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Kappa Mu Epsilon, national honorary mathematics society, was founded in 1931. The object of the fraternity is fourfold: to further the interests of mathematics in those schools which place their primary emphasis on the undergraduate program; to help the undergraduate realize the important role that mathematics has played in the development of western civilization; to develop an appreciation of the power and beauty possessed by mathematics, due, mainly, to its demands for logical and rigorous modes of thought; and to provide a society for the recognition of outstanding achievements in the study of mathematics at the undergraduate level. The official journal, THE PENTAGON, is designed to assist in achieving these objectives as well as to aid in establishing fraternal ties between the chapters.

The Tree of Mathematics in the Light of Group Theory*

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A leading University mathematician of the midwest area recently stated that the half century just passed is most properly termed "the golden era of mathematical research." The peculiar charm of this period lies in the fact that it does not invalidate the major theories that have been developed and accumulated during the 2000 years preceding it. Through some unexplainable aegis, this period abounds in brilliant and fresh ideas that cause the older discoveries to take on an irridescant glow.

One of the most versatile abstractions of modern mathematics is the algebraic concept of a "group." Through this group-notion one is able to see the whole skeleton of mathematics in a new light. One might say that it serves as a sort of X-ray to bring out the basic structure of mathematics.

The comparison frequently employed wherein mathematics is likened to a tree is a familiar one. In fact, a book just recently released from the press carries the title, *The Tree of Mathematics*. In this paper the writer aims to point out the ways the group concept unifies the many branches of mathematics. It is able to do so, because it reveals the basic structure that is common to all mathematical objects whether they be spaces, formulas, motions, functions, or pure numbers.

Many mathematicians contributed to the long and rich history that attaches to the theory of groups. Names that stand out prominently in the record are Gauss, Cauchy, Hamilton, and Cayley. It needed the peculiar genius of Evariste Galois, a young French mathematician, to detect in the group idea a new device for solving equations. However, the discovery of Galois lay unproductive and barren until almost a century later when modern mathematicians saw in the group concept a fertile means for delineating the intricate structure common to mathematical ideas.

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*A paper presented at the 1959 National Convention of KME and awarded first place by the Awards Committee.

With these preliminary remarks disposed of, a close examination of the mathematical entity known as a "group" would seem to be appropriate. A group is one of the simplest of all mathematical systems, since it need have only *one* operation. What then constitutes a group? It consists of a *set* of mathematical objects and a *way* of combining them. Invoking the time-honored prerogative of mathematicians, each member of the set of mathematical entities is represented by a letter; and in order to maintain a desirable abstractness, the combining process is designated by the symbol $*$. The postulates or rules of operation which control manipulation with these elements are four:

I. *Closure*: All combinations of elements must produce one of the elements.

II. *Associativity*: Stated symbolically, it is $A^*(B^*C) = (A^*B)^*C$. This merely states that priority of combination is *not* the privilege of any one pair exclusive of another pair.

III. *Identity*: As the term indicates, it means that among the mathematical objects there must be one which when combined with each of the others, leaves each unchanged. Thus, $0 + 1 = 1 + 0 = 1$, or zero added to unity leaves unity unchanged. Hence, zero is an identity element for addition.

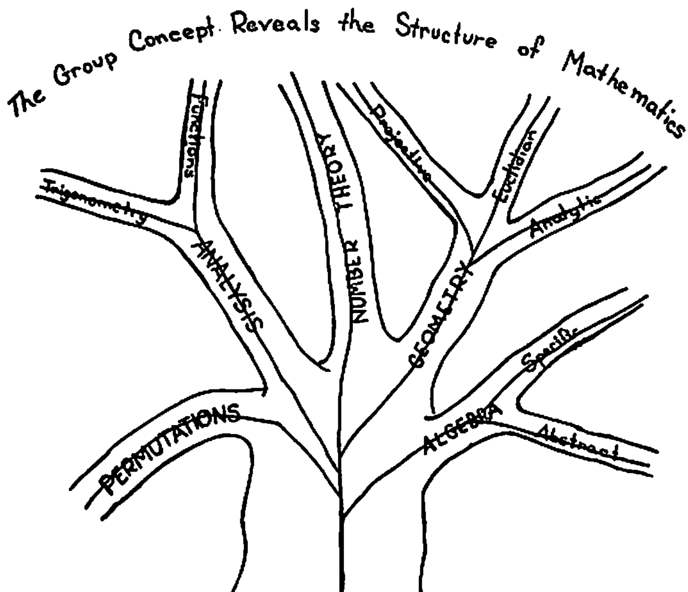
IV. *Inverse*: Each element of the group must have an inverse or mate, also an element in the system; the uniting of which pair produces the identity element of the group. Thus, when the integers 1 and -1 are combined under addition, the identity element zero is obtained.

To demonstrate these group postulates, the integers under addition (mod 5) will be examined. What is meant by mod 5? In the table (see page 7) all multiples of 5 have been omitted: e.g., $4 + 3 = 7$, but if the multiple of 5 is neglected, only the remainder of 2 appears in the chart.

One next inquires if the group postulates are satisfied. It is clearly manifest that *closure* is preserved, since no element other than 0, 1, 2, 3, or 4 appears in the chart. The postulate of *associativity* is satisfied. For example,

$$(2 + 4) + 3 = 2 + (4 + 3)$$
$$1 + 3 = 2 + 2$$
$$4 = 4$$

The table also exhibits the fact that zero is the *identity* element, for it is seen that zero added to each of the elements leaves the individual elements unchanged. Next, one looks for the *inverse* of each element. This becomes easy when the group is set up in tabular form. It is necessary merely to look for the zero element in each column and determine which two elements in combination produced the zero. Hence, 1 is observed to be the mate of 4; 4 is the mate, or inverse of 1. Therefore, since all four of the postulates are satisfied, the integers under addition (mod 5) form a group.



TREE OF MATHEMATICS

Now that a group and its postulates have been defined, the remainder of this paper will be devoted to a discussion of the tree of mathematics and to showing how the group concept is the unifying factor which holds all the branches to the trunk of the tree. In the diagram of the tree one notes that the group concept is designated by the central line passing up the trunk and into each of the branches; this is the "vascular system" which brings nourishment and life to all parts of the tree. The following examples which were taken from each of the branches are used to demonstrate how the group concept acts as the unifying factor in the tree of mathematics.

EXAMPLES OF ALGEBRAIC GROUPS

Abstract Algebra: The Klein Four Group

*	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	E	A
C	C	B	A	E

Postulates:

1. Closure is satisfied, since no element other than E, A, B, and C appears in the chart.
2. Associativity is satisfied.
   Example
   $$(B * A) * C = B * (A * C)$$
   $$C * C = B * B$$
   $$E = E$$
3. Identity element is E.
4. Inverse pairs are E & E, A & A, B & B, C & C.

Neither the elements nor the operation is defined.

Specific Algebra: The Four Fourth Roots of Unity

$\times$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

Postulates:

1. Closure is satisfied, since no element other than 1, i, -1, and $-i$ appears in the chart.
2. Associativity is satisfied.
   Example
   $$(1 \cdot 1) \cdot i = 1 \cdot (1 \cdot i)$$
   $$1 \cdot i = 1 \cdot i$$
   $$i = i$$
3. Identity element is 1.
4. Inverse pairs are 1 & 1, i & $-i$, -1 & -1, and $-i$ & $-i$.

EXAMPLES OF GROUPS UNDER NUMBER THEORY

A Finite Group: The Integers Under Addition (Mod 5)

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Postulates:

1. Closure is satisfied, since only 0, 1, 2, 3, and 4 appear in the table.
2. Associativity holds.
   Example
   $$(2 + 3) + 1 = 2 + (3 + 1)$$
   $$0 + 1 = 2 + 4$$
   $$1 = 1$$
3. Identity element is 0.
4. Inverse pairs are 0 & 0, 1 & 4, 2 & 3, 3 & 2, 4 & 1.

Examples of Infinite Groups:

1. The integers are a commutative group with respect to the operation of addition.
2. The rational numbers, with zero excluded, from a group under the operation of multiplication.
3. The real numbers form a commutative group with respect to addition.

EXAMPLES OF GROUPS UNDER ANALYSIS

Group of Functions: $(Z, -Z, 1/Z, -1/Z)$

*	Z	$-Z$	$1/Z$	$-1/Z$
Z	Z	$-Z$	$1/Z$	$-1/Z$
$-Z$	$-Z$	Z	$-1/Z$	$1/Z$
$1/Z$	$1/Z$	$-1/Z$	Z	$-Z$
$-1/Z$	$-1/Z$	$1/Z$	$-Z$	Z

The operation is a substitution of the horizontal element in place of the letter Z in the vertical element.

Postulates:

1. Closure is satisfied, since only Z, −Z, 1/Z, and −1/Z appear in the chart.
2. Associativity holds.
   Example:
$$\begin{aligned}(-Z * 1/Z) * -1/Z &= -Z * (1/Z * -1/Z)\\ -1/Z * -1/Z &= -Z * -Z\\ Z &= Z\end{aligned}$$
3. Identity element is Z.
4. Inverse pairs are Z & Z, −Z & −Z, 1/Z & 1/Z, −1/Z & −1/Z.

Group of Trigonometric Functions

*	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	a	d	c	f	e
c	c	f	e	b	a	d
d	d	e	f	a	b	c
e	e	d	a	f	c	b
f	f	c	b	e	d	a

Postulates:

1. Closure is satisfied since only the six given elements appear in the chart.
2. Associativity holds.
   Example
$$\begin{aligned}(c * b) * e &= c * (b * e)\\ f * e &= c * f\\ d &= d\end{aligned}$$
3. Identity element is a.
4. Inverse pairs are a & a, b & b, c & e, d & d, e & c, f & f.

Key:

$$\begin{aligned}a &= a = \sin^2 x\\ b &= 1/a = \csc^2 x\\ c &= 1 - 1/a = -\text{ctn}^2 x\\ d &= a/(a-1) = -\tan^2 x\\ e &= 1/(1-a) = \sec^2 x\\ f &= 1 - a = \cos^2 x\end{aligned}$$

The operation is a substitution of the horizontal element in place of the a in the vertical element.

Permutation Group of Four Numbers:

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Postulates:

1. Closure is preserved.
2. Associativity holds. Example
$$\begin{aligned}(b * c) * d &= b * (c * d)\\ d * d &= b * b\\ &= c\end{aligned}$$
3. Identity element is a.
4. Inverse pairs are a & a, b & d, c & c, d & b.

Key:

$$a = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix} \quad b = \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix} \quad c = \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix} \quad d = \begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{pmatrix}$$

The operation is one permutation followed by another permutation.

EXAMPLES OF GEOMETRIC GROUPS

Euclidean Geometry: The Rotations of a Wheel

*	R_{360}	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}
R_{360}	R_{360}	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}
R_{60}	R_{60}	R_{120}	R_{180}	R_{240}	R_{300}	R_{360}
R_{120}	R_{120}	R_{180}	R_{240}	R_{300}	R_{360}	R_{60}
R_{180}	R_{180}	R_{240}	R_{300}	R_{360}	R_{60}	R_{120}
R_{240}	R_{240}	R_{300}	R_{360}	R_{60}	R_{120}	R_{180}
R_{300}	R_{300}	R_{360}	R_{60}	R_{120}	R_{180}	R_{240}

An element is the number of degrees that a wheel is rotated in a counter-clockwise direction. The operation is one rotation followed by another.

Postulates:

1. Closure is satisfied, since only the given elements appear in the chart.
2. Associativity holds. Example
$$(R_{60} * R_{180}) * R_{120} = R_{360}$$
$$R_{60} * (R_{180} * R_{120}) = R_{360}$$
3. Identity element is R_{360}
4. Inverse pairs are R_{360} & R_{360}, R_{60} & R_{300}, R_{120} & R_{240}, R_{180} & R_{180}, R_{240} & R_{120}, R_{300} & R_{60}.

Analytic Geometry: The set of translations of a point form a commutative group. But since these translations do not readily lend themselves to graphical representation, a chart has not been included.

Projective Geometry: The Cross Ratio of Four Points on a Line

*	t	v	w	x	y	z
t	t	v	w	x	y	z
v	v	t	x	w	z	y
w	w	z	y	v	t	x
x	x	y	z	t	v	w
y	y	x	t	z	w	v
z	z	w	v	y	x	t

Postulates:

1. Closure is satisfied, since no element other than t, v, w, x, y, and z appears in the table.
2. Associativity holds.
   Example
   $$(z * x) * y = z * (x * y)$$
   $$y * y = z * v$$
   $$w = w$$
3. Identity element is t.
4. Inverse pairs are t & t, v & v, w & y, x & x, y & w, z & z.

Key: $t = t$ $\quad v = 1/t \quad w = 1 - 1/t = (t - 1)/t$
$x = t/(t-1) \quad y = 1 - t/(t - 1) = 1/1 - t \quad z = 1 - t$

The operation is a substitution of the horizontal element in place of the letter t in the vertical element.

In this paper it has been shown how the group concept acts as the unifying factor in the tree of mathematics. Granted that the group concept may be elemental—the space age we are entering demands proof of some formerly accepted truths. Perhaps proofs established today may serve as tools to probe the unknowns of tomorrow.

Business "Discovers" Probability*

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Flight 916 had landed and had discharged its passengers. An attendant was removing the remains of the dinner trays and their waste, for this had been a meal flight. With the waste were five untouched dinners, representing meals for passengers who had not claimed their reservations. The total value of about eight dollars was not much in itself, but it had been repeated over and over again on this flight and on others, until the airline was losing nearly $250,000 each year in its catering service.

Exactly a year later, when flight 916 landed, the loss had dropped to an average of less than two meals per flight, and the annual loss was less than $100,000. What had happened was that a simple, but nevertheless original, application had been made of some of the laws of probability.

A compilation had been made, over a period of months, of the number and the per cent of passengers who failed to 'show' each trip. The number varied, of course, both by the day of the week, and seasonably. When plotted in what statisticians call a histogram or distribution, the pattern was of a form closely resembling that known as a Poisson or binomial distribution. What was important was that this curve can be expressed mathematically, and that the form of the distribution was regularly reproduced for successive sets of data.

