$\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \cdot \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$(2) \quad A = V_{11}D \quad \text{and} \quad B = V_{21}D.$$

From equation (1) it is seen that every common right divisor of A and B is a right divisor of D , and from equations (2) it is seen that D is a right divisor of A and B . Therefore, D is a g.c.r.d. of A and B .

To illustrate the process of finding a g.c.r.d. of a pair of matrices in S_n , consider the matrices

$$A = \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}.$$

Form the matrix

$$H = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix},$$

and apply the elementary row transformations until H reduces to

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The remaining n by n block in the upper left corner is (A, B) . By performing the same row transformations on

$$I_{2n} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that are performed on H , W_{11} and W_{12} can be obtained.

I_{2n} reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 2 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 2 & -1 & 1 & 1 \\ 4 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

after the elementary row transformations are applied.

$$W_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_{12} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$W_{11}A + W_{12}B = (A, B) = D.$$

That is,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = D.$$

It can be shown that if D is a g.c.r.d of two matrices A and B , then UD for every unimodular matrix U is a g.c.r.d of A and B . Therefore, D is not a unique g.c.r.d

THEOREM 2. Every pair of matrices A and B in S_n has a g.c.l.d. D expressible in the form $D = AM + BN$ where D, M , and N are S_n .

From the proof of Theorem 1, it follows that the existence of a g.c.l.d. can be established in a similar manner using elementary column transformations. That is, there exists a unimodular matrix W with its inverse V such that

$$(3) \quad \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad AW_{11} + BW_{21} = D$$

$$(4) \quad \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}, \quad A = DV_{11}, \text{ and } B = DV_{12}.$$

From equations (3) it is seen that every common left divisor of A and B divides D , and from equations (4) it is seen that D is a left divisor of both A and B . Therefore, D is a g.c.l.d. of A and B .

To find a g.c.l.d. of two matrices A and B , form the matrix

$$\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

and apply the elementary column transformations until the last n columns are made zero. The remaining n by n block is D . W_{11} and W_{21} can be obtained by performing the same column transformations on I_{2n} . It can be shown that if D is a g.c.l.d. of two matrices A and B , then DU for every unimodular matrix U is a g.c.l.d. of A and B . Therefore, D is not a unique g.c.l.d.

S_n reduces to the set of integers when n equals 1. The reader may test the above theorems on the matrices

$$A = (21) \quad \text{and} \quad B = (9),$$

by finding (A, B) , W_{11} , and W_{12} such that $W_{11}A + W_{12}B = (A, B)$, and finding $((A, B), W_{11})$ and W_{21} such that $AW_{11} + BW_{21} = ((A, B))$.

A matrix A which is neither a zero matrix nor a unimodular matrix is called prime if every relation $A = BC$ implies that either B or C is unimodular. If the determinant of a matrix is a prime integer, then the matrix is prime. For if $A = BC$, then $|A| = |B| \cdot |C|$. Therefore, if the matrix B (or C) divides the matrix A , $|B|$ (or $|C|$) must divide $|A|$. However, if $|A|$ is a prime integer, then its only divisors are 1, -1 , and $|A|$. Hence, either $|B|$ or $|C|$ is equal to 1 or -1 , and either B or C is unimodular.

If the g.c.r.d.'s (or g.c.l.d.'s) of two matrices A and B are unimodular, then the g.c.l.d.'s (or g.c.r.d.'s) are also unimodular.

Let U be a unimodular g.c.r.d. of A and B , then there exist matrices A_1 and B_1 in S_n such that $A = A_1U$ and $B = B_1U$. Also, there are matrices A_2 and B_2 in S_n such that $A = UA_2$, $B = UB_2$, $|A| = |U| \cdot |A_2|$ and $|B| = |U| \cdot |B_2|$ since every unimodular matrix that is a right divisor of a matrix is a left divisor. The only common divisors of $|A_2|$ and $|B_2|$ are 1 and -1 since every matrix that is a common right divisor of A_2 and B_2 must be a right divisor of A , B , and U . However, the only right divisors of U are unimodular matrices. Therefore, for every $D = ((A, B))$ such that $|A| = |D| \cdot |A_3|$ and $|B| = |D| \cdot |B_3|$, it is seen that $|A_3| = |A_2| = |A_1| = |A|$, $|B_3| = |B_2| = |B_1| = |B|$, and $|D|$ is equal to 1 or -1 . That is, D is unimodular.

Two matrices A and B are called relatively prime if their g.c.r.d.'s and g.c.l.d.'s are unimodular.

THEOREM 3. If $(A, B) = D$ is non-singular, where $A = A_1D$ and $B = B_1D$, then A_1 and B_1 are relatively prime.

Since $D = W_{11}A + W_{12}B = W_{11}A_1D + W_{12}B_1D$, it is seen that

$$(5) \quad I = W_{11}A_1DD^{-1} + W_{12}B_1DD^{-1} \text{ and } I = W_{11}A_1 + W_{12}B_1.$$

Therefore, every common right divisor of A_1 and B_1 must be a right divisor of I . That is, if C is a right divisor of A_1 and B_1 , then $A_1 = A_2C$ and $B_1 = B_2C$ where A_2 and B_2 are in S_n . If this is the case, equations (5) may be written

$$I = W_{11}A_2C + W_{12}B_2C = (W_{11}A_2 + W_{12}B_2)C.$$

From the last equation it is seen that C must be a right divisor of I . However, the only divisors of I are unimodular. Therefore A_1 and B_1 are relatively prime.

THEOREM 4. If $((A, B) = D$ is non-singular, where $A = DA_1$ and $B = DB_1$, then A_1 and B_1 are relatively prime.

Theorem 4 can be established in a manner similar to the proof given for Theorem 3.

In number theory indeterminate equations with solutions restricted to integers are often called Diophantine equations. In dealing with the Diophantine equation $ax + by = c$, one assumes that a , b , and c are given integers where a and b are not zero. Therefore, in solving the matrix equation $AX + BY = C$, assume A , B , and C to be given matrices in S_n where A and B are not the zero matrix and X and Y are not solutions unless they are in S_n .

It is evident that the solutions of the Diophantine equations $ax + by = c$, $xa + yb = c$, $xa + by = c$, and $ax + yb = c$ are identical when a , b , and c are integers. This is not the case for corresponding matrix equations since multiplication of matrices is not always commutative. Therefore, it is necessary to consider the solutions of these various matrix equations separately.

The complete solution of a Diophantine equation consists of two parts. The first part is to determine under what conditions a solution exists. The second part, which is left unsolved for certain matrix equations in this paper, is to give a method which allows one to exhibit all solutions.

THEOREM 5. Let D be a non-singular g.c.l.d. of the matrices

A and B , and let A , B , and C be in S_n . Then a necessary and sufficient condition that the equation

$$(6) \quad AX + BY = C$$

has a solution X and Y in S_n is that D be a left divisor of C .

Let $A = DA_1$ and $B = DB_1$. If there exist matrices X_0 and Y_0 in S_n such that

$$(7) \quad AX_0 + BY_0 = C,$$

then every common left divisor of A and B , including D , is a left divisor of the right member of equation (7). Therefore, D must be a left divisor of C if a solution exists.

Conversely, suppose D is a left divisor of C such that $C = DC_1$, then $DA_1X + DB_1Y = DC_1$ and $A_1X + B_1Y = C_1$. However, there are matrices W_{11} and W_{21} in S_n such that

$$(8) \quad A_1W_{11} + B_1W_{21} = I.$$

Therefore, after multiplying both members of equation (8) on the right by C_1 ,

$$A_1W_{11}C_1 + B_1W_{21}C_1 = C_1,$$

and a solution of equation (6) is

$$X = W_{11}C_1 \text{ and } Y = W_{21}C_1.$$

To illustrate a method of finding a solution to equation (6), let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 1 \\ 2 & 6 \end{pmatrix}, \quad \text{and } C = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}.$$

After substituting these values in equation (6),

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} + \begin{pmatrix} 5 & 1 \\ 2 & 6 \end{pmatrix} \cdot \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}.$$

A g.c.l.d. of A and B is

$$((A, B)) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

when

$$W_{11} = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} \text{ and } W_{21} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Since

$$A_1 = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, B_1 = \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}, \text{ and } C_1 = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix},$$

$$A_1 W_{11} + B_1 W_{21} =$$

$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, upon multiplying the members of the above equation on the right by C_1 ,

$$\begin{aligned} \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \\ = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

From this equation it is seen that a solution to the given equation is

$$X = W_{11}C_1 = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -5 & -3 \\ -3 & -2 \end{pmatrix}$$

and

$$Y = W_{21}C_1 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}.$$

If A_1 (or B_1) in equation (8) is unimodular, then equation (6) has an infinite number of solutions in S_n .

Suppose A_1 is unimodular, then A_1^{-1} is contained in S_n . Since all solutions of equation (8) are solutions of equation (6), it is seen that

$$A_1^{-1}A_1X + A_1^{-1}B_1Y = A_1^{-1}C$$

and

$$X = A_1^{-1}C_1 - A_1^{-1}B_1Y$$

is a solution when Y is any matrix in S_n .

THEOREM 6. Let D be a non-singular g.c.r.d. of the matrices A and B , and let A , B , and C be in S_n . Then a necessary and sufficient condition that the equation

$$XA + YB = C$$

has a solution X and Y in S_n is that D be a right divisor of C .

Theorem 6 can be established in a manner similar to the proof given for Theorem 5.

The above discussion may be extended to the matrix equations of the form

$$AX + YB = C, XA + BY = C,$$

and the matrix equations with three or more unknown matrices. However, instead of continuing with matrix equations of this form, consider the equation

$$(9) \quad X^2 + Y^2 = Z^2.$$

The existence of infinitely many sets of integers which satisfy the quadratic Diophantine equation

$$(10) \quad x^2 + y^2 = z^2$$

was known by Pythagoras. If a , b , and c is any one of these many sets of integers which satisfy equation (10), then it is obvious that the corresponding set of scalar matrices A , B , and C will satisfy the matrix equation (9). For example, consider the set of integers 3, 4, and 5, which is a well known solution to equation (10). Then the corresponding scalar matrices

$$\begin{pmatrix} 3 & 0 & \cdots & 0 \\ 0 & 3 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 3 \end{pmatrix}, \begin{pmatrix} 4 & 0 & \cdots & 0 \\ 0 & 4 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 4 \end{pmatrix}, \text{ and } \begin{pmatrix} 5 & 0 & \cdots & 0 \\ 0 & 5 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 5 \end{pmatrix}$$

will satisfy equation (9).

Another set of matrices which will satisfy equation (9) is

$$X = N^2 - M^2, Y = 2NM, \text{ and } Z = N^2 + M^2$$

where M is any arbitrary matrix in S_n which commutes with N .

For equation (9) may be written in the form

$$(Z - X)(Z + X) = Y^2$$

whenever Z and X commute. For if Z and X commute, then $(Z - X)$ and $(Z + X)$ commute. Therefore if

$$Z = S + R \text{ and } X = S - R$$

then Z and X commute whenever R and S commute. Let

$$R = M^2 \text{ and } S = N^2.$$

If N and M commute, then N^2 and M^2 commute. Hence, a solution to equation (9) is

$$X = N^2 - M^2, Y = 2NM, \text{ and } Z = N^2 + M^2$$

where M and N are arbitrary matrices in S_n which commute.



The critical mathematician has abandoned the search for truth. He no longer flatters himself that his propositions are or can be known to him or to any other human being to be true; and he contents himself with aiming at the correct, or the consistent. The distinction is not annulled nor even blurred by the reflection that consistency contains immanently a kind of truth. He is not absolutely certain, but he believes profoundly that it is possible to find various sets of a few propositions each such that the propositions of each set are compatible, that the propositions of each such set imply other propositions, and that the latter can be deduced from the former with certainty. That is to say, he believes that there are systems of coherent or consistent propositions, and he regards it his business to discover such systems. Any such system is a branch of mathematics.

—C. J. KEYSER

The Tree of Math*

NYLE KARDATZKE
Student, Anderson College

We're learning 'bout a Tree of Math,
We love its structure dearly,
We work our problems every day,
And get right answers . . . nearly.

It's said the grounds for our dear tree
Cannot be proved for sure.
In spite of that, our motto is
"Have faith! It will endure!"

With our fine motto set in mind,
We set about to mount
That tree whose top we cannot see,
Whose marvels we cannot count.

Our start we made on little limbs
Of add, subtract, divide,
Of quotients over subtrahends—
Confusion we'd not hide.

