The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before December 31, 2022. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2022 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 891-900.

Problem 891. *Proposed by José Luis Díaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain*.

Let $1 \leq m, n \leq 2022$ be integers such that $(n^2 - mn - m^2)^2 = 1$. Determine the maximum value of $m^2 + n^2$.

Problem 892. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Find $\Omega = \int_{15\cos(2\arctan 3)}^{20\sin(2arctan\ 3)} \frac{sin^3 x + sin^5 x}{1 + cos^2 x + cos^4 x}\, dx$

Problem 893. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Solve for real numbers *x*,*y*,*z* the equations:

$$\sin x + \sin y = \sqrt{2 + 2\sin z}$$
$$\sin y + \sin z = \sqrt{2 + 2\sin x}$$
$$\sin z + \sin x = \sqrt{2 + 2\sin y}$$

Problem 894. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

(1) Prove that for any natural number n, the polynomial
$f_n(x) = [A(x)]^n - [B(x)]^n - [C(x)]^n + [D(x)]^n$ is divisible by
$g(x) = 2x^2 + x - 1$ where
$A(x) = 2x^3 + 10x^2 - 11x + 4$
$B(x) = x^2 - 2x + 2$
$C(x) = -x^3 + 14x^2 - 3x + 5$
$D(x) = 7x^3 + 8x^2 + 4$

(2) Prove that for any natural number n, the quantity
$\alpha_n = (-292)^n - 82^n - 1437^n + (-3068)^n$ is divisible by 119.

(3) Prove that for any natural number n, the quantity
$\beta_n = 65^n - 82^n - 9^n + 502^n$ is divisible by 119.

Problem 895. *Proposed by Vasile Mircea Popa, Lucian Blaga University, Sibiu, Romania.*

Calculate the following integral:

$$\int_0^\infty \frac{\sqrt{x}\ln x}{x^4 + x^2 + 1}\, dx$$

Problem 896. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

If $a,b,c,d,x_i y_i > 0$, for all i=1,…,m $x_{m+i} = x_i$ with $m \geq 2$ prove that

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{m}\frac{(ax_i + bx_{i+1})^2}{cx_{i+2} + dy_k}\right) \geq \frac{(a+b)^2 n^2 X_m^2}{cnX_m + dmY_n}$$

Where $X_m = \sum_{i=1}^{m} x_i$ and $Y_n = \sum_{k=1}^{n} y_k$

Problem 897. *Proposed by Toyesh Prakash Sharma (student), Agra College, Agra, India.*

With F_n and L_n being the Fibonacci and Lucas numbers, show that when $n > 0$

$$\frac{F_n^{F_n} + F_{n+1}^{F_{n+1}} + L_n^{L_n} + L_{n+1}^{L_{n+1}}}{4} \geq \left(\frac{F_{n+3}}{2}\right)^{\frac{F_{n+3}}{2}}$$

Problem 898. *Proposed by Florică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania.*

Let (b_n) be defined $b_n=\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!}}-\frac{n^2}{\sqrt[n]{n!}}$ Bătineţu's sequence. Find

$$L=\lim_{n\to\infty}\left(1+\frac{b_n}{n}\right)^{\frac{1}{n^{n-2}}\sum_{k=0}^{n}\frac{n^k}{k+1}\binom{n}{k}}$$

Problem 899. *Proposed by Mihaly Bencze, Brasov, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Solve in the the set of real numbers the following equation

$$8x^3+17x+\frac{4}{x}+\log_2^2\left(x+\frac{4}{x}\right)=x^4+20x^2+4+2^{-x^2+4x-2}$$

Problem 900. *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*

Prove that in any triangle ABC, the following inequality holds $\frac{2r}{R}+\sum\frac{a^2}{bc}\geq 4$.