The rest was simple to plan, even if it took much verification and trial to finally establish. It was decided to underestimate the number of meals by a fraction which could be represented by a point on the curve of the distribution. Suitable safeguards against people going hungry were made in the form of frozen meals even more expensive and attractive than the regular meals, such as shrimp or crabmeat salads, and these were held in reserve in the airport catering office. The planned number of regular meals were placed on board, and just before the flight left, any necessary number of the special meals were added. The stewardess would offer the first passengers a choice (and this was of particular interest to dieters for she could often pick them out) until the extra meals were disposed of,

*Presented at a meeting of the Ohio Alpha Chapter, Bowling Green, Ohio, February 25, 1959.

then serve the rest. Except when an unusual number failed to 'show,' there were never any leftover meals, to be thrown away at the end of the flight.

The airline catering business had discovered probability in a practical way that showed up on the profit and loss statement.

In another field, a buyer and a seller were in the process of agreeing on a sampling plan for the examination of shipments. The material, as supplied, was known to average rather closely to two per cent out of specification and since the cost of closer adherence would materially increase the price, and since the buyer knew that he could use up to four per cent off specifications, the latter figure was fixed upon for the limiting quality. It was agreed that from each shipment a sample of fifty items would be taken, and if no more than two items were out of the specification, the shipment would be accepted. The seller's experience that he averaged two per cent gave him assurance that there would be no difficulty.

Almost immediately the seller was in trouble. Nearly ten per cent of all shipments were returned, yet closer examination of every item in the shipment found only two per cent objectionable. The answer was in a law of probability, and the sample was not only too small, but the acceptance number was incorrectly chosen. Again we turn to the Poisson distribution, and it would have predicted the difficulty in advance. The appropriate distribution for a sample of fifty and an acceptance number of two would have found:

no defects	36.8%	of the time.
one or less defects	73.6%	of the time. (adding 36.8%)
two or less defects	92.0%	of the time. (adding 18.4%)
or 92.0% acceptance under the plan.		
three defects	6.1%	of the time.
four defects	1.5%	of the time.
five defects	.3%	of the time.
six or more defects	.1%	of the time.

These businessmen, particularly the seller, learned probability the hard way.

Another business made an item which was an assembly of five parts. The manufacturing division found that it could make the parts at a reasonable cost to a tolerance of .004 inches. Engineering said that the assembly must be held to .015 inches, or .003 inches per part. Manufacturing retorted that it had been meeting the .004 figure for years without trouble, and 'that was that.' It was up to the general manager to make a decision.

A statistician told him that the answer was that tolerances are not added arithmetically, but statistically. He squared the five tolerances, which were then measures on the order of what we call variance, added them, and took the square root. The answer was .00894 inches, or well within the .015 inch requirement from Engineering. The chances of interference, using Engineering's assumption that five pieces, each at the same extreme of tolerance, could be the largest in 1,000 such pieces, and would by chance be assembled together, was one in one thousand million million.

Then Manufacturing took the statistician to one side and asked, "What tolerance could we use—greater than .004 inches?" He set up the formula: t equals the square root of $5x^2$, set t to .015 and solved for x, which was .0067 inches. It was more than Manufacturing needed, so things were left as they were, with everybody satisfied.

Thus business had discovered that probability settles arguments.

As a final preliminary example, another manufacturer was having trouble with variability of the product from a filling operation where several filling mechanisms working in parallel filled separate packages. This was in spite of the fact that the operator continually took a package from the line, weighed it, and made adjustments according to the weight found.

The trouble was in over-correcting, and in correcting when not needed. The filling heads could not be adjusted precisely alike, and even if so, did not measure precisely the same amount into the package each time. There were statistical variations, not only between heads, but also in the products coming from one head. There was considerable variation from time to time. He was over-correcting because each time he happened to get a low fill from a head set low, he adjusted the whole line upward, or if he found a high fill from a high head, he adjusted the whole line downward.

The answer was in the statistical control chart procedure. A chart was set up on which the average weight could be plotted. The operator took, at such intervals as seemed necessary, one package from each of the filling heads and weighed them separately. On one portion of the chart he plotted the average weight of the packages, and on another the difference between the heaviest and the lightest package. After a few such weighings certain calculations were made, which resulted in lines being drawn horizontally on the chart. One was in an area generally above the individual plots of the average and one was below. One was above the general area of the plotted differences between high and low weights from a single sample. These became the control point lines.

Now a procedure could be set up. Whenever a new point was plotted between the lines bracketing the averages, nothing of any nature was to be done in the way of adjustment. If it was above the upper line or below the lower line, something very definitely was to be done. This adjustment was made to the filling process as a whole. The two areas of decision were the statistical decisions that the process was either 'in control,' and needed no adjustment, or that it was 'out of control,' and therefore needed attention.

If a point was above the upper line for differences between high and low weights (called ranges) it indicated that another aspect needed attention—that one or more of the heads may have changed relative to the others. Individual correction, rather than group correction, was needed.

Thus the control chart leads to two kinds of control, another discovery of probability.

In these four examples, business had not literally discovered probability. Mathematicians had known something about probability for two or three centuries. For half as long, so had actuaries and geneticists. But business could put a dollar sign on it, and, in doing so, create an incentive that amounted to discovery.

To explain further, and before giving other examples, we should develop something about the fundamentals of variation, and the mathematical laws that they follow. It is generally known, but not always admitted, that among data from a repetitive process there are variations whose limits depend on the process itself, and which can be measured by the data arising from the process. If no fundamental changes occur during the period of data collecting, as would occur if the machine were re-adjusted, or the raw material changed in character, or the like, then the resulting data represent the variations due only to chance causes. When the appropriate measure is applied, and the frequency of occurrence of each unit of measure is determined, the histogram or distribution (plotting number along the Y axis, and a continuous scale of the measurement along the X axis) takes on, generally, a bell shaped curve which is known as the normal probability curve of distribution. This curve can be expressed mathematically by several parameters, one of which we will now show.

For many years we have been accustomed to use the term 'average,' which could be obtained by adding up the individual values, and dividing by the number of terms added. Whether we realized it or not, the very use of the word average implied a variation, but we had no means of expressing that variation.

The statistician does this by a term he calls the standard deviation, and it is the fundamental building block of most probability calculations. Having the average, we convert each measurement, that went into the normal distribution curve, into a difference between it and the average, and call this its deviation. It does not matter whether the deviation is positive or negative, so far as calculation is concerned, because each is then squared, and the several squares added up, or totalled. The total is divided by the number of items, and the square root of the answer is obtained. The final result is called the standard deviation. We usually give it the symbol for sigma, and generally refer to it by that name.

The standard deviation, or sigma, is of particular importance to the statistician working in the field of probability. For the first time he has a measure of variation. If he marks off the point for the average along the continuous scale, and then marks off a distance equal to one sigma on either side, then, in the long run, 68.26% of all the measurements should fall between these limits. If he marks off a two sigma distance either way from the average, these limits should include 95.44% of all the measurements, and if he marks off the three sigma distances they would include 99.74%. Even when the process is slightly 'skewed' or in short runs, the agreement is usually close enough that the general laws can be applied.

The importance of the standard deviation as a general basis of all probability functions is very great, since it appears in almost every formula. Its square is called the 'variance' and, among other uses, enters into a system of comparing several variables, called the 'analysis of variance,' a technique which has powerful applications in the study of causes and their effects, as in research.

Next in line of application in probability uses is known as 'sampling' and it is the basis of studying the relation of an entire group, often called the population, and a few from that group, on which some estimation of the properties of the entire group is to be made. Quality must often be estimated by the use of samples, particularly when the test of quality is destructive, or otherwise makes the piece unusable, or for economy in decision procedure, or in speed of decision. In sampling, the statistician must know the laws of probability in sampling. This is because chance alone determines, or should determine, which item or items appear in the sample. He knows that unless pure chance selects the item that becomes part of the sample, the result can be in error, or 'biased,' and not be representative of the true condition of the population. In addition to the 'normal curve of probability,' used when measurement along a continuous scale is available, and as already discussed, there is often used another law, a probability based on the 'law of small numbers.'

In any sampling, where the decision as to whether each piece either conforms, or does not conform, to the standard must be made, and the result is expressed as a percentage of the sample, the usual method is to employ the expansion of the binomial, or its limiting form, the Poisson distribution as a method of estimating sampling results. Suppose that a sample of 100 items is used and that it is desired that we allow no more than one per cent non-conforming items. Do we expect that every sample of 100 items have exactly one non-conforming item—not by any means. We expect often to find none, even when we are sure that there are some of them in the population. Sometimes we will find one, or two, or even three or four. We only know that we are meeting the one percent figure by repeated samplings, but we can set up a procedure through which we can make assumptions of the true condition based on one, or a very few samples. What is the expected proportion of none, one, two, three and the like?

We first set up a formula for the probabilities, using the binomial, and express it as: $Q + P = 1$, where Q is the fraction of conforming items and P is the fraction of non-conforming items. Thus their sum must be unity. Where the proportion of non-conforming items is 1%, the formula becomes: $.99 + .01 = 1$.

What we are saying is that every time we examine one item, there is a probability of .99 that it conforms to the standard and a probability of .01 that it does not. We are also saying, for each individual item, that 99 to 1 it meets the standard. When we take a second item, the probability is the same for that item, and the probability is $.99 \times .99$ or .9801 that both conform. We have learned not to rely upon one item, or even two items as a sample upon which to make a decision. Experience has shown that the size of the sample must be chosen with reference to the proportion of non-conformity we expect to use as a criterion, with due respect to the cost of sampling and test. The size of the population from which we take the sample has some bearing, but not as a linear function.

For this discussion we will assume that a sample of 100 has been decided upon. Parenthetically we are introducing one of many 'decision functions,' the 'work-horse' of probability applications.

What are the probabilities of finding no non-conforming items, or one, or two, or more non-conforming items, when the true proportion is one per cent, and when a sample of 100 is taken as the basis of decision? We simply set up the proper binomial, with the sample size as the exponent, and expand:

$$(Q + P)^n = 1$$

or in this case:

$$(.99 + .01)^{100} = 1$$

The first term of the expansion is $.99^{100}$, or .36603, and this is the probability of finding no off-standard items in a sample of 100, even if it were known that there were one per cent present.

The second term is $100(.99)^{99}(.01)$ or .36973, and this is the probability of finding one non-conformist. The third term is .18467, or the probability for finding two. The expansion is continued in the same way until the following table can be constructed:

The probability of	none is	.36603
	one	.36973
	two	.18467
	three	.06010
	four	.01494
	five	.00290
	six	.00046
	seven	.00006
	eight	.00001

Thus one can expect to find none about 36.6% of the time and even three 6% of the time from the same population, while one in one hundred thousand times one could even find eight non-conformists. On the basis of a sample of 100 we could expect to make decisions fairly well between one population of one-half per cent quality and another of five per cent quality, while closer decisions would require larger or repeated samplings. We know there is uncertainty, but we have a pretty good idea of what that uncertainty is. Our business needs have discovered a better way to make decisions.

Having made a sufficient number of such tests, we can apply another decision mechanism called the Chi-square test of uniformity as a check on the continuing excellence of the sampling. We first make a frequency distribution from the data of the frequency of none, one, two and the like, and determine the grand average of all of the samplings. From this grand average we determine the theoretical frequency of none, one, two and the like. Now by appropriate mathematical procedures we determine whether the distribution we actually found could have reasonably come from that calculated average quality. The criterion of the decision comes from standard tables of Chi-square values. The decision is valid within the ability of the data we have accumulated, even to the extent of warning us to continue the study to resolve a doubt. The chi-square test can be applied to any distribution which can be expressed mathematically.

Usually applied to two separate determinations, such as two sigmas, two averages, two per cents defective, or the like, the value of the significance test is to measure by probability means, whether these two values represented data that could come from the same system of causes, or from two significantly different systems. Often these tests are used in evaluating research rather than routine control. The calculations result in a value, called 't,' for which reason some are often called 't-tests'; 't' is related to the distribution in the same way, and in the case of a normal distribution, is in units of standard deviation. From tables, using entries appropriate to the conditions of the test, one determines the values of '*t*' which represent the 5% probability, and the 1% probability that the two values could have come from the same system of causes. If the calculated value of '*t*' is less than the 5% value for '*t*' ('*t*' increases with decreasing probability) we do not take the chance of assuming that the two values from the data are significantly different. If the calculated value is greater than the 1% level for '*t*' we make the opposite decision. If '*t*' lies between the 5% and the 1% levels, we either adopt the decision based on the nearest '*t*' value, knowing the decision is borderline, or make further study in order to obtain more data.

Now that we must concede the fact of variation in repetitive processes, we must of necessity consider the important aspect of Engineering design, called specification. If things made must be allowed a variation, the allowable variation must be expressed by what we call a tolerance. Tolerances must allow a reasonable range in the items concerned, but must bar the unusable. Between these two extremes there often lies a gulf of guesswork, coercion and fantasy. Statistical studies may bridge this gulf with reason.

The control chart has a definite relationship, through logic, but not through statistics, to specification. That is to say, specifications arise out of the conditions that the item must meet, and are arbitrarily fixed. A process may be statistically 'in control,' but so centered that out-of-specification product is being made, or it may be 'out of control,' and so centered, and with so little variation, that the product is within specifications. There are three possible relationships between specification tolerances and the spread of variation of the product: where the normal distribution of the product far exceeds the tolerance, when they are equal, and when the distribution is smaller than the tolerance. The latter is the only desirable condition since allowances must be made for some shift in the process, as by wear of tools, and the like. This gives an operating range for the process.

Suppose we visualize a distribution such that it lies within tolerance, but with one three-sigma tail exactly on the tolerance limit. This, in effect, fixes the position of the center of the distribution, or grand average, and this in turn is the center of one position of the control line of a control chart for measurements, with sample averages allowed a variation either way. Now visualize the system moved toward the opposite tolerance line, with a similar positioning of the grand average. Between these two positions of the grand average is the working range of the process. This over-simplification is usually modified in practice by certain statistical procedures. The point is that we had to 'discover' probability in order to adopt this logical approach.