But when at last we mastered these,
We made our way up higher
To complex fractions, algebra,
And functions later we'd admire.

Through the leaves of algebra
We climbed without delay;
We factored, cubed, extracted roots,
The quadratic formula learned one
day.

We thought by now we must have gone
Beyond our first mistakes
In adding, dividing, multiplying;
But these their toll still take.

It seems that through the mighty limbs
Of this great tree we climb
There runs a sap of simple things
Which stays as sure as time.

Now since we've climbed through algebra
And entered calculus,
We're heading on through Math 14
With proofs we've learned to trust.

Now that we've climbed this far alone,
We're about to enter on
A climber's fellowship that's known
As Kappa Mu Epsilon.

We think that this is not a limb
Jutting from the trunk of Math,
That stays secure at one set height
And knows no rising path.

Instead, this seems to us to be
A call to upward go,
A helping hand that we, at Math,
Might more proficient grow.

So we'll not stop or e'en look down
To our early steps below,
But we'll look up and hope to reach
Heights of Math we can't yet know.

As said before, we cannot see
The top of this great tree.
But we'll approach the top, we hope,
With friends of KME.

*Written as a K.M.E. initiate for the Indiana Gamma initiation proceedings.

Mathematics in the Fertile Crescent

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Introduction. There is an intimate relation between the civilization of a nation and its progress in mathematical research. The importance of mathematics started at the time when the primitive man needed to differentiate between one thing and two things. This paper is a short resume of the mathematics of the Fertile Crescent people who established the first civilizations on earth some fifty centuries ago.

It is a common error to attribute Oriental mathematics only to the Egyptians and Babylonians. The different Semitic tribes of the Fertile Crescent, including the Babylonians, contributed to the advancement of the civilization in ancient times. The city of Babylon was the center of culture in that area for a long time, hence the general naming, the Babylonian Culture.

Historians agree that the Fertile Crescent civilization preceded all ancient civilizations. The late discoveries of January, 1957, in Jericho proved by the analysis of the element carbon 14 that the oldest world civilization had been founded on the banks of the river Jordan about 7800 B.C.¹

Discoveries in Crete revealed that there were two languages in use on the island, the first of which was Akkadian, a language of one of the semitic tribes of the Fertile Crescent.² This proves the infiltration of the Fertile Crescent culture to neighboring countries.

The early mathematics of the Hindus and the Chinese are uncertain as to date and importance. The Chinese mathematics has its tradition, practically unbroken until recent years, unlike the Middle Eastern which belong to vanished civilizations. There are no mathematical resources in India and in China which can be definitely dated to a certain period before the Christian era. However, some material contained in the Chinese mathematical texts of the 10th century A.D. is supposed, according to the opinion of some historians, to date back to the 11th century B.C. at the most.³

The geographic and strategic position of the Fertile Crescent

¹"The Oldest Known Civilization. . ." *The Asian Student*, January 27, 1957, p. 4.

²"Science and Citizen," *Scientific American*, October, 1957, p. 57.

³"Cajori, Florian, *A History of Mathematics*, The Macmillan Co., p. 7.

with its agricultural and mineral wealth made the people of the Fertile Crescent excel in the different fields of studies on any other contemporary nation. The Sumerians celebrated the beginning of the year at the vernal equinox as early as 5700 B.C. and probably earlier.⁴ The Egyptians' calendar, dating back to 4241 B.C., divided the year into twelve months, each thirty days, plus five feast days.⁵ Oriental mathematics originated as a practical science to facilitate computation of the calendar, administration of the harvest, organization of public works and collection of taxes. The initial emphasis was naturally on practical arithmetic and mensuration. Arithmetic evolved into Algebra and mensuration into theoretical geometry.

In the sixth century B.C. the Greeks came into direct contact with the people of the Fertile Crescent. The Fertile Crescent science acquired a new meaning and received new emphasis, leading to a more intense interest in these subjects. The mysticism of numbers was suggested to Greece by the Orient. Only for a certain period, Greece became a base for that spreading knowledge of the Fertile Crescent. India was another base. With the establishment of the Arabic Empire in the seventh century A.D. we find that the home of knowledge and wisdom was shifted again to the Fertile Crescent for a period of another eight centuries at which time people of knowledge and wisdom sought Baghdad the way they used to seek its nearby traditional city of Babylon.

Numbers. Our first concepts of numbers and form date back to the Old Stone Age. During that period man lived in caves under conditions very similar to those of animals. Little progress was made in understanding numerical values and space relations until the New Stone Age around 10,000 years B.C.

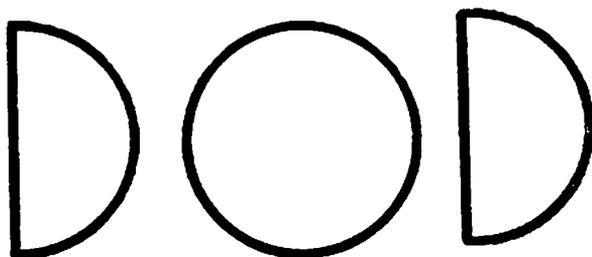
The first occurrence of numerical terms was qualitative rather than quantitative, making a distinction between one and two and many. Higher order numbers were first formed by repetition of unity. It is believed that the early Sumerians were inventors of the notation of numbers⁶ which dominated in the Fertile Crescent and were carried west by the Phoenicians, the sailors, and East by the caravans of the traders.

The wedge-shaped characters, cuneiform writing, of the eastern part of the Fertile Crescent was due to pressing the stylus on the soft

⁴"Smith, David, *History of Mathematics*. Vol. 1, Ginn and Co., 1932, p. 37.

⁵"Smith, *Ibid*, p. 37.

⁶"Cajori, Florian, *A History of Mathematics*, The Macmillan Co., p. 4.

FIGURE 1

$$60 \quad + \quad 10 \quad + \quad 1 \quad = \quad 71$$

This represents an early sexagesimal system. D is either one or sixty, depending on its place. The date is 3200 B. C.

clay, their media for writing, after which the clay was baked to hardness.

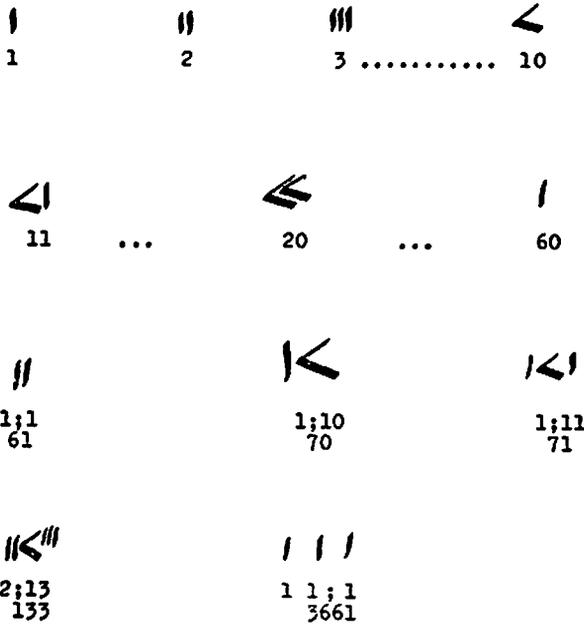
The Sumerians, the earliest of the Semitic Tribes, used two systems for writing numbers, the decimal with symbols for 1, 10, 100, 1,000, 10,000 and with ten as a base. Here the three types of numerals appear, all with the principle that the higher symbol precedes the lower one in the direction of writing.⁷

The sexagesimal system has 60 as base. Up to sixty, the symbols and the base were the same as in the decimal system, but sixty and its powers were represented by one. The most striking feature of the Fertile Crescent system of numbers is its uniqueness in the place value notation.⁸

⁷"Algebraic Development Among the Egyptians and the Babylonians", *The American Mathematical Monthly*, Vol. XXIV, p. 264.

⁸"Algebraic Development Among the Egyptians and the Babylonians", *The American Mathematical Monthly*, Vol. XXIV, p. 264.

FIGURE 2



The most striking feature of the Fertile Crescent system of numbers is the place value notation. e.g. I< is 70 and never 11. Sometimes difficulty is encountered in reading a number, wondering what place value it has. I< may be $60^2 + 10$ instead of $60 + 10$, or $70/60$ or $70/60^2$. Usually one can tell from the context of the problem the required value. This flexibility of numbers has an advantage in expressing their reciprocals. Reciprocal of 3 in our decimal system is 1.3 recurring which is in the sexagesimal system the regular << 20/60.

⁹Cajori, Florain, A History of Mathematics, The Macmillan Co., p. 5-6

Both the sexagesimal divisions and the place value notation remained the permanent possessions of mankind. Our present units of time, the division of the circle into 360 degrees and its divisions, the dozen and the 12 parts of a unit date back to the Sumerians.¹⁰

Zero was expressed in the Fertile Crescent by leaving a blank space. Not until the Selucidan era, was zero expressed with a symbol.¹¹

Arithmetic. Counting with reference to base was the origination of a primitive type of arithmetic. Multiplication began where 20 was expressed not as 10 plus 10, but as 2 times 10.

The Fertile Crescent people did much of their computative arithmetic with the aid of tables, which ranged from simple addition to lists of reciprocals and of exponents. They used units of length, width, area, volume, weight, time and capacity and showed their relation in tables of computation.¹²

Tables of reciprocals were used to reduce division into multiplication. As we expect from the level of civilization attained by these people they were familiar with all kinds of legal and domestic contracts, like bills, receipts, accounts, interest in both the simple and compound forms, mortgages, deeds of sale, guarantees and firms . . . and consequently with arithmetic processes to solve such problems. Exponential tables were used in compound interest computations. A problem dated 1700 B.C. asks for the time it would take a certain sum of money to double itself at compound interest of 20 percent. One of the tables gives a list of numbers of the form n^2 plus n^2 .¹³

There is some excellent approximation of square roots. Square root of 2 was indicated in the sexagesimal system by 1 : 24, 51, 10 which is approximately $\frac{17}{12}$ or 1.4167. The reciprocal of the square root of 2 is given as $\frac{22}{37}$ or 0.7083.

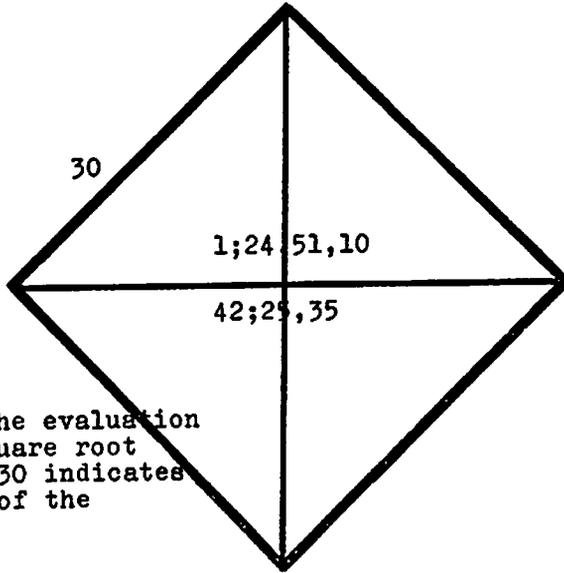
The semitic tribes were acquainted with the theory of proportion and even invented the so called musical proportion. They used negative numbers and were acquainted with the law of signs in multiplication. A late astronomical text was discovered in which before each of the twelve numbers the words 'tab' and 'lal', which mean

¹⁰Struik, D. J., *A Concise History of Mathematics*, Dover Publications, Inc., New York, 1948, p. 25.

¹¹"Babylonian Mathematics with Special References to Recent Discoveries," *The Mathematics Teacher*, Vol. XXIX, No. 5, p. 211.

¹²Struik, D. J., *A Concise History of Mathematics*, Dover Publications, Inc., New York, 1948, p. 29.

¹³Evos, Howard, *An Introduction to the History of Mathematics*, Rinehart and Co., New York, 1953, p. 32.

