The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before November 1, 2018. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2018 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 808-819

Problem 808. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a, b, c \in [1, \infty)$ then

$$\frac{e^{a+b+c}}{e^{b/a+c/b+a/c}} \leq a^b b^c c^a \leq \frac{e^{ab+bc+ca}}{e^{a+b+c}}$$

Problem 809. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Prove that if $a, b, c \in (2, \infty)$ then

$$\sqrt{2}\sum(\sqrt{a(b-2)}+\sqrt{b(a-2)}) < 3\sqrt{abc}$$

Problem 810. *Proposed by Daniel Sitaru, "Theodor Costescu" National Economic College, Drobeta Turnu – Severin, Mehedinti, Romania.*

Compute $L = \lim_{n\to\infty}\frac{1}{n}\int_1^n \frac{x^4+4x^3+12x^2+9x}{(x+3)^5-x^5-243}dx$

Problem 811. *Proposed by Jose Luis Diaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*

Let a, b, c be three positive integers. Prove that

$$\sqrt[ab+bc+ca]{a^{bc}b^{ca}c^{ab}} \leq \sqrt[3]{abc}$$

Problem 812. *Proposed by Jose Luis Diaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*

Let S be a finite set. Consider three partitions of it each one with n elements: $A_1,\ldots, A_n$; $B_1,\ldots, B_n$; $C_1,\ldots, C_n$. If for all $1\leq i, j, k \leq n$, it holds that $|A_i \cap B_j| + |B_j \cap C_k| + |C_k \cap A_i| \geq n,$ then prove that $|S| \geq \frac{n^3}{3}$. When does equality hold?

Problem 813. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Prove that if $a, b, c \in (1, \infty)$ then

$$\log_{ab^2c^2} a + \log_{a^2bc^2} b + \log_{a^2b^2c} c \geq \frac{3}{5}.$$

Problem 814. *Proposed by D.M. Batinetu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Let the sequence $(a_n)_{n\geq 1}$, defined by $a_1 = 1$, $a_{n+1} = (n+1)!\, a_n$ for any positive integer n. Compute $\lim_{n\to\infty} \frac{\sqrt[2n]{n!}}{\sqrt[n^2]{a_n}}$.

Problem 815. *Proposed byMarius Dragan, National College Mircea cel Batran, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzau, Romania.*

Let a, b, c be positive real number such that $a^4 + b^4 + c^4 = 3$. Show that

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq (a^2b^2 + b^2c^2 + c^2a^2 + abc)^2$$

Problem 816. *Proposed by Stanescu Florin, Serban Cioclescu School, Gaesti, Romania.*

Determine the largest real number α with the property that for any function $f:[0,1] \rightarrow [0,\infty)$ where the following hold:

i) f is convex and $f(0) = 0$;

ii) there exists $\varepsilon > 0$ with f differentiable on $[0,\varepsilon)$ and $f'(0) \neq 0$;

the following inequality holds:

$$\int_0^1 \frac{x^2}{\int_0^x f(t)dt} dx \geq \int_0^1 (x+\alpha)\cdot \frac{f(x)}{\left(\int_0^1 f(t)dt\right)^2} dx$$

Problem 817. *Proposed by Stanescu Florin, Serban Cioclescu School, Gaesti, Romania.*

Consider the complex numbers a, b, c, d all of modulus one which have the following properties:

a) $\arg a < \arg b < \arg c < \arg d$;

b) $2\sqrt{|(b+ai)(c+bi)(d+ci)(a+di)|} - \sqrt{\frac{[(a-c)(b-d)]^2}{abcd}} = 4.$

Show that $\max\left\{\left|1-\frac{i+4a^2}{i+8a(b-d)}\right|;\left|1-\frac{i+4b^2}{i+4b(a-c)}\right|\right\} \geq \frac{4}{\sqrt{17}}$

Problem 818. *Proposed by Titu Zvonaru, Comanesti, Romania and Neculai Stanciu, "George Emil Palade", Buzau, Romania.*

Determine all positive integers a,b,c,d,x,y,z,t such that $a \neq b \neq c \neq d$ and $a + b + c = td$, $b + c + d = xa$, $c + d + a = yb$, $d + a + b = zc$.

Problem 819. *Proposed by the editor.*

Billy has created a snowman with a base, torso, and a head all of which are spheres where the torso is smaller than the base and the head is smaller than the torso. The radius of each piece is a positive integer with the base having a 12 inch radius. He decides that he wants to use the snow to make 3 similar and identical snowmen for his younger siblings. He discovers that he can do this. What are the radii of the original and smaller snowmen?