Specifications have a direct relationship to cost—an all important business discovery in probability. The user of a product finds that his cost of using it is small if the product is almost unvarying, while costs due to waste, machine down-time, and re-work increases as the raw material he purchases becomes more variable. Careful study can express these increases as a mathematical curve. On the other hand the cost of procuring the raw material decreases as the variation is permitted to increase. This naturally shows up in the procurement price. It also can be expressed as a mathematical curve. The combination of these two cost lines, or more lines if other factors are evaluated such as efficiency, warehousing and the like, can be charted and added together graphically, or the functions, expressed by a common measure of the tolerance allowed, can be solved by the calculus, resulting in a minimum or a maximum curve, generally the former, where the location of zero slope is the best cost determination. Recognition of this property in specification writing allows a practical approach to the best cost specification.

Many specification writers fall into the trap of measuring a few items, then setting a specification based on their properties. This trap catches both the consumer and the producer. A more valid, and completely statistical, procedure is called a process capability study. Its object is to determine if the process can meet a specification by determining the combined effect of all the variables in the process. The procedure may logically determine whether a process or a machine can be used to make a product to specification.

There are several kinds of variation that can affect process capability. Consider a process where each machine has a multiple arrangement, with each separate unit in the arrangement performing the same operation simultaneously on different items of manufacture. It may be a machine filling similar packages from several filling heads, or a multiple stamping machine, a multiple spindle machine, or it may have several identical dies, or molds, or other devices. There may be two or more such machines making the same product, and, of course, each works over a span of time.

The basic variability is that of the product issuing from one of the heads or spindles. Next is the variation between the heads, then the overall variation between the machines, and finally their variation from time to time during the manufacture of that particular item. The machines may even be located in different factories. Nevertheless, the combined stream of product must meet the same specification requirement. The problem is one of measuring the individual and the collective variables.

One method of process capability study will introduce that area of business discovery of probability known as 'design of experiment.' Recently it was desired to establish whether a machine process could be improved. Call it a cutting operation where lengths were cut from bars in a twelve spindle machine, such as an automatic screw cutting machine. The cut-off not only was a matter of specification, but excess length resulted in fewer pieces per bar of metal fed to the machine, a matter of cost of material. It was needed to determine what the normal variation was in each cutting spindle at any given short interval, how much the spindles varied from each other, and how they changed during the day, as from one hour to the next.

It was arranged that three times during the morning, and again three times during the afternoon a sampling would be made. At each sampling four pieces would be taken consecutively from each head. The entire group of measurements were assembled in a table for a technique known as analysis of variance, in what would be termed a 4 by 12 by 6 experimental design (4 being the number of replications, six times during the test, from each of the 12 heads).

Since it is not the object of this discussion to go into formulation, all we will state are the fundamentals. Note that there are 72 sets of four measurements each, giving 72 estimations of the basic variation from any head at any short interval of time. This becomes the basic measure of variation. There are six sets of four measurements for each head, giving a measure of its change with time, which can be compared with the basic variation. There are 12 sets of four measurements for each short interval, which can be compared with the basic variation to see if there is a fundamental difference between heads. In each case, a question is answered: are the differences significant?

The answers were that the time-to-time variation was highly significant and that a control chart was needed. The differences between heads was less significant, and indicated that Engineering studies to control this were needed over a long-pull. Thus, two decisions of fundamental importance were made.

Analysis of variance is an important tool in the field of design of experiment. Other tools or techniques are available. Many of them come from the field of Operations Research. In all, one basic fact is notable. By carefully planning the experiment, in which the known variables are controlled in precise combination, a series of tests can be combined to yield usable information in a far more dependable and precise manner than by separate and sometimes unrelated explorations. Mathematical probability is made to function for the completely practical purposes of planning in advance that the answer obtained by the research is a usable and a dependable answer.

An entirely new approach has recently been made to the psychology of inspection, using the statistical approach called the Monte

Carlo method of simulating a process. Inspection is a process that can become too routine, mentally. Inspectors begin to make mental short cuts. They begin to superimpose desire, or indecision, or local influence, on statistical processes. Some of these may bias their judgement or decision—in other words make for wrong decision on quality or in other matters. They may be influenced by the borderline quality by avoiding the making of decisions by saying to themselves that they "won't count this one unless there are others like it," or not making decision if the deciding item can be called borderline, to the extent of passing bad items, or rejecting good items. It is the function of management to discover these errors and to provide proper training or retraining.

Many examples of applications of probability could be given. Several hundred technical and scientific papers are published each year in various journals, and in the proceedings of the American Society for Quality Control, the American Statistical Association, the Operations Research Society of America and other societies. We will close by one reference to a practical 'break through' in the field of accounting.

When an airline, or a railroad sells a ticket, or accepts a shipment which involves other transportation companies, the first line collects the entire fare or charge, and must forward to the succeeding line or lines their share of the combined sale. Calculating the exact shares for each transaction is a tedious and costly procedure. Abandoning the time honored accounting practice of 'to the penny,' several airlines studied applications of sampling. After trying both the exact and the sampling procedures side by side for some time, it was found that the savings in clerical cost between complete accounting and sampling was several times the cost of the greatest error introduced each month by the sampling method. Furthermore, the sampling errors, which were both positive and negative, tended to balance over a span of months. Therefore, it was simply good business to adopt the sampling procedure.

In a parallel study, railroads in certain 'transfer points' began to use the same method for the transfer of monies for freight interchanges. Sampling was also used to decrease the cost of the annual evaluation of transmission lines and equipment by a telephone company. Thus even the field of accounting began to discover probability.

We seem to have strayed far from the simple determination of

(Continued on page 37)

A Brief Study of Finite and Infinite Matrices from a Set-Theoretic Viewpoint*

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The first part of this paper assumes a working knowledge of the algebra of matrices and of the concept of matrix rank. A knowledge of elementary set theory helps understand the motivation. The second part assumes a background in analysis normally obtained in intermediate calculus, and some transfinite arithmetic. No effort has been made for this treatment to be self contained. Also the proofs of several theorems will be omitted due to lack of time.

We first consider finite matrices. Two matrices will be said to be equivalent if one is obtainable from the other by elementary row and column operations. Otherwise they will be said to be disjoint.

A matrix can be thought of as an ordered set of elements. Thus each row and each column is a subset of the matrix. For the present these sets will be finite, hence countable. We now make the two basic definitions which we shall use in matrix set theory.

Definition: Given two n by m *matrices* $A = (a_{ij})$ and $B = (b_{ij})$. The *union* of A and B, denoted $A \cup B$, will be defined to be the set of matrices having either 1) a_{ij} or 2) b_{ij} in the ijth position. The *intersection* of A and B, $A \cap B$, will be defined to be the matrix having 1) a_{ij}, in case $a_{ij} = b_{ij}$, or 2) 0, in case $a_{ij} \neq b_{ij}$, in the ijth position. Thus the matrix union yields a set of matrices, whereas the matrix intersection yields but a single matrix. However, all matrices concerned have the same dimension. Given $A = (a_{ij})$ and $B = (b_{ij})$ we will investigate the rank of the matrices belonging to set $A \cup B$ and the rank of the matrix $A \cap B$. The first theorem to be presented is as follows:

THEOREM 1. The number of matrices in the set $A \cup B$ is 2^{mn-q} where q is the number of elements common to A and B.

Proof: Each of the mn elements in the matrix C belonging to $A \cup B$ can be chosen in two ways, hence there are 2^{mn} possible choices. If q of the elements are common to both A and B, there will be $mn - q$ distinct elements to choose from and thus 2^{mn-q} distinct

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choices. Hence $A \cup B$ has 2^{mn-q} distinct members, one for each choice.

Considering the ranks of members of the set $A \cup B$, we know that if A has rank p and B has rank r, then at least one member of $A \cup B$ has rank p and at least one member of $A \cup B$ has rank r. For the matrix $D = A \cap B$, we have

THEOREM 2. If $A \cup B$ has 2^{mn-q} distinct members, then the maximum rank of $D = A \cap B$ is q.

Suppose A and B are n by m and disjoint. Let the rank of A be p and the rank of B be r. Then $r \neq p$. We may ask if the set $A \cup B$ contains matrices of all ranks between p and r inclusive. Assume that $r < p$. We will seek a matrix C belonging to $A \cup B$ whose rank is $r + 1$. For this purpose we consider a set ϕ which will represent $n!$ matrices obtained from A by permuting the rows of A. The following theorem results:

THEOREM 3. Let A be n by m and B be n by m. Assume that A is of rank p and that B is of rank r with $r < p$. Let ϕ be the set of $n!$ matrices obtained from A by permuting the rows of A. Then for at least one ϕ_1 belonging to ϕ, $\phi_1 \cup B$ contains a matrix, C, of rank $r + 1$.

Proof: There are r rows of B which are linearly independent. In the operation of taking the union, carry these r rows over into C. Now at least one of the n rows of A is independent of the n rows of B, since $r < p \leq n$. By permuting the rows of A, this row may be located so that it is not in a common position with any of the r independent rows of B. In the union operation carry this row into C. The remaining $n - r - 1$ rows, if any, may be carried into C from B. Hence C has $r + 1$ linearly independent rows and thus its row rank is $r + 1$ which is equal to its rank. This theorem may be extended as follows:

THEOREM 4. Let ϕ be the set of $n!$ matrices obtained from A by permuting the n rows of A. Under the assumptions of Theorem 3, there exist $p - r$ members of ϕ such that

1) ϕ_i belongs to ϕ, and

2) $\phi_i \cup B$ contains at least one matrix of rank $i + r$,
   $i = 1, 2, \cdots, p - r$.

These two theorems can be proved without permuting the rows of A. The motivations here will become evident later.

We now investigate the notion of space with respect to matrices A and B. We give the following definitions:

Definition: Given two n by m matrices A and B, and the set

$$\phi = \{\phi_1, \phi_2, \cdots, \phi_{n!}\}$$

consisting of the $n!$ matrices obtained from A by permuting the rows of A. We define the *union space* of A with respect to B, denoted $A * B$, to be the space spanned by the $n!$ unions, $\phi_i \cup B$.

If A has rank p and B has rank r, then $A * B$ contains $p - r$ matrices of ranks of all integers between p and r inclusive. This follows from Theorem 4. This definition obviously deals with a quite unwieldly number of matrices (unless, of course, severe restrictions are placed on A and B). However, we shall theoretically arrive at some generalizations of common matrix properties. Also these definitions will prove useful in our investigations of infinite matrices and matrix integration.

Using this definition of union space, and by introducing scalar multipliers, we may obtain p canonical (row reduced) forms if p is the rank of A. We will omit this construction, because, although simple, it is lengthy. At first glance it would seem as though we are here constructing canonical forms using only two elementary operations. This is actually an incomparable situation due to the definition of union space. However, this leads us to:

Definition: Given two n by m matrices $A = (a_{ij})$ and $B = (b_{ij})$, the *extended union* of A with respect to B, denoted $A \underline{\cup} B$, will be the set of matrices each member of which obtains each of its mn elements in one of the following three ways: The ijth position of each of the resulting matrices is to be 1) a_{ij}, or 2) b_{ij}, or 3) $a_{ij} + b_{ij}$. The new set, $A \underline{\cup} B$, will have 3^{mn-q} members.

We will now show that the three elementary operations on an n by m matrix A over the field of real numbers occur as special cases of the operations of introducing scalar multipliers and taking the extended union of members of the set ϕ with respect to a certain n by m matrix. This is done as follows.

Proof: Let us review the three elementary operations in light of the definition of the extended union. By doing this, the theorem will become obvious. First we are given an n by m matrix A. The set ϕ consists of the $n!$ n by m matrices obtained by permuting the rows of A. Thus the set ϕ accounts for the first of the three elementary operations. Then either we are given an n by m matrix B, or we must choose one as in this theorem. In this case we choose the matrix to be A. Now since A itself belongs to ϕ, let α be any non-zero scalar and take the extended union $(\alpha A) \underline{\cup} A$. If we desire a resulting matrix whose ith row has been multiplied by α, carry all but the ith row of the latter matrix A into the new matrix. This is equivalent to the second elementary operation. Lastly, to obtain a matrix as in the third elementary operation, by choosing the proper member of ϕ and applying the extended union, we obtain the desired result.

From the above discussion we may readily see that if the matrix B is restricted to be identically A, all forms obtained from introducing scalar multipliers and taking the extended union are also obtainable by successive applications of the three fundamental operations. However, the matrices obtained in the matrix union may be thought of as being functions of the matrices involved in the union. In this case we have a situation similar to the Riemann-Stieltjes integral in which we take the integrator to be the dummy variable corresponding to the case in which the second matrix in the union is taken to be the same as the first matrix.

Other results have been developed along these lines, however, due to the lack of time, they will be omitted.

We now consider briefly matrices whose dimensions are n by ∞. Since the elements of matrices under consideration are indexed by ordered pairs of integers and since there is a one-to-one correspondence between these pairs and a subset of the set of positive integers, an n by ∞ matrix contains a countable number of elements. The notion of an uncountable matrix was desirable, and the following definition seemed to be the most reasonable.

Definition: An n by ∞ matrix A is said to be uncountable if any row contains an infinite subset of an uncountable set. A matrix which is not uncountable is countable.

As before, let $A \underline{\cup} B$ denote the set of matrices in the extended union of two infinite matrices A and B. Then

THEOREM 6. $A \underline{\cup} B$ contains a countable number of infinite matrices.

THEOREM 7. If either A or B is uncountable, then $A \cup B$ contains an infinite number of uncountable matrices.

Many other results along this line have been developed, but due to the lack of time cannot be investigated. One result on convergent sequences will be mentioned.

THEOREM 8. Let A and B be n by ∞ matrices over the set of real numbers each of whose n rows is a convergent sequence. Then in the set $A \underline{\cup} B$ there exists a countably infinite number of distinct matrices each of which has within its n rows two convergent subsequences.