FIGURE 3

This is the evaluation
of the square root
of 2.¹⁴ 30 indicates
the side of the
square a.

$$\begin{aligned} \text{Diagonal}^2 &= 2 a^2 \\ \text{Diagonal} &= \sqrt{2} \times a \\ &= 1; 24, 51, 10 \times 30 \\ &= 42; 25, 35 \end{aligned}$$

$$\begin{aligned} \sqrt{2} &= 1; 24, 51, 10 \\ &= 1 + \frac{24}{60} + \frac{51}{3600} + \frac{10}{216000} \\ &\approx 1 \frac{25}{60} \\ &\approx 1.4162 \end{aligned}$$

¹⁴Neugebauer, O., The Exact Sciences in Antiquity,
Princeton University Press, 1952, p. 43

FIGURE 4

$$\begin{aligned}\sqrt{A} &= \sqrt{a^2 + h} \\ &\approx a + \frac{h}{2a} \\ &\approx \frac{1}{2} \left(a + \frac{A}{a} \right)\end{aligned}$$

$$\sqrt[3]{b} = \sqrt[3]{a} \times \sqrt[3]{\frac{b}{a}}$$

Square roots of numbers which are not a perfect square and could not be read from the tablets were found by means of the above formula.¹⁵

The lower formula is for the cube root.¹⁶

¹⁵Struik, D. J., A Concise History of Mathematics, Dover Publications, Inc., 1948, p. 29.

¹⁶Neugebauer, O., & A. Sachs, Mathematics Cuneiform Texts, American Society & The American Schools of Oriental Researches, New Haven, 1945, p. 42.

plus and minus, were placed, suggesting the arrangement of points above and below a line which cut a wave shaped curve.¹⁷

In the multiplication tablets all the products from 1 to 20 were given and then only the products of 30, 40, and 50. This was a means of saving space because all products up to 59 can be obtained from such a tablet by addition of two of these products.¹⁸

¹⁷The Mathematics Teacher, Babylonian Mathematics with Special References to Recent Discoveries, Ed. William, Vol. XXIX, No. 5, 1936, p. 211.

¹⁸Neugebauer, O., The Exact Sciences in Antiquity, Princeton University Press, 1952, pp. 19-24.

Beside slope problems we can say that they were acquainted with logarithms of numbers. Some tablets contained tables of a^n , where n is an integer between 2 and 10 and a is one of the square numbers 9, 16, 100, 225.

One of the problems asks for the power to which a certain number should be raised in order to yield a given number.¹⁹

Series showing the illumination of the face of the moon were given in geometric progression from 5 to 80 for the first five nights and then from 80 to 240 in arithmetic progression for the rest of the ten nights.²⁰

Algebra. Algebra was developed earlier than 2000 B.C. One of the specific reasons for its development seems to have been the use of old Sumerian script by the new Semitic rulers, the Babylonians. The ancient script was, like the hieroglyphics, a collection of ideograms, each sign denoting a single concept. The Semitic people used them for the phonetic rendition of their language and also took over some signs in their old meaning. These signs now expressed concepts, but were pronounced in a different way. Such ideograms were well fitted for an algebraic language as our present plus and minus signs which are ideograms. This algebraic language remained a traditional part of the curriculum in the schools of the Fertile Crescent.²¹

The Babylonians of Hammurabi's days were handling equations in many variables, quadratic equations and even problems involving cubic and biquadratic equations.²² They solved the quadratic equations by a method similar to the completion of the square. They arrived at the formula $x = \sqrt{c + (b/2)^2} - b/2$.²³ They also arrived at the law of expansion of the algebraic expression $(a + b)^2$.²⁴

They used to solve problems similar to those below by the aid of their tables:

$$1) \quad x^2 + x^3 = a$$

¹⁹Neugebauer, *Ibid.*, p. 35.

²⁰American Mathematical Monthly, *Algebraic Development Among the Egyptians and the Babylonians*, L. Karpinski, Vol. XXIV, p. 257.

²¹Struik, D. J., *A Concise History of Mathematics*, Dover Publications, Inc., New York, 1948, p. 30.

²²*Ibid.*, p. 26.

²³Claget, Marshall, *Greek Science in Antiquity*, Abelard-Schuman, Inc., 1955, p. 26.

²⁴Smith, D. E., *History of Mathematics*, V. 1 & 11, Ginn & Company, N.Y., 1923, p. 40.

$$2) \quad xy = a; c(x-y) - (x+y)^2 = -b$$

3) $xy = a; bx^2/y + cy^2/x + d = 0$. (This leads to an equation of the sixth degree, quadratic in x^2).

Geometry. The Neolithic man developed a keen feeling for geometric patterns. The baking and coloring of pottery, tiles, the plaining of rushes, the weaving of baskets and textiles and later the working of metals, led to the cultivation of plane and spatial relationships. Dancing patterns must have also played a role.

The Fertile Crescent geometry is intimately related to practical mensuration. It was more algebraic in nature. The Fertile Crescent people knew that the perpendicular through the vertex of an isosceles triangle bisects the base and that an angle inscribed in a semi-circle is a right angle.²⁵ They were the first to mention that six equilateral triangles completely fill the space around a point.²⁶ The circumference of a circle was taken as three times the diameter, and the area as one twelfth the square of the circumference.

They were also familiar with areas of right isosceles and equilateral triangles, the area of a parallelogram and the area of a trapezoid.

The volume of a parallelepiped was calculated as the product of the lengths of its three dimensions, and the volume of a right prism with a trapezoidal base to be equal to the products of the base times the altitude of the prism. Such a volume as the latter would be considered in estimating the amount of earth dug in a section of a canal. The volume of a right cylinder was obtained by finding the product of the base area and the altitude. The volume of a frustum of a cone or a square pyramid was sometimes incorrectly given the product of the altitudes and half the sum of the bases.²⁸

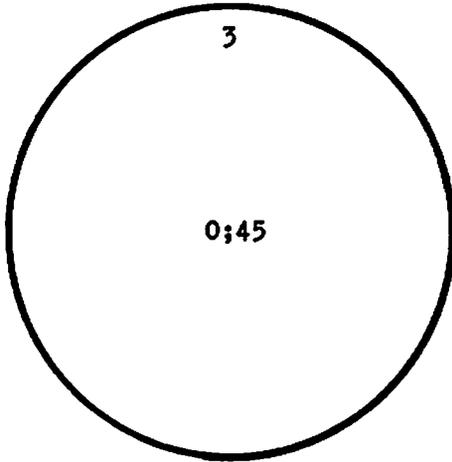
The determination of the square root of 2 and the diagonal of the square from its side is a sufficient proof that the Pythagorean theorem was known to the Fertile Crescent people a thousand years before Pythagoras. Moreover, they did elaborate work on the Pythagorean triple, the proportionality of 3:4:5 of the sides of a right

²⁵The Mathematics Teacher, *Babylonian Mathematics with Special References to Recent Discoveries*, Ed. William, Vol. XXIX, No. 5, p. 214.

²⁶American Mathematical Monthly, *Algebraic Development Among the Egyptians and the Babylonians*, L. Karpinski, Vol. XXIV, p. 265.

²⁸The Mathematics Teacher, *Babylonian Mathematics with Special References to Recent Discoveries*, Ed. William, Vol. XXIX, No. 5, p. 215.

The Pentagon
FIGURE 5²⁷



The diameter is 1
 The circumference is 3
 The area = $9 \times \frac{1}{12}$
 $= \frac{3}{4}$

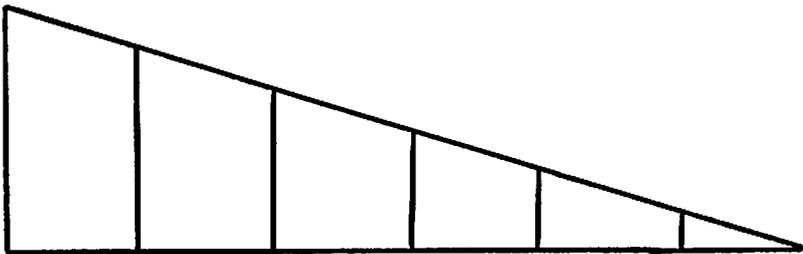
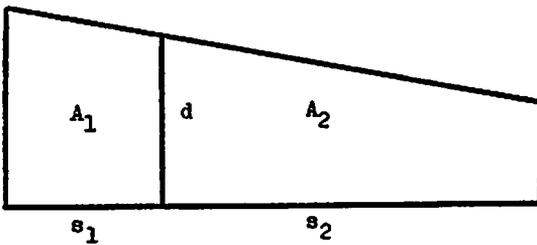
In sexagesimal system
 $= 9 \times 0;5$
 $= 0;45$

The diameter of this circle is 1, its circumference is 3 times 1 which is 3. The area was found by dividing the 12 into the square root of the circumference. The 12 could be interpreted as 4 times 3, the 4 to reduce the diameter squared into the radius squared and the 3 to cancel the power of π squared arising from squaring the circumference.

This shows that $\pi = 3$. Sometimes it was taken as $3 \frac{1}{8}$.

²⁷Neugebauer, O., The Exact Sciences in Antiquity, Princeton University Press, 1952, p. 44

FIGURE 6



The first problems on this tablet asks to divide a trapezoid with area A into two parts A_1 and A_2 such that the partial lengths s_1 and s_2 have the proportion 1:5. The partial areas and the common side are given.

The lower problem is an inheritance problem. A triangular field is divided among 6 brothers by equidistant lines parallel to one side of it. The question asks to find the difference between the allotments of the six brothers.

The steps of work and the correct answers were given in the same tablet but the figures were not.

Table 1

<u>Side <i>a</i></u>	<u>Side <i>b</i></u>	<u>Hypotenuse <i>c</i></u>
120	119	169
3456	3367	4825*
4800	4601	6649
13500	12709	18541
72	65	97
360	319	481
2700	2291	3541
960	799	1249
600	481*	769
6480	4961	8161
60	45	75
2400	1679	2929
240	161*	289
2700	1771	3229
90	56	106*

* are errors in original

angled triangle, and the Primitive Pythagorean triple in which the sides $a:b:c$ are not a multiple of the ratio $3:4:5$. The results of their constructing tables (Table 1) on this subject were reached from the relation: $a = 2uv$, $b = u^2 - v^2$ and $c = u^2 + v^2$ where u and v are relatively prime and u is greater than v .²⁹ There are many problems concerning a transversal parallel to a side of a right triangle and the proportionality of the sides. Moreover, they knew that the sides about corresponding angles of two similar right triangles are proportional.³⁰

Columns a and b denote the two sides of a right angled triangle and column c the hypotenuse. A fourth column which is practically destroyed, indicates the secant of one of the angles.

²⁹Eves, Howard, *An Introduction to the History of Mathematics*, Rinehart and Co., Inc. New York, 1953, p. 35.

³⁰The Mathematics Teacher, *op. cit.*, p. 216-17.

The people of the Fertile Crescent were indefatigable table makers and computers of high skill. The depth and diversity of the problems they considered are amazing. Research may still reveal that they were familiar with formulas of summation in the form.³¹

$$\sum_{i=0}^n r^i = (r^{n+1} - 1)/(r - 1)$$

and
$$\sum_{i=1}^n i^2 = (2n + 1)/3 \sum_{i=1}^n i = n(n+1)(2n + 1)/6.$$

In series problems they tackled such problems as

$$1 + 2 + 2^2 + \dots + 2^n = 2^n + 2^n - 1$$

and

$$1^2 + 2^2 + 3^2 + \dots + 10^2 = [1(1/3) + 10(2/3)]55 = 385$$

The mathematicians of the Fertile Crescent were not only able to establish the Pythagorean Theorem but adopted the solution to the further condition that the proportion of the sides c/a of the triangle ABC, decreases from step to step by a number deviating very little from one sixtieth; c/a denotes the secant of the angle B in the right angled triangle ABC.³²

Conclusion. The main criticism against the Oriental mathematics is that it was taught answering the question "HOW". Most of the credit in the antiquity of mathematics is given to the Greeks who supplied the "WHY" answer to many of the mathematical problems, and the Fertile Crescent people have had little recognition in this respect. Some reasons for this lack of recognition might be the following:

- 1) The Greek scientists and historians affirm the superiority of the Fertile Crescent Sciences but unfortunately most of the direct documents from the Fertile Crescent are missing.
- 2) Work in mathematics cannot be done without a knowledge of the principles behind it. Hence the "WHY" answer must have been known in certain circles in the Fertile Crescent.

³¹Evos, *op. cit.*, p. 33.

³²*Ibid.*, p. 36.

- 3) The priesthood of the Fertile Crescent monopolized work in sciences as a religious sanctuary for themselves. They might have been the ones who knew the "WHY" answer but it was not in their favour to give their secrets out to the layman.
- 4) To consider researches in sciences as spoils taken over by the Greeks after the defeat of the Fertile Crescent is similar to the case of Germany after the Second World War. Germany is getting little recognition today, although the achievements of the German scientists before and after the war cannot be denied.
- 5) There is an accelerating relation between the advancement of sciences and time. Probably we can classify the standard of the Greeks to be equivalent to a graduating high school student of the present time. Twenty centuries after the Greeks, until our present day, the whole of the world contributed an advancement of five or six years of college education. To consider that the Greeks started from very little and contributed in a period of only five centuries an equivalent of eleven years of education does not seem probable. There, in the Fertile Crescent mathematics was born, and there it grew to its youth, and from there it spread not only to Greece but to the whole world.



