The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before September 1, 2016. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall 2016 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859) 622-3051)

NEW PROBLEMS 769-779

Problem 769. *Proposed by the Northwest Missouri State University Problem Solving Group, Maryville, MO.*

Let $T_k = \dfrac{k(k+1)}{2}$ be the k^{th} triangular number.

(1) Under what condition(s) on $n \in \mathbb{N}$ does 13 divide $2(T_{3^n} - 1)$?

(2) Under what condition(s) on $n \in \mathbb{N}$ does 13 divide $2T_{3^n} + 1$?

Problem 770. *Proposed by Jose Luis Diaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*

Let f : [0,1] →ℝ be a continuous concave function. Prove that

$$\frac{3}{4}\int_0^{1/7} f(t)dt + \frac{1}{12}\int_0^{2/7} f(t)dt \le \frac{2}{3}\int_0^{3/14} f(t)dt$$

Problem 771. *Proposed by Jose Luis Diaz-Barrero, School of Civil Engineering, Barcelona Tech - UPC, Barcelona, Spain.*

Let $a < b$ be positive real numbers and let f : [a,b] →ℝ be a continuous function. Prove that there exists $c \in (a, b)$ such that

$$2f(c)=\frac{1}{\sqrt{c}}\left[\frac{\sqrt{a}+\sqrt{c}}{a-c}+\frac{\sqrt{b}+\sqrt{c}}{b-c}\right]\int_a^c f(t)dt$$

Problem 772. *Proposed by Marcel Chirita, Bucharest, Romania.*

Solve in positive integers the equation $x^2 - 97y! = 2015$.

Problem 773. *Proposed by Marcel Chirita, Bucharest, Romania.*

Let a, b, c be real numbers greater than or equal to 3. Prove that

$$\min\left(\frac{a^2b^2+3b^2}{b^2+27},\frac{b^2c^2+3c^2}{c^2+27},\frac{c^2a^2+3a^2}{a^2+27}\right)\le\frac{abc}{9}$$

Problem 774. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

If both x and y are positive real numbers, then find y as a function of x, provided

$$y' + (y+1)\ln(y+1)[1-(\ln(y+1))^{-2}((1/4)x^{-1} + x^{1/2})]x^{1/2} = 0.$$

Problem 775. *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Determine y explicitly as a function of x provided

$$x(1 + \sin x)y' + [\,(x^2 + y^2 + 4) - (\,-3 + \sin x)\,y - 2(1 + \sin x)] = 0$$

$y \neq -2$ and $x \neq k\pi$.

Problem 776. Proposed by Natanael Karjanto, University College, Suwon, Republic of Korea.

Show that for $\alpha > 0$ and $n \in \mathbb{N}$, the harmonic number H_n can be represented by the following integral:

$$H_n=\sum_{k=1}^{n}\frac{1}{k}=\frac{1}{2}\sum_{k=1}^{n}\left(\int_{-\infty}^{\infty}e^{-\alpha|x|}\operatorname{sech}^{k+1}x\,dx+\frac{\alpha-(k-1)}{n}\int_{-\infty}^{\infty}e^{-(\alpha+1)|x|}\operatorname{sech}^{k}x\,dx\right)$$

Problem. 777. *Proposed by Robert Gardner and William Ty Frazier (graduate student), East Tennessee State University, Johnson City, TN.*

Let $[x]$ represent the floor (or greatest integer) function. Let $n, m \in \mathbb{N}$ with $2 \le m \le n-1$ and $k \in \{0, 1, 2, ..., m-1\}$. Use the floor function to express the smallest integer N greater than or equal to n which is congruent to k modulo m.

Problem 778. *Proposed by Thomas Chu (graduate student), Western Illinois University, Macomb, IL.*

Let p_1 and p_2 be distinct odd primes both congruent to 1 or 3 mod 4. Prove that

$$\gcd\left(\frac{p_1+p_2}{2}, \frac{|p_1-p_2|}{4}\right)=1$$

Problem 779. *Proposed by the editor.*

Use all the digits 1,2,3,…,9 without repeats to create two primes such that their product is a maximum. Each digit should be used in only one of the two numbers.