We mention briefly matrix products interpreted as approximations to integrals. If A is 1 by m (a row matrix or row vector), and if B m by 1 (a column matrix or column vector), then the matrix product is a 1 by 1 matrix, C, where C is simply the inner or dot product of two compatible vectors. Let α be a real-valued function, defined and bounded on an interval, $[a, b]$, and let $P = \{a = x_0, x_1, \cdots, x_n = b\}$ be a partition of $[a, b]$. Also let

$$\Delta_k\alpha = \alpha\,(x_k) - \alpha\,(x_{k-1})$$

so that

$$\alpha\,(b) - \alpha\,(a) = \sum_{k=1}^{n} \Delta_k\alpha.$$

Let B be the n by 1 column matrix whose elements are $\Delta_i\alpha$, i.e.

$$B = \begin{bmatrix} \Delta_1\alpha \\ \Delta_2\alpha \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n\alpha \end{bmatrix}$$

Also let f be a real-valued function, defined and bounded on $[a, b]$. Let t_k be any point belonging to $[x_{k-1}, x_k]$. Let F_p be the infinite set of 1 by n matrices whose ith element is $f(t_i)$ for the partition P. Thus F_p is the set of matrices of the form $[f(t_1), \cdots, f(t_n)]$ where $x_{k-1} \leq t_k \leq x_k$. If A belongs to F_p, the matrix product AB now becomes a Riemann-Stieltjes sum of f with respect to α for the partition P, denoted $S(P,f,\alpha)$.

Note also that if

$$B' = \begin{bmatrix} \alpha(x_1) \\ \alpha(x_2) \\ \cdot \\ \cdot \\ \cdot \\ \alpha(x_n) \end{bmatrix} \quad \text{and} \quad B'' = \begin{bmatrix} \alpha(x_0) \\ \alpha(x_1) \\ \cdot \\ \cdot \\ \cdot \\ \alpha(x_{n-1}) \end{bmatrix}$$

then $S(P,f,\alpha) = A(B' - B'')$. Several theorems concerning the above concept follow.

The connection between infinite matrices and the Riemann integral is as follows:

THEOREM 9. Let f be a function defined and bounded on $[a, b]$. Let P be a partition of $[a, b]$ such that $x_i - x_{i-1} = (b - a)/n$. Then f is Riemann integrable on $[a, b]$ if and only if the following limit exists:

$$\lim_{n \to \infty} [f(t_1)\, f(t_2) \cdots f(t_n)] \begin{bmatrix} \Delta_1 x \\ \Delta_2 x \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n x \end{bmatrix}$$

In this case the limit will be equal to $\int_a^b f\, dx$.

We close with a few brief words concerning the concept of outer Lebesgue measure applied to matrices. We investigate measure from a matric, set-theoretic viewpoint. Let f be a real-valued function, defined and bounded on an interval $[a, b]$. Let P be a partition of $[a, b]$, and write P in column matrix form; i.e.

$$P = \begin{bmatrix} \Delta_1 x \\ \Delta_2 x \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n x \end{bmatrix}$$

Then P is composed of two parts; namely, those subintervals of P containing points of discontinuity of f, denoted here by P_D, and all other subintervals, to be denoted by P_C. In fact we may write

$$P_D = \begin{bmatrix} \Delta_1 x \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \Delta_2 x \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \Delta_n x \end{bmatrix}$$

by replacing in the matrix P those subintervals containing only points of continuity of f by zero. Now $P - P_D = P_C$ and thus $P = P_D + P_C$. This can also be written $P = P_D \cup P_C$ where we take the one matrix of the union having no zero elements. Considering P, P_D, and P_C as vectors and letting $\cdot$ stand for the inner or dot product between two vectors, we have

$$\begin{aligned} P &= P_D + P_C \\ P \cdot P &= (P_D + P_C) \cdot (P_D + P_C) \\ &= P_D \cdot P_D + P_C \cdot P_C \qquad \text{since } P_C \cdot P_D = 0. \end{aligned}$$

Now $P_D \cdot P_D$ gives the sum of the squares of the lengths of the subintervals containing points of discontinuity of f. In conclusion, one may prove the following theorem:

THEOREM 10. f is Riemann integrable on $[a, b]$, if and only if, for every given $\varepsilon > 0$, there exists a partition P_ε of $[a, b)$ such that if P is a refinement of P_ε and $P = P_D + P_C$ under the above notation, we have $P \cdot P - P_C \cdot P_C < \varepsilon$.

The Four-Dimensional Cube*

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Mathematics, in its trend toward abstract thinking, has, in the last century, been leading the mind of man to a study of four-dimensional geometry. Four-dimensional space is, of course, one in which there exists four mutually perpendicular lines. This is in contrast with our well-known three dimensional space in which only three mutually perpendicular lines exist.

It is the intention of this paper to give a better understanding of higher space geometry concepts, especially those dealing with the four-dimensional cube. Such a study will increase the understanding of the geometries of plane and space. I, personally, found my interest in four-dimensional space aroused by attempting to graph a two dimensional equation in complex coordinates.

The method to be used in analyzing this problem is one of comparing the geometries of figures in spaces from zero to and including three and inferring what the corresponding fourth-dimensional concept would be.

The two major techniques for studying four-dimensional figures are (1) examining three dimensional sections of the figure, and (2) considering projections into a three dimensional space or a hyperplane. Both methods have their disadvantages. When observing sections of a four-dimensional figure, one has no perception how these may be connected, and only a small portion of the figure may be seen at a time. In contrast with this, the projection study allows the observer to view the complete figure with the disadvantage of sacrificing a depth perception.

The method of revolving projections was chosen as the basis for this study. It should be noted, at this point, that all projection rays are perpendicular to the plane of projection. This is in contrast with the perspective projection in which all projection rays converge to a point. Orthographic projections are used since these show the exact form of the object, instead of permitting perspective distor-

*A paper presented at the 1959 National Convention of KME and awarded third place by the Awards Committee.

tions. Figure 1-a illustrates a sketch perspectively distorted while Figure 1-b is an orthographic projection of Figure 1-a.

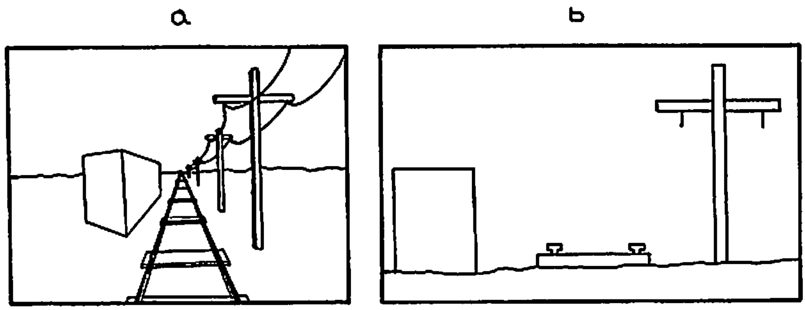


FIGURE I
PERSPECTIVE VERSES ORTHOGRAPHIC

Since a regular projection of a cube of three dimensions in standard position would be a square of two dimensions, we might compare these figures with their dimensional spaces by saying: a square is related to two dimensional space as a cube is related to three dimensional space. By this type of inference, the different dimensional figures may be compared in tabular form as in Table I.

Table I

n dimensions	figure			
0	point			
1	line	line segment		
2	plane	equilateral triangle	circle	square
3	hyperplane	tetrahedron	sphere	cube
4		pentatope	hyper-sphere	tesseract

The sequence: point, line, plane, hyperplane is obvious from the respective definitions. A point has no dimension, a line has one dimension, and so on. In zero dimensions, there is only one geometrical figure possible, the point. There is a great increase in complexity of the possible geometrical figures with each increase in dimensional representation.

The method of comparing dimensions may be illustrated with the simplest of the straight line figures, the two dimensional triangle series. It is known that for straight line figures of one dimension two points are necessary for their determination. Likewise, three points determine a two dimensional straight line figure, and four points determine the three dimensional figure. Therefore, five points should determine the corresponding four-dimensional figure. In the special case when the two dimensional figure is an equilateral triangle, the corresponding three and four dimensional figures are called, respectively, the tetrahedron, and the pentatope. Since the equilateral triangle consists of three equal line segments, and the tetrahedron consists of four equilateral triangles, we may conclude the pentatope should consist of five tetrahedrons. The combinations of numbers of points, lines, surfaces, and so on, also infer a definite sequence. Here these figures are regarded as consisting only of the bounding elements thus excluding the interior.

For the circle-sphere-hypersphere sequence, this type of analy-

Table II

	figure	parts				
n dimensions		0	1	2	3	4
0	point	P				
1	line segment	2P	L			
2	square	4P	4L	A		
3	cube	8P	12L	6A	S	
4	tesseract	16P	32L	24A	8S	H

P represents vertex points
L represents edge line segments
A represents face areas
S represents space volumes
H represents hyperspace content

sis cannot be used since the figures in this series have no vertices. However, each of these figures may be generated by rotation about a bisecting element; for example, a line bisecting a circle may act as the axis to generate a sphere. It may here be noted that a circle is generated by rotating a point in a given plane at a constant distance from a fixed point. Therefore three collinear points equally spaced may be considered as a one dimensional circle with its center. We may also infer that a hypersphere of four dimensions is generated by "rotating" a sphere in four-dimensional space about a plane through its center.

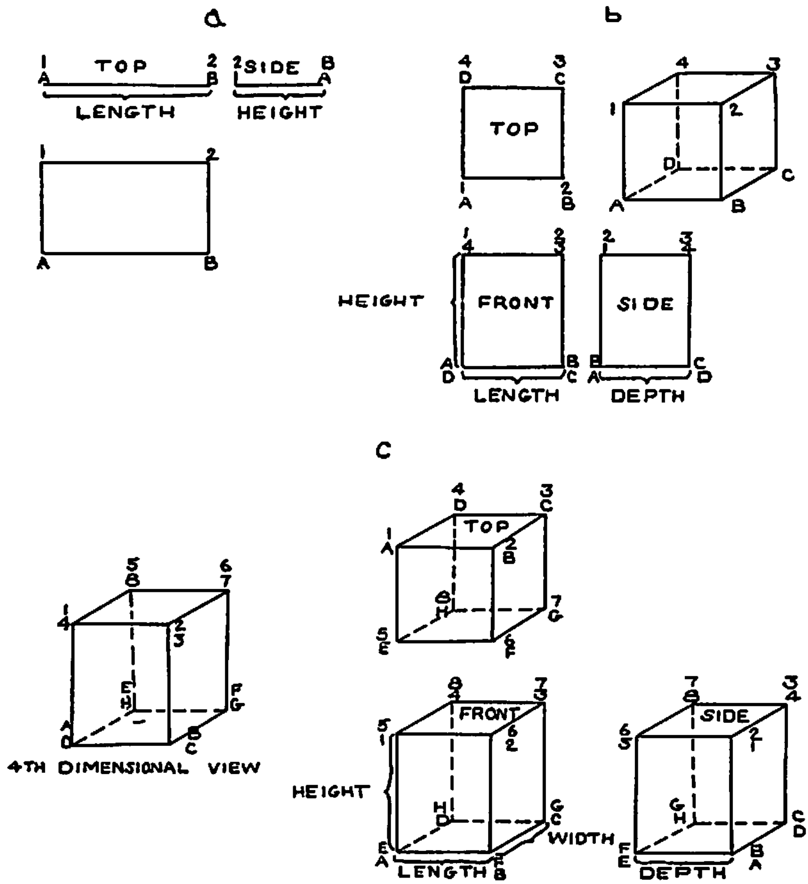


FIGURE 2
PRINCIPAL PROJECTIONS OF DIFFERENT DIMENSIONAL FIGURES

A square, which has four vertices and four equal line segments, when extended one dimension, produces a cube of 8 vertices and 12 equal line segments together with 6 squares of surface. Table II is a tabular form of these components as has been previously calculated.

It is customary to speak of a projection as being a representation of an "n" dimensional figure in "$n - 1$" dimensions. Thus, the principal projections of a two dimensional rectangle in one dimension would appear as in Figure 2-a. Figures 2-b and 2-c are the principal projections for the corresponding three and four dimensional figures. The consistency between these figures is readily noticed by labeling the vertices in Figure 2 in a set pattern. Notice the top of the two figures denoting each projected vertex is the closer to the actual unprojected vertex. This is most easily seen in the top projection of Figure 2-a. In the vertex "$\genfrac{}{}{0pt}{}{1}{A}$", the 1 denotes the closer of the two vertices coinciding by projection.

Notice that there are as many projections as there are dimensions in the original figures and these are always mutually perpendicular, thus the principal three dimensional projections of the four-dimensional rectangular figure would appear, as indicated, with 16 vertices. The measurement dimensions of the original figures are found in the projections as labeled (length, width, and so on).

In considering rotations of this four dimensional figure, the special case when it is a hypercube or a tesseract is used for simplicity. First the corresponding rotated projections of a cube are considered. When a cube is projected after being slightly rotated, two figures approximating squares, very close together, are seen. Their corresponding vertices are connected, as indicated in Figure 3-a.

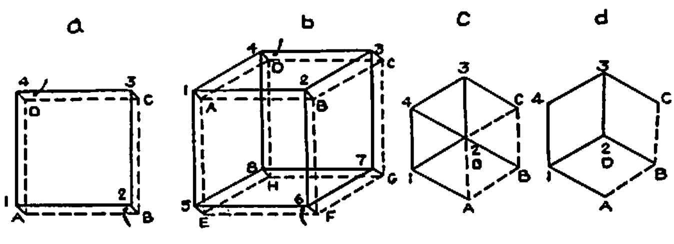


FIGURE 3
TOP PROJECTION ROTATIONS

Extending this concept by one dimension, we may say that a tesseract, after being rotated slightly in hyperspace, would appear, when projected on this hyperplane, as two approximate cubes closely interconnected with their corresponding vertices connected, as in Figure 3-b.

It may be noted that this figure of two interconnected cubes is very similar to the popular representation of the tesseract, as illustrated in Figure 4-b, consisting of a cube within a cube with the corresponding vertices connected.

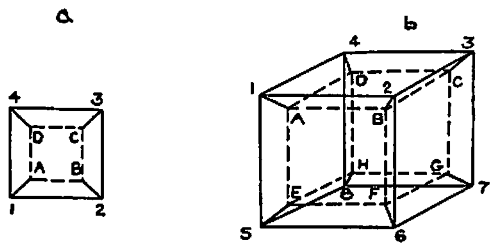


FIGURE 4

An analogous projection is obtained by the shadow of a cube produced by a light close to the center of a side. This is illustrated in Figure 4-a. Using orthographic projections, Figure 4-a could be considered as a top projection of a frustum of a square pyramid. From these observations we may conclude Figure 4-b is a perspective projection of the tesseract or an orthographic projection of a frustum of a cubical hyperpyramid. The frustum of a square pyramid has the same basic construction as the cube, thus it is possible to study the parts of the tesseract with this figure.