"It is impossible not to feel stirred at the thought of the emotions of men at certain historic moments of adventure and discovery—Columbus when he first saw the Western shore, Pizarro when he stared at the Pacific Ocean, Franklin when the electric spark came from the string of his kite, Galileo when he first turned his telescope to the heavens. Such moments are also granted to students in the abstract regions of thought, and high among them must be placed the morning when Descartes lay in bed and invented the method of co-ordinate geometry."

—ALFRED NORTH WHITEHEAD

Hyperbolic Functions

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Hyperbolic functions are found useful in the application of mathematics to varied types of problems, and in particular, to problems confronting electrical engineers.

Certain combinations of exponential functions which occur frequently in practice are given special names. Because of the fact that, in a certain sense, these functions are related to a hyperbola just as the circular functions, sine, cosine, etc., are related to the unit circle, it is conventional to call them hyperbolic functions, in particular hyperbolic sine, hyperbolic cosine, and hyperbolic tangent. Corresponding to the formulas of ordinary trigonometry, an analogous set of formulas can be developed in hyperbolic trigonometry. Rather than an angle, it is fundamentally essential that the concept of sector area be introduced.

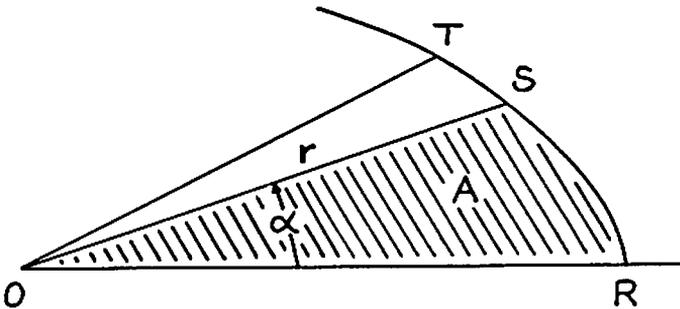


FIG. 1

Given a sector (Fig. 1) formed by two straight lines OR and OS , drawn from the origin O , and the arc RS of a curve. Let r be the length of OS , α be the angle ROS , and A the area of the sector. If the sector takes an increment SOT due to an infinitesimal increase in the angle α , the differential sector area is given by the formula

$$dA = \frac{1}{2}r^2d\alpha.$$

If OR is taken as the x -axis and (x,y) are the rectangular coordinates of the point S , we have

$$r^2 = x^2 + y^2, \quad d\alpha = d(\arctan \frac{y}{x}) = \frac{x dy - y dx}{x^2 + y^2},$$

and hence

$$dA = \frac{1}{2}(x dy - y dx).$$

Now consider (Fig. 2) a unit circle, $x^2 + y^2 = 1$, and a unit hyperbola, $x^2 - y^2 = 1$. Let u be the sector $OSRS$ with $OR = 1$. We shall express the rectangular coordinates, (x, y) , of S in terms of u . As the sector opens out, S moves to T and S' to T' . Thus the increment in the sector area, u , is twice the area of SOT , and the differential of sector area is

$$du = x dy - y dx.$$

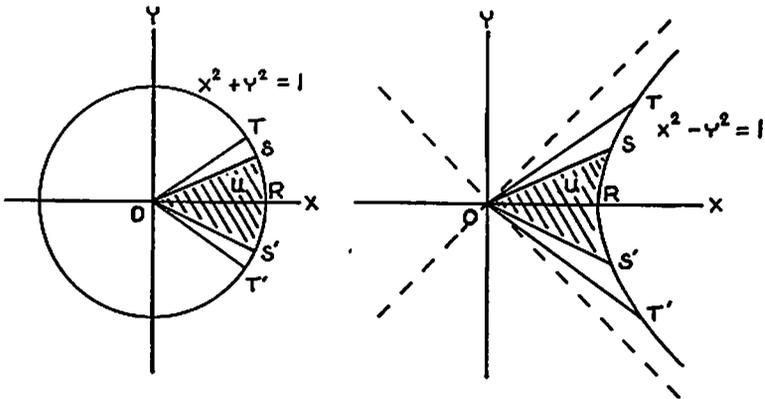


FIG. 2

Substituting first the value of y from the equation of the circle, then the value of y from the equation of the hyperbola, we have

For the Circle

$$\begin{aligned} du &= x d(\sqrt{1-x^2}) - \sqrt{1-x^2} dx \\ &= \left(\frac{-x^2}{\sqrt{1-x^2}} - \sqrt{1-x^2} \right) dx \\ &= \frac{-dx}{\sqrt{1-x^2}}, \end{aligned}$$

$$u = \int_1^x \frac{-dx}{\sqrt{x^2 - 1}} = \cos^{-1}x,$$

$$x = \cos u.$$

For the Hyperbola

$$du = x d(\sqrt{x^2 - 1}) - \sqrt{x^2 - 1} dx$$

$$= \left(\frac{x^2}{\sqrt{x^2 - 1}} - \sqrt{x^2 - 1} \right) dx$$

$$= \frac{dx}{\sqrt{x^2 - 1}},$$

$$u = \int_1^x \frac{dx}{\sqrt{x^2 - 1}} = \ln(x + \sqrt{x^2 - 1}),$$

$$e^u = x + \sqrt{x^2 - 1}$$

$$e^{2u} - 2xe^u + x^2 = x^2 - 1$$

$$x = \frac{1}{2}(e^u + e^{-u}) = \cosh u.$$

Following the case for the circle, we are led to a familiar circular function, cosine, to express x in terms of u . We can think of the cosine of a number as representing an area just as easily as we can think of it as being the cosine of an angle. For instance, if OR is one inch, the number of square inches in the circular sector, u , is equal to the number of radians in the angle SOR , so that the x coordinate of S can be regarded as the cosine of either number. In the case of the hyperbola we arrive at the fact that $x = \frac{1}{2}(e^u + e^{-u})$. It is natural to call this relation, by analogy, a hyperbolic function, the hyperbolic cosine of u . Thus we get our first definition in hyperbolic trigonometry:

$$\cosh u = \frac{1}{2}(e^u + e^{-u}).$$

Do these equations, $x = \cos u$, $x = \cosh u$, check dimension-

ally? Is it possible to take the cosine or the hyperbolic cosine of square feet and obtain feet? Suppose that instead of the unit circle and unit hyperbola we had used the equations

$$x^2 + y^2 = a^2 \qquad x^2 - y^2 = a^2.$$

Then we would obtain

$$x = a \cos \frac{u}{a^2} \qquad x = a \cosh \frac{u}{a^2}.$$

If a is in inches and u is in square inches, the ratio u/a^2 is a pure number whose cosine or hyperbolic cosine is also dimensionless, so that x has the dimensions of a . When $a = 1$ we can write

$$x = \cos u \qquad x = \cosh u$$

with the understanding that x is in units of OR and u is the number of square units in the sector area.

In order to express y in terms of u , we have

For the Circle	For the Hyperbola
$y = \sqrt{1 - x^2}$	$y = \sqrt{x^2 - 1}$
$= \sqrt{1 - \cos^2 u}$	$= \sqrt{\frac{e^{2u} + 2 + e^{-2u} - 4}{4}}$
$= \sin u$	$y = \frac{1}{2}(e^u - e^{-u}) = \sinh u$

This is the second definition of hyperbolic trigonometry, $\sinh u = \frac{1}{2}(e^u - e^{-u})$. The functions $\sinh u$ and $\cosh u$ are used more frequently than the other four functions.

In the foregoing definitions it may be noticed that the abbreviations of the names are formed by affixing the letter "h" to the corresponding abbreviations for the ordinary trigonometric functions. The reason for the name, "hyperbolic", is not of particular consequence. Another common notation is Ch u to replace $\cosh u$ and Sh u to replace $\sinh u$.

The analogy is carried still further with the defining of the function

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{\frac{1}{2}(e^u - e^{-u})}{\frac{1}{2}(e^u + e^{-u})}$$

$$= \frac{e^u - e^{-u}}{e^u + e^{-u}} .$$

The other hyperbolic functions; hyperbolic secant, hyperbolic cosecant, and hyperbolic cotangent are defined by analogy to the circular functions:

$$\operatorname{sech} u = \frac{1}{\cosh u} = \frac{2}{e^u + e^{-u}}$$

$$\operatorname{csch} u = \frac{1}{\sinh u} = \frac{2}{e^u - e^{-u}}$$

$$\operatorname{coth} u = \frac{1}{\tanh u} = \frac{\cosh u}{\sinh u} = \frac{e^u + e^{-u}}{e^u - e^{-u}} .$$

These six definitions are used to define the six functions when u is negative also. By substituting $-u$ for u in the definitions we find:

$$\sinh(-u) = -\sinh u$$

$$\cosh(-u) = \cosh u$$

$$\tanh(-u) = -\tanh u$$

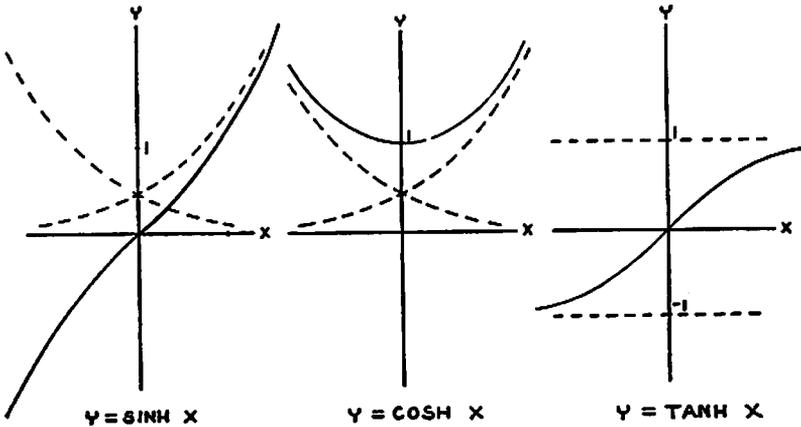


FIG. 3

The following information can be obtained directly from the definitions of the hyperbolic functions and is helpful in sketching the graphs (Fig. 3).

When $x = 0$, the function $\sinh x$ has the value zero; $\sinh x$ can take on any real value.

The function $\cosh x$ has the value one for $x = 0$; $\cosh x \geq 1$ for all real values of x .

The function $\tanh x$ has the value zero for $x = 0$; $-1 < \tanh x < 1$; $\lim_{x \rightarrow \infty} \tanh x = 1$; $\lim_{x \rightarrow -\infty} \tanh x = -1$. The hyperbolic functions have no real period, but corresponding to the period of 2π possessed by the circular functions there is a period of $2\pi i$ for the six hyperbolic functions when the domain is extended to the complex plane.

The graphs of Fig. 3 were obtained by drawing the two curves $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ and adding or subtracting ordinates. The graphs of the other three hyperbolic functions are shown in Fig. 4.

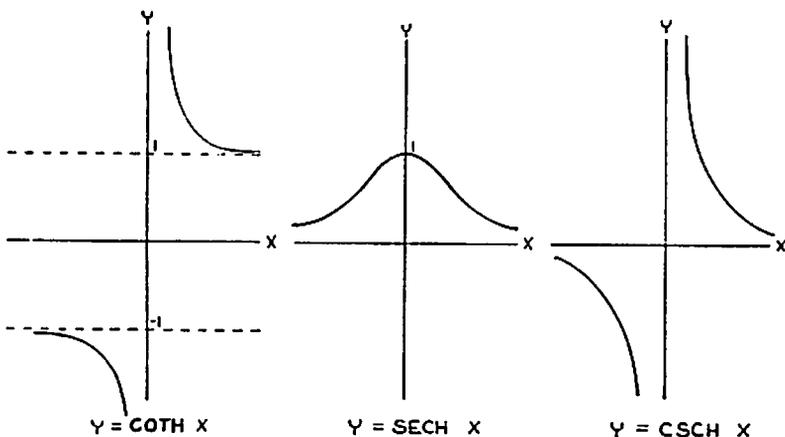


FIG. 4

From the definitions all the formulas of hyperbolic trigonometry may be derived. From the definitions of $\sinh x$ and $\cosh x$ it follows that

$$\sinh^2 x = \left(\frac{e^x - e^{-x}}{2} \right)^2 = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$\cosh^2 x = \left(\frac{e^x + e^{-x}}{2} \right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4}$$

From these two equations we can observe that

$$\cosh^2 x - \sinh^2 x = 1.$$

This is an identity similar to the well-known identity $\cos^2 x + \sin^2 x = 1$ in circular trigonometry.

From the definitions we also get the following useful formulas:

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$$

$$e^x = \cosh x + \sinh x$$

$$e^{-x} = \cosh x - \sinh x$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

These identities are quite analogous to those for the trigonometric functions and are easy to verify.

Directly from the definition we find that if

$$y = \sinh x = \frac{1}{2}(e^x - e^{-x})$$

then

$$\frac{dy}{dx} = \frac{1}{2}(e^x + e^{-x})$$

or

$$\frac{d \sinh x}{dx} = \cosh x.$$

In a similar manner

$$\frac{d \cosh x}{dx} = \frac{d}{dx} \left[\frac{1}{2}(e^x + e^{-x}) \right]$$

The Pentagon

$$\begin{aligned}
 &= \frac{1}{2}(e^x - e^{-x}) \\
 &= \sinh x.
 \end{aligned}$$

For the derivation of the formula for the derivative of the hyperbolic tangent,

$$\begin{aligned}
 \frac{d \tanh x}{dx} &= \frac{d}{dx} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right) \\
 &= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
 &= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{(e^x + e^{-x})^2} \\
 &= \frac{4}{(e^x + e^{-x})^2} \\
 &= \operatorname{sech}^2 x.
 \end{aligned}$$

In a similar manner the formulas for the derivatives of the other hyperbolic functions can be found.

$$\begin{aligned}
 \frac{d \operatorname{csch} x}{dx} &= -\operatorname{csch} x \coth x \\
 \frac{d \operatorname{sech} x}{dx} &= -\operatorname{sech} x \tanh x \\
 \frac{d \operatorname{coth} x}{dx} &= -\operatorname{csch}^2 x
 \end{aligned}$$

Some important integration formulas for the hyperbolic functions may be developed directly from the definitions.

$$\begin{aligned}
 \int \sinh x \, dx &= \int \frac{1}{2}(e^x - e^{-x}) \, dx \\
 &= \frac{1}{2} \int e^x \, dx - \frac{1}{2} \int e^{-x} \, dx \\
 &= \frac{1}{2} e^x + \frac{1}{2} e^{-x} + C \\
 &= \cosh x + C.
 \end{aligned}$$

In a like manner

$$\int \cosh x \, dx = \int \frac{1}{2}(e^x + e^{-x}) \, dx \\ = \sinh x + C.$$

Most standard mathematical tables contain these and other formulas for derivatives and integrals of hyperbolic functions and also the numerical values of the hyperbolic functions.

The arrangements of the LL and LLO scales as reciprocals on the modern slide rule make possible rapid evaluation of hyperbolic functions. The vector slide rule has scales especially for hyperbolic sines and tangents. By definition $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. To determine $\sinh x$ and $\cosh x$ where $x = 1.23$, for example, use the following procedures:

Runner to 1.23 on *D* scale.

Read $e^x = 3.42$ on *LL3*.

Read $e^{-x} = .292$ on *LLO3*.

$$\sinh 1.23 = \frac{3.42 - .292}{2} = 1.564; \quad \cosh 1.23 = \frac{3.42 + .292}{2}$$

= 1.856. All the other functions can rapidly be evaluated similarly.

The inverse hyperbolic functions are related to the hyperbolic functions in exactly the same way as the inverse trigonometric functions are related to the trigonometric functions. Thus $y = \cosh^{-1}x$, which is read as y equals the inverse hyperbolic cosine of x , means $x = \cosh y$. The same notation is used for the other inverse hyperbolic functions.

$$y = \sinh^{-1}x \longleftrightarrow \sinh y = x$$

$$y = \tanh^{-1}x \longleftrightarrow \tanh y = x$$

$$y = \operatorname{sech}^{-1}x \longleftrightarrow \operatorname{sech} y = x$$

$$y = \operatorname{csch}^{-1}x \longleftrightarrow \operatorname{csch} y = x$$

$$y = \operatorname{coth}^{-1}x \longleftrightarrow \operatorname{coth} y = x$$

The hyperbolic functions are actually simple combinations of exponential functions so it is not surprising that the inverse hyperbolic functions are easily expressed in logarithmic form. Consider

$$y = \cosh^{-1}x$$

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Then

$$\begin{aligned} 2x &= e^y + e^{-y} \\ e^y - 2x + e^{-y} &= 0. \end{aligned}$$

Multiplying by e^y ,

$$e^{2y} - 2xe^y + 1 = 0,$$

which is a quadratic equation in e^y . From the quadratic formula we obtain

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

We must keep both signs so

$$y = \ln(x \pm \sqrt{x^2 - 1})$$

hence

$$\cosh^{-1}x = \ln(x \pm \sqrt{x^2 - 1}), \text{ where } x \geq 1.$$

Similarly if

$$y = \sinh^{-1}x$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1},$$

where we must use only the positive sign since e^y cannot be negative. Taking natural logarithms of both sides we obtain

$$y = \ln(x + \sqrt{x^2 + 1}),$$

then

$$\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1}).$$

Using the same procedure it is found that

$$\tanh^{-1}x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right), \text{ where } |x| < 1.$$

The three graphs in Fig. 5 display the inverse hyperbolic sine, cosine, and tangent.

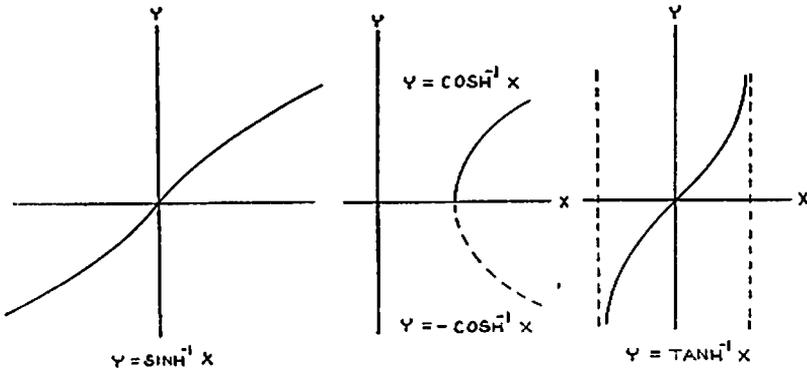


FIG. 5

The function $\sinh^{-1}x$ is a single-valued continuous function defined for all real values of x . This function may assume any value, positive or negative. Furthermore, $\sinh^{-1}(-x)$ is equal to $-\sinh^{-1}x$.

The function $\cosh^{-1}x$ is a double-valued function, the two values differing only in sign, and is defined for all real values of $x \geq 1$. This function may be split up into two single-valued continuous functions, using the notation $\text{Cosh}^{-1}x$ to stand for the non-negative value that is assumed by $\cosh^{-1}x$ for any value of $x \geq 1$. Then, the heavy line in the former graph would correspond to $y = \text{Cosh}^{-1}x$, and the dashed line to $y = -\text{Cosh}^{-1}x$. The function $\text{Cosh}^{-1}x$ may assume any non-negative value.

The function $\tanh^{-1}x$ is a single-valued continuous function for all values of x such that $|x| \leq 1$. This function may assume any real value. It also has the same symmetry property as $\sinh^{-1}x$, that is, $\tanh^{-1}(-x) = -\tanh^{-1}x$.

From the logarithmic equivalents of the inverse hyperbolic functions, we can easily derive the following formulas:

$$\frac{d(\sinh^{-1}x)}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{d(\cosh^{-1}x)}{dx} = \frac{1}{\sqrt{x^2 - 1}}, \text{ for } x > 1$$

$$\frac{d(\tanh^{-1}x)}{dx} = \frac{1}{1 - x^2}, \text{ for } |x| < 1.$$

Occasionally these two formulas are helpful

$$\coth^{-1}x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right), \text{ for } |x| > 1$$

$$\frac{d(\coth^{-1}x)}{dx} = \frac{1}{1 - x^2}, \text{ for } |x| > 1.$$

The other derivatives are:

$$\frac{d(\operatorname{sech}^{-1}x)}{dx} = \frac{-1}{x\sqrt{1-x^2}}, \text{ for } 0 < x < 1$$

$$\frac{d(\operatorname{csch}^{-1}x)}{dx} = \frac{-1}{|x|\sqrt{x^2+1}}, \text{ for } x \neq 0.$$

The hyperbolic cosine curve is that in which a piece of string, cable, high tension line, watch fob, etc., will hang between two points by which it is suspended. This arc is known as a catenary whose standard equation is

$$y = a \cosh \frac{x}{a}.$$

Using Fig. 6, we get the following working formulas where L is the number of feet between the supports of the cable.

$$y = a \cosh \frac{x}{a}, \quad a = \frac{F_0}{w}$$

$$\text{The dip is given by } d = a\left(\cosh \frac{L}{2a} - 1\right).$$

$$\text{The cable length is } s = 2a \sinh \frac{L}{2a}.$$

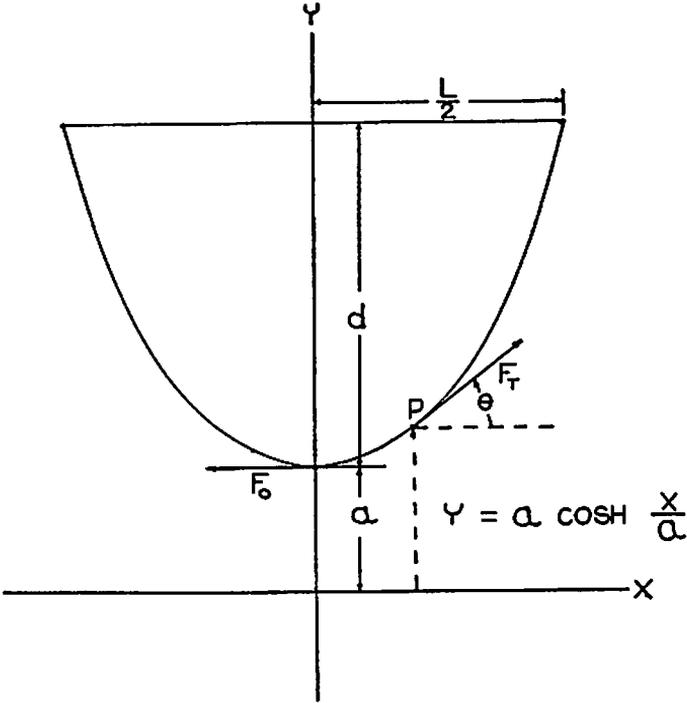


FIG. 6

The distance above the origin is $a = \frac{s^2}{8d} - \frac{d}{2}$.

The tension at P is $F_T = wy$.

The tension at the support is $F = w(a + d)$.

The weight of a unit length of the cable is represented by w .

One other important use of the hyperbolic functions is in the differential equation

$$\frac{d^2x}{dt^2} = k^2x.$$

This is the case of a repulsive force directed away from O; the force

and acceleration have the same sign as the displacement. The solution is given by

$$x = C_1 e^{kt} + C_2 e^{-kt}$$

or

$$x = C_1 \sinh kt + C_2 \cosh kt$$

and

$$v = k(C_1 \cosh kt + C_2 \sinh kt).$$

There are many more practical and theoretical applications of the hyperbolic functions. These applications are too long and detailed to include in this article. The relationship between these functions and logarithms gives a key to their practicalness.

It has been my intention in this article to bring together several aspects of the hyperbolic functions into one unit.



De Morgan was explaining to an actuary what was the chance that a certain proportion of some group of people would at the end of a given time be alive; and quoted the actuarial formula, involving π , which, in answer to a question, he explained stood for the ratio of the circumference of a circle to its diameter. His acquaintance, who had so far listened to the explanation with interest, interrupted him and exclaimed, "My dear friend, that must be a delusion, what can a circle have to do with the number of people alive at a given time?"

—W. W. R. BALL

The Problem Corner

EDITED BY J. D. HAGGARD

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before October 1, 1960. The best solutions submitted by students will be published in the Fall, 1960, issue of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to J. D. Haggard, Department of Mathematics, Kansas State College, Pittsburg, Kansas.

PROPOSED PROBLEMS

131. *Proposed by Mary Sworske, Mount Mary College, Milwaukee, Wisconsin.*

Find three natural numbers a, b, c such that $a^n + b^n + c^n$ is an integral multiple of 18 for any natural number n .