We may note that as the cube in Figure 3-a is rotated, the two marked vertices approach each other, if rotation is chosen about the diagonal of the front side. This diagonal must be perpendicular to the projectors or lines of sight. Assuming these conditions are fulfilled, the cube of Figure 3-a would appear as in Figure 3-c when the two marked vertices coincide in the center of the figure. For simplicity, we may regard this cube as a solid thus eliminating three lines from consideration, as illustrated in Figure 3-d. Figures 3 and 4 have vertices labeled according to Figure 2. These are all based upon the top projections. Also it is noted that these two figures consist of two similar elements. For clarity one of these elements is dashed throughout.

It is easily seen that Figure 3-d consists of three squares, each distorted until one diagonal equals a side. It may, therefore, be inferred that the corresponding special projection for the tesseract consists of four cubes each distorted until one diagonal equals a side. Figure 5 is a sketch illustrating this special projection of the tes-

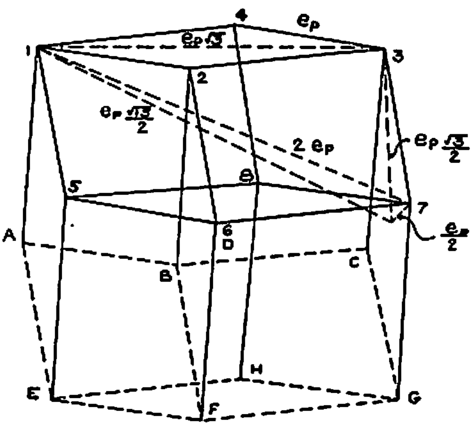


FIGURE 5
ISOMETRIC PROJECTION OF THE TESSERACT

seract. These vertices are also labeled the same as those of the tesseract being projected into Figure 2-c. Notice the four omitted lines connected to the vertex D: $\overline{4D}$, $\overline{AD}$, $\overline{DH}$ and $\overline{DC}$. The four lines meeting in the center of the figure correspond directly with the three lines meeting in the center of Figure 3-d. Actually there are two points coinciding at this point, but these projections were drawn only of the half of the figures closer to the planes of projection. The three lines intersecting at the center in Figure 3-d are mutually perpendicular. The same may be said about the four lines intersecting in the center of Figure 5.

Since the projection of an oblique line is shorter than the line itself, we know that the edge of a cube is longer than its projected edge in Figure 3-d. It is readily shown by geometry that the ratio of the projected length in this special projection, to the original line is $\sqrt{2/3}$. It is also easily seen that the corresponding ratio for the projection of a square on one dimension is $\sqrt{1/2}$ for each edge is located 40° from the line of projection.

The corresponding ratio for Figure 5 may be determined using the principle that the tesseract may be rotated about a diagonal of a bounding cube until two vertices coincide, as was before discussed. By solid geometry, the diagonal acting as the axis of rotation is found to have a length twice that of the edge projection. Also this diagonal length equals $\sqrt{3}$ times the edge of the tesseract, as may be seen by again considering the principal projections in Figure 2-c. By eliminating the revolving diagonal length from these two conditions, it is seen that the ratio of the projection lengths, to the tesseract edges is $\sqrt{3/4}$.

It is now seen that the tesseract edges themselves are positioned 30° from this hyperplane for this special projection. To better understand this statement, consider any vertex of the tesseract being projected into Figure 5 as located in this hyperplane. Then any of the lines of the tesseract, passing through this vertex, are 30° from this hyperplane.

Using these concepts and principles, a four-dimensional geometry may be created systematically but not exceptionally rigorously. However, one may more readily perceive the more complex hyperspace figures which is a great aid in such a study.

(Continued from page 22)

the standard deviation and binomial expansion of the exercise book, but this is not true since they have entered into all of the methods and examples given. Thus, we come back again and again to their fundamental aspect in all applications of statistical probability as the lowly building blocks in a growing structure of practical usage, through which is centered the fact that business has not only become aware of, but also, in its own mind, has 'discovered' probability. If chance is a game, business has now found that to play the game, one needs most to learn the rules of the game.

The Problem Corner

EDITED BY J. D. HAGGARD

The Problem Corner invites questions of interest to undergraduate students. As a rule the solution should not demand any tools beyond the calculus. Although new problems are preferred, old ones of particular interest or charm are welcome provided the source is given. Solutions of the following problems should be submitted on separate sheets before March 1, 1960. The best solutions submitted by students will be published in the Spring, 1960, issue of THE PENTAGON, with credit being given for other solutions received. To obtain credit, a solver should affirm that he is a student and give the name of his school. Address all communications to J. D. Haggard, Department of Mathematics, Kansas State College of Pittsburg, Pittsburg, Kansas.

PROPOSED PROBLEMS

126. *Proposed by Mark Bridger, student, High School of Science, Bronx, New York.*

Show that the quartic equation $x^4 - 13x^3 - 12x^2 - 17x + 37 = 0$ has no negative root.

127. *Proposed by the Editor* (From *The Foundations and Fundamental Concepts of Mathematics* by Eves and Newsom).

A man wishes to go from his house to the bank of a straight river for a pail of water, which he will then carry to his barn on the same side of the river as his house. Find the point on the riverbank from which he should take the water in order to minimize the distance he travels.

128. *Proposed by Paul R. Chernoff, student, Harvard College.*

The inverse $F^{-1}(t)$ of a real function $F(t)$ is defined so that $F^{-1}[F(t)] = F[F^{-1}(t)] = t$. If $F(t)$ and its inverse function $F^{-1}(t)$ are continuous functions of t on the entire interval concerned, prove that

$$\int_a^b F(t)\,dt = b \cdot F(b) - a \cdot F(a) - \int_{F(a)}^{F(b)} F^{-1}(t)\,dt$$

129. *Proposed by the Editor* (From *The Mathematical Monthly*).

Find the greatest (volume) right circular cylinder coaxal with, and inscribed in, the solid formed by rotating around the y-axis the area bounded by the two axes, the parabola $y = 9x^2 - 28x + 24$, and the parabola's minimum ordinate.

130. *Proposed by the Editor* (From *The Mathematical Monthly*).

Prove that $\sum_{n=1}^{\infty} \frac{(n-1)}{n!} = 1$

SOLUTIONS

108. *Proposed by the Editor.*

Show that $1/2 + 1/3 + \cdots\cdots + 1/n$ cannot be an integer for any integer n.

Solution by Mark Bridger, High School of Science, Bronx, New York.

Suppose $1/2 + 1/3 + \cdots\cdots + 1/n = I$ (an integer) for some integer n. Moreover suppose n is the smallest such n for which the sum is an integer. Let p be the largest prime smaller than than n. There is no other multiple of p between I and p, and no other multiple of p between p and n ("Bertrand's Conjecture" see Hardy and Wright "Number Theory" for proof).

Now $n!$ contains p as a factor only once, thus we can write $n! = Kp$ where K is the product of all the integers from 1 to n excluding p.

Multiplying both sides of $1/2 + 1/3 + \cdots + 1/n = I$ by Kp gives:

$$(1)\ Kp[1/2 + 1/3 + \cdot\cdot + 1/(p-1)] + Kp/p + Kp[1/(p+1) + \cdot\cdot\ 1/n] = IKp$$

The left side of (1) is an integer since $n!$ contains $1, 2, \cdots, n$ as factors. Now each term on the left of (1) is a multiple of p except Kp/p.

If $M(p)$ represents a multiple of p, we can now write (1) as follows:

$$(2) \qquad M(p) + K = IKp$$

p therefore divides the right side of (2) but not the left because K does not contain p as a factor. Thus a contradiction is reached and the theorem is proved.

115. *Proposed by R. G. Smith, Kansas State College of Pittsburg.*

Given two points A and B in three dimensioned Euclidean space with distances a and b respectively from a line l, show that no point interior to segment AB has distance to l greater than max. (a,b).

Solution by Mark Bridger, High School of Science, Bronx, New York.

Let the line l be the x-axis and denote the two fixed points as: $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Their distances from l would then be given by the formulas:

$$a^2 = y_1^2 + z_1^2, \quad b^2 = y_2^2 + z_2^2$$

No loss of generality results from taking $a =$ max. (a,b). That is $a > b > 0$ and $a^2 > b^2$.

Now take any point $P(x_0, y_0, z_0)$ interior to the segment AB. It will divide AB into AP and PB whose ratio we shall denote by m/n. Then:

$$y_0 = \frac{my_1 + ny_2}{m + n}, \quad z_0 = \frac{mz_1 + nz_2}{m + n}$$

If d is the distance of P from the x-axis (line l) then:

$$\begin{aligned} d^2 &= y_0^2 + z_0^2 \\ &= \frac{m^2(y_2^2 + z_2^2) + n^2(y_1^2 + z_1^2) + 2mn(y_1y_2 + z_1z_2)}{(m + n)^2} \\ &= \frac{m^2b^2 + n^2a^2 + 2mn(y_1y_2 + z_1z_2)}{(m + n)^2} \end{aligned}$$

For any real a and b, $(a^2 + b^2)/2 \geq ab$ and since $a^2 > b^2$ we may substitute $\frac{1}{2}(a^2 + b^2)$ for $(y_1y_2 + z_1z_2)$ and a^2 for b^2 obtaining:

$$d^2 < \frac{m^2a^2 + n^2a^2 + 2mna^2}{(m + n)^2} = a^2$$

Thus $d <$ max. (a,b).

121. *Proposed by the Editor.*

If A, B, and C are the angles formed by a diagonal of a rectangular parallelepiped with its edges, show that:

$$\sin^2 A + \sin^2 B + \sin^2 C = 2.$$

Solution by Mary Sworske, Mount Mary College, Milwaukee, Wisconsin.

Let d be the diagonal of the parallelepiped, and a, b, c the edges which form the angles A, B, C respectively with d. Then:

$$\sin A = (\sqrt{d^2 - a^2})/d$$

$$\sin B = (\sqrt{d^2 - b^2})/d$$

$$\sin C = (\sqrt{d^2 - c^2})/d$$

$$\text{Thus } \sin^2 A + \sin^2 B + \sin^2 C = \frac{d^2 - a^2 + d^2 - b^2 + d^2 - c^2}{d^2}$$

$$= \frac{3d^2 - (a^2 + b^2 + c^2)}{d^2}$$

$$= 2$$

since $d^2 = a^2 + b^2 + c^2$.

Editor's Note: Kay Dundas points out that also $\cos^2 A + \cos^2 B + \cos^2 C = 1$.

Also solved by Loretta Bauer, Mount Mary College, Milwaukee, Wisconsin; Mark Bridger, High School of Science, Bronx, New York; Paul R. Chernoff, Harvard College, Cambridge, Massachusetts; Marilyn Cook, Wake Forest College, Winston-Salem, North Carolina; Kay Dundas, Fort Hays Kansas State College; Don Hayler, Pomona College, Claremont, California; Warren Shreve, Iowa State Teachers College, Cedar Falls, Iowa.

122. *Proposed by George Mycroft, Kansas State College of Pittsburg.*

With each letter symbolizing a digit decode the following puzzle:

$$\begin{array}{r} \text{FIVE} \\ -\ \text{FOUR} \\ \hline \text{ONE} \\ +\ \text{ONE} \\ \hline \text{TWO} \end{array}$$

Solution by Gilbert Orozco, California State Polytechnic College, San Luis Obispo.

There are two solutions to the problem as follows:

$$\begin{array}{r} 3496 \\ -\ 3210 \\ \hline 286 \\ +\ 286 \\ \hline 572 \end{array} \qquad\qquad \begin{array}{r} 9516 \\ -\ 9280 \\ \hline 236 \\ +\ 236 \\ \hline 472 \end{array}$$

One of the two solutions was submitted by Loretta Bauer, Mount Mary College, Milwaukee, Wisconsin; Paul Chernoff, Harvard College, Cambridge, Massachusetts; Kay Dundas, Fort Hays Kansas State College; William Harnish, Drake University, Des Moines, Iowa.

123. *Proposed by the Editor.* (From *The American Mathematical Monthly*).

Let a real positive number n be split into x equal parts in such a manner that the product of the parts will be greatest. How many parts will there be?

Solution by Mark Bridger, High School of Science, Bronx, New York.

If n is split into x equal parts, each part must equal n/x. Their product is therefore:

$$p = (n/x)^x.$$

Since n is a positive constant, p will be a maximum when its derivative with respect to x is zero. This yields $\ln(n/x) = 1$ or $x = n/e$. Now if n/e is an integer this is the solution. If not, since we were to have equal parts, we must take $x = [n/e + 1/2]$, i.e. greatest integer in $n/e + 1/2$.

Also solved by Paul Chernoff, Harvard College, Cambridge, Massachusetts; Kay Dundas, Fort Hays Kansas State College; Don Hayler, Pomona College, Claremont, California; Dennis Hult, Nebraska State Teachers College, Wayne, Nebraska; Ralph Milano, Montclair State College, Upper Montclair, New Jersey; Warren Shreve, Iowa State Teachers College, Cedar Falls, Iowa.

124. *Proposed by the Editor.*

Show that the square of any odd integer is one more than an integral multiple of eight.

Solution by Ralph Milano, Montclair State College, Upper Montclair, New Jersey.

Let $2n + 1$ be any odd integer, where n is an integer. Then:

$$\begin{aligned}(2n + 1)^2 &= 4n^2 + 4n + 1\\ &= 4(n^2 + n) + 1\\ &= 8[n(n + 1)/2] + 1\end{aligned}$$

Since $n(n + 1)$ is even for any integral n then the expression in the bracket is an integer and the theorem is proved.

Also solved by Mark Bridger, High School of Science, Bronx, New York; Paul Chernoff, Harvard College, Cambridge, Massachusetts; Marilyn Cook, Wake Forest College, Winston-Salem, North Carolina; Kay Dundas, Fort Hays Kansas State College; Don Hayler, Pomona College, Claremont, California; Warren Shreve, Iowa State Teachers College, Cedar Falls, Iowa.

125. *Proposed by the Editor.* (From a Russian university entrance examination).

Two factories each received an order for an identical number of machines. The first factory started 20 days earlier and finished work 5 days earlier than the second factory. At the moment when the number of machines made by both factories taken together was equal to one-third of the total number on order, the number of machines made by the first factory was four times the number produced by the second.