132. *Proposed by Don Hayler, Pomona College, Claremont, California.*

Consider the series

$$S_k = 1^k + 2^k + 3^k + \cdots + n^k = \sum_{x=1}^n x^k$$

$$\text{For } k = 1, S_1 = \frac{n(n+1)}{2}$$

$$\text{For } k = 2, S_2 = \frac{n(n+1)(2n+1)}{6}$$

Compute S_3 and S_4 .

133. *Proposed by the Editor (From The Mathematical Monthly).*

Let M and N be two points one unit apart. With M and N as centers and with unit radii draw arcs ANB and AMB . Let Q be any point on arc AMB and P_1 and P_2 be points on arc ANB such that N is midpoint of arc P_1P_2 .

Show that $QP_1 + QP_2 \leq 2$

134. *Proposed by the Editor (From The Mathematical Monthly.)*

Evaluate the determinant.

$$\begin{vmatrix} n & n-1 & \cdots & n-m \\ n-1 & n & \cdots & n-m+1 \\ n-2 & n-1 & \cdots & n-m+2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ n-m & n-m+1 & \cdots & n \end{vmatrix}$$

135. *Proposed by the Editor (From The Scientific American).*

Can an obtuse triangle be cut into only acute triangles? If so what is the minimum number of acute triangles, if not give a proof.

SOLUTIONS

126. *Proposed by Mark Bridger, student, High School of Science, Bronx, New York.*

Show that the quartic equation $x^4 - 13x^3 - 12x^2 - 17x + 37 = 0$ has no negative root.

Solution by Edward Ross, High School of Science, Bronx, New York.

$$x^4 - 13x^3 - 12x^2 - 17x + 37 = 0 \text{ can be written} \\ (x^2 - 6)^2 = 13x^3 + 17x - 1$$

The left side is greater than or equal to zero for all x , while the right side is greater than or equal to zero only when x is greater than zero. Thus the original equation can have no negative roots.

Also solved by Paul R. Chernoff, Harvard College, Cambridge, Massachusetts; Don Hayler, Pomona College, Claremont, California; Robert Myers, Chicago Teachers College; Michael Rothkopt, Pomona College, Claremont, California; Edna Schwartz, Hunter College High School, New York, N.Y.; Vencil Skarda, Pomona College, Claremont, California; Gilbert Wood, California State Polytechnic College, San Luis Obispo, California.

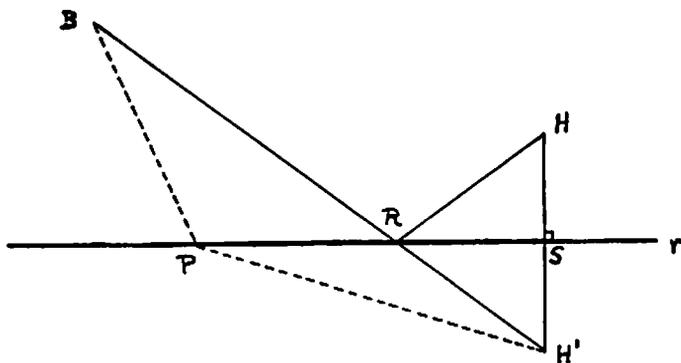
127. *Proposed by the Editor (From The Foundations and Fundamental Concepts of Mathematics by Eves and Newson.)*

A man wishes to go from his house to the bank of a straight river for a pail of water, which he will then carry to his barn on the same side of the river as his house. Find the point on the riverbank

from which he should take the water in order to minimize the distance he travels.

Solution by Patrick J. Boyle, San Jose State College, San Jose, California.

Let the house be represented by point H , the barn by B and the river by line r .



Locate the image H' , of H , in line r and let HH' intersect r in point S .

Draw BH' intersecting r at a point R .

Since H' is the image of H , $HS = H'S$, and angles HSR and $H'SR$ are each right angles. Thus triangles HSR and $H'SR$ are congruent: SAS.

Therefore $RH = RH'$ and $BR + RH = BH'$.

Now let P be any point on r other than R , then $BP + PH' > BH'$: triangle inequality.

Therefore R is the point on r which will minimize the distance.

Also solved by Paul R. Chernoff, Harvard College, Cambridge, Massachusetts; Don Hayler, Pomona College, Claremont, California; Robert Myers, Chicago Teachers College; Robert R. Poole, Pomona College, Claremont, California; Edward Ross, High School of Science, Bronx, New York; Michael Rothkopt, Pomona College, Claremont, California; Vencil Skarda, Pomona College, Claremont, California.

128. *Proposed by Paul R. Chernoff, student, Harvard College.*

The inverse $F^{-1}(t)$ of a real function $F(t)$ is defined so that $F^{-1}[F(t)] = F[F^{-1}(t)] = t$. If $F(t)$ and its inverse function $F^{-1}(t)$ are differentiable and continuous functions of t on the entire interval considered, prove that:

$$\int_a^b F(t)dt = b \cdot F(b) - a \cdot F(a) - \int_{F(a)}^{F(b)} F^{-1}(t)dt$$

Solution by Robert R. Poole, Pomona College, Claremont, California.

Since $F(t)$ is differentiable and continuous we use integration by parts with $u = F(t)$ and $v = t$ obtaining:

$$\int_a^b F(t)dt = t \cdot F(t) \Big|_a^b - \int_a^b t \cdot d[F(t)]$$

and since $F^{-1}(t)$ is continuous and $t = F^{-1}[F(t)]$

$$\int_a^b F(t)dt = b \cdot F(b) - a \cdot F(a) - \int_{F(a)}^{F(b)} F^{-1}[F(t)]d[F(t)].$$

Changing variables in the last integral above, letting $t = F(t)$ we obtain:

$$\int_a^b F(t) dt = b \cdot F(b) - a \cdot F(a) - \int_{F(a)}^{F(b)} F^{-1}(t)dt.$$

Also solved by Don Hayler, Pomona College, Claremont, California; Edward Ross, High School of Science, Bronx, New York; Vencil Skarda, Pomona College, Claremont, California; and the proposer.

129. *Proposed by the Editor (From The Mathematical Monthly.)*

Find the greatest (volume) right circular cylinder coaxial with and inscribed in the solid formed by rotating around the y -axis the area bounded by the two axes, the parabola $y = 9x^2 - 28x + 24$, and the parabola's minimum ordinate.

Solution by Vencil Skarda, Pomona College, Claremont, California.

Such a cylinder has a radius x and a height $9x^2 - 28x + 24$. Its volume, $v = \pi x^2 (9x^2 - 28x + 24)$ with $0 \leq x \leq 14/9$, $14/9$ occurring where $9x^2 - 28x + 24$ is a minimum.

By computing $\frac{dv}{dx}$ one obtains a relative maximum at $x = 1$ giving $v = 5\pi$, however the absolute maximum occurs at the end point of the interval where $v = \pi(14/9)^2 (20/9) = 3920\pi/729$, which is the true maximum volume.

This was the only complete solution received. The following persons submitted a partial solution by giving the relative maximum of 5π , thereby overlooking the absolute maximum volume: Don Hayler, Pomona College, Claremont, California; Robert Myers, Chicago Teachers College; Robert R. Poole, Pomona College, Claremont, California; Edward Ross, High School of Science, Bronx, New York; Michael Rothkopt, Pomona College, Claremont, California.

130. *Proposed by the Editor (From The Mathematical Monthly).*

$$\text{Prove that } \sum_{n=1}^{\infty} (n-1)/n! = 1$$

Solution by Michael Rothkopt, Pomona College, Claremont, California.

$$\begin{aligned} \sum_{n=1}^{\infty} (n-1)/n! &= \sum_{n=1}^{\infty} n/n! - \sum_{n=1}^{\infty} 1/n! \\ &= \sum_{n=1}^{\infty} 1/(n-1)! - \sum_{n=1}^{\infty} 1/n! \\ &= \sum_{n=0}^{\infty} 1/n! - \sum_{n=1}^{\infty} 1/n! \\ &= \sum_{n=0}^0 1/n! = 1/0! = 1. \end{aligned}$$

Also solved by Paul Chernoff, Harvard College, Cambridge, Massachusetts, who gave an interesting proof by induction; Don

(Continued on page 128)

Mathematical Curiosa

ALFRED MOESSNER
Gunzenhausen, Germany

1. a) $7 + 10 + 14 + 15 + 23 = 2 + 3 + 3 + 9 + 11 + 19 + 22$
b) $7^3 + 10^3 + 14^3 + 15^3 + 23^3 = 2^3 + 3^3 + 3^3 + 9^3 + 11^3 + 19^3 + 22^3$
c) $7^5 + 10^5 + 14^5 + 15^5 + 23^5 = 2^5 + 3^5 + 3^5 + 9^5 + 11^5 + 19^5 + 22^5$
d) $10^2 + 15^2 + 23^2 = 3^2 + 19^2 + 22^2$
e) $10^6 + 15^6 + 23^6 = 3^6 + 19^6 + 22^6$
2. a) $5(1 + 2 + 3 + 4) = 1 + 14 + 16 + 19$
b) $5^5(1^5 + 2^5 + 3^5 + 4^5) = 1^5 + 14^5 + 16^5 + 19^5$
c) $5(2 + 4) = 14 + 16$
3. a) $3^5 + 29^5 = 4^5 + 10^5 + 20^5 + 28^5$,
where $3 + 29 = 4 + 28$ and $10 + 20$ is divisible by 30.
- b) $12^5 + 38^5 = 13^5 + 37^5 + 5^5 + 25^5$,
where $12 + 38 = 13 + 37$, and $5 + 25$ is divisible by 30.
- c) $107^5 = 7^5 + 100^5 + 43^5 + 57^5 + 80^5$,
where $107 = 7 + 100$, and $43 + 57 + 80$ is divisible by 30.
- d) $17516^5 = 709^5 + 5366^5 + 11441^5 + 14175^5 + 15435^5$,
where $17516 = 709 + 5366 + 11441$, and $14175 + 15435$ is
divisible by 30.
- e) $84^5 + 159^5 = 109^5 + 134^5 + 75^5 + 135^5$,
where $84 + 159 = 109 + 134 = 3^5$; $109 - 84 = 134 - 109 = 5^2$;
 $159 - 84 = 75 = 3 \cdot 5^2$; $159 - 109 = 2 \cdot 5^2$; $135 - 134 = 1^5$;
 $84 - 75 = 3^2$; $159 - 134 = 5^2$; and $75 + 135$ is divisible by 30.

The Mathematical Scrapbook

EDITED BY J. M. SACHS

It is an error to believe that rigor in proof is an enemy of simplicity. On the contrary we find it confirmed by numerous examples that the rigorous method is at the same time, the simpler and the more easily comprehended. The very effort for rigor forces us to find out simpler methods of proof.

—D. HILBERT

= Δ =

Find pairs of numbers (a, b) such that the cube of the sum is 27 times the cube of the first number. Find pairs of numbers (a, b) such that the cube of the sum is 64 times the cube of the first number. Find pairs of numbers such that the cube of the sum is k^3 times the cube of the first.

The solution to the first problem proposed will show how all the others can be solved. Consider the cube of the sum of a and b :

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

If we look upon this as a cubic expression in b and set it equal, by the conditions of the problem, to $27a^3$ we get the equation:

$$b^3 + 3ab^2 + 3a^2b - 26a^3 = 0.$$

If we consider the possible solutions in the form $b = ka$, where k is an integer, we discover that the only such root is $b = 2a$. Dividing out the factor $b - 2a$ we obtain the quadratic $b^2 + 5ab + 13a^2 = 0$ which has two solutions

$$b = \frac{-5a \pm 3ai\sqrt{3}}{2} \quad \text{or} \quad b = \frac{1}{2}(-5 \pm 3\sqrt{3}i)a$$

Thus the first question proposed has three answers:

$$(a, 2a) ; (a, \frac{1}{2}[-5 + 3\sqrt{3}i]a) ; (a, \frac{1}{2}[-5 - 3\sqrt{3}i]a)$$

Try the same scheme for fourth and higher powers.

= Δ =

German intellect is an excellent thing, but when a German product is presented it must be analyzed. Most probably it is a combination of intellect(I) and tobacco-smoke(T). Certainly I_5T_1 and I_2T_3 occur; but I_1T_3 is more common and I_2T_{15} and I_1T_{20} occur. In

many cases metaphysics(M) occurs and I hold that $I_a T_b M_c$ never occurs without $b + c > 2a$.