The first factory worked on the order altogether x days, producing m machines per day; the second factory worked y days producing n machines per day. Find those of the quantities x, y, m and n and those of the ratios x/y and m/n which can be determined from the data given in the problem.

Solution by Mark Bridger, High School of Science, Bronx, New York.

Each factory produced the same number of machines, therefore:

1. $xm = yn$ and $m/n = y/x$
   Since the first factory started 20 days earlier than the second and finished 5 days sooner than the second:
2. $x = y + 15$ or $y = x - 15$
   When the first factory had worked z days it had produced 4 times the number of machines of the second:
3. $zm = 4(z - 20)n$ or $z = 80n/(4n - m)$.
   At this time, however, the machines produced by both factories equaled one-third the total order. $(1/3)(xm + yn) = (2/3)(xm)$. Thus:
4. $zm + (z - 20)n = 2xm/3$
   Substituting into this equation from 3 we obtain:
5. $zm + zm/4 = 2xm/3$ and upon eliminating z we obtain:
6. $(5m/4)(80n)/(4n - m) = 2xm/3$
   Simplifying and substituting for m/n from equation 1 we get:
7. $2/5 - (1/10)(y/x) = 15/x$
   Eliminating y using equation 2 we get:
8. $2/5 - (x - 15)/10x = 15/x$
   Which yields $x = 45$ and by substituting into 7 we obtain $y = 30$.
9. Since only three independent equations were given (1, 2, 6) we are unable to determine m and n, however, $m/n = y/x = 2/3$.

Editor's Note: Late solutions by Edward Ross, High School of Science, Bronx, New York, were received for problems 121, 122, 123, 124 and 125.

"And Lucy, dear child, mind your arithmetic. . . . What would life be without arithmetic, but a scene of horrors?"

—Sidney Smith

The Mathematical Scrapbook

EDITED BY J. M. SACHS

To think logically the logically thinkable—that is the mathematician's aim.

—C. J. KEYSER

=△=

Myth tells us that, in early times, the sage Yu, the enlightened emperor, saw on the calamitous Yellow River a divine tortoise whose back was decorated with the figure made up of the numbers from 1 to 9, arranged in the form of a magic square or lo-shu.

—F. CAJORI
A History of Mathematics

=△=

The history of magic squares goes back many years. A simple method of construction for magic squares of odd order can be described in the following rules:

1. Write a 1 in the center cell of the top row.
2. Continue writing the integers in increasing order, moving up and to the right to the next diagonal cell.
3. If the motion up carries you above the square, shift to the bottom cell in the same column. If the motion to the right carries you out of the square, shift to the cell on the left margin of the same row.
4. If a cell is already occupied, put the number which would go there in the cell just below the previous number written.

Following these rules we can write the familiar 3 by 3 and 5 by 5 magic squares or, if we choose, a magic square of any odd order such as 13.

8	1	6
3	5	7
4	9	2

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

In puzzling over the preceding rules for the construction of magic squares of odd order, your editor became curious about what kind of array would result if such a series of rules were applied to a cube in three-space. Some rather curious results were obtained from the first attempt to generalize these construction rules. Perhaps some of the readers would like to try modifying these to get more satisfying or more symmetric patterns. Suppose we call any horizontal triple in our cube a row, any vertical triple a column, and any one of the vertical slabs of nine cells, a slice. In this way we can represent our cube by showing the faces of three slices with the left one representing the slice closest to the viewer, the middle one the middle slice and the right one the slice farthest from the viewer.

Rules:

1. Write a 1 in the top middle cell of the middle slice.
2. Continue writing the integers in increasing order, moving back a slice, then right one column, then up one row.
3. If the motion back carries you out of the cube, shift to first slice. If the motion to the right carries you out of the cube, shift to left column in the same row of same slice. If the motion up carries you out of the cube, shift to bottom cell in same column of same slice.
4. If cell is already occupied, put the number which would have been written there in the first unoccupied cell obtained by reversing the motion in 2, starting at the cell in which the previous number was written. That is instead of back, right, up; go down, and if that cell is occupied go left, and if that cell is occupied, go forward.

The 3 by 3 by 3 cube which can be obtained by these rules is shown below:

8	26	17
3	21	12
4	22	13

10	1	19
14	5	23
18	9	27

24	15	6
25	16	7
20	11	2

In each slice the columns total to 15, 69, and 42. In every slice the diagonals each total 42. Is it true that every diagonal which can be obtained by summing a triple of numbers which lie on a straight line but not in a row or column is 42? Since 15 is 27 less than 42 and 69 is 27 more than 42, can cell interchanges be made to have all columns total 42? Could you vary the construction rules somewhat and perhaps get more startling results? Apply this to a 5 by 5 by 5 cube and see what happens.

=△=

The construction of magic squares with an even number of cells is more difficult than construction of such squares with an odd number of cells. A 4 by 4 square can be constructed by lining out the two diagonals and counting squares, writing in the number of the square if it is not on the diagonal. Then, going back to the starting point in the upper left corner, we begin with the largest number not written and go backwards filling in the numbers which were not written in the cells on the diagonals.

	2	3	
5			8
9			12
	14	15	

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

There does not seem to be an analogous method for a 4 by 4 by 4 cube. Can you convince yourself that every cell in such a cube is on some diagonal? The editor of this column would welcome letters on the subject of magic cubes and would like to publish in future issues accounts of results obtained by experimenting with these.

=△=

Suppose we try the construction we have used for odd ordered magic squares on the construction of magic rectangles. If we construct a rectangle with an odd number of rows and an odd number of columns and starting with 1 in the upper left hand corner, we follow the up and to the right diagonal order of numbering our cells, what will we get? The 5 by 3 rectangle will be

1	7	13	4	10
6	12	3	9	15
11	2	8	14	5

The rows sum to 35, 45, and 40. The columns sum to 18, 21, 24, 27, and 30. Can you explain this? Or is it so obvious as to defy explanation? Interchanging elements within a column will not affect the column totals but will change row totals. For example interchanging the 10 and the 15 in the fifth coumn will leave the column sums unchanged but make all three of the row sums be 40. Interchanging elements within a row will not affect the row totals. Can you now re-arrange the elements within the rows so that all of the column totals are 24? If you are unable to do this, can you prove that it is not possible? Try to generalize and imagine this for an M and N rectangle with M and N odd. What happens if you try this scheme on a rectangle with either M, N, or both even? Does the diagonally up to the right method of writing the integers in increasing order work here?

=Δ=

With all the attention now being given to the exploration of space, and to space medicine in particular, it seems appropriate to call attention to the following remark by Ernst Mach in a lecture delivered in 1910. "Mathematical and physiological researches have shown that the space of experience is simply an actual case of many conceivable cases, about whose peculiar properties experience alone can instruct us."

=Δ=

A group of friends meet at a party. The discussion involves birth months. One man notes that no two people present have the same birth month. What is the minimum number of people needed at a party if the probability that two shall have the same birth month is more than one-half?

=Δ=

When Plato wrote over the portal of his school, "Let no one ignorant of geometry enter here," he did not mean that questions relating to lines and surfaces would be discussed by his disciples. On the contrary the topics to which he directed their attention were some of the deepest problems—social, political, moral—on which the mind could exercise itself. Plato and his followers tried to think out together conclusions respecting the being, the duty, and the destiny of man, and the relation in which he stood to the gods and to the unseen world. What had geometry to do with these things? Simply this: That a man whose mind has not undergone a rigorous training in systemic thinking, and in the art of drawing legitimate inferences from premises, was unfitted to enter on the discussion of these high topics; and that sort of logical discipline which he needed was most likely to be obtained from geometry. . . .

—J. C. Fitch

=Δ=

Pick a favorite digit from the number 15,873. Multiply the given number by seven times the digit you have chosen. The product will consist entirely of your chosen digit. For example suppose you pick the digit 8. The product of 15,873 and 56 is 888,888. The same trick can be accomplished with the number 12,345,679 if you use as multiplier nine times the chosen digit. Suppose the chosen digit is 5. The product of 12,345,679 and 45 is 555,555,555. Can you explain this trick which seems mystifying at first glance but is really quite simple?

=Δ=

The science of pure mathematics, in its modern developments, may claim to be the most original creation of the human spirit.

—A. N. Whitehead

=Δ=

Three planes travel on the same course at the same altitude, departing from the same airport. The second leaves one hour after the first. The third leaves one hour after the second. The second and third planes overtake the first simultaneously. The second plane travels 60 mph faster than the first and the third plane travels 90 mph faster than the second. Find the speed of the three planes.

=Δ=

If a healthy minded person takes an interest in science, he gets busy with his mathematics and haunts the laboratory.

—W. S. FRANKLIN

=△=

The Pythagoreans were intrigued by triangular numbers and also by square numbers. A triangular number is any number which can be written in a triangular array such as 3, 6, 10, etc., as shown below.

$$\begin{array}{c} 1 \\ 1\;1 \end{array}\;(3) \qquad \begin{array}{c} 1 \\ 1\;1 \\ 1\;1\;1 \end{array}\;(6) \qquad \begin{array}{c} 1 \\ 1\;1 \\ 1\;1\;1 \\ 1\;1\;1\;1 \end{array}\;(10)$$

A square number is of course any number which can be written in a square array such as

$$\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\;(4) \qquad \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\;(9)$$

The question which recently came to the attention of this editor is, "Under what conditions is a square number also triangular?" We can find examples so we know that there are square numbers which are also triangular, for instance 36 and 1225. Since a triangular number can be written as the sum of consecutive integers beginning with unity, that is $1 + 2 + 3 + \cdots + n$, we can write it in the form $\frac{1}{2}n(n + 1)$. We seek positive integral solutions for the equation $n(n + 1) = 2N^2$. In looking at this problem certain conjectures come to mind. Can you verify that N cannot be a prime; that $1 + 8N^2$ must be a square; that the final digit in N must be 0, 1, 4, 5, 6, or 9; that the final digit in N^2 must be 0, 1, 3, 5, 6, or 8; that the final digit in $2N^2$ must be 0, 1, 2, or 6? Contributions on this topic are invited.

=△=

It may be said that the conceptions of differential quotient and integral, which in their origins certainly go back to Archimedes, were introduced into science by the investigations of Kepler, Descartes, Cavalieri, Fermat, and Wallis. . . The capital discovery that differentiation and integration are inverse operations belongs to Newton and Leibniz.

—SOPHUS LIE

The Book Shelf

EDITED BY R. H. MOORMAN

Dictionary of Astronomy and Astronautics, Armand Spitz and Frank Gaynor, Philosophical Library, Inc., (15 East 40th Street) New York 16, 1959, 439 pp., $6.00.

Over 2200 terms and concepts are defined and discussed in this dictionary, compiled by Armand Spitz, Coordinator of Visual Satellite Observations for the Smithsonian Astrophysical Observatory, Cambridge, Massachusetts, and Frank Gaynor, author of *Encyclopedia of Atomic Energy* and Contributing Editor of the *Encyclopedia Britannica.*

Astronomy is much more generously treated than is astronautics. Not only are there far more astronomical terms, but also these terms tend to be defined more clearly and completely. There are also quite a few terms from nuclear physics.

The book is highly informative and remarkably interesting. Along with all the scientific information, it purveys many intriguing bits of mythology and history as well as reassuring words, the last under such entries as "collision" and "cyanogen in comets."

Unfortunately, a number of the definitions and discussions are faulty. The grammar is not always above reproach, nor is the mathematics and physics. Some of the definitions as stated are meaningless, illogical, circular, or simply not clear. Examples are

> "matter . . . The atoms are composed of smaller particles, of three important kinds, protons, electrons and neutrons, which bear electrical charges which are positive, negative and neutral, respectively.
>
> These ultimate particles, with their electric charges, are all alike."
>
> "seasons . . . Differences in climate during the year at different latitudes on the earth."
>
> "selenocentric . . . Relating to the center of the moon; referring to the moon as a crater."

There are several obvious errors. Examples are

> "coordinates . . . Lines drawn perpendicular to two other lines that are usually perpendicular to each other, . . ."

> "hyperbola . . . It is an open curve with only one focus."
>
> "mass ratio . . . The ratio of the weight of a rocket to the weight of the fuel that it carries; it can be calculated by dividing the weight of the rocket carrying a full load by its weight when empty."

Although the preface mentions "lengthy years of work on this compilation," some of the faults seem to be due to haste and carelessness. There are quite a few misprints. Speed was no doubt necessary to keep a book in this rapidly moving field from becoming obsolete while being produced. The result is a timely volume, which, although not entirely reliable, contains much easily accessible information.

—MABEL S. BARNES
Occidental College

Principles and Techniques of Applied Mathematics, Bernard Friedman, John Wiley and Sons, Inc. (440 Fourth Avenue) New York 16, N.Y., 1956, 315 pp., $8.00.

The subject matter of this book has been used for several years as the basis of a one year course in the Graduate School of New York University. The student who expects to profit by reading it should be thoroughly familiar with such concepts as matrices, dyads, Lebesgue integration, Hilbert spaces, Green's Functions, and eigenvalues. This statement is not meant as a criticism of the book but is intended to indicate the level of the subject matter.

The author organized the book around two main themes. The first is to show how "the abstract theory of linear operators can be used to unify and systematize the techniques of applied mathematics," the second is to develop specific techniques which yield *explicit* solutions of partial differential equations. These themes are developed through theorems, examples, and problems. The organization of the material is good, the examples are well chosen, and the problems will present a real challenge to the serious reader. The book has considerable value because it presents a unified treatment of a body of material which has not been found in one place previously.

The use of "Rule" and "Method" as paragraph headings, and the use of appendices at the end of each chapter seems unfortunate. The use of "Rule" and "Method" reminds the reviewer of the old-fashioned "cook-book" texts in arithmetic and the appendices appear to be afterthoughts.

The book contains five chapters. In the first chapter, entitled "Linear Spaces," the author indicates two basic methods of solving the linear equation $Lx = m$ where L is a linear operator, m a given vector and x an unknown vector. In Chapters Two and Three these methods are developed in detail. Chapter Four is devoted to the theory of spectral representation of ordinary differential operators. In Chapter Five specific methods for solving equations of mathematical physics are developed.

The typography of the book is good. Very few misprints were found by the reviewer, and the printed material is not crowded.

—W. Toalson
Ft. Hays Kansas State College

Rocket, Sir Philip Joubert de la Ferté, Philosophical Library (15 East 40th Street) New York, 1957, 190 pp., $6.00.

The first part of this book is historical and traces the military development of rockets and rocket propulsion from the year 1232 down to 1944 when Hitler's V-1 and V-2 weapons were a threat to Britain's security. The latter part of the book attempts to analyze the military and political effects of long range rockets carrying atomic or thermo-nuclear warheads.

The author, Sir Joubert de la Ferté, Air Chief Marshal of Great Britain, is in a perfect position to draw facts and figures from official files. A large amount of data has also been gleaned from documents, diaries, and depositions of captured German scientists and military leaders. Taken together, these present an interesting account of the development of the V-1 and V-2 weapons as well as the counter measures taken against them. The story of Peenemunde is told dramatically albeit from the Allied viewpoint.

The V-1 was the flying bomb with the pulse (ram jet) motor. It can be likened to a slow-speed pilotless aircraft. Since it could be heard approaching, V-1 was hard on the nerves of its intended victims. The defense evolved against it included fighter planes, AA guns with proximity fuses, and a balloon barrage.

The V-2 was larger and a true rocket. Since its speed of flight was supersonic, there was no warning noise of its approach. Air Marshal Joubert admits there was no means of defense against it in England. Therefore it was necessary to destroy the missiles at their launching sites. The strategy and the missions that accomplished that are told in interesting detail.

It is stated that the V-1 and V-2 missiles would have changed the outcome of World War II had they become operational on their originally scheduled date. Hitler was not convinced of their real potential and delayed their development because of a dream he once had.

The author claims that as recently as 1935 there was no thought or research in England on the development of long range rocket missiles. Nor did they possess any knowledge of German activity along that line. In fact, when reports of German rockets trickled to England during the early days of World War II, they were roundly and openly scoffed at in high places. The manner in which this German veil of secrecy was finally penetrated and the launching sites revealed is a story within itself. It is an account of the unbelievable results obtained through the proper interpretation of aerial photographs.

In conclusion, attention is drawn to the many factors that affect the development of the defenses of the Free World in the years to come. Modern (1957) global military rockets and the strategy that might be employed in future wars are reviewed. Speculation as to the role of NATO and the March of Communism is aired. The book closes on a note of bewilderment that ". . .the U.S.A. which has done so much to protect the freedom of the world should have gone so far in becoming the biggest promoter of Russia's interests by backing the aspiration of every adventurer that harbors the flag of Nationalism."

—Ralph L. Dunckel
Tennessee Polytechnic Institute

"I have often admired the mystical way of Pythagoras and the secret magic of numbers."

—Sir Thomas Browne

Installation of New Chapters

EDITED BY MABEL S. BARNES

THE PENTAGON is pleased to report the installation of three new chapters.

NEW YORK GAMMA CHAPTER

State University of New York, Teachers College, Oswego, New York

New York Gamma Chapter was installed on May 21, 1959. The installation ceremony was conducted by Dr. Frank Hawthorne, Supervisor of Mathematics Education, New York State Education Department, and past National Historian of Kappa Mu Epsilon. It was held in the music room of the Union Building on the college campus and was followed by a dinner in the Union dining room. After the dinner Dr. Hawthorne presented a paper on "The Geometry of Projectiles."

Fifteen charter members were initiated. From the faculty were Mr. Gordon D. Mock, Dr. Roland F. Smith, Mr. John W. Walcott, and Mr. Fred W. Weiler. The students were Edith Fiske, Roger J. Friske, Walter J. Kersch, Patricia LeClair, Marilynn Nagy, Judith Patrick, Janette Scott, Ronald Silfer, Janice Stoutner, Lawrence Van Patten, and Norma Warchalaski. Dr. Emmet C. Stopher had already been initiated as a member of Kansas Epsilon Chapter at Fort Hayes Kansas State College.

The officers of New York Gamma are Judith Patrick, president; Lawrence Van Patten, vice-president; Patricia LeClair, recording secretary; Edith Fiske, treasurer; Dr. Roland F. Smith, corresponding secretary; and Dr. Emmet C. Stopher, faculty sponsor.

The College at Oswego is one of the rapidly growing units of the State University of New York. It has approximately 2200 students. For years it has trained general elementary teachers and specialists in the field of industrial arts. More recently, beginning with the fall semester of 1958-59, it was charged with the additional responsibility of training secondary teachers of mathematics and science. This new program, accompanied by an increase in the quantity and quality of mathematics, makes particularly timely the installation of the new chapter.

TENNESSEE BETA CHAPTER

East Tennessee State College, Johnson City, Tennessee

Tennessee Beta Chapter was installed on May 22, 1959, by Dr. Loyal F. Ollman, Chairman of the Division of Natural Sciences, Mathematics and Engineering, Hofstra College, and past National Treasurer of Kappa Mu Epsilon.

A banquet was held at Raymond's Restaurant. Bobby Lee McConnell, president of the Mathematics Club (the organization which had just become Tennessee Beta), served as toastmaster, and Lewis Waddell, the new president, gave the invocation. After being introduced by Dr. Lester Hartsell, Chairman of the Mathematics Department and a Kappa Mu Epsilon member of long standing, Dr. Ollman presented an interesting address on "The History and Activities of Kappa Mu Epsilon." Special recognition was given to Professor T. C. Carson, Chairman of the Mathematics Department from 1928 to 1958, and to Mrs. Joe McCormick, Assistant Professor of Mathematics, for their assistance in organizing the Math Club in 1953, and for sponsoring it since. Special guests representing the college were President and Mrs. Burgin E. Dossett; Miss Ella Ross, Dean of Women; and Dr. George Dove, Director of the School of Arts and Sciences.

The tables were decorated with arrangements of pink roses in silver bowls and ivy entwined about pink candles in silver holders. The pentagonal programs had rose covers lettered in silver.

Twenty-three charter members were initiated. The students were Charles Joe Allen, William Bowman, Callie Davis, Jane E. Davis, Anne K. DeVault, William Dickey, Andrew Francis, Robert Hale, Catherine Hillon, Francis Horne, Bobby Lee McConnell, Thomas Allen O'Dell, Royce E. Parman, and Lewis Waddell. The faculty were Miss Sally Pat Carson, Professor T. C. Carson, Miss Velma Cloyd, Mr. Ellison Jenkins, Mr. Stanford Johnson, Colonel Henry Linsert, Mrs. Joe McCormick, Mr. Robert Murdock, and Miss Vella Mae Smith.

The officers for the coming year are Lewis Waddell, president; William Bowman, vice-president; Anne K. DeVault, recording secretary; Robert Hale, treasurer; Catherine Hillon, historian; Mrs. Joe McCormick, corresponding secretary; Mr. Robert Murdock, faculty sponsor; and Professor T. C. Carson, honorary sponsor.

PENNSYLVANIA GAMMA CHAPTER

Waynesburg College, Waynesburg, Pennsylvania

The installation ceremony for Pennsylvania Gamma Chapter was held in the Fort Jackson Hotel on May 23, 1959, under the supervision of Professor Harry R. Mathias of Bowling Green State University, past National Vice-President of Kappa Mu Epsilon, with the assistance of the newly elected officers and of Dr. Lester T. Moston, Chairman of the Mathematics Department and Dean of Waynesburg College. Following the ceremony, the president receiving the charter from Professor Mathias and the new members were photographed.

The installation banquet was held in a private dining room. Several invited guests from the faculty and administration of Waynesburg College were present along with the members and their wives and friends. William Gardiner, as toastmaster, opened the program. President Paul R. Stewart of Waynesburg College in his inimitable manner reviewed the history of the Mathematics Department from its beginning. Dr. Moston continued with the history of Delta Pi Mu, the honorary mathematics fraternity which has just become the Pennsylvania Gamma Chapter of Kappa Mu Epsilon. Professor Mathias gave a talk on "Investing in Kappa Mu Epsilon," in which he acquainted the new members with the history, goals, and purposes of the society and pointed out what was expected of them.

The charter members were Louis Checchi, Kenneth Coley, Bert Craft, Wilma Franko, William Gardiner, Edmund Gwynne, Robert Haver, George Husk, James Klingensmith, Walter Lindsay, David Marks, Frank Matz, Thomas Munzak, Audrey Oberg, William Roos, Edward Sienicki, Paul Stewart, and Herbert Zaar; and from the faculty, Dr. Lester T. Moston and Mr. Arthur E. Stafford.

The officers of the new chapter are William Gardiner, president; Robert Haver, vice-president; Audrey Oberg, secretary; Walter Lindsay, treasurer; Dr. Lester T. Moston, faculty sponsor and corresponding secretary.

Waynesburg College is a liberal arts college, founded in 1849 and related to the Presbyterian Church. It is coeducational and has an enrollment of approximately 1500.

We are particularly happy to welcome three new chapters to our society. We wish each of them success in their activities.

Kappa Mu Epsilon News

EDITED BY FRANK C. GENTRY, HISTORIAN

The Twelfth Biennial Convention of Kappa Mu Epsilon was held May 7, 8, 9, 1959 in the Union of Bowling Green State University, with Ohio Alpha as host chapter.

Two hundred twenty three members and guests of Kappa Mu Epsilon attended the convention.

Chapter	No. of Students	No. of Faculty	Chapter	No. of Students	No. of Faculty
Alabama Beta	4	1	Michigan Beta	7	3
California Alpha	2		Michigan Gamma	2	
California Beta	1		Misouri Alpha	4	2
California Gamma	1		Missouri Beta	4	1
Colorado Alpha	1	1	Missouri Gamma	3	1
Illinois Alpha	2		Missouri Epsilon	11	2
Illinois Beta	2	2	Nebraska Alpha	12	3
Indiana Alpha	1		New Jersey Alpha	2	1
Indiana Beta	4	1	New York Alpha	2	1
Indiana Gamma	4		New York Beta	5	1
Iowa Alpha	5	1	N. Carolina Alpha	4	1
Kansas Alpha	2	3	Ohio Alpha	13	9
Kansas Beta	15	5	Ohio Gamma	6	2
Kansas Gamma	9		Oklahoma Alpha	3	1
Kansas Delta	3	1	Pennsylvania Alpha	6	
Kansas Epsilon	4	1	Texas Alpha	4	2
Louisiana Beta	1	1	Texas Epsilon	1	
Michigan Alpha	2	1	Wisconsin Alpha	6	

Guests: Bluffton College 12

FRIDAY, MAY 8, 1959

President C. C. Richtmeyer, National President opened the first general session in the Ballroom. Kenneth H. McFall, Provost, Bowling Green State University extended greetings and welcomed the convention members to the campus. Prof. R. G. Smith, National Vice-President, gave the response in behalf of Kappa Mu Epsilon. Margaret E. Martinson, Acting-Secretary, called the roll by chapters.

The petitions for New Chapters were read. After some discussion, it was voted to accept the following schools for chapters of Kappa Mu Epsilon: Eastern Tennessee State College, Johnson City, Tennessee; Nebraska State Teachers College, Kearney, Nebraska;

State University of New York Teachers College, Oswego, New York; Radford College, Radford, Virginia; Waynesburg College, Waynesburg, Pennsylvania.

At 10:00 a.m. the following student papers were read:

1. *Method of Least Squares*, Carol Cummings, Texas Alpha, Texas Technological College.
2. *So You're Going to Take a Chance*, Nancy Bowman, Missouri Beta, Central Missouri State College.
3. *Problems Whose Solutions Lead to Cycloids*, Theodore A. Mueller, Missouri Epsilon, Central College.
4. *The Four Dimensional Cube*, Norman Sellers, Kansas Beta, State Teachers College, Emporia.
5. *Matrices from a Set-Theoretic Viewpoint*, Phillip A. Griffiths, North Carolina Alpha, Wake Forest College.
12. *A Note on Pythagorean Triples*, Reginauld Mazares, Louisiana Beta, Southwest Louisiana Institute.

At 1:15, two informal discussion groups were held on the topic, "Let's Exchange Ideas." The student section met in the Ballroom while the faculty section met in the Pink Dogwood Room.

Following the discussion groups, the members boarded busses and cars in front of the Union for a trip to the Rossford Ordnance Depot.

At 6:00 p.m. the banquet was held in the Ballroom. Following announcements and introductions the group heard Professor Earl J. Mickle of Ohio State University speak on "Proofs in Mathematics."

SATURDAY, MAY 9, 1959

At 9:00 the following student papers were read:

6. *The Tree of Mathematics in the Light of Group Theory*, Patricia Nash, Kansas Gamma, Mount Saint Scholastica College.
7. *A Function with Range 0, −1, 0, 1*, Myron Williams, Indiana Gamma, Anderson College.
8. *Mathematics in the Fertile Crescent*, Molak Yunan, Ohio Alpha, Baldwin-Wallace College.
9. *Newton's Discovery of the Calculus*, Donna Jean Decker, Ohio Alpha, Bowling Green State University.

10. *On the Rank of a Matrix,* W. Stephen Zimmerman, Kansas Alpha, State Teachers College, Pittsburg.
11. *Poetical Mathematics,* Sarah Smith, Missouri Beta, Central Missouri State College.

At 11:00 Dr. Richtmeyer opened the second business meeting by calling for the reports of the National Officers.

Professor Keith Moore reported that the Auditing Committee (Profesor Keith Moore, Michigan Alpha; Professor Sherralyn Craven, Missouri Beta; Professor W. M. Perel, Texas Alpha) found the Treasurer's books correct and in excellent order. Professor Tucker moved that the report be accepted. The motion was seconded and carried.

Three invitations for the 1961 convention were extended by: Donald Dittmer of Missouri Beta, Ronald Nelson of Kansas Epsilon, and Karen Shaw of Kansas Beta.

At 1:30 the third business meeting was called to order by President Richtmeyer. Professor Charles Tucker gave the report of the Nominating Committee (Professor Charles Tucker, Kansas Beta; Professor Harry Mathias, Ohio Alpha; Professor E. H. Matthews, Missouri Alpha) and presented the following nominations:

President:	Carl Fronabarger, Missouri Alpha
	Loyal F. Ollman, New York Alpha
Vice-President:	Ronald Smith, Kansas Alpha
	Fred Sparks, Texas Alpha
Secretary:	Laura Greene, Kansas Delta
Treasurer:	Walter C. Butler, Colorado Alpha
	Basil Gillam, Iowa Beta
Historian:	Raymond Carpenter, Oklahoma Alpha
	Frank C. Gentry, New Mexico Alpha

President Richtmeyer asked for further nominations from the floor. There were none. The slate was accepted and ballots were distributed to the voting delegates.