—A. DEMORGAN

= Δ =

The problem of this science (the philosophy of mathematics) is nothing else than that of defining in its full scope the content of mathematical knowing which transcends logic, and to systematize this content according to its own principles. In order to solve this problem it is evidently necessary to lay down, within each mathematical discipline, a strict line of division between that which is logically provable and that which has the status of a presupposition to such proof and is got at by means of intuition. . . . This problem demands, on one hand, the limitations of the axioms to a minimum (to be exact, those presuppositions which are necessary for the logical construction of the theory in question) and the actual proving of those others which are provable. The problem requires, on the other hand and contrary to the usual practice of mathematics, an increase in the number of axioms; for the totality of axioms should suffice, without further presuppositions, to develop the whole theory in a purely logical way. As a matter of fact, if in accordance with this problem we formulate as special presuppositions all borrowings from intuition which are made in the course of a proof and which are usually introduced tacitly, we discover new axioms.

It is easily seen that this "critical mathematics" or "philosophy of mathematics", is nothing other than the axiomatics in which the modern mathematician has so much confidence.

—L. NELSON

= Δ =

Zeno was concerned with three problems These are the problem of the infinitesimal, the infinite, and continuity. . . . From him to our own day, the finest intellects of each generation in turn attacked these problems, but achieved broadly speaking nothing. . . . Weierstrass, Dedekind, and Cantor, . . . have completely solved them. Their solutions . . . are so clear as to leave no longer the slightest doubt of difficulty. This achievement is probably the greatest of which the age can boast The problem of the infinitesimal was solved by Weierstrass, the solutions of the other two are begun by Dedekind and definitely accomplished by Cantor.

—BERTRAND RUSSELL

= Δ =

The paradoxes of Zeno (5th century B.C.) mentioned in the preceding quotation were concerned with the infinitely fine divisibility of space and time. The paradox of Achilles and the tortoise lies in the argument that if the tortoise is given a head start Achilles can never overtake him no matter how fast (finite speed) he runs. Suppose the tortoise has a head start of 50 feet and Achilles runs ten times as fast as his opponent. When he has run the 50 feet the tortoise has moved on 5 feet. When Achilles has run the 5 feet, the tortoise has moved on .5 feet, etc. The arrow paradox states that at any instant the arrow is in a particular location, i.e., motionless, and since time is a collection of instants, there is no motion.

These puzzling problems raised philosophical doubts about the nature of time and space and continuity. Aristotle defined continuity, partly in answer to the questions raised by Zeno, as follows: "A thing is continuous when of any two successive parts the limits at which they touch, are one and the same, and are, as the word implies, held together." How would this definition apply to our modern meaning of continuity at a point on a curve? Is there any difficulty with the word *successive*?

$$= \Delta =$$

We have adroitly defined the infinite in arithmetic by a love-knot, in this manner ∞ ; but we possess not therefore the clearer notion of it.

—VOLTAIRE

$$= \Delta =$$

Given that the sum of the positive integers a , b , and c is less than 10 and $a < b < c$. Given that the product $a \cdot b \cdot c$ is known but that this is not sufficient information to determine a , b , and c , find the product $a \cdot b \cdot c$. If you know also that two of a , b , and c are even, can you find the smallest possible values for a , b , and c ?

$$= \Delta =$$

A number of problems of the form of the one above have come to the attention of your editor in the past few months. Perhaps the general form might be stated as follows:

Suppose the sum of the positive integers $a_1 < a_2 < a_3 < \dots < a_n$ is less than N . If the product $a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_n = H$, how many possible sets (a_1, a_2, \dots, a_n) can be found?

Consider the possibility of three integers as suggested by the problem above. The only possible values for N would then be 7 or greater. If $N = 7$ then H must be 6 and $a = 1, b = 2, c = 3$. If

$N = 8$ then H could be 6 or 8 for $a = 1, b = 2, c = 3$, or $a = 1, b = 2, c = 4$. Some other given information, such as the condition that two of the three factors are even, would enable one to choose between the two possibilities, if H were not given. For $N = 9$ we get the following possibilities for a, b , and c , one for each possible H :

a	b	c	H
1	2	3	6
1	2	4	8
1	2	5	10
1	3	4	12

It is with $N = 10$ that the subject begins to get more interesting.

a	b	c	H
1	2	3	6
1	2	4	8
1	2	5	10
1	2	6	12

We can call these the 2-chains since in each one the product $a \cdot b$ is 2 and H increases by 2 for an increase of 1 in c .

1	3	4	12
1	3	5	15

3-chains.

2	3	4	24
---	---	---	----

6-chains.

Note that in solving the original problem suggested we would note that $H = 12$ belongs to two chains, a 2-chain and a 3-chain. The condition that two of the three must be even would enable us to pick out the solution, $a = 1, b = 2, c = 6$.

In general with three numbers a, b , and c , how many chains can share a given H for a given N ? How much other information is necessary in order to determine a, b , and c ? Can you generalize to $a_1, a_2, a_3, \dots, a_n$? This looks like a fertile field for some investigation.

$$= \Delta =$$

Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions . . . A curve is the totality of points whose coordinates are functions of a parameter which may be differentiated as often as may be required.

—F. KLEIN

The Book Shelf

EDITED BY H. E. TINNAPPEL

From time to time there are published books of common interest to all students of mathematics. It is the object of this department to bring these books to the attention of readers of THE PENTAGON. In general, textbooks will not be reviewed and preference will be given to books written in English. When space permits, older books of proven value and interest will be described. Please send books for review to Professor Harold E. Tinnappel, Bowling Green State University, Bowling Green, Ohio.

Properties of Matter, Third Edition, F. C. Champion and N. Davy, Philosophical Library (15 East 40th Street) New York, 1959, 334 pages, \$10.00.

This book presents a number of topics in the general area indicated by the title. The chapters dealing with various subjects are independent of each other, except as indicated in the titles where, for example, Newton's treatment of gravitation is covered in more than one chapter.

There is an excellent treatment of dimensionality and dimensional integrity is observed throughout the book. Footnotes supply information as to the more extended treatments available.

The student of mathematics will find the book interesting in the extensive presentation of the applications of mathematics to a variety of physical problems. The author recognizes the necessity of making what he frankly calls "oversimplifying assumptions" in order to bring the treatment within practicable mathematical scope. There are certain cases where one could wish for more precision in usage. For example, in discussing the pendulum, the author states that if the amplitude is "small" we may use θ for $\sin \theta$. We might ask: How small is small enough for this simplification? There appears to be a tacit assumption that the functions dealt with have the necessary properties of differentiability, etc.

The examples of statistics at the end are of the "cook book" variety.

On the whole we may state that this is a very useful book, more perhaps to the physicist than to the mathematician. The latter is more likely to be impatient with the lapses in rigor than the former.

—F. C. OGG

Bowling Green State University

Chemistry of Nuclear Power, J. K. Dawson and G. Long, Philosophical Library, Inc., (15 East 40th Street) New York, 1959, 208 pp., \$10.00.

The authors of this compact little book have succeeded in accomplishing their proclaimed endeavor: describing the contributions of the chemist to the over-all development of nuclear power. In addition, the reader is given an excellent introduction to the current status of the entire atomic energy industry.

This book is recommended for those with some scientific training who are aware of the great impact of atomic power development upon the science and technology of today. The book is not highly technical; the background acquired in the average introductory physics or physical chemistry course, along with some knowledge of inorganic chemistry, should be adequate to enable the interested reader to benefit from its contents without undue difficulty.

The authors point out that the role of the chemists in the atomic energy industry cannot be properly appreciated without a background of knowledge of the entire field. To help supply this background, several chapters are included in which chemistry is only slightly involved. These include a chapter "Raw Materials" and fifty pages of discussion on the various types of nuclear reactors. These chapters give the monograph a more unified presentation than would otherwise have been possible.

The book begins with an excellent introductory chapter "The Role of the Chemist in the Atomic Energy Industry". The historical development of the industry is briefly outlined. Administrative line of the United Kingdom Atomic Energy Authority are also shown. Both authors are staff members of the Atomic Energy Research Establishment at Harwell; accordingly the illustrative material is drawn predominantly from British sources. The impact of nuclear power on chemistry, the versatility of the new radiochemical techniques, and the chemist's contributions to the development of atomic energy are discussed in concisely summarized fashion.

Subsequent chapters include "Fission and the Fission Products", "The New Heavy Elements", "Separation Processes", "The Handling of Radioactive Materials" and "Disposal of Radioactive Wastes". The salient points of each of these topics is presented, and general references are given at the end of each chapter for those who wish to read further. It is somewhat regrettable that specific references are not given for much of the data; presumably these may be

found in the general references. The book ends with an interesting chapter, concerning "Some Future Applications".

It is unfortunate that the exorbitant price of \$10.00 is asked for this book. It is very likely a great deterrent to many who would otherwise choose to add this book to their scientific library.

—NORMAN J. MEYER

Bowling Green State University

University Mathematics, Second Edition, Joseph Blakey, Philosophical Library (15 East 40th Street) New York, 1959, 581 pp., \$10.00.

According to the preface, this book is intended as a first-year course in mathematics and contains all the various branches required for a Science Degree at London University. To see what this entails requires a listing of the topics presented.

The book begins by listing formulas from algebra, trigonometry, and the co-ordinate geometry of the straight line and circle. Limits, series, exponential and hyperbolic functions, and partial fractions are developed and used to derive the standard differentiation and integration formulas. This is followed by expansion of functions in power series, maxima, minima, points of inflection, tangents, normals, curvature, and partial differentiation. Next is a short chapter on determinants and then roughly one-third of the book is devoted to a very detailed development of classical plane and solid analytic geometry, without any use of the calculus already presented. The book concludes with chapters on area under a curve, volume and surface of revolution, length of curves, differential equations, spherical trigonometry, moments of inertia and damped simple harmonic motion.

This text is devoted strictly to presentation of standard manipulative techniques for solving problems which appear in the London University examinations. Nearly all the exercises are taken from London University Science and Engineering Degree examination papers in Pure Mathematics. No attempt is made to motivate any of the material or to discuss any of the theoretical details. Many definitions lack precision and consequently are useless. Theorems frequently are stated incorrectly and no mention is made of the conditions which must be satisfied by functions if the manipulative techniques developed are to be applicable.

—LOUIS C. GRAUE

Bowling Green State University

Directions for Papers to be Presented at the Kappa Mu Epsilon Convention

EMPORIA, KANSAS
April 21 and 22, 1961

An important part of the convention program will be the presentation of papers by student members of KME. It is essential that chapter sponsors and members begin now to plan for papers to be presented at the next convention. Each student should choose the field of mathematics of greatest interest to him and then search for a topic area suitable for a paper. Faculty sponsors should counsel with students in the selection of topics and encourage competition within the chapter in order that high caliber papers will result.

Who may submit papers: Any member may submit a paper for use on the convention program. Papers may be submitted by graduates and undergraduates; however, undergraduates will not compete against graduates. Awards will be granted for the best papers presented by undergraduates. If enough papers are presented by graduates, special awards may be given for their best papers.

Subject: The material should be within the scope of the understanding of undergraduates, preferably the undergraduate who has completed differential and integral calculus. The Selection Committee will naturally favor papers that are within this limitation and which can be presented with reasonable completeness within the time limit prescribed.

Time limit: The usual time limit is twenty minutes but this may be changed on recommendation of the Selection Committee.

Paper: The paper to be presented or a complete outline of it must be submitted to the Selection Committee accompanied by a description of charts, models, or other visual aids that are to be used in presenting the paper. A carbon copy of the complete paper may be submitted if desired. Each paper must indicate that the author is a member of KME and whether he is a graduate or an undergraduate student.

Date and place due: The papers must be submitted before February 1, 1961, to the office of the National Vice-President.

Selection: The Selection Committee will choose about eight papers for presentation at the convention. All other papers will be listed by title on the convention program.

Prizes:

1. The author of each paper presented will be given a two-year extension of his subscription to THE PENTAGON.
2. Authors of the two or three best papers presented by undergraduates, according to the judgment of a committee composed of faculty and students, will be awarded copies of the James' *Mathematical Dictionary*, suitably inscribed.
3. If a sufficient number of papers submitted by graduate students are selected for presentation, then one or more similar prizes will be awarded for the best paper or papers from this group.

Ronald G. Smith
National Vice-President
Kansas State College
Pittsburg, Kansas



The ideal of mathematics should be to erect a calculus to facilitate reasoning in connection with every province of thought, or of external experience, in which the succession of thoughts, or of events can be definitely ascertained and precisely stated. So that all serious thought which is not philosophy, or inductive reasoning, or imaginative literature, shall be mathematics developed by means of a calculus.

—A. N. WHITEHEAD

Installation of New Chapters

EDITED BY MABEL S. BARNES

THE PENTAGON is pleased to report the installation of two new chapters.

VIRGINIA BETA CHAPTER *Radford College, Radford, Virginia*

Virginia Beta Chapter was installed on November 12, 1959, by Dr. Charles K. Martin, Jr., President of Radford College. The candidates were presented to Dr. Martin by Mrs. Joyce Smith. Miss Barbara A. Howard discussed the purpose of Kappa Mu Epsilon, and Miss Barbara Lyons the meaning of the crest. Following the installation ceremony Dr. Martin addressed the group on the subject of the importance of the newer ideas in mathematics to an adequate mathematical program in the public schools. The group then attended a banquet to celebrate the installation of the new chapter.

The officers of the chapter are Joyce A. Smith, president; Barbara A. Howard, vice-president; Barbra Lyons, recording secretary; Sally Tank, treasurer; Mrs. Barbara M. Thornton, corresponding secretary; and Dr. R. N. Pendergrass, faculty sponsor. Other charter members are Helen E. Bolt, Kim Bowman, Jane Brown, Dana L. Castle, Carol Lauffer, Sandra Le Sueur, Joan Mitchell, Mary C. Steger, Iris J. Sweeney, Mr. Robert H. Appleby, and Mr. Shiu Hon Wong. Both Dr. Martin and Dr. Pendergrass were members of Missouri Alpha Chapter. Dr. Martin was a charter member and Dr. Pendergrass was initiated in 1946.

Since 1928 Radford College has had a mathematics club, the Tri-M Club. Membership is open to anyone interested in mathematics.

Radford College was founded in 1913 in order to train women for careers in teaching. Since 1946 it has been the Women's Division of Virginia Polytechnic Institute. The curriculum is now that of the liberal arts college with professional courses available for training teachers.

NEBRASKA BETA CHAPTER

Nebraska State Teachers College, Kearney, Nebraska

Nebraska Beta Chapter was installed on December 11, 1959, by Dr. Carl V. Fronabarger, National President of Kappa Mu Epsilon. The ceremony took place in the Student Council Room of the Student Union and was followed by a banquet in the Mural Room of the Midway Hotel. Dr. Fronabarger addressed the mathematicians and their guests on "Some Aspects of the Theory of Games of Strategy".

Charter members are Richard Barlow, Dan Bennett, Myrtle Bowers, Marjorie Colton, Terry Cox, Stan Druse, Larry Forsberg, Betty Hale, Gary Lee Haller, Dallas Johnson, Roger Jurgens, Gene P. Kingsley, Roger Lahm, Larry E. Lechner, Fred Lees, Jr., Jasper LeRoy Melton, James McDermott, Warren C. Newbold, Lloyd J. Quaring, Gene Schlueter, Theodore Guy Sherbeck, Richard Shiers, Lewis Stevens, Daniel Stineman, Rolland Sturtevant, Richard J. Thomas, R. J. Thomazin, Donald Uerling, and Ronald L. Walters; and from the faculty Dayle G. Fitzke, L. M. Larsen, and Theodora Nelson.

The new chapter's officers are Warren C. Newbold, president; James McDermott, vice-president; Myrtle Bowers, recording secretary; Richard Shiers, treasurer; Larry Lechner, historian; and Theodora Nelson, corresponding secretary.

In 1956 the Mathematics Club, which became Nebraska Beta Chapter, was founded by a group which had been meeting informally for exchange of ideas and for recreation. It has sponsored an annual picnic, open to all who wished to attend. From the sale of mathematics handbooks and slide rules it derives income used for a scholarship for mathematics majors. It is planning a Mathematics Clinic for students in need of assistance with their work.

Nebraska State Teachers College, Kearney, was founded in 1905. It has over 1600 students and over 100 faculty members and administrators.

We welcome the two new chapters and extend sincere good wishes to both of them.

Kappa Mu Epsilon News

EDITED BY FRANK C. GENTRY, HISTORIAN

Oklahoma Alpha, Northeastern State College, Tahlequah.

We had the largest group of initiates to join last fall since the original 24 members of the Mathematics Club became the first chapter of Kappa Mu Epsilon in 1931. This year there were 23 new members, 22 students and one faculty member initiated. We still have the Mathematics Club, not all of whose members are eligible for Kappa Mu Epsilon. The two organizations share in the preparation of programs, most of which are presented by students.

Missouri Alpha, Southwest Missouri State College, Springfield.

We have had five regular meetings of the Chapter this year with most of the programs by students. The outstanding lecture was given by Professor Thomas Storvick of the University of Missouri. His subject was "Engineering in the Space Age".

Nebraska Alpha, State Teachers College, Wayne.

We initiated 11 new members in the spring of 1959. Twelve members and 2 sponsors attended the National Convention at Bowling Green. Ten members qualified for the spring honor roll, and five for the fall honor roll. In December ten members of our Chapter went to Kearney, Nebraska, for the installation of Nebraska Beta. We welcome another Chapter of Kappa Mu Epsilon in Nebraska.

New Mexico Alpha, University of New Mexico, Albuquerque.

Our Chapter contributed \$100 to purchase eight chairs for the Mathematics Department Library. The books and journals in the Library are loaned by the members of the Mathematics Department in order that students and staff may have access to them.

Alabama Beta, Florence State College, Florence.

Four of our members and the mother of our president attended the National Convention last spring. Six of our members have been offered assistantships and scholarships for graduate study. Last year we entered a float in the Homecoming Parade and won first prize. This year we had the honor of building the Queen's float.

Missouri Beta, Central Missouri State College, Warrensburg.

We require that each new member present an acceptable paper

on mathematics to the Chapter before he is initiated. This policy results in many interesting papers and we never want for a program. This year our annual Spring Tour will be a trip to Kansas City where we will visit the Business Men's Assurance Company, Linda Hall Technical Library, and the Kansas City Science Fair.

Kansas Gamma, Mount St. Scholastica College, Atchison.

Our Chapter planned 18 meetings for the present school year, about half of them somewhat social in nature. On February 9, four of our members appeared on Television Station KFEQ, St. Joseph, Missouri, with a program on the aims and purposes of Kappa Mu Epsilon. Sixteen of our 27 members and pledges made the first semester honor roll.

New Jersey Beta, Montclair State College, Montclair.

We expect to initiate about 18 new members this spring. We had our annual banquet in January and will participate in the College Carnival in May.

Michigan Gamma, Wayne State University, Detroit.

We have decided to relinquish our charter in Kappa Mu Epsilon and become a local graduate and undergraduate organization.

Ohio Gamma, Baldwin-Wallace College, Berea

We were proud to have one of our members present a paper before the National Convention last spring. A nice group of our members attended. We are glad that we are able to have members of the Staff of N.A.S.A.'s Lewis Research Center as guest speakers. In January, Dr. George Moshos discussed computing machines.

California Alpha, Pomona College, Claremont.

This year we had 6 or 8 students participating in the Putnam Competition. Our Chapter is sponsoring a series of lectures on digital computers by Dr. Thomas W. Kampe, a senior mathematician at Librascope Division in Glendale. The lectures emphasize the understanding of computers rather than the theory or design of them.

Missouri Epsilon, Central College, Fayette.

More than 23 members of our Chapter have received advanced degrees and are now working in mathematics or related fields. The Bulletin, our college newspaper, published a brief statement about each of them last fall.

Nebraska Beta, Nebraska State Teachers College, Kearney.

Our Chapter gives a \$25 award each semester to a junior or senior mathematics major who has shown high scholastic and moral standards. A free mathematical "help clinic" is operated by the members of the Chapter to give help to students in Freshman mathematics courses.



(Continued from page 113)

Hayler, Pomona College, Claremont, California; Robert R. Poole, Pomona College, Claremont, California; Edward Ross, High School of Science, Bronx, New York; Vencil Skarda, Pomona College, Claremont, California; Gilbert Wood, California Polytechnic College, San Luis Obispo, California.

Editor's note: The Problem Corner is in need of problems for future issues. Both faculty and students may submit what appears to them to be interesting problems that will challenge the reader.

ACTIVE CHAPTERS of KAPPA MU EPSILON*

Chapter	Location	Installation Date
Oklahoma Alpha	Northeastern State College, Tahlequah	April 18, 1931
Iowa Alpha	State Teachers College, Cedar Falls	May 27, 1931
Kansas Alpha	Kansas State College of Pittsburg, Pittsburg	Jan. 30, 1932
Missouri Alpha	Southwest Missouri State College, Springfield	May 20, 1932
Mississippi Alpha	State College for Women, Columbus	May 30, 1932
Mississippi Beta	State College, State College	Dec. 14, 1932
Nebraska Alpha	State Teachers College, Wayne	Jan. 17, 1933
Illinois Alpha	Illinois State Normal University, Normal	Jan. 26, 1933
Kansas Beta	State Teachers College, Emporia	May 12, 1934
New Mexico Alpha	University of New Mexico, Albuquerque	March 28, 1935
Illinois Beta	Eastern Illinois State College, Charleston	April 11, 1935
Alabama Beta	State Teachers College, Florence	May 20, 1935
Alabama Gamma	Alabama College, Montevallo	April 24, 1937
Ohio Alpha	Bowling Green State University, Bowling Green	April 24, 1937
Michigan Alpha	Albion College, Albion	May 29, 1937
Missouri Beta	Central Missouri State College, Warrensburg	June 10, 1939
South Carolina Alpha	Coker College, Hartsville	April 5, 1940
Texas Alpha	Texas Technological College, Lubbock	May 10, 1940
Texas Beta	Southern Methodist University, Dallas	May 15, 1940
Kansas Gamma	Mount St. Scholastica College, Atchison	May 26, 1940
Iowa Beta	Drake University, Des Moines	May 27, 1940
New Jersey Alpha	Upsala College, East Orange	June 3, 1940
Tennessee Alpha	Tennessee Polytechnic Institute, Cookeville	June 5, 1941
New York Alpha	Hofstra College, Hempstead	April 4, 1942
Michigan Beta	Central Michigan University, Mount Pleasant	April 25, 1942
Illinois Gamma	Chicago Teachers College, Chicago	June 19, 1942
New Jersey Beta	State Teachers College, Montclair	April 21, 1944
Illinois Delta	College of St. Francis, Joliet	May 21, 1945
Kansas Delta	Washburn Municipal University, Topeka	March 29, 1947
Missouri Gamma	William Jewell College, Liberty	May 7, 1947
Texas Gamma	Texas State College for Women, Denton	May 7, 1947
Wisconsin Alpha	Mount Mary College, Milwaukee	May 11, 1947
Ohio Gamma	Baldwin-Wallace College, Berea	June 6, 1947
Colorado Alpha	Colorado State University, Fort Collins	May 16, 1948
California Alpha	Pomona College, Claremont	June 6, 1948
Missouri Epsilon	Central College, Fayette	May 18, 1949
Mississippi Gamma	Mississippi Southern College, Hattiesburg	May 21, 1949
Indiana Alpha	Manchester College, North Manchester	May 16, 1950
Pennsylvania Alpha	Westminster College, New Wilmington	May 17, 1950
North Carolina Alpha	Wake Forest College, Wake Forest	Jan. 12, 1951
Louisiana Beta	Southwest Louisiana Institute, Lafayette	May 22, 1951
Texas Epsilon	North Texas State College, Denton	May 31, 1951
Indiana Beta	Butler University, Indianapolis	May 15, 1952
Kansas Epsilon	Fort Hays Kansas State College, Hays	Dec. 6, 1952
Pennsylvania Beta	LaSalle College, Philadelphia	May 19, 1953
California Beta	Occidental College, Los Angeles	May 28, 1954
Virginia Alpha	Virginia State College, Petersburg	Jan. 29, 1955
Indiana Gamma	Anderson College, Anderson	April 5, 1957
New York Beta	Albany State Teachers College	May 16, 1957
California Gamma	California State Polytechnic College, San Luis Obispo	May 23, 1958
New York Gamma	State University of New York, Oswego	May 21, 1959
Tennessee Beta	East Tennessee State College, Johnson City	May 22, 1959
Pennsylvania Gamma	Waynesburg College, Waynesburg	May 23, 1959
Virginia Beta	Radford College, Radford	Nov. 12, 1959
Nebraska Beta	Nebraska State Teachers College, Kearney	Dec. 11, 1959
Ohio Delta	Wittenberg University, Springfield	April 28, 1960

* Listed in order of date of installation.

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