Dr. Jerome Sachs gave the report of the Awards Committee (Professor Jerome Sachs, Illinois Alpha; Professor Elizabeth Glass, New York Beta; Professor Roland Lenz, Nebraska Alpha; Alan Franz, Indiana Alpha; Robert Austin, New York Beta; Charles Barnett, Nebraska Alpha). The committee presented a copy of *The Mathematics Dictionary* to each of the following:

First Place: Patricia Nash, Kansas Gamma, for *The Tree of Mathematics in the Light of Group Theory.*

Second Place: Phillip A. Griffiths, North Carolina Alpha, for *Matrices from a Set-Theoretic Viewpoint.*

Third Place: Norman Sellers, Kansas Beta, for *The Four-Dimensional Cube.*

Honorable Mention was given to:

Theodore Mueller, Missouri Epsilon, for *Problems Whose Solutions Lead to Cycloids.*

Sara Smith, Missouri Beta, for *Poetical Mathematics.*

Professor Tucker reported the election of the following officers for the next biennium: President, Carl Fronabarger, Missouri Alpha; Vice-President, Ronald G. Smith, Kansas Alpha; Secretary, Laura Z. Greene, Kansas Delta; Treasurer, Walter C. Butler, Colorado Alpha; Historian, Frank C. Gentry, New Mexico Alpha. Dr. Richtmeyer installed the newly-elected officers.

Professor J. D. Haggard of Kansas Alpha representing the Resolutions Committee (Professor John Burger, Kansas Beta; Professor Wilbur Ehrich, Missouri Epsilon; Gerald Linn, Kansas Beta; Richard Moreland, Missouri Epsilon; Patricia Nash, Kansas Gamma) presented the following report of the Resolutions Committee.

Whereas this, the Twelfth Biennial Convention of Kappa Mu Epsilon assembled, finds absent from its sessions three persons who for so many conventions past have been in the forefront of its leadership, be it resolved that through the national secretary we convey to Miss Laura Z. Greene, Miss E. Marie Hove, and Sister Helen Sullivan our regret that they are unable to attend this convention and our hope that the circumstances retaining each will soon be removed.

Whereas the activities of this convention have far surpassed the expectation of each of us, be it resolved that we express our appreciation:

1. To the host chapter, Ohio Alpha, and to Bowling Green State University, for their fine hospitality, for their provision of excellent facilities for our convention, for the food and refreshments, and all the many things that contribute to the success of a meeting such as this.

2. To each of the national officers whose work always in addition to a full load of responsibilities elsewhere provides the direction and continuity that has kept KME a growing fraternity. Especially to retiring President C. C. Richtmeyer for his untiring efforts over the past four years, and to Carl Fronabarger for his outstanding editorship of THE PENTAGON for the past six years.

3. To Professor Earl J. Mickle for an interesting, entertaining and provocative banquet address on "Proofs in Mathematics."

4. To the Editor and staff of THE PENTAGON whose work has contributed so much to our pride in KME.

5. To the many students who prepared and presented papers which contributed so significant a part to our program.

6. To all those and to many more unnamed who have worked in many ways to make this convention the success it has been, the delegates to the Twelfth Biennial Convention of Kappa Mu Epsilon say "Thank you."

After the acceptance of the committee report, the meeting adjourned.

REPORT OF THE NATIONAL PRESIDENT

The recent explosion of college enrollments, and the tremendous increase in interest in the fields of science and mathematics have been reflected in the growth and activities of our society.

Two years ago, I predicted that we would have over 1500 initiates in the 1957-59 biennium. The fact is that we have initiated 1812 into KME during this biennium as compared with 1328 in 1955-57, and 1168 in 1953-55. With the accelerating enrollments and the addition of new chapters it would not surprise me if the next biennium showed 2500 initiates.

Since our last convention at Pittsburg, Kansas, we have installed two new chapters, New York Beta at Albany in May, 1957, and California Gamma at San Luis Obispo in May, 1958. We now have 52 active chapters and the five additional chapters you have approved will bring the total to 57.

With continued growth the work of your national officers becomes correspondingly greater. At the last convention, the national council authorized some clerical help for the national secretary and the business manager of THE PENTAGON. It is probable that this clerical assistance will need to be increased for these and other national officers. In the not too distant future we may need to employ a part-time or full-time executive secretary to handle many of the details of the national organization.

As the number of chapters increases, we should give increased attention to regional meetings in the even-numbered years. One such regional meeting was held last Spring at Emporia, including chapters from Kansas, Missouri, and Nebraska. I hope that chapters in other areas will get together for a regional convention next spring.

In 1957, Kappa Mu Epsilon and the Science Teaching Improvement Program of the American Association for the Advancement of Science cooperated in sponsoring an essay contest on the subject, "Opportunities in Teaching Mathematics in Secondary Schools." Some of these essays have appeared in recent issues of THE PENTAGON. We might give some thought to the possibility of other contests and awards.

As you will note from the treasurer's report we are in a very solid financial condition. I suggest that some consideration be given to the establishment of a scholarship or a loan fund for worthy students to do graduate work in mathematics.

I should like to express my appreciation to the many people who have made this biennium a successful one for the society. In particular, I should like to commend Miss Greene and Mr. Madison who have been very efficient in carrying out the duties of the important offices of National Secretary and National Treasurer. My thanks also to Mr. Hawthorne for his work as National Historian, and to Mr. Smith, the National Vice President, who did an excellent job of organizing the program of student papers. I am especially grateful to past-president Charles Tucker for his helpful counsel and advice during my term of office. To Mr. Fronabarger, Editor of THE PENTAGON, and Mr. Waggoner, the business manager, my appreciation for a job well done.

I also wish to thank all of the corresponding secretaries and faculty sponsors who have given so much of their time to carrying on the work of the society at the chapter level. It is here that the real work of the organization is done and without the cooperation and diligent effort of these people, the society would not flourish.

To all of you my sincere thanks for having helped make possible a successful biennium for Kappa Mu Epsilon.

—CLEON C. RICHTMEYER

REPORT OF THE NATIONAL SECRETARY

Since the last convention of Kappa Mu Epsilon two new chapters have been installed. California Gamma was installed at California Polytechnic College, San Luis Obispo, May 23, 1958. Professor Dana Sudborough, past business manager of THE PENTAGON, served as installing officer.

New York Beta was installed May 16, 1957, at New York State College for Teachers at Albany by Mr. Frank Hawthorne, National Historian.

Kappa Mu Epsilon now has 52 active chapters and four inactive chapters, making a total of 56 chapters in 24 states. The total membership is now 14,130.

Each month we receive inquiries about the establishment of new chapters.

I appreciate very much the cooperation of all the corresponding secretaries in making the initiation reports. Your efficient work makes the work of the secretary much easier.

—Laura Z. Greene

REPORT OF THE BUSINESS MANAGER OF THE PENTAGON

I would like to take a few minutes to tell you some of the duties that go with the title Business Manager of THE PENTAGON. My most important responsibility, of course, is to see that each person who is entitled to receive our national magazine does so. To meet this responsibility I have two address cards on file for each subscriber. One card is filed alphabetically by name, the other card is filed by states according to the expiration date of the subscription. I endeavor to have on these address cards the correct, permanent address of each subscriber.

While I am on the subject of address cards, I would like to make what seems to be a biennial plea that you notify the business manager of any change of address on your subscription. Because of the recent raise in postal rates, it is more imperative than ever that I have the correct address before mailing a PENTAGON. It costs eight cents to mail a single PENTAGON. If the magazine is returned, it costs another eight cents. If the Postmaster supplies me with a change of address, it then costs still another eight cents to re-mail the PENTAGON. Including the cost of mailing envelopes, over one-half the cost of a magazine is expended in mailing it. If the Postmaster cannot supply a change of address, when the subscriber fails to do so, the subscriber's cards are pulled from the files, and he no longer receives the publication to which he is entitled.

In addition to the bulk mailing of each issue, as subscriptions to THE PENTAGON are received from the National Secretary or other sources, copies of THE PENTAGON are mailed until a reserve of 100 copies is reached. This spring, for example, 2300 PENTAGONS are being printed which leaves 300 magazines to be mailed at times other than the bulk mailing. Other duties of the Business Manager include filing sales tax returns with the State of Michigan, paying all bills for the PENTAGON except printing the magazine and the bulk mailing under our postal permit, which are paid by the National Treasurer, keeping a record of all financial transactions of the PENTAGON, and answering letters of inquiry concerning the magazine.

I thought perhaps that you would be interested in some data concerning your national magazine. It is mailed to every state in the union except Vermont, Delaware, North Dakota, Idaho, and Hawaii. The PENTAGON goes to ten foreign countries and two territories of the United States. More PENTAGONS are mailed to Kansas than any other state. The other five states to which more than 100 magazines are mailed ranked according to number of PENTAGONS sent to that state are California, Illinois, Texas, New York, and Missouri. As a Michigander, I should report that the next state in frequency of copies mailed is Michigan.

A word about complimentary copies. The library of every college which has a chapter of Kappa Mu Epsilon receives a complimentary copy. Each student who presented a paper at the Twelfth Biennial Convention will have his subscription extended two years. The authors of articles printed in the PENTAGON receive five copies of the issue in which their article is printed. Upon request of the President of Kappa Mu Epsilon to stimulate interest in the PENTAGON, sample copies are sent to individuals and mathematics clubs.

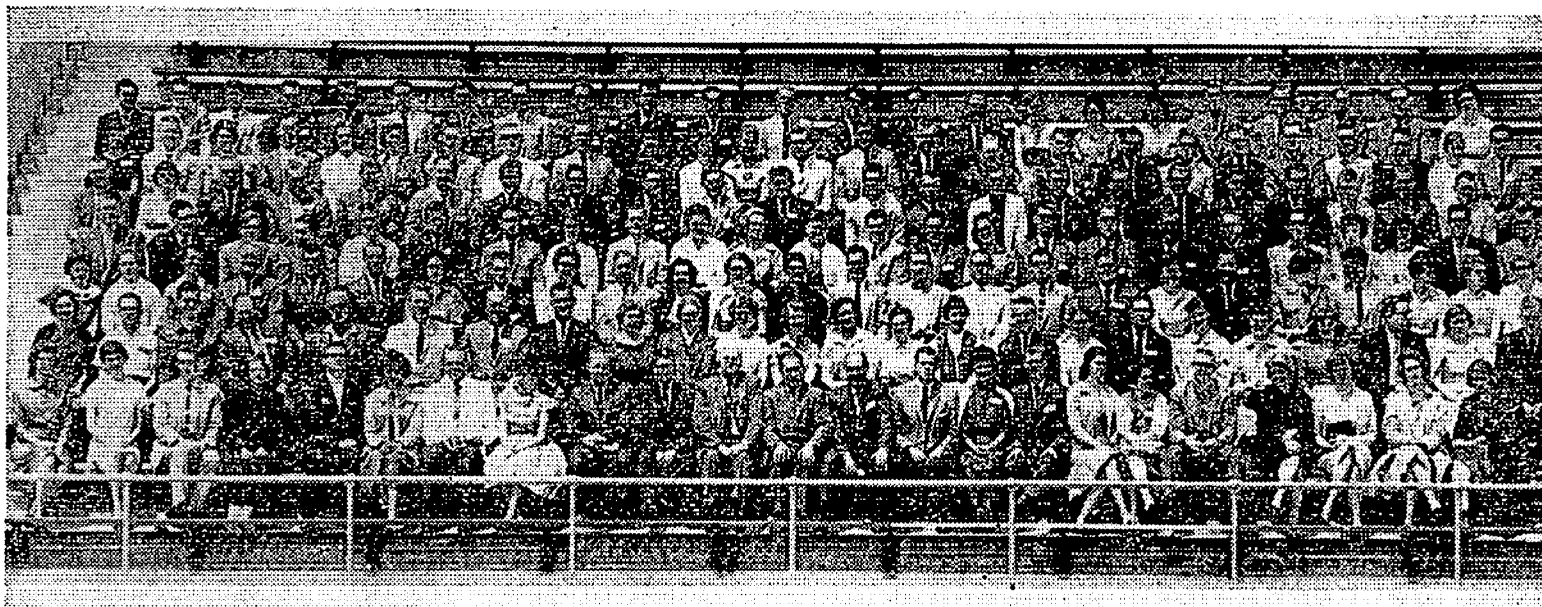
—WILBUR WAGGONER

FINANCIAL REPORT OF THE NATIONAL TREASURER

April 4, 1957 to April 3, 1959

Cash on hand April 4, 1957			$5828.64
Receipts from chapters			
Initiates (1812 at $5.00)	$9060.00		
Miscellaneous (Supplies, installations, etc.)	144.68		
Total receipts from chapters		$9204.68	
Miscellaneous receipts			
Interest on bonds and savings account	$287.65		
Balfour Company (Commissions)	157.50		
Sale of Ritual	1.95		
The Pentagon (Surplus)	226.86		
Total Miscellaneous Receipts		673.96	
Total Receipts			9878.64
Total receipts plus cash on hand			$15707.28
Expenditures			
National Convention, 1957			
Paid to chapter delegates	$1359.29		
Officers Expenses	512.09		
Miscellaneous (Prizes, host chapter, expenses, programs, etc.)	103.18		
Total Nation Convention		$1974.56	
Balfour Company (membership certificates, stationery, etc.)		988.11	
Pentagon (Printing and mailing four issues)		3764.78	
Placed In Savings Account		2627.25	
Installation Expense		58.30	
National Office Expense		496.34	
Total Expenditure			$9909.34
Cash Balance on Hand April 3, 1959			5797.94
Total Expenditure Plus Cash On Hand			$15707.28
Bonds On Hand April 3, 1959	$3000.00		
In Savings Account April 3, 1959	2627.25		
Total		$5627.25	
Total Assets as of April 3, 1959			$11425.19
Total Assets 1957			8828.64
Net Gain for Period			$2596.55

—M. LESLIE MADISON



Kappa Mu Epsilon Convention, May 8-9, 1959 - Bowling Green State University